Appendix B: Fourier Series

This appendix summarises the basic facts of Fourier series and interelates its various forms so that the reader understands how to recover the amplitude and phase of a Fourier component in each representation.

The most general form of the Fourier series in one dimension describing a function f(x) is

$$f_s(x) = \sum_{n=0}^{\infty} D_n \cos(nx + \Phi_n) \qquad (1)$$

so the function $f_s(x)$ represents the object f(x) on the range $[-\pi,+\pi]$ and is periodic with period 2π. The meaning of D_n and Φ_n is explained in appendix one. The equation for $f_s(x)$ may be expanded to give

$$f_s(x) = \sum_{n=0}^{\infty} D_n \cos(nx) \cos(\Phi_n) - D_n \sin(nx) \sin(\Phi_n)$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx) \qquad (2)$$

where A_0 is often referred to as the constant or `DC' term in the series and

$$A_n = D_n \cos(\Phi_n) \qquad (3)$$

$$B_n = -D_n \sin(\Phi_n) \qquad (4)$$

The amplitude D_n and phase Φ_n of each Fourier component may be found from

$$D_n = (A_n^2 + B_n^2)^{[1/1]} \qquad (5)$$

$$\tan(\Phi_n) = -\frac{B_n}{A_n} \qquad (6)$$

and the specific values of A_n and B_n for a particular object function f(x) are found from the integrations

$$A_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx)\, dx \qquad (7)$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin(nx)\, dx \qquad (8)$$

(The factor of one half in front of the DC term allows these equations to be generally applicable for all `n'.)

A complex Fourier series may be formed by setting

$$C_n = \frac{1}{2} (A_n - iB_n) \qquad (9)$$

to form a series representation of f(x) described by

$$f_s(x) = C_n e^{+i\, nx} \qquad (10)$$

Alternatively one may calculate C_n directly via the equation

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, e^{-i\,nx}\, dx \tag{11}$$

The complex form of the series thus incorporates A_n and B_n into the REAL and IMAGINARY parts of one complex amplitude C_n. Knowing this, one may easily recover the amplitude and phase of the Fourier component from the complex representation. In fact

$$D_n = 2\, |\, C_n\, | \tag{12}$$

with the factor of 2 arising due to the summation over $-\infty$ to $+\infty$ in the complex series but only from 0 to $+\infty$ in the 'standard' series representation. From equation 2.10 one sees that the phase of C_n is identical to Φ_n, and an advantage of the complex Fourier series is this clear identification of the phase of the component and the linear relationship between D_n and $|C_n|$.

Two Dimensional Series

Fourier's theorem may be extended into two dimensions to express a two dimensional function as a linear combination of 2-D sinusoidal basis functions. The series representation of f(x,y) is then

$$f_s(x,y) = \frac{1}{2} A_0 + \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} A_{mn} \cos(\, mx + ny + \Phi_{mn}\,) \tag{13}$$

which may be expanded as in the 1-D case, but this is not explicitly done here. The amplitude and phase of each component are found by performing the integrations

$$A_{mn} \cos(\, \Phi_{mn}\,) \;=\; \frac{1}{4\pi^2} \int\int_{-\pi}^{+\pi} f(x,y) \cos(\, mx + ny\,)\, dx\, dy \tag{14}$$

$$A_{mn} \sin(\, \Phi_{mn}\,) \;=\; -\frac{1}{4\pi^2} \int\int_{-\pi}^{+\pi} f(x,y) \sin(\, mx + ny\,)\, dx\, dy \tag{15}$$

$$A_{00}: \;=\; \frac{1}{4\pi^2} \int\int_{-\pi}^{+\pi} f(x,y)\, dx\, dy \tag{16}$$

from which A_{mn} and Φ_{mn} may be found.

Square Wave series: Evaluation

Functions may be represented on other ranges by the appropriate scaling of the cosine argument to be [(nπx)/L] where $f_s(x)$ is now periodic with period 2L, so that (in 1-D),

$$f_s(x) = \sum_{n=-\infty}^{+\infty} C_n\, e^{+i\,[(n\pi x)/L]} \tag{17}$$

where

$$C_n = \frac{1}{2L} \int_{-L}^{+L} f(x)\, e^{-i\,[(n\pi x)/L]}\, dx \tag{18}$$

A square wave with period 2L has, in general, pulses of width `a' separated by a distance `s', so that 2L=a+s. Figure B.1 shows a section of the square wave.

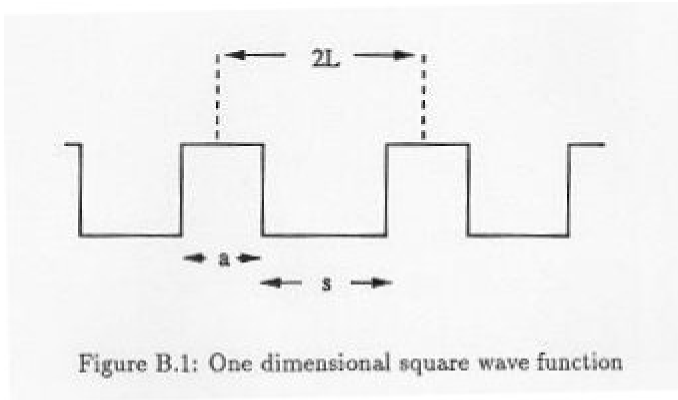

Figure B.1: One dimensional square wave function

The Fourier coefficients C_n are found from the integration

$$C_n = \frac{1}{2L} \int_{-[a/2]}^{+[a/2]} H e^{-i [(n\pi x)/L]} dx \tag{19}$$

where H represents the height of the pulse. This integration reduces to

$$H \frac{a}{n\pi} \sin\left(\frac{n\pi a}{2L} \right) \tag{20}$$

and, by dividing and multiplying by [2L/a], can be written as

$$C_n = H \frac{a}{2L} \operatorname{sinc}\left(\frac{na}{2L} \right) \tag{21}$$

File translated from TEX by TTH, version 3.02.
On 27 Oct 2001, 23:42.