THE UNIVERSITY of EDINBURGH

This thesis has been submitted in fulfilment of the requirements for a postgraduate degree (e.g. PhD, MPhil, DClinPsychol) at the University of Edinburgh. Please note the following terms and conditions of use:

- This work is protected by copyright and other intellectual property rights, which are retained by the thesis author, unless otherwise stated.
- A copy can be downloaded for personal non-commercial research or study, without prior permission or charge.
- This thesis cannot be reproduced or quoted extensively from without first obtaining permission in writing from the author.
- The content must not be changed in any way or sold commercially in any format or medium without the formal permission of the author.
- When referring to this work, full bibliographic details including the author, title, awarding institution and date of the thesis must be given.

Central Extensions of Current Groups and the Jacobi Group

Pamela Jane Docherty

Doctor of Philosophy
University of Edinburgh
October 9, 2012

Abstract

A current group G^X is an infinite-dimensional Lie group of smooth maps from a smooth manifold X to a finite-dimensional Lie group G, endowed with pointwise multiplication. This thesis concerns current groups G^Σ for compact Riemann surfaces Σ. We extend some results in the literature to discuss the topology of G^Σ where G has non-trivial fundamental group, and use these results to discuss the theory of central extensions of G^Σ. The second object of interest in the thesis is the Jacobi group, which we think of as being associated to a compact Riemann surface of genus one. A connection is made between the Jacobi group and a certain central extension of G^Σ. Finally, we define a generalisation of the Jacobi group that may be thought of as being associated to a compact Riemann surface of genus $g \geq 1$.

Declaration

I do hereby declare that this thesis was composed by myself and that the work described within is my own, except where explicitly stated otherwise.

Pamela Docherty
2012

Acknowledgements

This thesis would not have been possible without the help and support of many people. I would like to express my gratitude to my supervisor, Harry Braden, for his continued guidance, support and patience. I am grateful for his attention to detail and constructive criticisms while proof-reading the thesis. I would also like to thank Marco Bertola for helpful discussions via email. I am grateful for the pastoral advice from my second supervisor, Iain Gordon. I am indebted to the Engineering and Physical Science Research Council and to the School of Mathematics for the generous financial support, as well as many of my colleagues in the School of Mathematics, in particular Gill Law, who is always extremely willing to help PGs.

On a personal level, thanks are also due to my wonderful family and friends. To my many fellow PG students over the years: Elena, Daniele, Rosemary, Richard, David, Patricia, Antonella, George, Tim, Eric, Noel, Hari, Amy, Patrick to name but a few. I have been so lucky to have made such good friends over the past five years, for both the social aspect and the excellent advice and encouragement they have given me. In particular thanks to Paul and Spiros who proof-read some of the thesis and made helpful comments, and are always eager to talk maths. Thank you also to Helen, Sandy, Jane and Mhairi. Still mathematicians at heart, they encouraged me when the going got tough. Thanks also go to Colin, who made the final few months of writing a little more enjoyable.

Finally, I would like to give special thanks to my family, in particular my parents, for their continuing support and encouragement in everything I do; without their help and belief in me, I doubt that I would have made it this far.

Contents

List of symbols

Throughout the thesis, $\cdot$ denotes group multiplication, $\diamond$ denotes an action of a group element and $\circ$ denotes composition of maps.

Symbol	Meaning	Location
Σ	A compact Riemann surface of genus g	
E	A compact Riemann surface of genus one, an elliptic curve	
G	A simple, connected, compact or complex, finite-dimensional Lie group	
$\mathfrak{g}$	A finite-dimensional simple Lie algebra, $\mathfrak{g} = \mathrm{Lie}(G)$	
K	A (possibly infinite-dimensional) Lie group	
$\mathfrak{k}$	A (possibly infinite-dimensional) Lie algebra, $\mathfrak{k} = \mathrm{Lie}(K)$	
G^Σ	The current group $\mathrm{Map}_*(\Sigma, G)$	(1)
$\mathfrak{g}^X$	The Lie algebra of G^X; the current algebra $\mathrm{Map}(\Sigma, \mathfrak{g})$	(3.2)
UK	The universal central extension of K by $Z = H_2(K)$	
$U\mathfrak{k}$	The universal central extension of $\mathfrak{k}$ by $\mathfrak{z} = H_2(\mathfrak{k})$	(2.17)
LG	The loop group $\mathrm{Map}(S^1, G)$	
$L\mathfrak{g}$	The loop algebra $\mathrm{Map}(S^1, \mathfrak{g})$	(3.1)
$\widehat{L\mathfrak{g}}$	A central extension of $L\mathfrak{g}$; affine Lie algebra	(3.1.1)
$\dot{\widehat{L\mathfrak{g}}}$	An extension of $\widehat{L\mathfrak{g}}$; also called affine	(3.1.6)
H	Principal A-bundle over K	
$\widetilde{G}^\Sigma$	The simply connected covering group of G^Σ	(7.1)
UG^Σ	The universal central extension of G^Σ	(7.2)
$\widehat{G}^\Sigma_J$	An extension of G^Σ by the Jacobian J of Σ	(7.1.1)
$\widehat{G}^\Sigma_C$	An extension of G^Σ; trivial complex line bundle over G^Σ	(7.7)
$U\mathfrak{g}^X$	The universal central extension by $\mathfrak{a} = \Omega^1(X)/d\Omega^0(X)$	(3.9)
$\widehat{\mathfrak{g}}^\Sigma$	An extension of $\mathfrak{g}^\Sigma$ by $\mathscr{H}^*_\Sigma$ with cocycle Ω	(3.12)
$\widetilde{\mathfrak{g}}^\Sigma$	An extension of $\mathfrak{g}^\Sigma$ by $\bar{\mathscr{H}}^*_\Sigma$ with cocycle $\widetilde{\Omega}$	(3.25)
$\dot{\widehat{\mathfrak{g}}}^E$	A semidirect product of $\widehat{\mathfrak{g}}^\Sigma$ and δ	(3.3.2)

$\dot{\widetilde{\mathfrak{g}}}^E$	A semidirect product of $\widetilde{\mathfrak{g}}^\Sigma$ and δ	(3.3.3)
$\widehat{\mathfrak{g}}_c^\Sigma$	The Lie algebra of $\widehat{G}_C^\Sigma$	(7.1)
$\mathfrak{t}$	A (complex) Cartan subalgebra of $\mathfrak{g}$	
$\mathfrak{t}^*$	The dual of the Cartan subalgebra $\mathfrak{t}$	
$\alpha(h)$	The pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$	
Φ	A finite root system spanning $\mathfrak{t}^*$	(6.1)
$\triangle = \{\alpha_i\}$	A set of simple roots for Φ	(6.1)
$\triangle^\vee = \{h_i\}$	The set of simple coroots where $\alpha_i(h_i) = 2$	(6.1)
$Q = \bigoplus_i \mathbb{Z}\alpha_i$	The root lattice for Φ	(6.1)
$\langle\cdot,\cdot\rangle_K$	The (positive-definite) Killing form on $\mathfrak{t}$	(6.1)
$\langle\cdot,\cdot\rangle$	The normalised Killing form on $\mathfrak{t}$	(6.1)
t_α	Unique element of $\mathfrak{t}$ where $\alpha(h) = \langle t_\alpha, h\rangle$ for $h \in \mathfrak{t}$	(6.1)
$(\cdot,\cdot)$	Bilinear form on $\mathfrak{t}^*$ given by $(\alpha_i, \alpha_j) := \langle t_{\alpha_i}, t_{\alpha_j}\rangle$	(6.1)
h_α	The coroot in $\mathfrak{t}$ corresponding to $\alpha \in \mathfrak{t}^*$ $[e_\alpha, e_{-\alpha}] = h_\alpha = \frac{2t_\alpha}{\langle t_\alpha, t_\alpha\rangle}$	(6.1)
$Q^\vee$	The coroot lattice $Q^\vee = \bigoplus_i \mathbb{Z}h_{\alpha_i}$	(6.1)
Λ	The weight lattice, $\Lambda = \{\lambda \in \mathfrak{t}^* \mid \lambda(h_\alpha) \in \mathbb{Z} \forall \alpha \in \triangle\}$	(6.1)
Ψ	The isomorphism $\mathfrak{t}^* \to \mathfrak{t}$ given by $\alpha \mapsto t_\alpha$	(6.1)
$\Lambda^\vee$	Dual weight lattice, $\Lambda^\vee = \{h \in \mathfrak{t} \mid \alpha(h) \in \mathbb{Z} \forall \alpha \in \Phi\}$	(6.1)
$C = (C_{ij})$	The Cartan matrix, $C_{ij} = \alpha_i(h_{\alpha_j}) = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$	(6.1)
W	Weyl group of Φ	(6.1)
$\widetilde{\Phi}$	An affine root system with $\widetilde{\Phi}/\mathbb{R}\delta = \Phi$	
δ	The basic imaginary root of $\widetilde{\Phi}$	
c	The central element	
$\widetilde{Q} = Q \oplus \mathbb{Z}\delta$	The root lattice of $\widetilde{\Phi}$	
$\widetilde{C}$	The generalised Cartan matrix of $\widetilde{\Phi}$	
$\widetilde{W}$	Affine Weyl group of Φ	
$\mathscr{H}$	The upper half-plane $\{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > 0\}$	(6.1)
$\mathscr{H}_g$	The upper half-space $\{\tau \in \mathrm{Mat}_g\mathbb{C} \mid \mathbf{z}^T(\mathrm{Im}\tau)\mathbf{z} > 0 \,\forall \mathbf{z} \in \mathbb{R}^g\}$	(7.1)
Y	The Tits cone $Y = \mathbb{C} \oplus \mathfrak{t} \oplus \mathscr{H}$	(6.1)
Y_g	A space $Y_g = \mathbb{C} \oplus \mathfrak{t}^g \oplus \mathscr{H}_g$	(7.1)
$J_\mathfrak{g}$	Jacobi group associated to the Lie algebra $\mathfrak{g}$	(6.1)
$J_\mathfrak{g}^g$	Higher-dimensional Jacobi group associated to $\mathfrak{g}$	(7.1)

Chapter 1

Introduction

We are concerned with a class of the infinite-dimensional Lie groups and their Lie algebras, namely *current groups* and *current algebras*. Currents are smooth maps from a manifold X to a Lie group G, and these form an infinite-dimensional Lie group $G^X := \mathrm{Map}(X, G)$ under pointwise multiplication. The Lie algebra $\mathfrak{g}^X$ of G^X is the algebra of smooth maps from X to $\mathfrak{g} = \mathrm{Lie}(G)$ endowed with pointwise Lie bracket.

The most well-studied of current groups is the *loop group* $LG := G^{S^1}$ of smooth maps from the circle S^1 to G. The loop group is the subject of the famous monograph of Pressley and Segal [PS86], many theorems of which we shall recall in the thesis. We are most interested in the case where the manifold is a two-dimensional surface Σ. In this case we are able to study the current groups $\mathrm{Map}(\Sigma, G)$ from a topological viewpoint and produce results for their homology and homotopy groups, extending certain results which appear in *eg.* [PS86] and [EF94]. Indeed, we may describe the genus g surface Σ topologically by attaching a disk D^2 to a $2g$-wedge of circles $\bigvee_{i=1}^{2g} S^1$ via the commutator map $c = [a_1, b_1] \ldots [a_g, b_g]$ (here $\{a_i, b_i\}$ is a homology basis for Σ). Using this description we can obtain information from the homotopy and homology long exact sequences induced from the cofibration sequence $S^1 \to^c \bigvee_{i=1}^{2g} S^1 \to \Sigma$.

An important problem in the theory of current groups and algebras is the study of their *central extensions*. Central extensions often appear naturally in mathematical physics, the most classic example being the Heisenberg group and the Heisenberg algebra, which shall be referred to often over the course of this thesis. The study of central extensions for groups and algebras is also important in representation theory. In particular the study of *projective representations* for a given group give genuine representations of a central extension of the group.

The problem of central extensions for the loop group LG and its Lie algebra $L\mathfrak{g}$ has been completely solved in [PS86]. A central extension of the loop algebra $L\mathfrak{g} := \mathfrak{g}^{S^1}$ appears in the construction of affine Kac-Moody algebras, *cf.* [Kac90]. The case for the current group G^Σ (which we are chiefly interested in) and its Lie algebra $\mathfrak{g}^\Sigma$ is somewhat sparser; some important results were obtained in [EF94] and [FK96] and indeed also [PS86]. For a finite-dimensional Lie algebra $\mathfrak{g}$, every central extension of $\mathfrak{g}$ lifts to a corresponding Lie group extension, which is a result of Hochschild [Hoc51]. In the infinite-dimensional setting, this is no longer the case and the obstructions to lifting the Lie algebra extension to the Lie group appear in the work of Neeb [Nee96], [Nee02], Neeb and Wagemann [NW03] and Maier and Neeb [MN03]. We apply some results of Neeb to study the central extensions of current groups from a topological viewpoint, and obtain some results on the homology and homotopy groups of these extensions.

Stepping away from current groups and algebras, we are also interested in a different group, the *Jacobi group*. The Jacobi group appears in many guises in the literature but its study has been somewhat disjoint. A certain form of the Jacobi group was first studied systematically by Eichler and Zagier in the monograph [EZ85]. The Jacobi group is defined there as a semidirect product between a 3-dimensional integral Heisenberg group with centre $\mathbb{Z}$, $H_{\mathbb{Z}} = \mathbb{Z}^2 \times \mathbb{Z}$ as a set, and the full modular group $SL_2(\mathbb{Z})$. The invariants of the Jacobi group are called Jacobi forms, and as Eichler and Zagier suggest, one may think of them as a 'cross between modular forms and elliptic functions'.

It had been shown by Looijenga [Loo80] in the study of generalised root systems that a class of Jacobi groups (although without this name) can be constructed, each one of these associated to a root system Φ of rank l (and hence simple Lie algebra $\mathfrak{g}$) of type $A_i, B_i, C_i, D_i, E_6, E_7, E_8, F_4, G_2$. In this case the $(2l+1$-dimensional) Heisenberg group is $H_{\mathbb{Z}} = Q^\vee \times Q^\vee \times \mathbb{Z}$, where $Q^\vee$ is the coroot lattice of Φ and we define an extended Weyl group $\mathscr{W} = W \ltimes H_{\mathbb{Z}}$, where the semidirect product is constructed via the action of the finite Weyl group W on the coroot lattice $Q^\vee$ and hence the Heisenberg group $H_{\mathbb{Z}}$. It was noticed by Looijenga that $\mathscr{W}$ admits an action of $SL_2(\mathbb{Z})$, and a further semidirect product $J_{\mathfrak{g}} = \mathscr{W} \rtimes SL_2(\mathbb{Z})$ is constructed, which we call the *Jacobi group associated to the Lie algebra* $\mathfrak{g}$. When $\mathfrak{g}$ is of type A_1 the Jacobi group is exactly that defined by Eichler and Zagier. The class of Jacobi groups associated to root systems were then studied in more detail by Wirthmüller [Wir92] in the context of invariant theory. The Jacobi group $J_{\mathfrak{g}}$ admits a representation on the *Tits cone* (*cf.* [Kac90]), $Y = \mathbb{C} \oplus \mathfrak{t} \oplus \mathscr{H}$, where $\mathfrak{t}$ is the Cartan subalgebra of $\mathfrak{g}$ and $\mathscr{H}$ is the Poincaré upper half-plane. The Jacobi group finds application in the study of certain integrable systems, *cf.* [Ber00a], [Ber00b] and [Str10]. In particular, it is shown in [Ber00b] that the orbit space $Y/J_{\mathfrak{g}}$ admits the structure of a Frobenius manifold, which is intrinsically related to the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations.

We are interested in a result in the thesis of Bertola [Ber99] which purported to construct a certain central extension of a current algebra of maps from an elliptic curve E to a simple Lie algebra $\mathfrak{g}$ (in much the same way as the affine algebra is an extension of the loop algebra in the theory of Kac-Moody algebras) and show that the 'Weyl group' of the extension is exactly the Jacobi group. In the course of the thesis we shall show that this cannot be the case and attempt to modify the theorem appropriately. This leads us to define a projective representation of G^E, *ie.* a representation of a certain central extension of G^E. Indeed, the author defines a certain extension of the adjoint action of the group G^E on the aforementioned current algebra extension. We show that this is not an action of G^E, rather an extension thereof by a one-dimensional centre, which we construct in Chapter 7 and call

$\widehat{G}^E_C$.

We find a further relation between the central extension of the current group $\widehat{G}^E_C$ and the Jacobi group $J_{\mathfrak{g}}$. Namely, G^E contains a certain discrete subgroup of currents, and restricting the extension $\widehat{G}^E_C$ to this discrete subgroup becomes a Heisenberg group. We then find the (adjoint) action of this Heisenberg group on the extended algebra normalises a Cartan subalgebra and we identify this with the action of the Heisenberg group $H_{\mathbb{Z}}$ on the Tits cone Y.

We have mentioned that the Jacobi group can be associated to an elliptic curve E, *ie.* a Riemann surface of genus one. In light of this interpretation, we ask if we may generalise the definition to construct a Jacobi group that is associated to a Riemann surface of genus $g \geq 1$. Namely, we construct a semidirect product of a finite Weyl group W and a $(2lg+1)$-dimensional Heisenberg group $(Q^\vee)^g \times (Q^\vee)^g \times \mathbb{Z}$, (the coroot lattice $Q^\vee$ and Weyl group W are again associated to a simple Lie algebra $\mathfrak{g}$ of rank l) with a given action of the symplectic group $Sp_{2g}(\mathbb{Z})$. We also construct a space analogous to the Tits cone on which the Jacobi group $J^g_{\mathfrak{g}}$ acts naturally, namely a space $Y_g = \mathbb{C} \oplus \mathfrak{t}^g \oplus \mathscr{H}_g$, where $\mathscr{H}_g$ denotes g-dimensional Siegel upper-half space and $\mathfrak{t}^g$ denotes a vector space that is g copies of the finite-dimensional Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. We also define Jacobi forms as the objects that are invariant under this new Jacobi group. This is the extent of our study of these objects and this may be of interest for future research.

The outline of the thesis is as follows. In Chapter 2 we discuss the necessary preliminaries on extensions of discrete groups and Lie groups, as well as Lie algebras, and in Chapter 3 we introduce the theory of current algebras, in particular results for extensions of loop algebras and double loop (elliptic) algebras. In Chapter 4 we discuss the topology of the current groups $\mathrm{Map}(\Sigma, G)$ for Σ a two-dimensional surface, and in Chapter 5 we discuss the topology of central extensions of $\mathrm{Map}(\Sigma, G)$. In Chapter 6 we introduce the Jacobi group, discussing definitions of Wirthmüller [Wir92] and Bertola [Ber99]. In Chapter 7 we introduce a new central extension $\widehat{G}^\Sigma_C$ and demonstrate a connection between the Jacobi group and the extension $\widehat{G}^E_C$ for E an elliptic curve. In Chapter 8 we construct the generalisation of the Jacobi group associated to a Riemann surface of genus $g \geq 1$.

Chapter 2

Preliminaries on Extensions

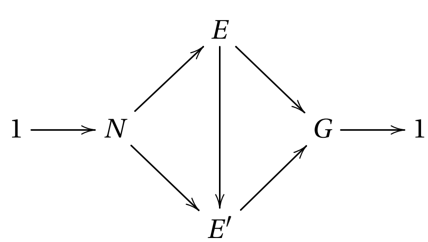
This thesis concerns extensions of current groups and algebras, which are infinite-dimensional Lie groups and Lie algebras, but we shall begin by reviewing the theory of extensions for discrete groups for comparison. This is an important topic in homological algebra, and will also be of use to us in the later chapters. We shall then review the theory for Lie groups and Lie algebras. For more details, see [Bro82] or [Wei94].

2.1 Extensions of Discrete Groups

This exposition closely follows Brown [Bro82]. An extension of a group G by a group N is a short exact sequence of groups

$$1 \longrightarrow N \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1\ . \tag{2.1}$$

A second extension $1 \to N \to E' \to G \to 1$ of G by N is said to be *equivalent* to (2.1) if there is a map $E \to E'$ making the diagram



commute. Note that such a map is necessarily an isomorphism. We would like to classify all group extensions of G by N up to equivalence. That is, we would like to understand the ways in which we can construct a group E with normal subgroup $N \triangleleft E$ and quotient group $E/N \cong G$.

Definition 2.1.1. *We shall denote by* $\mathrm{Ext}(G,N)$ *the set of equivalence classes of extensions of* G *by* N.

In the following we shall simplify the discussion slightly by only considering the case where the normal subgroup N is abelian, which we shall now call A. (For details on the situation where N is not abelian, see [Bro82].) From here on we shall always write the group A *additively* to emphasise its abelian nature, so in particular we write the identity element $1_A = 0$. In this case, the extension

$$0 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1 \tag{2.2}$$

gives rise to an action of G on A. Indeed, E acts on A by conjugation as $A \triangleleft E$, and the conjugation action of A on itself is trivial as A is abelian, so there is an induced action of $E/A = G$ on $\iota(A)$. This makes A a G-module. Explicitly, given $g \in G$ we choose $\tilde{g} \in E$ such that $\pi(\tilde{g}) = g$, and the action $G \times A \to A$ is then

$$\iota(ga) \mapsto \tilde{g}\iota(a)\tilde{g}^{-1},$$

which can be rewritten as

$$\tilde{g}\iota(a) = \iota(ga)\tilde{g}. \tag{2.3}$$

This shows that $\iota(A)$ is central in E if and only if the G-action on A is trivial. In this case the extension is called a *central extension*, which we will discuss in Section 2.1.3. First, however, we will discuss the simplest class of extensions, the *split extensions*.

2.1.1 Split Extensions

Let A be a G-module and consider the extension

$$0 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1.$$

We says the above exact sequence *splits* if there exists a group homomorphism $s : G \to E$ (called a *section*) satisfying $\pi \circ s = id_G$. We shall see that there is exactly one split extension (up to equivalence) associated to a given action of G on A, which is the *semidirect product* of A and G, whose definition we now recall.

Definition 2.1.2. *The semidirect product $A \rtimes G$ of two groups A and G is the set $A \times G$ with group law given by*

$$(a, g) \cdot (b, h) = (a + gb, gh)$$

for all $(a, g), (b, h) \in A \times G$.

Proposition 2.1.3. *The extension*

$$0 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1$$

splits if and only if it is equivalent to the extension

$$0 \longrightarrow A \xrightarrow{\iota'} A \rtimes G \xrightarrow{\pi'} G \longrightarrow 1.$$

Proof. It is trivial that the extension splits if it is equivalent to a semidirect product: the section $G \to A \rtimes G$ is $g \mapsto (0, g)$. Conversely, let $s : G \to E$ be a section (group homomorphism) for the extension, and we note that we have a bijection $A \times G \to E$ given by $(a, g) \mapsto \iota(a)s(g)$. There is then a unique group law on the set $A \times G$ such that the bijection is an isomorphism. We find

$$\iota(a)s(g)\iota(b)s(h) = \iota(a)\iota(gb)s(g)s(h) = \iota(a + gb)s(gh),$$

where the first equality holds from (2.3), and the second holds as ι and s are group homomorphisms. Then the group law on $A \times G$ is given by

$$(a, g) \cdot (b, h) = (a + gb, gh).$$

The set $A \times G$ with this group law is, by definition, the semidirect product $A \rtimes G$. □

In general, given a group homomorphism $\phi : G \to \mathrm{Aut}(A)$, $\phi : g \mapsto \phi_g$, we have the semidirect product $A \rtimes G$ (sometimes written $A \rtimes_\phi G$ to emphasise the dependence on the group homomorphism ϕ), with group law

$$(a, g) \cdot (b, h) = (a + \phi_g(b), gh).$$

The section $s : G \to A \rtimes_\phi G$ must be of the form $s(g) = (c(g), g)$ for some function $c : G \to A$. Then

$$s(g)s(h) = (c(g) + gc(h), gh),$$

so that s is a homomorphism if and only if $c(gh) = c(g) + gc(h)$, that is, c is a *1-cocycle* (or derivation[1]) For further details, see [Bro82], who shows the following.

Proposition 2.1.4. *For any G-module A, the A-conjugacy classes of splittings of the split extension*

$$0 \longrightarrow A \longrightarrow A \rtimes G \longrightarrow G \longrightarrow 1$$

are in one-one correspondence with the elements of $H^1(G, A)$.

The previous propositions show that, as we claimed at the start of the section, there is exactly one split extension (up to equivalence) for each possible action of G on A. For example, when the action of G on A is trivial, and so the extension is equivalent to the direct product $G \times A$, the possible splittings are in one-one correspondence with homomorphisms $G \to A$.

[1] In order to see that this coincides with the usual definition of a derivation $c(gh) = c(g)h + gc(h)$, we think of G acting trivially on the right of A as well as the aforementioned action on the left, which gives us the definition above.

2.1.2 Abelian Extensions

Let A be a G-module. All extensions of G by A will be assumed to give rise to a fixed action of G on A. Given an extension

$$0 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1, \tag{2.4}$$

we choose a function $s : G \to E$ such that $\pi \circ s = id_G$. For simplicity, we normalise s so that it satisfies $s(1_G) = 1_E$. However, we do not assume s is a group homomorphism. If this is the case, then the extension splits and we refer to the previous section for the structure of E. In general there exists a function $f : G \times G \to A$ that measures the failure of s to be a group homomorphism. For any $g, h \in G$, the elements $s(gh)$ and $s(g)s(h)$ of E are both mapped by π to gh in G, so they differ by an element of $\iota(A)$. Thus we can define f by the equation

$$s(g)s(h) = \iota(f(g,h))s(gh).$$

Note that because we have set $s(1_G) = 1_E$, we have

$$f(g,1_G) = 0 = f(1_G, g)$$

for all $g \in G$. The function f is called the *factor set* associated to the extension (2.4) and the function $s : G \to E$. We may recover the extension from the G-module structure on A and the function f. Indeed, since $s(G)$ is a set of coset representatives for $\iota(A)$ in E, we have a bijection $A \times G \to E$ given by $(a, g) \mapsto \iota(a)s(g)$. We compute the group law on $A \times G$ that makes the bijection an isomorphism of groups:

$$\begin{aligned}\iota(a)s(g)\iota(b)s(h) &= \iota(a)\iota(gb)s(g)s(h)\\ &= \iota(a+gb)\iota(f(g,h))s(gh)\\ &= \iota(a+gb+f(g,h))s(gh).\end{aligned}$$

Then the group law on $A \times G$ is given by

$$(a,g)\cdot(b,h) = (a+gb+f(g,h), gh). \tag{2.5}$$

We call the group with the group law (2.5) $E_f := A \times_f G$. It is of course a semidirect product exactly when f is trivial, i.e. s is a group homomorphism and so the extension splits. Since $\iota(a) = \iota(a)s(1_G)$ for any $a \in A$, the map $\iota : A \to E \simeq E_f$ is the canonical inclusion $a \mapsto (a, 1_G)$ and the map $\pi : E_f \simeq E \to G$ is the canonical projection $(a, g) \mapsto g$. Then our original extension is equivalent to the extension

$$0 \longrightarrow A \xrightarrow{\iota} E_f \xrightarrow{\pi} G \longrightarrow 1,$$

defined entirely by the group law (2.5), ι, π in terms of G, A and f.

Conversely, we may ask whether we could choose an arbitrary function $f : G\times G \to A$

and define a group E_f by means of the group law (2.5). We show below that this is the case exactly when $f : G \times G \to A$ is a *2-cocycle*, that is, f satisfies the cocycle condition

$$gf(h,k) - f(gh,k) + f(g,hk) - f(g,h) = 0 \tag{2.6}$$

for all $g,h,k \in G$. Indeed, a short calculation of the triple products $[(a,g)\cdot(b,h)]\cdot(c,k)$ and $(a,g)\cdot[(b,h)\cdot(c,k)]$ shows that the multiplication (2.5) on $A \times G$ is associative when

$$f(g,h) + f(gh,k) = gf(h,k) + f(g,hk),$$

which we see is exactly the cocycle condition (2.6). The set of all cocycles $f : G \times G \to A$ with respect to a given action on A form an abelian group under pointwise addition, which is denoted $Z(G,A)$. We now see that we have a correspondence between extensions given by (2.4) with normalised section s, and the group $Z(G,A)$.

Now let us consider a different choice of normalised section $s' : G \to E$ for the extension (2.4). An arbitrary normalised section s' is given by

$$s'(g) = \iota(c(g))s(g), \tag{2.7}$$

where $c : G \to A$ is an arbitrary function with $c(1_g) = 0$. We calculate the factor set associated to the new section s':

$$\begin{aligned}\iota(c(g))s(g)\iota(c(h))s(h) &= \iota(c(g))\iota(gc(h))s(g)s(h)\\ &= \iota(c(g)+gc(h))\iota(f(g,h))s(gh)\\ &= \iota(c(g)+gc(h)+f(g,h)-c(gh))\iota(c(gh))s(gh).\end{aligned}$$

Thus the factor set f' associated to the section s' is given by

$$f'(g,h) = f(g,h) + c(g) + gc(h) - c(gh).$$

In other words, $f' = f + \delta c$ where δc is a *2-coboundary*, that is, it satisfies

$$(\delta c)(g,h) = c(g) + gc(h) - c(gh)$$

for a 1-cocycle $c : G \to A$. The 2-cocycles which are 2-coboundaries form a subgroup $B(G,A) \subset Z(G,A)$.

Definition 2.1.5. *The quotient group of 2-cocycles modulo 2-coboundaries is called the second cohomology group of G with values in A. It is denoted*

$$H^2(G,A) = Z(G,A)/B(G,A).$$

We have proved the following theorem.

Theorem 2.1.6. *Given a G-module A, the set of equivalence classes of extensions of G by A*

$\mathrm{Ext}(G, A)$ *is exactly* $H^2(G, A)$.

2.1.3 Central Extensions

We now consider central extensions. A central extension is a particular case of an abelian extension where now $\iota(A) \subset Z(E)$, the centre of E. As we noted in Section 2.1, this is the case if and only if the G-action on A is trivial. (Note that a semidirect product $A \rtimes G$ which is a central extension is necessarily the direct product $A \times G$.) The group law (2.5) in this case becomes

$$(a, g) \cdot (b, h) = (a + b + f(g, h), gh), \tag{2.8}$$

with the cocycle $f : G \times G \to A$ satisfying

$$f(g, h) + f(gh, k) = f(h, k) + f(g, hk). \tag{2.9}$$

We note inversion and conjugation in the extended group is given by the identities

$$(a, g)^{-1} = (-a - f(g, g^{-1}), g^{-1}); \tag{2.10}$$

$$(a, g)(b, h)(a, g)^{-1} = (b + f(g, h) - f(ghg^{-1}, g), ghg^{-1}). \tag{2.11}$$

We make a short remark on the centres of the various groups. We have that

$$Z(E)/\iota(A) \lhd Z\left(E/\iota(A)\right).$$

Indeed if $z\iota(A) \in Z\left(E/\iota(A)\right)$ then $z\iota(A)p\iota(A) = p\iota(A)z\iota(A)$ for all $p \in E$ and so

$$zp = pz\iota(a) \tag{2.12}$$

for some $a \in A$, which clearly is true for $z \in Z(E)$. In general $Z(E)/\iota(A) \ncong Z\left(E/\iota(A)\right)$. We have

$$Z(E) = \{(a, g) \in A \times G \,|\, g \in Z(G), f(g, h) - f(h, g) = 0 \;\forall\, h \in G\}$$

and

$$Z(E)/\iota(A) = \{g \in Z(G) \,|\, f(g, h) - f(h, g) = 0 \;\forall\, h \in G\},$$

which is a normal subgroup in $Z(G) := Z(E/\iota(A))$. In particular we see that if $Z(E)/\iota(A) \ncong Z(G)$ then $Z(G)$ acts nontrivially on E. For example consider the realisation of the dihedral group D_8 as a central extension of $C_2 \times C_2$, where C_2 is the cyclic group of order two:

$$1 \longrightarrow C_2 \longrightarrow D_8 \longrightarrow C_2 \times C_2 \longrightarrow 1.$$

Here $Z(D_8/\iota(C_2)) \cong Z(C_2 \times C_2) = C_2 \times C_2$ whereas $Z(D_8) \cong C_2$ and $Z(D_8)/\iota(C_2) = \mathrm{id}$.

Universal Central Extensions

Definition 2.1.7. *A central extension UG of G is called* universal *if given any other central extension $E_f = A \times_f G$ of G by an abelian group A, there is a unique mapping $UG \to E_f$ such that the diagram*

$$\begin{array}{ccc} UG & & \\ \downarrow & \searrow & \\ & & G \\ \downarrow & \nearrow & \\ E_f & & \end{array} \tag{2.13}$$

commutes.

Now recall that G is called *perfect* if it is equal to its derived subgroup $[G, G]$, where the bracket denotes the group commutator.

Proposition 2.1.8. *The group G possesses a universal central extension if and only if G is perfect. The universal central extension UG of G is given by*

$$1 \longrightarrow H_2(G) \longrightarrow UG \longrightarrow G \longrightarrow 1, \tag{2.14}$$

where $H_2(G) = H_2(G, \mathbb{Z})$ is the second integral homology group of G. Furthermore, UG is also perfect.

For a proof, see [Wei94].

2.2 Central Extensions of Lie Groups and Lie Algebras

2.2.1 Extensions of Lie Algebras

A Lie algebra central extension $\mathfrak{h}$ of the Lie algebra $\mathfrak{k}$ by an abelian Lie algebra (vector space) may expressed in terms of a continuous 2-cocycle $\omega : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{a}$. Given

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{k} \longrightarrow 0 \tag{2.15}$$

we may construct $\mathfrak{h}$ in terms of pairs $(u, X) \in \mathfrak{a} \times \mathfrak{k}$ with commutator

$$[(u, X), (v, Y)]_{\mathfrak{h}} = (\omega(X, Y), [X, Y]_{\mathfrak{k}}), \tag{2.16}$$

where ω satisfies the Lie algebra cocycle identity

$$\omega([X, Y], Z) + \omega([Z, X], Y) + \omega([Y, Z], X) = 0$$

for all $X, Y, Z \in \mathfrak{k}$. A Lie algebra cocycle ω is called a 2-coboundary if $\omega(X, Y) = \alpha([X, Y])$ for a 1-cocycle $\alpha : \mathfrak{k} \to \mathfrak{a}$. A central extension defined by such a 2-coboundary becomes the trivial extension by the zero cocycle (*ie.* the direct sum of Lie algebras) after the change of coordinates $(u, X) \mapsto (u - \alpha(X), X)$.

Proposition 2.2.1. *The set of equivalence classes of Lie algebra central extensions of a Lie algebra* $\mathfrak{k}$ *by an abelian Lie algebra* $\mathfrak{a}$ *is given by the second cohomology group*

$$H^2(\mathfrak{k}, \mathfrak{a}) = Z(\mathfrak{k}, \mathfrak{a}) / B(\mathfrak{k}, \mathfrak{a})$$

where $Z(\mathfrak{k}, \mathfrak{a})$ *is the vector space of 2-cocycles on* $\mathfrak{k}$ *with values in* $\mathfrak{a}$*, and* $B(\mathfrak{k}, \mathfrak{a})$ *is the subspace of 2-coboundaries.*

Similarly to the property of extensions of groups given in Proposition 2.1.8, a (non-abelian) Lie algebra $\mathfrak{k}$ has a universal central extension if and only if it is perfect, $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}]$. We note that the condition of non-abelian is necessary: abelian Lie algebras are not perfect but may have a universal central extension. Finite-dimensional semisimple Lie algebras are perfect and in fact coincide with their universal central extension, *ie.* all central extensions of a finite-dimensional semisimple Lie algebra are trivial. The universal central extension of a Lie algebra is given by

$$0 \longrightarrow H_2(\mathfrak{k}) \longrightarrow U\mathfrak{k} \longrightarrow \mathfrak{k} \longrightarrow 0, \tag{2.17}$$

where $H_2(\mathfrak{g})$ is the second Lie algebra homology space, *cf.* [Wei94].

We mention briefly the form of a semidirect product of Lie algebras: given Lie algebras $\mathfrak{k}$ and $\mathfrak{g}$ with homomorphism $\sigma : \mathfrak{g} \to \mathrm{Der}(\mathfrak{k})$, we can form a semidirect product $\mathfrak{h} := \mathfrak{k} \rtimes \mathfrak{g}$ of Lie algebras with Lie structure

$$[(\xi, \eta), (\xi', \eta')] = ([\xi, \xi'] + \sigma(\eta)\xi' - \sigma(\eta')\xi, [\eta, \eta']), \tag{2.18}$$

with $\xi, \xi' \in \mathfrak{k}$ and $\eta, \eta' \in \mathfrak{g}$.

2.2.2 Central Extensions of Lie Groups

In order to discuss central extensions of Lie groups, we shall briefly consider the more general case of topological groups. Consider the sequence

$$0 \longrightarrow A \longrightarrow H \longrightarrow K \longrightarrow 1, \tag{2.19}$$

where K and A are both connected topological groups, with A in addition abelian. In studying extensions of discrete groups in the previous section we defined a set-theoretic section s which was not necessarily a group homomorphism. In the case of topological groups, we have the following lemma:

Lemma 2.2.2. *[HM06, A.2.26, A.2.27] Let* K *be a connected simply connected topological group and* H *a group. Let* U *be an open symmetric connected component neighbourhood* *of e_K and $s : U \to H$ a function such that*

$$s(xy) = s(x)s(y)$$

for all $x, y \in U$. Then there exists a unique group homomorphism extending s. If in addition K is a topological group and s is continuous, then its extension is also continuous.

Thus when K is a connected simply connected topological group and H a topological group then given any continuous section of the identity $s : U \to H$ we may extend this to all of K and consequently

$$0 \longrightarrow A \xrightarrow{\iota} H \xrightarrow{\pi} K \longrightarrow 1 \tag{2.20}$$

splits. Equivalently, H is a topologically trivial principal A-bundle over K. When K is connected but not simply connected we extend s to a global section $s : K \to H$ and introduce the cocycle $f(x, y) = s(x)s(y)s(xy)^{-1}$. Both the section s and cocycle f are continuous on a neighbourhood containing the identity but need not be globally continuous. Neeb [Nee02] shows such central extensions are classified by such locally *continuous* 2-cocycles modulo locally continuous 2-coboundaries,

$$H^2_c(K, A) := Z^2_c(K, A)/B^2_c(K, A).$$

Now let K and H be Lie groups. We replace the (locally) continuous maps in the preceding with (locally) smooth maps, that is smooth maps/sections defined in a neighbourhood containing the identity. Neeb [Nee02] describes these as *smooth central extensions* and the resulting principal A-bundles H over K as *smooth principal bundles*. These are now described by locally *smooth* 2-cocycles modulo locally smooth 2-coboundaries and we have the following proposition.

Proposition 2.2.3. *There is a one-one correspondence between $\mathrm{Ext}_{\mathrm{Lie}}(K, A)$, the set of central extensions of K by A that admit a smooth section, and the elements in the second cohomology group $H^2_s(K, A) = Z_s(K, A)/B_s(K, A)$, where $Z_s(K, A)$ and $B_s(K, A)$ denote the sets of smooth 2-cocycles and smooth 2-coboundaries on K respectively.*

We shall only discuss smooth central extensions from now on and shall omit the subscript s. We consider only central extensions which are locally trivial smooth principal bundles, that is, those which admit smooth local sections. This ensures the existence of a continuous linear section for the corresponding Lie algebra [KW09]. In what follows we always consider central extensions that arise from smooth local sections s in the neighbourhood of the identity. If the principal A bundle H over K is topologically nontrivial then there is no global section and there exist central extensions of Lie groups which cannot be defined by globally smooth 2-cocycles.

For K connected and simply connected, we have to following theorem of Hochschild [Hoc51]:

Theorem 2.2.4.

$$\mathrm{Ext}_{\mathrm{Lie}}(K, A) \simeq H^2(\mathfrak{k}, \mathfrak{a}).$$

For non-simply connected K, Neeb [Nee96] shows that the obstruction is the action of $\pi_1(K)$ on the Lie algebra central extension $\mathfrak{h}$:

Proposition 2.2.5. *[Nee96, I.2] A Lie algebra central extension $\mathfrak{a} \to \mathfrak{h} \to \mathfrak{k}$ integrates to a Lie group central extension $A \to H \to K$ if and only if the fundamental group $\pi_1(K)$ acts trivially on $\mathfrak{h}$.*

Let Γ be a discrete normal subgroup of the connected Lie group K. As K acts continuously on itself by conjugation, its action on Γ must be trivial and so $\Gamma \subset Z(K)$. In particular Γ is abelian. Thus the fundamental group of a connected Lie group is abelian. Suppose further that Γ be a discrete central subgroup of a connected topological group H. Then consideration of (2.12) shows $z \in Z(H/\Gamma)$ if, for all $h \in H$, there is some $\gamma \in \Gamma$ so that $zh = hz\gamma$. But now continuity in H and the fact that H is connected gives $\gamma = e_H$ and consequently $z \in Z(H)$. Thus we have

$$Z(H)/\Gamma \cong Z\left(H/\Gamma\right).$$

We remarked in the previous section that it is possible for $Z(H)/A \lhd Z(K) := Z(H/A)$ and $Z(H)/A \ncong Z(K)$. Suppose then we may have a discrete subgroup Γ in $Z(K)$ but not properly contained in $Z(H)/A$. Let $\tilde{\gamma} \in H$ be such that $\pi(\tilde{\gamma}) = \gamma \in K$, then

$$\gamma \diamond h = \tilde{\gamma} h \tilde{\gamma}^{-1}$$

gives a well-defined action acting trivially on $A \subset Z(H)$. Then Γ acts nontrivially on H and also acts on the Lie algebras

$$0 \to \mathfrak{a} \to \mathfrak{h} \to \mathfrak{k} \to 0$$

with trivial actions apart from that on $\mathfrak{h}$. Suppose there is a central extension

$$1 \longrightarrow A' \longrightarrow H' \longrightarrow K/\Gamma \longrightarrow 1$$

corresponding to this, where $\mathrm{Lie}(H') = \mathfrak{h}$ etc. Then $H' = \tilde{H}/\Gamma_H$ for a simply connected Lie group $\tilde{H}$ with $\mathrm{Lie}(\tilde{H}) = \mathfrak{h}$ and where the discrete group Γ_H is central. Then Γ acts nontrivially by conjugation on H' while acting trivially on A' and K/Γ. But this is not possible, and so no central extension can exist. Indeed, write H' in terms of the cosets A' and elements of K/Γ.

Then Γ fixes each element of A' and each element of K/Γ, and hence fixes H'. The sequence

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \pi_1(K) & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & H & \underset{s}{\overset{\pi}{\rightleftarrows}} & \widetilde{K} & \longrightarrow & 1 \\ & & & & & & \downarrow & & \\ & & & & & & K & & \\ & & & & & & \downarrow & & \\ & & & & & & 1 & & \end{array} \tag{2.21}$$

splits. Thus for a discrete central subgroup $\Gamma \subset H/A$ which is not contained in $Z(H)/A$, the central extension $\mathfrak{a} \to \mathfrak{h} \to \mathfrak{h}/\mathfrak{a}$ does not integrate to a central extension of the form

$$0 \longrightarrow A' \longrightarrow H' \longrightarrow (H/A)/\Gamma \longrightarrow 1.$$

Example 2.2.6. *Let us consider the most classic example of a central extension, namely the 3-dimensional Heisenberg group,* Heis_3*, given by*

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Heis}_3 \longrightarrow \mathbb{R}^2 \longrightarrow 1 \tag{2.22}$$

Now $Z(\mathrm{Heis}_3)/\mathbb{R}$ *is trivial but we have for example* $\Gamma = \mathbb{Z}^2$ *a normal subgroup in* $Z(\mathrm{Heis}_3/\mathbb{R}) = \mathbb{R}^2$*. Then as* $\mathbb{Z}^2$ *acts nontrivially via the adjoint action on* H*, there can be no central extension of the form*

$$0 \longrightarrow A \longrightarrow H \longrightarrow K \longrightarrow 1$$

for $K = \mathbb{R}^2/\mathbb{Z}^2$.

2.2.3 Adjoint Actions

We now record some calculations for the adjoint actions of these Lie group extensions on themselves and their corresponding Lie algebras. We shall require these results in Chapter 5. Suppose we have a Lie group central extension

$$0 \longrightarrow A \overset{\iota}{\longrightarrow} H \overset{\pi}{\longrightarrow} K \longrightarrow 1. \tag{2.23}$$

Then we may obtain the adjoint action Ad^H of H on $\mathfrak{h}$ and the adjoint action ad^H of $\mathfrak{h}$ on $\mathfrak{h}$ by explicit differentiation of the identity

$$(a,k)(b,l)(a,k)^{-1} = (b + f(k,l) - f(klk^{-1},k), klk^{-1})$$

for $(a,k),(b,l) \in A \times K$ and f an A-valued 2-cocycle on K.

Proposition 2.2.7. *The adjoint actions* Ad^H *of the Lie group* H *on its Lie algebra* $\mathfrak{h}$ *and* ad^H *of the Lie algebra* $\mathfrak{h}$ *on itself are given by*

$$\mathrm{Ad}^H_{(a,k)}(t,Y) = (t+\theta(k,Y), \mathrm{Ad}^K_k Y) \tag{2.24}$$

and

$$[(s,X),(t,Y)]_{\mathfrak{h}} = (\omega(X,Y),[X,Y]_{\mathfrak{k}}). \tag{2.25}$$

Here (s,X), $(t,Y) \in \mathfrak{a}\times\mathfrak{k}$, $(a,k) \in A\times K$, $\theta : K\times\mathfrak{h} \to \mathfrak{a}$ *and* $\omega : \mathfrak{k}\times\mathfrak{k} \to \mathfrak{a}$. *The requirement that*

$$\mathrm{Ad}^H_{(a,k)}\mathrm{Ad}^H_{(b,l)}(t,Y) = \mathrm{Ad}^H_{(a+b+f(k,l),kl)}(t,Y)$$

gives

$$\theta(kl,Y) = \theta(l,Y)+\theta(k,\mathrm{Ad}^K_l Y) \tag{2.26}$$

while the antisymmetry of the Lie bracket means ω *is skew and the Jacobi identity means it must satisfy the cocycle identity*

$$\omega([X,Y],Z)+\omega([Y,Z],X)+\omega([Z,X],Y) = 0. \tag{2.27}$$

Proof. Differentiating

$$(a,k)(b,l)(a,k)^{-1} = (b+f(k,l)-f(klk^{-1},k), klk^{-1})$$

with $l = e^{tY}$, we find that

$$\theta(k,Y) = \frac{d}{dt}\Big[f(k,l)f(klk^{-1},k)^{-1}\Big]_{t=0} = d_2 f(k,e)\cdot Y - d_1 f(e,k)\cdot\mathrm{Ad}^K_k Y. \tag{2.28}$$

Here $d_{1,2}f$ denotes differentiation of the first or second argument of f and $d_2 f(k,e)\cdot Y = Y^j d_{2j} f(k,e)$ denotes the sum of the derivatives; as $d_2 f(k,e)\in\mathfrak{a}$ this sum is in $\mathfrak{a}$. Then

$$\omega(X,Y) := \big(d^2 f(e,e)\big)(X,Y) - \big(d^2 f(e,e)\big)(Y,X)).$$

Here we have used that $f(k,e_K) = 0$ for all $k\in K$ to show that $d_1 f(e,e) = 0$ and so $d^2 f(l,k)|_{(e,e)} = d_{12} f(l,k)|_{(e,e)}$ is well defined. We note that if $f'(l,k) = c(l)c(k)c(lk)^{-1} f(l,k)$ for a function $c : H \to A$ smooth in the neighbourhood of the identity with $c(e_K) = c_A$, then

$$\theta'(k,Y) = \theta(k,Y) + Y\cdot\dot{c}(e) - \mathrm{Ad}^K_k Y\cdot\dot{c}(e)$$

and

$$\omega'(X,Y) = \omega(X,Y) - [X,Y]\cdot\dot{c}(e) = \omega(X,Y) - dc(e)([X,Y]).$$

Thus equivalent extensions arise from cocycles differing by 2-coboundaries, $\omega(X,Y) = \alpha([X,Y])$ for some $\alpha : \mathfrak{k}\to\mathfrak{a}$. As the cocycles we obtain are continuous on a neighbourhood of the identity we have a one-to-one correspondence between equivalence classes of central extensions of $\mathfrak{k}$ by $\mathfrak{a}$ and elements of $H^2_c(\mathfrak{k},\mathfrak{a})$. The map

$$D : H^2_s(K,A) \to H^2_c(\mathfrak{k},\mathfrak{a}); \qquad [f] \mapsto \omega$$

is a group homomorphism. To see that (2.26) holds we note

$$\begin{aligned}
&f(k_2,l)f(k_2lk_2^{-1},k_2)^{-1}f(k_1,k_2lk_2^{-1})f(k_1k_2lk_2^{-1}k_1^{-1},k_1)^{-1}\\
&= f(k_2,l)\left[f(k_2^{-1},k_2)f(k_2l,k_2^{-1})^{-1}\right]^{-1}\left[f(k_1k_2l,k_2^{-1})f(k_1,k_2l)f(k_2l,k_2^{-1})^{-1}\right]\\
&\quad\times\left[f(k_1k_2lk_2^{-1}k_1^{-1}k_2^{-1},k_1)^{-1}f(k_1k_2l,k_2^{-1})^{-1}f(k_1k_2,k_2^{-1})\right]\\
&= f(k_2,l)f(k_1,k_2l)f(k_1k_2,k_2^{-1})f(k_2^{-1},k_2)^{-1}f(k_1k_2lk_2^{-1}k_1^{-1}k_2^{-1},k_1)^{-1}\\
&= f(k_1k_2,l)f(k_1,k_2)f(k_1k_2,k_2^{-1})f(k_2^{-1},k_2)^{-1}f(k_1k_2lk_2^{-1}k_1^{-1}k_2^{-1},k_1)^{-1}\\
&= f(k_1k_2,l)f(k_1k_2lk_2^{-1}k_1^{-1}k_2^{-1},k_1)^{-1}
\end{aligned}$$

and the result follows upon differentiation. □

Now let λ_k and ρ_k denote left and right multiplication on the Lie group K for $k \in K$. Let X_R denote the right invariant vector field on K such that $X_R(e_K) = X \in \mathfrak{k}$.

Lemma 2.2.8. *Let Ω be a left invariant 2-form on K. Then*

$$\lambda_k^*\left(i_{X_R}\Omega\right) = i_{\left(\mathrm{Ad}^K_{k^{-1}}X\right)_R}\Omega.$$

Proof. To see this first note that

$$\lambda_k^*\left(i_{X_R}\Omega\right) = \lambda_k^*\circ i_{X_R}\circ\Omega = \lambda_k^*\circ i_{X_R}\circ\lambda_{k^{-1}}^*\circ\lambda_k^*\circ\Omega = \lambda_k^*\circ i_{X_R}\circ\lambda_{k^{-1}}^*\circ\Omega$$

using the left invariance of Ω. Now if α is any diffeomorphism and α^{-1} its inverse, then

$$\alpha^*\circ i_V\circ\alpha^{-1\,*} = i_{\alpha_*^{-1}V}$$

for if μ is any form, here a two form for simplicity, and if X is any vector field, then

$$\begin{aligned}
(\alpha^*\circ i_V\circ\alpha^{-1\,*}\mu)(X) &= (i_V\circ\alpha^{-1\,*}\mu)(\alpha_*X) = (\alpha^{-1\,*}\mu)(V,\alpha_*X) = \mu(\alpha^{-1}{}_*V,\alpha_*^{-1}\alpha_*X)\\
&= \mu(\alpha^{-1}{}_*V,X) = (i_{\alpha^{-1}{}_*V}\mu)(X).
\end{aligned}$$

Thus

$$\lambda_k^*\left(i_{X_R}\Omega\right) = i_{\lambda_{k^{-1}}{}_*X_R}\Omega$$

and the result follows upon establishing

$$\lambda_{g*}X_R = (\mathrm{Ad}^K_g X)_R.$$

Now we have

$$\begin{aligned} X_L(gk) &= [\lambda_{g*}X_L](gk) = \lambda_{g*}(k)X_L(k) \\ X_R(kg) &= [\rho_{g*}(k)X_R](kg) = \rho_{g*}(k)X_R(k) \\ [\lambda_{g*}\rho_{k*}W](glk) &= \lambda_{g*}(lk)[\rho_{h*}W](lk) = \lambda_{g*}(lk)\rho_{k*}(l)W(l) = \rho_{k*}(gl)\lambda_{g*}(l)W(l). \end{aligned}$$

Then

$$\begin{aligned} [\lambda_{g*}X_R](k) &= \lambda_{g*}(g^{-1}k)X_R(g^{-1}k) = \lambda_{g*}(g^{-1})\rho_{g^{-1}k*}(e)X_R(e) \\ &= \lambda_{g*}(g^{-1}k)\rho_{k*}(g^{-1})\rho_{g^{-1}*}(e)X_R(e) = \rho_{k*}(e)\lambda_{g*}(g^{-1}k)\rho_{g^{-1}*}(e)X_R(e) \\ &= \rho_{k*}(e)(\mathrm{Ad}^K_g X)(e) = (\mathrm{Ad}^K_g X)_R(k). \end{aligned}$$

□

Lemma 2.2.9. *Suppose that $i_{X_R}\Omega$ is exact, $i_{X_R}\Omega = df_X$, and f_X chosen so that $f_X(e_K) = 0$. Then $\Psi : K \times \mathfrak{k} \to \mathfrak{a}$ with*

$$\Psi(k, X) = f_X(k^{-1})$$

is a cocycle.

Proof. We must show that Ψ satisfies (2.26), or

$$f_X(k_2^{-1}k_1^{-1}) = f_X(k_2^{-1}) + f_{\mathrm{Ad}^K_{k_2} X}(k_1^{-1})$$

which may be rewritten $f_X \circ \lambda_{k_2^{-1}} = f_X(k_2^{-1}) + f_{\mathrm{Ad}^K_{k_2} X}$. Both sides agree at e_K. Considering the differential yields

$$d(f_X \circ \lambda_{k_2^{-1}}) = \lambda^*_{k_2^{-1}} df_X = \lambda^*_{k_2^{-1}}\left(i_{X_R}\Omega\right) = i_{\left(\mathrm{Ad}^K_{k_2} X\right)_R}\Omega = df_{\mathrm{Ad}^K_{k_2} X}$$

and so we have agreement. Thus Ψ is a cocycle. □

Finally, we solve

$$i_{X_R}\Omega = df_X$$

for f_X by evaluating both sides on arbitrary left invariant vector fields Y_L. Then

$$\begin{aligned} \Omega_g\left(\left(\mathrm{Ad}^K_{k^{-1}} X\right)_R(g), Y_L(g)\right) &= \left[i_{\left(\mathrm{Ad}^K_{k^{-1}} X\right)_R}\Omega\right](Y_L)(g) = \left[\lambda^*_k\left(i_{X_R}\Omega\right)\right]_g\left(Y_L(g)\right) \\ &= \left(i_{X_R}\Omega\right)_{kg}\left(\lambda_{k*}(g)Y_l(g)\right) = \left(i_{X_R}\Omega\right)_{kg}\left(Y_L(kg)\right). \end{aligned}$$

So upon taking $g = e_K$ we wish to solve

$$\omega\left(\mathrm{Ad}^K_{k^{-1}} X, Y\right) = \left(i_{X_R}\Omega\right)_k(Y_L(k)) = df_X(Y_L)(k) = (df_X)_k(\lambda_{k*}Y) = (\lambda^*_k df_X)(Y_L)(e)$$

for arbitrary left invariant vector fields Y_L with $Y_L(e) = Y$. With $\lambda^*_k df_X = d(\lambda^*_k f_X) = d(f_X \circ \lambda_k)$

we have

$$df_{(\mathrm{Ad}^K_{k^{-1}}X)_R}(Y)(e) = \omega\left(\mathrm{Ad}^K_{k^{-1}}X, Y\right) = d(f_X \circ \lambda_k)(Y)(e).$$

This may be written as

$$df_X(k) = \lambda^*_{k^{-1}} df_{(\mathrm{Ad}^K_{k^{-1}}X)_R}$$

so that again (noting $\lambda^*_a\lambda^*_b = \lambda^*_{ba}$)

$$\begin{aligned} df_X(kg) &= \lambda^*_{(kg)^{-1}} df_{(\mathrm{Ad}^K_{(kg)^{-1}}X)_R} = \lambda^*_{(kg)^{-1}}\lambda^*_g\lambda^*_{g^{-1}} i_{\left(\mathrm{Ad}^K_{g^{-1}}\mathrm{Ad}^K_{k^{-1}}X\right)_R}\Omega \\ &= \lambda^*_{(kg)^{-1}}\lambda^*_g df_{(\mathrm{Ad}^K_{k^{-1}}X)_R}(g) = \lambda^*_{k^{-1}} df_{(\mathrm{Ad}^K_{k^{-1}}X)_R}(g). \end{aligned}$$

From

$$f_X(kg) = f_X(k) + f_{\mathrm{Ad}^K_{k^{-1}}X}(g)$$

and $f_X(e) = 0$ we see

$$f_X(k) = -f_{\mathrm{Ad}^K_{k^{-1}}X}(k^{-1}).$$

Thus with $g = e^{tY}$ we have

$$\frac{d}{dt} f_{\mathrm{Ad}^K_{k^{-1}}X}(g) = \frac{d}{dt}\left[(\mathrm{Ad}^K_{k^{-1}}X)\cdot d_1 f(e,g) - (\mathrm{Ad}^K_{k^{-1}}X)\cdot d_2 f(g,e)\right] = \frac{d}{dt} f_X(kg)\Big|_{t=0}$$

We would like to show

$$f_X(k^{-1}) = \theta(k,X) + X\cdot\dot{c}(e) - \mathrm{Ad}^K_k X\cdot\dot{c}(e)$$

for appropriate functions c. Each of $f_X(k^{-1})$, $\theta(k,X)$ and $X\cdot\dot{c}(e) - \mathrm{Ad}^K_k X\cdot\dot{c}(e)$ satisfies the cocycle condition (2.26) and the same initial condition $\theta(e,X) = 0$. Here both $f_X(k^{-1})$ and $\theta(k,X)$ satisfy the same differential equation

$$df_X(Y) = \omega(X,Y) = -\frac{d}{dt}\theta(e^{tY},X)|_{t=0}$$

and so (*cf.* [Nee02, §3])

$$f_X(k^{-1}) = \theta(k,X).$$

Chapter 3

Current Algebras and their Central Extensions

As mentioned in the introduction, in this thesis we are interested in central extensions of *current groups* and *current algebras*. Let us focus solely on the current algebras and their central extensions in this chapter. We shall also be interested in the adjoint and coadjoint representations on these extensions.

We begin by discussing central extensions of the loop algebra $L\mathfrak{g} := \text{Map}(S^1, \mathfrak{g})$. It was said by Khesin and Wendt [KW09] that "the loop algebra is in a sense too simple to give rise to a rich theory", which is why the pass to a central extension is beneficial. For example, the construction of an *affine Kac-Moody Lie algebra* is by means of a one-dimensional central extension of the loop algebra, which is discussed in detail in [Kac90], and does indeed have a rich theory associated to it. A reasonable next step is to consider current groups for an elliptic curve $E = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. This is called the *double loop group*, and we define an analogous extension to the affine Lie algebra, which is called the *elliptic Lie algebra*. Although the theory of double loop groups is sparser than the theory of loop groups, some key results appear in the papers [EF94] and [FK96] and we recall these in this chapter. The book [KW09] is also an excellent reference for loop algebras and double loop algebras, and we follow their exposition. As well as double loop algebras, we are interested in current algebras for more general compact Riemann surfaces Σ of genus g. We shall see that the current algebra $\text{Map}(\Sigma, \mathfrak{g})$ does not require any specification of complex structure on Σ (*ie.* it is just a two real-dimensional surface), but when we pass to central extensions of $\text{Map}(\Sigma, \mathfrak{g})$ we see that the extension depends upon the choice of complex structure. Throughout the thesis, G is a finite-dimensional, simple, compact Lie group with simple Lie algebra $\mathfrak{g}$, but in this chapter G is in addition simply connected.

3.1 Central Extensions of Loop Algebras

Let us commence the chapter by considering central extensions of the loop algebra $L\mathfrak{g}$. As the finite-dimensional Lie algebra $\mathfrak{g}$ is simple, it possesses a unique (up to scalar multiple) nondegenerate symmetric invariant bilinear form $\langle\cdot,\cdot\rangle$. Then we have an induced nondegenerate bilinear form on $L\mathfrak{g}$, which we shall also denote $\langle\cdot,\cdot\rangle$. It is given by

$$\langle\xi,\eta\rangle := \frac{1}{2\pi}\int_0^{2\pi}\langle\xi(\theta),\eta(\theta)\rangle d\theta$$

for $\xi,\eta\in L\mathfrak{g}$.

Definition 3.1.1. *We construct a one-dimensional central extension $\widehat{L\mathfrak{g}}$ of the loop algebra $L\mathfrak{g}$ by means of the 2-cocycle $\omega : L\mathfrak{g}\times L\mathfrak{g}\to\mathbb{R}$ given by*

$$\omega(\xi,\eta) := \langle\xi,\eta'\rangle = \frac{1}{2\pi}\int_0^{2\pi}\langle\xi(\theta),\eta'(\theta)\rangle d\theta, \tag{3.1}$$

where $\eta' = \frac{d\eta}{d\theta}$. As a vector space $\widehat{L\mathfrak{g}} = L\mathfrak{g}\oplus\mathbb{R}$, and the Lie structure on $\widehat{L\mathfrak{g}}$ is given by the bracket

$$[(\xi(\theta),j_1),(\eta(\theta),j_2)] = \big([\xi(\theta),\eta(\theta)],\omega(\xi,\eta)\big) \tag{3.2}$$

for $\xi,\eta\in L\mathfrak{g}$ and $j_1,j_2\in\mathbb{R}$.

The Lie algebra extension $\widehat{L\mathfrak{g}}$ lifts to a Lie group extension $\widehat{LG}$, and we would like to consider the adjoint action of the corresponding group $\widehat{LG}$ on $\widehat{L\mathfrak{g}}$. However, specifying how an element $g\in LG$ acts in the adjoint representation of $\widehat{LG}$ is enough: the centre of a group acts trivially in its adjoint representation.

Remark 3.1.2. *The extended Lie group $\widehat{LG}$ is topologically nontrivial and so it may not be described via a cocycle. See [KW09] for its construction, which shall not be required here.*

Proposition 3.1.3. *The adjoint action of an element $g\in LG$ on $(\xi,j)\in\widehat{L\mathfrak{g}}$ is given by*

$$Ad_g(\xi,j) = \big(Ad_g(\xi), j-\langle g^{-1}g',\xi\rangle\big), \tag{3.3}$$

where $g' = \dfrac{dg}{d\theta}$.

Proof. We must first check that the above map does indeed define an action, that is $Ad_g(Ad_h(\xi,j)) = Ad_{gh}(\xi,j)$ for all $g,h\in LG$. For the first factor there is nothing to check, as it is just the pointwise adjoint action. For the second factor we calculate:

$$j-\langle h^{-1}h',\xi\rangle-\langle g^{-1}g',h\xi h^{-1}\rangle = j-\langle (gh)^{-1}(gh)',\xi\rangle,$$

which shows that Ad_g does define an action on the centrally extended algebra $\widehat{L\mathfrak{g}}$. In order to see that it is the adjoint action of the group $\widehat{LG}$, we must check that its infinitesimal action coincides with the adjoint action of $\widehat{L\mathfrak{g}}$ on itself. This is a short calculation. □

Now we shall consider the (smooth part of the)[1] dual space $\widehat{L\mathfrak{g}}^*$ of $\widehat{L\mathfrak{g}}$. We may

[1] Formally, the dual space $\widehat{L\mathfrak{g}}^*$ contains elements which are not necessarily represented by a smooth 1-form, *cf.* [KW09].

consider an element of $\widehat{L\mathfrak{g}}^*$ as a pair $(A,k) \in L\mathfrak{g} \oplus \mathbb{R}$:

$$\widehat{L\mathfrak{g}}^* = \{(A,k) | A \in L\mathfrak{g}, k \in \mathbb{R}\}.$$

There is a nondegenerate pairing between $\widehat{L\mathfrak{g}}^*$ and $\widehat{L\mathfrak{g}}$ given by

$$\langle (A,k),(\xi,j)\rangle = \langle A,\xi\rangle_{L\mathfrak{g}} + kj = \frac{1}{2\pi}\int_0^{2\pi} \langle A(\theta),\xi(\theta)\rangle d\theta + kj.$$

Proposition 3.1.4. *The coadjoint action of an element $g \in LG$ on $\widehat{L\mathfrak{g}}^*$ is as follows:*

$$Ad_g^*(A,k) = (Ad_g A + kg'g^{-1}, k). \tag{3.4}$$

Proof. The coadjoint action is defined via

$$\langle Ad_g^*(A,k),(\xi,j)\rangle = \langle (A,k), Ad_{g^{-1}}(\xi,j)\rangle.$$

Then

$$\begin{aligned}
\langle (A,k), Ad_{g^{-1}}(\xi,j)\rangle &= \langle (A,k), \big(g^{-1}\xi g, j - \langle \xi, g(g^{-1})'\rangle\big)\rangle \\
&= \langle (A,k), \big(g^{-1}\xi g, j + \langle \xi, g'g^{-1}\rangle\big)\rangle \\
&= \langle A, g^{-1}\xi g\rangle + kj + k\langle \xi, g'g^{-1}\rangle \\
&= \langle Ad_g A, \xi\rangle + kj + k\langle \xi, g'g^{-1}\rangle \\
&= \langle Ad_g A + kg'g^{-1}, \xi\rangle + kj.
\end{aligned}$$

□

Corollary 3.1.5. *In the coadjoint representation of $\widehat{L\mathfrak{g}}$ on its dual $\widehat{L\mathfrak{g}}^*$, an element $\xi \in L\mathfrak{g}$ acts via*

$$\mathrm{ad}_\xi^*(A,k) = ([\xi,A] + k\xi', 0).$$

We earlier defined a nondegenerate bilinear form on the loop algebra $L\mathfrak{g}$, given by

$$\langle \xi,\eta\rangle := \frac{1}{2\pi}\int_0^{2\pi} \langle \xi(\theta),\eta(\theta)\rangle_{\mathfrak{g}} d\theta$$

for $\xi,\eta \in L\mathfrak{g}$. However, there cannot be a nondegenerate invariant bilinear form on the central extension $\widehat{L\mathfrak{g}}$, as the centre of any Lie algebra is orthogonal to its commutator subalgebra with respect to any invariant form (given $x = [u,v] \in [\mathfrak{g},\mathfrak{g}]$ and $z \in Z(\mathfrak{g})$ we have $\langle z,x\rangle = \langle z,[u,v]\rangle = \langle [z,u],v\rangle = 0$), and $\widehat{L\mathfrak{g}}$ is perfect [PS86], that is, $[\widehat{L\mathfrak{g}},\widehat{L\mathfrak{g}}] = \widehat{L\mathfrak{g}}$. For this reason a further extension of $\widehat{L\mathfrak{g}}$ is introduced on which we may define an invariant bilinear form. This is also called an affine (Kac-Moody) Lie algebra and is discussed in detail in [Kac90, Chapter 7].

Definition 3.1.6. *We define an extension $\dot{\widehat{L\mathfrak{g}}}$ of $\widehat{L\mathfrak{g}}$ as a semidirect product of $\widehat{L\mathfrak{g}}$ and the* *derivation* $\xi \mapsto \dfrac{d}{d\theta}\xi$. *As a vector space* $\widehat{L\mathfrak{g}}^{\cdot} = \widehat{L\mathfrak{g}} \oplus \mathbb{R} \oplus \mathbb{R}$, *and its Lie bracket is given by*

$$[(\xi_1, j_1, k_1), (\xi_2, j_2, k_2)] = \left([\xi_1, \xi_2] + k_1 \frac{d}{d\theta}\xi_2 - k_2 \frac{d}{d\theta}\xi_1, \omega(\xi_1, \xi_2), 0\right) \tag{3.5}$$

for $(\xi_i, j_i, k_i) \in \dot{\widehat{L\mathfrak{g}}} = \widehat{L\mathfrak{g}} \oplus \mathbb{R} \oplus \mathbb{R}$.

On $\dot{\widehat{L\mathfrak{g}}}$ we may define a nondegenerate bilinear form given by

$$\langle (\xi_1, j_1, k_1), (\xi_2, j_2, k_2) \rangle = \langle \xi_1, \xi_2 \rangle_{L\mathfrak{g}} + j_1 k_2 + j_2 k_1. \tag{3.6}$$

Proposition 3.1.7. *[PS86, 4.9.4] For* $g \in LG$ *then the adjoint action of* g *on* $\dot{\widehat{L\mathfrak{g}}} = \widehat{L\mathfrak{g}} \oplus \mathbb{R}$ *is given by*

$$g : (\xi, j, k) \mapsto \left(g\xi g^{-1} + k g' g^{-1}, j - \langle g^{-1} g', \xi \rangle - \frac{1}{2} k \langle g^{-1} g', g^{-1} g' \rangle, k\right). \tag{3.7}$$

It can be seen via a short calculation that the bilinear form (3.6) is invariant under the action (3.7) of $\widehat{LG}$ on $\dot{\widehat{L\mathfrak{g}}}$:

$$\begin{aligned}
&\langle Ad_g(\xi_1, j_1, k_1), Ad_g(\xi_2, j_2, k_2) \rangle \\
&\quad = \langle \big(g\xi_1 g^{-1} + k_1 g' g^{-1}, j_1 - \langle \xi_1, g^{-1} g' \rangle - \tfrac{1}{2} k_1 \langle g^{-1} g', g^{-1} g' \rangle, k_1\big), \\
&\qquad \big(g\xi_2 g^{-1} + k_2 g' g^{-1}, j_2 - \langle \xi_2, g^{-1} g' \rangle - \tfrac{1}{2} k_2 \langle g^{-1} g', g^{-1} g' \rangle, k_2\big) \rangle \\
&\quad = \langle \xi_1, \xi_2 \rangle + k_1 \langle g' g^{-1}, g\xi_2 g^{-1} \rangle + k_2 \langle g' g^{-1}, g\xi_1 g^{-1} \rangle + k_1 k_2 \langle g' g^{-1}, g' g^{-1} \rangle \\
&\qquad + j_1 k_2 - k_2 \langle \xi_1, g^{-1} g' \rangle - \tfrac{1}{2} k_1 k_2 \langle g^{-1} g', g^{-1} g' \rangle \\
&\qquad + j_2 k_1 - k_1 \langle \xi_2, g^{-1} g' \rangle - \tfrac{1}{2} k_1 k_2 \langle g^{-1} g', g^{-1} g' \rangle \\
&\quad = \langle \xi_1, \xi_2 \rangle + j_1 k_2 + j_2 k_1 \\
&\quad = \langle (\xi_1, j_1, k_1), (\xi_2, j_2, k_2) \rangle.
\end{aligned}$$

We note that the definition of the adjoint action (3.7) is the only possible definition that preserves the bilinear form (3.6) (*cf.* [PS86]) and coincides with (3.3) when $k = 0$.

Remark 3.1.8. *Note that [Kac90]* $\mathfrak{t} \oplus \mathbb{R} \oplus \mathbb{R}$ *is an* $(l+2)$*-dimensional commutative subalgebra in* $\dot{\widehat{L\mathfrak{g}}}$, *where* $\mathfrak{t}$ *is the Cartan subalgebra of the finite-dimensional Lie algebra* $\mathfrak{g}$, *identified with the subalgebra of constant loops* $S^1 \to \mathfrak{t}$ *in* $L\mathfrak{g}$.

3.2 Central Extensions of Current Algebras

Now let X be a smooth compact oriented connected manifold and $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. From [PS86] we have the following description of the universal central extension of a current algebra, where $\langle \cdot, \cdot \rangle$ denotes the (unique up to scalar multiple) nondegenerate bilinear form on $\mathfrak{g}$.

Proposition 3.2.1. *[PS86, 4.2.8] The universal central extension of $\mathfrak{g}^X$ is an extension by the space $\mathfrak{a} = \Omega^1(X)/d\Omega^0(X)$ of complex-valued 1-forms on X modulo exact forms. The extension is defined by the cocycle*

$$(\xi,\eta) \mapsto \langle \xi, d\eta\rangle \mod d\Omega^0(X) \tag{3.8}$$

where $\xi,\eta \in \mathfrak{g}^X$. (Notice that $(\xi,\eta) \mapsto \langle \xi, d\eta\rangle$ is skew-symmetric only up to an exact form.)

For a proof of universality, see [PS86, 4.2.8]. The universal central extension is given in the short exact sequence

$$0 \longrightarrow \Omega^1(X)/d\Omega^0(X) \longrightarrow U\mathfrak{g}^X \longrightarrow \mathfrak{g}^X \longrightarrow 0. \tag{3.9}$$

For a general compact manifold X, the space $\Omega^1(X)/d\Omega^0(X)$ is infinite-dimensional. However, for the circle S^1, every one-form is closed and so $\Omega^1(X)/d\Omega^0(X)$ is exactly the first de Rham cohomology group of S^1, and so is one-dimensional, $H^1_{\mathrm{dR}}(S^1) \simeq \mathbb{R}[d\theta]$. Then the universal central extension $UL\mathfrak{g}$ for $L\mathfrak{g}$ is isomorphic to the central extension $\widehat{L\mathfrak{g}}$ given in Definition 3.1.1.

For n-dimensional compact X we visualise the universal central extension as follows, *cf.* [KW09, 1.7]. Fix an element γ in the (smooth) dual space of $\Omega^1(X)/d\Omega^0(X)$. This is the space of closed $(n-1)$-forms $Z^{n-1}(X)$. Indeed, there is a pairing between a one-form $u \in \Omega^1(X)$ and an $(n-1)$-form γ given by $\langle u,\gamma\rangle := \int_X u \wedge \gamma$. When γ is closed, the pairing is zero if and only if the one-form u is exact, *ie.* $u \in d\Omega^0(X)$. Now we may associate a real-valued 2-cocycle to the closed $(n-1)$-form γ by

$$\omega_\gamma(\xi,\eta) = \int_X \langle \xi, d\eta\rangle \wedge \gamma.$$

In the next section we shall see that when X is a compact Riemann surface of genus g, although the centre of the universal central extension is infinite-dimensional, we can in fact obtain a central extension with a *finite-dimensional* (in fact, g-dimensional) centre which relies on the complex structure of the surface.

3.3 Current Algebras for a Compact Riemann Surface

Let Σ be a compact Riemann surface of genus g. In choosing a complex structure on Σ, we may define a g-dimensional central extension as follows, *cf.* [EF94, Section 1]. Denote by $\mathscr{H}_\Sigma$ the space of holomorphic differential forms on Σ. This is g complex-dimensional. Now let Φ be the identity element in $\mathscr{H}_\Sigma \otimes \mathscr{H}_\Sigma^*$, which may be regarded as a holomorphic differential on Σ with values in $\mathscr{H}_\Sigma^*$, the dual of the space $\mathscr{H}_\Sigma$.

Definition 3.3.1. *Define a 2-cocycle Ω on $\mathfrak{g}^\Sigma$ with values in $\mathscr{H}_\Sigma^*$ by*

$$\Omega(\xi,\eta) = \int_\Sigma \Phi \wedge \langle \xi, d\eta\rangle, \tag{3.10}$$

where $\xi,\eta \in \mathfrak{g}^\Sigma$. In other words, given a holomorphic one-form α on Σ, and decomposing d

into its Dolbeault components $d = \partial + \bar\partial$, we set the value of the cocycle on α to be

$$\Omega(\xi,\eta)(\alpha) = \int_\Sigma \alpha \wedge \langle \xi, \bar\partial \eta \rangle.$$

This cocycle defines a g-dimensional central extension of $\mathfrak{g}^\Sigma$, which we shall call $\widehat{\mathfrak{g}}^\Sigma$. The Lie bracket on $\widehat{\mathfrak{g}}^\Sigma$ is given by

$$[(\xi_1, j_1),(\xi_2, j_2)] = ([\xi_1,\xi_2],\Omega(\xi_1,\xi_2)) \tag{3.11}$$

for $(\xi_1, j_1),(\xi_2, j_2) \in \mathfrak{g}^\Sigma \oplus \mathbb{C}$. This may be expressed in a short exact sequence

$$0 \longrightarrow \mathscr{H}^*_\Sigma \longrightarrow \widehat{\mathfrak{g}}^\Sigma \longrightarrow \mathfrak{g}^\Sigma \longrightarrow 0. \tag{3.12}$$

The centrally extended Lie algebra $\widehat{\mathfrak{g}}^\Sigma$ lifts to a centrally extended Lie group $\widehat{G}^\Sigma$, which we shall define in Chapter 7. The Lie algebra $\widehat{\mathfrak{g}}^\Sigma$ may be obtained as a quotient of the universal central extension $U\mathfrak{g}^\Sigma$ by factorising it by the subgroup of elements $\mu \in \mathfrak{a} = \Omega^1(X)/d\Omega^0(X)$ such that

$$\int_\Sigma \Phi \wedge \tilde\mu = 0, \tag{3.13}$$

where $\tilde\mu = \mu + df$. It can be seen that $\widehat{\mathfrak{g}}^\Sigma$ relies on the complex structure of Σ (that is, the choice of basis of $\mathscr{H}_\Sigma$), whereas the universal central extension $U\mathfrak{g}^\Sigma$ does not.

Now let E be a compact Riemann surface of genus one, that is, a complex torus $E = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ with τ a complex number with positive imaginary part, *ie.* $\tau \in \mathscr{H}$, the upper half-plane. Let z be a holomorphic coordinate on E. The extension defined in 3.3.1 has complex-dimension one:

$$\widehat{\mathfrak{g}}^E = \mathfrak{g}^E \oplus \mathbb{C}\omega^\vee, \tag{3.14}$$

where we define $\omega^\vee$ to be the single generator of $\mathscr{H}^*_E$ with $\omega^\vee(dz) = 1$, where dz is the single generator of $\mathscr{H}_E$. We may simplify the cocycle slightly in this case by redefining it as $\Omega : \mathfrak{g}^E \times \mathfrak{g}^E \to \mathbb{C}$ with

$$\Omega(\xi_1,\xi_2) = \langle \xi_1, \bar\partial\xi_2 \rangle = \int_E \langle \xi_1(z,\bar z), \bar\partial \xi_2(z,\bar z)\rangle \, dz \wedge d\bar z, \tag{3.15}$$

for $\xi_1, \xi_2 \in \mathfrak{g}^E$. In the above we have defined a bilinear form $\langle\cdot,\cdot\rangle := \int_E \langle\cdot,\cdot\rangle dz \wedge d\bar z$ on $\mathfrak{g}^E$, as for the loop algebra. The Lie structure on $\widehat{\mathfrak{g}}^E$ is given by the bracket

$$\big[(\xi_1(z,\bar z), j_1),(\xi_2(z,\bar z), j_2)\big] = \big([\xi_1(z,\bar z),\xi_2(z,\bar z)],\Omega(\xi_1,\xi_2)\big). \tag{3.16}$$

We call $\widehat{\mathfrak{g}}^E$ the *double loop algebra, cf.* [KW09]. The adjoint and coadjoint actions of the group $\widehat{G}^E$ on the Lie algebra $\widehat{\mathfrak{g}}^E$ and its dual are defined similarly to those of the loop group $\widehat{LG}$.

Definition 3.3.2. *In the adjoint representation of $\widehat{G}^E$, the action of an element $g \in G^E$ on $\widehat{\mathfrak{g}}^E$*

is given by

$$g : (\xi, j) \mapsto \left(Ad_g \xi, j - \int_E \langle \xi, g^{-1} \bar{\partial} g \rangle dz \wedge d\bar{z} \right) \tag{3.17}$$

for all $(\xi, j) \in \mathfrak{g}^E \oplus \mathbb{C}$.

In fact, this completely determines the adjoint action of the extended group $\widehat{G}^E$, as the centre acts trivially. The pairing between $\widehat{\mathfrak{g}}^E$ and $(\widehat{\mathfrak{g}}^E)^*$ is

$$\langle (A, k), (\xi, j) \rangle = \langle A, \xi \rangle + kj. \tag{3.18}$$

Proposition 3.3.3. *In the coadjoint representation of* $\widehat{G}^E$*, the action of an element* $g \in G^E$ *on* $(\widehat{\mathfrak{g}}^E)^*$ *is given by*

$$g : (A, k) \mapsto (Ad_g A + k \bar{\partial} g g^{-1}, k). \tag{3.19}$$

Proof. The coadjoint action is defined by

$$\langle Ad_g^*(A, k), (\xi, j) \rangle = \langle (A, k), Ad_{g^{-1}}(\xi, j) \rangle.$$

The calculation is exactly analogous to (3.1). □

3.3.1 Current Groups as Gauge Groups

In this section let G be in addition simply connected. Let us consider a principal G-bundle P over E. We define the *group of gauge transformations* $\mathrm{Gau}(P)$ to be the group of G-equivariant bundle diffeomorphisms Φ which preserve the fibres, i.e. the diagram

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & P \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi} \\ & & E \end{array}$$

commutes. The group of gauge transformations (or *gauge group*) $\mathrm{Gau}(P)$ coincides with the current group G^E when P is topologically trivial[2]. Let $\mathscr{D}^{(0,1)}$ denote the space of anti-holomorphic connections

$$\mathscr{D}^{(0,1)} := \{\nabla = k \frac{\partial}{\partial \bar{z}} + A | A \in \Omega^1(\Sigma)\}.$$

The group of gauge transformations $\mathrm{Gau}(P)$ acts naturally on the space of connections $\mathscr{D}^{(0,1)}$ on P via

$$g : \nabla \mapsto g \nabla g^{-1} + k \frac{\partial g}{\partial \bar{z}} g^{-1}.$$

The dual space $(\widehat{\mathfrak{g}}^E)^*$ may be identified with $\mathscr{D}^{(0,1)}$ and the coadjoint action of G^E on $(\widehat{\mathfrak{g}}^E)^*$ can then be thought of as the action of the gauge group on the connections. (We note that it is simply convention that *anti-holomorphic* connections are considered and indeed in

[2] See Appendix D.

Section 3.3.3 we shall consider another extension of $\mathfrak{g}^E$, the dual space of which may be thought of as the space of *holomorphic* connections.)

Let us fix a hyperplane $L_k \subset (\widehat{\mathfrak{g}}^E)^*$, with k constant. Note that these hyperplanes are invariant under the coadjoint action (3.19) of G^E. We have the following interesting theorem of Etingof and Frenkel:

Theorem 3.3.4. *[EF94] Coadjoint orbits of the group G^E in the hyperplane L_k are in one-to-one correspondence with equivalence classes of holomorphic G-bundles over E.*

See Appendix D for the proof and more details on the principal bundle and gauge group interpretation.

3.3.2 Bilinear Forms and the Elliptic Lie Algebra

As in the loop case, a nondegenerate invariant bilinear form cannot exist on the centrally extended Lie algebra $\widehat{\mathfrak{g}}^E$. It is reasonable therefore to construct a further extension of $\widehat{\mathfrak{g}}^E$, exactly analogous to the affine Lie algebra, for which there exists a nondegenerate bilinear form. We call this the *elliptic Lie algebra*, and define it as follows:

Definition 3.3.5. *The elliptic Lie algebra is an extension of $\widehat{\mathfrak{g}}^E$, constructed as a semidirect product by $\mathbb{C}$. Indeed, we define $\sigma : \mathbb{C} \to \mathrm{Der}(\widehat{\mathfrak{g}}^E)$ by $k \mapsto k\delta$ where $\delta := \dfrac{\partial}{\partial \bar{z}}$ and δ kills the centre of $\widehat{\mathfrak{g}}^E$. The Lie bracket is given by:*

$$[(\xi_1, j_1, k_1), (\xi_2, j_2, k_2)] = \Big([\xi_1, \xi_2] + k_1 \frac{\partial}{\partial \bar{z}} \xi_2 - k_2 \frac{\partial}{\partial \bar{z}} \xi_1, \Omega(\xi_1, \xi_2), 0\Big), \tag{3.20}$$

where $\Omega : \mathfrak{g}^E \times \mathfrak{g}^E \to \mathbb{C}\omega^\vee$ is as in (3.15).

Proposition 3.3.6. *There is an action of the Lie group G^E on the elliptic Lie algebra $\hat{\widehat{\mathfrak{g}}}^E$ analogous to (3.7) given by*

$$\begin{aligned} g : (\xi, j, k) \to \Big(\mathrm{Ad}_g \xi + k\bar{\partial} g g^{-1}, j - \int_E \langle g^{-1} \bar{\partial} g, \xi \rangle dz \wedge d\bar{z} \\ - \frac{1}{2} k \int_E \langle g^{-1} \bar{\partial} g, g^{-1} \bar{\partial} g \rangle dz \wedge d\bar{z}, k\Big), \end{aligned} \tag{3.21}$$

Proof. We have

$$\begin{aligned}
Ad_{gh}(\xi, j, k) &= \Big((gh)\xi(gh)^{-1} + k\bar{\partial}(gh)(gh)^{-1}, j - \langle (gh)^{-1}\bar{\partial}(gh), \xi\rangle \\
&\quad - \tfrac{1}{2}k\langle (gh)^{-1}\bar{\partial}(gh), (gh)^{-1}\bar{\partial}(gh)\rangle, k\Big) \\
&= \Big(gh\xi h^{-1}g^{-1} + k\bar{\partial}gg^{-1} + kg\bar{\partial}hh^{-1}g^{-1}, j - \langle h^{-1}g^{-1}\bar{\partial}gh + h^{-1}\bar{\partial}h, \xi\rangle \\
&\quad - \tfrac{1}{2}k\langle h^{-1}g^{-1}\bar{\partial}gh + h^{-1}\bar{\partial}h, h^{-1}g^{-1}\bar{\partial}gh + h^{-1}\bar{\partial}h\rangle, k\Big) \\
&= \Big(Ad_g(Ad_h\xi + k\bar{\partial}hh^{-1}) + \bar{\partial}gg^{-1}, j - \langle h^{-1}\bar{\partial}h, \xi\rangle \\
&\quad - \tfrac{1}{2}k\langle h^{-1}\bar{\partial}h, h^{-1}\bar{\partial}h\rangle - \langle g^{-1}\bar{\partial}g, Ad_h\xi + k\bar{\partial}hh^{-1}\rangle \\
&\quad - \tfrac{1}{2}k\langle g^{-1}\bar{\partial}g, g^{-1}\bar{\partial}g\rangle, k\Big) \\
&= Ad_g(Ad_h(\xi, j, k)),
\end{aligned}$$

and (3.21) does indeed define an action of G^E on the elliptic Lie algebra $\hat{\hat{\mathfrak{g}}}^E$. □

We now define a G^E-invariant bilinear form on $\hat{\hat{\mathfrak{g}}}^E$.

Proposition 3.3.7. *A G^E-invariant bilinear form on $\hat{\hat{\mathfrak{g}}}^E$ is given by*

$$\langle (\xi_1, j_1, k_1), (\xi_2, j_2, k_2)\rangle = \langle \xi_1, \xi_2\rangle + j_1k_2 + j_2k_1. \tag{3.22}$$

Proof. To show G^E- invariance of the bilinear form, we calculate

$$\begin{aligned}
&\langle Ad_g(\xi_1, j_1, k_1), Ad_g(\xi_2, j_2, k_2)\rangle \\
&\qquad = \langle \big(g\xi_1g^{-1} + k_1\bar{\partial}gg^{-1}, j_1 - \langle \xi_1, g^{-1}\bar{\partial}g\rangle \\
&\qquad\quad - \tfrac{1}{2}k_1\langle g^{-1}\bar{\partial}g, g^{-1}\bar{\partial}g\rangle, k_1\big), \\
&\qquad\quad \big(g\xi_2g^{-1} + k_2\bar{\partial}gg^{-1}, j_2 - \langle \xi_2, g^{-1}\bar{\partial}g\rangle \\
&\qquad\quad - \tfrac{1}{2}k_2\langle g^{-1}\bar{\partial}g, g^{-1}\bar{\partial}g\rangle, k_2\big)\rangle \\
&\qquad = \langle \xi_1, \xi_2\rangle + k_1\langle \bar{\partial}gg^{-1}, g\xi_2g^{-1}\rangle + k_2\langle \bar{\partial}gg^{-1}, g\xi_1g^{-1}\rangle \\
&\qquad\quad + k_1k_2\langle \bar{\partial}gg^{-1}, \bar{\partial}gg^{-1}\rangle + j_1k_2 - k_2\langle \xi_1, g^{-1}\bar{\partial}g\rangle \\
&\qquad\quad - \tfrac{1}{2}k_1k_2\langle g^{-1}\bar{\partial}g, g^{-1}\bar{\partial}g\rangle + j_2k_1 - k_1\langle \xi_2, g^{-1}\bar{\partial}g\rangle \\
&\qquad\quad - \tfrac{1}{2}k_1k_2\langle g^{-1}\bar{\partial}g, g^{-1}\bar{\partial}g\rangle \\
&\qquad = \langle \xi_1, \xi_2\rangle + j_1k_2 + j_2k_1,
\end{aligned} \tag{3.23}$$

which shows that the bilinear form is indeed G^E-invariant. □

3.3.3 Another Elliptic Lie Algebra

We now construct an alternative extension of $\mathfrak{g}^E$, following a construction of Bertola [Ber99]. The extension is a slight modification of the extension $\hat{\hat{\mathfrak{g}}}^E$, but is homotopy equivalent. The reason for the modification, as we shall see, is that the new extension admits a *finite-dimensional* maximal nilpotent subalgebra, which we shall make use of in future chapters.

Definition 3.3.8. *Let $\bar{\Phi}$ be the identity element in $\bar{\mathscr{H}}_\Sigma \otimes \bar{\mathscr{H}}_\Sigma^*$, where $\bar{\mathscr{H}}_\Sigma$ denotes the (g-dimensional) space of anti-holomorphic forms on Σ. The element $\bar{\Phi}$ may be regarded as an anti-holomorphic differential on Σ with values in $\bar{\mathscr{H}}_\Sigma^*$. Define a 2-cocycle on $\mathfrak{g}^\Sigma$ with values in $\bar{\mathscr{H}}_\Sigma^*$ by*

$$\tilde{\Omega}(\xi,\eta) = \int_\Sigma \bar{\Phi} \wedge \langle \xi, d\eta \rangle, \tag{3.24}$$

where $\xi,\eta \in \mathfrak{g}^\Sigma$. This cocycle defines a g-dimensional central extension of $\mathfrak{g}^\Sigma$, which we shall call $\tilde{\mathfrak{g}}^\Sigma$. This may be expressed in a short exact sequence

$$0 \longrightarrow \bar{\mathscr{H}}_\Sigma^* \longrightarrow \tilde{\mathfrak{g}}^\Sigma \longrightarrow \mathfrak{g}^\Sigma \longrightarrow 0. \tag{3.25}$$

We see this is exactly analogous to the elliptic algebra defined in 3.12, with holomorphic being replaced by anti-holomorphic. Now for an elliptic curve E, we obtain another one-dimensional central extension $\tilde{\mathfrak{g}}^E = \mathfrak{g}^E \oplus \mathbb{C}\bar{\omega}^\vee$, where $\bar{\omega}^\vee$ denotes the generator for $\bar{\mathscr{H}}_E^*$, the dual of the space of anti-holomorphic differential forms on Σ. The cocycle may be written

$$\tilde{\Omega}(\xi,\eta) = \int_\Sigma d\bar{z} \wedge \langle \xi, \partial\eta \rangle. \tag{3.26}$$

The adjoint and coadjoint actions in the previous section may also be defined analogously for $\tilde{\mathfrak{g}}^E$, exchanging $\bar{\partial}$ for ∂ in the definitions. In this context however the dual space $(\tilde{\mathfrak{g}}^E)^*$ may not of course be identified with the space of anti-holomorphic connections, but rather with the space of *holomorphic* connections.

Remark 3.3.9. *We have defined two cocycles for holomorphic and anti-holomorphic one-forms; we could have equally defined a cocycle using harmonic forms.*

Let us now construct an extension of $\tilde{\mathfrak{g}}^E$ in exactly the same way as we constructed the elliptic Lie algebra $\hat{\mathfrak{g}}^E$ from $\widehat{\mathfrak{g}}^E$. Namely, we again take a semidirect product of $\tilde{\mathfrak{g}}^E$ and $\mathbb{C}$ via the representation $\sigma : \mathbb{C} \to \mathrm{Der}(\tilde{\mathfrak{g}}^E)$, $k \mapsto k\delta$ where $\delta := \frac{\partial}{\partial \bar{z}}$, and δ kills the centre of $\tilde{\mathfrak{g}}^E$.

Definition 3.3.10. *The elliptic Lie algebra $\tilde{\mathfrak{g}}^E$ is the vector space $\dot{\tilde{\mathfrak{g}}}^E = \mathfrak{g}^E \oplus \mathbb{C} \oplus \mathbb{C}$ with Lie bracket is given by*

$$[(\xi_1, j_1, k_1), (\xi_2, j_2, k_2)] = \left([\xi_1, \xi_2] + k_1 \frac{\partial}{\partial \bar{z}} \xi_2 - k_2 \frac{\partial}{\partial \bar{z}} \xi_1, \tilde{\Omega}(\xi_1, \xi_2), 0 \right), \tag{3.27}$$

where $(\xi_i, j_i, k_i) \in \dot{\tilde{\mathfrak{g}}}^E$ and $\tilde{\Omega} : \tilde{\mathfrak{g}}^E \times \tilde{\mathfrak{g}}^E \to \mathbb{C}\bar{\omega}^\vee$ is given in (3.26).

3.4 Weyl Group of the Elliptic Lie algebra

We now discuss a section of the thesis of Bertola [Ber99, 1.2]. The author constructs the aforementioned elliptic Lie algebra $\dot{\tilde{\mathfrak{g}}}^E$ and seeks to define its 'Weyl group'. In this context the Weyl group of an infinite-dimensional Lie algebra is defined to be the subgroup of the group of automorphisms normalising a Cartan subalgebra, modulo the centraliser. This is not a common definition in the literature. The aforementioned Weyl group is claimed to be isomorphic to the *Jacobi group*, an object which is central to this thesis and will be defined in Chapter 6. We shall show that there are some errors in the claim that the Weyl group of the elliptic Lie algebra $\dot{\tilde{\mathfrak{g}}}^E$ is the Jacobi group, but we nevertheless make a connection between the Jacobi group and $\dot{\tilde{\mathfrak{g}}}^E$ in Chapter 7.

3.4.1 Cartan Subalgebras

Recall that a Cartan subalgebra of a Lie algebra is a self-normalising nilpotent subalgebra. Below $\mathfrak{t}$ is a Cartan subalgebra of the finite-dimensional Lie algebra $\mathfrak{g}$.

Proposition 3.4.1. $\mathfrak{t}^E \oplus \mathbb{C}\omega^\vee$ *is a Cartan subalgebra of* $\widehat{\mathfrak{g}}^E = \mathfrak{g}^E \oplus \mathbb{C}\omega^\vee$.

Proof. We want to show that $\mathfrak{t}^E \oplus \mathbb{C}\omega^\vee$ is a self-normalising nilpotent subalgebra of $\widehat{\mathfrak{g}}^E$. Let $(\xi_i, j_i) \in \mathfrak{t}^E \oplus \mathbb{C}\omega^\vee$. Now $[[(\xi_1, j_1), (\xi_2, j_2)], (\xi_3, j_3)] = [(0, \Omega(\xi_1, \xi_2)), (\xi_3, j_3)] = (0,0)$ so $\mathfrak{t}^E \oplus \mathbb{C}\omega^\vee$ is nilpotent. Furthermore $[(\xi_1, j_1), (\xi_2, j_2)] \in \mathfrak{t}^E \oplus \mathbb{C}\omega^\vee$ for all $(\xi_i, j_i) \in \mathfrak{t}^E \oplus \mathbb{C}\omega^\vee$ so certainly $\mathfrak{t}^E \oplus \mathbb{C}\omega^\vee$ is contained in the normaliser of $\mathfrak{t}^E \oplus \mathbb{C}\omega^\vee$ in $\widehat{\mathfrak{g}}^E$. To show $\mathfrak{t}^E \oplus \mathbb{C}\omega^\vee$ is in fact equal to the normaliser of $\mathfrak{t}^E \oplus \mathbb{C}\omega^\vee$, we require that given $(x, j) \in \mathfrak{g}^E \oplus \mathbb{C}\omega^\vee$ and $[(x, j), (\xi, j_1)] \in \mathfrak{t}^E \oplus \mathbb{C}\omega^\vee$ for all $(\xi, j_1) \in \mathfrak{t}^E \oplus \mathbb{C}\omega^\vee$, then $(x, j) \in \mathfrak{t}^E \oplus \mathbb{C}\omega^\vee$. But as the Lie bracket is defined pointwise, this must be the case. □

The same argument may also be applied to show that $\mathfrak{t}^E \oplus \mathbb{C}\bar{\omega}^\vee$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}^E = \mathfrak{g}^E \oplus \mathbb{C}\bar{\omega}^\vee$.

Proposition 3.4.2. $\mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$ *is a Cartan subalgebra of* $\dot{\tilde{\mathfrak{g}}}^E$*, where* $\mathfrak{t}$ *is regarded as a subalgebra of* $\mathfrak{t}^E$ *of constant maps* $E \to \mathfrak{t}$.

Proof. We want to show that $\mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$ is a self-normalising nilpotent subalgebra of $\dot{\tilde{\mathfrak{g}}}^E$. Now $[(\xi_1, j_1, k_1), (\xi_2, j_2, k_2)] = 0$ as ξ_1, ξ_2 are constant maps so all derivatives are zero. Then $\mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$ is abelian, and so nilpotent. Furthermore $[(\xi_1, j_1, k_1), (\xi_2, j_2, k_2)] \in \mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$ for all $(\xi_i, j_i, k_i) \in \mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$ so certainly $\mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$ normalises $\mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$ inside $\dot{\tilde{\mathfrak{g}}}^E$. To show $\mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$ is exactly the normaliser of $\mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$, we require that given $(x, j, k) \in \mathfrak{g}^E \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$ and $[(x, j), (\xi, j_1)] \in \mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$ for all $(\xi, j_1, k_1) \in \mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$, then $(x, j, k) \in \mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$. But $\mathfrak{t}$ is self-normalising in $\mathfrak{g}$ so $\mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$ must be self-normalising in $\dot{\tilde{\mathfrak{g}}}^E$. □

The previous proposition shows that the infinite-dimensional Lie algebra $\dot{\tilde{\mathfrak{g}}}^E$ admits a *finite-dimensional* Cartan subalgebra, which is not the case in $\dot{\widehat{\mathfrak{g}}}^E$ (as purely holomorphic maps $E \to \mathfrak{t}$ also normalise $\mathfrak{t} \oplus \omega^\vee \oplus \mathbb{C}\delta \subset \dot{\widehat{\mathfrak{g}}}^E$). This is the reason why [Ber99] insists upon the unconventional choice of cocycle $\tilde{\Omega}$, in order to obtain a finite-dimensional Cartan subalgebra.

We now discuss a transformation on $\dot{\tilde{\mathfrak{g}}}^E$, as defined in [Ber99, Proposition 1.2.4].

$$(\xi, j, k) \mapsto \big(\mathrm{Ad}_g \xi + k\bar{\partial} g g^{-1}, j - \int_E \langle g^{-1}\partial g, \xi\rangle dz \wedge d\bar{z} \\ - \frac{1}{2} k \int_E \langle g^{-1}\partial g, g^{-1}\bar{\partial} g\rangle dz \wedge d\bar{z}, k\big), \tag{3.28}$$

where $(\xi, j, k) \in \mathfrak{g}^E \oplus \mathbb{C} \oplus \mathbb{C}$, and $g \in G^E$. Note that this transformation differs from the action of G^E on $\hat{\mathfrak{g}}^E$ in (3.21). The difference is in the central extension (j-coordinate) where the operator $\bar{\partial}$ is replaced by ∂.

Proposition 3.4.3. *The above definition does not describe an action of the current group G^E, as claimed in [Ber99, 1.2.4].*

Proof. We see from the following calculation that (3.28) does not describe an action of G^E.

$$
\begin{aligned}
(gh)\diamond(\xi,j,k) &= \big(Ad_{gh}\xi + k\bar{\partial}(gh)(gh)^{-1}, j - \langle (gh)^{-1}\partial(gh), \xi\rangle \\
&\quad -\tfrac{1}{2}k\langle (gh)^{-1}\partial(gh), (gh)^{-1}\bar{\partial}(gh)\rangle, k\big) \\
&= \big(Ad_{gh}\xi + k\bar{\partial}gg^{-1} + kg\bar{\partial}hh^{-1}g^{-1}, j - \langle h^{-1}g^{-1}\partial gh + h^{-1}\partial h, \xi\rangle \\
&\quad -\tfrac{1}{2}k\langle h^{-1}g^{-1}\partial gh + h^{-1}\partial h, h^{-1}g^{-1}\bar{\partial}gh + h^{-1}\bar{\partial}h\rangle, k\big) \\
&= \big(Ad_g(Ad_h\xi + k\bar{\partial}hh^{-1}) + \bar{\partial}gg^{-1}, j - \langle h^{-1}\partial h, \xi\rangle - \tfrac{1}{2}k\langle h^{-1}\partial h, h^{-1}\bar{\partial}h\rangle \\
&\quad -\langle g^{-1}\partial g, Ad_h\xi + k\bar{\partial}hh^{-1}\rangle - \tfrac{1}{2}k\langle g^{-1}\partial g, g^{-1}\bar{\partial}g\rangle, k\big),
\end{aligned}
$$

and

$$
\begin{aligned}
g\diamond(h\diamond(\xi,j,k)) &= g(Ad_h\xi + k\bar{\partial}hh^{-1}, j - \langle h^{-1}\partial h, \xi\rangle \\
&\quad -\tfrac{1}{2}k\langle h^{-1}\partial h, h^{-1}\bar{\partial}h\rangle, k) \\
&= (Ad_g(Ad_h\xi + k\bar{\partial}hh^{-1}) + k\bar{\partial}gg^{-1}, j - \langle h^{-1}\partial h, \xi\rangle \\
&\quad -\tfrac{1}{2}k\langle h^{-1}\partial h, h^{-1}\bar{\partial}h\rangle \\
&\quad -\langle g^{-1}\partial g, Ad_h\xi + k\bar{\partial}hh^{-1}\rangle - \tfrac{1}{2}k\langle g^{-1}\partial g, g^{-1}\bar{\partial}g\rangle, k).
\end{aligned}
$$

The above statements differ by a factor of

$$\frac{1}{2}k\left[\langle g^{-1}\partial g, \bar{\partial}hh^{-1}\rangle - \langle \partial hh^{-1}, g^{-1}\bar{\partial}g\rangle\right] \tag{3.29}$$

which may also be written as

$$\frac{1}{2}k\int_E \langle g^{-1}dg \wedge dhh^{-1}\rangle. \tag{3.30}$$

□

We claim that the definition described in (3.28) is in fact a *projective representation* coming from a certain central extension of G^E. In Chapter 7 we define a new one complex-dimensional central extension of the group G^E, which we shall call $\widehat{G}^E_C$, and we modify (3.28) appropriately to give an action of an element $(g,0) \in G^E \oplus \mathbb{C}$ on the space $\dot{\hat{\mathfrak{g}}}^E$.

Further, we note that (3.28) is not an automorphism of $\dot{\hat{\mathfrak{g}}}^E$, that is

$$[g\diamond(\xi_1,j_1,k_1), g\diamond(\xi_2,j_2,k_2)] \neq g\diamond[(\xi_1,j_1,k_1),(\xi_2,j_2,k_2)]$$

and so cannot be contained in the Weyl group.

Remark 3.4.4. *There is no Lie group corresponding to the Lie algebra* $\dot{\tilde{\mathfrak{g}}}^E$*, so there is no group action whose infinitesimal action corresponds to the Lie bracket (3.20).*

Chapter 4

Topology of Current Groups

In order to better understand the possible group central extensions of G^{Σ} it is important to learn more about the topology of G^{Σ}. In this chapter we shall discuss the homotopy and homology groups of G^{Σ}, considering both based and unbased maps. We use the notation $\mathrm{Map}(\Sigma, G)$ for unbased maps and $\mathrm{Map}_*(\Sigma, G)$ for based maps. When it is necessary to specify base points we refer to the base point on Σ as z_0 (or x_0 if we are considering a general compact smooth manifold X), and the base point (identity element) on G as e.

The homotopy and homology groups of G^{Σ} were given in [PS86] for the case where G is a simply connected Lie group. In particular, it was found that when G is simply connected, the infinite-dimensional Lie group G^{Σ} is connected. We extend these results to derive the homotopy and homology groups where G is not necessarily simply connected, including the description for the set of connected components $\pi_0(G^{\Sigma})$.

Topologically, a current group is a *mapping space* endowed with the *compact open topology*:

Definition 4.0.5. *For X, Y topological spaces, the* mapping space $\mathrm{Map}(X, Y)$ *is defined to be the set of all continuous maps $f : X \to Y$ with the* compact open *topology defined as follows. Given a compact subset $K \subset X$ and an open subset $U \subset Y$, let*

$$V(K, U) = \{f \in \mathrm{Map}(X, Y) | f(K) \subset U\}.$$

The collection of all such $V(K, U)$ forms a sub-basis of $\mathrm{Map}(X, Y)$.

For more information on mapping spaces, see *eg.* [Mau80].

4.1 Preliminaries

We first review some of the topological definitions we shall use in this chapter. For reference, see for example [Mau80]. If X and Y are based topological spaces, we use $[X, Y]$ to denote the set of homotopy classes of base point preserving maps from X to Y. We use the notation $X \sim Y$ for homotopy equivalent spaces and $X \simeq Y$ for homeomorphic spaces.

Definition 4.1.1. *The* smash product $X \wedge Y$ *of two topological spaces* X *and* Y*, with base points* x_0 *and* y_0 *respectively, is the quotient of the space* $X \times Y$ *under the identification* $(x_0, y) \sim (x, y_0)$ *for all* $x \in X$ *and* $y \in Y$*:*

$$X \wedge Y := X \times Y / X \vee Y. \tag{4.1}$$

We have, for example, $S^m \wedge S^n$ homeomorphic to S^{m+n}.

Definition 4.1.2. *The* reduced suspension sX *of a topological space* X *is defined as the smash product* $sX := X \wedge S^1$*, and so for example, the reduced suspension of* S^n *is homeomorphic to* S^{n+1}.

We have that

$$[sX, Y] \simeq [X, \Omega Y], \tag{4.2}$$

where ΩY is the loop space of a based space Y, also denoted Y^{S^1}. In fact, in general, we have that

$$[X \wedge Y, Z] \simeq [X, Z^Y] \tag{4.3}$$

for a locally compact Hausdorff space Y. We also define $s^k X := X \wedge S^k$, which follows from the definition of sX and the fact that $S^n \wedge S^m \simeq S^{n+m}$. Now (from e.g. [Mau80, 6.2.22]) we have the homeomorphism

$$(X \vee Y) \wedge Z \simeq (X \wedge Z) \vee (Y \wedge Z), \tag{4.4}$$

and so $s(X \vee Y) = (X \vee Y) \wedge S^1 \simeq (X \wedge S^1) \vee (Y \wedge S^1) = sX \vee sY$. We also note that when Z is an H-space[1] (so, in particular, a topological group or mapping space) that

$$[X \vee Y, Z] = [X, Z] \times [Y, Z] \tag{4.5}$$

and so

$$[s(X \vee Y), Z] = [X \vee Y, \Omega Z] = [X, \Omega Z] \times [Y, \Omega Z] = [sX, Z] \times [sY, Z] = [sX \vee sY, Z]. \tag{4.6}$$

We now introduce fibrations and cofibrations.

Definition 4.1.3. *A continuous mapping* $f : A \to B$ *is called a* fibration *if the pair* (A, B) *satisfies the* homotopy lifting property *with respect to all spaces* Y*. That is, given any homotopy* $Y \to B$ *can be lifted to a homotopy* $Y \to A$.

Definition 4.1.4. *A continuous mapping* $f : A \to B$ *is called a* cofibration *if the pair* (A, B) *satisfies the* homotopy extension property *with respect to all spaces* Z*. That is, any homotopy* $A \to Z$ *can be extended to a homotopy* $B \to Z$.

[1] An H-space is a topological space Z with continuous mapping $\mu : Z \times Z \to Z$ and identity element z_0 such that $\mu(z, z_0) = z = \mu(z_0, z)$ for all $z \in Z$.

Thus if $A \subset B$, the inclusion map $i : A \to B$ is a cofibration if the pair (A, B) has the homotopy extension property.

Definition 4.1.5. *Given a map $f : A \to B$, the* mapping cone C_f *is defined to be the space obtained from B and $cA := A \wedge I$ (the reduced cone of A) by identifying for each $a \in A$ the point $a \wedge 0$ in cA and $f(a) \in B$.*

Note that when $f(A) \subset B$ is null-homotopic, then we have the homotopy

$$C_f \sim C_* = sA \vee B. \tag{4.7}$$

The map $f : A \to B$ may then be extended to a cofibration sequence

$$A \xrightarrow{f} B \xrightarrow{i} C_f \longrightarrow sA \xrightarrow{sf} sB \longrightarrow \cdots, \tag{4.8}$$

where sf is the canonical map $sf : sA \to sB$ given by $sf([a,t]) \mapsto [f(a), t]$ where $[a,t] \in sA$ is the equivalence class of $(a,t) \in A \times S^1$.

Theorem 4.1.6. *[Mau80, 6.4.3] The cofibration sequence (4.8) induces an exact sequence of sets*

$$\longrightarrow [sB, Z] \longrightarrow [sA, Z] \longrightarrow [C_f, Z] \xrightarrow{i^*} [B, Z] \xrightarrow{f^*} [A, Z] \tag{4.9}$$

for any space Z.

Lemma 4.1.7. *From the above exact sequence we see that if we have maps $f : A \to B$ and $g : B \to Z$, then g will extend to a map $\hat{g} : C_f \to Z$ if and only if $g \circ f$ is null-homotopic.*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{i} & C_f \\ & & & \searrow^{g} & \vdots\,{\scriptstyle \hat{g}} \\ & & & & Z \end{array} \tag{4.10}$$

Proof. Suppose there is a map $\hat{g} \in [C_f, Z]$ extending $g \in [B, Z]$. Then $g \in \mathrm{Im}(i^*)$. But

$$\begin{aligned} g \in \mathrm{Im}(i^*) \quad &\Leftrightarrow \quad g \in \ker(f^*) \\ &\Leftrightarrow \quad f^* g = \mathrm{id} \in [A, Z] \\ &\Leftrightarrow \quad g \circ f \text{ is null-homotopic.} \end{aligned}$$

□

4.2 Homotopy of $\mathrm{Map}_*(\Sigma, G)$

We would now like to consider the homotopy classes of $\mathrm{Map}_*(\Sigma, G)$. We have

$$\pi_k(X) := [S^k, X] \tag{4.11}$$

as sets. Using the properties of sets $[X, Y]$ discussed in the previous section, we find

$$\pi_k(\mathrm{Map}_*(\Sigma, G)) := [S^k, \mathrm{Map}_*(\Sigma, G)] = [S^k, G^\Sigma] = [S^k \wedge \Sigma, G] = [s^k\Sigma, G], \tag{4.12}$$

and so in order to discover the homotopy groups of $\mathrm{Map}_*(\Sigma, G)$ we seek to understand the class of maps from the reduced suspension $s^k\Sigma$ to G.

Now let us consider the topological structure of Σ and its suspensions. A compact Riemann surface Σ of genus g may be obtained by attaching a 2-cell D^2 to a $2g$-wedge of circles via the commutator map $c : S^1 \to \bigvee_{i=1}^{2g} S^1$. Let the canonical generators of $\pi_1(\Sigma)$ be denoted by $\{a_i, b_i\}_{i=1}^{g}$, with each a_i, b_i representing a copy of S^1 inside the one-skeleton $\bigvee_{i=1}^{2g} S^1 \subset \Sigma$. Then we may write

$$\Sigma \sim \left(\bigvee_{i=1}^{2g} S^1\right) \bigcup\nolimits_{[a_1,b_1]\ldots[a_g,b_g]} D^2. \tag{4.13}$$

We have the cofibration sequence

$$S^1 \xrightarrow{c} \bigvee\nolimits_{i=1}^{2g} S^1 \xrightarrow{i} \Sigma \tag{4.14}$$

where i is the inclusion of the one-skeleton inside Σ and c is the product of the commutators $[a_1, b_1]\cdots[a_g, b_g]$. We may extend the cofibration sequence as follows:

$$S^1 \xrightarrow{c} \bigvee\nolimits_{i=1}^{2g} S^1 \xrightarrow{i} \Sigma \longrightarrow sS^1 \xrightarrow{sc} s\bigvee\nolimits_{i=1}^{2g} S^1 \longrightarrow s\Sigma \longrightarrow \cdots, \tag{4.15}$$

and using the properties of the reduced suspension as discussed previously we rewrite this as

$$S^1 \xrightarrow{c} \bigvee\nolimits_{i=1}^{2g} S^1 \xrightarrow{i} \Sigma \longrightarrow S^2 \xrightarrow{sc} \bigvee\nolimits_{i=1}^{2g} S^2 \longrightarrow s\Sigma \longrightarrow \cdots \tag{4.16}$$

$$\cdots \longrightarrow S^{k+1} \xrightarrow{s^k c} \bigvee\nolimits_{i=1}^{2g} S^{k+1} \longrightarrow s^k\Sigma \longrightarrow \cdots.$$

Then in homology we have

$$H_*(\Sigma) \cong H_*\left(\left(\bigvee_{i=1}^{2g} S^1\right) \bigvee S^2\right).$$

Now suppose we have $m \in \mathrm{Map}_*(\Sigma, G)$. We may restrict m to a map $f : \bigvee_{i=1}^{2g} S^1 \to G$ by applying i^*. From Lemma 4.1.7 we may deduce that such a map f extends to a map $\Sigma \to G$ if and only if $f \circ c$ is null-homotopic and we have the diagram

$$\begin{array}{ccccc} S^1 & \xrightarrow{c} & \bigvee_{i=1}^{2g} S^1 & \xrightarrow{i} & \Sigma \\ & & & \searrow^{f} & \big\downarrow m \\ & & & & G \end{array} \tag{4.17}$$

Let us consider the commutator map c in more detail. It is a (based) map from S^1 to the wedge of circles $\bigvee_{i=1}^{2g} S^1$, hence is an element of the fundamental group $\pi_1(\bigvee_{i=1}^{2g} S^1) \cong F^{2g}$, the free group on $2g$ generators. Then $f \circ c : S^1 \to G$, i.e. $[f \circ c] \in \pi_1(G)$ and f extends to a map $m : \Sigma \to G$ if and only if $[f \circ c]$ is the identity in $\pi_1(G)$. Now consider the following diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & F^1 & \longrightarrow & F^{2g} & \longrightarrow & F^{2g} / < [a_1, b_1] \cdots [a_g, b_g] > & \longrightarrow & 1 \\
 & & \| & & \| & & \| & & \\
 & & \pi_1(S^1) & \xrightarrow{c_*} & \pi_1(\bigvee_{i=1}^{2g} S^1) & \xrightarrow{i_*} & \pi_1(\Sigma) & \longrightarrow & H_1(\Sigma) \\
 & & & & & \searrow^{f_*} & \downarrow & & \downarrow \\
 & & & & & & \pi_1(G) & \xrightarrow{\simeq} & H_1(G)
\end{array} \tag{4.18}$$

As G is a topological group, $\pi_1(G)$ is abelian, so $\pi_1(G) \simeq H_1(G)$. A map from $\pi_1(\Sigma)$ to $\pi_1(G)$ then factors through $H_1(\Sigma)$. Then elements of $[\Sigma, G]$ are labelled by maps $f_* : F^{2g} \to \pi_1(G)$ such that $f_* c = \mathrm{id} \in \pi_1(G)$, or equivalently,

$$[\Sigma, G] = \{f_* \in \mathrm{Hom}(F^{2g}, \pi_1(G)) \mid f_*([a_1, b_1] \cdots [a_g, b_g]) = \mathrm{id}\} \tag{4.19}$$
$$= \{f_* \in \mathrm{Hom}(H_1(\Sigma), H_1(G))\} \tag{4.20}$$
$$\tag{4.21}$$

Then we have obtained

Theorem 4.2.1. *The set of connected components of* $\mathrm{Map}_*(\Sigma, G)$ *is*

$$\pi_0(\mathrm{Map}_*(\Sigma, G)) = H_1(G) \otimes H^1(\Sigma). \tag{4.22}$$

Corollary 4.2.2. *[PS86] When G is simply connected,* $\mathrm{Map}_*(\Sigma, G)$ *is connected.*

Note that the previous corollary states exactly that any map from a surface Σ to a simply connected Lie group G is homotopic to the identity map. When G is not simply connected however $\mathrm{Map}_*(\Sigma, G)$ is not path connected and in order to talk about homotopy groups we need to specify a base point $f_0 : \Sigma \to G$. Then we shall be discussing the component of $\mathrm{Map}_*(\Sigma, G)$ containing f_0. We are now in a position to discuss the homotopy groups of $\mathrm{Map}_*(\Sigma, G)$. We require the following

Proposition 4.2.3. *For* $k \geq 1$ *we have the homotopy equivalence*

$$s^k\Sigma \sim \left(\bigvee_{i=1}^{2g} S^{k+1}\right) \bigvee S^{k+2} \tag{4.23}$$

Proof. Note that $s^k\Sigma$ is the mapping cone $C_{s^k c}$. We consider the map $s^k c : S^{k+1} \to \bigvee_{i=1}^{2g} S^{k+1}$.

But for $k \geq 1$ we have $\pi_{k+1}(\bigvee_{i=1}^{2g} S^{k+1})$ is free abelian[2], and so the image of $s^k c$ is the identity (as $s^k c$ is a commutator), so $s^k c$ is null-homotopic. Then from (4.7) we have

$$C_{s^k c} \sim \bigvee_{i=1}^{2g} S^{k+1} \bigvee sS^{k+1}. \tag{4.24}$$

□

We note that the above proposition does not hold for $k = 0$, *ie.* Σ and $\left(\bigvee_{i=1}^{2g} S^1\right) \vee S^2$ are not homotopically equivalent.

We recall that if $f : A \to B$ is a cofibration, then $Y^B \to Y^A$ is a fibration. Hence the cofibration (4.14) yields the fibration

$$\mathrm{Map}_*(S^2, G) \longrightarrow \mathrm{Map}_*(\Sigma, G) \longrightarrow \mathrm{Map}_*(\bigvee_{i=1}^{2g} S^1, G) \longrightarrow \mathrm{Map}_*(S^1, G)\ , \tag{4.25}$$

which induces a long exact sequence in homotopy

$$\cdots \xrightarrow{(s^{k+1}c)^*} \pi_k(\mathrm{Map}_*(S^2, G)) \longrightarrow \pi_k(\mathrm{Map}_*(\Sigma, G)) \longrightarrow \pi_k(\mathrm{Map}_*(\bigvee_{i=1}^{2g} S^1, G)) \xrightarrow{(s^k c)^*} \cdots. \tag{4.26}$$

We note that

$$\pi_k(\mathrm{Map}_*(S^2, G)) = [S^k, G^{S^2}] = [S^k \wedge S^2, G] = [S^{k+2}, G] = \pi_{k+2}(G), \tag{4.27}$$

and

$$\pi_k(\mathrm{Map}_*(\bigvee_{i=1}^{2g} S^1, G)) = [S^k, G^{\vee_{i=1}^{2g} S^1}] = [S^k, \Pi_{i=1}^{2g} G^{S^1}] = \Pi_{i=1}^{2g}[S^k, G^{S^1}] = [\pi_{k+1}(G)]^{2g}, \tag{4.28}$$

where $[\pi_{k+1}(G)]^{2g}$ denotes the direct sum of $2g$ copies of $\pi_{k+1}(G)$ and the second equality holds because $Z^{X\vee Y}$ is homeomorphic to $Z^X \times Z^Y$. We noted previously that the image of the map $s^k c$ is the identity, hence we obtain the short exact sequence of groups and homomorphisms

$$1 \longrightarrow \pi_{k+2}(G) \longrightarrow \pi_k(\mathrm{Map}_*(\Sigma, G)) \longrightarrow [\pi_{k+1}(G)]^{2g} \longrightarrow 1 \tag{4.29}$$

for $k \geq 1$.

Lemma 4.2.4. *The k^{th} homotopy group of* $\mathrm{Map}_*(\Sigma, G)$ *has the following form:*

$$\begin{aligned}\pi_k(\mathrm{Map}_*(\Sigma, G)) = [S^k, G^\Sigma] = [S^k \wedge \Sigma, G] = [s^k\Sigma, G] &= \left[\left(\bigvee_{i=1}^{2g} S^{k+1}\right) \bigvee S^{k+2}, G\right] \\ &= [\pi_{k+1}(G)]^{2g} \times \pi_{k+2}(G).\end{aligned} \tag{4.30}$$

Then from the previous lemma we see that the exact sequence (4.29) splits. Let us find the splitting $r^* : [\pi_{k+1}(G)]^{2g} \to \pi_k\big(\mathrm{Maps}_*(\Sigma, G)\big)$ such that with the projection

[2] For any topological spaces X, Y, we have $\pi_1(X \vee Y) = \pi_1(X) \star \pi_1(Y)$, where $\star$ denotes the free product.

$\iota^* : \pi_k\left(\mathrm{Maps}_*(\Sigma, G)\right) \to [\pi_{k+1}(G)]^{2g}$ we have $\iota^* \circ r^* = \mathrm{id}$. Dually we would like a map $\iota : \left(\bigvee_{i=1}^{2g} S^{k+1}\right) \to s^k\Sigma$ and a map $r : s^k\Sigma \to \bigvee_{i=1}^{2g} S^{k+1}$ such that $r \circ \iota = \mathrm{id}$. But we have $\iota = s^k i$ giving our inclusion and with

$$s^k\Sigma = C_{s^k c} = \left(\bigvee_{i=1}^{2g} S^{k+1}\right) \bigcup_{s^k c} D^{k+2} \sim \left(\bigvee_{i=1}^{2g} S^{k+1}\right) \bigvee S^{k+2}$$

we obtain r by collapsing the S^{k+2} which is continuous. In general we could not have continuously mapped all of $D^{k+2} = cS^{k+1}$ to a point if $s^k c$ had not been homotopically trivial.

Remark 4.2.5. *A similar argument is used in [Han83] to compute the homotopy groups of* $\mathrm{Map}_*(\Sigma, S^2)$.

Proposition 4.2.6. *Let G be a compact, connected, simple, non-abelian, finite-dimensional Lie group and Σ a compact Riemann surface of genus g. Then*

$$\pi_0(\mathrm{Map}_*(\Sigma, G)) \simeq [\pi_1(G)]^{2g}. \tag{4.31}$$

Proof. From Theorem (4.1.6) we have that the cofibration $S^1 \to \bigvee_{i=1}^{2g} S^1 \to \Sigma$ induces an exact sequence of sets

$$[S^2, G] \longrightarrow [\Sigma, G] \longrightarrow [\textstyle\bigvee_{i=1}^{2g} S^1, G] \xrightarrow{c^*} [S^1, G], \tag{4.32}$$

giving

$$\pi_2(G) \longrightarrow \pi_0(\mathrm{Map}_*(\Sigma, G)) \longrightarrow [\pi_1(G)]^{2g} \xrightarrow{c^*} \pi_1(G). \tag{4.33}$$

Using the classical theorem of E. Cartan that $\pi_2(G) = 1$ and that the image of c^* in the abelian group $\pi_1(G)$ must be the identity, we obtain

$$\pi_0(\mathrm{Map}_*(\Sigma, G)) \simeq [\pi_1(G)]^{2g}. \tag{4.34}$$

□

We combine our previous results in the following theorem:

Theorem 4.2.7. *For Σ a compact Riemann surface of genus g and G a connected, simple, compact or complex Lie group, we have*

- $\pi_0(\mathrm{Map}_*(\Sigma, G)) \simeq [\pi_1(G)]^{2g}$
- $\pi_1(\mathrm{Map}_*(\Sigma, G)) = \mathbb{Z}$
- $\pi_2(\mathrm{Map}_*(\Sigma, G)) = \mathbb{Z}^{2g} \oplus \pi_4(G)$,

where $\pi_4(G)$ is torsion. Furthermore, we have $\pi_k(\mathrm{Map}_*(\Sigma, G)) = [\pi_{k+1}(G)]^{2g} \times \pi_{k+2}(G)$ for all $k \geq 1$.

In the previous theorem we have used E. Cartan's theorem that $\pi_2(G) = 1$ and the fact that $\pi_3(G) = \mathbb{Z}$, a theorem of Bott [Bot54]. Note that our theorem agrees with [PS86] when $\pi_1(G) = 0$.

4.3 Homotopy of $\mathrm{Map}(\Sigma, G)$

We now consider maps from Σ to G with no distinguished base point. We have a fibration

$$\begin{array}{ccc} \mathrm{Map}_*(\Sigma, G) & \xrightarrow{i} & \mathrm{Map}(\Sigma, G) \\ & & \downarrow{\scriptstyle ev} \\ & & G, \end{array} \tag{4.35}$$

where ev is the evaluation map at the image of the base point, $ev : f \mapsto f(z_0)$ for $f \in \mathrm{Map}(\Sigma, G)$, and

$$\mathrm{Map}(\Sigma, G) \simeq \mathrm{Map}_*(\Sigma, G) \times G. \tag{4.36}$$

Indeed, given a point $(h(z), g) \in \mathrm{Map}_*(\Sigma, G) \times G$ we may form an unbased map $g \cdot h(z) : \Sigma \to G$, and given an unbased map $f(z)$ we can form the element $(f(z_0)^{-1} f(z), f(z_0)) \in \mathrm{Map}_*(\Sigma, G) \times G$. Then the fibration (4.35) is the trivial bundle and we have

$$\pi_k(\mathrm{Map}(\Sigma, G)) = \pi_k(\mathrm{Map}_*(\Sigma, G)) \times \pi_k(G). \tag{4.37}$$

Let $\tilde{G}$ denote the universal cover of G. The universal cover sits in a short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \tilde{G} \xrightarrow{\pi} G \longrightarrow 1 \tag{4.38}$$

with $\Gamma = \pi_1(G)$ central and discrete.

Proposition 4.3.1. *The sequence*

$$1 \longrightarrow \Gamma \longrightarrow \mathrm{Map}(\Sigma, \tilde{G}) \xrightarrow{\pi\circ} \mathrm{Map}(\Sigma, G) \longrightarrow \mathrm{Hom}(\pi_1(\Sigma), \pi_1(G)) \longrightarrow 1 \tag{4.39}$$

is exact.

Proof. From above, we have that $\mathrm{Map}(\Sigma, G) = \mathrm{Map}_*(\Sigma, G) \times G$ and similarly $\mathrm{Map}(\Sigma, \tilde{G}) = \mathrm{Map}_*(\Sigma, \tilde{G}) \times \tilde{G}$. We choose the base point of both the based groups to be the constant map $1 : \Sigma \to e \in G \subset \tilde{G}$, but note that any choice of base point $f_0 : \Sigma \to G$ is a conjugation $f_0 \mapsto g f_0 g^{-1}$. The map that sends an element $\tilde{f} \in \mathrm{Map}(\Sigma, \tilde{G})$ to an element $f \in \mathrm{Map}(\Sigma, G)$ is composition with $\pi : \tilde{G} \to G$. The kernel of this map then contains all constant maps sending all of Σ to a single element $\gamma \in \Gamma \subset \tilde{G}$, and so we conclude that the sequence is exact at $\mathrm{Map}(\Sigma, \tilde{G})$. We then may consider the sequence

$$1 \longrightarrow \mathrm{Map}(\Sigma, \tilde{G}) \xrightarrow{\pi\circ} \mathrm{Map}_*(\Sigma, G) \longrightarrow \mathrm{Hom}(\pi_1(\Sigma), \pi_1(G)) \longrightarrow 1.$$

Now the map $\psi : \mathrm{Map}_*(\Sigma, G) \to \mathrm{Hom}(\pi_1(\Sigma), \pi_1(G))$ is given by $\psi : f \mapsto f_*$. From the lifting property of covering spaces, we have that $f : \Sigma \to G$ extends to a map $\tilde{f} : \Sigma \to \tilde{G}$ if and only if $f_*(\pi_1(\Sigma)) \subset \pi_1(\tilde{G}) = 1$, that is $f \in \ker\psi$, and the sequence (4.39) is exact at $\mathrm{Map}(\Sigma, G)$.

Now we showed in (4.21) that

$$\mathrm{Hom}(\pi_1(\Sigma), \pi_1(G)) \simeq \mathrm{Hom}(H_1(\Sigma), \pi_1(G)) \simeq [\pi_1(G)]^{2g}$$

is the set of connected components of $\mathrm{Map}_*(\Sigma, G)$ and so the map from $\mathrm{Map}_*(\Sigma, G)$ to $\mathrm{Hom}(\pi_1(\Sigma), \pi_1(G))$ may be viewed as the inclusion of an element $f : \Sigma \to G$ into its corresponding connected component. Let $\alpha \in \mathrm{Hom}(\pi_1(\Sigma), \pi_1(G))$. Then $\alpha(a_i), \alpha(b_i) \in \pi_1(G)$ for $i = 1, \ldots, g$ and so we have $2g$ maps from the circle to G, or equivalently one map $\bigvee_{i=1}^{2g} S^1 \to G$. Such a map may be extended to one in $\mathrm{Map}_*(\Sigma, G)$, and so the map $\mathrm{Map}_*(\Sigma, G) \to \mathrm{Hom}(\pi_1(\Sigma), \pi_1(G))$ is surjective. The sequence (4.39) is consequently exact at $\mathrm{Hom}(\pi_1(\Sigma), \pi_1(G))$. □

4.4 Homology and Cohomology

We now calculate the integral homology and cohomology groups of $\mathrm{Map}_*(\Sigma, G)_e$, the identity component of $\mathrm{Map}_*(\Sigma, G)$, where G is a finite-dimensional, simple, connected Lie group. From (4.30) we have

$$\pi_1(\mathrm{Map}_*(\Sigma, G)_e) = \pi_3(G) \times [\pi_2(G)]^{2g} = \mathbb{Z},$$

and $H_1(\mathrm{Map}_*(\Sigma, G)_e, \mathbb{Z})$ is the abelianisation of $\pi_1(\mathrm{Map}_*(\Sigma, G))$ and so

$$H_1(\mathrm{Map}_*(\Sigma, G)_e, \mathbb{Z}) = \mathbb{Z}.$$

Theorem 4.4.1. *Let G be a connected, simple and non-abelian Lie group. Then*

$$H_2(\mathrm{Map}_*(\Sigma, G)_e, \mathbb{Z}) \cong \pi_2(\mathrm{Map}_*(\Sigma, G)) \cong \mathbb{Z}^{2g} \oplus \pi_4(G);$$
$$H^2(\mathrm{Map}_*(\Sigma, G)_e, \mathbb{Z}) \cong \mathrm{Hom}(\pi_2(\mathrm{Map}_*(\Sigma, G)), \mathbb{Z}) \cong \mathbb{Z}^{2g}.$$

When G is abelian we have $H_2(\mathrm{Map}_*(\Sigma, G)_e, \mathbb{Z}) = H^2(\mathrm{Map}_*(\Sigma, G)_e, \mathbb{Z}) = 0.$

To prove this theorem, we require the following lemma[3].

Lemma 4.4.2. *For a path-connected space X with* $\pi_1(X) = \mathbb{Z}$*, we have*

$$H_2(X) \cong \pi_2(X)_{\pi_1(X)},$$

where $\pi_2(X)_{\pi_1(X)}$ *is the largest quotient of* $\pi_2(X)$ *on which* $\pi_1(X)$ *acts trivially, i.e. the group of coinvariants.*

[3] The result is an exercise in [Bro82].

Proof. Consider first the Serre fibration

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & S^1 \end{array}$$

with the connected fibre F simply connected. Now from the associated long exact sequence in homotopy we have $\pi_1(E) = \pi_1(S^1)$ as F is simply connected, and $\pi_2(F) = \pi_2(E)$ as the higher homotopy groups of S^1 are trivial. By the Hurewicz theorem we have that $\pi_2(F) \cong H_2(F)$. Now from the long exact sequence in homology we deduce that the map $H_2(F) \to H_2(E)$ is surjective, again as the higher homology groups of S^1 are trivial. Let us consider the Wang exact sequence [Wan49]

$$H_2(F) \longrightarrow H_2(F) \longrightarrow H_2(E) \longrightarrow H_1(F) \longrightarrow \cdots .$$

Then $H_2(E) \cong H_2(F)_{\pi_1(S^1)}$ and so $H_2(E) \cong \pi_2(E)_{\pi_1(S^1)} \cong \pi_2(E)_{\pi_1(E)}$. Now suppose we have a path-connected space X with $\pi_1(X) = \mathbb{Z}$. Its universal cover is classified by a map $X \to B\mathbb{Z} = S^1$, the classifying space of $\mathbb{Z}$, and we convert this map to a fibration by appropriate composition. Then X is homotopy equivalent to E and we have the result. □

We now prove Theorem 4.4.1.

Proof. For the homology, we use Lemma 4.4.2 and as $\mathrm{Map}_*(\Sigma, G)$ is an H-space, [Ser51] tells us that the action of $\pi_1(\mathrm{Map}_*(\Sigma, G))$ on $\pi_2(\mathrm{Map}_*(\Sigma, G))$ is trivial, and so $H_2(\mathrm{Map}_*(\Sigma, G)_e, \mathbb{Z}) \cong \pi_2(\mathrm{Map}_*(\Sigma, G))_{\pi_1(\mathrm{Map}_*(\Sigma,G))} = \pi_2(\mathrm{Map}_*(\Sigma, G))$.

Now for the cohomology we recall the universal coefficient theorem for cohomology from *eg.* [Hat02]:

$$0 \longrightarrow \mathrm{Ext}(H_1(X, \mathbb{Z}), \mathbb{Z}) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow \mathrm{Hom}(H_2(X), \mathbb{Z}) \longrightarrow 0.$$

We have $H_1(\mathrm{Map}_*(\Sigma, G)_e, \mathbb{Z}) = \mathbb{Z}$ and $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$ so we obtain

$$H^2(\mathrm{Map}_*(\Sigma, G)_e, \mathbb{Z}) \cong \mathrm{Hom}(H_2(\mathrm{Map}_*(\Sigma, G)_e), \mathbb{Z}).$$

We now derive the result for G abelian. In this case we have

$$\pi_1(\mathrm{Map}_*(\Sigma, G)) = \pi_3(G) \times [\pi_2(G)]^{2g} = 0,$$

and $\mathrm{Ext}(H_1(\mathrm{Map}_*(\Sigma, G)_e, \mathbb{Z}), \mathbb{Z}) = \mathrm{Ext}(0, \mathbb{Z}) = 0$. Hurewicz's theorem then gives

$$H_2(\mathrm{Map}_*(\Sigma, G)_e, \mathbb{Z}) \cong \pi_2(\mathrm{Map}_*(\Sigma, G)) \cong \pi_4(G) \times [\pi_3(G)]^{2g} = 0$$

and $H^2(\mathrm{Map}_*(\Sigma, G)_e, \mathbb{Z}) = 0$. □

4.5 Principal G-bundles over Σ

Let us recall some facts about principal G-bundles over a manifold X. We have some theorems from [AB83]. For G a topological group, there exists a *classifying space* BG so that isomorphism classes of principal G-bundles over X are in natural bijective correspondence with homotopy classes $[X, BG]$. The classifying space BG has the properties that $\pi_1(BG) \cong G$ and its higher homotopy groups are trivial. In other words, it is an Eilenberg-Maclane space of type $K(G,1)$. The correspondence between the principal G-bundles over X and the homotopy classes $[X, BG]$ is given by pulling back a universal principal G-bundle EG over BG. The following fibration describes the universal G-bundle EG over BG,

$$\begin{array}{ccc} G & \longrightarrow & EG \\ & & \downarrow \\ & & BG, \end{array} \tag{4.40}$$

which induces the long exact sequence in homotopy

$$\cdots \to \pi_{k+1}(G) \to \pi_{k+1}(EG) \to \pi_{k+1}(BG) \to \pi_k(G) \to \pi_k(EG) \to \pi_k(BG) \to \cdots, \tag{4.41}$$

and the fact that EG is contractible gives us

$$\pi_{k+1}(BG) \cong \pi_k(G).$$

In particular we have that

$$1 \longrightarrow [S^2, BG] \longrightarrow [\Sigma, BG] \longrightarrow [\bigvee_{i=1}^{2g} S^1, BG] \xrightarrow{c^*} [S^1, BG].$$

Assuming G is connected we have $1 = \pi_0(G) = \pi_1(BG)$ and $[\bigvee_{i=1}^{2g} S^1, BG] = (\pi_1(BG))^{2g} = 1$. Therefore we have that

$$[S^2, BG] \cong [\Sigma, BG]$$

and so principal G-bundles are classified by $\pi_2(BG) \cong \pi_1(G)$.

Theorem 4.5.1. *Let G be a connected finite-dimensional Lie group, then (as sets)*

$$[\Sigma, BG] = \pi_1(G). \tag{4.42}$$

We may view this result in different ways. Suppose we have a principal G-bundle over Σ. Upon removing a disk the G-bundle over $\Sigma \setminus D^2$ is homotopic to one over the 1-skeleton $\bigvee_{i=1}^{2g} S^1$ of Σ. Now as $[\bigvee_{i=1}^{2g} S^1, BG] = 1$, this bundle is trivial on $\Sigma \setminus D^2$. Similarly a G-bundle over D^2 is trivial. Thus we have trivial bundles on $\Sigma \setminus D^2$ and D^2 and as we glue these together we get nontrivial bundles characterized by the clutching function from the overlap $S^1 = \partial D^2$ to G, or $\pi_1(G)$.

The same result may be obtained by obstruction theory. Again we have a trivial bundle on the 1-skeleton $\bigvee_{i=1}^{2g} S^1$ of Σ. The obstructions to a section of our G-bundle P are then given by $H^2(\Sigma, \pi_1(G))$, the obstruction in extending from the 1-skeleton to the 2-cell. The universal coefficient theorem yields

$$\begin{aligned} H^2(\Sigma, \pi_1(G)) &\cong \mathrm{Hom}(H_2(\Sigma, \mathbb{Z}), \pi_1(G)) \oplus \mathrm{Ext}(H_1(\Sigma, \mathbb{Z}), \pi_1(G)) \\ &\cong \mathrm{Hom}(\mathbb{Z}, \pi_1(G)) \oplus \mathrm{Ext}(\mathbb{Z}^{2g}, \pi_1(G)) \\ &\cong \pi_1(G), \end{aligned}$$

where we have used that $0 = \mathrm{Ext}(F, \pi_1(G))$ for a free abelian group F and that $\mathrm{Hom}(\mathbb{Z}, \pi_1(G)) \cong \pi_1(G)$ for the abelian group $\pi_1(G)$. This is a particular case of the Hopf-Whitney theorem.

Theorem 4.5.2 (Hopf-Whitney)**.** *Let K be a complex of dimension n and Y be $(n-1)$-connected. Then there is a one-to-one correspondence between $[K, Y]$ and $H^n(K, \pi_n(Y))$.*

It is important to note that the preceding construction was possible due to the topological description of Σ as

$$\Sigma \sim \left(\bigvee_{i=1}^{2g} S^1 \right) \bigcup_c D^2,$$

with $c = [a_1, b_1] \cdots [a_g, b_g]$ the commutator map. In particular, as c is a commutator its suspensions $s^k c$ are null-homotopic. For a general two-dimensional manifold X when sc need not be null-homotopic we find that

$$* \longrightarrow [S^2, BG] \longrightarrow [X, BG] \longrightarrow 1$$

and we only know we have a surjection.

Chapter 5

Topology of Central Extensions of Current Groups

In Chapter 2, we discussed central extensions of Lie algebras and Lie groups. When the Lie group is finite-dimensional, we may always integrate a Lie algebra central extension to a Lie group central extension, however for infinite-dimensional Lie groups this is no longer the case. In the first section of this chapter we discuss the obstructions to integrability for infinite-dimensional extensions, and we shall see that the key result is a long exact sequence constructed by Neeb [Nee02]. We then use the results on the topology of current groups obtained in Chapter 4 to discuss the topology of the universal central extension of G^{Σ}. Many results on central extensions of current groups are given in [PS86] and [EF94]. We note in [EF94] the group G is always taken to be simply connected.

5.1 The Long Exact Sequence of Neeb

In the following section let K be a connected Lie group (not necessarily finite-dimensional) with Lie algebra $\mathfrak{k}$, $\widetilde{K}$ its universal covering group with $K = \widetilde{K}/\pi_1(K)$, with $\pi_1(K)$ discrete and central, A a connected abelian Lie group with Lie algebra $\mathfrak{a}$ and $A = \mathfrak{a}/\Gamma$ for a discrete group Γ. An important result in the obstruction to integrating Lie algebra extensions to Lie group extensions is due to Neeb [Nee02, 7.12], who shows that the sequence

$$\begin{aligned}\operatorname{Hom}(K,A) \hookrightarrow \operatorname{Hom}(\widetilde{K},A) \longrightarrow \operatorname{Hom}(\pi_1(K),A) \longrightarrow \operatorname{Ext}_{\mathrm{Lie}}(K,A)\\ \xrightarrow{D} H^2_c(\mathfrak{k},\mathfrak{a}) \xrightarrow{P} \operatorname{Hom}(\pi_2(K),A)\times\operatorname{Hom}(\pi_1(K),\operatorname{Lin}(\mathfrak{k},\mathfrak{a})),\end{aligned} \tag{5.1}$$

is exact. Here $\operatorname{Lin}(\mathfrak{k},\mathfrak{a})$ is the space of continuous linear maps $\mathfrak{k}\to\mathfrak{a}$, and D is the map that assigns a Lie group extension its corresponding Lie algebra extension. The image of the map P will give us Lie algebra extensions that do not integrate to Lie group extensions. Before describing P, let us introduce the *period homomorphism* of a 2-cocycle.

Definition 5.1.1. *Let $\omega \in Z^2_c(\mathfrak{k},\mathfrak{a})$ be an $\mathfrak{a}$-valued 2-cocycle on $\mathfrak{k}$ and denote by Ω the left-invariant $\mathfrak{a}$-valued 2-form on K which equals ω at the identity. The* period homomorphism *per*$_\omega : \pi_2(K) \to \mathfrak{a}$ *of* ω *is the homomorphism obtained by integrating* Ω *over sufficiently smooth representatives of homotopy classes*[1] $[\sigma] \in \pi_2(K)$:

$$[\sigma] \mapsto \int_\sigma \Omega.$$

Note that there is some difficulty with the construction of the period homomorphism for an infinite-dimensional Lie group K. The main issue is that one need not assume that K is smoothly paracompact (that is, not every open cover has a subordinate smooth partition of unity), and so the de Rham isomorphisms $H^n_{dR}(K,\mathbb{R}) \cong H^n_{\text{sing}}(K,\mathbb{R})$ need not hold. See [Nee02] for a direct construction that does not require the de Rham theorems.

We shall now see that the period homomorphism associated to the cocycle ω in the Lie algebra extension $\mathfrak{h} = \mathfrak{k} \oplus_\omega \mathfrak{a}$ is one of the obstructions to the integrability of the Lie algebra extension. Namely, the group extension exists if and only if the period group $\Pi_\omega := \text{im}(\text{per}_\omega)$ is discrete. Let us describe the map

$$P : H^2_c(\mathfrak{k},\mathfrak{a}) \to \text{Hom}(\pi_2(K), A) \times \text{Hom}(\pi_1(K), \text{Lin}(\mathfrak{k},\mathfrak{a})),$$

denoting by P_1 and P_2 the map of P onto the first and second factors respectively. We define $P_1([\omega]) : \pi_2(K) \to A$ by

$$P_1([\omega]) := \pi_\Gamma \circ \text{per}_\omega,$$

where $\pi_\Gamma : \mathfrak{a} \to A = \mathfrak{a}/\Gamma$ is the quotient map. To define the second component of P let $X \in \mathfrak{k}$ and denote by X_R the corresponding right invariant vector field on K. Then $i_{X_R}\Omega$ is a closed $\mathfrak{a}$-valued one-form on K and to any $[\gamma] \in \pi_1(K)$ we may associate $\int_\gamma i_{X_R}\Omega$. Neeb [Nee02, 3.6] shows the latter exists. Then $P_2([\omega])([\gamma]) : \mathfrak{k} \to \mathfrak{a}$ is given by

$$P_2([\omega])([\gamma]) : X \mapsto \int_\gamma i_{X_R}\Omega. \tag{5.2}$$

There are several well-known results for finite-dimensional groups which are essentially consequences of the exact sequence (5.1). For example, when K is a finite-dimensional, connected, simply connected Lie group then we have $\pi_1(K) = 1$ and $\pi_2(K) = 0$ and we obtain from the exact sequence (5.1) that

$$\text{Ext}_{\text{Lie}}(K, A) \cong H^2_c(\mathfrak{k},\mathfrak{a}),$$

the result of Hochschild [Hoc51] stated in Chapter 2.

Another consequence of the long exact sequence (5.1) is that the adjoint action of $\mathfrak{k}$ on $\mathfrak{h} = \mathfrak{a} \oplus_\omega \mathfrak{k}$ integrates to a smooth action of K on $\mathfrak{h}$ if and only if $P_2([\omega]) = 0$. In particular when we have such an adjoint action then we show below that

$$f_X(k) := \theta(k^{-1}, X) = p_{\mathfrak{a}}(\text{Ad}^H_{k^{-1}}(0, X))$$

[1] See [Nee02] for proofs of the facts that each homotopy class does indeed contain a smooth representative and that $\int_\sigma \Omega$ is independent of the choice of representative.

is a cocycle such that $df_X = i_{X_R}\Omega$ and so is exact, hence $P_2([\omega]) = 0$. (Here $p_{\mathfrak{a}}$ is projection onto $\mathfrak{a}$.) It is interesting to note that this means that the adjoint action of K on $\mathfrak{h}$ may exist even when the group central extension H does not, *cf.* Section 2.2.3.

Theorem 5.1.2. *[Nee02, 7.6] Let K be a connected Lie group, $\omega \in Z^2_c(\mathfrak{k},\mathfrak{a})$ where $\mathfrak{a}$ is a sequentially complete locally convex space. Then the adjoint action of $\mathfrak{k}$ on $\mathfrak{h} := \mathfrak{a}\oplus_\omega \mathfrak{k} \to \mathfrak{a}$ integrates to a smooth action Ad^H of K if and only if $P_2([\omega]) = 0$.*

Proof. Suppose $P_2([\omega]) = 0$. Then from (5.2) we have that $\int_\gamma i_{X_R}\Omega = 0$ for all γ and so $i_{X_R}\Omega$ is exact. Put $i_{X_R}\Omega = df_X$ where $X \in \mathfrak{k}$. We have from Lemma 2.2.9 that when $i_{X_R}\Omega = df_X$ is exact and the smooth function $f_X : K \to \mathfrak{a}$ is chosen so that $f_X(e_K) = 0$, then $\theta(k,X) = f_X(k^{-1})$ is a smooth cocycle such that the adjoint action

$$\mathrm{Ad}^H_{(a,k)}(u,X) = (u+\theta(k,X),\mathrm{Ad}^K_k X) \tag{5.3}$$

is smooth. The corresponding derived action of $\mathfrak{h}$ is

$$\mathrm{ad}^{\mathfrak{h}}_{(a,Y)}(u,X) = (\omega(Y,X),[Y,X]_{\mathfrak{k}}) = [(a,Y),(u,X)]_{\mathfrak{h}}. \tag{5.4}$$

Then by restriction we have a smooth action of K on $\mathfrak{h}$. Conversely, suppose that we have a smooth action Ad^H of K on $\mathfrak{h}$ with this derived action of $\mathfrak{h}$. As $\mathrm{ad}^{\mathfrak{h}}$ acts by derivations then K acts on $\mathfrak{h}$ by automorphisms. Let $X \in \mathfrak{k} \subseteq \mathfrak{h}$ and define $f_X : K \to \mathfrak{a}$ by

$$f_X(k) := \theta(k^{-1},X) = p_{\mathfrak{a}}(\mathrm{Ad}^H_{k^{-1}}(0,X)).$$

Then

$$\begin{aligned} df_X(e)(Y) &= p_{\mathfrak{a}}(\mathrm{ad}^{\mathfrak{h}}_{(0,-Y)}(0,X)) \\ &= p_{\mathfrak{a}}([(0,-Y),(0,X)]_{\mathfrak{h}}) \\ &= p_{\mathfrak{a}}(\omega(X,Y),[X,Y]_{\mathfrak{k}}) \\ &= \omega(X,Y). \end{aligned}$$

But we also have, with the right invariant vector field (generated by left translations) $Y_R(k) =$

$\frac{d}{dt}|_{t=0} \exp(tY)k$, with $k \in K$ and $Y \in \mathfrak{k}$, that

$$\begin{aligned}
df_X(k)\,(Y_R(k)) &= \frac{d}{dt}\Big|_{t=0} p_{\mathfrak{a}}\left(\mathrm{Ad}^H_{(\exp(tY)k)^{-1}}(0,X)\right) \\
&= \frac{d}{dt}\Big|_{t=0} p_{\mathfrak{a}}\left(\mathrm{Ad}^H_{k^{-1}}\,\mathrm{Ad}^H_{\exp(-tY)}(0,X)\right) \\
&= p_{\mathfrak{a}}\left(\mathrm{Ad}^H_{k^{-1}}\,\mathrm{ad}^{\mathfrak{h}}_{(0,-Y)}(0,X)\right) \\
&= p_{\mathfrak{a}}\left(\mathrm{Ad}^H_{k^{-1}}[X,Y]_{\mathfrak{h}}\right) \\
&= p_{\mathfrak{a}}\left([\mathrm{Ad}^H_{k^{-1}}X, \mathrm{Ad}^H_{k^{-1}}Y]_{\mathfrak{h}}\right) \\
&= \omega(\mathrm{Ad}^K_{k^{-1}}X, \mathrm{Ad}^K_{k^{-1}}Y) \\
&= \lambda^*_{k^{-1}}\omega(\rho_{k*}X, \rho_{k*}Y) \\
&= \lambda^*_{k^{-1}}\omega(\rho_{k*}X_R(e), \rho_{k*}Y_R(e)) \\
&= \lambda^*_{k^{-1}}\omega(X_R(k), Y_R(k)) \\
&= \Omega_k(X_R(k), Y_R(k)) \\
&= i_{X_R}\Omega_k(Y_R(k)).
\end{aligned}$$

Here we have used the left invariance $\lambda^*_{k^{-1}}\omega = \Omega_k$ in the penultimate step. Thus we see that $i_{X_r}\Omega$ is exact and so $P_2([\omega]) = 0$. □

Now consider the case where G is a finite-dimensional semisimple Lie group and $\mathfrak{a} = \Omega^1(\Sigma)/d\Omega^0(\Sigma)$. Set

$$\theta(k, X) = -\langle X, k^{-1}dk\rangle.$$

Then

$$\theta(k_1k_2, X) = \theta(k_2, X) + \theta(k_1, \mathrm{Ad}^K_{k_2}X)$$

and so yields a cocycle. Then

$$\Omega(X,Y) = \langle X, dY\rangle$$

and we get an action Ad^H of K on $\mathfrak{h}$ as in the theorem. Thus for this class of examples we have $P_2([\omega]) = 0$. This is an example of what Maier and Neeb [MN03] refer to as a cocycle of product type.

5.2 The Connecting Map

In this section we assume that a central extension H of K by A exists. The extension H has a corresponding Lie algebra extension $\mathfrak{h} = \mathfrak{k} \oplus_\omega \mathfrak{a}$ for some 2-cocycle $\omega \in Z^2_c(\mathfrak{k}, \mathfrak{a})$. Following [Nee02], we shall consider only central Lie group extensions which are smooth principal bundles, that is, admit a smooth *local* section. This ensures the existence of a continuous linear section on the corresponding Lie algebra extension. We have seen in Chapter 2 that the group extensions which admit a *smooth section* (and hence are topologically trivial principal bundles) are those which may be represented by a smooth 2-cocycle. We set up some preliminary notation. Let

$$\begin{array}{ccc} A & \xrightarrow{i} & H \\ & & \downarrow q \\ & & K \end{array} \tag{5.5}$$

be a smooth principal bundle. We have induced long exact sequences in homotopy

$$\cdots \to \pi_{k+1}(A) \to \pi_{k+1}(H) \to \pi_{k+1}(K) \to \pi_k(A) \to \pi_k(H) \to \pi_k(K) \to \cdots, \tag{5.6}$$

and homology

$$\cdots \to H_{k+1}(A) \to H_{k+1}(H) \to H_{k+1}(K) \to H_k(A) \to H_k(H) \to H_k(K) \to \cdots. \tag{5.7}$$

We shall assume our extension is by a connected abelian Lie group A that can be written as $A = \mathfrak{a}/\Gamma$ for a discrete subgroup Γ. As A is abelian we have $\pi_k(A) = 1$ for $k \geq 2$ and so $\pi_k(H) \cong \pi_k(K)$ for $k \geq 3$, and the sequence becomes

$$\begin{array}{l} 0 = \pi_2(A) \longrightarrow \pi_2(H) \xrightarrow{q_*} \pi_2(K) \xrightarrow{\delta} \pi_1(A) \xrightarrow{i_*} \pi_1(H) \\ \longrightarrow \pi_1(K) \longrightarrow \pi_0(A) \longrightarrow \pi_0(H) \longrightarrow \pi_0(K) \longrightarrow 1. \end{array} \tag{5.8}$$

Further, as A is connected, we have $\pi_0(H) \cong \pi_0(K)$. In order to proceed we shall require more information about the connecting homomorphism $\delta : \pi_2(K) \to \pi_1(A)$. We use a result of Neeb [Nee02] which relates δ to the period homomorphism per_ω of the 2-cocycle ω of the corresponding Lie algebra extension $\mathfrak{h} = \mathfrak{k} \oplus_\omega \mathfrak{a}$. This will show that whenever the central extension H exists, δ will be surjective, and this will enable us to determine the remaining homotopy groups for H.

Theorem 5.2.1. *[Nee02, 5.11] Let H be a central extension of K by A as above, with induced exact sequence in homotopy (5.8). Then $\delta = -\mathrm{per}_\omega$.*

Proof. Consider $[\sigma] \in \pi_2(K)$ and $[\gamma] \in \pi_1(A)$ such that $\delta([\sigma]) = [\gamma]$. Then as the sequence (5.6) is exact and $[\gamma] \in \operatorname{im}\delta$ we must have $[\gamma] \in \ker i_*$, that is $i_*[\gamma] = 1 \in \pi_1(H)$. Therefore there exists a $D^2 \subset H$ such that $\partial D^2 = \gamma$. Then $\sigma := q(D^2) \in \pi_2(K)$. Suppose we have a 2-form $\Omega \in \Omega^2(K, \mathfrak{a})$ such that $q^*\Omega = -d\theta$, with $\theta \in \Omega^1(H, \mathfrak{a})$. Now as H is a smooth principal A-bundle over a smooth manifold K we have a connection 1-form $\theta \in \Omega^1(H, \mathfrak{a})$ such that $\mathrm{Ad}_a(R_a^*\theta) = \theta$ and $\theta(e_A) = \mathrm{id}_{\mathfrak{k}}$. Associated to θ we have its curvature $d\theta + \frac{1}{2}[\theta, \theta] = d\theta$ as θ is an abelian connection, and this is horizontal $q^*\Omega = -d\theta$. Here we may get θ from the canonical invariant 1-form θ_A on A with $\theta_A(e_A) = \mathrm{id}_{\mathfrak{a}}$ by extending using the left multiplication, $\eta_g : A \to H$, $a \mapsto ga$. Then $\eta^*\theta = \theta_A$. Then

$$\int_\sigma \Omega = \int_{q(D^2)} \Omega = \int_{D^2} q^*\Omega = -\int_{D^2} d\theta = -\int_{\partial D^2} \theta = -\int_\gamma \theta_A,$$

where $\theta_A := \theta|_A$. When $\gamma \in C_*^\infty(S^1, A)$ (*ie.* a smooth based map from S^1 to A) we have the natural identification

$$\int_\gamma \theta_A := [\gamma] \in \pi_1(A) \subseteq \mathfrak{a}.$$

Thus

$$\mathrm{per}_\omega([\sigma]) = \int_\sigma \Omega = -\int_\gamma \theta = -\int_\gamma \theta_{\mathrm{A}} = -[\gamma] = -\delta([\sigma]) \in \pi_1(\mathrm{A}).$$

□

We have seen that the long exact sequence (5.1) tells us there is a central extension if and only if $P([\omega]) = 0$. Thus

$$P_1([\omega])([\sigma]) = \pi_\Gamma \circ \mathrm{per}_\omega([\sigma]) = \pi_\Gamma([\gamma]) = 0,$$

as $[\gamma] \in \pi_1(A)$. That is, $\mathrm{per}_\omega([\sigma]) \in \pi_1(\mathrm{A})$ for all $[\sigma] \in \pi_2(K)$, and so $\delta = -\mathrm{per}_\omega : \pi_2(\mathrm{K}) \to \pi_1(\mathrm{A})$ is surjective. We have shown that the connecting map $\delta : \pi_2(K) \to \pi_1(A)$ is surjective whenever we have a central extension.

5.3 Application

We would now like to apply the previous results to the current group $\mathrm{Map}_*(\Sigma, G)$ where G is a compact, connected, simple finite-dimensional Lie group. As we have seen in Chapter 4, $\mathrm{Map}_*(\Sigma, G)$ need not be connected (when G is not simply connected) and so we set $K = \mathrm{Map}_*(\Sigma, G)_e$, the connected component of the identity. Let H be an extension of K by a connected abelian Lie group A with $\pi_1(A) = \mathbb{Z}^{2g}$. Recalling briefly our results on the homotopy of K in Chapter 4, we have $\pi_1(K) = \mathbb{Z}$ and $\pi_2(K) = \mathbb{Z}^{2g} \oplus \pi_4(G)$. We obtain the long exact sequence in homotopy from (5.8)

$$0 \longrightarrow \pi_2(H) \xrightarrow{q_*} \mathbb{Z}^{2g} \oplus \pi_4(G) \xrightarrow{\delta} \mathbb{Z}^{2g} \xrightarrow{i_*} \pi_1(H) \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{5.9}$$

Now using the result from the previous section that the connecting map δ is surjective whenever the central extension exists, we obtain from (5.9)

$$0 \longrightarrow \pi_2(H) \longrightarrow \mathbb{Z}^{2g} \oplus \pi_4(G) \xrightarrow{\delta} \mathbb{Z}^{2g} \longrightarrow 0\ , \tag{5.10}$$

and

$$0 \longrightarrow \pi_1(H) \longrightarrow \mathbb{Z} \longrightarrow 0\ . \tag{5.11}$$

Then $\pi_1(H) = \pi_1(K) = \mathbb{Z}$. Further, $i_*([\phi]) = 0 \in \pi_1(H)$ for all $[\phi] \in \pi_1(A)$, so $i(A) \subset H$ is contractible. Then the sequence (5.10) splits, and the splitting lemma for abelian groups tells us that if a short exact sequence $0 \to A \to B \to C \to 0$ splits, then $B \cong A \oplus C$. Thus we have $\pi_2(H) \oplus \mathbb{Z}^{2g} \cong \mathbb{Z}^{2g} \oplus \pi_4(G)$ whence $\pi_2(H) = \pi_4(G)$. We have obtained the following theorem.

Theorem 5.3.1. *Let G be a compact connected simple finite-dimensional Lie group, $K = \mathrm{Map}_*(\Sigma, G)_e$ and H a central extension of K by a connected finite-dimensional abelian group*

$A = \mathfrak{a}/\pi_1(A)$, *with* $\pi_1(A) = \mathbb{Z}^{2g}$. *Then*

$$\pi_0(H) = \pi_0(K) = [\pi_1(G)]^{2g},$$
$$\pi_1(H) = \pi_1(K) = \mathbb{Z},$$
$$\pi_2(H) = \pi_4(G),$$
$$\pi_k(H) = \pi_k(K) = \pi_{k+2}(G) \times [\pi_{k+1}(G)]^{2g}, \qquad k \geq 3.$$

5.4 Topology of the Universal Central Extension

Let us consider the universal central extension of $K = \mathrm{Map}_*(\Sigma, G)_e$ from [MN03, Section IV]

$$1 \longrightarrow Z \longrightarrow UK \longrightarrow K \longrightarrow 1, \tag{5.12}$$

where

$$Z = \mathscr{A} \times \pi_1(K)$$

with $\mathscr{A} = (\mathfrak{z}/\Pi_\omega)$ and $\Pi_\omega := \mathrm{im}(\mathrm{per}_\omega) \cong \mathbb{Z}^{2g}$. The universal extension UK is simply connected [MN03, IV.4] and has corresponding Lie algebra central extension $\mathfrak{z} \to U\mathfrak{k} \to \mathfrak{k}$. We have seen in Chapter 3, Proposition 3.2.1 that the universal central extension of $\mathfrak{k}$ is an extension by the space $\mathfrak{z} = \Omega^1(\Sigma)/d\Omega^0(\Sigma)$. Recall from the previous section that

$$\pi_0(\mathrm{Map}_*(\Sigma, G)) = [\pi_1(G)]^{2g}, \qquad \pi_1(K) = \mathbb{Z}, \qquad \pi_2(K) = \mathbb{Z}^{2g} \oplus \pi_4(G),$$

where $\pi_4(G)$ is torsion. (Recall that for an H-space each component has the same homotopy type, see *eg.* [Mau80].) Now let us consider the long exact sequence in homotopy.

$$\begin{array}{l} \longrightarrow \pi_2(Z) \longrightarrow \pi_2(UK) \xrightarrow{q_*} \pi_2(K) \xrightarrow{\delta} \pi_1(Z) \xrightarrow{i_*} \pi_1(UK) \\ \longrightarrow \pi_1(K) \longrightarrow \pi_0(Z) \longrightarrow \pi_0(UK) \longrightarrow \pi_0(K) \longrightarrow 1. \end{array} \tag{5.13}$$

As Z is abelian, we have that

$$\pi_2(Z) = 0, \qquad \pi_1(Z) = \pi_1(\mathfrak{z}/\Pi_\omega) = \mathbb{Z}^{2g} \qquad \pi_0(Z) = \pi_1(K) = \mathbb{Z}.$$

Further, as UK is connected and simply connected we find that $\pi_2(UK) = \pi_4(G)$, $\pi_1(UK) = 0$ and $\pi_0(UK) = 0$. We record the previous information in a table:

	Z	UK	K
π_0	$\mathbb{Z}$	0	0
π_1	$\mathbb{Z}^{2g}$	0	$\mathbb{Z}$
π_2	0	$\pi_4(G)$	$\mathbb{Z}^{2g} \oplus \pi_4(G)$

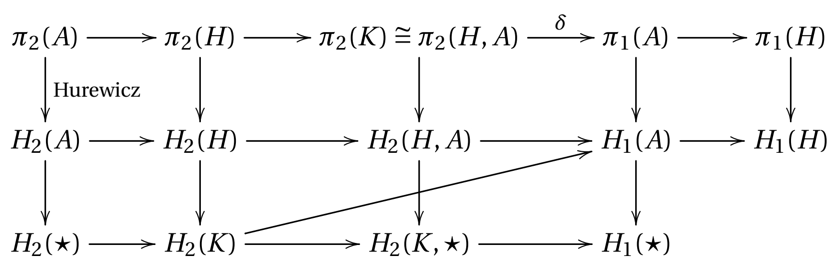
and for $k \geq 3, \pi_k(UK) = \pi_k(K)$. We note that the result for $\pi_2(UK)$ agrees with [MN03, III.7] which in turn contradicts a statement in [PS86, 4.10.1] saying that $\pi_2(UK)$ is trivial. We now turn to the homology groups. For a general principal A-bundle H

$$\begin{array}{ccc} A & \longrightarrow & H \\ & & \downarrow \\ & & K \end{array} \tag{5.14}$$

we have the long exact sequence in homology

$$\cdots \to H_{k+1}(A) \to H_{k+1}(H) \to H_{k+1}(K) \to H_k(A) \to H_k(H) \to H_k(K) \to \cdots. \tag{5.15}$$

Combining with the long exact sequence in homotopy (5.6) we have



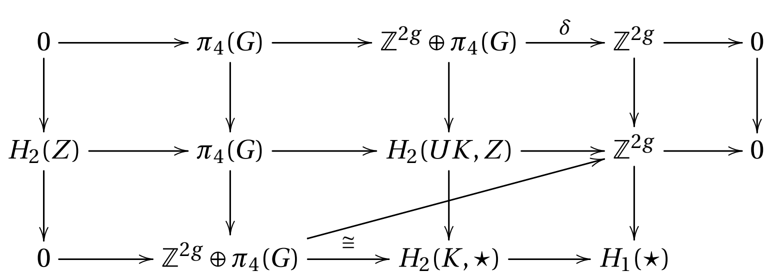
Recall from Theorem 4.4.1 that

$$H_2(K, \mathbb{Z}) = \pi_2(K) = \mathbb{Z}^{2g} \oplus \pi_4(G).$$

Further, we may use Hurewicz's theorem to deduce

$$H_2(UK, \mathbb{Z}) \cong \pi_2(UK) = \pi_4(G).$$

Now it remains to find $H_2(Z, \mathbb{Z})$. From Lemma 4.4.2 we find $H_2(Z, \mathbb{Z}) = \pi_2(Z) = 0$. Now using the fact that $H_1(X, \mathbb{Z})$ is the abelianisation of $\pi_1(X)$ and



(5.16)

we obtain

	Z	UK	K
H_1	$\mathbb{Z}^{2g}$	0	$\mathbb{Z}$
H_2	0	$\pi_4(G)$	$\mathbb{Z}^{2g} \oplus \pi_4(G)$

Now let us find the cohomology groups. We use the universal coefficient theorem:

$$0 \longrightarrow \text{Ext}(H_{n-1}(X,\mathbb{Z}),\mathbb{Z}) \longrightarrow H^n(X,\mathbb{Z}) \longrightarrow \text{Hom}(H_n(X),\mathbb{Z}) \longrightarrow 0\ .$$

Now if H is free we have $\text{Ext}(H,G)=0$, and using this fact we find

$$\begin{aligned}
&\text{Ext}(H_1(Z),\mathbb{Z}) = \text{Ext}(\mathbb{Z}^{2g},\mathbb{Z}) = 0\\
&\text{Hom}(H_2(Z),\mathbb{Z}) = \text{Hom}(0,\mathbb{Z}) = 0\\
&\Rightarrow H^2(Z) = 0
\end{aligned}$$

$$\begin{aligned}
&\text{Ext}(H_1(K),\mathbb{Z}) = \text{Ext}(\mathbb{Z},\mathbb{Z}) = 0\\
&\text{Hom}(H_2(K),\mathbb{Z}) = \text{Hom}(\mathbb{Z}^{2g}\oplus\pi_4(G),\mathbb{Z}) = \mathbb{Z}^{2g}\\
&\Rightarrow H^2(K) = \mathbb{Z}^{2g}
\end{aligned}$$

$$\begin{aligned}
&\text{Ext}(H_1(UK),\mathbb{Z}) = \text{Ext}(0,\mathbb{Z}) = 0\\
&\text{Hom}(H_2(UK),\mathbb{Z}) = \text{Hom}(\pi_4(G),\mathbb{Z}) = 0\\
&\Rightarrow H^2(UK) = 0
\end{aligned}$$

$$\begin{aligned}
&\text{Hom}(H_1(Z),\mathbb{Z}) = \text{Hom}(\mathbb{Z}^{2g},\mathbb{Z}) = \mathbb{Z}^{2g}\\
&\Rightarrow H^1(Z) = \mathbb{Z}^{2g}
\end{aligned}$$

$$\begin{aligned}
&\text{Hom}(H_1(K),\mathbb{Z}) = \text{Hom}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}\\
&\Rightarrow H^1(K) = \mathbb{Z}
\end{aligned}$$

$$\begin{aligned}
&\text{Hom}(H_2(UK),\mathbb{Z}) = \text{Hom}(0,\mathbb{Z}) = 0\\
&\Rightarrow H^2(UK) = 0
\end{aligned}$$

The results obtained previously are summarised in the table below.

Theorem 5.4.1. *The topology of the universal extension UK of $K = \text{Map}_*(\Sigma,G)_e$ is as follows.*

	Z	UK	K
π_0	$\mathbb{Z}$	0	0
π_1	$\mathbb{Z}^{2g}$	0	$\mathbb{Z}$
π_2	0	$\pi_4(G)$	$\mathbb{Z}^{2g}\oplus\pi_4(G)$
H_1	$\mathbb{Z}^{2g}$	0	$\mathbb{Z}$
H_2	0	$\pi_4(G)$	$\mathbb{Z}^{2g}\oplus\pi_4(G)$
H^1	$\mathbb{Z}^{2g}$	0	$\mathbb{Z}$
H^2	0	0	$\mathbb{Z}^{2g}$

We record the theorem of [EF94] for comparison. Note that here G is assumed to be simply connected, so that $\text{Map}_*(\Sigma,G)$ is connected.

Theorem 5.4.2. *[EF94, 2.3] The universal central extension UG^{Σ} of* $\mathrm{Map}_*(\Sigma, G)$ *has*

$$\pi_n(UG^{\Sigma}) \otimes \mathbb{R} = 0$$

for $n < 3$. The kernel of the extension is the infinite-dimensional abelian group $\mathscr{A} = \Omega^1(\Sigma)/\Omega^1_{\mathbb{Z}}(\Sigma)$.

Chapter 6

The Jacobi Group

We now leave current groups and their extensions and introduce another object of interest: the Jacobi group. The Jacobi group was first systematically studied in the monograph of Eichler and Zagier [EZ85], where it is defined as a semidirect product of $SL_2(\mathbb{Z})$ and a 3-dimensional Heisenberg group $\mathbb{Z}^2 \times \mathbb{Z}$. It is the group of transformations of *Jacobi forms*, which Eichler and Zagier describe as 'a cross between elliptic functions and modular forms in one variable.' Indeed, a Jacobi form (of *weight* k and *index* m) is defined to be a holomorphic function $\phi : \mathbb{C} \times \mathscr{H} \to \mathbb{C}$ satisfying the two transformation equations

$$\phi(z,\tau) = e^{2\pi i m(\mu^2\tau + 2\mu z + \lambda\mu + n)}\phi(z + \lambda + \tau\mu, \tau)$$

and

$$\phi(z,\tau) = (c\tau + d)^{-k} e^{\left(\frac{-2\pi i b z^2}{a + b\tau}\right)} \phi\left(\frac{z}{a + b\tau}, \frac{c + d\tau}{a + b\tau}\right).$$

Later, a different form of the group appeared in the study of generalised root systems in the work of Saito [ST97]. It was noted by [Loo80] that the *extended affine Weyl group* of a generalised root system associated to a simple Lie algebra $\mathfrak{g}$ carried a natural $SL_2(\mathbb{Z})$-action, and the resulting semidirect product was called the Jacobi group by Wirthmüller [Wir92], as when the Lie algebra is of type A_1 it coincides with the Jacobi group as defined by Eichler and Zagier. This is the form of the Jacobi group we shall introduce below.

The Jacobi group $J_{\mathfrak{g}}$ enjoys strong links to the theory of integrable systems, in particular the WDVV equations, as demonstrated by [Ber99] and [Str10]. It has a natural representation on a space Y called the *Tits cone* [Loo80], and the quotient space $Y/J_{\mathfrak{g}}$ has the structure of a *Frobenius manifold* [Ber00a].

In this chapter we introduce the Jacobi group from the viewpoint of [Loo80] and [Wir92], which shall lead the way to the discussion in Chapter 7 of an association to a certain extension of the current group G^E as hinted at in Chapter 3. Note we have modified conventions in the group laws and actions for agreement in later chapters. The monograph [EZ85] and the main papers on the subject [Loo80], [Wir92] and [Ber99] all have differing conventions.

6.1 Construction of the Jacobi Group

We construct the Jacobi group as an extension of the affine Weyl group, following [Wir92] and [Loo80]. We also note that this construction appears in [Kac90, Chapter 13], although without this name. The notation will be set up here but is summarised in the preamble for convenience. More details on root systems may be found in Appendix A, where we follow the conventions of Carter [Car05]. Let Φ denote an irreducible finite root system associated to a simple complex Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{t}$, and let $\Phi^\vee$ denote the dual root system. Fix a set of simple roots $\triangle = \{\alpha_1, \ldots, \alpha_l\}$ for Φ, where $l = \operatorname{rank}\Phi$, and a set of simple coroots, $\triangle^\vee = \{h_1, \ldots, h_l\}$ for $\Phi^\vee$, where $\alpha_i(h_i) = 2$. Let $Q \subset \mathfrak{t}^*$ denote the root lattice generated by Φ and $Q^\vee \subset \mathfrak{t}$ denote the lattice generated by $\Phi^\vee$, which we call the coroot lattice. Associated to Φ we have the Cartan matrix C, with $C_{ij} = \alpha_j(h_i)$ for $i, j = 1, \ldots, l$. Let W be the Weyl group of Φ, generated by the fundamental reflections $s_1, \ldots, s_l$ where

$$s_i(\lambda) = \lambda - \alpha_i(\lambda) h_i$$

for $\lambda \in \mathfrak{t}$. On $\mathfrak{t}$ we naturally have a W-invariant bilinear form, the Killing form $\langle \cdot, \cdot \rangle_K$. We normalise the Killing form (dropping the subscript K) so that $\langle h_\alpha, h_\alpha \rangle = 2$ for h_α a long coroot, which implies that $\langle \lambda, \lambda \rangle \in 2\mathbb{Z}$ for all $\lambda \in Q^\vee$. This induces a bilinear form on $\mathfrak{t}^*$ defined by $(\alpha_i, \alpha_j) := \langle t_{\alpha_i}, t_{\alpha_j} \rangle_K$, where t_{α_i} is the unique element of $\mathfrak{t}$ such that $\alpha(h) = \langle h, t_\alpha \rangle$ for all $h \in \mathfrak{t}$. We have an isomorphism $\Psi : \mathfrak{t}^* \to \mathfrak{t}$ given by $\alpha \mapsto t_\alpha$ (and $\frac{2\alpha}{(\alpha,\alpha)} \mapsto h_\alpha$).

We introduce a Heisenberg group which is a central extension of $\mathfrak{t} \times \mathfrak{t}$ as follows:

Definition 6.1.1. *The set* $\mathfrak{t} \times \mathfrak{t} \times \mathbb{R}$ *defines a Heisenberg group which we denote by H with group law given by*

$$(\lambda, \mu, n) \cdot (\lambda', \mu', n') = (\lambda + \lambda', \mu + \mu', n + n' + \langle \lambda', \mu \rangle - \langle \lambda, \mu' \rangle). \tag{6.1}$$

We shall always write the group law in the Heisenberg group additively.

Remark 6.1.2. *The above group law also demonstrates that H is a Heisenberg group with centre $\mathbb{R}$ and skew-symmetric cocycle $f_1 : (\mathfrak{t} \times \mathfrak{t}) \times (\mathfrak{t} \times \mathfrak{t}) \to \mathbb{R}$ given by $f_1\big((\lambda, \mu), (\lambda', \mu')\big) = \langle \lambda', \mu \rangle - \langle \lambda, \mu' \rangle$. In [Ber99], a different cocycle is used. However, as we noted in Chapter 2, two cocycles differing by a coboundary give rise to isomorphic groups. Bertola uses the cocycle $f_2((\lambda, \mu), (\lambda', \mu')) = 2\langle \lambda', \mu \rangle$, and we find $f_1 = f_2 + \delta c$, where c is the coboundary $c : (\lambda, \mu) \mapsto \langle \lambda, \mu \rangle$.*

Now denote by $H_\mathbb{Z}$ the subgroup of H generated by $(\lambda, 0, 0), (0, \mu, 0)$ for $\lambda, \mu \in Q^\vee$, and $(0, 0, n)$ for $n \in \mathbb{Z}$. Then

$$H_\mathbb{Z} = \{(\lambda, \mu, n) \in H \,|\, \lambda, \mu, \in Q^\vee, n + \langle \lambda, \mu \rangle \in \mathbb{Z}\}.$$

Now the finite Weyl group W acts naturally on the coroot lattice $Q^\vee$, and we extend the action to an action on the Heisenberg group $H_\mathbb{Z}$ by acting trivially on the centre:

$$w(\lambda, \mu, n) = (w\lambda, w\mu, n),$$

for $w \in W$, $(\lambda, \mu, n) \in H_{\mathbb{Z}}$. This allows us to define a semidirect product of W and $H_{\mathbb{Z}}$ which we call the extended affine Weyl group.

Definition 6.1.3. *The* extended affine Weyl group $\mathscr{W}$ *is the semidirect product* $W \ltimes H_{\mathbb{Z}}$*, given in the exact sequence*

$$1 \longrightarrow H_{\mathbb{Z}} \longrightarrow \mathscr{W} \longrightarrow W \longrightarrow 1. \tag{6.2}$$

The group law may be written as

$$\begin{aligned}(w, \mathbf{t}) \cdot (w', \mathbf{t}) &= (ww', \mathbf{t} \cdot w(\mathbf{t}')) \\ &= (ww', (\lambda, \mu, n) \cdot (w\lambda', w\mu', n)) \\ &= (ww', (\lambda + w\lambda', \mu + w\mu', n + n' + \langle w\lambda', \mu\rangle - \langle \lambda, w\mu'\rangle))\end{aligned}$$

where $w \in W$ *and* $\mathbf{t} = (\lambda, \mu, n) \in Q^\vee \times Q^\vee \times \mathbb{Z}$.

Let Y denote the space

$$Y = \mathbb{C} \oplus \mathfrak{t} \oplus \mathscr{H} \ni (u, x, \tau),$$

where $\mathscr{H}$ denotes the upper half-plane, $\mathscr{H} = \{\tau \in \mathbb{C} \mid \operatorname{im} \tau > 0\}$. This is called the *Tits cone* [Loo80].

Proposition 6.1.4. *The Heisenberg group* $H_{\mathbb{Z}}$ *acts on the space* Y *in the following way:*

$$(\lambda, \mu, n) \diamond (u, x, \tau) = \big(u + n + 2\langle x, \mu\rangle + \langle \mu, \mu\rangle\tau + \langle \lambda, \mu\rangle, x + \lambda + \mu\tau, \tau\big), \tag{6.3}$$

for $(\lambda, \mu, n) \in H$, $(u, x, \tau) \in Y$.

It was then noted by [Loo80] that Y also carries an action of $SL_2(\mathbb{Z})$ expressed by the formula

$$\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (u, x, \tau) \mapsto \left(u - \frac{b\langle x, x\rangle}{a + b\tau}, \frac{x}{a + b\tau}, \frac{c + d\tau}{a + b\tau}\right) \tag{6.4}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Note that the $SL_2(\mathbb{Z})$-action on the coordinates $(x, \tau) \in \mathbb{C} \times \mathscr{H}$ is natural and the action on the u-coordinate is defined as such to be compatible with the action of $\mathscr{W}$, allowing us to form a semidirect product $\mathscr{W} \rtimes SL_2(\mathbb{Z})$ equipped with an action on Y, which we shall call the Jacobi group. We explicitly state the action of $SL_2(\mathbb{Z})$ on $\mathscr{W}$:

Lemma 6.1.5. *An element* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *acts on an element* $(w, \mathbf{t}) \in \mathscr{W}$ *as follows:*

$$\begin{aligned}\gamma \diamond w &= w \\ \gamma \diamond (\lambda, \mu, n) &= \big(d\lambda - c\mu, -b\lambda + a\mu, n - \tfrac{1}{2}\langle d\lambda - c\mu, -b\lambda + a\mu\rangle + \tfrac{1}{2}\langle \lambda, \mu\rangle\big).\end{aligned} \tag{6.5}$$

Remark 6.1.6. *Note that we have* $\langle \lambda, \lambda\rangle \in 2\mathbb{Z}$ *for all* $\lambda \in Q^\vee$ *and so* $-\frac{1}{2}\langle d\lambda - c\mu, -b\lambda + a\mu\rangle + \frac{1}{2}\langle \lambda, \mu\rangle = \frac{1}{2}ac\langle \mu, \mu\rangle - bc\langle \lambda, \mu\rangle + \frac{1}{2}bd\langle \lambda, \lambda\rangle \in \mathbb{Z}$.

Definition 6.1.7. *The* Jacobi group $J_{\mathfrak{g}}$ *is the semidirect product* $J_{\mathfrak{g}} := \mathscr{W} \rtimes SL_2(\mathbb{Z})$ *with the action of* $SL_2(\mathbb{Z})$ *on* $\mathscr{W}$ *given in (6.5) and group law*

$$(w, \mathbf{t}, \gamma) \cdot (w', \mathbf{t}', \gamma') = (ww', \mathbf{t} \cdot w(\gamma \diamond \mathbf{t}'), \gamma\gamma') \tag{6.6}$$

for all $(w, \mathbf{t}), (w', \mathbf{t}') \in \mathscr{W}$, $\gamma, \gamma' \in SL_2(\mathbb{Z})$.

Indeed, the group law comes from the definition as nested semidirect products: $((w,\mathbf{t}),\gamma) \cdot ((w',\mathbf{t}'),\gamma') = ((w,\mathbf{t}) \cdot \gamma \diamond (w',\mathbf{t}'), \gamma\gamma') = (ww', \mathbf{t} \cdot w(\gamma \diamond \mathbf{t}'), \gamma\gamma')$.

Proposition 6.1.8. *The Tits cone* Y *admits a (faithful) action of the Jacobi group* $J_{\mathfrak{g}}$ *as follows:*

$$\begin{aligned} w &: (u, x, \tau) \mapsto (u, wx, \tau) \\ \mathbf{t} &: (u, x, \tau) \mapsto \left(u + n + 2\langle x, \mu\rangle + \langle \mu, \mu\rangle \tau + \langle \lambda, \mu\rangle, x + \lambda + \mu\tau, \tau\right) \\ \gamma &: (u, x, \tau) \mapsto \left(u - \frac{b\langle x, x\rangle}{a + b\tau}, \frac{x}{a + b\tau}, \frac{c + d\tau}{a + b\tau}\right). \end{aligned} \tag{6.7}$$

An element $(w, \mathbf{t}, \gamma) \in J_{\mathfrak{g}}$ *acts as*

$$(w, \mathbf{t}, \gamma) : (u, x, \tau) \mapsto (\mathrm{id}_W, \mathbf{t}, I)[(\mathrm{id}_W, \mathrm{id}_H, \gamma)[(w, 0, I)(u, x, \tau)]]. \tag{6.8}$$

6.2 Jacobi Forms

We mention briefly the definition of *Jacobi forms*, which are holomorphic functions which are invariant under the action of the Jacobi group $J_{\mathfrak{g}}$.

Definition 6.2.1. *A* Jacobi form of weight k and index m *(*$k, m \in \mathbb{N}$*) is a holomorphic function* $\phi : \mathfrak{t} \times \mathscr{H} \to \mathbb{C}$ *satisfying*

1. $\phi\left(\frac{x}{a+b\tau}, \frac{c+d\tau}{a+b\tau}\right) = (a + b\tau)^k e^{\frac{2\pi i b\langle x,x\rangle}{a+b\tau}} \phi(x, \tau)$;
2. $\phi(x + \lambda, \tau) = \phi(x, \tau)$;
3. $\phi(x + \mu\tau, \tau) = e^{-2\pi i m(\langle \mu,\mu\rangle\tau + 2\langle \mu, x\rangle)} \phi(x, \tau)$;

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \lambda, \mu \in Q^\vee$.

Following [Ber99], and similarly to [Str10] we introduce a new function $\phi : \mathbb{C} \times \mathfrak{t} \times \mathscr{H} \to \mathbb{C}$ defined by

$$\phi(u, x, \tau) = e^{mu} \phi(x, \tau), \tag{6.9}$$

which we also call a Jacobi form. The invariance properties of this Jacobi form are then given by

1. $\phi\left(u - \frac{b\langle x,x\rangle}{a+b\tau}, \frac{x}{a+b\tau}, \frac{c+d\tau}{a+b\tau}\right) = (a + b\tau)^k \phi(u, x, \tau)$;
2. $\phi(u, x + \lambda, \tau) = \phi(u, x, \tau)$;

3. $\phi(u + 2\langle x, \mu\rangle + \tau\langle \mu, \mu\rangle, x + \tau\mu, \tau) = \phi(u, x, \tau)$;

4. $\phi(u, wx, \tau) = \phi(u, x, \tau)$,

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \lambda, \mu \in Q^\vee, w \in W$.

Chapter 7

Extensions of G^Σ and the Jacobi Group

We shall now discuss interesting extensions of (the identity component of) G^Σ, including the lift of the Lie algebra extension (3.12) and a new extension which we shall relate to the Jacobi group from Chapter 6. In particular, this extension arose in modifying the theorem of Bertola discussed in Chapter 3. Indeed, we will show that the Heisenberg part of the Jacobi group can be interpreted as a discrete subgroup of a central extension of G^E, and the Heisenberg action on the Tits cone can be interpreted as coming from a projective representation of G^E on an extension of its Lie algebra.

We shall use several results of Chapter 4 on the topology of G^Σ. Recall from Theorem 4.2.7 that for a simple, complex Lie group G and a compact Riemann surface Σ of genus g, we have that $\pi_0(G^\Sigma) = [\pi_1(G)]^{2g}$ and $\pi_1(G^\Sigma) = \mathbb{Z}$.

7.1 Extensions of G^Σ

Theorem 7.1.1. *[EF94, Theorem 2.2] There exists a central extension $\widehat{G}^\Sigma_J$ of the current group G^Σ by the Jacobian variety of Σ whose Lie algebra is $\widehat{\mathfrak{g}}^\Sigma$.*

Note this construction depends on the complex structure of Σ, and is a finite-dimensional extension. When viewed as a principal bundle, this extension is topologically non-trivial, which (as we noted in Chapter 2) implies that we may not write down a continuous group 2-cocycle for this extension. In the case where Σ is a complex torus E, the extension is by means of E itself, and the identification $\Sigma = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ gives E the structure of an abelian group.

Recall that the simply connected covering group $\widetilde{G}^\Sigma$ of G^Σ is a central extension as follows

$$1 \longrightarrow \Gamma \longrightarrow \widetilde{G}^\Sigma \longrightarrow G^\Sigma \longrightarrow 1, \tag{7.1}$$

where $\Gamma = \pi_1(G^\Sigma)$ is discrete and central. We have from Theorem 4.2.7 that $\pi_1(G^\Sigma) = \mathbb{Z}$.

Theorem 7.1.2. *[EF94, Theorem 2.3] The universal central extension UG^Σ of the group G^Σ is an extension of $\widetilde{G}^\Sigma$ by $\mathscr{A} = \Omega^1(\Sigma)/\Omega^1_{\mathbb{Z}}(\Sigma)$, the space of complex-valued one-forms on Σ modulo closed one-forms with integral periods.*

$$1 \longrightarrow \mathscr{A} \longrightarrow UG^\Sigma \longrightarrow \widetilde{G}^\Sigma \longrightarrow 1 \tag{7.2}$$

For a proof of universality, see for example [MN03].

In light of [Nee02, 7.14], the central extension (7.2) is the pullback of some central extension H of G^Σ to the universal covering group $\widetilde{G}^\Sigma$ of G^Σ:

$$1 \longrightarrow \mathscr{A} \longrightarrow H \longrightarrow G^\Sigma \longrightarrow 1. \tag{7.3}$$

Then UG^Σ is also a central extension of G^Σ, as the kernel acts trivially on the Lie algebra $\mathfrak{h}$. The kernel isomorphic to $\mathscr{A} \times \pi_1(G^\Sigma)$. We write the universal central extension in the short exact sequence

$$1 \longrightarrow Z \longrightarrow UG^\Sigma \longrightarrow G^\Sigma \longrightarrow 1, \tag{7.4}$$

where $Z = \mathscr{A} \times \pi_1(G^\Sigma)$. We studied the topology of this extension in Section 5.4.

We shall construct a different central extension of G^Σ from the one described in Theorem 7.1.1. In fact, we shall see that the extension we construct is in a sense orthogonal to $\widehat{G}^\Sigma_J$, in that it comes from the $\pi_1(G^\Sigma)$-part of the universal central extension. Unlike $\widehat{G}^\Sigma_J$, our extension (viewed as a principal bundle) is topologically trivial and we can thus define it via a cocycle. Let $g \in G^\Sigma$ be a map from the surface Σ to G. Then $g^{-1}dg$ and dgg^{-1} are both $\mathfrak{g}$-valued one-forms on Σ.

Lemma 7.1.3. *The map $\rho : G^\Sigma \times G^\Sigma \to \Omega^2(\Sigma, \mathbb{C})$ given by*

$$\rho(g,h) = \langle g^{-1}dg \wedge dhh^{-1} \rangle \tag{7.5}$$

for all $g, h \in G^\Sigma$ is a 2-cocycle on G^Σ. Here $\langle g^{-1}dg \wedge dhh^{-1}\rangle$ denotes both the wedge product of the $\mathfrak{g}$-valued one-forms on Σ and evaluation of the invariant bilinear form $\langle \cdot, \cdot \rangle$ defined on $\mathfrak{g}$.

Proof. We check that the cocycle identity $\rho(gh,k) + \rho(g,h) = \rho(g,hk) + \rho(h,k)$ holds:

$$\begin{aligned}
\rho(gh,k) &+ \rho(g,h) - \rho(g,hk) - \rho(h,k) \\
&= (gh)^{-1}d(gh) \wedge dkk^{-1} + g^{-1}dg \wedge dhh^{-1} \\
&\quad - g^{-1}dg \wedge d(hk)(hk)^{-1} - h^{-1}dh \wedge dkk^{-1} \\
&= h^{-1}g^{-1}dgh \wedge dkk^{-1} + h^{-1}dh \wedge dkk^{-1} + g^{-1}dg \wedge dhh^{-1} \\
&\quad - g^{-1}dg \wedge dhh^{-1} - g^{-1}dg \wedge hdkk^{-1}h^{-1} - h^{-1}dh \wedge dkk^{-1} \\
&= 0.
\end{aligned}$$

□

We now integrate $\rho(g,h) \in \Omega^2(\Sigma)$ over the surface Σ to obtain a complex-valued cocycle on G^Σ:

$$C(g,h) = \int_\Sigma \rho(g,h) = \int_\Sigma \langle g^{-1}dg \wedge dhh^{-1}\rangle. \tag{7.6}$$

Definition 7.1.4. *The cocycle (7.6) yields a central extension of G^Σ by $\mathbb{C}$, which we denote $\widehat{G}^\Sigma_C$. The group law on $\widehat{G}^\Sigma_C$ is written explicitly as*

$$(g,a)\cdot(h,b) = (gh, a+b+C(g,h)) \tag{7.7}$$

for all $(g,a),(h,b) \in G^\Sigma \oplus \mathbb{C}$. We write the centrally extended group $\widehat{G}^\Sigma_C$ in a short exact sequence:

$$1 \longrightarrow \mathbb{C} \longrightarrow \widehat{G}^\Sigma_C \longrightarrow G^\Sigma \longrightarrow 1. \tag{7.8}$$

Let us consider the Lie algebra of this extension. The Lie algebra cocycle corresponding to $C \in \Omega^2(G^\Sigma)$ is given by

$$c(\xi,\eta) = \int_\Sigma \langle d\xi \wedge d\eta\rangle, \tag{7.9}$$

and the central extension splits: $\widehat{\mathfrak{g}}^\Sigma_c = \mathfrak{g}^\Sigma \oplus \mathbb{C}$. From the universal property of the universal central extension UG^Σ, we require a group homomorphism ψ such that the diagram

$$\begin{array}{ccc} UG^\Sigma & & \\ \downarrow{\scriptstyle\psi} & \searrow & \\ \widehat{G}^\Sigma_C & \longrightarrow & G^\Sigma \end{array} \tag{7.10}$$

commutes. Now as product sets, $UG^\Sigma = (\mathscr{A} \times \mathbb{Z}) \times G^\Sigma$, and $\widehat{G}^\Sigma_C = \mathbb{C} \times G^\Sigma$. Then $\psi(z,f) = (\chi(z), f)$ for $f \in G^\Sigma, z \in Z$ and a homomorphism $\chi : \mathscr{A} \times \mathbb{Z} \to \mathbb{C}$. The fact that the Lie algebra cocycle is trivial means that $\mathscr{A} \subset \ker\chi$, so χ factors through a homomorphism $\pi_1(G^\Sigma) \to \mathbb{C}$. Then there exists a Lie algebra homomorphism (section) $\theta : \mathfrak{g}^\Sigma \to \widehat{\mathfrak{g}}^\Sigma_C$ associated to the splitting of the central extension. The section θ integrates to a group homomorphism $\theta_{G^\Sigma} : \widetilde{G^\Sigma} \to \widehat{G}^\Sigma_C$. Again writing the groups as product sets we see $\theta_{G^\Sigma} : G^\Sigma \times \pi_1(G^\Sigma) \to G^\Sigma \times \mathbb{C}$. Then restricting θ_{G^Σ} to $\pi_1(G^\Sigma)$ yields the required homomorphism $\chi : \pi_1(G^\Sigma) \to \mathbb{C}$.

Proposition 7.1.5. *[KW09] The left-invariant closed 3-form*

$$\kappa(g) = \frac{1}{4\pi^2}\langle g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\rangle, \tag{7.11}$$

is the pullback of an integral closed 3-form on the simply connected compact simple Lie group G. It is the single generator for $H^3(G,\mathbb{Z})$.

In Chapter 4 we noted that we have an isomorphism $\pi_1(G^\Sigma) \simeq \pi_3(G)$. The required homomorphism $\pi_3(G) \to \mathbb{C}$ is then

$$\chi : \gamma \mapsto \int_\gamma \kappa(g), \tag{7.12}$$

for $\gamma \in \pi_3(G)$.

As we are able to write the group law explicitly using a cocycle, we may interpret $\widehat{G}_C^\Sigma$ as a topologically trivial complex line bundle over G^Σ

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \widehat{G}_C^\Sigma \\ & & \downarrow \\ & & G^\Sigma. \end{array}$$

The fact that we have a continuous cocycle implies the existence of a global section, and hence the triviality of the bundle. In the principal bundle viewpoint we see that the Lie algebra section $\theta : \mathfrak{g}^\Sigma \to \widehat{\mathfrak{g}}_c^\Sigma$ provides a closed connection one-form, and $\chi : \pi_1(G^\Sigma) \to \mathbb{C}$ is its holonomy. (A connection tells us how a path in G^Σ can be lifted to a horizontal path in $\widehat{G}_C^\Sigma$, with the difference between the start point and end point being given by the holonomy.)

Let us consider the cocycle $c : G^\Sigma \times G^\Sigma \to \Omega^2(\Sigma)$ in more detail. Using Hodge decomposition we may write $c(g,h)$ as

$$c(g,h) = d\alpha + \beta$$

where $\alpha \in \Omega^1(\Sigma)$ and β harmonic. Then

$$C(g,h) = \int_\Sigma c(g,h) = \int_\Sigma \beta.$$

7.2 A Representation for $\widehat{G}_C^E$

Here we return to the idea of Bertola [Ber99] as discussed in Section 3.3.3. We restrict our attention to the central extension for the current group G^E, where E is a compact Riemann surface of genus one, which we may think of as $E = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. In this case the space $\mathscr{H}_E$ (respectively $\bar{\mathscr{H}}_E$) of holomorphic (resp. anti-holomorphic) one-forms on E is one-dimensional, with unique generator dz (respectively $d\bar{z}$). Decomposing the operator d into its Dolbeault components $d = \partial + \bar{\partial}$, we write the cocycle $C(g,h)$ as:

$$C(g,h) = \int_E \langle g^{-1} dg \wedge dhh^{-1} \rangle = \int_E \left[\langle g^{-1}\partial g, \bar{\partial} hh^{-1} \rangle - \langle g^{-1}\bar{\partial} g, \partial hh^{-1} \rangle \right] dz \wedge d\bar{z},$$

for $g,h \in G^E$. We would like to relate $\widehat{G}_C^E$ to our previous discussion of the Jacobi group as inspired by Bertola. Indeed, we shall construct an action of $\widehat{G}_C^E$ which is an appropriate modification of the transformation (3.28) on the elliptic Lie algebra $\dot{\mathfrak{g}}^E$, which we record again here:

$$(\xi, j, k) \mapsto \big(\mathrm{Ad}_g \xi + k\bar{\partial} g g^{-1}, j - \langle g^{-1}\partial g, \xi \rangle - \tfrac{1}{2} k \langle g^{-1}\partial g, g^{-1}\bar{\partial} g \rangle, k \big). \tag{7.13}$$

Recall $\dot{\widehat{\mathfrak{g}}}^E = \mathfrak{g}^E \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$ with Lie bracket given by

$$[(\xi_1, j_1, k_1), (\xi_2, j_2, k_2)] = \left([\xi_1, \xi_2] + k_2\bar{\partial}\xi_1 - k_1\bar{\partial}\xi_2, \widetilde{\Omega}(\xi_1, \xi_2), 0\right),$$

where $(\xi_i, j_i, k_i) \in \mathfrak{g}^E \oplus \mathbb{C}\bar{\omega}^\vee$ and Ω is the 2-cocycle on $\mathfrak{g}^E$ with values in $\mathscr{H}_E^*$ given by

$$\widetilde{\Omega}(\xi_1, \xi_2)(dz) = \int_E dz \wedge d\bar{z}\langle\xi_1, \partial\xi_2\rangle.$$

We now give our action of $\widehat{G}_C^E$ on $\dot{\widehat{\mathfrak{g}}}^E$.

Proposition 7.2.1. *An element $(g, n) \in \widehat{G}_C^E$ acts on the space $\dot{\widehat{\mathfrak{g}}}^E$ as follows:*

$$(g, n) : (\xi, j, k) \mapsto \left(Ad_g\xi + (\tau - \bar{\tau})\bar{\partial}gg^{-1}, j + n + 2\langle\xi, g^{-1}\partial g\rangle + (\tau - \bar{\tau})\langle g^{-1}\partial g, g^{-1}\bar{\partial}g\rangle, k\right). \tag{7.14}$$

Note that the scalar $(\tau - \bar{\tau})$ is simply for future convenience.

Proof. We first recall the group law in $\widehat{G}_C^E$:

$$\begin{aligned}(g, n) \cdot (h, m) &= \left(gh, n + m + \int_E \langle g^{-1}dg \wedge dhh^{-1}\rangle\right)\\ &= \left(gh, n + m + \langle g^{-1}\partial g, \bar{\partial}hh^{-1}\rangle - \langle g^{-1}\bar{\partial}g, \partial hh^{-1}\rangle\right).\end{aligned}$$

Then we have

$$\begin{aligned}\left[(g, n) \cdot (h, m)\right] \diamond (\xi, j, k) = \big(&Ad_{gh}\xi + \bar{\partial}(gh)(gh)^{-1}, j + n + m + \langle g^{-1}\partial g, \bar{\partial}hh^{-1}\rangle\\ &- \langle\partial hh^{-1}, g^{-1}\bar{\partial}g\rangle + 2\langle\xi, (gh)^{-1}\partial(gh)\rangle\\ &+ \langle(gh)^{-1}\partial(gh), (gh)^{-1}\bar{\partial}(gh)\rangle, k\big)\end{aligned}$$

and

$$\begin{aligned}&(g, n) \diamond [(h, m)\diamond(\xi, j, k)]\\ &\quad= (g, n) \diamond \left(Ad_h\xi + \bar{\partial}hh^{-1}, j + m + 2\langle\xi, h^{-1}\partial h\rangle + \langle h^{-1}\partial h, h^{-1}\bar{\partial}h\rangle, k\right)\\ &\quad= \big(Ad_g Ad_h\xi + g\bar{\partial}hh^{-1}g^{-1} + \bar{\partial}gg^{-1}, j + m + 2\langle\xi, h^{-1}\partial h\rangle\\ &\qquad + \langle h^{-1}\partial h, h^{-1}\bar{\partial}h\rangle + n + 2\langle Ad_h\xi + \bar{\partial}hh^{-1}, g^{-1}\partial g\rangle\\ &\qquad + \langle g^{-1}\partial g, g^{-1}\bar{\partial}g\rangle, k\big)\\ &\quad= \big(Ad_{gh}\xi + \bar{\partial}(gh)(gh)^{-1}, j + n + m + 2\langle\xi, (gh)^{-1}\partial(gh)\rangle\\ &\qquad + \langle(gh)^{-1}\partial(gh), (gh)^{-1}\bar{\partial}(gh)\rangle + \langle g^{-1}\partial g, \bar{\partial}hh^{-1}\rangle\\ &\qquad - \langle g^{-1}\bar{\partial}g, \partial hh^{-1}\rangle, k\big)\\ &\quad= \left[(g, n) \cdot (h, m)\right] \diamond (\xi, j, k).\end{aligned}$$

Then the extended group $\widehat{G}_C^E$ admits an action on the extended Lie algebra $\dot{\widehat{\mathfrak{g}}}^E$. □

We note that the bilinear form on $\dot{\widehat{\mathfrak{g}}}^E$ is not invariant under the action of $\widehat{G}_C^E$. In the next two sections we shall relate the extension to $\widehat{G}_C^E$ to the Jacobi group introduced in Chapter 6.

7.2.1 A Discrete Subgroup of $\widehat{G}_C^E$

We recall from Proposition 3.4.2 that $\dot{\mathfrak{g}}^E$ has a finite-dimensional Cartan subalgebra $\dot{\mathfrak{t}}^E = \mathfrak{t} \oplus \mathbb{C}\bar{\omega}^\vee \oplus \mathbb{C}\delta$, where $\mathfrak{t}$ is the Cartan subalgebra of the finite-dimensional Lie algebra $\mathfrak{g}$.

Let us consider a discrete subgroup of G^E, consisting of elements of the form

$$F_{(\lambda,\mu)}(z,\bar{z}) = \exp \frac{1}{\tau - \bar{\tau}} \left[(\lambda + \tau\mu)\bar{z} - (\lambda + \bar{\tau}\mu)z \right], \tag{7.15}$$

where $\lambda + \tau\mu \in Q^\vee \oplus \tau Q^\vee \subset \mathfrak{t}$, where $Q^\vee$ is the coroot lattice in $\mathfrak{t}_\mathbb{R}$. (Recall $\mathfrak{t}$ is a complex Cartan subalgebra of $\mathfrak{g}$.) We consider the extension $\widehat{G}_C^E$ restricted to this subgroup. We calculate the cocycle between two elements $F = F_{(\lambda,\mu)}$ and $F' = F_{(\lambda',\mu')}$.

$$\begin{aligned} c(F,F') = F^{-1}dF \wedge dF'F'^{-1} &= \frac{1}{\tau-\bar{\tau}} \left[(\lambda+\tau\mu)d\bar{z} - (\lambda+\bar{\tau}\mu)dz \right] \wedge \frac{1}{\tau-\bar{\tau}} \left[(\lambda'+\tau\mu')d\bar{z} - (\lambda'+\bar{\tau}\mu')dz \right] \\ &= \frac{1}{(\tau-\bar{\tau})^2} \left[(\lambda+\tau\mu)(\lambda'+\bar{\tau}\mu') - (\lambda+\bar{\tau}\mu)(\lambda'+\tau\mu') \right] dz \wedge d\bar{z} \\ &= \frac{1}{\tau-\bar{\tau}} \left[\langle \lambda', \mu \rangle - \langle \lambda, \mu' \rangle \right] dz \wedge d\bar{z}. \end{aligned}$$

Then we have

$$C(F,F') = \frac{1}{\tau-\bar{\tau}} \int_E \left[\langle \mu, \lambda' \rangle - \langle \lambda, \mu' \rangle \right] dz \wedge d\bar{z} = \langle \mu, \lambda' \rangle - \langle \lambda, \mu' \rangle, \tag{7.16}$$

where we have used that $\int_E dz \wedge d\bar{z} = \tau - \bar{\tau}$ and we find that $C(F,F')$ takes integer values for all $F, F' \in G^E$, as the bilinear form is integral on the coroot lattice.

Lemma 7.2.2. *The extension $\widehat{G}_C^E$ restricted to F^E is a Heisenberg group with centre $\mathbb{Z}$. As a set, it is $\widehat{F}_\mathbb{Z}^E = F^E \oplus \mathbb{Z}$, and the group law is given by*

$$(F_{(\lambda,\mu)}, n) \cdot (F_{(\lambda',\mu')}, n') = \left(F_{(\lambda+\lambda',\mu+\mu')}, n + n' + \langle \mu, \lambda' \rangle - \langle \lambda, \mu' \rangle \right). \tag{7.17}$$

The action of an element $(F,n) = (F_{(\lambda,\mu)}, n) \in \widehat{G}_C^E$ on $\dot{\mathfrak{g}}^E$ is as follows:

$$\begin{aligned} &(F,n) \diamond (\xi, j, k) \\ &\quad = \left(Ad_F^{G^E} \xi + (\tau-\bar{\tau})\bar{\partial}FF^{-1}, j + n + 2\langle \xi, F^{-1}\partial F \rangle + (\tau-\bar{\tau})\langle F^{-1}\partial F, F^{-1}\bar{\partial}F \rangle, k \right) \end{aligned} \tag{7.18}$$

For an element F of the form (7.15), we note $F^{-1}\partial F = -\frac{\lambda+\bar{\tau}\mu}{\tau-\bar{\tau}}$ and $F^{-1}\bar{\partial}F = \frac{\lambda+\tau\mu}{\tau-\bar{\tau}}$. Then the action (7.18) may be written as:

$$(F,n)\diamond(\xi,j,k)=\Big(\xi+\lambda+\tau\mu,\, j+n-\frac{2}{\tau-\bar\tau}\langle\xi,\lambda+\bar\tau\mu\rangle \\ -\frac{1}{\tau-\bar\tau}\langle\lambda+\bar\tau\mu,\lambda+\tau\mu\rangle, k\Big). \tag{7.19}$$

Let us perform the change of coordinates $j\mapsto m=j+\frac{\langle\xi,\xi\rangle}{\tau-\bar\tau}$. Then the action of (F,n) on the new coordinates is:

$$\begin{aligned}(F,n)\diamond(\xi,m,k)&=\Big(F\xi F^{-1}+(\tau-\bar\tau)\bar\partial FF^{-1}, m+n+2\langle\xi,F^{-1}\partial F\rangle+ \\ &\qquad(\tau-\bar\tau)\langle F^{-1}\partial F,F^{-1}\bar\partial F\rangle+2\langle\xi,F^{-1}\bar\partial F\rangle+(\tau-\bar\tau)\langle F^{-1}\bar\partial F,F^{-1}\bar\partial F\rangle,k\Big)\\ &=\Big(F\xi F^{-1}+(\tau-\bar\tau)\bar\partial FF^{-1}, m+n+2\langle\xi,F^{-1}\partial F+F^{-1}\bar\partial F\rangle\\ &\qquad+(\tau-\bar\tau)\langle F^{-1}\partial F,F^{-1}\bar\partial F+F^{-1}\partial F\rangle,k\Big),\end{aligned}$$

and (7.19) becomes

$$(F,n)\diamond(\xi,m,k)=(\xi+\lambda+\tau\mu, m+n+2\langle\xi,\mu\rangle+\tau\langle\mu,\mu\rangle+\langle\lambda,\mu\rangle,k). \tag{7.20}$$

We note this action bears a strong similarity to the action of the Heisenberg action of the Jacobi group on the Tits cone, which we shall make precise in the next section.

Proposition 7.2.3. *The action of the subgroup $\widehat{F}^E_{\mathbb{Z}}\subset\widehat{G}^E_C$ preserves the Cartan subalgebra $\dot{\hat{\mathfrak{t}}}^E\subset\dot{\hat{\mathfrak{g}}}^E$.*

Proposition 7.2.4. *The space $\dot{\hat{\mathfrak{t}}}^E$ admits an action of $SL_2(\mathbb{Z})$ via*

$$\gamma\diamond(x,j,k)=\Big(\frac{x}{a+b\tau},j,k\Big).$$

Under the change of coordinates $j\mapsto m=j+\frac{\langle x,x\rangle}{\tau-\bar\tau}$ this becomes

$$\gamma\diamond(x,m,k)=\Big(\frac{x}{a+b\tau},m-\frac{b\langle x,x\rangle}{a+b\tau},k\Big).$$

Here $x\in\mathfrak{t}$ is viewed as the constant map $z\mapsto x$ for all $z\in E$. Under a conformal change of coordinates, $x\mapsto\frac{x}{a+b\tau}$.

7.3 Geometric Interpretation of the Jacobi Group

We now return to the Jacobi group and look to relate it to extended current group $\widehat{G}^E_C$. We noted in the previous section that G^E contains a discrete subgroup F^E of gauge transformations of the form

$$F_{(\lambda,\mu)}(z,\bar z)=\exp\frac{1}{\tau-\bar\tau}\left[(\lambda+\tau\mu)\bar z-(\lambda+\bar\tau\mu)z\right], \tag{7.21}$$

for $\lambda, \mu \in Q^\vee$. We see that $F_{(\lambda,\mu)}(z+1, \bar{z}+1) = F_{(\lambda,\mu)}(z,\bar{z}) = F_{(\lambda,\mu)}(z+\tau, \bar{z}+\bar{\tau})$ exactly when $\exp\lambda = \exp\mu = \mathrm{id} \in G$. When G is simply connected, the elements that satisfy $\exp\lambda = \mathrm{id} \in G$ are exactly the coroots, $\lambda \in Q^\vee$. When restricting the extension to the subgroup F^E, we find the cocycle takes integer values.

Proposition 7.3.1. *The subgroup $\widehat{F}^E_{\mathbb{Z}} \subset \widehat{G}^E_C$ is isomorphic to the Heisenberg group $H_{\mathbb{Z}}$ in the definition of the Jacobi group. The isomorphism is given by $(F_{(\lambda,\mu)}, n) \mapsto (\lambda, \mu, n)$.*

Let us now compare the Cartan subalgebra $\dot{\mathfrak{t}}^E$ of $\dot{\mathfrak{g}}^E$ and the Tits cone $Y = \mathbb{C} \oplus \mathfrak{t} \oplus \mathscr{H}$. We shall attach the parameter τ of the complex torus E to $\dot{\mathfrak{t}}^E = \mathfrak{t} \oplus \mathbb{C} \oplus \mathbb{C}$, to make clear how $\tau \in \mathscr{H}$ transforms under the various actions. As k remains fixed by all transformations we are therefore working in the hyperplane L_k. Then we shall work in the coordinates (x, m, τ), omitting the k-coordinate. The Heisenberg action on the Tits cone Y may then be interpreted as an action of an element $(F_{(\lambda,\mu)}, n) \in \widehat{F}^E_{\mathbb{Z}}$ on $\dot{\mathfrak{t}}^E$.

$$
\begin{aligned}
(F_{(\lambda,\mu)}, n) : (x, m, \tau) \mapsto& \\
&\left(\mathrm{Ad}_F(x) + \bar{\partial}FF^{-1}, m + n + 2\langle x, F^{-1}\partial F\rangle + \langle F^{-1}\partial F, F^{-1}\bar{\partial}F\rangle, \tau\right),
\end{aligned}
$$

where $x \in \mathfrak{t}$ is regarded as a constant element of $\mathfrak{g}^E$. We reformulate the discussion of semidirect products in this setting: the Weyl group W acts on $\widehat{F}^E_{\mathbb{Z}}$ as

$$
w : (F_{(\lambda,\mu)}, n) \mapsto (F_{(w\lambda, w\mu)}, n),
$$

setting up a semidirect product exactly as before. We extend the action of $SL_2(\mathbb{Z})$ on F^E to an action on $\widehat{F}^E_{\mathbb{Z}}$, which is defined as follows:

$$
\gamma \diamond (F_{(\lambda,\mu)}, n) = \left(F_{(d\lambda - c\mu, -b\lambda + a\mu)}, n - \tfrac{1}{2}\langle d\lambda - c\mu, -b\lambda + a\mu\rangle + \tfrac{1}{2}\langle \lambda, \mu\rangle\right), \tag{7.22}
$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We see that with the isomorphism $(F_{(\lambda,\mu)}, n) \mapsto (\lambda, \mu, n)$, this is exactly the action of the Heisenberg group $H_{\mathbb{Z}}$ on the Tits cone Y given in (6.5). We have therefore made a connection with a projective representation of G^E (namely, a genuine representation of our central extension $\widehat{G}^E_C$) and the Jacobi group equipped with its natural representation on the space Y. We rewrite the action of the Jacobi group on Y in this new setting:

$$
\begin{aligned}
(F_{(\lambda,\mu)}, n) \diamond (x, m, \tau) &= \left(\mathrm{Ad}_F(x) + \bar{\partial}FF^{-1}, m + n + 2\langle x, F^{-1}\partial F\rangle + \langle F^{-1}\partial F, F^{-1}\bar{\partial}F\rangle, \tau\right) \\
w \diamond (x, m, \tau) &= (wx, m, \tau) \\
\gamma \diamond (x, m, \tau) &= \left(\frac{x}{a + b\tau}, m - \frac{b\langle x, x\rangle}{a + b\tau}, \frac{c + d\tau}{a + b\tau}\right).
\end{aligned}
$$

for $(x, m, \tau) \in \mathfrak{t} \oplus \mathbb{C} \oplus \mathscr{H}$, $w \in W$, $(F_{(\lambda,\mu)}, n) \in \widehat{F}^E_{\mathbb{Z}}$, and $\gamma \in SL_2(\mathbb{Z})$, which agrees with the action stated in Proposition 6.1.8.

Chapter 8

A Generalisation of the Jacobi Group

We now construct a generalisation of the Jacobi group $J_{\mathfrak{g}}$ introduced in Chapter 6. The motivation for this is the coordinate $\tau \in \mathcal{H}$ is interpreted as the modular parameter of an elliptic curve, *ie.* a Riemann surface of genus one. It is then natural to ask if we may construct a group with coordinate $\tau \in \mathcal{H}_g$, which may be interpreted as the period matrix of a Riemann surface of genus g. (See Appendix B for details on our conventions for Riemann surface calculations.) Although a generalisation of the Eichler-Zagier Jacobi group (i.e. for $\mathfrak{g}$ of type A_1 only) has been considered by several authors (see, for example, [Yan96]), a higher-genus generalisation of the Jacobi group associated to a general simple root system has not. We think of the Eichler-Zagier Jacobi group as being associated to the elliptic curve E via the modular parameter $\tau \in \mathcal{H}$, and the generalised Jacobi group as being associated to a genus g compact Riemann surface via an element $\tau \in \mathcal{H}_g$. The Jacobi group $J_{\mathfrak{g}}^g$ we construct will act naturally on a space analogous to the Tits cone Y.

8.1 Construction of Higher-dimensional Jacobi Group

We proceed as follows. First, choose an element $\tau \in \mathcal{H}_g$, where $\mathcal{H}_g$ denotes Siegel upper-half space, that is, τ is a $g \times g$ complex symmetric matrix with positive definite imaginary part. We write $\boldsymbol{\lambda} + \tau\boldsymbol{\mu} \in (Q^\vee)^g \oplus \tau(Q^\vee)^g \in \mathfrak{t}_{\mathbb{R}}^g \oplus \tau\mathfrak{t}_{\mathbb{R}}^g$, with $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^g)^T$ and $\boldsymbol{\mu} = (\mu^1, \dots, \mu^g)^T$, $\lambda^i, \mu^i \in Q^\vee$ for $i = 1, \dots, g$. We form a $(2gl+1)$-dimensional Heisenberg group which is $(Q^\vee)^g \times (Q^\vee)^g \times \mathbb{Z}$ as a set. We denote this Heisenberg group by $H_{\mathbb{Z}}^g$ and write the group structure for all $(\boldsymbol{\lambda}, \boldsymbol{\mu}, n), (\boldsymbol{\lambda}', \boldsymbol{\mu}', n') \in (Q^\vee)^g \times (Q^\vee)^g \times \mathbb{Z}$:

$$(\boldsymbol{\lambda}, \boldsymbol{\mu}, n) \cdot (\boldsymbol{\lambda}', \boldsymbol{\mu}', n') = (\boldsymbol{\lambda} + \boldsymbol{\lambda}', \boldsymbol{\mu} + \boldsymbol{\mu}', n + n' + \boldsymbol{\mu}^T \boldsymbol{\lambda}' - \boldsymbol{\lambda}^T \boldsymbol{\mu}'), \tag{8.1}$$

where $\boldsymbol{\lambda}^T \boldsymbol{\mu} = \sum_{i=1}^g \langle \lambda^i, \mu^i \rangle$, with $\langle \cdot, \cdot \rangle$ denoting the W-invariant bilinear form on $\mathfrak{t}_{\mathbb{R}}$ which is integral on the coroot lattice $Q^\vee$.

Further, as the finite Weyl group W acts naturally on $(Q^\vee)^g$ as

$$w\boldsymbol{\lambda} = (w\lambda^1, \dots, w\lambda^l)^T,$$

we may form an action of W on the Heisenberg group $H^g_{\mathbb{Z}}$:

$$w : (\boldsymbol{\lambda}, \boldsymbol{\mu}, n) \mapsto (w\boldsymbol{\lambda}, w\boldsymbol{\mu}, n). \tag{8.2}$$

This allows us to form the semidirect product $\mathscr{W} := W \ltimes H^g_Q$, given by the exact sequence

$$1 \longrightarrow H^g_Q \longrightarrow \mathscr{W} \longrightarrow W \longrightarrow 1. \tag{8.3}$$

The group law is as before:

$$(w, \mathbf{t}) \cdot (w', \mathbf{t}') = (ww', \mathbf{t} + w\mathbf{t}'), \tag{8.4}$$

for $w, w' \in W$, $\mathbf{t} = (\boldsymbol{\lambda}, \boldsymbol{\mu}, n), \mathbf{t}' = (\boldsymbol{\lambda}', \boldsymbol{\mu}', n') \in H^g_{\mathbb{Z}}$.

Proposition 8.1.1. *The space* $Y_g = \mathbb{C} \oplus \mathfrak{t}^g \oplus \mathscr{H}_g \ni (u, \mathbf{x}, \tau)$ *admits an action of the group* $\mathscr{W} = W \ltimes H^g_Q$ *given by*

$$\begin{aligned} \mathbf{t} &: (u, \mathbf{x}, \tau) \mapsto (u + n + 2\boldsymbol{\mu}^T\mathbf{x} + \boldsymbol{\mu}^T\tau\boldsymbol{\mu} + \boldsymbol{\mu}^T\boldsymbol{\lambda}, \mathbf{x} + \boldsymbol{\lambda} + \tau\boldsymbol{\mu}, \tau), \\ w &: (u, \mathbf{x}, \tau) \mapsto (u, w\mathbf{x}, \tau), \end{aligned} \tag{8.5}$$

where $\mathbf{t} = (\boldsymbol{\lambda}, \boldsymbol{\mu}, n) \in H^g_{\mathbb{Z}}$, $w \in W$.

Proof. The W-action is immediate. For the Heisenberg action, we calculate the resulting u-coordinate, where $\mathbf{t} = (\boldsymbol{\lambda}, \boldsymbol{\mu}, n)$, $\mathbf{t}' = (\boldsymbol{\lambda}', \boldsymbol{\mu}', n')$:

$$\begin{aligned} \mathbf{t} \diamond (\mathbf{t}' \diamond u) &= \mathbf{t} \diamond \left(u + n' + 2\boldsymbol{\mu}'^T\mathbf{x} + \boldsymbol{\mu}'^T\tau\boldsymbol{\mu}' + \boldsymbol{\mu}'^T\boldsymbol{\lambda}'\right) \\ &= \big(u + n' + 2\boldsymbol{\mu}'^T\mathbf{x} + \boldsymbol{\mu}'^T\tau\boldsymbol{\mu}' + \boldsymbol{\mu}'^T\boldsymbol{\lambda}' + n + 2\boldsymbol{\mu}^T(\mathbf{x} + \boldsymbol{\lambda}' + \tau\boldsymbol{\mu}') \\ &\quad + \boldsymbol{\mu}^T\tau\boldsymbol{\mu} + \boldsymbol{\mu}^T\boldsymbol{\lambda}\big) \\ &= \big(u + n + n' + +2(\boldsymbol{\mu} + \boldsymbol{\mu}')^T\mathbf{x} + (\boldsymbol{\mu} + \boldsymbol{\mu}')^T\tau(\boldsymbol{\mu} + \boldsymbol{\mu}') + (\boldsymbol{\mu} + \boldsymbol{\mu}')^T(\boldsymbol{\lambda} + \boldsymbol{\lambda}') \\ &\quad + \boldsymbol{\mu}^T\boldsymbol{\lambda}' - \boldsymbol{\lambda}^T\boldsymbol{\mu}'\big) \\ &= (\mathbf{t} \cdot \mathbf{t}') \diamond u, \end{aligned}$$

and the $\mathbf{x}$- and τ-coordinates are clear. □

Let $J = \left(\begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix}\right)$. The symplectic group $Sp_{2g}(\mathbb{Z})$ is defined as follows:

$$Sp_{2g}(\mathbb{Z}) := \{\gamma \in \mathrm{Mat}_{2g}(\mathbb{Z}) | \gamma^T J \gamma = J\}. \tag{8.6}$$

Recall the symplectic group $Sp_{2g}(\mathbb{Z})$ acts naturally on $\mathscr{H}_g$ via[1]

$$\gamma : \tau \mapsto (C + D\tau)(A + B\tau)^{-1} \qquad \text{for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}). \tag{8.7}$$

Now it is straightforward to define an action of $Sp_{2g}(\mathbb{Z})$ on Y_g, by generalising the $SL_2(\mathbb{Z})$ action (6.4):

[1] For details of our conventions, see Appendix B.

Proposition 8.1.2. *The space $Y_g = \mathbb{C} \oplus \mathfrak{t}^g \oplus \mathscr{H}_g$ admits an action of $Sp_{2g}(\mathbb{Z})$ as follows:*

$$\gamma \diamond (u, \mathbf{x}, \tau) = \left(u - \mathbf{x}^T (A + B\tau)^{-1} B\mathbf{x}, (A + B\tau)^{-T}\mathbf{x}, (C + D\tau)(A + B\tau)^{-1}\right), \tag{8.8}$$

for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$.

Proof. We require $(\gamma\gamma') \diamond (u, \mathbf{x}, \tau) = \gamma \diamond (\gamma' \diamond (u, \mathbf{x}, \tau))$. We write

$$\gamma\gamma' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} AA' + BC' & AB' + BD' \\ CA' + DC' & CB' + DD' \end{pmatrix} =: \begin{pmatrix} R & S \\ T & V \end{pmatrix}. \tag{8.9}$$

Now $\gamma \in Sp_{2g}(\mathbb{Z})$ acts on $\tau \in \mathscr{H}_g$ as

$$\tau \mapsto \tau' := \gamma' \diamond \tau = (C' + D'\tau)(A' + B'\tau)^{-1}$$

and it is immediate that $(\gamma\gamma') \diamond \tau = \gamma \diamond (\gamma' \diamond \tau)$. We define (*cf.* Appendix B) $\mathbf{x}' := \gamma' \diamond \mathbf{x} = (A' + B'\tau)^{-T}\mathbf{x}$. A short calculation shows that $(A + B\tau') = (R + S\tau)(A' + B'\tau)^{-1}$, and so the $\mathbf{x}$-coordinate transforms as $\gamma \diamond (\gamma' \diamond \mathbf{x}) = \gamma((A' + B'\tau)^{-T}\mathbf{x}) = (A + B\tau')^{-T}(A' + B'\tau)^{-T}\mathbf{x} = (R + S\tau)^{-T}\mathbf{x}$. Finally, we calculate for the u-coordinate:

$$\begin{aligned}
\gamma \diamond (\gamma' \diamond u) &= \gamma \diamond (u - \mathbf{x}^T (A' + B'\tau)^{-1} B'\mathbf{x}) \\
&= u - \mathbf{x}^T (A' + B'\tau)^{-1} B'\mathbf{x} - \mathbf{x}'^T (A + B\tau')^{-1} B\mathbf{x}' \\
&= u - \mathbf{x}^T (R + S\tau)^{-1} \left[(R + S\tau)B'^T + B\right] (A' + B'\tau)^{-T}\mathbf{x} \\
&= u - \mathbf{x}^T (R + S\tau)^{-1} \left[S(A' + B'\tau)^T\right] (A' + B'\tau)^{-T}\mathbf{x} \\
&= u - \mathbf{x}^T (R + S\tau)^{-1} S\mathbf{x} \\
&= (\gamma\gamma') \diamond u,
\end{aligned}$$

where in the fourth equality we have calculated $RB'^T + B = [AA' + BC']B'^T + B[D'A'^T - C'B'^T] = [AB' + BD']A'^T = SA'^T$ using[2] $D'A'^T - C'B'^T = I$ and $A'B'^T = B'A'^T$. □

Now we have actions of $Sp_{2g}(\mathbb{Z})$ and of the Heisenberg group $H_{\mathbb{Z}}^g$ on Y_g. We would like to combine these to produce a semidirect product of the two, together with a compatible action of the semidirect product on Y_g. As such, we require an action of $Sp_{2g}(\mathbb{Z})$ on $H_{\mathbb{Z}}^g$.

Proposition 8.1.3. *The Heisenberg group $H_{\mathbb{Z}}^g$ admits an action of $Sp_{2g}(\mathbb{Z})$ as follows:*

$$\gamma \diamond (\boldsymbol{\lambda}, \boldsymbol{\mu}, n) = \left(D\boldsymbol{\lambda} - C\boldsymbol{\mu}, -B\boldsymbol{\lambda} + A\boldsymbol{\mu}, n - \tfrac{1}{2}(D\boldsymbol{\lambda} - C\boldsymbol{\mu})^T(-B\boldsymbol{\lambda} + A\boldsymbol{\mu}) + \tfrac{1}{2}\boldsymbol{\lambda}^T\boldsymbol{\mu}\right), \tag{8.10}$$

for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}), (\boldsymbol{\lambda}, \boldsymbol{\mu}, n) \in H_{\mathbb{Z}}^g$.

Proof. We have

$$(\gamma\gamma') \diamond (\boldsymbol{\lambda}, \boldsymbol{\mu}, n) = \left(V\boldsymbol{\lambda} - T\boldsymbol{\mu}, -S\boldsymbol{\lambda} + R\boldsymbol{\mu}, n - \tfrac{1}{2}(V\boldsymbol{\lambda} - T\boldsymbol{\mu})^T(-S\boldsymbol{\lambda} + R\boldsymbol{\mu}) + \tfrac{1}{2}\boldsymbol{\lambda}^T\boldsymbol{\mu}\right).$$

[2] See Appendix B for all the identities that are used.

Now it is clear that $D(D'\boldsymbol{\lambda} - C'\boldsymbol{\mu}) - C(-B'\boldsymbol{\lambda} + A'\boldsymbol{\mu}) = V\boldsymbol{\lambda} - T\boldsymbol{\mu}$ and $-B(D'\boldsymbol{\lambda} - C'\boldsymbol{\mu}) + A(-B'\boldsymbol{\lambda} + A'\boldsymbol{\mu}) = -S\boldsymbol{\lambda} + R\boldsymbol{\mu}$, so $\gamma \diamond (\gamma' \diamond (\boldsymbol{\lambda}, \boldsymbol{\mu})) = (\gamma\gamma') \diamond (\boldsymbol{\lambda}, \boldsymbol{\mu})$. We then check the action on the central extension:

$$\begin{aligned}\gamma \diamond (\gamma' \diamond n) &= \gamma \diamond (n - \tfrac{1}{2}(D'\boldsymbol{\lambda} - C'\boldsymbol{\mu})^T(-B'\boldsymbol{\lambda} + A'\boldsymbol{\mu}) + \tfrac{1}{2}\boldsymbol{\lambda}^T\boldsymbol{\mu}) \\ &= n - \tfrac{1}{2}(D'\boldsymbol{\lambda} - C'\boldsymbol{\mu})^T(-B'\boldsymbol{\lambda} + A'\boldsymbol{\mu}) + \tfrac{1}{2}\boldsymbol{\lambda}^T\boldsymbol{\mu} - \tfrac{1}{2}(V\boldsymbol{\lambda} - T\boldsymbol{\mu})^T(-S\boldsymbol{\lambda} + R\boldsymbol{\mu}) \\ &\quad + \tfrac{1}{2}(D'\boldsymbol{\lambda} - C'\boldsymbol{\mu})^T(-B'\boldsymbol{\lambda} + A'\boldsymbol{\mu}) \\ &= n - \tfrac{1}{2}(V\boldsymbol{\lambda} - T\boldsymbol{\mu})^T(-S\boldsymbol{\lambda} + R\boldsymbol{\mu}) + \tfrac{1}{2}\boldsymbol{\lambda}^T\boldsymbol{\mu} \\ &= (\gamma\gamma') \diamond n.\end{aligned}$$

□

We then define an action of $Sp_{2g}(\mathbb{Z})$ on $\mathscr{W} \ni (w, \mathbf{t})$ by $\gamma : (w, \mathbf{t}) \mapsto (w, \gamma \diamond \mathbf{t})$. Propositions 8.1.1 and 8.1.2 allow us to form a semidirect product $\mathscr{W} \rtimes Sp_{2g}(\mathbb{Z}) = (W \ltimes H_{\mathbb{Z}}^g) \rtimes Sp_{2g}(\mathbb{Z})$, which admits an action on the space Y_g.

Definition 8.1.4. *The Jacobi group $J_{\mathfrak{g}}^g$ is the semidirect product $\mathscr{W} \rtimes Sp_{2g}(\mathbb{Z})$. The group law is*

$$(w, \boldsymbol{t}, \gamma) \cdot (w', \boldsymbol{t}', \gamma') = (ww', \boldsymbol{t} + w(\gamma\boldsymbol{t}'), \gamma\gamma'), \tag{8.11}$$

for all $(w, \mathbf{t}, \gamma), (w', \mathbf{t}, \gamma') \in W \times H_{\mathbb{Z}}^g \times Sp_{2g}(\mathbb{Z})$.

Proposition 8.1.5. *The Jacobi group $J_{\mathfrak{g}}^g$ has a representation on the space Y_g via the actions of $w \in W$, $\mathbf{t} = (\boldsymbol{\lambda}, \boldsymbol{\mu}, n) \in H_{\mathbb{Z}}^g$ and $\gamma \in Sp_{2g}(\mathbb{Z})$ as follows:*

$$\begin{aligned} w &: (u, \mathbf{x}, \tau) \mapsto (u, w\mathbf{x}, \tau) \\ \mathbf{t} &: (u, \mathbf{x}, \tau) \mapsto (u + n + 2\boldsymbol{\mu}^T\mathbf{x} + \boldsymbol{\mu}^T\tau\boldsymbol{\mu} + \boldsymbol{\mu}^T\boldsymbol{\lambda}, \mathbf{x} + \boldsymbol{\lambda} + \tau\boldsymbol{\mu}, \tau) \\ \gamma &: (u, \mathbf{x}, \tau) \mapsto \left(u - \mathbf{x}^T(A + B\tau)^{-1}B\mathbf{x}, (A + B\tau)^{-T}\mathbf{x}, (C + D\tau)(A + B\tau)^{-1}\right), \end{aligned} \tag{8.12}$$

for all $(u, \mathbf{x}, \tau) \in Y_g$.

Proof. The previous propositions show that this is an action. □

8.2 Generalisation of Jacobi Forms

We shall now postulate a definition of Jacobi forms that are invariant under the action of the generalised Jacobi group $J_{\mathfrak{g}}^g$. We make no claim of the existence of such objects, which may be an interesting topic for further study.

Definition 8.2.1. *A generalised Jacobi form is a map $\phi : \mathbb{C} \times \mathfrak{t}^g \times \mathscr{H}_g \to \mathbb{C}$ subject to the* *following transformation properties:*

$$\begin{aligned}\phi(u,\mathbf{x},\tau) &= \phi(u,w\mathbf{x},\tau)\\ \phi(u,\mathbf{x},\tau) &= \phi\left(u+n+2\boldsymbol{\mu}^T\mathbf{x}+\boldsymbol{\mu}^T\tau\boldsymbol{\mu},\mathbf{x}+\boldsymbol{\lambda}+\tau\boldsymbol{\mu},\tau\right)\\ \phi(u,\mathbf{x},\tau) &= \phi\left(u-\mathbf{x}^T(A+B\tau)^{-1}B\mathbf{x},(A+B\tau)^{-T}\mathbf{x},(C+D\tau)(A+B\tau)^{-1}\right)\end{aligned}$$

for $w\in W$, $(\boldsymbol{\lambda},\boldsymbol{\mu},n)\in H^g_{\mathbb{Z}}$ *and* $\gamma\in Sp_{2g}(\mathbb{Z})$.

When $g=1$ we find that this definition coincides with the Jacobi forms defined in Section 6.2.

8.3 A Discrete Subgroup of $\widehat{G}_C^\Sigma$

Recall from Section 7.2.1 the definition of the distinguished elements of G^E

$$F_{(\lambda,\mu)}(z,\bar{z})=\exp\frac{1}{\tau-\bar{\tau}}[(\lambda+\tau\mu)\bar{z}-(\lambda+\bar{\tau}\mu)z].$$

We take the opportunity to define the generalisation of these elements in the group G^Σ, where Σ is a compact Riemann surface of genus $g\geq 1$. We now require smooth maps $\Sigma\to G$, that is, maps that are periodic with respect to the homology basis of Σ. From the calculation in Appendix C, we find that these maps take the form

$$\begin{aligned}g_{(\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{a},\mathbf{b})}(P) &= \exp\frac{1}{2i}Z_{ij}\left[\int_\star^P\bar{\omega}^j(\lambda^i+\tau^i_k\mu^k)-\int_\star^P\omega^j(\lambda^i+\bar{\tau}^i_l\mu^l)\right]\\ &= \exp\frac{1}{2i}\left[\bar{\mathbf{v}}^T(P)Z(\boldsymbol{\lambda}+\tau\boldsymbol{\mu})-\mathbf{v}^T(P)Z(\boldsymbol{\lambda}+\bar{\tau}\boldsymbol{\mu})\right],\end{aligned}\tag{8.13}$$

where $Z=(\operatorname{im}\tau)^{-1}$, $v:\Sigma\to J(\Sigma)$ is the Abel map from the Riemann surface Σ to its Jacobian $J(\Sigma)$ via $\mathbf{v}(P)=(\int_\star^P\omega_1,\ldots,\int_\star^P\omega_g)^T$, $\bar{\mathbf{v}}(P):=(\int_\star^P\bar{\omega}_1,\ldots,\int_\star^P\bar{\omega}_g)^T$, $\boldsymbol{\lambda}=(\lambda^1,\ldots,\lambda^g)^T\in(Q^\vee)^g$, $\boldsymbol{\mu}=(\mu^1,\ldots,\mu^g)^T\in(Q^\vee)^g$. These currents are well-defined in G^Σ provided $\exp\lambda^i=I=\exp-\mu^i$, that is, $\lambda^i,\mu^i\in\ker\varepsilon$, where ε is the exponential map.

The current depends on a choice of homology basis $\{\mathbf{a},\mathbf{b}\}$ of Σ and an element $\boldsymbol{\lambda}+\tau\boldsymbol{\mu}\in(Q^\vee)^g\oplus\tau(Q^\vee)^g\subset\mathfrak{t}^g$, and are defined as follows:

Proposition 8.3.1. *The current* $g:\Sigma\to G$ *defined in (8.13) is invariant under the action of the symplectic group* $Sp_{2g}(\mathbb{Z})$.

Proof. Let $\gamma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in Sp_{2g}(\mathbb{Z})$. From the calculations in Appendix B, we have (as τ and $\gamma\diamond\tau$ are symmetric)

$$(\gamma\diamond\tau)^T=(A^T+\tau B^T)^{-1}(C^T+\tau D^T)=(\gamma\diamond\tau)$$

and

$$\gamma\diamond Z=(A+B\tau)Z(A+B\bar{\tau})^T=(A+B\bar{\tau})Z(A+B\tau)^T.$$

Let $X = \bar{\mathbf{v}}^T Z(\boldsymbol{\lambda}+\tau\boldsymbol{\mu}) - \mathbf{v}^T Z(\boldsymbol{\lambda}+\bar{\tau}\boldsymbol{\mu})$, where $\mathbf{v} = (\int\omega_1,\dots,\int\omega_g)^T, \bar{\mathbf{v}} = \int(\bar{\omega}_1,\dots,\int\bar{\omega}_g)^T$. Then

$$\begin{aligned}
X' &= \bar{\mathbf{v}}^{T'} Z'(\boldsymbol{\lambda}'+\tau'\boldsymbol{\mu}') - \mathbf{v}^{T'} Z'(\boldsymbol{\lambda}'+\bar{\tau}'\boldsymbol{\mu}') \\
&= \bar{\mathbf{v}}^T (A+B\bar{\tau})^{-1}(A+B\bar{\tau})Z(A+B\tau)^T\left[(D\boldsymbol{\lambda}-C\boldsymbol{\mu})+(C+D\tau)(A+B\tau)^{-1}(-B\boldsymbol{\lambda}+A\boldsymbol{\mu})\right] \\
&\quad - \mathbf{v}^T Z(A+B\bar{\tau})^T\left[(D\boldsymbol{\lambda}-C\boldsymbol{\mu})+(C+D\bar{\tau})(A+B\bar{\tau})^{-1}(-B\boldsymbol{\lambda}+A\boldsymbol{\mu})\right] \\
&= \bar{\mathbf{v}}^T Z(A+B\tau)^T(A+B\tau)^{-T}\left[(A+B\tau)^T(D\boldsymbol{\lambda}-C\boldsymbol{\mu})+(C+D\tau)^T(-B\boldsymbol{\lambda}+A\boldsymbol{\mu})\right] \\
&\quad - \mathbf{v}^T\cdot Z(A+B\bar{\tau})^T(A+B\bar{\tau})^{-T}\left[(A+B\bar{\tau})^T(D\boldsymbol{\lambda}-C\boldsymbol{\mu})+(C+D\bar{\tau})^T(-B\boldsymbol{\lambda}+A\boldsymbol{\mu})\right] \\
&= \bar{\mathbf{v}}^T Z\left[(A^TC-C^TA)\boldsymbol{\mu}+\tau(D^TA-B^TC)\boldsymbol{\mu}+(A^TD-C^TB)\boldsymbol{\lambda}+\tau(B^TD-D^TB)\boldsymbol{\lambda}\right] \\
&\quad - \mathbf{v}^T\cdot Z\left[(A^TC-C^TA)\boldsymbol{\mu}+\bar{\tau}(D^TA-B^TC)\boldsymbol{\mu}+(A^TD-C^TB)\boldsymbol{\lambda}+\bar{\tau}(B^TD-D^TB)\boldsymbol{\lambda}\right] \\
&= \bar{\mathbf{v}}^T Z(\boldsymbol{\lambda}+\tau\boldsymbol{\mu}) - \mathbf{v}^T Z(\boldsymbol{\lambda}+\bar{\tau}\boldsymbol{\mu}).
\end{aligned}$$

□

The previous proposition shows that our definition of the elements $g_{(\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{a},\mathbf{b})}$ does not depend on the choice of homology basis. We may therefore fix the choice of homology basis and refer to $g_{(\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{a},\mathbf{b})}$ as $g_{(\boldsymbol{\lambda},\boldsymbol{\mu})}$.

Proposition 8.3.2. *Elements of the form $(g,n)\in\widehat{G}_C^\Sigma$ where $g\in G^\Sigma$ is of the form (8.13) and $n\in\mathbb{Z}$ form a discrete subgroup $\widehat{F}_{\mathbb{Z}}^\Sigma$ of $\widehat{G}_C^\Sigma$.*

When $g,h\in G^\Sigma$ are of the form (8.13) we see from the calculation in Appendix C that

$$\int_\Sigma \langle g^{-1}dg\wedge dhh^{-1}\rangle = -2\left(\langle\lambda_1^a,\mu_{2a}\rangle - \langle\lambda_2^a,\mu_{1a}\rangle\right), \tag{8.14}$$

and the latter is an integer for all $\lambda_i^a,\mu_i^a\in Q^\vee$.

We see that all calculations for the Jacobi group in Chapter 6 carry through to the new Jacobi group we have constructed.

Appendix A

Root System Conventions

Our conventions follow [Car05]. Let Φ denote an irreducible finite root system associated to a simple complex Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{t}$, and let $\Phi^\vee$ denote the dual root system. Fix a set of simple roots $\triangle = \{\alpha_1, \dots, \alpha_l\}$ for Φ, where $l = \operatorname{rank}\Phi$, and a set of simple coroots, $\triangle^\vee = \{h_1, \dots, h_l\}$ for $\Phi^\vee$, where $\alpha_i(h_i) = 2$. Let $Q \subset \mathfrak{t}^*$ denote the root lattice generated by Φ and $Q^\vee \subset \mathfrak{t}$ denote the lattice generated by $\Phi^\vee$, which we call the coroot lattice. Their respective dual lattices are $\Lambda^\vee = \{h \in \mathfrak{t} |\ \alpha(h) \in \mathbb{Z}\ \forall \alpha \in \triangle\} \subset \mathfrak{t}$ and $\Lambda = \{\lambda \in \mathfrak{t}^* | \lambda(h_\alpha) \in \mathbb{Z}\ \forall \alpha \in \triangle\} \subset \mathfrak{t}^*$. These are called the dual weight lattice and weight lattice, respectively. Let W be the Weyl group of Φ, generated by the fundamental reflections $s_1, \dots, s_l$ where $s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ for $\lambda \in \mathfrak{t}^*$. Associated to Φ we have the Cartan matrix C, with $C_{ij} = \alpha_j(h_i)$ for $i, j = 1, \dots, l$.

On $\mathfrak{t}$ we naturally have a W-invariant bilinear form, the Killing form $\langle \cdot, \cdot \rangle_K$. This induces a bilinear form on $\mathfrak{t}^*$ defined by $(\alpha_i, \alpha_j) := \langle t_{\alpha_i}, t_{\alpha_j} \rangle_K$, where t_{α_i} is the unique element of $\mathfrak{t}$ such that $\alpha(h) = \langle h, t_\alpha \rangle_K$ for all $h \in \mathfrak{t}$. We have an isomorphism $\Psi : \mathfrak{t}^* \to \mathfrak{t}$ given by $\alpha \mapsto t_\alpha$ (and $\frac{2\alpha}{(\alpha,\alpha)} \mapsto h_\alpha$). Let us now consider an affine root system $\widetilde{\Phi}$, with base roots $\alpha_0, \alpha_1, \dots, \alpha_l$, and associated generalised Cartan matrix $\widetilde{C}$, where $\widetilde{C}_{ij} = \alpha_j(h_i)$ for $i, j = 0, 1, \dots, l$. The Cartan matrix C of finite type may be obtained from $\widetilde{C}$ by removing the 0^{th} row and 0^{th} column.

The finite root system Φ may be obtained from $\widetilde{\Phi}$ by taking the quotient by $\mathbb{R}\delta$. We continue extending our notation from the finite case by denoting the dual root system $\widetilde{\Phi}^\vee$ with basis $h_0, h_1, \dots, h_l$, and denoting the root lattice $\widetilde{Q} = \mathbb{Z}\widetilde{\Phi}$. We write an element of $\widetilde{Q}$ as $q + n\delta$, where $q \in Q$ and $n \in \mathbb{Z}$. We consider $\widetilde{\Phi}$ as realised in a real vector space $V_\mathbb{R}$ of dimension $l + 2$. In order to describe $V_\mathbb{R}$, introduce a vector $\beta \in V_\mathbb{R}$ defined by $\beta(h_i) = \delta_{0i}$ for $i = 0, 1, \dots l$, so $\{\alpha_0, \alpha_1, \dots, \alpha_l, \beta\}$ is a basis of $V_\mathbb{R}$. Let V denote $V_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$, and on V we may also define a W-invariant bilinear form (which we also call $(\cdot, \cdot)$) which restricts to $(\cdot, \cdot)$ on $V_\mathbb{R}$. Note that Q is integral, so that $(a, b) \in \mathbb{Z}$ for all $a, b \in Q$. We also have $(\delta, \delta) = 0$ and $(\delta, Q) = 0$. The canonical central element in V is defined as

$$c = c_0 h_0 + \dots c_l h_l,$$

where $\mathbf{c} = (c_0, c_1, \dots, c_l)^T$ is the unique vector having positive integer coordinates for which

$\mathbf{c}^T \widetilde{C} = \mathbf{0}$. The basic imaginary root in V^* is defined as

$$\delta = a_0\alpha_0 + \ldots + a_l\alpha_l,$$

where $\mathbf{a} = (a_0, a_1, \ldots, a_l)^T$ is the unique vector having positive integer coordinates for which $\widetilde{C}\mathbf{a} = \mathbf{0}$.

We list some properties of the elements of V and V^* for convenience.

- A basis for V is $\{h_0, h_1, \ldots, h_l, d\}$. The standard invariant form $\langle\cdot,\cdot\rangle$ on V^* is given by

$$\begin{aligned}
&\langle h_i, h_j\rangle = a_j c_j^{-1} C_{ij} \qquad i, j = 0, 1, \ldots, l\\
&\langle h_0, d\rangle = a_0\\
&\langle h_i, d\rangle = 0 \qquad i = 1, \ldots, l\\
&\langle d, d\rangle = 0
\end{aligned}$$

- A basis for V^* is $\{\alpha_0, \alpha_1, \ldots, \alpha_l, \beta\}$. The standard invariant form on $(\cdot,\cdot)$ on V is given by

$$\begin{aligned}
&(\alpha_i, \alpha_j) = a_i^{-1} c_i C_{ij} \qquad i, j = 0, 1, \ldots, l\\
&(\alpha_0, \beta) = a_0^{-1}\\
&(\alpha_i, \beta) = 0 \qquad i = 1, \ldots, l\\
&(\beta, \beta) = 0
\end{aligned}$$

- The pairing between V and V^* is given by

$$\begin{aligned}
\alpha_j(h_i) &= C_{ij} \qquad i, j = 0, 1, \ldots, l\\
\alpha_0(d) &= 1\\
\alpha_i(d) &= 0 \qquad i = 1, \ldots, l\\
\beta(h_0) &= 1\\
\beta(h_i) &= 0 \qquad i = 1, \ldots, l\\
\beta(d) &= 0
\end{aligned}$$

- Some properties of the central element $c = c_0h_0 + c_1h_1 + \cdots + c_lh_l \in V^*$ are

$$\begin{aligned}
\langle h_i, c\rangle &= 0 \qquad i = 0, 1, \ldots, l\\
\langle d, c\rangle &= a_0\\
\langle c, c\rangle &= 0\\
\alpha_j(c) &= 0 \qquad j = 0, 1, \ldots, l\\
\beta(c) &= 1
\end{aligned}$$

- Some properties of the imaginary root $\delta = a_0\alpha_0 + a_1\alpha_1 + \cdots a_l\alpha_l \in V$ are

$$
\begin{aligned}
(\alpha_j, \delta) &= 0 \qquad i = 0, 1, \ldots, l \\
(\beta, \delta) &= 1 \\
(\delta, \delta) &= 0 \\
\delta(h_i) &= 0 \qquad i = 0, 1, \ldots, l \\
\delta(d) &= a_0 \\
\delta(c) &= 0.
\end{aligned}
$$

Appendix B

Riemann Surface Conventions

We use the following conventions in Chapter 8. Let Σ be a compact Riemann surface of genus g. We note some homological conventions for Σ. Choose a canonical homology basis $\pi = \{\mathbf{a},\mathbf{b}\} = \{a_1,\dots a_g, b_1,\dots b_g\}$ for $H_1(\Sigma,\mathbb{Z})$. (The fact that π is canonical means that $a_j \cdot b_k = \delta_{jk}$ and $a_j \cdot a_k = 0 = b_j \cdot b_k$ for $j,k = 1,\dots,g$.) Now we choose a basis $\boldsymbol{\xi} = \{\xi_1,\dots,\xi_g\}$ for $H^1(\Sigma)$. The period matrix of Σ is defined as

$$\Pi := \begin{pmatrix} \mathscr{A} \\ \mathscr{B} \end{pmatrix} = \begin{pmatrix} \int_{a_1}\xi_1 & \dots & \int_{a_1}\xi_g \\ \vdots & & \vdots \\ \int_{a_g}\xi_1 & \dots & \int_{a_g}\xi_g \\ \int_{b_1}\xi_1 & \dots & \int_{b_1}\xi_g \\ \vdots & & \vdots \\ \int_{b_g}\xi_1 & \dots & \int_{b_g}\xi_g \end{pmatrix},$$

where $\mathscr{A}$ denotes the $g\times g$ matrix of a-periods of $\{\xi_1,\dots\xi_g\}$ and $\mathscr{B}$ denotes the $g\times g$ matrix of b-periods of $\{\xi_1,\dots\xi_g\}$. We note that the first g rows of Π are linearly independent, so we may find another basis $\boldsymbol{\omega} = \{\omega_1,\dots\omega_g\}$ of $\mathscr{H}^1(\Sigma)$ such that $\int_{a_k}\omega_j = \delta_{jk}$. The period matrix is then of the form

$$\Pi = \begin{pmatrix} I_g \\ \tau \end{pmatrix},$$

and $\tau = \mathscr{B}\mathscr{A}^{-1}$. It can be shown that $\tau \in \mathscr{H}_g$, the Siegel upper-half space. That is, τ is a complex symmetric $g\times g$ matrix with $\operatorname{im}\tau$ positive definite.

Let $J = \left(\begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix}\right)$. The symplectic group $Sp_{2g}(\mathbb{Z})$ is defined as follows:

$$Sp_{2g}(\mathbb{Z}) := \{\gamma \in \mathrm{Mat}_{2g}(\mathbb{Z}) | \gamma^T J\gamma = J\}. \tag{B.1}$$

If $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ for $A,B,C,D \in \mathrm{Mat}_g(\mathbb{Z})$, we note that $\gamma^T J\gamma = J$ implies $A^T C = C^T A, B^T D = D^T B$ and $A^T D - C^T B = I_g$. We find also that $\gamma^{-1} = -J\gamma^T J$ and $\gamma^{-T} = -J\gamma J$, and the fact that γ^{-1} and γ^{-T} are in $Sp_{2g}(\mathbb{Z})$ gives us the identities $D^T A - B^T C = I$, $AD^T - BC^T = I = DA^T - CB^T$, $A^T B = B^T A$ and $D^T C = C^T D$.

Recall the symplectic group $Sp_{2g}(\mathbb{Z})$ acts naturally on $\mathscr{H}_g$ via

$$\gamma : \tau \mapsto (C + D\tau)(A + B\tau)^{-1} \qquad \text{for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}) \tag{B.2}$$

We state that $\mathbf{z}^T = (\mathbf{z_1}, \mathbf{z_2})^T \begin{pmatrix} 1 \\ \tau \end{pmatrix} \in \mathfrak{t}^g$ transforms as follows under the action of $Sp_{2g}(\mathbb{Z})$:

$$\gamma : \mathbf{z}^T \mapsto \mathbf{z}'^T := \mathbf{z}^T (A + B\tau)^{-1}.$$

Indeed

$$\gamma : \mathbf{z}^T \mapsto \mathbf{z}'^T \quad = (\mathbf{z_1}', \mathbf{z_2}')^T \begin{pmatrix} 1 \\ \tau' \end{pmatrix}$$

Then $\mathbf{z}'^T (A + B\tau) = \mathbf{z}^T$ and so $\mathbf{z}' = (A + B\tau)^{-T}\mathbf{z}$. We note also that an element $\boldsymbol{\lambda} + \tau\boldsymbol{\mu} \in Q^\vee \oplus \tau Q^\vee \subset \mathfrak{t}^g_{\mathbb{C}}$ transforms under the action of $\gamma \in Sp_{2g}(\mathbb{Z})$ as

$$\begin{aligned} \gamma : \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{\lambda}' \\ \boldsymbol{\mu}' \end{pmatrix} &= \gamma^{-T} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{pmatrix} \\ &= -J\gamma J \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{pmatrix} \\ &= \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{pmatrix} \\ &= \begin{pmatrix} D\boldsymbol{\lambda} - C\boldsymbol{\mu} \\ -B\boldsymbol{\lambda} + A\boldsymbol{\mu} \end{pmatrix}. \end{aligned}$$

Let $Z = (\operatorname{im}\tau)^{-1}$. We calculate also the transformation $\gamma : Z \mapsto (A + B\tau) Z (A + B\bar{\tau})^T$ by calculating $\gamma(\tau - \bar{\tau})$.

$$\begin{aligned} \gamma(\tau - \bar{\tau}) &= (C + D\tau)(A + B\tau)^{-1} - (C + D\bar{\tau})(A + B\bar{\tau})^{-1} \\ &= (C + D\tau)(A + B\tau)^{-1} - (A + B\bar{\tau})^{-T}(C + D\bar{\tau})^T \\ &= (A + B\bar{\tau})^{-T} \left[(A + B\bar{\tau})^T (C + D\tau) - (C + D\bar{\tau})^T (A + B\tau) \right] (A + B\tau)^{-1} \\ &= (A + B\bar{\tau})^{-T} \left[\tau - \bar{\tau} \right] (A + B\tau)^{-1}, \end{aligned} \tag{B.3}$$

where we have used the symmetry of τ in the second equality and the identities $A^T C = C^T A$, $B^T D = D^T B$ and $A^T D - C^T B = I_g$ in the last. We have $\tau - \bar{\tau} = 2i \operatorname{im}\tau$ and so we find

$$\gamma : Z \mapsto (A + B\tau) Z (A + B\bar{\tau})^T.$$

Appendix C

Calculations

In the genus one case, we have been interested in currents in G^E of the form

$$F(z,\bar{z}) = \exp \frac{1}{\tau-\bar{\tau}}\left[(\lambda+\tau\mu)\bar{z}-(\lambda+\bar{\tau}\mu)z\right], \tag{C.1}$$

where $\lambda, \mu \in Q^\vee$.

Lemma C.0.3. *Elements of the form $F_{(\lambda,\mu)}$ are invariant under the action of $SL_2(\mathbb{Z})$.*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $SL_2(\mathbb{Z})$ acts naturally on the upper-half plane $\mathscr{H}$ via

$$\tau \mapsto \tau' := \gamma \diamond \tau = \frac{c+d\tau}{a+b\tau}$$

for all $\tau \in \mathscr{H}$. We also have

$$x \mapsto x' := \gamma \diamond x = \frac{x}{a+b\tau}.$$

Then

$$\begin{aligned} F_{(\lambda',\mu')}(z',\bar{z}') &= \exp \frac{1}{\tau'-\bar{\tau}'}\left[(\lambda'+\tau'\mu')\bar{z}'-(\lambda'+\bar{\tau}'\mu')z'\right] \\ &= \exp \frac{|a+b\tau|^2}{\tau-\bar{\tau}}\Bigg[\left((d\lambda-c\mu)+\frac{c+d\tau}{a+b\tau}(-b\lambda+a\mu)\right)\frac{\bar{z}}{a+b\bar{\tau}} \\ &\qquad -\left((d\lambda-c\mu)+\frac{c+d\bar{\tau}}{a+b\bar{\tau}}(-b\lambda+a\mu)\right)\frac{z}{a+b\tau}\Bigg] \\ &= \exp \frac{1}{\tau-\bar{\tau}}\left[(\lambda+\tau\mu)\bar{z}-(\lambda+\bar{\tau}\mu)z\right]. \end{aligned} \tag{C.2}$$

□

We calculate the appropriate analogous elements of G^Σ, for Σ a Riemann surface of genus g. We shall use this definition in Chapter 8. For an element of the form (C.1), we note $F^{-1}dF$ has the form

$$F^{-1}dF = \frac{1}{\tau-\bar{\tau}}\left[(\lambda+\tau\mu)d\bar{z}-(\lambda+\bar{\tau}\mu)dz\right] \tag{C.3}$$

Then we have

$$\int_E \langle F^{-1}dF \wedge dF'F'^{-1}\rangle = \langle \lambda', \mu\rangle - \langle \lambda, \mu'\rangle.$$

We calculate the analogous statement for Σ a Riemann surface of genus g. Suppose $v, w \in H^1(\Sigma;\mathbb{R}) \otimes \mathfrak{g}_{\mathbb{R}}$. We write

$$v = \sum_{a=1}^{g} (\bar{\alpha}^a \omega_a + \alpha^a \bar{\omega}_a), \tag{C.4}$$

and

$$w = \sum_{b=1}^{g} (\bar{\beta}^b \omega_b + \beta^b \bar{\omega}_b), \tag{C.5}$$

where $\omega_a(z) \in H^{(1,0)}(\Sigma,\mathbb{C})$ and $\alpha^a, \beta^b \in \mathfrak{g}$. We would like to find $g, h \in G^\Sigma$ with $v = g^{-1}dg$ and $w = dhh^{-1}$ such that

$$\int_\Sigma \langle v \wedge w\rangle \in \mathbb{Z}.$$

We obtain

$$\begin{aligned}\langle v\wedge w\rangle = \langle g^{-1}dg \wedge dhh^{-1}\rangle &= \langle(\bar{\alpha}^a\omega_a + \alpha^a\bar{\omega}_a)\wedge(\bar{\beta}^b\omega_b + \beta^b\bar{\omega}_b)\rangle \\ &= \langle \bar{\alpha}^a, \beta^b\rangle[\omega_a \wedge \bar{\omega}_b] + \langle \alpha^a, \bar{\beta}^b\rangle[\bar{\omega}_a \wedge \omega_b].\end{aligned}$$

Now $\int_\Sigma \omega_a \wedge \omega_b = \tau_{ab} - \bar{\tau}_{ab} = 2i \operatorname{im}\tau_{ab}$ and integrating the above 2-form over Σ we have

$$\begin{aligned}\int_\Sigma \langle g^{-1}dg \wedge dhh^{-1}\rangle &= \langle \bar{\alpha}^a, \beta^b\rangle(\bar{\tau}_{ab} - \tau_{ab}) + \langle \alpha^a, \bar{\beta}^b\rangle(\tau_{ab} - \bar{\tau}_{ab}) \\ &= 2i \operatorname{im}\tau_{ab}\left[\langle \alpha^a, \bar{\beta}^b\rangle - \langle \bar{\alpha}^a, \beta^b\rangle\right],\end{aligned}$$

so let us write $\alpha^a = \frac{1}{2i}[(\operatorname{im}\tau)^{-1}]^a_f(\lambda_1^f + \tau^f_h\mu_1^h)$ Then the above formula reads:

$$\begin{aligned}\int_\Sigma \langle g^{-1}dg \wedge dhh^{-1}\rangle &= \delta_{ab}[(\operatorname{im}\tau)^{-1}]^b_d\left(\langle \lambda_1^a + \tau^a_e\mu_1^e, \lambda_2^d + \bar{\tau}^d_f\mu_2^f\rangle - \langle \lambda_1^a + \bar{\tau}^a_e\mu_1^e, \lambda_2^d + \tau^d_f\mu_2^f\rangle\right) \\ &= [(\operatorname{im}\tau)^{-1}]_{ad}[\langle \lambda_1^a, \mu_2^f\rangle(\bar{\tau}^d_f - \tau^f_d) + \langle \lambda_2^d, \mu_1^e\rangle(\tau^a_e - \bar{\tau}^a_e)] \\ &= 2\left(\langle \lambda_1^a, \mu_{2a}\rangle - \langle \lambda_2^a, \mu_{1a}\rangle\right),\end{aligned}$$

where in the previous we have used the fact that τ is symmetric. In the following calculations, we use the notation $Z = (\operatorname{im}\tau)^{-1}$. We have

$$g(P) = \exp\frac{1}{2i}Z_{ab}\left[(\lambda^a + \tau^a_d\mu^d)\int_\star^P \bar{\omega}^b - (\lambda^a + \bar{\tau}^a_c\mu^c)\int_\star^P \omega^b\right] \tag{C.6}$$

$$
\begin{aligned}
g(P+a_f) &= g(P)\exp\frac{1}{2i}Z_{af}\left[(\lambda^a+\tau^a_d\mu^d)\bar\tau^b_f-(\lambda^a+\bar\tau^a_c\mu^c)\right]\\
&= g(P)\exp Z_{af}(\mathrm{im}\,\tau)^a_d\mu^d\\
&= g(P)\exp\mu^f
\end{aligned}
$$

$$
\begin{aligned}
g(P+b_f) &= g(P)\exp\frac{1}{2i}Z_{ab}\left[(\lambda^a+\tau^a_d\mu^d)\bar\tau^b_f-(\lambda^a+\bar\tau^a_c\mu^c)\tau^b_f\right]\\
&= g(P)\exp\left[-\lambda^f+Z_{ab}(\tau^a_d\bar\tau^b_f-\tau^b_f\bar\tau^a_d)\mu^d\right]\\
&= g(P)\exp-\lambda^f
\end{aligned}
$$

The latter equality is because

$$
\begin{aligned}
\tau^a_d\bar\tau^b_f-\tau^b_f\bar\tau^a_d &= \tau^a_d(\bar\tau^b_f-\tau^b_f)+\tau^b_f(\tau^a_d-\bar\tau^a_d)\\
&= -2i\tau^a_d(Z^{-1})^b_f+2i\tau^b_f(Z^{-1})^a_d,
\end{aligned}
$$

so

$$
Z_{ab}[\tau^a_d\bar\tau^b_f-\tau^b_f\bar\tau^a_d]=-2i\tau_{fd}+2i\tau_{df}=0, \tag{C.7}
$$

and $g(P)$ is well-defined on Σ provided $\exp\mu^f=\exp-\lambda^f=\mathrm{id}\in G$.

Appendix D

Gauge Groups and Principal Bundles

D.1 A Discussion of Gauge Groups and Current Groups

Let $\pi : P \to \Sigma$ be a principal G-bundle on the compact Riemann surface Σ of genus g. Let $\{U_\alpha\}$ be an open cover for Σ. We have local trivialisations $\Psi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times G$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}U_\alpha & \xrightarrow{\Psi_\alpha} & U_\alpha \times G \\ & \searrow^{\pi} & \downarrow{\scriptstyle \mathrm{pr}_1} \\ & & U_\alpha \end{array} \tag{D.1}$$

commutes. Then Ψ_α is of the form $\Psi_\alpha(p) = (\pi(p), g_\alpha(p))$ for some G-equivariant map $\psi_\alpha : \pi^{-1}U_\alpha \to G$, which is a fibrewise diffeomorphism.

Definition D.1.1. *A gauge transformation of the principal G-bundle P is a G-equivariant bundle diffeomorphism $\Phi : P \to P$ which preserves the fibres, that is the diagram*

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & P \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi} \\ & & \Sigma \end{array} \tag{D.2}$$

commutes. The gauge transformations of the bundle P form a group under composition, denoted $Gau(P)$.

We may think of gauge transformations in terms of local trivialisations, as they map fibres to themselves. Consider the following diagram:

$$\begin{array}{ccccc} \pi^{-1}U_\alpha & \xrightarrow{\Phi|_{\pi^{-1}U_\alpha}} & \pi^{-1}U_\alpha & \xrightarrow{\Psi_\alpha} & U_\alpha \times G \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi} & \swarrow_{\mathrm{pr}_1} & \\ & & \Sigma & & \end{array} \tag{D.3}$$

The commutativity of the diagram implies that

$$\Psi_\alpha(\Phi(p)) = \big(\pi(\Phi(p)), \psi_\alpha(\Phi(p))\big), \tag{D.4}$$

so let us define a map $\tilde{\phi}_\alpha : \pi^{-1}U_\alpha \to G$ by

$$\tilde{\phi}_\alpha(p) = \psi_\alpha(\Phi(p))\psi_\alpha(p)^{-1}. \tag{D.5}$$

Equivariance of ψ_α and of Φ means

$$\begin{aligned}\tilde{\phi}_\alpha(pg) &= \psi_\alpha(\Phi(pg))\psi_\alpha(pg)^{-1}\\ &= \psi_\alpha(\Phi(p)g)g^{-1}\psi_\alpha(p)^{-1}\\ &= \psi_\alpha(\Phi(p))\psi_\alpha(p)^{-1}\\ &= \tilde{\phi}_\alpha(p),\end{aligned} \tag{D.6}$$

so $\tilde{\phi}_\alpha$ is G-invariant.

Now suppose $\pi : P \to \Sigma$ is a trivial principal G-bundle. Then we have a global trivialisation, that is, a diffeomorphism $\Psi : P \to \Sigma \times G$ such that

$$\begin{array}{ccc} P & \xrightarrow{\Psi} & \Sigma \times G \\ & \underset{\pi}{\searrow} & \downarrow \mathrm{pr}_1 \\ & & \Sigma \end{array} \tag{D.7}$$

commutes. Then the map $\Psi : P \to \Sigma \times G$ must be of the form

$$\Psi(p) = (\pi(p), \psi(p)) \tag{D.8}$$

for some $\psi : P \to G$. Now consider the group of gauge transformations $\mathrm{Gau}(P)$ of this trivial principal G-bundle P. Applying the trivialisation, we have

$$\begin{array}{ccccc} P & \xrightarrow{\Phi} & P & \xrightarrow{\Psi} & \Sigma \times G \\ & \underset{\pi}{\searrow} & \downarrow \pi & \underset{\mathrm{pr}_1}{\swarrow} & \\ & & \Sigma & & \end{array}, \tag{D.9}$$

so that $\Psi(\Phi(p)) = (\pi(\Phi(p)), \psi(\Phi(p)))$. As in (D.5), we shall define a G-invariant map $\tilde{\phi} : P \to G$ given by

$$\tilde{\phi}(p) := \psi(\Phi(p))\psi(p)^{-1}. \tag{D.10}$$

The same calculation as (D.6) shows that $\tilde{\phi}$ is G-invariant, using the G-equivariance of ψ and of Φ. This implies that $\tilde{\phi}(p) = \phi(\pi(p))$ for some $\phi : \Sigma \to G$. Then the gauge transformations $\Phi : P \to P$ are uniquely determined by maps $\phi : \Sigma \to G$. We have shown:

Lemma D.1.2. *The group of gauge transformations* $\mathrm{Gau}(P)$ *of a trivial principal* G*-bundle on a compact Riemann surface* Σ *coincides with the current group* G^Σ *of maps from* Σ *to* G*.*

D.2 The Moduli Space of Flat and Unitary Principal G-bundles over Σ

Let X be some smooth manifold. We note that a representation $\chi : \pi_1(X) \to G$ of the fundamental group of X determines a flat principal G-bundle on X. Indeed, consider the universal cover $\tilde{X}$ of X. It may be regarded as a principal $\pi_1(X)$-bundle over X, with constant transition functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \pi_1(X)$. Now we may construct a principal G-bundle over X by defining the transition functions to be $g_{\alpha\beta} = \chi \circ h_{\alpha\beta}$. Moreover the bundle is flat as the transition functions are constant. Conversely, given a flat principal G-bundle we have a natural homomorphism $\pi_1(X) \to G$ given by the holonomy. (The fact that the bundle is flat means that the holonomy depends only on the homotopy class of the loop.) Furthermore, we call a flat principal G-bundle *unitary* if it is induced by a representation $\pi_1(X) \to K$, where K is the maximal compact subgroup of G.

Below Σ is a compact Riemann surface of genus g and L_k is the hyperplane in $(\widehat{\mathfrak{g}}^\Sigma)^*$ where the level k is fixed.

Proposition D.2.1. *[EF94, Proposition 3.2] Coadjoint orbits of G^Σ in the hyperplane L_k are in one-to-one correspondence with equivalence classes of holomorphic G-bundles over Σ.*

Proof. We identify an element $(A, k) \in (\widehat{\mathfrak{g}}^\Sigma)^*$, with the operator $\nabla = k\bar{\partial} + A$. We associate ∇ to a holomorphic principal G-bundle $P(A)$ as follows. Consider the partial differential equation

$$\nabla\psi = A\psi + k\frac{\partial\psi}{\partial\bar{z}} = 0. \tag{D.11}$$

Here, A is a $\mathfrak{g}$-valued $(0,1)$-form, the operator D is a $(0,1)$-connection and ψ is a horizontal section for that connection. We can find an open covering $\{U_i\}_{i\in I}$ of Σ such that there exist local solutions $\psi_i : U_i \to G$ of the equation (D.11). We construct a principal G-bundle by defining transition functions $\phi_{ij} : U_i \cap U_j \to G$ by $\phi_{ij} := \psi_i^{-1}\psi_j$. These functions are holomorphic and satisfy the cocycle condition $\phi_{ij}\phi_{jk}\phi_{ki} = \psi_i^{-1}\psi_j\psi_j^{-1}\psi_k\psi_k^{-1}\psi_i = \mathrm{id}$ on $U_i \cap U_j \cap U_k$. Then they can be interpreted as the gluing functions of a holomorphic principal G-bundle $P(A)$ on Σ, associated to the $\mathfrak{g}$-valued one-form A on Σ.

Conversely, suppose we have a holomorphic principal G-bundle P on Σ. We shall find an operator $\nabla = k\bar{\partial} + A$ such that $P = P(A)$ via the construction above. Now as G is simply connected, the principal G-bundle P is topologically trivial. We choose a global holomorphic trivialisation $\Psi : P \to \Sigma \times G$, and the local holomorphic trivialisations over the open sets U_i will be expressed by smooth functions $\psi_i : U_i \to G$ such that the transition functions $\phi_{ij} : U_i \cap U_j \to G$ given by $\phi_{ij} = \psi_i^{-1}\psi_j$ are holomorphic on $U_i \cap U_j$. Further, we have

$$\bar{\partial}\psi_i\psi_i^{-1} = \psi_j\bar{\partial}(\psi_j^{-1}\psi_i)\psi_i^{-1} - \psi_j\bar{\partial}\psi_j^{-1} = \bar{\partial}\psi_j\psi_j^{-1}$$

on $U_i \cap U_j$, as $\psi_j^{-1}\psi_i$ is holomorphic. Therefore there exists a $\mathfrak{g}$-valued 1-form A on Σ such that $A = -k\bar{\partial}\psi_i\psi_i^{-1}$ so that (D.11) holds on each U_i. Setting $\nabla = k\bar{\partial} + A$, we see that the bundle P is exactly $P(A)$.

Finally, two holomorphic principal G-bundles $P(A_1)$ and $P(A_2)$ are equivalent if and only if there exists a current $g \in G^\Sigma$ such that $A_2 = gA_1g^{-1} + k\bar{\partial}gg^{-1}$, that is, the operators $D_1 = k\bar{\partial} + A_1$ and $D_2 = k\bar{\partial} + A_2$ are related by a gauge transformation. The correspondence between coadjoint orbits of G^Σ in L_k and equivalence classes of holomorphic principal G-bundles over Σ is then $(A,k) \leftrightarrow P(A)$. □

We record another interesting result of Etingof and Frenkel for a Riemann surface $E = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ of genus one.

Proposition D.2.2. *[EF94, Proposition 4.1] The space $\mathcal{M}$ of equivalence classes of flat and unitary principal G-bundles on E is isomorphic to* $\mathfrak{t}_\mathbb{C}/(W \ltimes (Q^\vee \oplus \tau Q^\vee))$.

Proof. Let $\nabla = k\bar{\partial} + A$. From Proposition D.2.1 we have that ∇ determines a principal G-bundle $P(A)$ over E. If the bundle $P(A)$ is flat[1] then the equation $\nabla\psi = 0$ has a solution $\psi(z)$ with $\psi(z+1) = \psi(z)$ and $\psi(z+\tau) = \psi(z)L$, where L is a fixed element of G. If the bundle $P(A)$ is unitary, then $L \in G$ is semisimple and we may assume it lies in a maximal torus $T \subset G$. We claim that the element L completely determines the bundle $P(A)$. Fix $l \in \mathfrak{g}$ such that $\exp(l) = L$ and set $F(z) = \psi(z)\exp\left(-l\frac{z-\bar{z}}{\tau-\bar{\tau}}\right)$. Now F is doubly periodic:

$$
\begin{aligned}
F(z+1) &= \psi(z+1)\exp\left(-l\frac{(z+1)-(\bar{z}+1)}{\tau-\bar{\tau}}\right) = \psi(z)\exp\left(-l\frac{z-\bar{z}}{\tau-\bar{\tau}}\right) = F(z),\\
F(z+\tau) &= \psi(z+\tau)\exp\left(-l\frac{(z+\tau)-(\bar{z}+\bar{\tau})}{\tau-\bar{\tau}}\right) = \psi(z)L\exp(-l)\exp\left(-l\frac{z-\bar{z}}{\tau-\bar{\tau}}\right)\\
&= F(z),
\end{aligned}
$$

that is, $F \in G^E$. Then F acts on ∇ via the coadjoint representation:

$$
\begin{aligned}
F \diamond (k\bar{\partial} + A) &= k\bar{\partial} + kF^{-1}\bar{\partial}F + Ad_F(A)\\
&= k\bar{\partial} + k\exp\left(l\frac{z-\bar{z}}{\tau-\bar{\tau}}\right)\psi^{-1}\bar{\partial}\psi\exp\left(-l\frac{z-\bar{z}}{\tau-\bar{\tau}}\right) + k\frac{l}{\tau-\bar{\tau}}\\
&\quad + \exp\left(l\frac{z-\bar{z}}{\tau-\bar{\tau}}\right)\psi^{-1}A\psi\exp\left(-l\frac{z-\bar{z}}{\tau-\bar{\tau}}\right)\\
&= k\bar{\partial} + k\frac{l}{\tau-\bar{\tau}} + \exp\left(l\frac{z-\bar{z}}{\tau-\bar{\tau}}\right)[k\psi^{-1}\bar{\partial}\psi + \psi^{-1}A\psi]\exp\left(-l\frac{z-\bar{z}}{\tau-\bar{\tau}}\right).
\end{aligned}
\tag{D.12}
$$

But $\nabla\psi = 0$, *ie.* $k\bar{\partial}\psi + A\psi = 0$. Then

$$
F \diamond (k\bar{\partial} + A) = k\left(\bar{\partial} + \frac{l}{\tau-\bar{\tau}}\right). \tag{D.13}
$$

This shows that the action of the function F on the hyperplane L_k is constant, and depends only on the element $L \in G$.

Now we ask which elements of T correspond to equivalent bundles. Let r be the rank of G (ie $\dim T = r$). If $h_i, 1 \le i \le r$ is the standard basis for $\mathfrak{t}$ then $\exp(2\pi i h_i) = 1$ and so

[1] Almost all holomorphic principal bundles are flat and unitary, *cf.* [EF94, 3.3].

we find other solutions to the equation $\nabla\psi = 0$ as follows:

$$\psi_n(z) = \psi(z) \exp\left[2\pi i z \sum_{i=1}^{r} n_i h_i\right], \qquad n_i \in \mathbb{Z}. \tag{D.14}$$

These solutions satisfy the periodicity conditions $\psi_n(z+1) = \psi_n(z)$ and $\psi_n(z+\tau) = \psi_n(z)L_n$, where $L_n = L \exp 2\pi i \tau \sum n_i h_i$, and L_n corresponds to the same bundle as L. Furthermore, if we conjugate L by an element of the Weyl group W of G, we get an element $L' \in T$ that also corresponds to the same bundle as L. We have thus shown that the principal G-bundle $P(A)$ is completely determined by the equivalence class of the element L in the complex space $T/(W \ltimes \exp 2\pi i \tau Q^\vee) = \mathfrak{t}/W \ltimes (Q^\vee \oplus \tau Q^\vee)$, using the fact that $T = \mathfrak{t}/Q^\vee$. □

Bibliography

[AB83] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615.

[Ber99] Marco Bertola, *Jacobi groups, Jacobi forms and their applications*, Ph.D. thesis, S.I.S.S.A, Trieste, 1999.

[Ber00a] ______, *Frobenius manifold structure on orbit space of Jacobi groups. I*, Differential Geom. Appl. **13** (2000), 19–41.

[Ber00b] ______, *Frobenius manifold structure on orbit space of Jacobi groups. II*, Differential Geom. Appl. **13** (2000), no. 3, 213–233.

[Bot54] Raoul Bott, *On torsion in Lie groups*, Proc. Nat. Acad. Sci. U. S. A. **40** (1954), 586–588.

[Bro82] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1982.

[Car05] R. W. Carter, *Lie algebras of finite and affine type*, Cambridge Studies in Advanced Mathematics, vol. 96, Cambridge University Press, Cambridge, 2005.

[EF94] Pavel I. Etingof and Igor B. Frenkel, *Central extensions of current groups in two dimensions*, Comm. Math. Phys. **165** (1994), no. 3, 429–444.

[EZ85] Martin Eichler and Don Zagier, *The theory of Jacobi forms*, Progress in Mathematics, vol. 55, Birkhäuser Boston Inc., Boston, MA, 1985.

[FK96] Igor B. Frenkel and Boris A. Khesin, *Four-dimensional realization of two-dimensional current groups*, Comm. Math. Phys. **178** (1996), no. 3, 541–562.

[Han83] Vagn Lundsgaard Hansen, *The homotopy groups of a space of maps between oriented closed surfaces*, Bull. London Math. Soc. **15** (1983), no. 4, 360–364.

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[HM06] Karl H. Hofmann and Sidney A. Morris, *The structure of compact groups*, de Gruyter Studies in Mathematics, vol. 25, Walter de Gruyter & Co., Berlin, 2006.

[Hoc51] G. Hochschild, *Group extensions of Lie groups*, Ann. of Math. (2) **54** (1951), 96–109.

[Kac90] Victor G. Kac, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990.

[KW09] Boris Khesin and Robert Wendt, *The geometry of infinite-dimensional groups*, Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics, vol. 51, Springer-Verlag, Berlin, 2009.

[Loo80] Eduard Looijenga, *Invariant theory for generalized root systems*, Invent. Math. **61** (1980), no. 1, 1–32.

[Mau80] C. R. F. Maunder, *Algebraic topology*, Cambridge University Press, Cambridge, 1980.

[MN03] Peter Maier and Karl-Hermann Neeb, *Central extensions of current groups*, Math. Ann. **326** (2003), no. 2, 367–415.

[Nee96] Karl-Hermann Neeb, *A note on central extensions of Lie groups*, J. Lie Theory **6** (1996), no. 2, 207–213.

[Nee02] ______, *Central extensions of infinite-dimensional Lie groups*, Ann. Inst. Fourier (Grenoble) **52** (2002), no. 5, 1365–1442.

[NW03] Karl-Hermann Neeb and Friedrich Wagemann, *The universal central extension of the holomorphic current algebra*, Manuscripta Math. **112** (2003), no. 4, 441–458.

[PS86] Andrew Pressley and Graeme Segal, *Loop groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1986, Oxford Science Publications.

[Ser51] Jean-Pierre Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. (2) **54** (1951), 425–505.

[ST97] Kyoji Saito and Tadayoshi Takebayashi, *Extended affine root systems. III. Elliptic Weyl groups*, Publ. Res. Inst. Math. Sci. **33** (1997), no. 2, 301–329.

[Str10] Ian A. B. Strachan, *Weyl groups and elliptic solutions of the WDVV equations*, Adv. Math. **224** (2010), no. 5, 1801–1838.

[Wan49] Hsien-Chung Wang, *The homology groups of the fibre bundles over a sphere*, Duke Math. J. **16** (1949), 33–38.

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

[Wir92] Klaus Wirthmüller, *Root systems and Jacobi forms*, Compositio Math. **82** (1992), no. 3, 293–354.

[Yan96] Jae-Hyun Yang, *Kac-Moody algebras, the Monstrous Moonshine, Jacobi forms and infinite products*, Number theory, geometry and related topics (Iksan City, 1995), Pyungsan Inst. Math. Sci., Seoul, 1996, pp. 13–82.