Two Topics in Commutative Ring Theory

by

Andrew James Duncan

Presented for the degree of Doctor of Philosophy in Mathematics at the University of Edinburgh, October 1988.
Dedicated
to

Cathy, Anna and Helen,
who have put up with so much,
for so long.
The following record of research work is submitted as a thesis in support of an application for the degree of Doctor of Philosophy at the University of Edinburgh, having been submitted for no other degree. Except where acknowledgement is made, the work is original.

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In Chapter 1, L. O'Carroll suggested that the study of infinite intersections and unions of ideals could lead to the type of results that were eventually found in Sections 1.3 and 1.4. It was his proposal that this study should begin with McAdam's paper, [McA], and Sharp and Vamos's paper [ShVa]. In particular he prompted me to use Baire's Category Theorem, as Sharp and Vamos had done, to obtain the results of Section 1.4. He also turned my attention to the results of [Kap], which it proved possible to generalise in Sections 1.8 and 1.9, and showed me the work of Griffith and Bartijn ([Gr1], [Gr2] and [Ba]) which was used in Section 1.9 and 1.10. He provided insights, overviews, hints, improvements and corrections too numerous to catalogue.

Chapter 2 builds directly on L. O'Carroll's paper [O'C1] and is the result of many conversations with L. O'Carroll over several months. It is, therefore, difficult to make fair acknowledgement. I have made some attributions in the chapter but in most cases I felt that the pattern of contributions was too complex to describe, even
if I could remember it. I can only say that my main contributions were to generalise the uniform Artin-Rees theorem from affine rings to those of the type implied by the hypotheses of Theorem 2.3.4, and to generalise Theorem 2 and the Main Lemma of [Zar], and that in all other parts of the chapter, where no acknowledgement is made in the text, the contribution of one author is inextricable from that of the other. I am grateful to T. H. Lenagan and J. Howie for suggesting the use of König's graph Lemma to prove Theorem 2.3.4.

I owe many thanks to Cathy Duncan, for cooking my supper and washing up the dishes, for being both mother and father to Anna and Helen over the last six months, for unwavering support, sympathy and love.

Most of the results of Chapter 2, Sections 2.1, 2.2 and 2.3 are to appear elsewhere. Theorem 2.3.4 is, essentially, contained in [DO'C1] as are many of the arguments in Sections 2.1 and 2.2. The results of Section 2.1, up to Theorem 2.1.14, and the whole of Section 2.2 are recorded in [DO'C2].

Throughout the course of this research the author was funded by a postgraduate studentship from the Science and Engineering Research Council.
Summary

This thesis falls into two parts, namely Chapter 1 and Chapter 2, which are completely separate. A brief description of each part follows.

In Chapter 1 we study infinite unions and intersections of cosets of ideals, with a view to obtaining infinite analogues of the finite "prime avoidance" results. In Section 1.1 we examine possible extensions of intersection results from the finite to the infinite. We find that the kind of result that would be useful later in the chapter is not, in general, true. In section 1.2, we establish results that are known so far about ideals which are contained in unions of cosets of ideals. Sections 1.3 and 1.4 generalise these known results, in particular circumstances. In Section 1.5 we describe a "prime avoidance" property that we should like a set of ideals to have, and examine the implications of this property. Section 1.6 uses the results of Sections 1.3 and 1.4 to give examples of the behaviour described in Section 1.5. Sections 1.7 to 1.11 are concerned with applications of the results obtained earlier. In Section 1.7 we give infinite analogues of the finite "prime avoidance" results. In Sections 1.8 and 1.9 we generalise standard results from [Kap] on zero-divisors and regular sequences. In Section 1.10 we apply the results of Section 1.9 to big Cohen-Macaulay modules.

In Chapter 2 we develop the idea of Zariski regularity to
prove a uniform Artin-Rees theorem and to generalise the Main Lemma of [Zar]. We begin by defining and characterising the notion of Zariski regularity, in Section 2.1. We show, in Section 2.2, that Zariski regularity is an open condition and move on to prove the uniform Artin-Rees theorem, in Section 2.3. Finally, in Section 2.4, we generalise Zariski's Main Lemma to a wider class of rings and to modules over these rings.
Notes to the reader

Throughout this work I shall use the standard notation and conventions of commutative algebra, wherever possible. Since these vary from author to author, those that are used in this thesis are described here. I have, on the whole, followed Matsumura's book, Commutative ring theory ([Matt]), and sometimes also Kaplansky's Commutative Rings ([Kap]).

Sets and functions

I use "\(\subseteq\)" to denote "contains" and "\(\subset\)" to denote "contains but is not equal to". A similar distinction is made between "\(\subseteq\)" and "\(\subset\)".

I take it that symbols will be read in context. For example; "\(\in\)" will be used to mean "an element of", "is an element of", "be an element of", "elements of", etc., etc..

Set and class theory is used, in general, without explicit reference.

Let \(X\) and \(Y\) be sets, and \(f : X \longrightarrow Y\) a function. Let \(W\) be a subset of \(X\). I usually abuse notation, writing \(f(W)\) when I mean
\[ \bigcup_{w \in W} f(w) \]. However, in some parts of Section 2.3 it is necessary to be more careful.

Let \( X \) and \( \Lambda \) be sets. Given a fixed map \( \Lambda \to X : \lambda \mapsto x_\lambda \), we say that \( (x_\lambda)_{\lambda \in \Lambda} \) is a family of elements of \( X \) indexed by \( \Lambda \).

I use the following symbols when dealing with numbers:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Denotes</th>
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<tbody>
<tr>
<td>( Z )</td>
<td>the integers</td>
</tr>
<tr>
<td>( N )</td>
<td>the positive integers</td>
</tr>
<tr>
<td>( R )</td>
<td>the real numbers</td>
</tr>
<tr>
<td>( Z_n )</td>
<td>( \mathbb{Z}/n\mathbb{Z} ), for any ( n \in \mathbb{N} )</td>
</tr>
<tr>
<td>( \text{Card} )</td>
<td>the class of cardinal numbers</td>
</tr>
<tr>
<td>( \text{card}(X) ) or (</td>
<td>X</td>
</tr>
<tr>
<td>( \text{Ord} )</td>
<td>the class of ordinal numbers</td>
</tr>
<tr>
<td>( \text{ord}(X) )</td>
<td>the ordinal of the set ( X )</td>
</tr>
<tr>
<td>( \aleph_0 )</td>
<td>( \text{card}(\mathbb{N}) )</td>
</tr>
<tr>
<td>( \aleph_1 )</td>
<td>the first uncountable cardinal</td>
</tr>
</tbody>
</table>

Rings and modules

\( R \) always denotes a commutative ring with non-zero identity. I do not assume that rings are Noetherian. In particular, a local ring is not necessarily Noetherian.
R[X_1, \ldots , X_n] denotes the polynomial ring in n indeterminates over R. This notation may be extended to infinitely many indeterminates. R[x_1, \ldots , x_n] denotes the R-algebra generated by x_1, \ldots , x_n over R. R[[X_1, \ldots , X_n]] denotes the power series ring in n indeterminates over R.

I use both \( \oplus M_i \) and \( \bigsqcup M_i \) to denote the direct sum, and \( \prod M_i \) to denote the direct product, of the family of modules \( \{M_i : i \in I\} \). A similar notation is used for direct sums and products of rings.

I refer the reader to [Mat2], Section 13 or [ZS2], Chapter VII, Sections 2 and 12, and Chapter VIII, Section 1 for the definitions graded ring and graded module and associated graded ring and module. I shall henceforward use the term graded to mean, in the terminology of [Mat2], Z-graded. Let R be a ring, I an ideal of R and M an R-module. Then I use \( \text{gr}^k_I(M) \) to denote \( (I^kM)/(I^{k+1}M) \), for \( k \geq 0 \), and \( \text{gr}_I(M) \) to denote \( \bigoplus_{k \geq 0} \text{gr}^k_I(M) \). Let S be a graded ring, \( d \in \mathbb{Z} \) and \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) and \( N = \bigoplus_{i \in \mathbb{Z}} N_i \) graded S-modules. An R-module homomorphism \( \varphi : M \rightarrow N \) is said to be homogenous of degree \( d \) if \( \varphi(M_j) \subseteq N_{j+d} \), for all \( j \in \mathbb{Z} \).

Let R be a domain and M an R-module such that for all non-zero
elements $s$ of $R$, $sm = 0$ implies that $m = 0$, where $m$ is any element of $M$; then $M$ is said to be a torsion-free $R$-module.

Primes, associated primes and zero divisors

Let $R$ be a ring, $E$ a subset of $R$, $t$ an element of $R$, $M$ an $R$-module and $X$ and $Y$ subsets of $M$. I use the following notation:

<table>
<thead>
<tr>
<th>Notation</th>
<th>Meaning</th>
</tr>
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<tbody>
<tr>
<td>$\text{Spec}(R)$</td>
<td>the set of prime ideals of $R$</td>
</tr>
<tr>
<td>$\text{Max-Spec}(R)$</td>
<td>the set of maximal ideals of $R$</td>
</tr>
<tr>
<td>$V(E)$</td>
<td>${P \in \text{Spec}(R) : P \supseteq E}$</td>
</tr>
<tr>
<td>$D(t)$</td>
<td>${Q \in \text{Spec}(R) : t \notin Q}$</td>
</tr>
<tr>
<td>$(X :_R Y)$</td>
<td>${r \in R : rY \subseteq X}$</td>
</tr>
<tr>
<td>$\text{Ann}_R(X)$</td>
<td>${0 :_R X}$</td>
</tr>
<tr>
<td>$Z(M)$</td>
<td>${\text{zero divisors of } M} = \bigsqcup_{m \in M} \text{Ann}_R(m)$</td>
</tr>
<tr>
<td>$\text{Ass}_R(M)$</td>
<td>${P \in \text{Spec}(R) : P = \text{Ann}_R(x), \text{ for some } x \in M}$</td>
</tr>
<tr>
<td>${\text{maximal primes of } M}$</td>
<td>${P \in \text{Spec}(R) : P \text{ is maximal among ideals contained in } Z(M)}$</td>
</tr>
<tr>
<td>${\text{maximal primes of } I}$</td>
<td>${\text{maximal primes of } R/I} \text{ (by abuse of notation)}$</td>
</tr>
<tr>
<td>$\text{Supp}_R(M)$</td>
<td>${P \in \text{Spec}(R) : M_p \neq 0}$</td>
</tr>
<tr>
<td>$r(I) = \sqrt{I}$</td>
<td>the radical of $I$ [= {r \in R : r^n \in I \text{ for some } n \in \mathbb{N}} ]</td>
</tr>
</tbody>
</table>
dim(R) the Krull dimension of R
= supremum of lengths of strictly
decreasing chains of prime ideals
of R (which may be transfinite)

In the situation described above, if X and Y are submodules of
M, then (X :_R Y) is an ideal of R. From Theorem 2 of [Kap],
\[ Z(M) = \bigcup \{ P : P \text{ is a maximal prime of } M \} \].
If R is Noetherian and M is finitely generated, then Ass_R(M) is a
finite set and
\[ Z(M) = \bigcup \{ P : P \in \text{Ass}_R(M) \} \],
(from [Mat2] Theorems 6.1 and 6.5, for example).

If M is a finitely generated R-module, then dim_R(M) is
defined to be \( \dim_R(R/\text{Ann}_R(M)) \).

I use basic properties of Ass_R(M), Supp_R(M) and Z(M) without
reference. These facts are to be found in, for example, [Mat2].

Systems of parameters

Let R be a ring and M an R-module. Then \( \ell(M) \) denotes the
length of a composition series of M. If M does not have a
composition series, we set \( \ell(M) = \infty \). (Composition series are
defined in (for example) [Mat2] on page 12.)

Now let R be a semi-local Noetherian ring and M a finitely
generated R-module. Then it follows from [Mat2], Theorem 13.4, that
\( \dim_R(M) \) is finite; \( \dim_R(M) = d \), say. Given \( d \) elements
\( a_1, \ldots, a_d \) in the Jacobson radical of \( R \) such that
\( \ell(M/(a_1 M + \cdots + a_d M)) < \infty \), we say that \( a_1, \ldots, a_d \) is a system
of parameters for \( M \). It follows from [Mat2], Theorem 13.4, that, in
the situation described, a system of parameters for \( M \) always exists.

Let \( \mathfrak{m} \) denote the Jacobson radical of \( R \). Let \( I \) be an ideal of \( R \)
such that \( \mathfrak{m}^v \subseteq I \subseteq \mathfrak{m} \), for some \( v \in \mathbb{N} \). Then \( I \) is said to be an
ideal of definition of \( R \). It is easy to see that
every system of parameters of \( R \) generates an ideal of definition of
\( R \). Basic properties of system of parameters are covered in [Ser],
Chapitre III, B), 3.

Regular sequences and Cohen–Macaulay modules

Let \( R \) be a ring and \( M \) an \( R \)-module. An element \( r \in R \) is said
to be \( M \)-regular if \( r \notin Z(M) \). A sequence \( r_1, \ldots, r_n \) of elements
of \( R \) is said to be an \( M \)-sequence if the following two conditions
hold:

1. \( r_1 \) is \( M \)-regular and
2. \( r_{i+1} \) is \( M/(r_1 M + \cdots + r_i M) \)-regular, for
   \( i = 1, \ldots, n - 1 \);

3. \( r_1 M + \cdots + r_n M \neq M \).
For basic facts about $M$-sequences, when $M$ is a finitely generated module, I refer the reader to [Mat2], Section 16 and [Kap], Section 3-1. The $I$-depth of the module $M$, where $I$ is an ideal of $R$, is defined to be

$$\text{depth}_R(I,M) = \inf \{i : \text{Ext}^i_R(R/I,M) \neq 0\},$$

if this exists, and $\infty$ otherwise. Note that, in this definition, there is no requirement that $R$ be Noetherian or $M$ finitely generated. However, if $R$ is Noetherian, $M$ is finitely generated and $IM \neq M$, it follows from [Mat2], Theorem 16.7, that $\text{depth}_R(I,M)$ is the length of any maximal $M$-sequence in $I$.

Now let $R$ be a Noetherian local ring, with maximal ideal $m$, and $M$ a non-zero $R$-module such that $\text{depth}_R(m,M) = \dim_R(M)$. Then we say that $M$ is a Cohen-Macaulay $R$-module. If $R$ is a Cohen-Macaulay $R$-module, then $R$ is said to be a Cohen-Macaulay ring. The basic properties of finitely generated Cohen-Macaulay modules appear in [Ser], Chapitre IV, B).

**Regular rings**

Let $R$ be a Noetherian local ring, with maximal ideal $m$. If $m$ can be generated by a system of parameters of $R$, then $R$ is said to be a regular local ring. A system of parameters generating $m$ is called a regular system of parameters.

A Noetherian ring $R$ such that $R_p$ is a regular local ring, for all primes $P \in \text{Spec}(R)$, is called a regular ring. This condition is equivalent to $R_m$ being a regular local ring for all
Basic properties of regular rings are covered in [Mat2]. In particular I need the facts that:

a) a regular local ring is a Cohen-Macaulay ring ([Mat2], Theorem 17.8) and

b) in a Cohen-Macaulay local ring any subset of a system of parameters is an R-sequence ([Mat2], Theorem 17.4).

Maps and exactness

I shall sometimes use the term R-module map instead of R-module homomorphism.

If E is the short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

then $$E \otimes D$$ is the sequence

$$0 \longrightarrow A \otimes D \longrightarrow B \otimes D \longrightarrow C \otimes D \longrightarrow 0,$$

which is not necessarily exact at $$A \otimes D$$.

Ext and Tor are used without introduction (see [Rot] or [Mat2]).
Topologies

Let $X$ be a topological space and $Y$ a non-empty subspace of $X$. $Y$ is said to be irreducible if the following condition holds:

$Y = Y_1 \cup Y_2$, where $Y_1$ and $Y_2$ are closed in $Y$, implies that $Y = Y_1$ or $Y = Y_2$.

Let $\mathcal{F} = \{V(I) : I$ is an ideal of $R\}$. There is a topology on $\text{Spec}(R)$ for which $\mathcal{F}$ is the set of closed sets. This is called the Zariski topology. I shall always consider $\text{Spec}(R)$ with its Zariski topology. If $X$ is a subset of $\text{Spec}(R)$ I shall also call the induced topology of $X$ the Zariski topology. If $R$ is a domain then $\text{Spec}(R)$ is an irreducible topological space.

If a semi-local ring is said to be complete, this means complete with respect to the Jacobson radical topology, defined in Section 1.4.

I have given here only as much as I deem necessary to avoid ambiguity. Any unexplained terminology can be found in the texts mentioned. From now on I use "we" and never "I" as it often makes more sense and it is easier not to use both.
CHAPTER 1

This chapter is devoted to the study of unions and intersections of ideals. The starting point for this work is a diverse body of results on coverings and intersections; coverings, that is, of ideals by unions of ideals, and intersections of ideals which are contained in ideals. The aim is to combine and generalise these known results. In this chapter a ring is Noetherian only if it is explicitly said to be. In particular, local and semi-local rings are not necessarily Noetherian.

We begin with the classical "prime avoidance" results which appear frequently throughout commutative algebra. We distinguish two main types of result:

1.0.1. If \( R \) is a ring and \( I \) is an ideal of \( R \) such that
\[
I \subseteq P_1 \cup \cdots \cup P_n, \quad \text{where} \quad P_1, \ldots, P_n \quad \text{are prime ideals of} \quad R, \quad \text{then} \quad I \subseteq P_i, \quad \text{for some} \quad i = 1, \ldots, n;
\]

1.0.2. If \( J_1, \ldots, J_m \) are ideals of \( R \) and \( Q \) is a prime ideal of \( R \) such that \( Q \supseteq J_1 \cap \cdots \cap J_m \) then \( Q \supseteq J_i \) for some \( i = 1, 2, \ldots, m. \)

(See for example [AM] Proposition 1.11.)

We shall refer to results in the pattern of 1.0.1 and 1.0.2 as results of type (1) and type (2) respectively. Type (1) has been developed to include more general finite unions, and infinite unions. These developments are the subject of Section 1.2. In Sections 1.3 and 1.4 we develop type (1) to more general infinite
unions. First however in Section 1.1 we shall attempt to extend type (2) to infinite intersections.

1.1 Infinite Intersections

In the development of results of type (1) from 1.0.1 much use is made of 1.0.2. Thus it seems appropriate to begin by trying to extend 1.0.2 to infinite intersections of ideals in order to be able to extend 1.0.1 in the same way. However, as soon becomes clear, this does not lead very far.

Let \( R \) be a ring, \( \alpha \) an infinite cardinal, \( \Lambda \) a set such that \( |\Lambda| = \alpha \) and \( P \) a prime ideal of \( R \). Suppose that given any family \( (J_\lambda)_{\lambda \in \Lambda} \) of ideals of \( R \), indexed by \( \Lambda \), such that \( P \supseteq \bigcap_{\lambda \in \Lambda} J_\lambda \), then \( P \supseteq J_\mu \), for some \( \mu \in \Lambda \). Then \( R \) is said to have the \( \alpha \)-intersection property. If \( R \) has the \( \alpha \)-intersection property for all cardinals \( \alpha \), then \( R \) is said to have the strong intersection property.

Examples abound to show that the strong intersection property very rarely occurs. For instance if \( R \) is a Noetherian domain with a non-zero proper ideal, \( I \), then \( \bigcap_{n=0}^{\infty} I^n = (0) \), by the Krull intersection theorem. (See, for example, [Kap] Theorem 77.) Again, let \( R = k[X_0, X_1, \ldots] \), the polynomial ring in an infinite number of indeterminates; then \( \bigcap_{n=0}^{\infty} (X_n) = (0) \). Also any Hilbert domain with an infinite number of prime ideals fails the strong
intersection property. In fact in such a ring,
\[ \bigcap \{ P : P \in \text{Spec}(R) ; P \neq (0) \} = (0). \]

In the Noetherian case the Krull intersection theorem is the last word on the problem, as shown by the following corollary (to the aforesaid theorem):

Corollary 1.1.1. Let R be a Noetherian ring and let \( \alpha \) be a cardinal such that \( \alpha \geq \aleph_0 \). Then the following are equivalent:

(i) R has the strong intersection property;
(ii) R has the \( \alpha \)-intersection property;
(iii) R is Artinian.

Proof. In an Artinian ring all intersections reduce to finite intersections by the descending chain condition (d.c.c.), so (iii) \( \Rightarrow \) (i) \( \Rightarrow \) (ii), and it remains to show that (ii) \( \Rightarrow \) (iii). Let \( P \) be any prime ideal of R and \( M \) a maximal ideal of R containing \( P \). Then
\[
\bigcap_{n=0}^{\infty} (M^n + P) / P \subseteq \bigcap_{n=0}^{\infty} (M^n / P) = (0),
\]
by the Krull intersection theorem. Thus \( P \nsubseteq \bigcap_{n=0}^{\infty} M^n \) and so, by hypothesis, \( P \nsubseteq M^r \) for some \( r \geq 1 \), which means that \( P \) must be maximal. Therefore R is Artinian. \( \Box \)

Following notation in Kaplansky ([Kap] ; § 1-3), an integral domain, R, is a \( G \)-domain if the quotient field of R is finitely
generated as an \( R \)-algebra. If \( P \) is a prime ideal of \( R \) such that \( R/P \) is a \( G \)-domain then \( P \) is a \( G \)-ideal.

Lemma 1.1.2. Let \( R \) be an integral domain and \( X \) an indeterminate. If \( |\text{Spec}(R[X])| = \alpha \geq \aleph_0 \) then \( R[X] \) does not have the \( \alpha \)-intersection property.

\textbf{Proof.} \( R[X] \) is not a \( G \)-domain ([Kap], Theorem 21), and from [Kap], Theorem 19, we have
\[
\bigcap \{P : P \in \text{Spec}(R[X]) ; P \neq (0)\} = (0).
\]
Since this is an intersection of \( \alpha \) non-zero prime ideals contained in the prime ideal \( (0) \) the lemma is proved. \( \square \)

Note. As the first example above shows, the lemma is trivial if \( R \) is a Noetherian ring, since \( R[X] \) is, in this case, a Noetherian domain.

Intersection properties do not seem to extend very easily from the finite to infinite case, even in the best-behaved of rings. Therefore we narrow our attention to particular intersections of ideals. In the non-Noetherian case we have only the following restatement of [Kap], Theorem 27 to offer:

\textbf{Theorem 1.1.3.} Let \( R \) be a ring and \( P \) a prime ideal of \( R \). Then the following are equivalent:

\begin{enumerate}
\item Given any subset \( \mathcal{M} \) of \( \text{Spec}(R) \) such that \( \bigcap_{Q \in \mathcal{M}} Q = P \)
\end{enumerate}

then \( P \in \mathcal{M} \);
(2) P is the contraction of a maximal prime of R[X], X an indeterminate.

**Proof.** Given (1), suppose that P is not the contraction of a maximal ideal of R[X]. Then R/P is not a G-domain ([Kap], Theorem 27). If u is any non-zero element of R/P it follows that there is some non-zero prime, Q', of R/P such that u ∉ Q' ([Kap], Theorem 19). Therefore

\[ a \in R \setminus P \implies 0 \neq \bar{a} \in R/P, \]

\[ \exists q_a \in \text{Spec}(R/P) \text{ such that } q_a \neq (0) \text{ and } \bar{a} \notin q_a, \]

\[ \exists q_a \in \text{Spec}(R) \text{ such that } q_a \supset P \text{ and } a \notin q_a, \]

\[ q = \bigcap \{q \in \text{Spec}(R) : q \supset P ; q \neq P\} = P, \]

contradicting (1). Thus P must be the contraction of a maximal ideal of R[X].

Conversely, given that P is the contraction of a maximal ideal of R[X], R/P is a G-domain ([Kap], Theorem 27). Suppose then that we have \( \mathcal{M} \subseteq \text{Spec}(R) \) such that P = \( \bigcap_{Q \in \mathcal{M}} Q \).

Then

\[ \bigcap_{Q \in \mathcal{M}} (Q/P) = (0) \text{ in } R/P, \]

so that Q/P = (0) for some Q ∈ \( \mathcal{M} \) ([Kap], Theorem 19), and Q = P for some Q ∈ \( \mathcal{M} \). □

If R is a Noetherian ring then we can do slightly better:

**Theorem 1.1.4.** Let R be a Noetherian ring and \( (P_\lambda)_{\lambda \in \Lambda} \) a family of
prime ideals of $R$ indexed by an infinite set $\Lambda$. Set $I = \bigcap_{\lambda \in \Lambda} P_{\lambda}$ and let $Q_1, \ldots, Q_r$ be the minimal elements of $\text{Ass}_R(R/I)$. Let $p$ be a prime ideal of $R$ such that $p \supseteq \bigcap_{\lambda \in \Lambda} P_{\lambda}$. Then $p \supseteq Q_k$, for some $k = 1, \ldots, r$, and

either (i) $P_{\lambda} \subseteq p$ for some $\lambda \in \Lambda$,

or (ii) $P_{\lambda} \nsubseteq p$ for any $\lambda \in \Lambda$, and $p \supseteq Q_k \Rightarrow Q_k$ is not a $G$-ideal.

**Proof.** Note that $\sqrt{I} = I$, so that $I = \bigcap_{j=1}^r Q_j$. Let $A_j = \{\lambda \in \Lambda : P_{\lambda} \supseteq Q_j\}$ and let $J_j = \bigcap_{\lambda \in A_j} P_{\lambda}$ for each $j = 1, \ldots, r$. Since $P_{\lambda} \supseteq I = \bigcap_{j=1}^r Q_j$, for all $\lambda \in \Lambda$, it follows from 1.0.2 that, for each $\mu \in \Lambda$, $P_{\mu} \supseteq Q_i$, for some $i$ such that $1 \leq i \leq r$, so that, for each $\mu \in \Lambda$, $\mu \in A_i$, for some $i$ such that $1 \leq i \leq r$, and therefore

$$\bigcap_{j=1}^r J_j = \bigcap_{\lambda \in A_j} P_{\lambda} = \bigcap_{j=1}^r Q_j \subseteq Q_k,$$

for each $k = 1, \ldots, r$.

Thus, for each $k = 1, \ldots, r$, $Q_k \supseteq J_j$ for some $j = 1, \ldots, r$ (by 1.0.2). If $k \neq j$ then, by definition of $J_j$, $Q_k \nsubseteq Q_j$,

contradicting the hypothesis that all the $Q$'s are minimal in $\text{Ass}_R(R/I)$. Thus $Q_k \supseteq J_k$, giving
\[ Q_k = J_k, \text{ for all } k \text{ such that } 1 \leq k \leq r. \] (1.1.5)

Now \( p \supseteq I = \bigcap_{j=1}^{r} Q_j \supseteq p \supseteq Q_k, \) where \( 1 \leq k \leq r. \) Let \( k \) be such that \( p \supseteq Q_k. \) Suppose that \( P_\mu \nsubseteq p \) for any \( \mu \in A. \) (Note that if \( |\Lambda_k| < \infty \) then \( p \supseteq P_\mu \) for some \( \mu \in \Lambda_k, \) by (1.1.5) and 1.0.2, so, under the assumption that \( P_\mu \nsubseteq p \) for any \( \mu \in A, \) we must have \( |\Lambda_k| \geq \aleph_0. \)) Then

\[ Q_k = \bigcap_{\lambda \in \Lambda_k} P_\lambda, \] by (1.1.5),

so either \( P_\mu = Q_k, \) for some \( \mu \in \Lambda_k \subseteq A, \) or \( Q_k \) is not a \( G \)-ideal (Theorem 1.1.3 and [Kap], Theorem 27). But if \( P_\mu = Q_k \) then

\( p \supseteq Q_k \supseteq P_\mu, \) a contradiction. Thus we have, as required, that \( Q_k \) is not a \( G \)-ideal. \( \square \)

Corollary 1.1.6. Let \( R \) be a Noetherian ring and \( (P_\lambda)_{\lambda \in \Lambda} \) a family of distinct prime ideals of \( R \) indexed by an infinite set \( \Lambda. \) Let \( p \) be a prime ideal of \( R \) such that \( p \supseteq \bigcap_{\lambda \in \Lambda} P_\lambda. \) Then

either (i) \( p \supseteq P_\mu \) for some \( \mu \in \Lambda, \)

or (ii) given any finite subset, \( \Gamma \subseteq \Lambda, \)

\( p \supseteq \bigcap \{ P_\lambda : \lambda \in \Lambda \setminus \Gamma \}. \)

Proof. With the notation of Theorem 1.1.4 and its proof; suppose that \( P_\mu \nsubseteq p \) for any \( \mu \in \Lambda. \) Then from the proof of the theorem we have \( k \) such that \( 1 \leq k \leq r, \) \( p \supseteq Q_k \) and \( |\Lambda_k| \geq \aleph_0. \) Let \( \Gamma \) be
1.2 Coverings so far

In [McC] McCoy considers the case of an ideal $I$ contained in a union of ideals which are not necessarily prime. McCoy begins with the observation that if $I \subseteq J_1 \cup J_2$ then $I \subseteq J_i$, for $i = 1$ or 2, where $J_1$ and $J_2$ are ideals. That this result does not carry over to a union of three ideals is shown by an example in a ring with trivial multiplication. The following example shows that the same result fails even when $1 \neq 0$ is in the ring. Let $R = \mathbb{Z}_2[X,Y]$, $I = (X,Y)$, $J_1 = (X,XY,Y^2)$, $J_2 = (Y, X^2, XY)$ and $J_3 = (X+Y, X^2, XY, Y^2)$. Then $I \subseteq J_1 \cup J_2 \cup J_3$. In fact, if $a \in I$, a finite subset of $\Lambda$, and let $\Delta_j = \{ \lambda \in \Lambda \setminus \Gamma : P_\lambda \supseteq Q_j \}$, for $j = 1, \ldots, r$. Let $D_j = \bigcap_{\lambda \in \Delta_j} P_\lambda$ (if $\Delta_j = \emptyset$, then $D_j = R$); so

$$\bigcap_{\lambda \in \Delta_j} P_\lambda = \left( \bigcap_{j=1}^r D_j \right) \cap \left( \bigcap_{\lambda \in \Gamma} P_\lambda \right) = \bigcap_{j=1}^r Q_j \subseteq Q_k \subseteq P,$$

by (1.1.5). Thus $p \supseteq D_q$ for some $q$ such that $1 \leq q \leq r$, or $p \supseteq P_\mu$ for some $\mu \in \Gamma$ (by 1.0.2). Thus $p \supseteq D_q$, by the assumption on $p$, and hence $p \supseteq \bigcap_{\lambda \in \Delta} P_\lambda$, where $\Delta = \Lambda \setminus \Gamma$. 

This is all we have to say about problems of type (2) and these results have not proved helpful in solving problems of type (1) discussed below.
1.2 Coverings so far

then \( a = rX + sY \), for some \( r, s \in \mathbb{R} \). Writing \( r = r_0 + r' \) and \( s = s_0 + s' \), where \( r_0, s_0 \in \mathbb{Z}_2 \) and \( r', s' \in \langle X, Y \rangle \), we have \( a = r_0X + r'X + s_0Y + s'Y \). Since \( r'X + s'Y \in \langle X^2, XY, Y^2 \rangle \subseteq J_1 \), for \( 1 \leq i \leq 3 \), it is enough to show that \( r_0X + s_0Y \in J_1 \cup J_2 \cup J_3 \), for all possible choices of \( r_0 \) and \( s_0 \). Letting \( r_0 \) and \( s_0 \) take all values in \( \mathbb{Z}_2 \) we have \( r_0X + s_0Y = X + Y, X, Y \) or 0, which are all in \( J_1 \cup J_2 \cup J_3 \). Thus \( a \in J_1 \cup J_2 \cup J_3 \), and so 
\( I \subseteq J_1 \cup J_2 \cup J_3 \). However \( X \notin J_3 \) and \( X + Y \notin J_1 \cup J_2 \) and these elements are both in \( I \), so that \( I \notin J_1, I \notin J_2 \) and \( I \notin J_3 \).

McCoy finds that if \( I \subseteq J_1 \cup \cdots \cup J_m \) and \( I \) is not contained in the union of any \( m - 1 \) of these ideals, then there exists an integer, \( k \), such that \( I^k \subseteq J_1 \cap \cdots \cap J_m \). As a corollary of this result McCoy obtains a strengthening of 1.0.1; if \( n - 2 \) of the ideals, \( P_1, \ldots, P_n \), are prime, then \( I \subseteq P_i \) for some \( i \), \( 1 \leq i \leq n \). In fact McCoy requires only that \( I \) is an additive subgroup of \( \mathbb{R} \), not necessarily an ideal.

Building on McCoy's work, Davis, Gilmer and McAdam obtain the following results. Assume that \( I \subseteq P_1 + c_1 U \cdots U P_n + c_n \), where \( P_i \) is prime and \( c_i \in \mathbb{R} \), for \( 1 \leq i \leq n \). Then, given the "right conditions", \( (I, c_i) \subseteq P_i \) for some \( i \) such that \( 1 \leq i \leq n \). The "right conditions" are that one of the following is true:

1.2.1. all \( c_i \) are equal (Davis [Kap], Theorem 124);
1.2.2. all \( P_i \) are distinct (Gilmer [Gil]);
1.2.3. \( I + P_i \neq R \) for \( i = 1, 2, \ldots, n \) (McAdam [McA]).

Note: (1) 1.2.2 is a generalisation of 1.2.1.
(2) To see that 1.2.1 can be deduced from [Kap], Theorem 124 and
First note that 1.2.1 says that for some \( i \) such that \( 1 < i < n \), and for any \( c \in \mathbb{R} \),
\[
I \subseteq P_1 + c \cup \cdots \cup P_n + c \Rightarrow (I, c) \subseteq P_i. \tag{*}
\]
Given 1.2.1, suppose that
\[
(I, c) = (I, (- c)) \not\subseteq P_1 \cup \cdots \cup P_n; \text{ then}
I \not\subseteq (P_1 - c) \cup \cdots \cup (P_n - c) \quad \text{(by (*))},
\]
so
\[
\exists x \in I \text{ such that } x \not\in (P_1 - c) \cup \cdots \cup (P_n - c). \quad \text{Hence}
\]
x + c \not\in P_1 \cup \cdots \cup P_n, \text{ which gives [Kap], Theorem 124.}

Conversely given [Kap], Theorem 124, suppose we have
\[
I \subseteq P_1 + c \cup \cdots \cup P_n + c.
\]
Assume there is \( x \in I \) such that \( x - c \notin P_1 \cup \cdots \cup P_n \). Since
\( x \in I \), \( x = y + c \) where \( y \in P_i \) for some \( i \) such that \( 1 \leq i \leq n \).
Thus \( x - c = y \in P_i \), which is a contradiction. Therefore we have
\[
(I, c) = (I, (- c)) \subseteq P_1 \cup \cdots \cup P_n \quad \text{([Kap], Theorem 124), so}
(I, c) \subseteq P_i \text{ for some } i, 1 \leq i \leq n \quad \text{(by 1.0.1)}.
\]

(3) A weaker version of [Kap], Theorem 124 appears as a lemma in the appendix of [Dav].

Finally (so far) McAdam [McA] combines (1.2.1), (1.2.2) and (1.2.3) and, as well, provides some answers to (more general versions of) McCoy's original question about non-prime covers. The main theorem in this paper is the following:

**Theorem 1.2.4.** [McA] Let \( J_1, \ldots, J_n \) be, not necessarily distinct, ideals of \( R \) and let \( c_1, \ldots, c_n \) be elements of \( R \). If \( I \) is an
ideal such that \( I \subseteq J_1 + c_1 \cup \cdots \cup J_n + c_n \) and if any of the following conditions hold, then for some \( i \) such that \( 1 \leq i \leq n \), \((I, c_i) \subseteq J_i\).

(a) For each \( j = 1, \ldots, n \) and each maximal prime, \( P \), of \( J_j \), \( R/P \) is infinite;

(b) \( R/M \) is infinite for all maximal ideals \( M \) of \( R \);

(c) For each \( j = 1, \ldots, n \) and each maximal prime, \( P \), of \( J_j \), \( P \) is not a maximal ideal of \( R \);

(d) For each \( j = 1, \ldots, n \) and each maximal prime, \( P \), of \( J_j \), \( P \) is not a maximal prime of \( J_k \) for any \( k \neq j \);

(e) For all \( k \neq j \), \( J_k \subseteq Z(R/J_j) \).

The ideas and arguments of McAdam's proof of Theorem 1.2.4 will arise in the following sections and so are not given here. The same goes for the consequences of the theorem, which comprise all the above results of Type (1). In Sections 1.3 and 1.4 we aim to provide non-finite analogues of Theorem 1.2.4. We now state known non-finite analogues of results of Type (1). Burch [Bur], and Sharp and Vámos [ShVá] have shown that under certain circumstances many of the above results extend to covers of ideals by countable unions (of ideals).

**Theorem 1.2.5.** [ShVá] Let \( R \) be a complete Noetherian local ring. Let \{\( P_i : i = 1, 2, \ldots \)\} be a countable family of prime ideals of \( R \), let \( I \) be an ideal of \( R \) and let \( x \in R \) be such that \( I + x \subseteq U \{P_i : i = 1, 2, \ldots \} \). Then \( I + x \subseteq P_j \) for some \( j \geq 1 \). □
1.2 Coverings so far

We omit the proof since similar arguments are used in a later section.

Note: Burch [Bur] proved (a weaker version of) Theorem 1.2.5 by construction of a Cauchy sequence. Sharp and Vámos subsequently proved the (full) theorem using Baire's Category Theorem. Burch's original proof does however go through to give the full result.

Theorem 1.2.6. [ShVá] Let $R$ be a ring and assume that there exists an uncountable family $\{u_\lambda : \lambda \in \Lambda\}$ of elements of $R$ such that $u_\lambda - u_\mu$ is a unit of $R$ whenever $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$. Let $I$ be a finitely generated ideal of $R$ and let $\{J_i : i = 1, 2, \ldots\}$ be a countable family of ideals of $R$ such that $I \subseteq \bigcup \{J_i : i = 1, 2, \ldots\}$. Then $I \subseteq J_r$ for some $r \geq 1$.

We reproduce the proof of Theorem 1.2.6 for comparison with the proof of Theorem 1.3.5.

Proof. Let $x_1, \ldots, x_k \in R$ generate $I$. For each $\lambda \in \Lambda$, set

$$y_\lambda = x_1 + u_\lambda x_2 + \ldots + u_\lambda^{k-1} x_k, \quad \in I.$$

Now there is an uncountable subset $\Lambda'$ of $\Lambda$ and an integer $r \geq 1$ such that $y_\lambda \in J_r$ for all $\lambda \in \Lambda'$. In particular, there exist $k$ different members $\lambda_1, \ldots, \lambda_k$ of $\Lambda$ such that $y_{\lambda_1}, \ldots, y_{\lambda_k} \in J_r$.

The $k \times k$ matrix $H = (u_{\lambda_i}^{j-1})$ has determinant which is a unit of $R$. Since
In this section we consider extensions of results of type (1) in rings containing a "large" set of elements with differences which lie "well outside" the ideals we are considering. This provides us with an analogue to, at least, the second condition of Theorem 1.2.4.

The next two preparatory lemmas enable us to prove the main results of both the present section and Section 1.4. The first of these is due to McAdam ([McA] Lemma 4) and the second is based on an idea in the proof of [McA] Theorem 5.

**Lemma 1.3.1.** [McA] Let \( R \) be a ring and \( I \) an ideal of \( R \) and let \( a \) and \( b \) be elements of \( R \). Then

\[
(I + a :_R b) = (I :_R b) + s,
\]

where \( s \) is any element of \( (I + a :_R b) \).
Proof. Let \( r \in (I :_R b) \) and fix \( s \in (I + a :_R b) \). Then
\[
rb + sb \in I + (I + a) = I + a,
\]
so \( r + s \in (I + a :_R b), \forall r \in (I :_R b), \)
therefore \( (I :_R b) + s \subseteq (I + a :_R b). \)

Conversely if \( t \in (I + a :_R b) \) then
\[
(t - s)b \in I,
\]
which implies \( t - s \in (I :_R b), \)
so that \( t \in (I :_R b) + s, \)
hence \( (I + a :_R b) \subseteq (I :_R b) + s. \) \( \square \)

Lemma 1.3.2. Let \( R \) be a ring and \( I \) a finitely generated ideal of \( R \).

Let \( (J_\lambda)_{\lambda \in \Lambda} \) be a family of, not necessarily distinct, ideals of \( R \),
and \( (c_\lambda)_{\lambda \in \Lambda} \) a family of elements of \( R \), both indexed by a set \( \Lambda \).

Suppose that \( I \subseteq \bigcup (J_\lambda + c_\lambda) \) and \( I \notin J_\mu \) for any \( \mu \in \Lambda \). Then, for each \( \lambda \in \Lambda \), there is a maximal prime \( P_\lambda \) of \( J_\lambda \) and an element \( s_\lambda \in R \), such that
\[
R \subseteq \bigcup_{\lambda \in \Lambda} (P_\lambda + s_\lambda).
\]

Proof. Step (1): Suppose firstly that \( I \subseteq \bigcup J_\lambda \). Choose \( x \in I \)
\( \lambda \in \Lambda \)
such that \( x \notin \bigcup J_\lambda \). For each \( r \in R, \lambda \in \Lambda \)
\[
rx \in I \subseteq \bigcup (J_\lambda + c_\lambda), \quad \lambda \in \Lambda
\]
so \( r \in \bigcup_{\lambda \in \Lambda} ((J_{\lambda} + c_{\lambda}) : R x) = \bigcup_{\lambda \in \Lambda} ((J_{\lambda} : R x) + s_{\lambda}) \),

where \( s_{\lambda} \in R \), for all \( \lambda \in \Lambda \) (Lemma 1.3.1).

Since \( x \notin J_{\mu} \), \( (J_{\mu} : R x) \) can be expanded to a maximal prime \( P_{\mu} \) of \( J_{\mu} \) for each \( \mu \in \Lambda \). Thus \( R \subseteq \bigcup_{\lambda \in \Lambda} (P_{\lambda} + s_{\lambda}) \) as required.

Step (2): In the light of Step (1) we may assume that

\[ I \subseteq \bigcup_{\lambda \in \Lambda} J_{\lambda}. \]

The ideal \( I \) must be generated by at least two elements, otherwise the current assumption leads to a contradiction. We use induction on \( n \), the number of generators of \( I \), starting with the case \( n = 2 \). The inductive hypothesis is the following: given an ideal \( L \), generated by fewer than \( n \) elements, a family \( (A_{\nu})_{\nu \in \Omega} \) of, not necessarily distinct, ideals of \( R \) and a family \( (b_{\nu})_{\nu \in \Omega} \) of elements of \( R \), both indexed by a set \( \Omega \), such that \( L \subseteq \bigcup_{\nu \in \Omega} (A_{\nu} + b_{\nu}) \) and \( L \nsubseteq A_{\omega} \), for any \( \omega \in \Omega \); then \( R \subseteq \bigcup_{\nu \in \Omega} (Q_{\nu} + t_{\nu}) \), where \( Q_{\omega} \) is some maximal prime of \( A_{\omega} \) and \( t_{\omega} \in R \), for each \( \omega \in \Omega \). Suppose that \( I \) is generated by \( n \geq 2 \) elements \( x_1, \ldots, x_n \). For \( j = 1, 2, \ldots, n \) let \( a_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \). If there is some \( j \) such that \( 1 \leq j \leq n \) and \( a_j \nsubseteq J_{\lambda} \) for any \( \lambda \in \Lambda \), then the result follows from the inductive hypothesis. In fact, we have

\[ a_j \subseteq I \subseteq \bigcup_{\lambda \in \Lambda} (J_{\lambda} + c_{\lambda}), \quad a_j \notin J_{\mu} \text{ for any } \mu \in \Lambda \text{ and } a_j \text{ generated} \]
by fewer than \( n \) elements. On the other hand suppose that, for every 
j such that \( 1 \leq j \leq n \), \( a_j \subseteq J_\mu \) for some \( \mu \in \Lambda \). Then

\[
a_n - x_n \subseteq \bigcup_{\lambda \in \Lambda} J_\lambda.
\]

For each \( y \in a_n \) choose \( \lambda(y) \in \Lambda \) such that \( y - x_n \in J_\lambda(y) \). Let \( \Gamma = \{ \mu \in \Lambda : \mu = \lambda(y) \text{ for some } y \in a_n \} \). If \( a_n \subseteq J_\mu \) for any \( \mu \in \Gamma \) then \( y - x_n \in J_\mu \), for some \( y \in a_n \), so \( x_n \in J_\mu \) and this gives \( I = (a_n, x_n) \subseteq J_\mu \), a contradiction. Thus \( a_n \not\subseteq J_\mu \), for any \( \mu \in \Gamma \). For all \( y \in a_n \), there is \( \mu \in \Gamma \) such that \( y - x_n \in J_\mu \), and therefore we have

\[
a_n - x_n \subseteq \bigcup_{\lambda \in \Gamma} J_\lambda,
\]

and so

\[
a_n \subseteq \bigcup_{\lambda \in \Gamma} (J_\lambda + x_n).
\]

Since \( a_n \not\subseteq J_\mu \) for any \( \mu \in \Gamma \), the result follows by induction. \( \square \)

We record the following observation as a lemma for future reference:

**Lemma 1.3.3.** Let \( R \) be a ring, \( I \) and \( J \) ideals of \( R \) and \( c \) an element of \( R \). Then \( (I, c) \subseteq J \) if and only if \( I \subseteq J \) and \( I \cap (J + c) \neq \emptyset \).

**Proof.** If \( (I, c) \subseteq J \) then \( -c \in J \), so \( 0 \in I \cap (J + c) \neq \emptyset \).

Conversely, if \( i \in I \cap (J + c) \) then \( i - j = c \) for some \( j \in J \); if also \( I \subseteq J \) then \( c \in J \), so \( (I, c) \subseteq J \). \( \square \)
1.3 Infinite covers and difference sets

The next definition is motivated by the hypothesis of Theorem 1.2.6:

Definition 1.3.4. Let $R$ be a ring, $X$ a subset of $R$, and $D$ a subset of $R$, of cardinality $\beta$. Suppose that whenever $a, b \in D$ and $a \neq b$, $a - b$ is (is not) in $X$. Then we say that $R$ has $\beta$-differences in ($outside$) $X$ and that $D$ is an $\beta$-difference set in ($outside$) $X$.

Theorem 1.3.5. Let $R$ be a ring, $(J_\lambda)_{\lambda \in \Lambda}$ a family of ideals and $(c_\lambda)_{\lambda \in \Lambda}$ a family of elements of $R$, both indexed by a set $\Lambda$ of cardinality $\alpha$. Let $\beta$ be a cardinal such that $\beta > \alpha$. Assume that for any family $(P_\lambda)_{\lambda \in \Lambda}$ such that, for each $\mu \in \Lambda$, $P_\mu \in \{P : P$ is a maximal prime of $J_\mu \}$. $R$ has a $\beta$-difference set $D$ outside $\bigcup_{\lambda \in \Lambda} P_\lambda$.

Let $I$ be a finitely generated ideal of $R$ such that

$I \subseteq \bigcup_{\lambda \in \Lambda} (J_\lambda + c_\lambda)$.

Then

$(I, c_\mu) \subseteq J_\mu$ for some $\mu \in \Lambda$.

Proof. First we may drop all cosets $J_\lambda + c_\lambda$ such that

$I \cap (J_\lambda + c_\lambda) = \emptyset$, without affecting the hypotheses of the theorem.

Then, by Lemma 1.3.3, it is enough to show that $I \subseteq J_\mu$ for some $\mu \in \Lambda$.

Now if $I \not\subseteq J_\mu$ for any $\mu \in \Lambda$ then from Lemma 1.3.2 we have

$R \subseteq \bigcup_{\lambda \in \Lambda} (P_\lambda + s_\lambda)$,

where for each $\mu \in \Lambda$, $P_\mu$ is a maximal prime of $J_\mu$ and $s_\mu \in R$. As $|D| = \beta > \alpha = |\Lambda|$ there is $\mu \in \Lambda$ such that $P_\mu + s_\mu$ contains at
least one pair $a, b \in D$ such that $a \neq b$. Then $a - b \in P_\mu$, which
is a contradiction since $a - b \notin \bigcup_{\lambda \in A} P_\lambda$. \(\square\)

Note that the possibility that $\beta$ is finite is not excluded in the above theorem.

**Examples. 1.3.6.** Let $R$ be an algebra over a field $k$ of cardinality $\beta \geq \aleph_0$. The non-zero elements of $k$ form a $\beta$-difference set outside any union of proper ideals of $R$.

1.3.7. Let $R$ be an algebra over a ring $S$ of cardinality $\beta \geq \aleph_0$, such that $R = S[\bar{X}]$ for some set of indeterminates $\bar{X} \subseteq R$. Let

\[\mathcal{M} = \{J : J \text{ is an ideal of } R; J \subseteq \bar{X}R\}.\]

Then, since $S \cap \bar{X}R = \{0\}$, $S \setminus \{0\}$ is an $\beta$-difference set outside any union of primes in $\mathcal{M}$.

1.3.8. Let $R$ be a ring, $\beta$ an infinite cardinal and $I_1, \ldots, I_r$ pairwise coprime ideals of $R$, such that $R/I_i$ has cardinality at least $\beta$, for $i = 1, \ldots, r$. Let $J = I_1 \cap I_2 \cdots \cap I_r = \prod_{i=1}^{r} I_i$. Then

\[R/J \cong R/I_1 \oplus \cdots \oplus R/I_r,\]

and the diagonal map $\phi : R \to R/I_1 \oplus \cdots \oplus R/I_r$ is surjective.

Let $\Lambda$ be a set of cardinality $\beta$, and for $i = 1, \ldots, r$ let

\[\left( u^i_\lambda \right)_{\lambda \in \Lambda} \]

be a set of distinct non-zero elements of $R/I_i$, indexed by
1.8 Infinite covers and difference sets

A. Let

\[ D = \{ u_\lambda = (u_1^\lambda, \ldots, u_r^\lambda) \in R/I_1 \oplus \ldots \oplus R/I_r : \lambda \in \Lambda \}. \]

Then, if \( \lambda, \mu \in \Lambda \) such that \( \lambda \neq \mu \), we have

\[ u_\lambda - u_\mu = (u_1^\lambda - u_1^\mu, \ldots, u_r^\lambda - u_r^\mu), \]

and \( u_1^\lambda - u_1^\mu \neq 0 \) for \( i = 1, \ldots, r \). (We set \((x_1^i, \ldots, x_r^i)\).) Since \( \phi \) is surjective we may choose a transversal \( E \) of \( D \) in \( R \). Then \(|E| = \beta\), and for any two distinct elements \( a \) and \( b \) of \( E \), we have distinct elements \( p, \tau \in \Lambda \) such that \( \phi(a) = u_p \) and \( \phi(b) = u_\tau \). Suppose \( a - b \in I_j \), for some \( j = 1, \ldots, r \). Then

\[ \phi(a - b) = u_p - u_\tau = (u_1^p - u_1^\tau, \ldots, u_r^p - u_r^\tau) \in \phi(I_j), \]

so \( u_1^p - u_1^\tau = 0 \), a contradiction. Thus \( a - b \notin I_j \), for any \( j = 1, \ldots, r \). It follows that \( E \) is a \( \beta \)-difference set outside \( \bigcup_{i=1}^r I_i \).

1.3.9. Let \( R \) be a semi-local ring with maximal ideals \( m_1, \ldots, m_r \) such that \( R/m_i \) has cardinality at least \( \beta \), for \( i = 1, \ldots, r \) and \( \beta \) an infinite cardinal. This is a particular case of example 1.3.8, in which the \( \beta \)-difference set \( E \) is outside any union of proper ideals.
1.4 Infinite covers in complete semi-local Noetherian rings

We shall now follow the path taken by McAdam in proving Theorem 1.2.4 and, using the methods of Sharp and Vámos, find an infinite analogue of this theorem. Throughout this section $R$ will be a semi-local Noetherian ring complete in the Jacobson radical topology. We first give a description of this topology in so far as we require it. For a fuller treatment see, for example, [ZS2], Chapter VIII.

A ring, $R$, is a topological ring, with respect to a given topology, if the ring operations, $R \times R \to R : (r,s) \to r - s$, and $(r,s) \to rs$, are continuous. A topological ring is a homogeneous space, that is, if $X \subseteq R$ is an open (closed) set then for any $x \in R$ the subset $X + x$ is also open (closed) in $R$. This follows since the map $R \to R : a \to a + x$ is a homeomorphism for all $x \in R$. Let $R$ be a topological ring and let $\Sigma$ be a collection of open sets of $R$ which contain 0. If every open set in $R$ contains a set in $\Sigma$ then we say that $\Sigma$ is a basis of neighbourhoods of 0 for $R$. If $\Sigma$ is a basis of neighbourhoods of 0 then the sets of the form $x + U$, where $x \in R$ and $U \in \Sigma$, form a basis for the open sets of the topological ring $R$. Let $R$ be a ring and let $(B_\lambda)_{\lambda \in \Lambda}$ be a family of ideals of $R$, indexed by a directed set $\Lambda$, such that if $\lambda, \mu \in \Lambda$ and $\lambda < \mu$ then $B_\lambda \supseteq B_\mu$. Then there exists a unique topology on $R$ such that $(B_\lambda)_{\lambda \in \Lambda}$ is a basis of neighbourhoods for 0, and $R$ is a topological ring. Furthermore it is easy to see that $R$ is Hausdorff, with respect to this topology, iff $\bigcap_{\lambda \in \Lambda} B_\lambda = (0)$. 
Let I be an ideal of R. Taking $\Lambda = \{0, 1, 2, \ldots \}$ and $B_n = I^n$, we define the $I$-adic topology to be that having $(B_n)_{n \in \Lambda}$ as a basis of neighbourhoods for 0. If $S$ is any subset of R let $\bar{S}$ denote the closure of $S$ in the I-adic topology. Then

$x \in \bar{S} \iff (x + I^n) \cap S \neq \emptyset$ for all $n \geq 0$,

$\iff x \in S + I^n$ for all $n \geq 0$,

$\iff x \in \bigcap_{n=0}^{\infty} (S + I^n)$.

Hence $\bar{S} = \bigcap_{n=0}^{\infty} (S + I^n)$.

Now suppose that R is a Noetherian ring and that I is contained in the Jacobson radical of R. Let $a$ be any ideal of R, $a \neq R$. Then $\bar{a} = \bigcap_{n=0}^{\infty} (a + I^n)$, and we have,

$\left( \bigcap_{n=0}^{\infty} (a + I^n) \right)/a = \bigcap_{n=0}^{\infty} ((a + I^n)/a) = \bigcap_{n=0}^{\infty} (I^n(R/a))$.

Let $C = \bigcap_{n=0}^{\infty} (I^n(R/a))$; then C is a submodule of $R/a$ and so is finitely generated as an R-module. Therefore $C = 0$, by the Krull Intersection theorem and Nakayama's lemma ([Kap], Theorems 74 and 78). Thus $\bar{a} = \bigcap_{n=0}^{\infty} (a + I^n) = a$, that is, all ideals of R are closed in the I-adic topology.

If $a$ is any ideal of R then $a$ is open in the I-adic topology iff $a \supseteq I^n$ for some $n \geq 0$. In fact if $a \supseteq I^n$ then we have $x \in a \implies x + I^n \subseteq a$, so $a$ is open. Conversely if $a$ is open then,
since \(0 \in a\), we have \(I^n \subseteq a\) for some \(n \geq 0\).

For \(x \in R\) define \(\nu(x) = \max \{n \in \mathbb{Z} : x \in I^n\}\) if \(x \notin \bigcap_{n=0}^{\infty} (I^n)\), and \(\nu(x) = \infty\) if \(x \in \bigcap_{n=0}^{\infty} (I^n)\). Then for any \(x, y \in R\) we can define the distance
\[
d(x, y) = e^{-\nu(x-y)} , \quad \text{where} \quad e \in \mathbb{R}, \quad e > 1.
\]

If \(\bigcap_{n=0}^{\infty} (I^n) = (0)\) then \(d\) defines a metric on \(R\), and it is clear that the topology induced by this metric coincides with the \(I\)-adic topology. Thus when \(R\) is Noetherian and \(I\) is contained in the Jacobson radical, \(R\) is a metric space with the \(I\)-adic topology and, for instance, we may consider its completion with respect to this topology.

Throughout this section \(\mathcal{J}\) denotes the Jacobson radical of \(R\). If \(R\) is complete with respect to the \(\mathcal{J}\)-adic topology we say, merely, that \(R\) is complete. The \(I\)-adic topology is defined similarly for a general \(R\)-module. We shall sometimes refer to the \(I\)-adic topology simply as the \(I\)-topology.

The idea of Sharp and Vámos, which we shall use, is to apply Baire's Category theorem, which we now state, to \(R\) as a metric space with the \(I\)-adic topology:

**Baire's Category Theorem.** If \(X\) is a complete metric space and \(\{A_i : i = 1,2,\ldots\}\) is a family of nowhere dense subsets of \(X\),
\[
\text{then} \quad X \nsubseteq \bigcup_{i=1}^{\infty} A_i. \tag*{\square}
\]
This will be used in the following lemma, which is required to prove the main result of this section.

Lemma 1.4.1. Let $R$ be a complete semi-local Noetherian ring, let $I$ be an ideal of $R$ and let $\{P_i : i \in \mathcal{N}\}$ be a family of distinct primes of $R$, where $\mathcal{N} = \mathbb{N}$, or $\mathcal{N} = \{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$. Let $\{c_{i,g} : g = 1, 2, \ldots \}$ be a family of elements of $R$ for every $i \in \mathcal{N}$. Suppose that $I \not\subseteq P_j$ for any $j \in \mathcal{N}$. If

\[ I \subseteq \bigcup_{i \in \mathcal{N}} \bigcup_{g=1}^{\infty} (P_i + c_{i,g}), \]

then there is a finite subset $\mathcal{K}$, of $\mathcal{N}$, such that

\[ I \subseteq \bigcup_{k \in \mathcal{K}} \bigcup_{g=1}^{\infty} (P_k + c_{k,g}). \]

Furthermore $\mathcal{K}$ can be chosen such that, for each $k \in \mathcal{K}$, $P_k$ is a maximal ideal of $R$, $R/P_k$ is a countable field and there is at least one pair of integers, $u, v$, such that $1 < u < v$ and $c_{k,u} \neq c_{k,v}$.

**Proof.** Suppose that $I \not\subseteq \bigcup_{k \in \mathcal{K}} \bigcup_{g=1}^{\infty} (P_k + c_{k,g})$, for any finite subset, $\mathcal{K}$, of $\mathcal{N}$ such that

$\{P_k : k \in \mathcal{K}\} \subseteq \{P : P = P_i \text{ for some } i = 1, 2, \ldots\} \cap \text{Max-Spec}(R)$.

Let $s \in \mathbb{Z}$ such that $s > 0$. Assume inductively that maximal ideals $P_{i_1}, \ldots, P_{i_{s-1}}$ have been found in $\{P_i : i = 1, 2, \ldots\}$. 

To simplify notation we renumber, setting $P'_k = P_i$ for $1 \leq k \leq s - 1$, $P'_{j+s-1} = P_j$ for $1 < j < t$, and $P_{j+t}, \ldots, P_{j+s-1}$.

Then dropping ', thus we have maximal ideals $P_1, \ldots, P_{s-1}$. Let

$$D = \bigcup_{k=1}^{s-1} \bigcup_{g=1}^{\infty} (P_k + c_k, g);$$

then by assumption, $I \cap D^c \neq \emptyset$

(where $S^c$ denotes the complement of $S$ in $R$, for any subset $S$ of $R$).

Since $P_k$ is maximal, $P_k$ contains $J$ and is therefore open (in the $J$-adic topology), for $k = 1, \ldots, s-1$. Thus $P_k + c_k, f$ is open, for each $k$ and for all $f \geq 1$, and so $D$ is open (as a subset of $R$ in the $J$-adic topology). Since $I$ is an ideal of $R$, $I$ is closed and we have $I \cap D^c$ closed and non-empty in $R$. Thus $I \cap D^c$ is a complete metric space, with the induced metric. Now, setting $\mathcal{L} = \mathcal{M} \setminus \{1, \ldots, s-1\}$, from

$$I \subseteq \bigcup_{i \in \mathcal{L}} \bigcup_{g=1}^{\infty} (P_i + c_i, g),$$

we have

$$I \cap D^c \subseteq \bigcup_{i \in \mathcal{L}} \bigcup_{g=1}^{\infty} (P_i + c_i, g)$$

(by definition of $D$), giving

$$I \cap D^c = \bigcup_{i \in \mathcal{L}} \bigcup_{g=1}^{\infty} ((P_i + c_i, g) \cap I \cap D^c).$$

Applying Baire's Category theorem, we can choose $j \in \mathcal{L}$, and $d \geq 1$, such that $(P_j + c_j, d) \cap I \cap D^c$ is not nowhere dense in $I \cap D^c$. Let $z$ be an interior point of the closure of $(P_j + c_j, d) \cap I \cap D^c$ (which is already closed) in $I \cap D^c$; then
there is \( t \in \mathbb{Z}, \ t \geq 1, \) and \( r \in R \) such that
\[
z \in (I \cap D^c) \cap (r + J^t) \subseteq (P_j + c_j, d) \cap I \cap D^c.
\]

Let \( x \in I \cap J^t; \) then \( x \in P_k \) for \( 1 \leq k \leq s-1, \) since each of these ideals is maximal. If \( z + x \in P_k + c_k, f \) for any \( 1 \leq k \leq s-1 \) and \( f \geq 1, \) then \( z \in D, \) a contradiction. Thus
\[
z + x \in (I \cap D^c) \cap (r + J^t) \subseteq (P_j + c_j, d) \cap I \cap D^c,
\]
so that
\[
z + x \in (P_j + c_j, d),
\]
and thus \( I \cap J^t \subseteq P_j, \) since \( z \in P_j + c_j, d. \)

By hypothesis \( I \not\subseteq P_j, \) so \( J^t \not\subseteq P_j \) (by 1.0.2), and since \( P_j \) is prime and \( R \) is semi-local we have \( P_j \supseteq m, \) for some maximal ideal \( m \) of \( R. \) By induction there are infinitely many maximal ideals of \( R \) in \( \{P_i : i = 1, 2, \ldots \}, \) contradicting the semi-locality of \( R. \)

Therefore there is a finite subset \( \mathcal{A}, \) of \( \mathfrak{M}, \) such that
\[
I \subseteq \bigcup_{k \in \mathcal{A}} \bigcup_{g=1}^{\infty} (P_k + c_k, g), \tag{*}
\]
and \( P_k \) is a maximal ideal of \( R, \) for all \( k \in \mathcal{A}. \) We may now assume that, given \( s \in \mathcal{A}, \)
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\[ I \cap \left( \bigcup_{g=1}^{\infty} (P_s + c_{s,g}) \right) \]

\[ \subseteq I \cap \left( \bigcup_{g=1}^{\infty} \bigcup_{s \neq k} (P_k + c_{k,g}) \right). \]  \hspace{1cm} (1.4.2)

(Otherwise we may drop all cosets of \( P_s \) and still retain a cover of \( I \).)

Fix \( s \in \mathfrak{R} \) and choose any element \( a \) in the left hand side of 1.4.2 which is not in the right hand side. Let \( J = I \cap (\bigcap_{s \neq k} P_k) \). Let \( y \in J \); then \( y \equiv 0 \pmod{P_j} \) for any \( j \neq s, j \in \mathfrak{R} \). Thus \( a + y \equiv a \not\equiv c_{j,f} \pmod{P_j} \) for any \( f \geq 1 \) and \( j \in \mathfrak{R}, j \neq s \). (If \( a \equiv c_{j,f} \) then \( a \in P_j + c_{j,f} \)). Since \( a + y \in I \),

\[ a + y \in \bigcup_{g=1}^{\infty} P_s + c_{s,g} \] \hspace{1cm} (using (*)),

and so \( y \in \bigcup_{g=1}^{\infty} P_s + (c_{s,g} - a) \).

Hence \( J \) is covered by a countable union of cosets of \( P_s \).

\( P_s \) cannot contain \( J \) since \( I \notin P_s \), all the \( P_i \) are distinct and \( P_k \) is maximal for all \( k \in \mathfrak{R} \). Therefore, since \( P_s \) is maximal, it follows that \( P_s + J = R \) and \( (P_s + J)/P_s = R/P_s \). Since \( J \) is covered by a countable union of cosets of \( P_s \), we have a countable field \( R/P_s \).
Finally, if \( c = c_s,1 = c_s,2 = \ldots \), then we have

\[ J \subseteq (P_s + (c - a)), \]

where \( a \) is in the left hand side of 1.4.2. This implies that \( J \subseteq P_s \), which cannot happen. Thus there are positive integers, \( u, v \in \mathbb{Z} \), such that \( c_s,u \neq c_s,v \). The result follows since \( s \) was arbitrary in \( \mathbb{R} \).

**Remark.** From this lemma we can obtain countable analogues of all the usual prime avoidance theorems; (that is, results of type (1) proved by Davis, Gilmer and McAdam; see Section 1.2.) However we defer these until we have the main theorem of this section on "not necessarily prime" ideals.

**Theorem 1.4.3.** Let \( R \) be a complete semi-local Noetherian ring. Let \{\( J_i \) : \( i = 1,2,\ldots \)\} be a countable family of, not necessarily distinct, ideals of \( R \), and \{\( c_i \) : \( i = 1,2,\ldots \)\} a countable family of elements of \( R \). If \( I \) is an ideal of \( R \) such that

\[ I \subseteq \bigcup_{i=1}^{\infty} (J_i + c_i), \]

and any of the following conditions (a) through (f) hold, then

\[ (I, c_k) \subseteq J_k, \text{ for some } k \geq 1. \]

(a) For each \( k \geq 1 \) and for every maximal prime, \( P \), of \( J_k \), \( R/P \) is uncountable;

(b) \( R/m \) is uncountable, for all maximal ideals, \( m \), of \( R \);

(c) For each \( k \geq 1 \) and for every maximal prime, \( P \), of...
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\( J_k, \ P \notin \text{Max-Spec}(R); \)

(d) For each \( j,k \geq 1 \) such that \( j \neq k, \)
\( \{P : P \text{ is a maximal prime of } J_j\} \cap \{P : P \text{ is a maximal prime of } J_k\} = \emptyset; \)

(e) For each \( j,k \geq 1 \) such that \( j \neq k, \)
\( \text{Ass}_R(R/J_j) \cap \text{Ass}_R(R/J_k) = \emptyset; \)

(f) For each \( j,k \geq 1, \) if \( j \neq k \) then \( J_k \subseteq Z(R/J_j). \)

Proof. We may assume, without loss of generality, that, for each \( k \geq 1, \) \( I \cap (J_k + c_k) \neq \emptyset. \) Then by Lemma 1.3.3 it is enough to show that, for some \( k \geq 1, \) we have \( I \subseteq J_k. \)

Assume that \( I \nsubseteq J_k \) for any \( k \geq 1. \) Then, using Lemma 1.3.2,
\[ R \subseteq \bigcup_{i=1}^{\infty} (P_i + s_i), \]
where, for each \( k \geq 1, \) \( P_k \) is a maximal prime of \( J_k \) and \( s_k \in R. \)

Collecting together all like primes in the above union we have, with suitable renaming,
\[ R \subseteq \bigcup_{d \in \mathcal{D}} \bigcup_{g=1}^{\infty} (Q_d + t_{d,g}), \quad (1.4.4) \]
where each \( Q_d \) is a maximal prime of some \( J_k, \) \( \mathcal{D} \) is a countable set, \( t_{d,g} \in R, \ \forall \ g \geq 1, \) and if \( e \neq d \) then \( Q_e \neq Q_d. \) To be precise:

let \( \mathbb{N} = \{Q \in \text{Spec}(R) : \exists \ k \geq 1 \text{ such that } Q = P_k\}; \) then \( \mathbb{N} \)
is denumerable, and so may be written as \( \mathbb{N} = \{Q_d : d \in \mathcal{D} \}, \) for some countable (possibly finite) set \( \mathcal{D}. \) Let \( N = \{s_i \in R : i = 1, 2, \ldots \} \) be the family found above, using

Lemma 1.3.2. Given any \( d \in \mathcal{D} \) we have a countable set
\[ \Sigma_d = \{ t \in R : \exists m \geq 1 \text{ such that } t = s_m \text{ and } Q_d = P_m \}, \]

which may be written as \(\{ t_{d,g} : g = 1, 2, \cdots \}\). (If \(|\Sigma_d| = n < \infty \) then set \( t_{d,h} = t_{d,n} \).) Now

\[
\bigcup_{i=1}^{\infty} (P_i + s_i) = \bigcup_{d \in \mathcal{D}, g=1}^{\infty} (Q_d + t_{d,g}),
\]

and the \(Q_d\) are all distinct, as required.

Using Lemma 1.4.1, the cover of \(R\) given by 1.4.4 reduces to

\[
R \subseteq \bigcup_{f \in \mathcal{F}} \bigcup_{g=1}^{\infty} (Q_f + t_{f,g}),
\]

where \(\mathcal{F}\) is finite, \(Q_f\) is a maximal ideal of \(R\), \(R/Q_f\) is a countable field and there are integers, \(u, v\) such that \(t_{f,u} \neq t_{f,v}\) for each \(f \in \mathcal{F}\). Since each \(Q_f\) is a maximal prime of \(J_k\) for some \(k \geq 1\), (a) and (c) certainly give contradictions.

Furthermore (b) \(\rightarrow\) (a), so (b) gives a contradiction. Given (d) then each set \(\Sigma_d = \{ s_m \}\) for a unique integer \(m\). Hence, for all \(g \geq 1\), \(t_{d,g} = t_{d,1}\) for every \(d \in \mathcal{D}\), and in particular for each \(d \in \mathcal{D}\), again a contradiction. Obviously (e) \(\rightarrow\) (d). Finally, given (f), suppose \(P\) is a maximal prime of both \(J_i\) and \(J_k\), where \(i, k \geq 1\), and \(i \neq k\). Then

\[
J_k \subseteq P \subseteq U (P : P \text{ is a maximal prime of } J_i) = Z(R/J_i),
\]

a contradiction, so (f) \(\rightarrow\) (d). Thus, given any of conditions (a) to (f), the assumption that \(I \notin J_k\) for any \(k \geq 1\) leads to a contradiction. \(\square\)

Note 1.4.5. Condition (a) is satisfied if, for some cardinal
\[ \beta > \aleph_0, \quad R \text{ has } \beta \text{-differences outside } \bigcup_{i=1}^{\infty} P_i \text{ for all choices of maximal ideals } P_i \text{ of } J_i. \]

In fact in this case, let \( D \) be a \( \beta \)-difference set outside \( \bigcup_{i=1}^{\infty} P_i \). Given \( a, b \in D, \ a \neq b, \) then

\[ a - b \notin P_j, \text{ so that } a + P_j \neq b + P_j, \] and

\[ |R/P_j| \geq |(D + P_j)/P_j| \geq \beta > \aleph_0. \]

Furthermore condition (b) is equivalent to the condition that \( R \) has an \( \beta \)-difference set outside \( U \{ m : m \in \text{Max-Spec}(R) \} \), for some cardinal \( \beta > \aleph_0 \). This follows from the above argument and Example 1.3.9. Therefore, in the presence of condition (b), we do not require \( R \) to be complete.

**Corollary 1.4.6.** (cf. 1.2.2) Let \( R \) be a complete semi-local Noetherian ring. Let \( \{ P_i : i = 1, 2, \ldots \} \) be a countable family of distinct prime ideals of \( R \) and \( \{ c_i : i = 1, 2, \ldots \} \) a countable family of elements of \( R \). If \( I \) is an ideal of \( R \) such that

\[ I \subseteq \bigcup_{i=1}^{\infty} (P_i + c_i), \]

then

\[ (I, c_k) \subseteq P_k, \text{ for some } k \geq 1. \]

**Proof.** See condition (d) of the theorem. \( \square \)

**Corollary 1.4.7.** (cf. 1.2.3) Let \( R \) be a complete semi-local Noetherian ring. Let \( \{ P_i : i = 1, 2, \ldots \} \) be a countable family of, not necessarily distinct, prime ideals of \( R \) and \( \{ c_i : i = 1, 2, \ldots \} \) a countable family of elements of \( R \). If \( I \) is an ideal of \( R \) such that \( I + P_i \neq R \), for any \( i \geq 1 \), and
1.5 Covers and sieves

When we come to examine applications of our results we shall wish to refer to a property that is required in order that the application should work, rather than giving, for each application, an exhaustive list of those situations in which we know, for one reason or another, that the property required holds. We now define the most general property we require and examine its behaviour under various operations. Before moving on to applications we shall summarize our results in terms of this definition.

Definition 1.5.1. Let $\mathcal{C}$ be a subset of the set of ideals of $R$. Let $\mathcal{F} = \{ J_\lambda \}_{\lambda \in \Lambda}$ be a family of ideals of $R$ and $\mathcal{C} = \{ C_\lambda \}_{\lambda \in \Lambda}$ a family
of elements of \( R \), both indexed by a set \( \Lambda \). The ordered pair \((\mathcal{F}, \mathcal{C})\) is a cover of \( \mathcal{L} \) if for any ideal \( I \in \mathcal{L} \),
\[
I \subseteq \bigcup_{\lambda \in \Lambda} (J_\lambda + c_\lambda) = (I, c_\mu) \subseteq J_\mu, \text{ for some } \mu \in \Lambda.
\]

If \( \mathcal{L} = \{\text{finitely generated ideals of } R\} \) we shall refer to a cover of \( \mathcal{L} \) as a cover. If \((\mathcal{F}, \mathcal{C})\) is a cover of \( \mathcal{L} \) for every family \( \mathcal{C} = (c_\lambda)_{\lambda \in \Lambda} \), then we shall say that \( \mathcal{F} \) is a cover of \( \mathcal{L} \). If, for all \( \mu \in \Lambda, c_\mu = c \) for some fixed \( c \in R \), then we shall say that \((\mathcal{F}, \{c\})\) is a cover of \( \mathcal{L} \). Note that we allow \( J_\mu = J_\lambda \) when \( \mu \neq \lambda, \mu, \lambda \in \Lambda \).

For example suppose we are given a set \( \Lambda \), a family \( \mathcal{F} = (P_\lambda)_{\lambda \in \Lambda} \) of prime ideals and a family \( \mathcal{C} = (c_\lambda)_{\lambda \in \Lambda} \) of elements of \( R \), both families indexed by \( \Lambda \). If for any prime ideal \( Q \) of \( R \), \( Q \subseteq \bigcup_{\lambda \in \Lambda} (P_\lambda + c_\lambda) \) implies \( (Q, c_\mu) \subseteq P_\mu \), for some \( \mu \in \Lambda \), then \((\mathcal{F}, \mathcal{C})\) is a cover of \( \text{Spec}(R) \).

We remark that the definition could be formulated in terms of subsets of \( \{\text{ideals of } R\} \times \{\text{elements of } R\} \). The introduction of the set \( \Lambda \) and the associated indexing is notationally convenient and gives compatibility with the format of results. However the object defined does not depend on \( \Lambda \) and therefore there is no reference to \( \Lambda \) in the expression "(\( \mathcal{F}, \mathcal{C} \) is a cover)".

**Definition 1.5.2.** Let \( \mathcal{M} \) and \( \mathcal{L} \) be subsets of the set of ideals of \( R \). For each \( J \in \mathcal{M} \), let \( \mathcal{S}_J \) be a subset of \( R \) and set
Let \( \mathcal{X} = \{ X_j \}_{j \in \mathbb{N}} \). Let \( \alpha \) be a cardinal and \( \Lambda \) a set of cardinality \( \alpha \).

\((\mathcal{M}, \mathcal{X})\) is an \( \alpha \)-sieve for \( \mathcal{L} \) if the following condition is satisfied: given any family \( \mathcal{F} = (J_\lambda)_{\lambda \in \Lambda} \) of elements of \( \mathcal{M} \), indexed by \( \Lambda \), and any family \( \mathcal{C} = (c_\lambda)_{\lambda \in \Lambda} \) of elements of \( \mathbb{R} \), indexed by \( \Lambda \), such that, for each \( \mu \in \Lambda \), \( c_\mu \in X_\mu \), then \((\mathcal{F}, \mathcal{C})\) is a cover of \( \mathcal{L} \).

If \( \mathcal{L} = \{ \text{finitely generated ideals of } \mathbb{R} \} \) we shall refer to an \( \alpha \)-sieve for \( \mathcal{L} \) as an \( \alpha \)-sieve. If, for some fixed subset \( Y \) of \( \mathbb{R} \), \( X_j = Y \) for all \( j \in \mathbb{N} \), then we shall say that \((\mathcal{M}, Y)\) is an \( \alpha \)-sieve for \( \mathcal{L} \). If \((\mathcal{M}, R)\) is an \( \alpha \)-sieve for \( \mathcal{L} \) then we shall say that \( \mathcal{M} \) is an \( \alpha \)-sieve for \( \mathcal{L} \). Note that if \((\mathcal{M}, \mathcal{X})\) is an \( \alpha \)-sieve for \( \mathcal{L} \), \( \mathcal{N} \) is any subset of \( \mathcal{M} \) and \( \mathcal{Y} = \{ X_j : j \in \mathcal{N} \} \), then \((\mathcal{N}, \mathcal{Y})\) is also an \( \alpha \)-sieve for \( \mathcal{L} \).

Let us consider some particular cases of this definition. Let \( \Lambda \) be a set, let \( A = \{(P_\lambda)_{\lambda \in \Lambda} : P_\mu \in \text{Spec}(\mathbb{R}) \text{ for all } \mu \in \Lambda\} \) and let \( B = \{(c_\lambda)_{\lambda \in \Lambda} : c_\mu \in \mathbb{R} \text{ for all } \mu \in \Lambda\} \). Let \( \mathcal{L} = \{ \text{finitely generated ideals of } \mathbb{R} \} \). Suppose that, for any \((P_\lambda)_{\lambda \in \Lambda}\) in \( A \), any \((c_\lambda)_{\lambda \in \Lambda}\) in \( B \) and any \( I \in \mathcal{L} \), the following condition is satisfied:

\[
(i) \quad \bigcup_{\lambda \in \Lambda} (P_\lambda + c_\lambda) \supseteq (I, c_\mu) \subseteq P_\mu, \quad \text{for some } \mu \in \Lambda.
\]

Then \( \text{Spec}(\mathbb{R}) \) is an \( \alpha \)-sieve for \( \mathcal{L} \). In this case it is also true that, for any fixed \( c \in \mathbb{R} \), \((\text{Spec}(\mathbb{R}), \{c\})\) is an \( \alpha \)-sieve for \( \mathcal{L} \).

Now, for each \( P \in \text{Spec}(\mathbb{R}) \), choose \( c_P \in \mathbb{R} \) and set
\( \mathcal{X} = \{ \{c_P\} : P \in \text{Spec}(R) \} \). Suppose that, for any \((P_\lambda)_{\lambda \in \Lambda} \in A\) and any \( I \in \mathcal{L} \), the following condition is satisfied:

\[
(ii) \quad I \subseteq \bigcup_{\lambda \in \Lambda} (P_\lambda + c_{P_\lambda}) \subseteq (I, c_{P_\mu}) \subseteq P_\mu, \quad \text{for some } \mu \in \Lambda.
\]

Then \((\text{Spec}(R), \mathcal{X})\) is an \( \alpha \)-sieve for \( \mathcal{L} \). If, for some fixed \( c \in R \), \( c = c_P \) for all \( P \in \text{Spec}(R) \) then, if condition (ii) holds, \((\text{Spec}(R), \{c\})\) is an \( \alpha \)-sieve for \( \mathcal{L} \).

The first of these two definitions is easy to work with, and is compatible with the results of previous sections. The second, while unpleasant, is the most general version of what is required in applications.

We now examine the behaviour of covers and sieves under localisation and epimorphisms. We consider first localisation of covers. We require a preliminary lemma which is a generalisation of Lemma 1.3.1. If \( S \) is a multiplicatively closed subset of \( R \) and \( X \) is a subset of \( S^{-1}R \) then the contraction of \( X \) to \( R \) is denoted \( X \cap S R \). Furthermore, when no ambiguity arises, we will use \( x \) to denote both an element of \( R \) and its image \( x/1 \) in \( S^{-1}R \).

**Lemma 1.5.3.** Let \( R \) be a ring, \( I \) an ideal of \( R \) and \( S \) a multiplicatively closed subset of \( R \). Then for any \( x, y \in R \),

\[
(S^{-1}I + x : S^{-1}_R y) \cap S R = ((S^{-1}I \cap S R) : R y) + q,
\]

for any \( q \in ((S^{-1}I \cap S R) + x : R y) \).

If furthermore \( y \in S \), then
Proof. Let \( a \in (S^{-1}I + x : S^{-1}R y) \cap S R \), where \( x, y \in R \). Then

\[ ay - x \in S^{-1}I \cap S R, \]

and so

\[ a \in ((S^{-1}I \cap S R) + x : R y), \]

so that

\[ a \in ((S^{-1}I \cap S R) : R y) + q, \]

for any \( q \in ((S^{-1}I \cap S R) + x : R y) \) (by Lemma 1.3.1).

Conversely if \( b \in ((S^{-1}I \cap S R) : R y) + q \) for any \( q \) as given above,

then

\[ b \in ((S^{-1}I \cap S R) + x : R y), \]

(Lemma 1.3.1 again),

and so

\[ by \in (S^{-1}I \cap S R) + x, \]

giving

\[ by \in S^{-1}I + x, \text{ in } S^{-1}R, \]

so that

\[ b \in (S^{-1}I + x : S^{-1}R y) \cap S R, \]

giving the first part of the lemma.

Now suppose that \( y \in S \). By the argument above we have,

\[ (S^{-1}I + x : S^{-1}R y) \cap S R = ((S^{-1}I \cap S R) : R y) + q, \]

and the second assertion of the lemma follows if it can be shown
1.5 Covers and sieves

that

\[(S^{-1} I \cap_S R :_R y) = S^{-1} I \cap_S R.\]

From the first part of the lemma

\[(S^{-1} I \cap_S R :_R y) = (S^{-1} I :_{S^{-1} R} y) \cap_S R.\]

As \(y\) is a unit in \(S^{-1} R\) we have \((S^{-1} I :_{S^{-1} R} y) = S^{-1} I\), giving the required equality. □

Theorem 1.5.4. Let \(R\) be a ring, \(S\) a multiplicatively closed subset of \(R\) and \(\mathcal{L}\) a subset of the set of ideals of \(R\). Let \(\mathcal{F} = (J_\lambda)_{\lambda \in \Lambda}\) be a family of ideals of \(R\) indexed by a set \(\Lambda\). Suppose that \(S\) and \(\mathcal{F}\) satisfy the following condition:

if \(\lambda \in \Lambda\) then there is \(\mu \in \Lambda\) such that,

\[S^{-1} J_\lambda \cap_S R = J_\mu.\] \hfill (1.5.5)

Let \(\Psi = \{\lambda \in \Lambda : S^{-1} J_\lambda \cap_S R = J_\lambda\}\), \(\mathcal{F} = (J_\lambda)_{\lambda \in \Psi}\),

\[S^{-1} \mathcal{F} = \{S^{-1} J_\lambda : \lambda \in \Psi\}\] and \(S^{-1} \mathcal{L} = \{S^{-1} I : I \in \mathcal{L}\}\). Then

(1) if \(\mathcal{F}\) is a cover of \(\mathcal{L}\) then \(S^{-1} \mathcal{F}\) is a cover of \(S^{-1} \mathcal{L}\);

(2) if \(\mathcal{C} = (c_\lambda)_{\lambda \in \Psi}\) is a family of elements of \(R\) indexed by \(\Psi\) such that \((\mathcal{F}, \mathcal{C})\) is a cover of \(\mathcal{L}\), then
(\(S^{-1} \mathcal{H} , \mathcal{C}\)) is a cover of \(S^{-1} \mathcal{L}\), where \(\mathcal{C}\) is regarded as \((c_{\lambda}/1)_{\lambda \in \Psi}\) in \(S^{-1} \mathcal{R}\).

Proof. We prove (1) first. Let \((z_{\lambda})_{\lambda \in \Psi}\) be a family of elements of \(S^{-1} \mathcal{R}\), and for each \(\lambda \in \Psi\) suppose that \(z_{\lambda} = d_{\lambda}/t_{\lambda}\), where \(d_{\lambda}, t_{\lambda} \in \mathcal{R}\) and \(t_{\lambda} \in \mathcal{S}\). Given \(I \in \mathcal{L}\) such that

\[
S^{-1} I \subseteq \bigcup_{\lambda \in \Psi} (S^{-1} J_{\lambda} + z_{\lambda}),
\]
we wish to show that \((S^{-1} I, z_{\mu}) \subseteq S^{-1} J_{\mu}\) for some \(\mu \in \Psi\).

Suppose \(x \in I\); then

\[
x/1 \in S^{-1} I \subseteq \bigcup_{\lambda \in \Psi} (S^{-1} J_{\lambda} + z_{\lambda}),
\]
so \(x/1 \in S^{-1} J_{\mu} + d_{\mu}/t_{\mu}\) for some \(\mu \in \Psi\). Thus

\[xt_{\mu}/1 \in S^{-1} J_{\mu} + d_{\mu}/1,\]
and so

\[
x/1 \in (S^{-1} J_{\mu} + d_{\mu}/1 : S^{-1} \mathcal{R} t_{\mu}/1),
\]
so that \(x \in (S^{-1} J_{\mu} + d_{\mu} : S^{-1} \mathcal{R} t_{\mu}) \cap \mathcal{S} \mathcal{R}\),

hence \(x \in (S^{-1} J_{\mu} \cap \mathcal{S} \mathcal{R}) + q_{\mu}\),

for any \(q_{\mu} \in ((S^{-1} J_{\mu} \cap \mathcal{S} \mathcal{R}) + d_{\mu} : t_{\mu})\), by Lemma 1.5.3.

For each \(\tau \in \Psi\) fix \(q_{\tau} \in ((S^{-1} J_{\tau} \cap \mathcal{S} \mathcal{R}) + d_{\tau} : t_{\tau})\); then
\[
I \subseteq \bigsqcup_{\lambda \in \Psi} (S^{-1}J_{\lambda} \cap S \mathcal{R}) + q_{\lambda} = \bigsqcup_{\lambda \in \Psi} J_{\lambda} + q_{\lambda},
\]

by definition of \( \Psi \). We define a map \( \varphi : \Lambda \rightarrow \Psi \) as follows: for \( \rho \in \Lambda \setminus \Psi \) choose \( \mu \in \Psi \) such that \( J_{\mu} = S^{-1}J_{\rho} \cap S \mathcal{R} \) and set \( \varphi(\rho) = \mu \); for \( \tau \in \Psi \) set \( \varphi(\tau) = \tau \). Note that if \( \rho \in \Lambda \) then \( S^{-1}J_{\rho} = S^{-1}J_{\varphi(\rho)} \). We may now write

\[
I \subseteq \bigsqcup_{\lambda \in \Lambda} J_{\lambda} + q_{\lambda}',
\]

where, for all \( \mu \in \Lambda \), \( q_{\mu}' = q_{\varphi(\mu)} \). By the hypothesis of (1),

\[
I \subseteq (I, q_{\nu}') \subseteq J_{\nu} \quad \text{for some } \nu \in \Lambda,
\]

so that \( S^{-1}I \subseteq S^{-1}J_{\nu} = S^{-1}J_{\varphi(\nu)} \), for some \( \nu \in \Lambda \).

Since \( (I, q_{\nu}') \subseteq J_{\nu} \) we have, by Lemma 1.3.3, \( I \cap (J_{\nu} + q_{\nu}') \neq \emptyset \).

Let \( i \in I \cap (J_{\nu} + q_{\nu}') \) and write \( \mu = \varphi(\nu) \); then

\[
it_{\mu} \in J_{\nu} + q_{\nu}'t_{\mu}.
\]

Now \( \mu \in \Psi \), \( q_{\nu}' = q_{\mu} \), \( q_{\mu} \in ((S^{-1}J_{\mu} \cap S \mathcal{R}) + d_{\mu : \mathbb{R} \mu}) \) and

\[
S^{-1}J_{\nu} = S^{-1}J_{\varphi(\nu)} = S^{-1}J_{\mu}.
\]

Therefore we have, in \( S^{-1} \mathbb{R} \),

\[
it_{\mu} \in S^{-1}J_{\nu} + S^{-1}J_{\mu} + d_{\mu} = S^{-1}J_{\mu} + d_{\mu}.
\]

Hence

\[
i = (it_{\mu})(1/t_{\mu}) \in S^{-1}J_{\mu} + z_{\mu}.
\]

Thus \( S^{-1}I \subseteq S^{-1}J_{\mu} \) and \( S^{-1}I \cap (S^{-1}J_{\mu} + z_{\mu}) \neq \emptyset \). By Lemma 1.3.3
(S^{-1}I, z_\mu) \subseteq S^{-1}J_\mu, and this proves (1), since \mu \in \Psi.

For (2) we can follow the same argument replacing each t_\lambda by 1, and d_\lambda by c_\lambda throughout. Then

I \subseteq \bigcup_{\lambda \in \Psi} (S^{-1}I_\lambda \cap S R) + c_\lambda = \bigcup_{\lambda \in \Psi} (J_\lambda + c_\lambda),

and the result follows using the hypothesis of (2). 

Corollary 1.5.6. Let R be a ring, S a multiplicatively closed subset of R and \mathcal{L} a subset of the set of ideals of R. Let \mathcal{F} = (P_\lambda)_{\lambda \in \Lambda} be a family of prime ideals of R and \mathcal{C} = (c_\lambda)_{\lambda \in \Lambda} be a family of elements of R, both families indexed by a set \Lambda. Let

S^{-1}\mathcal{F} = \{S^{-1}P_\lambda : \lambda \in \Lambda\}. If (\mathcal{F}, \mathcal{C}), or \mathcal{F}, is a cover of \mathcal{L},

then, with the notation of the theorem, (S^{-1}\mathcal{F}, \mathcal{C}), or S^{-1}\mathcal{F}, respectively, is a cover of S^{-1}\mathcal{L}. 

Corollary 1.5.7. Let R be a ring, S a multiplicatively closed subset of R, \mathcal{L} a subset of the set of ideals of R and \alpha a cardinal. Let

\mathcal{M} = \{I : I is an ideal of R ; I \cap S = \emptyset\}, and suppose that \mathcal{M} is
an $\alpha$-sieve for $\mathcal{L}$. Then \{ideals of $S^{-1}R$\} is an $\alpha$-sieve for $S^{-1}\mathcal{L}$.

**Proof.** Let $\Lambda$ be a set of cardinality $\alpha$, and $(a_{\lambda})_{\lambda \in \Lambda}$ a family of ideals of $S^{-1}R$ indexed by $\Lambda$. If $a_{\mu} = S^{-1}R$ for some $\mu \in \Lambda$, then $(a_{\lambda})_{\lambda \in \Lambda}$ is certainly a cover, so we assume that $a_{\mu} \neq S^{-1}R$ for any $\mu \in \Lambda$. For each $\mu \in \Lambda$ set $J_\mu = a_{\mu} \cap S$. Then $J_\mu \cap S = \emptyset$, and $S^{-1}J_\mu = S^{-1}(a_{\mu} \cap S) = a_{\mu}$, so $S^{-1}J_\mu \cap S = a_{\mu} \cap S = J_\mu$, for all $\mu \in \Lambda$. Thus $(J_\lambda)_{\lambda \in \Lambda}$ satisfies condition (1.5.5) and in the notation of Theorem 1.5.4 we have $\Lambda = \psi$. By hypothesis $(J_\lambda)_{\lambda \in \Lambda}$ is a cover of $\mathcal{L}$, so by Theorem 1.5.4, $(S^{-1}J_\lambda)_{\lambda \in \Lambda} = (a_{\lambda})_{\lambda \in \Lambda}$ is a cover of $S^{-1}\mathcal{L}$. The result follows. \(\square\)

Using Theorem 1.5.4 we can extend the results of Section 1.4.

**Corollary 1.5.8** Let $A$ be a ring and suppose that $A = S^{-1}R$, where $R$ is a complete semi-local Noetherian ring and $S$ is a multiplicatively closed subset of $R$. Let $\Lambda$ be a set of cardinality...
and let \((J_{\lambda})_{\lambda \in \Lambda}\) be a family of (not necessarily distinct) ideals of \(A\). Suppose that one of the following conditions is satisfied:

either the family \((J_{\lambda} \cap S R)_{\lambda \in \Lambda}\) of ideals of \(R\) satisfies one of conditions (a), (c), (d), (e) or (f) of Theorem 1.4.3,

or \(R\) satisfies condition (b) of Theorem 1.4.3.

Let \((d_{\lambda})_{\lambda \in \Lambda}\) be a family of elements of \(A\) and a an ideal of \(A\) such that

\[
a \subseteq \bigcup_{\lambda \in \Lambda} (J_{\lambda} + d_{\lambda});
\]

then \((a, d_{\mu}) \subseteq J_{\mu}\) for some \(\mu \in \Lambda\).

**Proof.** By hypothesis \(\mathcal{F} = (J_{\lambda} \cap S R)_{\lambda \in \Lambda}\) is a cover for \(\mathcal{L} = \{\text{ideals of } R\}\). For any \(\mu \in \Lambda\) we have

\[
J_{\mu} \cap S R = S^{-1}(J_{\mu} \cap S R) \cap S R,
\]

and so \(\mathcal{F}\) satisfies (1.5.5). With the notation of the theorem we have in this case \(\mathcal{W} = \Lambda\), and so, from (1) of the theorem, \(S^{-1}\mathcal{F} = (J_{\lambda})_{\lambda \in \Lambda}\) is a cover for \(S^{-1}\mathcal{L} = \{\text{ideals of } S^{-1}R\}\). \(\square\)
In the ring $A$ above the family of ideals $(J_\lambda)_{\lambda \in \Lambda}$ may not satisfy condition (c) of Theorem 1.4.3 even when the family $(J_{\rho_s} R)_{\lambda \in \Lambda}$ does. Furthermore $A$ is not necessarily complete or even semi-local.

Suppose we perform the same operation in the situation studied in Section 1.3, in the hope of finding more covers. That is, let $R$ be a ring, $\Lambda$ a set such that $|\Lambda| = \alpha$ for some fixed cardinal $\alpha$ and

$J = (J_\lambda)_{\lambda \in \Lambda}$ a family of ideals of $R$, indexed by $\Lambda$. Assume, for any choice $p_\mu$ of a maximal prime of $J_\mu$, for every $\mu \in \Lambda$, that $R$ has a $\beta$-difference set outside $\bigcup_{\lambda \in \Lambda} p_\lambda$ where $\beta > \alpha$, $\beta \in \text{Card}$. Assume further that we have a multiplicatively closed subset $S$ of $R$, such that $R$, $J$ and $S$ satisfy condition (1.5.5) of Theorem 1.5.4. Let $\Upsilon$, $\Upsilon'$ and $S^{-1}J'$ be defined as in Theorem 1.5.4. For each $\mu \in \Upsilon$ choose a maximal prime $Q_\mu$ of $S^{-1}J_\mu$ and let $P_\mu = Q_\mu \cap S R$. Let $x \in P_\mu$. Then $x/1 \in Q_\mu$ and $Q_\mu$ is, by definition, contained in the zero divisors of $S^{-1}R/S^{-1}J_\mu$. Thus there is $r/t \in S^{-1}R\setminus S^{-1}J_\mu$ such that $xr/t \in S^{-1}J_\mu$, where $r, t \in R$ and $t \in S$. Therefore
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\[
xru \in J_\mu \text{ for some } u \in S. \text{ If } ru \in J_\mu \text{ then } r/t \in S^{-1}J_\mu, \text{ a contradiction. Thus } x \in (J_\mu : ru) \subseteq Z(R/J_\mu). \text{ Hence } P_\mu \subseteq Z(R/J_\mu)
\]
and \( P_\mu \) can be expanded to a maximal prime \( p_\mu \) of \( J_\mu \). If we choose for \( \rho \in A \setminus \Psi \) any maximal prime \( p_\rho \) of \( J_\rho \), then by hypothesis \( R \) has a \( \beta \)-difference set, \( D \) say, outside \( \bigcup_{\lambda \in \Lambda} p_\lambda \), which is certainly outside \( \bigcup_{\lambda \in \Psi} p_\lambda \). Let \( a \) and \( b \) be distinct elements of \( D \); then
\[
a/1 - b/1 \text{ cannot be in } S^{-1}P_\mu, \text{ for any } \mu \in \Psi, \text{ since by construction } S \cap P_\mu = \emptyset. \text{ Therefore } a/1 - b/1 \text{ can certainly not be zero, and } \{ d/1 : d \in D \} \text{ is a } \beta \text{-difference set outside}
\]
\[
\bigcup_{\lambda \in \Psi} S^{-1}P_\lambda = \bigcup_{\lambda \in \Psi} Q_\lambda. \text{ We have the same situation after performing the localisation as before. Therefore no new situations have been found in which covers exist.}
\]

Covers also behave well under epimorphisms. For the remainder of this section, if \( R \) is a ring and \( a \) is an ideal of \( R \), and if \( Y \subseteq R \), then we let \( \bar{Y} = Y + a/a \), and if \( y \in R \) we let
\[
\bar{y} = y + a/a.
\]
Theorem 1.5.9. Let \( R \) be a ring, \( a \) an ideal of \( R \) and \( \mathcal{L} \) a subset of the set of ideals of \( R \). Let \( \mathcal{F} = (J_{\lambda})_{\lambda \in \Lambda} \) be a family of ideals of \( R \) indexed by a set \( \Lambda \). Assume that \( \mathcal{F} \) and \( a \) satisfy the following condition:

For all \( \rho \in \Lambda \), there is \( \mu \in \Lambda \) such that

\[
J_{\mu} = J_{\rho} + a. \tag{1.5.10}
\]

Let \( \Psi = \{\lambda \in \Lambda : J_{\lambda} \supseteq a\} \), \( \mathcal{I} = (J_{\lambda})_{\lambda \in \Psi} \), \( \mathcal{J} = (J_{\lambda})_{\lambda \in \Psi} \), and \( \mathcal{L} = \{I : I \in \mathcal{L}\} \). Then

1. if \( \mathcal{F} \) is a cover of \( \mathcal{L} \) then \( \mathcal{I} \) is a cover of \( \mathcal{L} \);

2. if \( \mathcal{C} = (c_{\lambda})_{\lambda \in \Psi} \) is a family of elements of \( R \) indexed by \( \Psi \) such that \( (\mathcal{I}, \mathcal{C}) \) is a cover of \( \mathcal{L} \), then \( (\mathcal{I}, \mathcal{C}) \) is a cover of \( \mathcal{L} \), where \( \mathcal{C} = (c_{\lambda})_{\lambda \in \Psi} \).

Proof. Let \( I \in \mathcal{L} \) and \( (c_{\lambda})_{\lambda \in \Psi} \) be a family of elements of \( R \) indexed by \( \Psi \) such that

\[
\mathcal{I} \subseteq \bigcup_{\lambda \in \Psi} (J_{\lambda} + c_{\lambda});
\]

then

\[
I + a \subseteq \bigcup_{\lambda \in \Psi} (J_{\lambda} + a) + (c_{\lambda} + a),
\]

and
so \[ I \subseteq \bigcup_{\lambda \in \Psi} (J_\lambda + c_\lambda) \] (by definition of \( \Psi \)), \((*)\)

from which (2) follows immediately.

For each \( \rho \in \Lambda \setminus \Psi \) we may choose \( \tau_\rho \in \Psi \) such that

\[ J_{\tau_\rho} = J_\rho + a. \]

Then we have, from \((*)\),

\[ I \subseteq \bigcup_{\lambda \in \Lambda} (J_\lambda + d_\lambda), \]

where, for \( \mu \in \Psi \), \( d_\mu = c_\mu \) and, for \( \rho \in \Lambda \setminus \Psi \), we set \( d_\rho = c_{\tau_\rho} \).

Then, given the hypothesis of (1),

\[ (I, d_\nu) \subseteq J_\nu, \]

for some \( \nu \in \Lambda \). Write \( \tau_\nu = \mu \in \Psi \) so that \( J_\mu = J_\nu + a \) and

\[ c_\mu = d_\nu. \]

Then

\[ (I, c_\mu) = (I, d_\nu) \subseteq J_\nu = J_\mu, \]

for some \( \mu \in \Psi \). \( \square \)

Corollary 1.5.11. Let \( R \) be a ring, \( a \) an ideal of \( R \) and \( \mathcal{J} \) a subset of the set of ideals of \( R \). Let \( \mathcal{J} = (J_\lambda)_{\lambda \in \Lambda} \) be a family of ideals of \( R \) and \( \mathcal{C} = (c_\lambda)_{\lambda \in \Lambda} \) be a family of elements of \( R \), both families indexed by a set \( \Lambda \). Assume that the following properties are
satisfied:

(i) \((\mathcal{F}, \mathcal{C})\) is a cover of \(\mathcal{L}\);

(ii) \(a \subseteq J_\mu\) for all \(\mu \in \Lambda\).

Let \(\mathcal{F} = (J_\lambda)_{\lambda \in \Lambda}\) and \(\mathcal{C} = (\bar{c}_\lambda)_{\lambda \in \Lambda}\). Then \((\mathcal{F}, \mathcal{C})\) is a cover of \(\mathcal{L}\). \(\square\)

Corollary 1.5.12. Let \(R\) be a ring, \(a\) an ideal of \(R\), \(\alpha\) a cardinal, \(\mathcal{M}\) the set of all ideals of \(R\) and \(\mathcal{L}\) a subset of \(\mathcal{M}\). For each \(J \in \mathcal{M}\), let \(\mathcal{A}_J\) be a subset of \(R\) and set \(\mathcal{X} = \{\mathcal{A}_J\}_{J \in \mathcal{M}}\). Let \(\mathcal{N} = \{\text{ideals of } R/a\}\) and for each \(n \in \mathcal{N}\) define

\[\mathcal{W}_n = \{c : c \in \mathcal{A}_J ; J \in \mathcal{M} ; J \geq a ; J = n\}\].

Set \(\mathcal{Y} = \{\mathcal{W}_n\}_{n \in \mathcal{N}}\).

If \((\mathcal{M}, \mathcal{X})\) is an \(\alpha\)-sieve for \(\mathcal{L}\), then \((\mathcal{N}, \mathcal{Y})\) is an \(\alpha\)-sieve for \(\mathcal{L}\).

Proof. Let \(\Lambda\) be a set of cardinality \(\alpha\). Let \((\mathcal{n}_\lambda)_{\lambda \in \Lambda}\) be a family of elements of \(\mathcal{N}\) and \((\bar{c}_\lambda)_{\lambda \in \Lambda}\) a family of elements of \(\bar{R}\) such that,

for each \(\mu \in \Lambda\), \(\bar{c}_\mu \in \mathcal{W}_{n_\mu}\). For each \(\mu \in \Lambda\), let \(J_\mu\) be the unique element of \(\mathcal{M}\) such that \(J_\mu \geq a\) and \(J_\mu = n_\mu\), and choose \(c_\mu \in \mathcal{A}_{J_\mu}\), a representative of \(\bar{c}_\mu\). Set \(\mathcal{F} = (J_\lambda)_{\lambda \in \Lambda}\) and set
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\[ \mathcal{C} = (c_{\lambda})_{\lambda \in \Lambda}. \]
Then, since \((\mathcal{M}, \mathcal{X})\) is an \(\alpha\)-sieve for \(\mathcal{L}\), \((\mathcal{F}, \mathcal{C})\) is a cover of \(\mathcal{L}\). Using Corollary 1.5.11 \((\mathcal{F}, \mathcal{C})\) is a cover of \(\mathcal{L}\) and, since \(\mathcal{F} = (n_{\lambda})_{\lambda \in \Lambda}\) and \(\mathcal{C} = (c_{\lambda})_{\lambda \in \Lambda}\), the result follows. \(\square\)

Notes 1.5.13. In the situation of the hypothesis or conclusion of the last corollary the ideal (1) is included in the set of ideals which have the \(\alpha\)-sieving property.

1.5.14. Again, performing this operation in the situation of Section 1.3 gives no new information. Let \(R\) be a ring, \(A\) a set such that
\[ |A| = \alpha, \] for some fixed cardinal \(\alpha\), and \(\mathcal{F} = (J_{\lambda})_{\lambda \in \Lambda}\) a family of ideals of \(R\). Assume, for any choice \(P_{\mu}\) of a maximal prime of \(J_{\mu}\), for every \(\mu \in \Lambda\), that \(R\) has a \(\beta\)-difference set outside \(\bigcup_{\lambda \in \Lambda} P_{\lambda}\), where \(\beta\) is an infinite cardinal such that \(\beta > \alpha\). Assume further that we have an ideal \(a\) of \(R\) such that \(a\) and \(\mathcal{A}\) satisfy (1.5.10) of Theorem 1.5.9. Let \(\Psi\) be the set defined in that theorem and let \(Q_{\mu}\) be any choice of maximal prime for \(J_{\mu}\), as \(\mu\) ranges over \(\Psi\). Then \(Q_{\mu} = P_{\mu}\) for some maximal prime, \(P_{\mu}\), of \(J_{\mu}\), and \(R\) has a \(\beta\)-difference set, \(D\) say, outside \(\bigcup_{\lambda \in \Psi} P_{\lambda}\). If \(a\) and \(b\) are distinct elements of \(D\) such that \(\tilde{a} - \tilde{b} \in \tilde{P}_{\mu}\), for some \(\mu \in \Psi\), then
\[ a - b \in (P_{\mu} + a) = P_{\mu}, \] since \(\mu \in \Psi\). This contradiction implies that \(\tilde{a} - \tilde{b} \notin \tilde{P}_{\mu} = Q_{\mu}\), for any \(\mu \in \Psi\), and a fortiori \(\tilde{a} - \tilde{b} \neq 0\).
Therefore the image of \(D\) in \(R/a\) is a \(\beta\)-difference set outside \(\bigcup_{\lambda \in \Psi} Q_{\lambda}\). We have returned to the original situation.
1.5.15. This time the operation is also trivial in the situation of Section 1.4. If \( R \) is a complete semi-local Noetherian ring it follows, from Section 1.4 and [ZS2], Chapter VIII Section 2 Theorem 6, that \( R/a \) is also a complete semi-local Noetherian ring. Suppose we are given a countable family \( \{J_i : i = 1, 2, \ldots\} \) of ideals of \( R \) and an ideal \( a \) such that \( a \subseteq J_j \) for all \( j \geq 1 \). Then

(a) \( |\bar{R}/\bar{P}| \geq \aleph_1 \) if and only if \( |R/P| \geq \aleph_1 \), for any maximal prime, \( P \), of any \( J_j \);

(b) \( |\bar{R}/\bar{m}| \geq \aleph_1 \) if and only if \( |R/m| \geq \aleph_1 \), for any maximal ideal, \( m \), of \( R \) containing \( a \);

(c) \( \bar{P} \) is non-maximal if and only if \( P \) is non-maximal for any maximal prime, \( P \), of any \( J_j \);

(d) \( \bar{P} \) is a maximal prime of \( \bar{J}_j \) if and only if \( P \) is a maximal prime of \( J_j \), for any \( j \geq 1 \);

(e) \( \text{Ass}_R(\bar{R}/\bar{J}_j) = \{\bar{P} : P \in \text{Ass}_R(R/J_j)\} \), since \( J_j \supseteq a \), for all \( j \geq 1 \);

(f) \( Z(\bar{R}/\bar{J}_j) = Z(R/J_j) \) since \( J_j \supseteq a \) for all \( j \geq 1 \).

Therefore no new types of cover arise as a result of lifting back to a ring \( R \) in which a condition of Theorem 1.4.3 holds (cf. Corollary 1.5.8).
1.6 Examples of $\alpha$-sieves

Here we collect the results of previous sections and express them in terms of the definition of an $\alpha$-sieve in Section 1.5. The plan, in what follows, is to give the most general conditions under which sieves are found, and then to narrow attention onto particular instances of these conditions. For the duration of this section let $R$ be a ring, $\mathcal{M}$ and $\mathcal{L}$ subsets of the set of ideals of $R$ and let $\alpha$ and $\beta$ be cardinals such that $\alpha < \beta$. For each $J \in \mathcal{M}$, let $\mathcal{N}_J$ be a subset of $R$ and set $\mathcal{X} = \{\mathcal{N}_J\}_{J \in \mathcal{M}}$. In the following examples $\mathcal{M}$, $\mathcal{L}$, $\alpha$, $\beta$ and $\mathcal{X}$ may be assumed to be arbitrary unless they are explicitly defined. First, for ease or reference, we reformulate Theorem 1.3.5 in terms of an $\alpha$-sieve.

1.6.1. Let $\mathcal{L} = \{\text{finitely generated ideals of } R\}$ and consider the following condition on $\mathcal{M}$: given a set $\Lambda$ of cardinality $\alpha$ and any family $(J_\lambda)_{\lambda \in \Lambda}$ of elements of $\mathcal{M}$, indexed by $\Lambda$, $R$ has $\beta$-differences outside every family $(P_\lambda)_{\lambda \in \Lambda}$ such that $P_\mu$ is a maximal prime of $J_\mu$ for each $\mu \in \Lambda$. If $\mathcal{M}$ satisfies this condition it is clear, in the light of Theorem 1.3.5, that $\mathcal{M}$ is an $\alpha$-sieve for $\mathcal{L}$. In particular, $\mathcal{M}$ satisfies this condition when $\beta \geq \aleph_0$ and

(i) $R$ is described by Example 1.3.6 and $\mathcal{M} \subseteq \{\text{proper ideals of } R\}$, or

(ii) $R$ is described as in Example 1.3.7 and $\mathcal{M} \subseteq \{J \text{ an ideal of } R: Z(R/J) \subseteq XR\}$, or

(iii) $R$ and $I_1, \ldots, I_r$ are described by Example 1.3.8 and
1.6 Examples of α-sieves

\[ \mathcal{M} \subseteq \{ J \text{ an ideal of } R : Z(R/J) \subseteq \bigsqcup_{i=1}^{r} I_i \}, \text{ or} \]

(iv) \( R \) is described in Example 1.3.9 and \( \mathcal{M} \subseteq \{ \text{proper ideals of } R \} \).

1.6.2. \( \mathcal{M} \) is an α-sieve for \( \mathcal{L} \) whenever one of the following conditions (1) to (4) holds:

1. \( R \) contains a β-difference set outside \( \bigsqcup_{J \in \mathcal{M}} Z(R/J) \) and \( \mathcal{L} \subseteq \{ \text{finitely generated ideals of } R \} \); this is a particular case of 1.6.1 which in fact covers items (i) to (iv) of 1.6.1;

2. \( R \) is a complete semi-local Noetherian ring, \( \mathcal{L} \) is arbitrary, \( \alpha = \aleph_0 \), and \( \mathcal{M} \) is such that any family \( (J_\lambda)_{\lambda \in \Lambda} \) of elements of \( \mathcal{M} \), indexed by a set \( \Lambda \) of cardinality \( \alpha \), satisfies one of conditions (a), (c), (d), (e) or (f) of Theorem 1.4.3; (note that condition (b) is covered in 1.6.1 (iv) above, as shown by Note 1.4.5;)

3. \( \alpha \) is finite and \( \mathcal{M} \) is such that any family \( (J_\lambda)_{\lambda \in \Lambda} \) of elements of \( \mathcal{M} \), indexed by a set \( \Lambda \) of cardinality \( \alpha \), satisfies one of conditions (a), (c), (d) or (e) of Theorem 1.2.4;

4. \( \alpha \) is finite, and \( R \) satisfies condition (b) of Theorem 1.2.4.

1.6.3. If \( \mathcal{M} \) is such that for any \( J \in \mathcal{M} \),

\[ \{ \text{maximal primes of } J \} \cap \text{Max-Spec}(R) = \emptyset, \]
then $\mathcal{M}$ is an $\alpha$-sieve for $\mathcal{L}$ whenever one of the following conditions (1) to (3) holds:

1. we have any of Examples 1.6.1 or 1.6.2;
2. $R$ is a complete semi-local Noetherian ring and $\alpha = R_0$
   (by Theorem 1.4.3 (c));
3. $\alpha$ is finite (by Theorem 1.2.4 (c)).

1.6.4. If $\mathcal{M}$ has the property that for two distinct elements $J_1$ and $J_2$ of $\mathcal{M}$, the set of maximal primes of $J_1$ does not meet the set of maximal primes of $J_2$, then $\mathcal{M}$ is an $\alpha$-sieve for $\mathcal{L}$ whenever one of the following conditions (1) to (3) holds:

1. we have any of Examples 1.6.1 or 1.6.2;
2. $R$ is a complete semi-local Noetherian ring and $\alpha = R_0$
   (by Theorem 1.4.3 (d));
3. $\alpha$ is finite (by Theorem 1.2.4 (d)).

1.6.5. If $\mathcal{M} \subseteq \text{Spec}(R)$ and $\mathcal{M}_p$ has exactly one element for every $p \in \mathcal{M}$, then $(\mathcal{M}, \mathcal{X})$ is an $\alpha$-sieve for $\mathcal{L}$ whenever one of the following conditions (1) to (3) holds:

1. we have any of Examples 1.6.1 or 1.6.2;
2. $R$ is a complete semi-local Noetherian ring and $\alpha = R_0$
   (by Corollary 1.4.6);
3. $\alpha$ is finite (this follows from Theorem 1.2.4 ([McA], Corollary 2)).

1.6.6. Suppose that $\mathcal{M} \subseteq \text{Spec}(R)$ and, for any $P \in \mathcal{M}$ and $I \in \mathcal{L}$, $I + P \neq R$. Then $\mathcal{M}$ is an $\alpha$-sieve for $\mathcal{L}$ whenever one of the
following conditions (1) to (3) holds:

1. we have any of Examples 1.6.1 or 1.6.2;
2. $R$ is a complete semi-local Noetherian ring and $\alpha = \mathbb{N}_0$ (by Corollary 1.4.7);
3. $\alpha$ is finite (this follows Theorem 1.2.4 ([McA], Corollary 3)).

1.6.7. If $\mathcal{M} \subseteq \text{Spec}(R)$ then $(\mathcal{M}, \mathcal{X})$ is an $\alpha$-sieve for $\mathcal{L}$ whenever one of the following conditions (1) to (3) holds:

1. we have any of Examples 1.6.1 or 1.6.2;
2. $R$ is the ring of Example 1.3.7, $\mathcal{L} \subseteq \{\text{finitely generated ideals of } R\}$ and, for all $\mathcal{P} \subseteq \mathcal{X}R$; this follows from Example 1.6.1 (ii);
3. $R$ and $I_1, \ldots, I_r$ are described in Example 1.3.8, $\mathcal{L} \subseteq \{\text{finitely generated ideals of } R\}$ and, for all $\mathcal{P} \in \mathcal{M}$, $\mathcal{P} \subseteq \bigcup_{i=1}^{r} I_i$; this follows from Example 1.6.1 (iii).

1.6.8. Let $A$ be a ring and let $\mathcal{R} = \{\text{ideals of } A\}$. Let $R, \mathcal{M}, \mathcal{X}$ and $\alpha$ be described as in any of Examples 1.6.2 (2), 1.6.3 (2) or 1.6.4 (2). Let $S$ be a multiplicatively closed subset of $R$ such that $A = S^{-1}R$ and let $\mathcal{N} \subseteq \mathcal{R}$ such that $m \cap S \subseteq R \in \mathcal{M}$ for all $m \in \mathcal{N}$. Then it follows, from Corollary 1.5.8, that $\mathcal{N}$ is an $\alpha$-sieve for $\mathcal{R}$. 
1.7 Sieves and prime avoidance

Corollary 1.7.1. (cf. Theorem 1.2.5; Theorem 1.2.6) Let $R$ be a ring, $\mathcal{M}$ and $\mathcal{L}$ subsets of the set of ideals of $R$ and $\alpha$ a cardinal. Assume that for any $x \in R$, $(\mathcal{M}, \{x\})$ is an $\alpha$-sieve for $\mathcal{L}$. Let $I$ be an element of $\mathcal{L}$, let $\Lambda$ be a set such that $|\Lambda| = \alpha$ and let $(J_{\lambda})_{\lambda \in \Lambda}$ be a family of elements of $\mathcal{M}$, indexed by $\Lambda$. Let $c$ be an element of $R$ such that

$$I + c \subseteq \bigcup_{\lambda \in \Lambda} J_{\lambda}.$$ 

Then

$$(I, c) \subseteq J_\mu \text{ for some } \mu \in \Lambda.$$ 

Proof. Let $a \in I$. Then

$$a + c \in J_\mu \text{ for some } \mu \in \Lambda,$$

so

$$a \in J_\mu + (-c) \text{ for some } \mu \in \Lambda,$$

hence

$$I \subseteq \bigcup_{\lambda \in \Lambda} (J_\lambda + (-c)),$$

and

$$(I, c) = (I, (-c)) \subseteq J_\mu \text{ for some } \mu \in \Lambda,$$

since $(\mathcal{M}, \{-c\})$ is, by hypothesis, an $\alpha$-sieve for $\mathcal{L}$. $\square$
Remark 1.7.2. All the examples of Section 1.6 meet the hypotheses of this corollary. In particular with \( R, \mathcal{L, M} \) and \( \alpha \) as in 1.6.5 (2), we have Theorem 1.2.5. Again with \( R, \mathcal{L, M, \alpha} \) and \( \beta \) as in 1.6.1, \( c = 0 \) and \( \alpha = \aleph_0 \), we have Theorem 1.2.6. Note that, in Example 1.6.5, \( \mathcal{M} \subseteq \text{Spec}(R) \); hence "prime" appears in the heading of this section.

Corollary 1.7.3. (cf. 1.2.1; [ShVa], Corollary (2.3)) Let \( R \) be a ring, \( \mathcal{M} \) and \( \mathcal{L} \) subsets of the set of ideals of \( R \) and \( \alpha \) a cardinal. Assume that for any \( x \in R \), \( (\mathcal{M}, \{x\}) \) is an \( \alpha \)-sieve for \( \mathcal{L} \). Let \( I \) be an element of \( \mathcal{L} \), let \( \Lambda \) be a set such that \( |\Lambda| = \alpha \) and let \( (J_\lambda)_{\lambda \in \Lambda} \) be a family of elements of \( \mathcal{M} \) indexed by \( \Lambda \). Let \( c \) be an element of \( R \) such that \( (I, c) \subseteq \bigcup_{\lambda \in \Lambda} J_\lambda \). Then there exists \( a \in I \) such that \( a + c \notin \bigcup_{\lambda \in \Lambda} J_\lambda \).

Proof. Suppose that, for all \( a \in I \), \( a + c \in \bigcup_{\lambda \in \Lambda} J_\lambda \). Then for each \( a \in I \) there is \( \mu \in \Lambda \) such that \( a + c \in J_\mu \), so \( a \in J_\mu + (-c) \). Thus we have

\[
I \subseteq \bigcup_{\lambda \in \Lambda} (J_\lambda + (-c)),
\]

and so, using the \( \alpha \)-sieveing property, \( (I, c) \subseteq J_\mu \), for some \( \mu \in \Lambda \), a contradiction. \( \square \)
1.8 Sieves and zero divisors

The aim here is to prove an analogue of [Kap] Theorem 82 for modules which are not necessarily finitely generated. If a module \( M \) can be generated by a set of cardinality \( \alpha \) we say that \( M \) is \( \alpha \)-generated. We begin by generalising [ShVa] Lemma (3.2) to give:

\[ \text{Lemma 1.8.1. Let } R \text{ be a Noetherian ring and } M \text{ a } \alpha \text{-generated } R\text{-module. Then} \]
\[ \text{(i) every } R\text{-homomorphic image of } M \text{ is } \alpha \text{-generated;} \]
\[ \text{(ii) every } R\text{-submodule of } M \text{ is } \alpha \text{-generated;} \]
\[ \text{(iii) } \text{Ass}_R(N) \text{ has cardinality at most } \alpha \text{ whenever } N \text{ is a homomorphic image or a submodule of } M. \]

\[ \text{Proof. Let } (x_{\lambda})_{\lambda \in \Lambda} \text{ be a generating set for } M, \text{ where } \Lambda \text{ is some well ordered set such that } |\Lambda| = \alpha. \text{ Then (i) is obvious. For (ii) and (iii) we use trans-finite induction on } \text{ord}(\alpha) = \tau. \text{ Let } < \text{ be the ordering in } \text{Ord, then we may write, in an obvious notation,} \]
\[ (x_{\lambda})_{\lambda \in \Lambda} = (x_{\sigma})_{\sigma < \tau}. \text{ We may safely assume that } \alpha \geq \aleph_0. \text{ Suppose that } \]
\[ \sigma \in \text{Ord, } \sigma < \tau. \text{ Then define } M_\sigma = \sum_{\rho < \sigma} Rx_\rho, \text{ that is} \]
\[ M_\sigma = \{ \sum_{i=1}^{n} r_i x_{\rho_i} : r_i \in R; \rho_i < \sigma; n \in \mathbb{N} \}. \text{ For the proof of (ii),} \]
let $N$ be a submodule of $M$ and set $N_\sigma = M_\sigma \cap N$. If $\tau$ is a limit ordinal, then $\tau = \bigcup_{\sigma < \tau} \sigma$ and so $N = \bigcup_{\sigma < \tau} N_\sigma$ (since if $\sigma < \tau$ the successor of $\sigma$ is less than $\tau$). By the inductive hypothesis, if $\sigma < \tau$ then $N_\sigma$ is generated by at most $|\sigma|$ elements (as it is a submodule of $M_\sigma$). Thus $N$ is generated by at most $\sum_{\sigma < \tau} |\sigma| = |\tau| = \alpha$ elements.

If on the other hand $\tau = \sigma + 1$ for some $\sigma \in \text{Ord}$, then

$$N/N_\sigma = N/(N \cap N_\sigma) = (N + M_\sigma)/M_\sigma \subseteq M/M_\sigma = (M_\sigma + R\sigma)/M_\sigma$$

(since $\nu < \tau \Rightarrow (\nu < \sigma \text{ or } \nu = \sigma)$).

Now $(M_\sigma + R\sigma)/M_\sigma$ is finitely generated (by the image of $x_\sigma$ in $M/M_\sigma$) over $R$, so that, as $R$ is Noetherian, $N/N_\sigma$ is also finitely generated, by $\bar{x}_1, \ldots, \bar{x}_r$, say, over $R$. Choose representatives $x_i$ of $\bar{x}_i$, in $N$, for $i = 1, \ldots, r$. Then

$$N = \sum_{i=1}^{r} Rx_i + N_\sigma,$$

and since $N_\sigma \subseteq M_\sigma$ and $\sigma < \tau$, $N$ is generated by at most $|\sigma| + r = |\sigma| = |\tau| = \alpha$ elements, as required.

To prove (iii) it suffices, in the light of (i) and (ii), to show that $|\text{Ass}_R(M)| \leq \alpha$. Let $P \in \text{Ass}_R(M)$; then $P = \text{Ann}_R(x)$ for some $x \in M$. If $\tau$ is a limit ordinal then $M = \bigcup_{\sigma < \tau} M_\sigma$, so that $x \in M_\rho$ for some $\rho < \tau$ and $P \in \text{Ass}_R(M_\rho)$. Conversely, if $P \in \text{Ass}_R(M_\rho)$, for some $\rho < \tau$, then $P = \text{Ann}_R(x)$ for some
x ∈ Mₚ ⊆ M, so P ∈ Assₐ(M). Thus, when τ is a limit ordinal, Assₐ(M) = ∪₀<τ Assₐ(Mₗ). By the inductive assumption,

|Assₐ(Mₗ)| ≤ |σ|, for σ < τ, so

|Assₐ(M)| = ∑₀<τ |Assₐ(Mₗ)| ≤ ∑₀<τ |Assₐ(Mₗ)| ≤ ∑₀<τ |σ| = α.

On the other hand if τ = σ + 1 for some σ ∈ Ord, we have an exact sequence of R-modules,

0 → Mₗ → M → M/Mₗ → 0,

so that Assₐ(M) ⊆ Assₐ(Mₗ) ∪ Assₐ(M/Mₗ) ([Mat2] Theorem 6.3). As in the proof of (ii) (with M instead of N) M/Mₗ is a finitely generated R-module so Assₐ(M/Mₗ) is finite. By the inductive assumption |Assₐ(Mₗ)| ≤ |σ|, so

|Assₐ(M)| ≤ |Assₐ(Mₗ)| + |Assₐ(M/Mₗ)| ≤ max{N₀, |Assₐ(Mₗ)|} ≤ |σ| = |τ| = α. □

We can now generalise [Kap] Theorem 82.

Theorem 1.8.2. Let R be a Noetherian ring, M an α-generated R-module and I an ideal of R such that I ⊆ Z(M). If (Assₐ(M),{0}) is an α-sieve then there is a non zero element m ∈ M such that Im = 0.
Proof. Let $I \subseteq Z(M) \Rightarrow I \subseteq \bigcap \{P \in \text{Ass}_R(M) : P \text{ is maximal in Ass}_R(M)\}$, which is a union of primes from a set of cardinality at most $\alpha$, by Lemma 1.8.1. The result follows. 

Note: Any of the examples of Section 1.6 fulfil the requirements of this theorem as long as $R$ is Noetherian and $\text{Ass}_R(M) \subseteq \mathcal{M}$, where $\mathcal{M}$ is defined in the example in question. In particular $\text{Ass}_R(M) \subseteq \mathcal{M}$ in 1.6.5.

1.9 Sieves and regular sequences

In this section we shall make use of the definitions and basic properties of regular sequences that are given in [Mat2], chapter 6, Section 16. In particular we use the property that if $M$ is a module over a Noetherian ring $R$, and $I$ is any ideal of $R$ such that $IM \neq M$, then any $M$-sequence in $I$ can be extended to a maximal $M$-sequence in $I$. In each of the results of this section the examples of 1.6 meet the $\alpha$-sieving requirements whenever $\text{Spec}(R) \subseteq \mathcal{M}$. This is always true for 1.6.4 and 1.6.5, and does not exclude 1.6.1, 1.6.2 or 1.6.8.

Theorem 1.9.1. (cf. [Kap], Theorem 121) Let $R$ be a Noetherian ring, $M$ an $\alpha$-generated $R$-module and $I$ an ideal of $R$ such that $IM \neq M$. Suppose that $(\text{Spec}(R), \{0\})$ is an $\alpha$-sieve. Then all maximal $M$-sequences in $I$ have the same length, namely
Proof. We show first that, given any \( M \)-sequence in \( I \) of length \( n \),

\[ \text{Ext}_R^i (R/I, M) = 0 \] for all \( i < n \). Let \( x_1, \ldots, x_n \) be an \( M \)-sequence

in \( I \). Set \( M_j = M/(x_1, \ldots, x_j)M \) for \( 1 \leq j \leq n \). Then \( x_1 \notin Z(M) \) and

the following sequence is exact:

\[
0 \rightarrow M \xrightarrow{x_1} M \rightarrow M_1 \rightarrow 0.
\]

Therefore we have a long exact sequence

\[
\cdots \rightarrow \text{Ext}_R^i (R/I, M) \xrightarrow{x_1} \text{Ext}_R^i (R/I, M) \rightarrow \text{Ext}_R^i (R/I, M_1) \rightarrow \cdots,
\]

where \( i \) ranges over \( \mathbb{N} \). We make the inductive assumption that, when

\( n > 1 \), given any \( \alpha \)-generated \( R \)-module \( N \) and an \( N \)-sequence in \( I \) of

length \( n - 1 \), then \( \text{Ext}_R^j (R/I, N) = 0 \) for all \( j < n - 1 \). Since

\( x_2, \ldots, x_n \) is an \( M_1 \)-sequence we have from the inductive hypothesis

that \( \text{Ext}_R^{n-2} (R/I, M_1) = 0 \). Thus

\[
0 \rightarrow \text{Ext}_R^{n-1}(R/I, M) \xrightarrow{x_1} \text{Ext}_R^{n-1}(R/I, M), \quad (*)
\]

is exact. Note that \( (*) \) is also exact when \( n = 1 \). Since \( x_1 \)

annihilates \( R/I \), \( x_1 \) acts as the zero endomorphism on \( \text{Ext}_R^1 (R/I, ) \)

so, from exactness of \( (*) \), \( \text{Ext}_R^{n-1}(R/I, M) = 0 \).

Now suppose that \( x_1, \ldots, x_n \) is a maximal \( M \)-sequence in \( I \). Let

\( 1 \leq i \leq n \) and set \( k = n - i + 1 \). Consider the short exact

sequence
1.9 Sieves and regular sequences

\[ 0 \longrightarrow M_{n-1} \xrightarrow{x_k} M_{n-1} \longrightarrow M_k \longrightarrow 0. \]

Hom\(_R(R/I, \cdot)\) induces a long exact sequence of Ext's, and in particular,

\[ \text{Ext}^i_R(R/I, M_{n-i}) \longrightarrow \text{Ext}^i_R(R/I, M_k) \longrightarrow \text{Ext}^i_R(R/I, M_{n-i}) \]

is exact. Since \(x_k, \ldots, x_n\) is an \(M_{n-1}\)-sequence of length \(i\), by the argument above \(\text{Ext}^i_R(R/I, M_{n-i}) = 0\). Again \(x_k\) annihilates \(R/I\) so acts as the zero endomorphism, and so we have

\[ \text{Ext}^i_R(R/I, M_{n-i+1}) \cong \text{Ext}^i_R(R/I, M_{n-1}) \text{ for } 1 \leq i \leq n. \]

Therefore

\[ \text{Hom}_R(R/I, M_n) \cong \text{Ext}^n_R(R/I, M). \] (**)

Now \(I \subseteq \mathcal{Z}(M_n)\) so, using Lemma 1.8.1 (i) and Theorem 1.8.2, there is a non-zero element \(m \in M_n\) such that \(Im = 0\). This gives us a non-zero map \(R/I \longrightarrow M_n : 1 \mapsto m\). Then (**') implies

\[ \text{Ext}^n_R(R/I, M) \neq 0. \] Since \(\text{Ext}^n_R(R/I, M)\) is independent of the choice of maximal \(M\)-sequence in \(I\), the result follows. \(\square\)

For a module \(M\) over a ring \(R\), with an ideal \(I\), we define depth\((I, M)\) to be \(\inf\{n \in \mathbb{Z} : \text{Ext}^n_R(R/I, M) \neq 0\}\), if this exists, and \(\infty\) otherwise.

Lemma 1.9.2. (cf. [Kap], Theorem 122) Let \(R\) be a Noetherian ring, \(M\) an \(\alpha\)-generated \(R\)-module and \(I\) an ideal of \(R\) such that \(\mathfrak{m}M \neq M\) for any maximal ideal \(\mathfrak{m}\) of \(R\) containing \(I\). Suppose that \((\text{Spec}(R), \{0\})\)
is an $\alpha$-sieve. Then there is a prime ideal $P$ of $R$ containing $I$ such that $\text{depth}(I,M) = \text{depth}(P,M)$.

**Proof.** Let $x_1, \ldots, x_k$ be a maximal $M$-sequence in $I$ and set $J = (x_1, \ldots, x_k)$. Then $I \subseteq Z(M/JM)$, so we have $I \subseteq P$ for some $P \in \text{Ass}_R(M/JM)$, using Lemma 1.8.1 (iii) and the $\alpha$-sieving property.

Thus $P \subseteq Z(M/JM)$ and so if $PM \neq M$, $x_1, \ldots, x_k$ is a maximal $M$-sequence in $P$. Since $P \subseteq M$ for some maximal ideal $M$ of $R$ which necessarily contains $I$, we have $PM \subseteq mM \neq M$ as required.

**Theorem 1.9.3.** (cf. [Kap], Theorem 125) Let $R$ be a Noetherian ring, $M$ an $\alpha$-generated $R$-module and $I$ an ideal of $R$ generated by $k$ elements, $x_1, \ldots, x_k$, such that $IM \neq M$. Assume that $(\text{Spec}(R), \{x\})$ is an $\alpha$-sieve for all $x \in R$. Then

(i) $\text{depth}(I,M) \leq k$, and

(ii) if $\text{depth}(I,M) = k$ then $I$ can be generated by $k$ elements forming an $M$-sequence.

**Proof.** Note first that, from Theorem 1.9.1, $\text{depth}(I,M)$ is finite, and equal to the length of any maximal $M$-sequence in $I$. Suppose $\text{depth}(I,M) \geq k$. Let $i$ be such that $0 \leq i < k$ and assume inductively that we have found elements

$$y_1 = x_1 + h_{12}x_2 + \cdots + h_{1k}x_k,$$

$$y_2 = x_2 + h_{23}x_3 + \cdots + h_{2k}x_k,$$

$$\vdots$$

$$y_i = x_i + \cdots + h_{ik}x_k,$$
where \( y_1, \ldots, y_i \) is an \( M \)-sequence. This is true if

\( i = 0 \). Set \( M_i = M/(y_1, \ldots, y_i)M \). As \( i < \text{depth}(I, M) \), we have

\( I \not\subseteq Z(M_i) \). Assume \( (x_{i+1}, \ldots, x_k) \subseteq Z(M_i) \) and let \( z \) be any element of \( I \). Using the above expressions for \( y_1, \ldots, y_i \), write

\[
z = a_1 x_1 + \cdots + a_k x_k
\]

\[
= a_1 y_1 + b_2 y_2 + \cdots + b_i y_i + b_{i+1} x_{i+1} + \cdots + b_k x_k,
\]

where \( a_r, b_s \in \mathbb{R} \), for \( 1 \leq r \leq k \) and \( 2 \leq s \leq k \). Since

\[
(x_{i+1}, \ldots, x_k) \subseteq Z(M_i)
\]

there is some non-zero element \( m \in M_i \) such that

\[
(b_{i+1} x_{i+1} + \cdots + b_k x_k) m = 0
\]

(Lemma 1.8.1 and Theorem 1.8.2). As \( (y_1, \ldots, y_i) \) annihilates \( M_i \), this implies that \( zm = 0 \), and therefore \( I \subseteq Z(M_i) \), a contradiction. Thus,

\[
(x_{i+1}, \ldots, x_k) = (x_{i+1}, (x_{i+2}, \ldots, x_k)) \not\subseteq Z(M_i),
\]

and using Lemma 1.8.1 (iii), the \( \alpha \)-sieving property and

Corollary 1.7.3, there is an element \( w \in (x_{i+2}, \ldots, x_k) \) such that

\( x_{i+1} + w \notin Z(M_i) \). Set \( y_{i+1} = x_{i+1} + w \). Then \( y_{i+1} \) is of the

required form and \( y_1, \ldots, y_i, y_{i+1} \) is an \( M \)-sequence. By induction we have elements

\[
y_j = x_j + h_j, j+1 x_{j+1} + \cdots + h_{j, k} x_k,
\]

for \( 1 \leq j \leq k \), such that \( y_1, \ldots, y_k \) is an \( M \)-sequence. By construction then, \( I = (y_1, \ldots, y_k) \), so that \( y_1, \ldots, y_k \) is a

maximal \( M \)-sequence in \( I \), and \( \text{depth}(I, M) = k \). The result follows. \( \Box \)

Theorem 1.9.4. (cf. [Kap], Theorem 127) Let \( R \) be a Noetherian ring, \( M \) an \( \alpha \)-generated \( R \)-module and \( I \) an ideal of \( R \). Suppose that \( x \) is an
element of $R$ such that $(I, x)M \neq M$. Set $J = (I, x)$ and assume that $aM$ is closed in the $J$-topology of $M$, for every ideal $a$, of $R$, contained in $I$. Assume further that $(\text{Spec}(R), \{r\})$ is an $\alpha$-sieve for all $r \in R$. Then $\text{depth}(J, M) \leq d + \text{depth}(I, M)$.

**Proof.** Let $\text{depth}(I, M) = k$ and let $x_1, \ldots, x_k$ be a maximal $M$-sequence in $I$. Set $M_k = M/(x_1, \ldots, x_k)M$, so that we have $I \subseteq Z(M_k)$. If $J \subseteq Z(M_k)$ then $\text{depth}(J, M) = k$, and there is nothing further to do. Assume that $(I, x) \notin Z(M_k)$ and use Lemma 1.8.1, the $\alpha$-sieving property and Corollary 1.7.3 to find an element $i \in I$ such that $x + i \notin Z(M_k)$. Let $y = x + i$. Then we have $J = (I, y)$ and $x_1, \ldots, x_k, y$ an $M$-sequence, so that $\text{depth}(J, M) \geq k + 1$. If $J \subseteq Z(M_k/yM_k)$ then $\text{depth}(J, M) = k + 1 = d + \text{depth}(I, M)$, and again there is nothing left to do. Assume that $J \notin Z(M_k/yM_k)$.

Let $S = \{s \in M_k : Is = 0\}$ and note that, since $I \subseteq Z(M_k)$, $S$ is a non-zero submodule of $M_k$ (Theorem 1.8.2). $S$ must be contained in $yM_k$ since otherwise there is some $t \in S \setminus yM_k$, and so $\bar{t} \neq 0$ in $M_k/yM_k$ and $J\bar{t} = (I, y)\bar{t} = 0$, contradicting the assumption that $J \notin Z(M_k/yM_k)$. (Here $\bar{t}$ is the image of $t$ in $M_k/yM_k$.) If $s$ is any element of $S$ then $s = ym$ for some $m \in M_k$. Thus $yM = yym = Is = 0$ and $y \notin Z(M_k)$, so $Im$ must be zero. Therefore $m \in S$, and it follows that $S = yS = y^2S = \ldots = y^nS = \ldots$. This gives,

$$S = \bigcap_{n=0}^{\infty} (y)^n S \subseteq \bigcap_{n=0}^{\infty} J^n S \subseteq \bigcap_{n=0}^{\infty} \left[ \frac{J^n M + (x_1, \ldots, x_k)M}{(x_1, \ldots, x_k)M} \right].$$
hence \[ S \subseteq \bigcap_{n=0}^{\infty} (J^n M + (x_1, \ldots, x_k)M) \]

and therefore \( S = 0 \), since \((x_1, \ldots, x_k)M\) is, by hypothesis, closed in the \( J \)-topology of \( M \). Thus the assumption \( J \notin Z(M_k/yM_k) \) leads to a contradiction, and the result is proved. \( \Box \)

Note: The requirement that \( aM \) be closed in the \( J \)-topology of \( M \) for every ideal \( a \subseteq I \) is a concoction designed to replace Nakayama's lemma. From the proof it is clear that we require only that there is some maximal \( M \)-sequence \( x_1, \ldots, x_k \) in \( I \) such that \((x_1, \ldots, x_k)M\) is closed in the \( J \)-topology of \( M \). For the origin of this condition see Section 1.10.

Theorem 1.9.5. (cf. [Kap], Theorem 129) Let \( R \) be a Noetherian ring, \( M \) an \( \alpha \)-generated \( R \)-module and \( I \) an ideal of \( R \) generated by \( n \) elements \( x_1, \ldots, x_n \). Suppose that \( IM \neq M \) and that \( aM \) is closed in the \( I \)-topology of \( M \), for every ideal \( a \), of \( R \), contained in \( I \). Assume that \( (Spec(R), \{x\}) \) is an \( \alpha \)-sieve for all \( x \in R \). Then

\[ \text{depth}(I,M) = n \iff x_1, \ldots, x_n \text{ is an } M\text{-sequence}. \]

Proof. The implication from right to left is obvious. To see the converse we use induction on \( n \). The result is obvious for \( n = 1 \).

Set \( J = (x_1, \ldots, x_{n-1}) \) and suppose that \( \text{depth}(J,M) < n - 1 \). From Theorem 1.9.4 \( \text{depth}(I,M) < n \), a contradiction. Therefore, using Theorem 1.9.3, \( \text{depth}(J,M) \) must be \( n - 1 \). If \( a \) is any ideal contained in \( J \subseteq I \), then
so that $aM$ is closed in the $J$-topology of $M$. Thus the hypotheses of the theorem are satisfied for $J$ instead of $I$ and we may apply the inductive hypothesis to see that $x_1, \ldots, x_{n-1}$ is an $M$-sequence.

Suppose $x_n \in Z(M/JM)$. Then there is a non zero element $m \in M/JM$ such that $x_nm = 0$. In this case $Im = 0$, so that $I \subseteq Z(M/JM)$ and $\text{depth}(1, M) = n - 1$. This contradiction implies that $x_1, \ldots, x_n$ is an $M$-sequence as required. □

1.10 Sieves and balanced big Cohen-Macaulay modules

In this section we examine modules of maximal depth over a local ring. Throughout the section $R$ will denote a local Noetherian ring and $m$ its maximal ideal. As usual $\alpha$ will denote an arbitrary cardinal. Let $M$ be an $R$-module. The longest possible $M$-sequence in $R$ is of length at most $\dim(R)$ ([Kap], Section 3-1, Exercise 22). If $M$ is $\alpha$-generated, $mM \neq M$ and $(\text{Spec}(R), \{0\})$ is an $\alpha$-sieve then this means that $\text{depth}(m, R) \leq \dim(R)$ (using Theorem 1.9.1). In fact this follows from Corollary (1.5) of [Fo] without the restriction that $(\text{Spec}(R), \{0\})$ is an $\alpha$-sieve. We consider here modules $M$ such that $\text{depth}(m, M) = \dim(R)$.

Given a module $M$ over a local ring $R$ and a system of parameters $a_1, \ldots, a_d$ for $R$, we say that $M$ is a big Cohen-Macaulay
(bCM) module with respect to \( a_1, \ldots, a_d \) if \( a_1, \ldots, a_d \) is an \( M \)-sequence. These modules are introduced and their importance is discussed in [Ho]. If \( M \) is a bCM module with respect to some system of parameters we shall say simply that \( M \) is a bCM module. If \( M \) is a finitely generated bCM \( R \)-module, then every system of parameters for \( R \) is an \( M \)-sequence ([AuBu], Proposition 5.7). As shown, for instance, by an example of P. Griffith (Remark 3.3 of [Grl]) this is not true, in general, for bCM modules which are not finitely generated. A class of modules which displays less drastic differences between finitely and infinitely generated objects has been defined by R.Y Sharp; \( M \) is a balanced big Cohen-Macaulay (bbCM) module if it is a bCM module with respect to every system of parameters for \( R \). In [Sh] it is shown that bbCM modules have many of the properties of finitely generated Cohen-Macaulay modules. If Spec(\( R \)) is a \( \alpha \)-sieve we should like to be able to assert that all \( \alpha \)-generated bCM modules are bbCM modules. However in order to achieve this we have had to impose the extra condition on closure of submodules that appears in the next theorem.

**Theorem 1.10.1.** Let \( R \) be a local Noetherian ring, \( \mathfrak{m} \) the maximal ideal of \( R \) and \( M \) an \( \alpha \)-generated \( R \)-module. Assume that (Spec(\( R \)),\{x\}) is an \( \alpha \)-sieve for all \( x \in R \), depth(\( \mathfrak{m}, M \)) = dim(\( R \)) and \( aM \) is closed in the \( \mathfrak{m} \)-topology of \( M \) for all proper ideals \( a \) of \( R \). Then \( M \) is a bbCM \( R \)-module.

**Proof.** Let \( x_1, \ldots, x_d \) be a system of parameters for \( R \). Set \( I = (x_1, \ldots, x_d) \). By hypothesis, \( M \) is separated in the \( \mathfrak{m} \)-adic
topology so that $IM \subseteq mM + M$. As $I$ is $m$-primary the only prime ideal containing $I$ is $m$. Therefore, from Lemma 1.9.2, we have $\text{depth}(I, M) = \text{depth}(m, M)$. That $x_1, \ldots, x_d$ is an $M$-sequence follows from Theorem 1.9.5. □

Remarks 1.10.2. The condition that $aM$ be closed in the $m$-topology of $M$, for all ideals $a$ of $R$, has been previously employed by Griffith. In what follows we shall refer to this condition as "condition (C)." In [Gr2], Theorem 1.1 it is shown that if $R$ is a complete regular local ring and $M$ is a countably generated $R$-module then the following are equivalent:

(i) $M$ is $R$-free;

(ii) $M$ is a bbCM module and $aM$ is closed for all ideals $a$ of $R$; 

(iii) $M$ is a flat $R$-module and $aM$ is closed for all ideals $a$ of $R$.

(Note that [Gr2], Theorem 1.1 must be read in conjunction with "note added in proof" at the end of [Gr2], in which "1.10" should read "2.10"). Furthermore an example is given ([Gr2], Example 1.3) to show that, in the above result, condition (C) cannot be replaced by the condition that $M$ is separated in the $m$-topology. We modify an example, due to J. Bartijn, to show that this is also the case in Theorem 1.10.1. Let $k$ be a field. Let $R = k[[X, Y, Z]]$, $m = (X, Y, Z)$, $L = \bigsqcup_{n \geq 0} R_n$ and $F = \prod_{n \geq 0} R_n$. We write the element

$$(f_0, f_1, f_2, \ldots)$$

of $F$ as $(f_n)$ and the element

$$(\ell_0, \ldots, \ell_\nu, 0, 0, 0, \ldots)$$

of $L$ as $(\ell_0, \ldots, \ell_\nu)$. Note that, for any ideal $I$ of $R$, $L \cap IF = IL$. Let $\bar{L}$ denote the closure of $L$ in
F in the $m$-topology. Bartijn shows that $\tilde{L} = \{(a_n) \in F : \lim_{n \to \infty} a_n = 0\}$ and that $\tilde{L} = \hat{L}$, where $\hat{L}$ is the $m$-adic completion of $L$ ([Ba], Chapter 1, Proposition 2.8). It follows from this that $\tilde{L}$ is separated in the $m$-adic topology and that $\tilde{L}$ is $R$-flat (loc. cit. Theorem 2.9 (i) and (ii); see also [Gr1], Section 1). Hence, if $r \in R$ and $r$ is not a zero divisor in $R$, the multiplication map $R \longrightarrow R$ remains injective on tensoring by $\tilde{L}$, and so $r$ is an $\tilde{L}$-regular element of $R$. Thus

\[ r \in R \text{ is } \tilde{L}\text{-regular if } r \text{ is } R\text{-regular.} \]  

Given any $k \in \mathbb{N}$, let $g_k = (f_k, n) \in F$ where $f_k, n = 0$, for $0 \leq n < k$, and $f_k, n = Z^{n-k}$, for $n \geq k$. Let $G = \{g_k \in F : k = 0, 1, 2, \ldots \}$. Let $T$ be the submodule of $F$ generated by the elements of $G$ over $R$. Since $G \subseteq \tilde{L}$, it is clear that $T \subseteq \tilde{L}$. Let $t$ be any element of $T$. Write $t = \sum_{i=0}^{s} r_i g_{k_i}$, where $r_i \in R$ and $g_{k_i} \in G$, for $0 \leq i \leq s$. We may assume, without loss of generality, that $0 \leq k_1 < k_2 < \ldots < k_s$. Then $t = (t_n)$ where $t_n = 0$, if $0 \leq n < k_1$, \[ t_n = r_1 Z^{n-k_1} + r_2 Z^{n-k_2} + \ldots + r_j Z^{n-k_j}, \] if $k_j \leq n < k_{j+1}$ and $1 \leq j < s$, and \[ t_n = r_1 Z^{n-k_1} + r_2 Z^{n-k_2} + \ldots + r_s Z^{n-k_s}, \] if $n \geq k_s$. Let $\ell = (t_0, t_1, \ldots, t_{k_s})$; $\ell \in L$ and we have

\[ t = \ell + Z(r_1 Z^{k_s-k_1} + \ldots + r_j Z^{k_s-k_j} + \ldots + r_s)g_{k_s+1}, \] so that $t \in L + ZT$. We have shown that
1.10 Balanced big Cohen-Macaulay modules

Now define $M = L + (X, Y)T$. Facts a) to d) below show that we have the example we require.

a) $R$ is a complete (regular) local Noetherian ring. It follows from Example 1.6.5 (2) that $(\text{Spec}(R), \{r\})$ is an $N_0$-sieve for all $r \in R$. $M$ is an $N_0$-generated $R$-module, since both $L$ and $T$ are $N_0$-generated. Furthermore $M \subseteq L$ so that $M$ is separated in its $m$-topology.

$$T \subseteq L + ZT. \quad (***)$$

b) $Z, X, Y$ is an $M$-sequence so that $M$ is a bCM $R$-module. That $Z$ is $M$-regular follows from $(*)$, since $M \subseteq L$ and $Z$ is $R$-regular. Given $m \in M$ such that $Xm \in ZM$ we wish to show that $m \notin ZM$. Write

$$m = \ell + Xt_1 + Yt_2,$$

where $\ell \in L$ and $t_1, t_2 \in T$. From $(***)$ we have $t_1 = \ell_1 + Zt_3$ and $t_2 = \ell_2 + Zt_4$, where $\ell_1, \ell_2 \in L$ and $t_3, t_4 \in T$. Thus

$$m = \ell + X\ell_1 + Y\ell_2 + Z(Xt_3 + Yt_4). \quad (***)$$

Since $Z(Xt_3 + Yt_4) \in ZM$ and by assumption

$$Xm = X(\ell + X\ell_1 + Y\ell_2) + ZX(Xt_3 + Yt_4) \in ZM,$$

it follows that $X(\ell + X\ell_1 + Y\ell_2) \in ZM \subseteq ZF$. Therefore

$$X(\ell + X\ell_1 + Y\ell_2) \in L \cap ZF = ZL. \quad \text{Since } L \text{ is } R\text{-free it follows that } \ell + X\ell_1 + Y\ell_2 \in ZL. \quad \text{Substitution in (***) gives } m \in ZL + ZM = ZM.$$

Therefore $X$ is $(M/ZM)$-regular. To see that $Y$ is $(M/(Z,X)M)$-regular we argue similarly. Suppose $m \in M$ is such that $Ym \in (Z,X)M$. As before, express $m$ in the form of (**). This time

$$X\ell_1 + Z(Xt_3 + Yt_4) \in (Z,X)M. \quad \text{Therefore, as in the above, it is enough to show that } Y(\ell + Y\ell_2) \in (Z,X)M \text{ implies that}$$
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\[ \ell + Y\ell_2 \in (Z,X)M. \] This is immediate, since if
\[ \ell + Y\ell_2 \in (Z,X)M \subseteq (Z,X)F \] then we must have
\[ \ell + Y\ell_2 \in L \cap (Z,X)F = (Z,X)L, \] and as \( L \) is free this implies
that \( \ell + Y\ell_2 \in (Z,X)L \), as required. Thus \( Y \) is \((M/(Z,X)M)\)-regular.

Finally, note that \( e_1 \in L \subseteq M \), and \( e_1 \notin (Z,X,Y)M \), so that
\( M \neq (Z,X,Y)M \). Hence \( Z,X,Y \) is an \( M \)-sequence. Since \( Z,X,Y \) is a
system of parameters for \( R \), \( M \) is a \( bCM \) module.

c) \( X,Y,Z \) is not an \( M \)-sequence, so \( M \) is not a \( bbCM \) module. As \( X \)
is \( R \)-regular, \( X \) is also \( \overline{L} \)-regular (using (\( * \))). Therefore \( X \) is
\( M \)-regular. Let \( u = g_0 \). Then \( u \in \overline{L} \) and \( u \notin M \). Suppose \( Xu \in XM \).
Then \( Xu = Xn \) for some \( n \in M \). Thus \( X(u - n) = 0 \) and \( X \) is \( \overline{L} \)
regular, so that \( u \in M \), a contradiction. Hence \( Xu \notin XM \). However
\( Y \) is \((X,Y)T \subseteq M \) so that \( Y(Xu) = X(Yu) \in XM \). As \( Xu \in (X,Y)T \subseteq M \),
this shows that \( Y \) is not \((M/XM)\)-regular, and so \( X,Y,Z \) cannot be an
\( M \)-sequence. As \( X,Y,Z \) is a system of parameters for \( R \), \( M \) cannot be
a \( bbCM \) module.

d) \( XM \) is not closed in the \( m \)-topology of \( M \). To see this consider
the element \( u \) defined in c) above. Given any integer \( \omega \geq 0 \) we can
write
\[ Xu = X(1,Z,Z^2, \ldots ,Z^{\omega-1}) + Z^\omega Xg_\omega, \]
so
\[ Xu \in XL + (X,Y,Z)^\omega XT \subseteq XM + m^\omega M, \]
for all \( \omega \geq 0 \). Therefore
\[ Xu \in \bigcap_{\omega=0}^{\infty} (XM + m^\omega M), \]
which is the closure of \( XM \) in the \( m \)-topology of \( M \). Since, from c)
above, \( Xu \notin XM \), \( XM \) is not closed in the \( m \)-adic topology.
As we have seen, all the hypotheses of Theorem 1.10.1 are satisfied by \( R \) and \( M \), except condition (C). Instead of condition (C) we have \( M \) separated in the \( m \)-topology. However the conclusion of the theorem does not hold.

1.10.3. We remark that the example of 1.10.2 is a counter-example to [Gr1], Proposition 2.9, for the case of \( M \) a countably generated module. We do not wish to become involved in more definitions at this stage so we shall use terminology from [Ba] and [Gr1] without explanation. For the definition of purity see [Mat2] or [Rot]. To see that \( L \) is a basic submodule of \( M \) (using the definition of basic submodule given by [Gr1], Section 2), note first that, since \( L \subseteq F \) is pure (using [Ba], Proposition 2.5 (i) and Proposition 2.2 (iii)), \( L \subseteq M \) is pure (using, for example, [Mat2], Theorem 7.13). Furthermore, for any integer \( k \geq 0 \),

\[
M = L + (X,Y)^T \subseteq L + Z^k(X,Y)^T,
\]

by repeated applications of \((*)\), so that \( M = L + m^kM \), for all \( k > 0 \). Since \( L \) is free, \( L \) is a basic submodule of \( M \) as claimed. In fact this follows from the discussion of [Ba], Chapter V, Section 2, Paragraph 1, and [Ba], Chapter I, Definition 4.9. As shown by 1.10.2 d), Griffith's claim, in the proof of [Gr1], Proposition 2.9, that his \( xM \) is closed in the \( m \)-adic topology, is false. This does not affect [Gr1], Proposition 2.10, which stands for countably generated modules (cf. [Ba], Chapter V, page 113, and [Gr2], Theorem 1.1 and "Note added in proof").
1.11 Sieves and localisation

The final application of this chapter demonstrates that sieves are in a sense "Noetherian" subsets of Spec(R); that is if $\mathcal{M}$ is a subset of Spec(R) of cardinality $\alpha$ and $\mathcal{M}$ is an $\alpha$-sieve then $\mathcal{M}$ has maximal elements.

Theorem 1.11.1. Let $R$ be a ring, $\alpha$ a cardinal and $\mathcal{M}$ be a subset of Spec(R) such that $|\mathcal{M}| \leq \alpha$ and $(\mathcal{M}, \{0\})$ is an $\alpha$-sieve. Let $F = \bigcup \{P : P \in \mathcal{M}\}$, and set $S = R \setminus F$. Then $S$ is a multiplicatively closed subset of $R$ and

$$\text{Max-Spec}(S^{-1}R) = \{S^{-1}P : P \text{ is maximal in } \mathcal{M}\}.$$ 

Proof. Let $P$ be in $\mathcal{M}$. Then $P \cap S = \emptyset$, so $S^{-1}P \in \text{Spec}(S^{-1}R)$.
Suppose $P$ is maximal in $\mathcal{M}$ and $\mathcal{M}$ is a maximal ideal of $S^{-1}R$ containing $S^{-1}P$. Then $\mathcal{M} = S^{-1}Q$ for some $Q \in \text{Spec}(R)$ such that $Q \cap S = \emptyset$. Thus $Q \subseteq F = \bigcup \{P : P \in \mathcal{M}\}$, and using the $\alpha$-sieving property, $Q \subseteq P'$ for some $P' \in \mathcal{M}$. Since $S^{-1}P \subseteq S^{-1}Q$, we have $P \subseteq Q \subseteq P'$, and so $P = Q = P'$, using the maximality of $P$ in $\mathcal{M}$. Thus $S^{-1}P = \mathcal{M} \in \text{Max-Spec}(S^{-1}R)$.
Conversely if $M$ is in $\text{Max-Spec}(S^{-1}R)$, then $M = S^{-1}Q$ for some $Q \in \text{Spec}(R)$, and as before $Q \subseteq P$, for some $P \in \mathfrak{M}$. Thus $S^{-1}Q \subseteq S^{-1}P$ and, since $S^{-1}Q$ is maximal, $S^{-1}Q = S^{-1}P$. Clearly $P$ is maximal in $\mathfrak{M}$. \qed
The theme of this chapter is the development of an idea of Zariski to prove, in the first instance, a uniform Artin-Rees theorem, and generally as a useful and interesting notion. The results of this chapter have been found whilst working in collaboration with L. O'Carroll. Indeed, presented herein is a development of [O'C1], in which paper a weak form of the uniform Artin-Rees theorem is proved using Zariski's idea.

The essence of the idea which forms our subject is contained in Zariski's definition of $(W, \nu)$-regular points of an irreducible algebraic variety ([Zar], Section 4). This gives Zariski his "key" to the proof of his Main Lemma ([Zar], Sections 2 and 5). A generalisation of this Main Lemma, with a very concise proof, is demonstrated by Eisenbud and Hochster in [EiHo]. In [EiHo] the authors pose the following question: Let $R$ be an affine ring, and suppose that $M \subseteq N$ are finitely generated $R$-modules. Is there an integer $k_0$ such that for all $k > k_0$ and all maximal ideals $m$ of $R$

\[ M \cap m^kN = m^{k-k_0}(M \cap m^{k_0}N) \ ? \]

This question is partially answered by the result of O'Carroll, in [O'C1], mentioned above. The main result of [O'C1] is proved using Zariski's concept of $(W, \nu)$-regularity (and is sufficient to prove the theorem of [EiHo]). Developing the idea of $(W, \nu)$-regularity and applying techniques similar to those in [O'C1] we obtain the results
of [DO'C1] (and Section 2.1 of this thesis), which settle Eisenbud and Hochster's conjecture as fact. Following the route taken by Zariski, in [Zar], we obtain a generalisation of the Main Lemma of [Zar], which has the theorem of [EiHo] as a corollary. Further examples of the utility of the theory can be found in [DO'C2].

2.1 Zariski regularity

The first definition below is a generalisation of Zariski's concept of \((W, ν)\)-regularity. The precise connection with Zariski's original definition will become clear as we investigate the properties of our own definition.

Definition 2.1.1. Let \(R\) be a ring, \(P\) and \(Q\) prime ideals of \(R\), and \(M\) and \(N\) \(R\)-modules such that \(P \subseteq Q\) and \(M \subseteq N\). Let \(r\) be a non-negative integer. Then \(Q\) is said to be \((M, N; P; r)\)-regular if

\[
\left[ \frac{P^r N}{(M \cap P^r N) + P^{r+1} N} \right]_Q \text{ is } [R/P]_Q\text{-free.}
\]

Definition 2.1.2 If \(Q\) is \((M, N; P; r)\)-regular for all integers \(r \geq 0\), then \(Q\) is said to be \((M, N; P)\)-regular.

We shall informally refer to both \((M, N; P; r)\)-regular and \((M, N; P)\)-regular primes as Zariski regular primes.
Standing notation 2.1.3. For simplicity we introduce some notation that is to endure the length of this chapter. Given a ring $R$, prime ideals $P$ and $Q$ of $R$ such that $P \subseteq Q$, and $R$-modules $M$ and $N$ such that $M \subseteq N$ we set

\[
E_r = \frac{P^r N}{(M \cap P^r N) + P^{r+1} N},
\]

\[
F_r = \frac{(M \cap P^r N) + P^{r+1} N}{P^{r+1} N},
\]

\[
G_r = \frac{P^r N}{P^{r+1} N},
\]

\[
H_r = \frac{(M \cap P^r N) + Q^{r+1} N}{Q^{r+1} N},
\]

and $A = R/P$.

2.1.4. From time to time we shall use the expressions "after localisation at $q$" or "supressing localisation at $q$", when $q$ is a prime ideal. By this we mean that we have written "$D_q$" as "D" for any module $D$ occurring in the text referred to.

2.1.5. Given a ring $R$ and prime ideals $p \subseteq q$ of $R$, we shall often write $\frac{R_q}{pR_q}$ as $[R/pR]_q$. It should always be clear from the context (when it makes any odds) whether we are localising the $R$-module at $q$, or the $R/p$ ring at $q/p$. We shall also write
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In Section 2.4 we shall reconsider this notation.

We shall now give several characterisations of Zariski regularity. First we need a definition. Let $R$ be a Noetherian ring, $I$ an ideal of $R$ and $L$ a finitely generated $R$-module. We say that $L$ is normally flat along $\mathcal{V}(I)$ if the $R/I$-module $\text{gr}_1(L)$ is $R/I$-flat. If $R/I$ is a local ring then $L$ is normally flat along $\mathcal{V}(I)$ if and only if

$$\frac{l^iL}{l^{i+1}L}$$

is $R/I$-free, for all $i \geq 0$. In fact, given that $R/I$ is local, we have:

L is normally flat along $\mathcal{V}(I)$

$\iff \frac{l^iL}{l^{i+1}L}$ is $R/I$-flat, for all $i \geq 0$,

$\iff \frac{l^iL}{l^{i+1}L}$ is $R/I$-free, for all $i \geq 0$, ([Mat2], Theorem 7.10),

$\iff \text{gr}_1(L)$ is $R/I$-free, ([Mat2], Theorem 2.5). (2.1.6)

Normal flatness is defined and used by Hironaka in [Hir].

Lemma 2.1.7. Let $R$ be a Noetherian ring, $P \subseteq Q$ prime ideals of $R$, $M \subseteq N$ finitely generated $R$-modules and $r$ a non-negative integer. Then, in the notation of 2.1.3,

1. $Q$ is $(M,N;P;r)$-regular if and only if

$$\text{Tor}_1^R([E_r]_Q, \kappa(Q)) = 0;$$

2. $Q$ is $(M,N;P)$-regular if and only if $[N/M]_Q$ is normally flat along the closed subset $\mathcal{V}(P_Q)$ of
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Spec(RQ);

(3) if, in addition to the given hypotheses,

\( Q/P \in \text{Reg-Spec}(A) \), then \( Q \) is \((M,N;P;r)\)-regular if and only if there exists a \( A_Q \)-module \( L \), of depth zero, such that \( \text{Tor}_1^{A_Q}(\mathbb{E}_r Q, L) = 0 \).

Proof. (1) follows immediately from Definition 2.1.1 and [Ser], Chapitre IV, Proposition 20. Since

\[
E_r = \frac{P^{r}N}{(M \cap P^{r}N) + P^{r+1}N} = \frac{P^{r}N + M}{P^{r+1}N + M} = \frac{P^{r}(N/M)}{P^{r+1}(N/M)},
\]

(2) follows from 2.1.6. In the light of (1) to prove (3) we need only show that if there exists an \( A_Q \)-module \( L \), of depth zero, such that \( \text{Tor}_1^{A_Q}(\mathbb{E}_r Q, L) = 0 \), then \( Q \) is \((M,N;P;r)\)-regular. This follows from [Lic], Corollary 6. \( \square \)

In the proof of the next theorem and at intervals thereafter we shall use the following fact:

Fact 2.1.8. Let \( R \) be a Noetherian ring, \( P \subset Q \) prime ideals of \( R \), \( N \) a finitely generated \( R \)-module and \( r \) a non-negative integer. Assume

that \( Q/P \in \text{Reg-Spec}(R/P) \), that \( \left[ \frac{P^i}{P^{i+1}} \right]_Q \) is \( A_Q \)-free, for \( i = 0,1, \ldots, r-1 \), and that \( N_Q \) is \( R_Q \)-free. Then

\[
[P^{r+s}N]_Q = [P^{r}N \cap Q^{r+s}N]_Q \text{ for all } s \geq 0.
\]
2.1 Zariski regularity

Proof. The case \( r = 0 \) is trivial so we may assume that \( r > 0 \). It follows from Lemma 1.3 of [RobVa] that a regular system of parameters for \( [R/P]_Q \) form a regular sequence modulo \( P^r \). Thus, since \( N_Q \) is \( R_Q \)-free, 2.1.8 follows from [RobVa], Lemma 1.1. □

Theorem 2.1.9. Let \( R \) be a Noetherian ring, \( P \subset Q \) prime ideals of \( R \), \( M \subseteq N \) finitely generated \( R \)-modules and \( r \) a non-negative integer. Assume that \( Q/P \in \text{Reg-Spec}(R/P) \), that \( [R/P]_Q \)-free, for \( i = 0,1, \ldots, r \), and that \( N_Q \) is \( R_Q \)-free. Then the following are equivalent:

1. \( Q \) is \((M,N;P;r)\)-regular;

2. there exists an integer \( s \geq 1 \) such that
   \[
   [(M \cap ((P^rN \cap Q^{r+s}N) + P^{r+1}N)) + P^{r+1}N]_Q = [Q^s(M \cap P^rN) + P^{r+1}N]_Q;
   \]

3. \[
   [(M \cap P^rN \cap Q^{r+1}N) + P^{r+1}N]_Q = [Q(M \cap P^rN) + P^{r+1}N]_Q;
   \]

4. \[
   [(M \cap JP^rN) + P^{r+1}N]_Q = [J(M \cap P^rN) + P^{r+1}N]_Q \text{ for all ideals } J \text{ of } R \text{ such that } J \supseteq P.
   \]

Proof. We show that \( (1) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \). To see that \( (1) \Rightarrow (4) \) recall the notation of 2.1.3 and consider the exact sequence of \( A_Q \)-modules
Let $J$ be any ideal of $R$ containing $P$. Then $R/J$ is an $A$-module and we can apply $\oplus_A R/J$ to 2.1.10. This gives a sequence
\[
0 \longrightarrow \frac{(M \cap P^r N) + P^{r+1} N}{J(M \cap P^r N) + P^{r+1} N} \longrightarrow \frac{P^r N}{JP^r N} \longrightarrow \frac{P^r N}{JP^r N + (M \cap P^r N)} \longrightarrow 0,
\]
where localisation at $Q$ has been suppressed. Now suppose that (1) is true. Then $A_Q\text{Tor}_1([E_r]_Q, [R/J]_Q) = 0$ (since $[E_r]_Q$ is $A_Q$-flat), so that the above sequence is exact. Therefore
\[
[((M \cap P^r N) + P^{r+1} N) \cap JP^r N]_Q \subseteq [J(M \cap P^r N) + P^{r+1} N]_Q,
\]
which implies that
\[
[M \cap JP^r N + P^{r+1} N]_Q \subseteq [J(M \cap P^r N) + P^{r+1} N]_Q,
\]
since $J \supseteq P$. The reverse inclusion is obvious, giving (4). Given (4), let $J = Q$. Then it is easy to see that (4) $\Rightarrow$ (3), since, from 2.1.8, $[P^r Q^r N]_Q = [P^r N \cap Q^{r+1} N]_Q$. If we set $s = 1$ in (2) we see that (3) $\Rightarrow$ (2). Given (2), let $s \geq 1$ be such that
\[
[((M \cap P^r N) \cap (Q^{r+s} N + P^{r+1} N)) + P^{r+1} N]_Q = [Q^s (M \cap P^r N) + P^{r+1} N]_Q,
\]
so that
\[
[((M \cap P^r N) \cap (Q^{r+s} N + P^{r+1} N)) + P^{r+1} N]_Q = [Q^s (M \cap P^r N) + P^{r+1} N]_Q,
\]
Using 2.1.8, this implies that
\[
[((M \cap P^r N) \cap (P^r Q^s N + P^{r+1} N)) + P^{r+1} N]_Q \subseteq [Q^s (M \cap P^r N) + P^{r+1} N]_Q,
\]
and therefore
\[
[((M \cap P^r N) + P^{r+1} N) \cap (P^r Q^s N + P^{r+1} N)]_Q \subseteq [Q^s (M \cap P^r N) + P^{r+1} N]_Q.
\]
After localisation at $Q$, we now have an injective map
The next theorem will relate Zariski's original definition of $(W,\nu)$-regularity to Definition 2.1.1.

**Theorem 21.12.** Let $R$ be a Noetherian ring, $P \subseteq Q$ prime ideals of $R$, $M \subseteq N$ finitely generated $R$-modules and $r$ a non-negative integer. Assume that $Q/P \in {\text{Reg-Spec}}(R/P)$, that $\left[\frac{P^i}{P^{i+1}}\right]_Q$ is $[R/P]_Q$-free, for $i = 0, 1, \ldots, r$, and that $N_Q$ is $R_Q$-free.

Recall the notation of 2.1.3 and consider the following conditions:

1. $\dim_{\kappa(Q)}(H_r \otimes_{R/Q} \kappa(Q)) = \dim_{\kappa(Q)}(F_r \otimes_A \kappa(Q))$;

2. $\dim_{\kappa(Q)}(H_r \otimes_{R/Q} \kappa(Q)) = \dim_{\kappa(P)}(F_r \otimes_A \kappa(P))$;
(3) \[ \dim_{\kappa(Q)}(E_r \otimes_A \kappa(Q)) = \dim_{\kappa(P)}(E_r \otimes_A \kappa(P)); \]

(4) \[ \dim_{\kappa(Q)}(F_r \otimes_A \kappa(Q)) = \dim_{\kappa(P)}(F_r \otimes_A \kappa(P)); \]

(5) \( Q \) is \((M,N;P;r)-regular.\)

We have (1) \( \iff \) (2) \( \iff \) (3) \( \iff \) (5) \( \rightarrow \) (4).

**Proof.** By hypothesis \([G_r]_Q\) is \(A_Q\)-free; \([G_r]_Q \cong (A_Q)^e\), say, for some integer \(e\). Hence, for any \(q \in \text{Spec}(R)\) such that \(P \subseteq q \subseteq Q\),

\[ G_r \otimes_A \kappa(q) \cong [G_r]_Q \otimes_{A_Q} \kappa(q) \cong (A_Q)^e \otimes_{A_Q} \kappa(q) \cong (\kappa(q))^e. \]

In particular, therefore

\[ \dim_{\kappa(Q)}(G_r \otimes_A \kappa(Q)) = \dim_{\kappa(P)}(G_r \otimes_A \kappa(P)). \tag{*} \]

Let \(a_1, \ldots, a_c\) be elements of \(F_r\) such that their images in \(F_r \otimes_A \kappa(Q)\) form a basis for \(F_r \otimes_A \kappa(Q)\) over \(\kappa(Q)\). Then \(a_1, \ldots, a_c\) generate \([F_r]_Q\) (using the minimal basis theorem; [Mat2], Theorem 2.3). Hence the images of \(a_1, \ldots, a_c\) span \([F_r]_P \cong F_r \otimes_A \kappa(P)\) over \(\kappa(P)\). Therefore

\[ \dim_{\kappa(P)}(F_r \otimes_A \kappa(P)) \leq \dim_{\kappa(Q)}(F_r \otimes_A \kappa(Q)). \tag{†} \]

Let \(b_1, \ldots, b_d\) be elements of \(M \cap P^rN\) such that their images in \([H_r]_Q \cong H_r \otimes_{R/Q} \kappa(Q)\) form a basis for \([H_r]_Q\) over \(\kappa(Q)\). Then there are elements \(b_{d+1}, \ldots, b_f\) in \(P^rN\) such that the images of \(b_1, \ldots, b_f\) in \(\left[ \frac{P^rN + Q^{r+1}N}{Q^{r+1}N} \right]_Q\) form a basis for
Hence the images of $b_1, \ldots, b_f$ in $[G_R]_Q$ form a minimal generating set for $[G_R]_Q$ over $A_Q$ ([Mat2], Theorem 2.3) (and $f$ here must equal $e$ above). Thus the images of these elements span $[G_R]_P \cong G_R \otimes_A \kappa(P)$ and, using (*), they must form a basis for $[G_R]_P \cong G_R \otimes_A \kappa(P)$ over $\kappa(P)$. Therefore the images of $b_1, \ldots, b_d$ in $[F_R]_P \cong F_R \otimes_A \kappa(P)$ are linearly independent over $\kappa(P)$. It follows that

$$\dim_{\kappa(Q)}(H_R \otimes_{R/Q} \kappa(Q)) = d$$

$$\leq \dim_{\kappa(P)}(F_R \otimes_A \kappa(P)).$$

(††)

Combining (†) and (††) we see that (1) $\rightarrow$ (2) and (1) $\rightarrow$ (4). We show now that (2) $\leftrightarrow$ (3). We have an exact sequence of $\kappa(Q)$–modules

$$0 \rightarrow \begin{bmatrix} P^r N + Q^{r+1} N \\ Q^{r+1} N \end{bmatrix}_Q \rightarrow \begin{bmatrix} P^r N + Q^{r+1} N \\ Q^{r+1} N \end{bmatrix}_Q \rightarrow \begin{bmatrix} P^r N + Q^{r+1} N \\ (M \cap P^r N) + Q^{r+1} N \end{bmatrix}_Q \rightarrow 0.$$
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\[ \dim_\kappa(Q) \left[ \frac{(M \cap P^N) + Q^{r+1}N}{Q^{r+1}N} \right] = \dim_\kappa(Q) \left[ \frac{P^N + Q^{r+1}N}{Q^{r+1}N} \right] \]

Moreover

\[ \left[ \frac{(M \cap P^N) + Q^{r+1}N}{Q^{r+1}N} \right] = R_{\Theta R/Q} \kappa(Q) \]

and, using 2.1.8,

\[ \left[ \frac{P^N + Q^{r+1}N}{(M \cap P^N) + Q^{r+1}N} \right] = R_{\Theta R/A} \kappa(Q) \]

Hence, using (\*) and (**)\n
\[ \dim_\kappa(Q)\left( H_{\Theta R/Q} \kappa(Q) \right) = \dim_\kappa(P)\left( G_{\Theta A} \kappa(P) \right) \]

\[ + \dim_\kappa(Q)\left( E_{\Theta A} \kappa(Q) \right). \quad (***) \]

From the exact sequence of \(\kappa(P)\)-modules

\[ 0 \rightarrow F_{\Theta A} \kappa(P) \rightarrow G_{\Theta A} \kappa(P) \rightarrow E_{\Theta A} \kappa(P) \rightarrow 0 \]

we have

\[ \dim_\kappa(P)\left( G_{\Theta A} \kappa(P) \right) = \dim_\kappa(P)\left( E_{\Theta A} \kappa(P) \right) + \dim_\kappa(P)\left( F_{\Theta A} \kappa(P) \right). \]

Substitution in (*** \(\star\)) gives

\[ \dim_\kappa(Q)\left( H_{\Theta R/Q} \kappa(Q) \right) = \dim_\kappa(P)\left( F_{\Theta A} \kappa(P) \right) + \dim_\kappa(P)\left( E_{\Theta A} \kappa(P) \right) \]

\[ + \dim_\kappa(P)\left( E_{\Theta A} \kappa(P) \right) \]

\[ - \dim_\kappa(Q)\left( E_{\Theta A} \kappa(Q) \right). \]

This shows (2) \(\Leftrightarrow\) (3). Given (3) we have
\[ \text{dim}_{\kappa(P)}([E_r]_{\mathbb{Q}} \otimes_{\mathbb{Q}} \kappa(P)) = \text{dim}_{\kappa(P)}(E_r \otimes_A \kappa(P)) \]

\[ = \text{dim}_{\kappa(Q)}(E_r \otimes_A \kappa(Q)) \]

\[ = \text{dim}_{\kappa(Q)}([E_r]_{\mathbb{Q}} \otimes_{\mathbb{Q}} \kappa(Q)). \]

Therefore (5) follows from (3), using [Har], Chapter II, Lemma 8.9.

It is easy to see that (5) \( \rightarrow \) (3). We have shown (2) \( \iff \) (3) \( \iff \) (5).

We shall now show that (5) \( \rightarrow \) (4). We shall then have (5) \( \rightarrow \) (1), since, (2) and (4) together imply (1). Given (5) the short exact sequence of \( A_{\mathbb{Q}} \)-modules (2.1.10) splits. Therefore \([F_r]_{\mathbb{Q}}\) is a direct summand of the \( A_{\mathbb{Q}} \)-free module \([G_r]_{\mathbb{Q}}\). Since \( A_{\mathbb{Q}} \) is local, this implies that \([F_r]_{\mathbb{Q}}\) is a free \( A_{\mathbb{Q}} \)-module. Tensoring \([F_r]_{\mathbb{Q}}\) over \( A_{\mathbb{Q}} \), first with \( \kappa(P) \) and then with \( \kappa(Q) \), we obtain (4).

This completes the proof of the theorem. \( \square \)

Remarks 2.1.13. (2) is a generalised version of Zariski's original regularity condition. That (5) and (2) are equivalent is shown by L. O'Carroll in [O'C1].

The properties that we require of \((M,N;P;r)\)-regularity in the proof of the uniform Artin-Rees theorem also turn out to characterise it, in perhaps a more useful way. This characterisation is the content of the next theorem.

Theorem 2.1.14. Let \( R \) be a Noetherian ring, \( P \subseteq Q \) prime ideals of \( R \), \( M \subseteq N \) finitely generated \( R \)-modules and \( r \) a non-negative integer. Assume that \( Q/P \in \text{Reg-Spec}(R/P) \), that \( \left[ \frac{p^i}{p^{i+1}} \right]_Q \) is
[R/P]_Q-free, for i = 0, 1, \ldots, r, and that N_0 is R_0-free. Then the following are equivalent:

1. Q is (M, N; P; r)-regular;

2. \[M \cap P^N \cap Q^{r+s}N]_Q = [Q^S(M \cap P^N) + (M \cap P^{r+1}N \cap Q^{r+s}N)]_Q \text{ for all } s \geq 0;

3. there exists an integer s \geq 1 such that
   a. \[M \cap P^N \cap Q^{r+s}N]_Q = [Q^S(M \cap P^N) + (M \cap P^{r+1}N \cap Q^{r+s}N)]_Q \text{ and }
   b. \[(M \cap P^Q^S N) + P^{r+1}N]_Q = [(M \cap P^Q^S (P + N)) + P^{r+1}N]_Q.

Furthermore if any of these three conditions holds then 3 (b) holds for all s \geq 0.

Proof. We show first that (1) \rightarrow (2). Suppose (1) is true. Let s be any non-negative integer. Set J = P + Q^S. Then, from (4) of Theorem 2.1.9,

\[[(M \cap (Q^S P^N + P^{r+1}N)) + P^{r+1}N]_Q = [Q^S(M \cap P^N) + P^{r+1}N]_Q,\]

and so, using 2.1.8,

\[[(M \cap P^N \cap Q^{r+s}N) + P^{r+1}N]_Q \subseteq [Q^S(M \cap P^N) + P^{r+1}N]_Q.\]

Intersecting with \[M \cap Q^{r+s}N]_Q we obtain
so that

\[ (M \cap PrN \cap Q^{r+s}N) \subseteq [Q^s(M \cap PrN) + (M \cap Pr^{r+1}N \cap Q^{r+s}N)]_Q. \]

The reverse inclusion is obvious, giving (2). If (2) holds (3) (a) is trivial, and in particular holds in the case \( s = 1 \). Set \( s = 1 \) and (3) (b) becomes trivial. Hence (2) \( \Rightarrow \) (3).

Suppose that (3) is true. Let \( s > 1 \) be such that (3) (a) and (b) hold. Then, supressing localisation at \( Q \),

\[ (M \cap ((PrN \cap Q^{r+s}N) + Pr^{r+1}N)) + Pr^{r+1}N \]

\[ = (M \cap Pr(Q^s + P)N) + Pr^{r+1}N \]

(using 2.1.8)

\[ = (M \cap PrQ^sN) + Pr^{r+1}N \] (by (3) (b))

\[ = Q^s(M \cap PrN) + (M \cap Pr^{r+1}N \cap Q^{r+s}N)) + Pr^{r+1}N \]

(by (3) (a))

\[ = Q^s(M \cap PrN) + Pr^{r+1}N. \]

This is condition (2) of Theorem 2.1.9, so (1) follows.

Finally, given that one of these conditions holds, after localisation at \( Q \), for any \( s \geq 0 \),

\[ (M \cap PrQ^sN) + Pr^{r+1}N = Q^s(M \cap PrN) + Pr^{r+1}N, \]

from (2) and 2.1.8, so that, after localisation at \( Q \),

\[ (M \cap PrQ^sN) + Pr^{r+1}N = (M \cap Pr(Q^s + P)N) + Pr^{r+1}N, \]

using (1) of this theorem and (4) of Theorem 2.1.9, with \( J = Q^s + P \). Hence (3) (b) holds for all \( s \geq 0 \).
Remark 2.1.15. That (1) $\Rightarrow$ (2) is the property of Zariski regularity crucial to the proof of the uniform Artin-Rees theorem and is due to L. O'Carroll, who also realised that the reverse implication would be valid.

To prove his Main Lemma, Zariski used a "key" property which is a direct result of the final statement of the previous theorem. The next corollary is a generalised version of Zariski's "key".

Corollary 2.1.16. Let $R$ be a Noetherian ring, $P \subseteq Q$ prime ideals of $R$, $M \subseteq N$ finitely generated $R$-modules and $r$ a non-negative integer. Assume that $Q/P \in \text{Reg-Spec}(R/P)$, that $\left[\frac{p^i}{p^{i+1}}\right]_Q$ is $[R/P]_Q$-free, for $i = 0, 1, \ldots, r$, and that $N_Q$ is $R_Q$-free. Suppose that $Q$ is $(M,N;P; r)$-regular. Then for all $s \geq t \geq 0$

$$[((M \cap P^TN \cap Q^{r+tN}) \cap Q^{r+sN}) \cap ((P^{r+1}N \cap Q^{r+tN}) + Q^{r+sN})]_Q$$

$$= [(M \cap P^{r+1}N \cap Q^{r+tN}) + Q^{r+sN}]_Q.$$

Proof. After localisation at $Q$

$$(M \cap P^TN \cap Q^{r+tN}) \cap ((P^{r+1}N \cap Q^{r+tN}) + Q^{r+sN})$$

$$= ((M \cap Q^{r+tN}) \cap P^TN) \cap ((P^{r+1}N \cap Q^{r+sN}), (Q^{r+tN} \cap P^{r+1}N + Q^{r+sN}))$$

$$= (M \cap Q^{r+tN}) \cap M \cap P^TN \cap (P^{r+1}N + Q^{r+sN})$$

$$= M \cap Q^{r+tN} \cap (M \cap (P^{r+1}N + P^TQ^SN))$$

$$\subseteq M \cap Q^{r+tN} \cap ((M \cap P^T(Q^SN + P)N + P^{r+1}N)$$

$$= M \cap Q^{r+tN} \cap ((M \cap P^TQ^SN) + P^{r+1}N)$$ (using the last
statement of Theorem 2.1.14)

\[(M \cap P^r Q^s N) + (M \cap P^{r+1} N \cap Q^{r+t} N)\]

\[\subseteq (M \cap P^{r+1} N \cap Q^{r+t} N) + Q^{r+s} N.\]

The result follows easily. \(\square\)

Note. In the case \(s = t + 1\) we have a generalisation of [Zar], Theorem 2.

Somewhat more memorable is the following derivative of the previous corollary:

Corollary 2.1.17. Let \(R\) be a Noetherian ring, \(P \subseteq Q\) prime ideals of \(R\), \(M \subseteq N\) finitely generated \(R\)-modules and \(r\) a non-negative integer. Assume that \(Q/P \in \text{Reg-Spec}(R/P)\), that \(\left[\frac{P^i}{P^{i+1}}\right]_Q\) is \([R/P]_Q\)-free, for \(i = 0, 1, \ldots, r\), and that \(N_Q\) is \(R_Q\)-free.

Suppose that \(Q\) is \((M, N; P; i)-regular\) for \(i = 0, 1, \ldots, r - 1\). Let \(L = N/M\). Then, for all \(s \geq 0\),

\[Q^{r+s} L_Q \cap P^r L_Q = Q^s P^r L_Q.\]

Proof. If \(r = 0\) there is nothing to prove, so we assume that \(r > 0\). Then

\[Q^{r+s} L_Q \cap P^r L_Q = \left[\frac{Q^{r+s} N + M}{M}\right]_Q \cap \left[\frac{P^r N + M}{M}\right]_Q\]
\[ (Q^{r+s}N + M) \cap P^rN = (Q^{r+s}N + M) \cap Q^rN \cap P^rN \cap (P \cap Q^rN + Q^{r+s}N), \]
\[ = ((M \cap P^0N \cap Q^rN) + Q^{r+s}N) \cap (P^0N \cap Q^rN + Q^{r+s}N) \cap P^rN, \]
\[ = ((M \cap PN \cap Q^rN) + Q^{r+s}N) \cap P^rN \quad \text{from Corollary 2.1.16}, \]
\[ = ((M \cap P^2N \cap Q^rN) + Q^{r+s}N) \cap P^rN \quad \text{from Corollary 2.1.16 again}, \]
\[ = ((M \cap P^iN \cap Q^rN) + Q^{r+s}N) \cap P^rN \quad \text{where } 0 \leq i < r, \]
\[ = ((M \cap P^iN \cap Q^rN) + Q^{r+s}N) \cap (P^{i+1}N \cap Q^rN + Q^{r+s}N) \cap P^rN, \]
Thus, from (*),

\[ Q^{r+s}L_Q \cap P^rL_Q = \left[ \frac{M}{P^rQ^sN + M} \right]_Q = P^rQ^sL_Q \] 

as required. \( \Box \)

Remark. This corollary generalises 2.1.8, which follows from Lemmas 1.3 and 1.1 (ii) of [RobVa] and a particular case of which is formula (7) of [Zar]. We could also have deduced the result straight from Korollar 1.2 of [AcScVo]. However the fact that it is a consequence of Zariski's work seems of interest in the present context. We remark that part (a) of [AcScVo], Korollar 1.2, gives rise to a further formula involving \( L_Q \) and \( P_Q \) which can also be proved by the methods above.

In the theorems above we have imposed conditions that have made Zariski regularity behave well. We have insisted that

\[ Q/P \in \text{Reg-Spec}(R/P), \text{ that } \left[ \frac{P^i}{P^{i+1}} \right]_Q \text{ is } A_Q \text{-free, for} \]
i = 0,1, \ldots ,r, and that N_Q is R_Q-free. The condition that each
\[ \left[ \frac{p^i}{p^{i+1}} \right]_Q \] is A_Q-free is an "open condition" and so, as we shall see,
does not restrict our use of the theory. We shall make this clear in
Section 2.2. In the applications that we consider, the problems that
arise can often be reduced to questions concerning free modules. The
requirement that N_Q is R_Q-free does not, in this case, restrict the
use of the theory. To reduce the implications of the condition that
A_Q must be regular is not so easy, and this requirement is reflected
in the restrictions that have to be placed on our applications. That
this condition cannot simply be removed is shown by the example of
Remark (2) in [EiHo], since in this example the uniform Artin-Rees
does not hold for maximal ideals (cf. Corollary 2.3.11).

It might appear that the conditions we have imposed in the
results above are so strong as to make these results trivial or that
the conditions could be weakened. We attempt to clarify the position
by giving some examples.

Observe first that if \((R,m)\) is a regular local ring and
P \in \text{Spec}(R) such that m/P \in \text{Reg-Spec}(R), then \(\frac{p^r}{p^{r+1}}\) is R/P-free,
for all \(r \geq 0\) ([Mat2], Theorem 16.2 (i)). Therefore if we take, in
our results above, Q \in \text{Reg-Spec}(R) we have immediately that
\[ \left[ \frac{p^i}{p^{i+1}} \right]_Q \] is A_Q-free, for \(i = 0, \ldots ,r\). However, as the
following example shows, we may have a non-regular local ring \((R,m)\)
which has a prime ideal $P$ such that $m/P \in \text{Reg-Spec}(R/P)$ and \( \frac{p^r}{p^{r+1}} \) is $R/P$-free for all $r \geq 0$.

**Example 2.1.18.** Let $B = k[X,Y]$, $n = XB + YB$ and $a = X^2B$. Let $R = B_n/aB_n$, so that $R$ is local with maximal ideal $m = nB_n/aB_n$, and let $P = \overline{X}R$, where $\overline{X}$ is the image of $X$ in $R$. We have $R/P \cong B_n/XB_n \cong k[Y](Y)$ and $m/P \cong nB_n/XB_n \cong (Y)k[Y](Y)$. Therefore $m/P \in \text{Reg-Spec}(R/P)$. Moreover

\[
\frac{p^r}{p^{r+1}} = \begin{cases} 
X^rB_n + X^2B_n & \text{if } r = 0 \\
X^{r+1}B_n + X^2B_n & \text{if } r = 1 \\
0 & \text{if } r > 1 
\end{cases}
\]

So \( \frac{p^r}{p^{r+1}} \) is certainly $R/P$-free if $r \neq 1$. Since $B_n$ is regular and $B_n/XB_n$ is regular, we know that $\frac{X^rB_n}{X^{r+1}B_n}$ is $B_n/XB_n$-free for all $r \geq 0$. Therefore \( \frac{p^r}{p^{r+1}} \) is $R/P$-free for all $r \geq 0$, as required. However, as $R$ is not a domain, $R$ cannot be regular.

Thus the conditions imposed in the results of this section do not force $Q$ to be in the regular spectrum of $R$. We now demonstrate that $Q$ is not forced to be $(M,N;P;r)$-regular either.

**Example 2.1.19.** Let $R = k[X,Y]$, $Q =XR + YR$, $P = XR$, $N = R$ and
$M = (Y - X^2)R$. $R$ is a regular ring and $R/P \cong k[Y]$. Thus $R/P$ is also a regular ring, and so \[ \frac{p^r}{p^{r+1}} \] is $[R/P]_Q$-free for all $r \geq 0$. We shall show that $Q$ is not $(M,N,P;r)$-regular for any $r \geq 0$.

Theorem 2.1.14 (3) (b) is trivially true when $s = 1$. Hence, from Theorem 2.1.14, $Q$ is $(M,N,P;r)$-regular if and only if
\[
[M \cap P^r N \cap Q^{r+1} N]_Q = [Q(M \cap P^r N) + (M \cap P^{r+1} N)]_Q.
\]

We have, using 2.1.8,
\[
[M \cap P^r N \cap Q^{r+1} N]_Q = [(Y - X^2)R \cap X^r R \cap (XR + YR)^{r+1}]_Q = [(Y - X^2)R \cap X^r (XR + YR)]_Q. \tag{2.1.20}
\]

Also
\[
[Q(M \cap P^r N) + (M \cap P^{r+1} N)]_Q = [(XR + YR)((Y - X^2)R \cap X^r R) + ((Y - X^2)R \cap X^{r+1} R)]_Q.
\]

Since $R$ is a unique factorisation domain and $X$ and $Y - X^2$ are distinct irreducible elements, we have
\[
(Y - X^2)R \cap X^r R = X^r (Y - X^2)R,
\]
for all $r \geq 0$. Thus
\[
[Q(M \cap P^r N) + (M \cap P^{r+1} N)]_Q = [(XR + YR)X^r (Y - X^2) + X^{r+1} (Y - X^2)R]_Q = [(XR + YR)X^r (Y - X^2)]_Q. \tag{2.1.21}
\]

Fix $r \geq 0$. Then
\[
X^r (Y - X^2) \in (Y - X^2)R \cap X^r (XR + YR) \subseteq [M \cap P^r N \cap Q^{r+1} N]_Q
\]
by 2.1.20. Suppose

\[ X^r(Y - X^2) \in [Q(M \cap P^rN) + (M \cap P^{r+1}N)]_Q \cap [(XR + YR)X^r(Y - X^2)]_Q, \]

using (2.1.21). Then

\[ X^r(Y - X^2) = (gX + hY)X^r(Y - X^2), \]

for some \( g, h, s \in R \) such that \( s \notin XR + YR \). Therefore

\[ sX^r(Y - X^2) = (gX + hY)X^r(Y - X^2), \]

so that \( s = gX + hY \), a contradiction. Hence

\[ [M \cap P^rN \cap Q^{r+1}N]_Q \neq [Q(M \cap P^rN) + (M \cap P^{r+1}N)]_Q, \]

and \( Q \) cannot be \( (M, N; P; r) \)-regular, for any \( r \geq 0 \).

2.2 Zariski regularity and open loci

In the previous section we defined Zariski regularity and demonstrated consequences of the definition that will be useful in applications. In this section we show that Zariski regularity occurs "almost everywhere"; that is, if \( R \) is a ring, given a pair of \( R \)-modules \( M \subseteq N \) and a prime ideal \( P \) of \( R \), the subset of \( \text{Spec}(R) \) consisting of primes that are \( (M, N; P) \)-regular is non-empty and open in \( V(P) \). Since \( V(P) \) is an irreducible subspace of \( \text{Spec}(R) \), this means that this subset is dense in \( V(P) \). We refer here always to the Zariski topology of \( \text{Spec}(R) \). We begin by generalising results of Zariski.
Theorem 2.2.1. (cf. [Zar], Theorem 3) Let $R$ be a Noetherian ring, $P \in \text{Spec}(R)$ and $N$ a finitely generated $R$-module with submodule $M$. Then for any $r \in \mathbb{Z}$ such that $r \geq 0$ the set
\[ U_r = \{ Q \in \text{Spec}(R) : Q \text{ is } (M,N,P;r)-\text{regular} \} \]
is a non-empty open subset of $\mathcal{V}(P)$.

Proof. That $U_r$ is open follows from Definition 2.1.1 and [Mat2], Theorem 4.10 (ii). Since $\kappa(P)$ is a field we certainly have $P \in U_r$, so that $U_r \neq \emptyset$. □

Corollary 2.2.2. (cf. [Zar], Corollary 1 to Theorem 3) Let $R$ be a Noetherian ring, $P \in \text{Spec}(R)$ and $N$ a finitely generated $R$-module with submodule $M$. Then for any $r \in \mathbb{Z}$ such that $r \geq 0$ the set
\[ V_r = \{ Q \in \text{Spec}(R) : Q \text{ is } (M,N,P;r)-\text{regular for } s = 0,1, \ldots ,r \} \]
is a non-empty open subset of $\mathcal{V}(P)$.

Proof. $V_r = U_0 \cap U_1 \cap \cdots \cap U_r$ and so is obviously non-empty and open. □

The two results above are weaker than the next theorem, which in practice is easier to use. One reason for giving them independently is the simplicity of their proofs in comparison to the proof of Theorem 2.2.3.

Theorem 2.2.3. Let $R$ be a Noetherian ring, $P \in \text{Spec}(R)$ and $N$ a finitely generated $R$-module with submodule $M$. Then the set
2.2 Zariski regularity and open loci

\[ W = \{ Q \in \text{Spec}(R) : Q \text{ is } (M,N;P)-\text{regular} \} \]

is a non-empty open subset of \( \mathcal{V}(P) \).

**Proof.** Let \( S = \bigoplus_{r \geq 0} (P^r/P^{r+1}) \). Then \( S \) is a graded \( R/P \)-algebra. If \( P^r \) is generated by a set \( x_1, \ldots, x_k \), as an ideal of \( R \), then the images of \( x_1, \ldots, x_k \) in \( P/P^2 \) generate \( S \) as an \( R/P \)-algebra. Therefore \( S \) is generated as an \( R/P \)-algebra by a finite set of homogenous elements of positive degree. In the notation of 2.1.3 we have

\[
E_r = \frac{P^r N}{(M \cap P^r N) + P^{r+1} N} = \frac{P^r N}{(M + P^{r+1} N) \cap P^r N} \approx \frac{P^r (N/M)}{P^{r+1} (N/M)} \quad \text{for all } r \geq 0.
\]

Let \( \mathcal{E} = \bigoplus_{r \geq 0} E_r \). If \( n_1, \ldots, n_q \) generate \( N \) as an \( R \)-module, then the images of \( n_1, \ldots, n_q \) in \( \mathcal{E} \) generate \( \mathcal{E} \) as an \( S \)-module. Therefore \( \mathcal{E} \) is a finitely generated graded \( S \)-module. It follows from Theorem 1, Chapter II, Section 1 (page 188), of [Hir] that the set

\[ W' = \{ p \in \text{Spec}(R/P) : \mathcal{E}_p \text{ is a flat } (R/P)_p \text{-module} \} \]

is an open set of \( \text{Spec}(R/P) \). From 2.1.6, with \( L = N/M, \ R = R_p, \) where \( p \in \text{Spec}(R) \) such that \( p \geq P \) and \( P/P = P' \) and \( I = pP \), it follows that
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\begin{align*}
W' &= \{ p \in \text{Spec}(R/P) : [E_r]_p \text{ is a free } (R/P)_p\text{-module for all } r \geq 0 \}. \\
\text{It is now clear that } Q \in W \text{ if and only if } Q/P \in W'. \text{ Since, under the canonical homeomorphism from } \mathcal{V}(P) \text{ to } \text{Spec}(R/P), \ W \text{ is the image of } W', \ W \text{ must be an open set of } \mathcal{V}(P). \text{ As } P \in W \text{ the result is proved.} \quad \square
\end{align*}

Corollary 2.2.4. Let \( R \) be a Noetherian ring, \( P \in \text{Spec}(R) \) and \( N \) a finitely generated \( R \)-module with submodule \( W \). Then the set

\[ T = \{ Q \in \text{Spec}(R) : Q \text{ is } (M,N;P)\text{-regular and } Q \text{ is } ((0),R;P)\text{-regular} \} \]

is a non-empty open subset of \( \mathcal{V}(P) \). \( \square \)

Note. If \( N_Q \) is \( R_Q \)-free, \( Q/P \in \text{Reg-Spec}(R/P) \) and \( Q \in T \) the hypotheses and conditions of Theorem 2.1.6 are satisfied for any \( r \geq 0 \). This corollary allows us, in practice, to assume that

\[ \left[ \frac{p^i}{p^{i+1}} \right]_Q \text{ is a free } [R/P]_Q\text{-module, for all } i \geq 0, \text{ for all } Q \text{ in} \]

the non-empty open subset \( W \) of \( \mathcal{V}(P) \), found in Theorem 2.2.3, on which we have Zariski regularity. This is what we meant in Section 2.1 when we said that this is an "open condition".
2.3 The uniform Artin-Rees theorem

The main result of this section is a uniform Artin-Rees theorem, proved using the results of the two previous sections. We first of all set up some notation and make some preliminary observations. A statement and proof of the (non-uniform) Artin-Rees lemma can be found, for instance, in [AM], Corollary 10.10, or [Mat2], Theorem 8.5.

Definition 2.3.1. Let R be a ring, N an R-module and M a submodule of N. A set $\mathfrak{G}$ of ideals of R is said to be uniform Artin-Rees (uAR) with respect to $M \subseteq N$ if there is a positive integer $k$ such that, for all integers $n \geq k$ and for all ideals $I \in \mathfrak{G}$,

$$M \cap I^n = I^n \cap (M \cap I^k).$$

The integer $k$ is said to be a uniform Artin-Rees number (for $\mathfrak{G}$ with respect to $M \subseteq N$). If $\mathfrak{G}$ is uAR with respect to $M \subseteq N$ for all submodules $M$ of $N$, then $\mathfrak{G}$ is said to be uAR with respect to $N$. If $I$ is the unique element of $\mathfrak{G}$, then we call $k$ an Artin-Rees number for $I$ with respect to $M \subseteq N$.

It should be emphasised that, if $\mathfrak{G}$ is uAR with respect to $N$, we have a uniform Artin-Rees number $k_M$ for $\mathfrak{G}$ with respect to $M \subseteq N$, for every finitely generated submodule $M$ of $N$; we do not mean to say that we have an upper bound on the set $\{k_M : M \subseteq N ; M$ is finitely generated$\}$. 
The next lemma is the natural extension of a result in [O'C1] (Section 1, page 3). The proof given here is L. O'Carroll's.

**Lemma 2.3.2.** Let $R$ be a Noetherian ring, $L$ an $R$-module and $N$ an $R$-homomorphic image of $L$. Let $\mathcal{G}$ be a set of ideals of $R$ that is uAR with respect to $L$. Then $\mathcal{G}$ is uAR with respect to $N$.

**Proof.** Let $M$ be any submodule of $N$. We have a surjective map $\varphi : L \rightarrow N$ with kernel $K$, say. Let $D = \varphi^{-1}(M)$. Then $D$ is a submodule of $L$ containing $K$. The restriction of $\varphi$ to $D$ is then a surjective map from $D$ to $M$ with kernel $K$. Therefore we have $N \cong L/K$ and $M \cong D/K$. By hypothesis we have a uAR number, $k$, for $\mathcal{G}$ with respect to $D \subseteq L$. Let $n$ be any integer greater than $k$ and $I$ any ideal in $\mathcal{G}$. Then

\[
M \cap I^N \cong (D/K) \cap I^n(L/K)
\]

\[
= (D/K) \cap (I^nL + K / K)
\]

\[
= D \cap (I^nL + K) / K,
\]

\[
= (D \cap I^nL) + K, \text{ since } D \supseteq K,
\]

\[
= I^{n-k}(D \cap I^kL) + K, \text{ by hypothesis},
\]

\[
= I^{n-k}[D \cap I^k(L/K)] + K,
\]

\[
= I^{n-k}((D/K) \cap I^k(L/K)), \text{ by the reverse of the argument above. Therefore}
\]

\[
M \cap I^N = I^{n-k}(M \cap I^kN).
\]
Thus $k$ is a uAR number for $\mathfrak{G}$ with respect to $M \subseteq N$, and the result follows. $\square$

Recall that a non-empty topological space $X$ is irreducible if $X = C_1 \cup C_2$, where $C_1$ and $C_2$ are closed subsets of $X$, implies that $X = C_1$ or $X = C_2$. For ease of reference we record the following as a lemma:

**Lemma 2.3.3.** Let $X$ be an irreducible topological space, $U$ a non-empty open subspace of $X$ and $D$ a dense subspace of $X$. Then $D \cap U$ is a dense subspace of $X$.

**Proof.** For any subspace $Y$ of $X$, let $\overline{Y}$ denote the closure of $Y$ in $X$ and $Y^c$ the complement of $Y$ in $X$. Then

$$X = \overline{D} = (D \cap U) \cup (D \cap U^c)$$

$$= D \cap U \cup D \cap U^c$$

$$\subseteq D \cap U \cup U^c.$$  

Therefore, either $X = D \cap U$ or $X = U^c$. As $U$ is non-empty, this means that $X = D \cap U$, as required. $\square$

Following [Matl], we say that a Noetherian ring $R$ is J-0 if Reg-Spec($R$) contains a non-empty open subset of Spec($R$) ([Matl], (32.B), page 246). In the proof of the following theorem we shall use some set theoretic notation and König's graph lemma. Let

$\{F_i : i = 1,2, \ldots \}$ be a countable sequence of finite sets and let
\( \mathcal{F} = \bigcup_{i=1}^{\infty} F_i \). Suppose that for each \( i \in \mathbb{N} \) we have a map \( \Phi_i : F_i \rightarrow \{ \text{subsets of } F_{i+1} \} \) and we denote by \( \Phi \) the set 
\[ \{ \Phi_i : i = 1, 2, \ldots \} \]. A path in the ordered pair \((\mathcal{F}, \Phi)\) is a sequence \( a_1, a_2, \ldots \) of elements of \( \mathcal{F} \) such that \( a_i \in F_i \) and \( a_{i+1} \in \Phi_i(a_i) \) for all \( i \in \mathbb{N} \). If the sequence is finite, with \( m \) elements, we say that the length of the path is \( m \). Otherwise we say that the length of the path is infinite.

**König's graph lemma.** In the situation described above: if \((\mathcal{F}, \Phi)\) has a path of length \( n \), for each \( n \in \mathbb{N} \), then \((\mathcal{F}, \Phi)\) has a path of infinite length.

*Proof.* Note that, for any \( i \in \mathbb{N} \), we can extend \( \Phi_i \) to a map 
\[ \Phi_i^P : \{ \text{subsets of } F_i \} \rightarrow \{ \text{subsets of } F_{i+1} \}; \text{ define } \Phi_i^P \text{ such that} \]
\[ \Phi_i^P(X) = \bigcup_{x \in X} \Phi_i(x), \text{ for any } X \subseteq F_i \text{ and for all } i \in \mathbb{N}. \text{ Then, for any } i \in \mathbb{N}, \Phi_i^P(\{a\}) = \Phi_i(a), \text{ for any } a \in F_i, \text{ and } \Phi_i^P(\emptyset) = \emptyset. \]

Suppose that, for every \( n \in \mathbb{N} \), there is a path of length \( n \) in \((\mathcal{F}, \Phi)\). Let \( t \) be any positive integer and let \( \{c_1, \ldots, c_t\} \subseteq \mathcal{F} \).

Consider the following condition on \( \{c_1, \ldots, c_t\} \):
\[ c_1, \ldots, c_j \text{ is a path in } (\mathcal{F}, \Phi) \text{ and} \]
\[ \Phi_{r+j}^P \Phi_{r+j-1}^P \ldots \Phi_j^P(\{c_j\}) \neq \emptyset, \text{ for each } j = 1, \ldots, t \text{ and all } r \geq 0. \]

\((*)\)

Now let \( s > 1 \) be an integer and suppose that we have a set \( \{c_1, \ldots, c_{s-1}\} \) that satisfies \((*)\), with \( t = s - 1 \). Then
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2.3

Let \( R \) be a Noetherian ring and \( C \) any non-empty closed subset of \( \text{Spec}(R) \). Then there is a proper ideal \( a \) of \( R \) such that \( C = \mathcal{V}(a) \).

C may be written as \( C = \mathcal{V}(P_1) \cup \cdots \cup \mathcal{V}(P_\nu) \), where \( P_1, \ldots, P_\nu \) are the minimal elements of \( \text{Ass}_R(A/a) \). Moreover \( P_1, \ldots, P_\nu \) are dependent only on \( C \); if \( b \) is any other ideal of \( R \) such that \( C = \mathcal{V}(b) \), the minimal elements of \( \text{Ass}_R(A/b) \) are, again, \( P_1, \ldots, P_\nu \). We shall call \( P_1, \ldots, P_\nu \) the prime components of \( C \).

Theorem 2.3.4. Let \( R \) be a Noetherian ring and \( I \) an ideal of \( R \) such that

\[ \phi_{s-1}^p(\{c_{s-1}\}) \] is a non-empty subset of \( F_s \). (When \( s = 1 \) we replace this statement with \( F_1 \neq \emptyset \).) Let

\[ \phi_{s-1}^p(\{c_{s-1}\}) = \phi_{s-1}(c_{s-1}) = \{a_{s,1}, \ldots, a_{s,k_s}\}. \] (When \( s = 1 \) we consider \( F_1 \) instead of \( \phi_{s-1}^p(\{c_{s-1}\}) \).) Suppose that, for each \( i = 1, \ldots, k_s \), there is some \( \mathcal{M}_i \in \mathbb{N} \) such that

\[ \phi_{s-1}^p(\{a_{s,i}\}) = \emptyset. \] Let \( M = \max\{\mathcal{M}_i : i = 1, \ldots, k_s\} \).

Then \( \phi_{s-1}^p(\{a_{s,i}\}) = \emptyset, \) for \( i = 1, \ldots, k_s \). Hence

\[ \phi_{s-1}^p(\{c_{s-1}\}) = \emptyset. \] If \( s > 1 \) this contradicts (*). If \( s = 1 \) let \( b_1, \ldots, b_{M+1} \) be a path in \( (\mathcal{S}, \mathcal{F}) \) of length \( M + 1 \).

Then \( b_{M+1} \in \phi_N(b_M) = \phi_M^p(\{b_M\}) \subseteq \phi_{M-1}^p(\{b_{M-1}\}) \subseteq \cdots \subseteq \phi_{s-1}^p(\{b_1\}) = \emptyset, \) which is impossible. Therefore there is some \( i \) such that \( 1 \leq i \leq k_s \) and \( \phi_{s+r}^p(\{a_{s,i}\}) \neq \emptyset, \) for all \( r \geq 0 \). Let \( c_s = a_{s,i} \) for some such \( i \). Then \( \{c_1, \ldots, c_s\} \) satisfies (*), with \( t = s \). By induction \( c_1, c_2, \ldots \) is a path of infinite length in \( (\mathcal{S}, \mathcal{F}) \). \( \square \)
that, for all primes \( Q \) of \( R \) containing \( I \), \( R/Q \) is \( J_0 \). Then \( \text{Max-Spec}(R) \cap \mathcal{V}(I) \) is uAR with respect to \( N \), for any finitely generated \( R \)-module \( N \).

**Proof.** Let \( N \) be any finitely generated \( R \)-module. In the light of Lemma 2.3.2 we may assume without loss of generality that \( N \) is a free \( R \)-module. Let \( M \) be any submodule of \( N \). We must demonstrate the existence of a uAR number for \( \text{Max-Spec}(R) \), with respect to \( M \subseteq N \). Let \( P \) be some fixed element of \( \mathcal{V}(I) \). The Artin-Rees lemma asserts the existence of a positive integer \( k = k_p \) such that, for all \( n \geq k \),

\[
M \cap P^nN = P^{n-k}(M \cap P^kN). \tag{2.3.5}
\]

Let \( T_p = \{ Q \in \text{Spec}(R) : Q \text{ is } ((0), R; P)\text{-regular and } Q \text{ is } (M, N; P)\text{-regular} \} \). Then \( T_p \) is a non-empty open subset of \( \mathcal{V}(P) \), by Corollary 2.2.4. For any \( Q \in T_p \) we have, by definition of

\((0), R; P)\text{-regularity}, that \[
\left[ \frac{pr}{pr+1} \right]_Q \text{ is } [R/P]_Q\text{-free, for all } r \geq 0.
\]

By hypothesis, \( R/P \) is \( J_0 \), so there exists a non-empty open subset \( U' \) of \( \text{Spec}(R/P) \) such that \( U' \subseteq \text{Reg-Spec}(R/P) \). Let \( U_p = \{ Q \in \mathcal{V}(P) : Q/P \in U' \} \).

Then \( U_p \) is a non-empty open subset of \( \mathcal{V}(P) \) and so \( S_p = T_p \cap U_p \) is a non-empty open subset of \( \mathcal{V}(P) \). For any \( Q \in S_p \) and any non-negative integer \( r \), the hypotheses and condition (1) of Theorem 2.1.14 are satisfied.

Now fix \( Q \in S_p \). From Theorem 2.1.14 (2), we have

\[
[M \cap P^rN \cap Q^{r+s}N]_Q = [Q^s(M \cap P^rN) + (M \cap P^{r+1}N \cap Q^{r+s}N)]_Q \text{ for all } r, s \geq 0. \tag{2.3.6}
\]
Thus, after supressing localisation at Q, we have, for any \( n \geq k \),

\[
M \cap Q^nN = M \cap P^0N \cap Q^nN = Q^n(M \cap P^0N) + (M \cap PN \cap Q^nN),
\]

using 2.3.6, so, after localisation at Q,

\[
M \cap Q^nN = Q^nM + Q^{n-1}(M \cap PN) + (M \cap P^2N \cap Q^nN),
\]

again. Repeating this process we obtain eventually, after localisation at Q,

\[
M \cap Q^nN = Q^nM + Q^{n-1}(M \cap PN) + Q^{n-2}(M \cap P^2N) + \cdots
\]

\[
\cdots + Q^{n-k+1}(M \cap P^{k-1}N) + Q^{n-k}(M \cap P^kN)
\]

\[
+ Q^{n-k-1}(M \cap P^{k+1}N) + \cdots + Q^2(M \cap P^{n-2}N)
\]

\[
+ Q(M \cap P^{n-1}N) + (M \cap P^nN).
\]

(2.3.7)

Now, for any \( m \) such that \( 0 \leq m \leq k-1 \),

\[
Q^{n-m}(M \cap P^mN) = Q^{n-k}Q^{k-m}(M \cap P^mN)
\]

\[
\subseteq Q^{n-k}(M \cap Q^{k-m}P^mN) \subseteq Q^{n-k}(M \cap Q^kN).
\]

Thus

\[
Q^nM + Q^{n-1}(M \cap PN) + \cdots
\]

\[
\cdots + Q^{n-k+1}(M \cap P^{k-1}N) \subseteq Q^{n-k}(M \cap Q^kN).
\]

(2.3.8)

Also, supressing localisation at Q,

\[
Q^{n-k}(M \cap P^kN) + Q^{n-k-1}(M \cap P^{k+1}N) + \cdots
\]

\[
\cdots + Q(M \cap P^{n-1}N) + (M \cap P^nN)
\]

\[
= Q^{n-k}(M \cap P^kN) + Q^{n-k-1}P(M \cap P^kN) + \cdots
\]

\[
\cdots + QP^{n-k-1}(M \cap P^kN) + P^{n-k}(M \cap P^kN)
\]

using 2.3.5,
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\[(Q + P)^{n-k}(M \cap P^k N) = Q^{n-k}(M \cap P^k N) \subseteq Q^{n-k}(M \cap Q^k N). \tag{2.3.9}\]

Therefore, combining 2.3.7, 2.3.8 and 2.3.9, for any \( n > k \),

\[[M \cap Q^n N]_Q \subseteq [Q^{n-k}(M \cap Q^k N)]_Q.\]

The reverse inclusion is obvious so, for any \( Q \in S_p \) and any \( n > k \),

\[[M \cap Q^n N]_Q = [Q^{n-k}(M \cap Q^k N)]_Q. \tag{2.3.10}\]

We have shown that there is a non-empty open subset \( S_p \) of \( \mathcal{V}(P) \) such that, for all \( Q \in S_p \), 2.3.10 holds, and so \( k = k_p \) is an Artin-Rees number for \( Q \) with respect to \( M \subseteq N \). (Recall that \( k = k_p \) is an Artin-Rees number for \( P \) with respect to \( M \subseteq N \).) Since \( P \) was arbitrary in \( \mathcal{V}(I) \) there is a non-empty open subset \( S_p \) of \( \mathcal{V}(p) \) satisfying the above property, relative to \( k_p \), for all \( p \in \mathcal{V}(I) \).

For each \( p \in \mathcal{V}(I) \), choose a set \( S_p \) of the type described above. We define a map \( \phi' : \mathcal{V}(I) \to \{ \text{subsets of } \mathcal{V}(I) \} \) by

\[\phi'(p) = \{ \text{prime components of } (\mathcal{V}(p) - S_p) \} \text{ if } S_p \neq \mathcal{V}(p), \text{ and by } \]

\[\phi'(p) = \emptyset \text{ if } S_p = \mathcal{V}(p). \]

We can extend \( \phi' \) to a map

\[\phi : \{ \text{subsets of } \mathcal{V}(I) \} \to \{ \text{subsets of } \mathcal{V}(I) \}\]

by setting \( \phi(\mathcal{A}) = \bigcup_{p \in \mathcal{A}} \phi'(p) \), for any subset \( \mathcal{A} \subseteq \mathcal{V}(I) \). By ordinary recursion there is a unique map

\[\theta : N \to \{ \text{subsets of } \mathcal{V}(I) \}\]

such that \( \theta(1) = \{ \text{prime components of } \mathcal{V}(I) \} \) and

\[\theta(n + 1) = \phi(\theta(n)). \]

By induction on \( n \), \( \theta(n) \) is a finite set for all
n \in \mathbb{N}. Define \( \phi_n : \Theta(n) \rightarrow \{\text{subsets of } \Theta(n + 1)\} \) by
\[
\phi_n(P) = \phi'(P), \quad \text{for any } P \in \Theta(n) \text{ and for all } n \in \mathbb{N}. \]

Let
\[
\Theta = \bigcup_{n \geq 1} \Theta(n) \quad \text{and} \quad \Phi = \{\phi_n : n = 1, 2, \ldots \}.
\]

We may assume that \( \Theta(1) \neq \emptyset \), since \( \Theta(1) = \emptyset \iff I = \mathbb{R} \), and if \( I = \mathbb{R} \) there is nothing to prove. Given any \( n \in \mathbb{N} \) such that \( n > 1 \), suppose that \( \Theta(n) \neq \emptyset \). Then choose \( P_n \in \Theta(n) \) so that, by construction, \( P_n \in \phi(\Theta(n - 1)) \), which implies that \( P_n \) is a prime component of \( \mathcal{V}(P_{n-1}) - S_{P_{n-1}} \), for some \( P_{n-1} \in \Theta(n - 1) \).

Moreover \( P_n \in \phi_{n-1}(P_{n-1}) \). Since \( P_{n-1} \in \Theta(n - 1) \), we have, by induction a path \( P_1, \ldots, P_{n-1} \) in \( (\Theta, \Phi) \). Thus \( P_1, \ldots, P_n \) is a path in \( (\Theta, \Phi) \) of length \( n \). If \( \Theta(n) \neq \emptyset \), for any \( n \in \mathbb{N} \), there is, by König's graph lemma, a path in \( (\Theta, \Phi) \) of infinite length. However, given any path \( P_1', P_2', \ldots \) in \( (\Theta, \Phi) \), we have \( P_i' \subseteq P_{i+1}' \), for all \( i \geq 1 \), since \( P_{i+1}' \in \phi_i(P_i') \). Thus the Noetherian assumption implies that \( \Theta(n) = \emptyset \), for some \( n \in \mathbb{N} \). Let \( \mu \) be the smallest positive integer such that \( \Theta(\mu) = \emptyset \). Then \( \Theta(n) = \emptyset \), for all \( n \geq \mu \). Hence \( \Theta \) is a finite set. Consequently \( \{S_P : P \in \Theta\} \) is a finite set.

Now suppose that \( q \) is any element of \( \mathcal{V}(I) \). Then there is some prime component \( p \) of \( \mathcal{V}(I) \) such that \( q \supseteq p \). In other words, \( q \supseteq p \) for some \( p \in \Theta(1) \). Let \( r = \max\{n \in \mathbb{N} : q \supseteq p \text{ for some } p \in \Theta(n)\} \). Then \( q \supseteq P_r \), for some \( P_r \in \Theta(r) \subseteq \Theta \), so that \( q \in \mathcal{V}(P_r) \). If \( q \notin S_{P_r} \), then \( q \in \mathcal{V}(P_r) - S_{P_r} \), so that \( q \) contains a prime component \( P_{r+1} \) of \( \mathcal{V}(P_r) - S_{P_r} \). Since \( P_{r+1} \in \Theta(r + 1) \) this is a contradiction. Hence \( q \in S_{P_r} \) and so \( \mathcal{V}(I) \subseteq \bigcup \{S_P : P \in \Theta\} \).
For any $P \in \mathcal{V}(I)$, denote by $k_P$ the Artin-Rees number of $P$ with respect to $M \subseteq N$. Then, since $\Theta$ is finite,

$$s = \max\{k_P : P \in \Theta\}$$

is a well-defined integer. Let $m$ be any element of $\text{Max-Spec}(R) \cap \mathcal{V}(I)$. Choose $P \in \Theta$ such that $m \in S_P$.

Let $k = k_P$. From 2.3.10, for any $n \geq s \geq k_P = k$,

$$[M \cap m^nN]_m = [m^{n-k}(M \cap m^kN)]_m = [m^{n-s}m^{s-k}(M \cap m^kN)]_m \subseteq [m^{n-s}(M \cap m^sN)]_m,$$

and the opposite inclusion is trivial. Furthermore, if $n$ is any other maximal ideal of $R$,

$$[M \cap m^nN]_n = [m^{n-s}(M \cap m^sN)]_n.$$

Hence $M \cap m^nN = m^{n-s}(M \cap m^sN)$ for all $n \geq s$. This is true for all $m \in \text{Max-Spec}(R) \cap \mathcal{V}(I)$ so that $s$ is a uAR number for $\text{Max-Spec}(R) \cap \mathcal{V}(I)$ with respect to $M \subseteq N$. Since $M$ was an arbitrary submodule of $N$ the result follows. □

Corollary 2.3.11. Let $R$ be a Noetherian ring and $N$ a finitely generated $R$-module with submodule $M$. If, for all $Q \in \text{Supp}(N/M)$, $R/Q$ is $J$-0, then $\text{Max-Spec}(R)$ is uAR with respect to $M \subseteq N$.

**Proof.** Let $I = \text{Ann}_R(N/M)$. Then $\text{Supp}(N/M) = \mathcal{V}(I)$. From the theorem, $\text{Max-Spec}(R) \cap \mathcal{V}(I)$ is uAR with respect to $M \subseteq N$.

Suppose that $a \in \text{Max-Spec}(R)$ and $a \notin \mathcal{V}(I)$. Given $a \leq b \in N$, we have $N_a = N_b$ so that

$$[M \cap n^bN]_a = [n^bN]_a = [n^{b-a}(M \cap n^aN)]_a.$$

Furthermore if $m \in \text{Max-Spec}(R)$ and $m \neq a$, then $n^b_m = n^a_m = R_m$, and
The uniform Artin-Rees theorem

\[ [M \cap n^bN]_m = [n^{b-a}(M \cap n^aN)]_m. \]

Therefore

\[ M \cap n^bN = n^{b-a}(M \cap n^aN). \]

In particular this is true when \( a = s \) is a uAR number for Max-Spec(R) ∩ \( \mathcal{V}(I) \) with respect to \( M \subseteq N \). Hence, if \( s \) is a uAR number for Max-Spec(R) ∩ \( \mathcal{V}(I) \), \( s \) is a uAR number for Max-Spec(R) with respect to \( M \subseteq N \).

The proof of the next corollary was shown to me by L. O'Carroll (and is several times shorter than my own attempt).

Corollary 2.3.12. Let \( R \) be a Noetherian ring, \( N \) a finitely generated \( R \)-module, \( M \) a submodule of \( N \) and \( I \) an ideal of \( R \) such that, for all primes \( Q \in \mathcal{V}(I) \), \( R/Q \) is \( J \)-0. Then there is a non-negative integer \( s \) and, for all \( Q \in \mathcal{V}(I) \), there is a non-empty open set \( U_Q \), containing \( Q \), such that, for all \( q \in U_Q \) and all \( n \geq s \),

\[ [M \cap Q^sN]_q = [Q^{n-s}(M \cap Q^sN)]_q. \]

Furthermore if \( I \subseteq \text{Ann}_R(N/M) \) then such an open set exists for every \( Q \in \text{Spec}(R) \).

Proof. From the proof of the theorem we have \( s \in N \) such that 2.3.10 holds, with \( k = s \) and \( n \geq s \), for all \( Q \in \mathcal{V}(I) \). Fix \( Q \in \mathcal{V}(I) \). Let

\[ L = \bigoplus_{r \geq 0} \left[ \frac{M \cap Q^{r+s}N}{Q^r(M \cap Q^sN)} \right]. \]

Then \( L \) is an \( R \)-module and \( L_Q = 0 \). Therefore, for any \( x \in L \), we
have an element \( t \in R \setminus Q \) such that \( xt = 0 \). Now consider the 

\[ R \text{-module } B = \bigoplus_{r \geq 0} Q^r. \]

\( B \) is a Noetherian \( R \)-algebra containing \( R \). The \( R \)-module \( \bigoplus_{r \geq 0} Q^r(Q^sN) \) is a finitely generated \( B \)-module with finitely generated \( B \)-module. As \( \bigoplus_{r \geq 0} Q^r(M \cap Q^sN) \) is also a \( B \)-module, \( L \) is a finitely generated \( B \)-module. Let \( \omega_1, \ldots, \omega_q \) be elements of \( L \) such that \( L = \sum_{i=1}^{q} B\omega_i \). Then we have an element \( t_1 \in R \setminus Q \) such that \( \omega_i t_1 = 0 \), for \( i = 1, \ldots, q \). Let 

\[ t = t_1 t_2 \cdots t_q. \]

Then \( t \notin Q \) and \( tL = t(\sum_{i=1}^{q} B\omega_i) = 0 \). Hence \( L_t = 0 \) as an \( R_t \)-module, and, for all \( q \in D(t) \subseteq \text{Spec}(R) \) and all \( n \geq s \), we have

\[ [M \cap Q^sN]_q = [Q^{n-s}(M \cap Q^sN)]_q. \]

Since \( t \notin Q \), \( U_q = D(t) \) meets the requirements of the first part of the theorem.

Now suppose we have found \( s \) as above and let \( Q \in \text{Spec}(R) \) such that \( Q \notin \mathcal{V}(I) \). If \( I \subseteq \text{Ann}_R(N/M) \) then \( Q \notin \mathcal{V}(\text{Ann}_R(N/M)) \). Since \( N/M \) is finitely generated, this implies that \[ [N/M]_Q = 0 \] and the final statement follows easily. \( \square \)
Before proceeding to the results of this section we need to establish some notation and conventions. We also take the opportunity to emphasise some elementary facts that are required in the sequel. Let $R$ be a ring, $P$ a prime ideal of $R$ and $L$ an $R$-module. If $X$ is a subset of $L_P$ we denote the inverse image of $X$ in $L$ by $X \cap L$, extending the notation of Chapter 1, Section 1.5.

For any $Q \in \text{Spec}(R)$, let $\varphi_Q : L \longrightarrow L_Q$ be the $R$-module map such that $\varphi_Q(\ell) = \ell/1 \in L_Q$, for any $\ell \in L$. Suppose $P \subseteq Q \in \text{Spec}(R)$.

Then the $R_Q$-module map $\varphi_P^Q : L_Q \longrightarrow L_P$ such that $\varphi_P^Q(\ell/s) = \varphi_P(\ell)(1/s) = \ell/s$, for any $\ell \in L$ and $s \in R \setminus Q$, makes the following diagram commute:

```
\begin{array}{c}
L \xrightarrow{\varphi_P} L_P \\
\downarrow \varphi_Q \downarrow \varphi_P^Q \\
L_Q \xrightarrow{\varphi_P^Q} L_P
\end{array}
```

We shall refer to the maps $\varphi_P$, $\varphi_Q$ and $\varphi_P^Q$ as "the" canonical maps. If $M$ is an $R$-module and $f : L \longrightarrow M$ is an $R$-module map, then we have an $R_P$-module map $f_P : L_P \longrightarrow M_P$, given by $f_P(\ell/s) = f(\ell)/s$, for any $\ell \in L$ and $s \in R \setminus Q$. Moreover the following diagram commutes:

```
\begin{array}{c}
L \xrightarrow{f} M \\
\downarrow \varphi_P \downarrow \mu_P \\
L_P \xrightarrow{f_P} M_P
\end{array}
```
where \( \varphi_p \) and \( \mu_p \) are the canonical maps. We shall distinguish between the modules \([N/M]_p\) and \(N_p/M_p\). Although these modules are isomorphic, we do not assume that they are equal, and specify all our isomorphisms explicitly. Elements of \([N/M]_p\) are of the form \(\frac{n \mod(M)}{s}\), whereas elements of \(N_p/M_p\) are of the form \(\frac{n}{s} \mod(M_p)\), where \(n \in N\) and \(s \in R \setminus P\).

### Lemma 2.4.1

Let \( R \) be a ring, \( P \in \text{Spec}(R) \), \( M \) an \( R \)-module and \( r \) a non-negative integer such that \( \frac{P^i M}{P^{i+1} M} \) is a torsion-free \( R/P \)-module, for \( i = 1, \ldots, r - 1 \). Then

\[ \mathcal{P}^r M \cap P M = P^r M. \]

**Proof.** Suppose \( x \in \mathcal{P}^r M \cap P M \). Then \( xs \in P^r M \) for some \( s \in R \setminus P \).

If \( x \notin P^r M \), let \( j = \max\{i \in \mathbb{Z} : x \in P^i M\} \), so that \( 0 \leq j < r \).

Then \( x \mod(P^{j+1} M) \neq 0 \) but \( s(x \mod(P^{j+1} M)) = 0 \). Since \( \frac{P^j M}{P^{j+1} M} \) is a torsion-free \( R/P \)-module, \( s \) must be in \( P \). This contradiction shows that \( x \in P^r M \), so that \( \mathcal{P}^r M \cap P M \subseteq P^r M \). The reverse inclusion is obvious, giving the lemma. \( \square \)

### Lemma 2.4.2

Let \( R \) be a Noetherian local ring, \( P \in \text{Spec}(R) \) and \( L \) a finitely generated \( R \)-module such that \( \text{gr}_P(L) \) is a free \( R/P \)-module. Then the \( R \)-module map \( \varphi_P : L \rightarrow L_P : \ell \mapsto \ell / \ell P \) is injective and there is a canonically induced, injective, \( R/P \)-module map

\[ \text{gr}_P^r(L) \rightarrow \text{gr}_P^r(L_P), \]
for all $r \geq 0$.

**Proof.** Let $x \in \ker(\varphi_p)$, so that $x/1 = 0$ in $L_p$. Then

$$(x/1) \in \bigcap_{k \geq 0} P^k L_p.$$ Therefore

$$x \in \bigcap_{k \geq 0} (P^k L_p \cap L) = \bigcap_{k \geq 0} P^k L,$$

from Lemma 2.4.1. Using Krull's intersection theorem, $\bigcap_{k \geq 0} P^k L = 0$ ([Mat2], Theorem 8.9, for example). Thus $\ker(\varphi_p) = 0$ and $\varphi_p$ is injective as claimed.

Let $r$ be a non-negative integer and $y \in P^r L$. Then $\varphi_p(y) \in P^r L_p$, so we have, by restriction of $\varphi_p$ to $P^r L$, an injective map $P^r L \rightarrow P^r L_p$. We define

$$gr^r(\varphi_p) : \frac{P^r L}{p^{r+1} L} \rightarrow \frac{P^r L_p}{p^{r+1} L_p}$$

by

$$y \mod(p^{r+1} L) \mapsto \varphi_p(y) \mod(p^{r+1} L_p),$$

where $y \in P^r L$. $gr^r(\varphi_p)$ is our candidate for the canonically induced, injective, $R/P$-module map required. $gr^r(\varphi_p)$ is an $R/P$-module map, since if $y \in P^r L$ then $\varphi_p(y) \in p^{r+1} L_p$.

Furthermore, given $r \geq 0$ and $y \in P^r L \cap P^r L_p$, then

$$y \in P^r L_p \cap L \cap P^r L = p^{r+1} L,$$

using Lemma 2.4.1. Hence $gr^r(\varphi_p)$ is injective. This is true for all $r \geq 0$, as required. □

**Lemma 2.4.3.** Let $R$ be a Noetherian ring, $P \in \text{Spec}(R)$ and $\mathcal{N}$ a dense subset of $V(P)$. For each $Q \in \mathcal{N}$, let $\rho^Q_p$ denote the
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canonical map \( R_Q \to R_P \). Then

\[
\bigcap_{Q \in \mathcal{N}} \rho_Q^{R_Q} \subseteq P_{R_P}.
\]

Proof. Let \( V = \{ q \in \mathcal{V}(P) : q \) is \(((0), R; P)\)-regular\}. Then \( V \) is a non-empty open subset of \( \mathcal{V}(P) \), by Theorem 2.2.3. Hence \( V \cap \mathcal{N} \) is dense in \( \mathcal{V}(P) \), by Lemma 2.3.3. It is therefore sufficient to prove the result under the assumption that \( \mathcal{N} \subseteq V \). Under this assumption \( \rho_Q^P \) is injective, by Lemma 2.4.2. Let \( x \in \bigcap_{Q \in \mathcal{N}} \rho_Q^{R_Q} \).

Then \( x \in R_P \), so we can write \( x = a/s \), where \( a, s \in R \) and \( s \notin P \). Let \( U_x = \{ Q \in \mathcal{V}(P) : s \notin Q \} \). Then \( U_x \) is non-empty and open in \( \mathcal{V}(P) \). Therefore \( \mathcal{N}_x = U_x \cap \mathcal{N} \) is dense in \( \mathcal{V}(P) \), using Lemma 2.3.3. Let \( Q \in \mathcal{N}_x \). Then \( x = a/s \) and \( s \notin Q \), so that \( a/s \in R_Q \). Hence \( x = \rho_Q^P(a/s) \). Also \( x \in \rho_Q^{R_Q} \), so \( x = \rho_Q^P(y/t) \), where \( y \in Q \) and \( t \in R \setminus Q \). As \( \rho_Q^P \) is injective, this implies that, in \( R_Q \), \( a/s = y/t \). Hence \( atv = ysv \), for some \( v \in R \setminus Q \). Then \( atv \in Q \) and \( tv \notin Q \), which means that \( a \in Q \). Thus \( a \in \bigcap_{Q \in \mathcal{N}_x} Q = P \), so

that \( x = a/s \in P_{R_P} \), as required. \( \square \)

We can now prove a generalisation of Zariski's Main Lemma ([Zar], Section 2). The proof is essentially Zariski's.

Theorem 2.4.4. Let \( R \) be a Noetherian ring, \( P \in \text{Spec}(R) \) and \( \mathcal{N} \) a dense subset of \( \mathcal{V}(P) \) such that, for all \( Q \in \mathcal{N} \), \( [R/P]_Q \) is a regular ring. Let \( L \) be a finitely generated \( R \)-module and, for each
2.4 Zariski's Main Lemma

Q ∈ \mathcal{N}, let \( \varphi_p^Q \) denote the canonical map \( L_Q \rightarrow L_p \). Then

\[
\bigcap_{Q \in \mathcal{N}} \varphi_p^Q(Q^rL_Q) \subseteq P^rL_p,
\]

for all \( r \geq 0 \).

Proof. Let \( \ell_1, \ldots, \ell_t \) be a minimal generating set for \( L \) and let

\[ N = \bigoplus_{i=1}^t R \ell_i \]

be the free \( R \)-module on the \( t \) generators \( \ell_1, \ldots, \ell_t \).

Let \( M \) be the kernel of the surjective \( R \)-module map \( N \rightarrow L \) given by mapping \( e_i \) to \( \ell_i \), for \( i = 1, \ldots, t \). Then \( L = N/M \). Let

\[ T = \{ Q \in \text{Spec}(R) : Q \text{ is } (M,N;P)\text{-regular and } Q \text{ is } ((0),R;P)\text{-regular} \} \]

Then, by Corollary 2.2.4, \( T \) is non-empty and open in \( \mathcal{V}(P) \). By Lemma 2.3.3, \( T \cap \mathcal{N} \) is dense in \( \mathcal{V}(P) \). It is therefore sufficient to prove the result under the assumption that \( \mathcal{N} \) is contained in \( T \). Under this assumption, \( \varphi_p^Q \) is injective for all \( Q \in \mathcal{N} \), by Lemma 2.4.2. We proceed by induction on \( r \). In the case \( r = 0 \) there is nothing to prove. Suppose inductively that the result has been proved for \( r \geq 0 \). From the inductive assumption, we have

\[
\bigcap_{Q \in \mathcal{N}} \varphi_p^Q(Q^{r+1}L_Q) \subseteq \bigcap_{Q \in \mathcal{N}} \varphi_p^Q(Q^rL_Q) \subseteq P^rL_p.
\]

Thus

\[
\bigcap_{Q \in \mathcal{N}} \varphi_p^Q(Q^{r+1}L_Q) = \bigcap_{Q \in \mathcal{N}} (\varphi_p^Q(Q^{r+1}L_Q) \cap P^rL_p).
\]

If \( x \in \varphi_p^Q(Q^{r+1}L_Q) \cap P^rL_p \), then \( x = \varphi_p^Q(y) \), for some element \( y \) of \( Q^{r+1}L_Q \). As \( \varphi_p^Q(y) = x \in P^rL_p \), we have \( y \in P^rL_p \cap P_Q L_Q = P^rL_Q \).
using Lemma 2.4.1. Therefore \( \phi^Q_p(y) = x \in \phi^Q_p(Q^{r+1}L_Q \cap P^rL_Q) \).

It follows that

\[
\bigcap_{Q \in \mathcal{M}} \phi^Q_p(Q^{r+1}L_Q) = \bigcap_{Q \in \mathcal{M}} \phi^Q_p(Q^{r+1}L_Q \cap P^rL_Q).
\]

Hence

\[
\bigcap_{Q \in \mathcal{M}} \phi^Q_p(Q^{r+1}L_Q) = \bigcap_{Q \in \mathcal{M}} \phi^Q_p(P^rQL_Q),
\]

using Corollary 2.1.17. It is therefore sufficient to show that

\[
\bigcap_{Q \in \mathcal{M}} \phi^Q_p(P^rQL_Q) \subseteq P^{r+1}L_P.
\]

Recall the notation of 2.1.3. Note that, if \( q \in \mathcal{V}(P) \), then by \( A_q \) we mean \( A \) localised at \( q/P \). Given any \( q \in \mathcal{V}(P) \), let \( \theta_q \) be the canonical \( A \)-module map \( \theta_q : E_r \rightarrow [E_r]_q \), and let \( \theta^q_p \) be the canonical \( A_q \)-module map \( \theta^q_p : [E_r]_q \rightarrow [E_r]_p \). Consider the \( A_p \)-vector space \( [E_r]_p \). Suppose the dimension of this vector space over \( A_p \) is \( d \). Choose elements \( \varepsilon_1, \ldots, \varepsilon_d \) of \( E_r \) such that \( \theta_p(\varepsilon_1), \ldots, \theta_p(\varepsilon_d) \) form a basis for \( [E_r]_p \). Set \( \check{\varepsilon}_1 = \theta_p(\varepsilon_1), \ldots, \check{\varepsilon}_d = \theta_p(\varepsilon_d) \). Let \( U = \{ q \in \mathcal{V}(P) : [E_r]_q \text{ is generated by } \theta_q(\varepsilon_1), \ldots, \theta_q(\varepsilon_d) \} \). Then \( U \) is a non-empty open subset of \( \mathcal{V}(P) \) (using [Mat2], Theorem 4.10 (i) (proof), and identifying \( U \) with its homeomorphic image in \( \text{Spec}(R/P) \)). Since \( P \in U \), if we set \( W = U \cap T \), then \( W \) is a non-empty open subset of \( \mathcal{V}(P) \). Thus, setting \( \mathcal{M} = \mathcal{N} \cap W \), \( \mathcal{M} \) is dense in \( \mathcal{V}(P) \), by Lemma 2.3.3.

Now fix \( Q \in \mathcal{M} \). Let \( \theta_q(\varepsilon_k) = \check{\varepsilon}_k \), for \( k = 1, \ldots, d \).
Since \( Q \in U \), \( \tilde{e}_1, \ldots, \tilde{e}_d \) generate \([E_r]_Q\). Consider the surjective \( A_Q\)-module map

\[
\oplus_{k=1}^d (A_Q)x_k \longrightarrow \Sigma_{k=1}^d (A_Q)\tilde{e}_k = [E_r]_Q
\]

where the \( x_i \) is an indeterminate, for \( i = 1, \ldots, d \). If the kernel of this map is \( C \), then \( C \otimes_{A_Q} A_P = 0 \), since \( \{\tilde{e}_1, \ldots, \tilde{e}_d\} \) is a basis for \([E_r]_P\). Since \( C \subseteq \oplus_{k=1}^d (A_Q)x_k \), \( C \) is a torsion-free \( A_Q\)-module. Therefore \( C \otimes_{A_Q} A_P = 0 \) implies that \( C = 0 \). Hence

\[
[E_r]_Q = \oplus_{k=1}^d (A_Q)\tilde{e}_k. \quad \text{(This fact is implicit in [Mat2], Theorem 4.10 (ii).)}
\]

Let \( \rho^P_P : A_Q \longrightarrow A_P \) be the canonical \( A_Q\)-module map. We now define an \( A_Q\)-module map

\[
\alpha_Q : [E_r]_Q = \oplus_{k=1}^d (A_Q)\tilde{e}_k \longrightarrow \oplus_{k=1}^d (A_P)\tilde{e}_k = [E_r]_P,
\]

where \( a_k \in A_Q \), for \( k = 1, \ldots, d \). Since \( \rho^P_P \) is injective, \( \alpha_Q \) is injective. Note that

\[
\alpha_Q(Q[E_r]_Q) = \alpha_Q(Q(\oplus_{k=1}^d (A_Q)\tilde{e}_k))
\]
For any \( q \in \mathcal{M} \), we can define the corresponding \( A_q \)-module map \( \alpha_q \). Moreover, \( \mathcal{M} \) is dense in \( V(P) \), so that
\[
\mathcal{M} = \{ q/P \in \text{Spec}(A) : q \in \mathcal{M} \}
\]
is dense in \( \text{Spec}(A) \). Hence, from Lemma 2.4.3, 
\[
\bigcap_{q \in \mathcal{M}} \rho_q^P([q/P]_q) = 0. 
\]
Thus, using (i),
\[
\bigcap_{q \in \mathcal{M}} \alpha_q(q[E_r]_q) = \bigcap_{q \in \mathcal{M}} \left( \bigoplus_{k=1}^{d} \rho_q^P([q/P]_q)\bar{e}_k \right) = \bigoplus_{k=1}^{d} \bigcap_{q \in \mathcal{M}} \rho_q^P([q/P]_q)\bar{e}_k = 0. \tag{ii}
\]

Going back to our fixed \( \mathcal{M} \); if \( x \in [E_r]_Q \) then
\[
x = \sum_{k=1}^{d} a_k \bar{e}_k, \text{ where } a_k \in A_Q, \text{ for } k = 1, \ldots, d. \text{ Therefore}
\]
\[
\theta_P^Q(x) = \theta_P^Q(\sum_{k=1}^{d} a_k \bar{e}_k) = \sum_{k=1}^{d} a_k \theta_P^Q(\bar{e}_k) \quad \text{(since } \theta_P^Q \text{ is an } A_Q \text{-module map).}
\]
\[
\begin{align*}
&= \sum_{k=1}^{d} a_k \theta_p^Q(\theta_p(\epsilon_k)) \\
&= \sum_{k=1}^{d} a_k \theta_p(\epsilon_k) \\
&= \sum_{k=1}^{d} a_k \tilde{\epsilon}_k \quad \text{(by definition)} \\
&= \sum_{k=1}^{d} \rho_p^Q(a_k) \tilde{\epsilon}_k \\
&= \alpha_q(\sum_{k=1}^{d} a_k \tilde{\epsilon}_k) = \alpha_q(x).
\end{align*}
\]

This holds for all \( q \in \mathcal{M} \), hence

\[
\alpha_q = \theta_p^q, \quad \text{for all } q \in \mathcal{M}. \quad \text{(iii)}
\]

Now let \( f \) be the isomorphism (given by the isomorphism theorems; line 3 of the proof of Lemma 2.1.7)

\[
f : \frac{p^r L}{p^{r+1} L} \rightarrow E_r.
\]

Let \( f_p \) and \( f_Q \) be the isomorphisms induced, on the relevant modules, by localising at \( P \) and \( Q \), respectively. Note that \( f_Q \) is an \( A_Q \)-module map, so
2.4 Zariski's Main Lemma

For any \( q \in \mathcal{V}(P) \), let \( \delta_q : \frac{P^r L}{P^{r+1} L} \rightarrow \left[ \frac{P^r L}{P^{r+1} L} \right]_q \) and \( \delta'^q : \left[ \frac{P^r L}{P^{r+1} L} \right]_q \rightarrow \left[ \frac{P^r L}{P^{r+1} L} \right]_p \) be the canonical maps. As we remarked at the beginning of this section \( f_p \delta^q = \phi^q \). Therefore, using (iii),

\[
\left( f_p \right) \delta^q = \alpha f_q, \quad \text{for all } q \in \mathcal{V}.
\]

Given any \( q \in \mathcal{V}(P) \), let \( \tau_q \) be the \( A_q \)-module isomorphism

\[
\tau_q : \left[ \frac{P^r L}{P^{r+1} L} \right]_q \rightarrow \frac{P^r L}{P^{r+1} L}_q,
\]

\[
(x \mod(P^{r+1} L)/s) \rightarrow (\varphi(x)(1/s) \mod(P^{r+1} L)_q),
\]

where \( x \) is any element of \( P^r L \). Then, given any \( x \in P^r L \),

\[
\tau_p(\delta'^q((x \mod(P^{r+1} L)/s)) = \tau_p(\delta_p(x \mod(P^{r+1} L))(1/s))
\]

\[
= \tau_p((x \mod(P^{r+1} L))/s)
\]

\[
= (1/s)(\varphi_p(x)) \mod(P^{r+1} L)_p).
\]

Also, using the notation of Lemma 2.4.2,

\[
gr^r(\varphi^q_p)(\tau_q(x \mod(P^{r+1} L)/s)) = gr^r(\varphi^q_p)(\varphi(x)(1/s) \mod(P^{r+1} L)_q)
\]

\[
= (1/s)(\varphi^q_p(\varphi(x))) \mod(P^{r+1} L)_p
\]
Hence

\[ \tau_p \delta^q_p = \text{gr}^r (\varphi^q_p) \tau^q_q, \quad \text{for all } q \in \mathcal{M}. \]  

(vi)

Combining (v) and (vi) and using the notation of Lemma 2.4.2, we have, since \( \tau^q_q \) is an isomorphism, for all \( q \in \mathcal{M} \),

\[ \alpha_{q} f_{q}^{-1} \tau^{-1} = f_p \tau_p^{-1} \text{gr}^r (\varphi^q_p), \quad \text{for all } q \in \mathcal{M}. \]  

(vii)

Now

\[
\begin{align*}
&f_p \tau_p^{-1} \left( \bigcap_{Q \in \mathcal{M}} \text{gr}^r (\varphi^q_p) \left[ \frac{Q^r L_Q}{p^{r+1} L_Q} \right] \right) \\
&\subseteq f_p \tau_p^{-1} \left[ \bigcap_{Q \in \mathcal{M}} \text{gr}^r (\varphi^q_p) \left[ \frac{Q^r L_Q}{p^{r+1} L_Q} \right] \right] \\
&= \bigcap_{Q \in \mathcal{M}} \alpha_{q} f_{q}^{-1} \left[ \tau_q^{-1} \left[ \frac{Q^r L_Q}{p^{r+1} L_Q} \right] \right] \\
&= \bigcap_{Q \in \mathcal{M}} \alpha_{q} f_{q}^{-1} \left[ Q^r Q^{-1} \left[ \frac{L_Q}{p^{r+1} L_Q} \right] \right] \\
&= \bigcap_{Q \in \mathcal{M}} \alpha_{q} \left[ f_{q} \left[ \frac{Q^r L_Q}{p^{r+1} L_Q} \right] \right] \\
&= \bigcap_{Q \in \mathcal{M}} \alpha_{q} (Q^{E = r} L_Q) \quad \text{(by (iv))} \\
&= 0, \quad \text{by (ii)}. 
\end{align*}
\]

Since \( f_p \tau_p^{-1} \) is injective, this means that, using the notation of
Lemma 2.4.2,

\[ 0 = \bigcap_{Q \in \mathcal{M}} \text{gr}^r_{\varphi_p}(\text{gr}^r_{\varphi_p}(Q\mathcal{P}L_Q)_{p^{r+1}L_Q}) \]

\[ = \bigcap_{Q \in \mathcal{M}} \left( \frac{\varphi_p^{Q}(Q\mathcal{P}L_Q) + p^{r+1}L_p}{p^{r+1}L_p} \right) \]

\[ \supseteq \frac{\left( \bigcap_{Q \in \mathcal{M}} \varphi_p^{Q}(Q\mathcal{P}L_Q) \right) + p^{r+1}L_p}{p^{r+1}L_p} \]

Hence

\[ \bigcap_{Q \in \mathcal{N}} \varphi_p^{Q}(Q\mathcal{P}L_Q) \subseteq \bigcap_{Q \in \mathcal{M}} \varphi_p^{Q}(Q\mathcal{P}L_Q) \subseteq p^{r+1}L_p, \]

as required. The theorem follows by induction. \( \Box \)

In [EiHo], Eisenbud and Hochster prove a theorem which has Zariski's Main Lemma as a corollary. The next result shows that Eisenbud and Hochster's theorem ([EiHo], Theorem) is a corollary of the above generalisation of Zariski's Main Lemma. Recall that if \( p \) is a prime ideal of a ring \( R \) then we denote \( p^r \cap \mathcal{P} \) by \( p^{(r)} \), for any \( r \in \mathbb{N} \).

Corollary 2.4.5. Let \( R \) be a Noetherian ring, \( P \in \text{Spec}(R) \) and \( \mathcal{N} \) a dense subset of \( \mathcal{V}(P) \) such that, for all \( Q \in \mathcal{N}, \) \( [R/P]_Q \) is a regular ring. Let \( L \) be a finitely generated \( P \)-coprimary \( R \)-module such that \( P^rL = 0 \), for some positive integer \( r \). Then
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\[ \bigcap_{Q \in \mathcal{N}} q^{(r)}_L = 0. \]

Proof. Suppose \( x \in \bigcap_{Q \in \mathcal{N}} q^{(r)}_L = \bigcap_{Q \in \mathcal{N}} (Q^r R_Q \cap q_R) L. \) Let \( Q \) be any element of \( \mathcal{N}. \) Then \( x = \sum_{j=1}^{n} r_j \ell_j, \) for some \( r_j \in q^{(r)} \) and \( \ell_j \in L, \) where \( j = 1, \ldots, n. \)

Let \( \varphi_p, \varphi_Q, \) and \( \varphi_Q^Q \) be the canonical maps, for the module \( L, \) described before Lemma 2.4.3. In \( L_Q \) we have

\[ \varphi_Q(x) = \sum_{j=1}^{n} r_j \varphi_Q(\ell_j) \in Q^r L_Q. \]

Thus \( \varphi_p(x) = \varphi_p \varphi_Q(x) \in \varphi_p(Q^r L_Q). \) This is true for all \( Q \in \mathcal{N}. \)

Hence

\[ \varphi_p(x) \in \bigcap_{Q \in \mathcal{N}} \varphi_p(Q^r L_Q) \subseteq p^r L_P = 0. \]

Since \( L \) is \( P \)-coprimary, \( Z(L) \subseteq P. \) Therefore \( \varphi_p \) is injective and \( x = 0. \) \( \Box \)
References


[DO'C2] A. Duncan and L. O'Carroll, On Zariski regularity, the vanishing of Tor and a uniform Artin-Rees theorem. To appear in the proceedings of the Mini-Semester on Commutative Algebra and Algebraic Geometry, held at the Stefan Banach International Mathematical Centre in Warsaw, 1988


References


