An Absolute Geopotential Height System for Ethiopia

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This study used airborne gravity data, the 2008 Earth Gravity Model (EGM08) and Shuttle Radar Topographic Mission (SRTM) digital elevation data in a ‘Remove-Compute-Restore’ process to determine absolute vertical reference system for Ethiopia. This gives a geopotential height at any isolated field point where there is a Global Navigation Satellite System (GNSS) measurement without reference to a vertical network or a regional datum point. Previously, height was determined conventionally by connecting the desired field point physically to a nearby benchmark of a vertical network using co-located measurements of gravity and spirit levelling. With the use of precise GNSS positioning and a gravity model this method becomes obsolete.

The new approach uses the ‘Remove-Restore’ process to eliminate longer to shorter wavelengths from the measured gravity data using EGM08 and geometrical and condensed gravity models of the SRTM data. This provides small, smooth and localised residuals so that the interpolation and integration involved is reliable and the Stokes-like integral can be legitimately restricted to a spherical cap. A very fast, stable and accurate computational algorithm has been formulated by combining ‘hedgehog’ and ‘multipoint’ models in order to make tractable an unavoidably huge computational task required to remove the effects of about 1.5 billion! SRTM topographic mass elements representing Ethiopia and its immediate surroundings at 92433 point airborne gravity observations.

The compute stage first uses an iterative Fast Fourier Transform (FFT) to predict residual gravity at aircraft height as a regular grid on to the surface of the ellipsoidal Earth and then it used a Fourier operation equivalent to Stokes’ integral to transform the localised gravity disturbance to residual potential. The restore process determines the geopotential number on or above the Earth’s surface where practitioners need it by restoring the potential effects of the removed masses. The accuracy of the geopotential number computed from gravity and topography was evaluated by comparing it with the one derived directly from EGM08 and precise geodetic levelling. The new model is in a good agreement across 100 km baseline with a standard deviation of $56 \times 10^{-2} \text{m}^2 \text{s}^{-2}$ and $39 \times 10^{-2} \text{m}^2 \text{s}^{-2}$ relative to EGM08 and levelling, respectively ($10^{-2} \text{m}^2 \text{s}^{-2}$ is approximately equivalent to 1mm of height). The new method provides an absolute geopotential height of a point on or above the Earth’s surface in a global sense by interpolating from geopotential models prepared as the digital grids carried in a chip for use with the GNSS receiver in the field.
Declaration

This thesis has been composed by myself and has not been submitted in any previous application for a degree. The work reported within was executed by myself, unless otherwise stated.

August 2010
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List of Acronyms

1-D One-Dimensional
2-D Two-Dimensional
3-D Three-Dimensional
DFT Discrete Fourier Transform
EGM Earth Gravity Model
ERS European Remote-Sensing Satellite
ESA European Space Agency
FFT Fast Fourier Transform
FT Fourier Transform
GEOSAT GEOdetic SATellite
GMT Generic Mapping Tools
GNSS Global Navigation Satellite System
GOCE Gravity field and steady-state Ocean Circulation Explorer
GPS Global Positioning System
GRACE Gravity Recovery and Climate Experiment
IAG International Association of Geodesy
IERS International Earth Rotation Service
IGS International GNSS service
IGSN71 The 1971 International Gravity Standardization Net
LSC Least Square Collocation
P-S Pizzetti-Somigliana
RMS Root Mean Square
SRTM Shuttle Radar Topographic Mission
WGS World Geodetic System
List of Symbols

\[ c \] geopotential number
\[ \delta \Omega \] differential angular element
\[ \exp \] exponential function
\[ r \] geocentric radius
\[ \eta \] angle between ellipsoidal normal and geocentric radius
\[ \psi \] geocentric angle
\[ \xi \] \( \sin \left( \frac{\psi}{2} \right) \)
\[ \xi_{1/2} \] angular correlation length
\[ \rho_{1/2} \] spatial correlation length on a map projection plane
\[ d\lambda \] angular size of the SRTM grid cell in the east direction
\[ d\varphi \] angular size of the SRTM grid cell in the north direction
\[ \omega \] angular velocity of the earth
\[ \phi_{ss} \] autocorrelation function
\[ \Delta g \] gravity anomaly
\[ \delta g \] gravity disturbance
\[ \overline{\Delta g} \] average value of gravity anomaly around a ring
\[ \overline{\delta g} \] average value of gravity disturbance around a ring
\[ \alpha \] azimuth angle
\[ w(n) \] butterworth filter
\[ x, y, z \] cartesian coordinates
\[ u \] confocal ellipsoidal coordinate (semi-minor axis)
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<tr>
<td>$C_{st}$</td>
<td>covariance function</td>
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<tr>
<td>$n, m$</td>
<td>degree and order of spherical harmonic coefficients</td>
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<tr>
<td>$\ell$</td>
<td>distance between the computation point and mass element</td>
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<tr>
<td>$J_2$</td>
<td>Earth’s gravitational flattening, second degree zonal harmonic coefficient of the gravity potential</td>
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<td>$h$</td>
<td>ellipsoidal height</td>
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<td>$F(k_x,k_y)$</td>
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<td>$\overline{C}<em>{mn}, \overline{S}</em>{mn}$</td>
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<td>$p_{n}^{m}$</td>
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<td>$GM$</td>
<td>geocentric gravitational constant</td>
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<td>$\vartheta$</td>
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<td>$\gamma$</td>
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\( \gamma_b \)  
normal gravity at the pole

\( T \)  
anomalous potential

\( \delta g_T \)  
gravitational attraction of the topography

\( \delta g_T^c \)  
topographic gravity at the computation point in the direction of normal gravity at the mass element

\( \delta g_T^h \)  
topographic gravity at the computation point in the direction perpendicular to normal gravity at the vertical line mass element

\( \delta g_T^v \)  
vertical component of topographic attraction at the computation point

\( \delta v_T \)  
gravitational potential of the topography

\( \delta v_T \)  
potential of the condensed topography

\( \delta g_T \)  
gravity of the condensed topography

\( K \)  
Hotine-Koch kernel

\( S \)  
Stokes’ kernel

\( H^* \)  
Modified Hotine-Koch kernel

\( F^* \)  
Lambert-Stokes kernel

\( j \)  
imaginary number, \( \sqrt{-1} \)

\( E \)  
linear eccentricity

\( e \)  
first eccentricity

\( e' \)  
second eccentricity

\( \ln \)  
Natural logarithm

\( k_2 \)  
Potential field Love number for second degree harmonics

\( dm \)  
mass element

\( M_c \)  
mass of celestial body

\( \varsigma \)  
mass per unit length

\( G \)  
Newton’s universal gravitational constant
$M$ principal radii of curvature in the direction of meridian
$N$ normal radius of curvature in the direction of prime vertical
$n_0$ cut-off degree
$\lambda_0$ cut-off wavelength
$K_{0L}$ “low-pass” cut-off wavenumber
$K_{0H}$ “high-pass” cut-off wavenumber
$k_N$ Nyquist wavenumber
$A, \beta$ parameters of the stochastic gravity power spectrum
$\pi$ pi
$\rho, \alpha$ polar coordinates, radius and azimuth
$P_n$ power spectrum of a global gravity model
$r_B$ radius of an internal Bjerhammer sphere
$W_2$ second degree tide-generated potential
$a$ semi-major axis of the reference ellipsoid
$b$ semi-minor axis of the reference ellipsoid
$\sigma$ surface density of the condensed topography
$g_2$ tide-generated gravity
$V$ volume
$\kappa$ wavenumber
$k_x, k_y$ wavenumbers in the $x$ and $y$ directions, respectively
$H$ orthometric height
$R$ radius of the Earth
$\beta$ reduced latitude
$C_{20}$ second degree zonal term
Chapter 1

Introduction

1.1 Background

A vertical reference system can be defined by directly computing the gravitational potential on or above the Earth’s surface from observations of gravity. The difference between the potential \( W_0 \) of a regional datum point and the potential \( W \) of any point on or above the Earth’s surface determines the geopotential height of that point.

\[
e = W_0 - W
\]  

(1.1)

where \( e \) is a geopotential height.

In 1849, Stokes devised a method of doing this. He computed a potential for a particular point by integrating surface gravity measurements over the whole globe. To simplify the integration process the Stokes-like integrals operate on disturbing part of the Earth’s gravity field. Figure 1.1 shows how Stokes-like integrals relate the anomalous geopotential \( T \) at point \( D \) (the origin) to a global integral of gravity anomalies, \( \Delta g \), or gravity disturbances \( \delta g \). (Hofmann-Wellenhof & Moritz, 2006, pp. 102-116). They are written as

\[
T_D = \frac{R}{4\pi} \int_{\psi=0}^{\pi} \left( \int_{\alpha=0}^{2\pi} \Delta g(\psi, \alpha) d\alpha \right) S(\psi) \sin \psi d\psi
\]  

(1.2)

\[
T_D = \frac{R}{4\pi} \int_{\psi=0}^{\pi} \left( \int_{\alpha=0}^{2\pi} \delta g(\psi, \alpha) d\alpha \right) K(\psi) \sin \psi \sin d\psi
\]  

(1.3)

where the gravity disturbance \( \delta g = g - \gamma \) is the difference between observed gravity \( g \) and normal gravity \( \gamma \) evaluated at the same point, while the gravity anomaly \( \Delta g = g - \gamma \) is the difference between observed gravity \( g \) at height \( H \) above sea level and normal gravity \( \gamma \) evaluated at a height \( H \) above the reference ellipsoid, \( R \) is the radius of the Earth, \( \psi \) is the geocentric spherical angle, \( S(\psi) \) is the Stokes’ kernel, \( K(\psi) \) is Hotine-Koch kernel, \( \alpha \) is azimuth angle.
In principle, the Stokes-like integral transform involves a continuous representation of gravity that is globally complete. In reality, we observe the magnitude of the gravity vector at discrete irregular points on or above the Earth’s surface. Practical evaluation of Stokes’ integral requires an interpolation process. The integral transform converts irregularly distributed gravity observations into surface spherical (or ellipsoidal) harmonic potential coefficients in a series expansion of Legendre polynomials or associated Legendre polynomials. Stokes integral comes from an analytical sum of this series but alternative approaches may work directly with these coefficients. Satellite gravity models provide a gravity model described by spherical harmonic coefficients.

Although satellite gravity and altimetry missions provide complete global coverage of the Earth’s gravity field at long wavelengths, the determination of a precise geopotential number that can meet engineering accuracies still requires methods that can limit the Stokes integral to a local region of the Earth’s surface where precise gravity data are available. The issue of limiting the Stokes’ integral to
a local scale has been examined by Molodensky et al., (1962) and Wong & Gore (1969), who sought a solution through kernel modification. Their approach has been developed further by many authors since (e.g. Featherstone et al., 1998; Jekeli, 1980; Meissl, 1971; Sjöberg, 1984; Vaníček & Kleusberg, 1987; Vaníček & Sjöberg, 1991).

One part of this thesis explores an alternative approach: we justify limiting the integral to a spherical cap from stochastic properties of the local gravity field, rather than any property of the kernel. In this study, Hipkin’s (2001) stochastic autocorrelation function has been used to study the properties of the Earth Gravity Model 2008 (EGM08, Pavlis et al., 2008). A localisation of the Stokes-like integral uses a combination of local gravity with satellite gravity and Shuttle Radar Topographic Mission (SRTM) elevation data to produce a small and smoothly varying residual so that the interpolation and integration involved in transforming irregularly distributed gravity residual to a potential is reliable and no far field omission error occurs by restricting the integral to a spherical cap.

Modern remove-compute-restore approaches for computing a geopotential number combine analysis of the satellite gravity, terrestrial gravity (land, shipborne or airborne) and SRTM data. They have become a standard procedure in physical geodesy in order to improve the accuracy of the model prediction. Hipkin (2004) discussed the classical regularisation approach and suggested that the modern remove-restore technique should use three components: the Pizzetti-Somigliana (P-S) ellipsoidal reference Earth model (Heiskanen & Moritz, 1967); a global gravity model described by a high-degree and order series of spherical or ellipsoidal harmonic coefficients and the difference between a geometrical model of the in situ topography and identical masses condensed to some coating on or below the reference ellipsoid. Satellite gravity data or a global gravity model provides long wavelength components of the Earth’s gravity field caused by all forms of lateral density variation in both far zone global topography and deep subsurface geological structures. Since satellites orbit the Earth at high altitude of about 200 – 500 km, shorter wavelength variations in the Earth’s gravity field caused by local perturbations will be attenuated and cannot be recovered from space-based observations. Terrestrial gravity data provide medium to shorter wavelength
components of the Earth’s gravity field caused by lateral variations in local
topography and near subsurface geological structures and they are therefore
complementary to satellite data. Marine gravity and satellite altimetry data
complement the satellite gravity data over the ocean surface. For geodetic
applications the land, airborne and marine gravimetry measurements can be made at
sufficient accuracy (~ 1 mgal to ~3 mgal) and at a resolution that can meet the
geodetic requirements (Forsberg & Olesen, 2006). SRTM data give very short
wavelength components of the Earth’s gravity field that cannot be recovered from
either satellite or airborne gravimetry observations. They provide unprecedented
resolution for a gravity model that varies laterally due to topographic irregularity and
it can be analytically computed without introducing any conceptual error. In this
research, the satellite, airborne and SRTM data were combined in a remove-restore
process to give complementary information about the Earth’s gravitational potential
thereby improving the accuracy of the geopotential prediction.

Much active research has been conducted to evaluate the gravitational
attraction of in situ topography on a spherical Earth (Heck & Seitz, 2007; Smith,
2000; Tsoulis, 1999) in order to suppress high frequency components from
observations so that the resulting Bouguer disturbance is smoother and reflects local
density variations. The exploration geophysicist needs it to explore the density
anomaly with respect to country-rock due to for example low density sediments or
high density minerals. For geodesists, the purpose of removing the gravity effect of
the terrain is to create a band-limited residual gravity field that is much smoother,
smaller and trend-free. This justifies simplifying approximations like using the Fast
Fourier Transform (FFT) algorithm for both downward continuation and the
transformation of gravity to potential.

However, the Bouguer disturbance is neither small nor trend-free and a
representation restricted to longer wavelength components of the topographic
gravitational attraction has to be added back to create gravity residual that is trend-
free and has small amplitudes at both the longest and shortest wavelengths. In
practice, this has been done by converting the geometrical topography into its
equivalent surface density represented by spherical harmonics, the procedure
described by Stokes and now described as Helmert’s second method of condensation
These modern techniques for gravity smoothing will be applied in this study to explore if any improvement in the Earth Gravity Model (EGM08) can be achieved. A new contribution of this work explores an alternative very fast algorithm used for computing the topographic attraction from SRTM 3 arc-seconds data, with the objective of improving the accuracy and resolution of the local gravity, exemplified by the Ethiopian airborne gravity survey.

The task of computing the gravitational attraction of the topography needs a mass model whose shape is as complex as the Earth’s topography. In order to remove the effect of topography from free air gravity or gravity disturbances, the attraction must be calculated at the point where gravity is measured. In this research, a new algorithm was implemented based on representing each prism by a vertical line mass – making the Earth’s topography a structure analogous to the spines of a hedgehog or porcupine, hence the algorithm’s name ‘hedgehog’. In the immediate locality of the measurement point, the fast vertical line mass is replaced by a more stable algorithm, a vertical array of point masses (multipoint).

Section 1.2 present the main objectives of the research and then section 1.3 gives a general outline of the thesis. The next two sections discuss the major data sets used to create a vertical reference system. Section 1.4 deals with the contribution of the Earth Gravity Model to computing a geopotential number and section 1.5 presents the Ethiopian airborne gravity survey and the quality of precise local gravity data used for subsequent computations. Finally in this chapter, section 1.6 highlights the principle of Global Navigation Satellite System measurements required to deliver geocentric coordinates needed as an input to determine geopotential height of a point on or above the Earth’s surface.

1.2 Thesis Aims

The aim of this research is to introduce a new geodetic approach to computing a vertical reference system. Nowadays, Global Navigation Satellite System (GNSS) technology has revolutionized the way absolute geocentric coordinates are determined with high accuracy, so it is worth asking why we need another, different vertical reference system for ‘height’.
“GNSS directly provides the geocentric radius \( r \) or equivalently ellipsoidal height for most positioning applications: \textit{we no longer need conventional height to tell us where we are}” (Hipkin, 2002).

However, some residual sets of tasks, mostly related to hydrodynamics and energetics of transporting mass, and some civil engineering works, still requires a physical height system defined by the Earth’s gravity field. In a strict sense, this sort of height must exactly determine the shape of level surfaces on or above the Earth’s surface; this is why we need them to identify what is ‘uphill’. Traditionally, a physical height system is determined using geodetic levelling by connecting the continental topography to the mean sea level through one or more tide gauge stations. With the greater accuracy now available from gravity and GNSS measurements, this technique becomes obsolete because mean sea level is not an equipotential surface and cannot be used for unification of height systems. Also, in practice, geodetic levelling is time consuming and unpredictably sensitive to the accumulation of errors. Since satellite-based positioning systems became available, a gravimetric geoid height is now subtracted from ellipsoidal height to determine orthometric height of a point above the geoid. However, a surface of constant orthometric height is not strictly a level surface.

The modern approach of determining a global vertical reference system needs the gravitational potential energy to be directly computed on or above the Earth’s surface from gravity observables. The further objective is to compute a geopotential number in a way that it is absolute in a global sense so that it can be used for a world vertical reference system. The geopotential number is a constant over any level surface and it is the only way to characterize a level surface uniquely. Any conversion of the geopotential number to a geometrical height (e.g. orthometric height, normal height) forms a derivative rather than a definitive part of a vertical reference system.

(Hipkin, 2002; Meyer et al., 2006) state that “there is, in fact, no single height system that is both geometric and honours level surfaces simultaneously because these two concepts are physically incompatible due to non-parallelism of the equipotential surfaces of the Earth’s gravity field.”

This research aims to present methods and show that they improve the accuracy and resolution of the EGM08 model in Ethiopia, even though most of airborne gravity data was already incorporated in its development. The data consist
of 92,433 along-track gravity measurements, corresponding to 90,000 km flight distance and the Shuttle Radar Topographic Mission (SRTM) data. The result was tested against 16 precise levelling control points belonging to the Ethiopian Mapping Agency. The research aims to validate the accuracy of the three gravity models: airborne gravity, EGM08 and the gravity effects of SRTM topography. The EGM08 gravity disturbance is expected to recover the main features of the measured gravity disturbance. The difference between hedgehog model for the in situ topography and identical masses condensed on the 1984 World Geodetic System (WGS84) reference ellipsoid provides small and localised residual gravity preserving the shortest wavelength components of the topography.

There are nine objectives for the thesis. These objectives are:

(i) To compute the direct SRTM topographic mass attraction at along-track airborne gravity measurement positions using a fast new vertical line-mass-element “hedgehog” algorithm so as to improve the reliability of interpolating the residual “Bouguer” gravity.

(ii) To evaluate how well the EGM08-derived gravity disturbance predicts the gravity disturbance measured in the aircraft.

(iii) To evaluate how well the spherical harmonic synthesis of the SRTM gravity effect compares with the hedgehog computation.

Then it presents the computational methods that combine the EGM08, airborne gravity and geometrical and condensed gravity models of the SRTM topography so as to improve the accuracy and resolution of EGM08. These objectives are:

(iv) To investigate the stability and computational accuracy of Fast Fourier Transform analytical continuation (Hipkin, 1988; Hipkin et al., 2010).

(v) To prove that any legitimate justification of the Stokes-like global integral to a limited spherical cap depends on the properties of the Earth’s gravity field, not by any kernel properties (Hipkin et al., 2010).

(vi) To experiment whether a planar Fast Fourier Transform algorithm using map projection coordinates can be successfully applied for regional geopotential computation (Hipkin et al., 2010; Hipkin & Hussain, 1982).

(vii) To assess the accuracy of the geopotential number computed from gravity and topographic data and EGM08 spherical harmonic coefficients,
compared with a geopotential number derived from geodetic levelling. This assessment includes changing the width of the band-pass Butterworth filter used to combine the three gravity models.

The final objectives are:

(viii) To establish a new geopotential vertical reference system for Ethiopia.

(ix) To demonstrate that the computational methods used in this research provide an absolute vertical reference system for a particular field point in a global sense without connecting to any local height network: ideal for the unification of world height systems.

In more practical terms, this research presents a new rationale for a real-time computation of the geopotential number during the GNSS surveying. A globally absolute geopotential number of a point on or above the Earth’s surface should be determined by interpolating from the pre-computed digital grids carried in a chip for use with a GNSS receiver in the field supplemented by real time computation in the field on a conventional laptop. This gives a geopotential height at any isolated point where there is a GNSS measurement without reference to a network or regional datum point.

1.3 Thesis Outline

Chapter 2 presents how to synthesise a small, smooth and localised residual gravity disturbance at irregular gravity observation points so that the resulting gravity residual can be reliably interpolated and integrated. Subtracting the EGM08 derived gravity disturbance and the difference between the \textit{in situ} and \textit{condensed} topographic gravitational attraction from the measured gravity disturbances gives small and smoothly varying localised residual gravity disturbances at gravity observation points. Chapter 2 develops an analytical continuation method to predict the EGM08 derived gravity disturbance and a gravity model of the condensed topography at airborne gravity measurement points and also introduces a new algorithm to calculate the gravitational attraction of the \textit{in situ} topography. Together, these eliminate both low and high frequency components from observed gravity. Now, with the availability of the SRTM 3 arc seconds (~90 m depending on latitude) digital elevation data, the development of a very fast algorithm is demanding for topographic computation. The new algorithm is based on representing the
topography by array of vertical line mass elements distributed on a surface of a spherical Earth model but all directed radially outward along the geocentric radius vector. Near the locality of a measurement point, each vertical line mass element is approximated by a vertical array of point masses.

Chapter 3 discusses the analytical downward continuation of the residual gravity disturbances synthesised in chapter 2. Downward continuation is an inverse problem – it seeks to determine gravity on a surface below the measurement points from observed gravity. By transforming the ellipsoidal surface of the Earth onto the planar surface of a map projection, the downward continuation problem can be solved using Fourier Transforms. The residual gravity disturbances can be determined at variable geocentric radii by multiplying the Fourier Transform of a priori residual gravity disturbance given on a planar surface of map projection by the upward continuation operator $\exp(-\kappa z)$. The downward continuation effect is then determined by vertically interpolating gravity at irregular observation points from results pre-computed at different geocentric radii using three-dimensional (3-D) cubic interpolation. This process needs to be done iteratively until the residual gravity reproduced at the measurement points adequately represents the original residual gravity. The iterative FFT algorithm can predict gravity residuals that are directly computed from spherical harmonic coefficients at a constant ellipsoidal terrain height of 2000 m onto the ellipsoidal surface of the Earth with accuracies better than 1 mgal.

Chapter 4 computes a gravitational potential from the downward continued residual gravity disturbances determined in chapter 3 using the Stokes-like integral transform. This chapter shows that the structure of classical kernel functions themselves does not justify localisation of the Stokes’ integral – this must be based on a property of the Earth’s gravity field. It presents a stochastic model of the global gravity field derived from EGM08 and then shows that the Stokes’ integral can only be localised for some wavelength components of the Earth’s gravity field, whatever the properties of the kernel. Short wavelength errors of omission in the global gravity field do not result in by restricting Stokes’ integral to a spherical cap whose radius is less than $2\degree$. This proposition was tested by approximating the Stokes’ integral by planar FFT. In the planar approximation, the Stokes’ integral is simply found by
multiplying the Fourier Transform of residual gravity by the Fourier Transform of the kernel. Applying a planar Fast Fourier Transform to gravity residuals distributed over the ellipsoidal surface of the Earth recovers their potential with millimetric accuracy.

Chapter 5 discusses the problems with the realisation and definition of the present height systems. It also introduces possible scenarios for the realisation of a globally absolute vertical reference system. It will answer the following key questions: “What is meant by physical heights?”, “Why we need them?”, and “how they are defined and realised?”. A vertical reference system is defined by the Earth’s gravity field using the Stokes-like integration or a FFT whose output is the geopotential number computed on or above the Earth’s surface and the GNSS geocentric coordinates are the inputs: the geopotential model, \( C \) becomes the vertical reference system.

Chapter 6 is discussion of the results and conclusion and future works. Figure 1.2 gives a summary of the computational stages involved in constructing geopotential number from surface gravity data.
Figure 1.2. Flow chart showing the scheme for computing geopotential numbers.

\[ \delta g_{\text{obs}}(h) = g(h) - \gamma(h) \]

\[ \delta g(h) = \delta g_{\text{obs}}(h) - \delta g_{\text{EGM08}}(h) \]

\[ \delta g_B(h) = \delta g(h) - \delta g_T(h) \]

\[ \delta g_{\text{residual}}(h) = \delta g_B(h) + \delta g_{Tc}(h) \]

\[ \delta g_{\text{residual}}(h) \Rightarrow \delta g(0) \]

\[ \delta g(0) \Rightarrow T(0) \]

\[ T(0) \Rightarrow T(h_D) \]

\[ C_D = U_0 - \left[ U(h_D) + T(h_D) + T_{\text{EGM08}}(h_D) + \delta g_{Tc}(h_D) - \delta g_{Tc}(h_D) \right] \]

Restore potential of Pizzetti-Somigliana ellipsoid, global model and \textit{in situ} topography and remove potential of condensed topography.

\[ \text{geopotential number at point } D \]
1.4 Earth Gravity Model

Much of the Earth’s gravitational field component comes from the low degree spherical harmonics of the Earth gravity model. The launch of the satellite gravity mission, Gravity Recovery and Climate Experiment (GRACE) in 2002, has improved the accuracy of the global gravity models (EIGEN-GRACE02S (Reigber et al., 2005), GGM02 (Tapley et al., 2005) and EGM08) significantly in comparison with the 1996 Earth Gravity Model (EGM96). The accuracy of the satellite geoid models has been reported to be better than 1 cm to degree and order 70 (Tapley et al., 2005). EGM96 incorporated surface gravity data, over three decades of precise satellite tracking data and altimeter measurements of the ocean surface from the TOPEX/POSEIDON, ERS-1 and GEOSAT missions (Lemoine et al., 1998). EGM96 provides a spatial resolution of about ~ 55 km globally and therefore local gravity field variations at wavelengths shorter than 111 km will not be recovered from its spherical harmonic coefficients. In contrast, EGM08 is represented by ultra-high spherical harmonic coefficients complete to degree and order 2159, with additional coefficients extending up to degree 2190 and order 2159 (Pavlis et al., 2008). It provides an unprecedented level of spatial resolution and accuracy for the recovery of gravity and potential fields over the whole globe. Now, the accuracy of the EGM08 derived geoid model is at the decimetre level: ±15 cm with 5 arcminute (~9 km) resolution at global scale. In general, its accuracy varies geographically depending on the quality and completeness of the terrestrial, airborne and marine gravity data included in the model development.

In this research the long wavelength components of the EGM08 model were used in combination with the airborne gravity and SRTM data. The higher degree spherical harmonic coefficients of EGM08 have to be suppressed by using a “low-pass” Butterworth filter in a spherical domain to synthesise gravity disturbance and its first and second derivatives. The low-pass property of the filter is necessary to eliminate the contribution of the high degree spherical harmonic coefficients from the global model so that medium to shorter wavelength components of the regional gravity field can be replaced by the airborne gravity and SRTM data. The gravity disturbance \( \delta g \) on the ellipsoid \( r = R(\vartheta) \) is computed as
$\delta g(\vartheta, \lambda, R) = \frac{GM}{a^2} \sum_{n=0}^{N} w(n) \left( \frac{a}{R(\vartheta)} \right)^{n+2} \sum_{m=0}^{n} \left( C_n^m \cos m\lambda + S_n^m \sin m\lambda \right) \\
\times \left[ (n+1) \cos \eta \bar{P}_n^m (\cos \vartheta) + \sin \eta \frac{\partial \bar{P}_n^m (\cos \vartheta)}{\partial \vartheta} \right] \tag{1.4}$

where $\eta$ is the angle between the ellipsoidal normal and the geocentric radius, $a$ is the semi-major axis of the reference ellipsoid, $R$ is the geocentric radius of the ellipsoid, $C_n^m, S_n^m$ are fully normalised dimensionless spherical harmonic coefficients, $n \& m$ are degree and order of spherical harmonic coefficients, $\bar{P}_n^m$ is fully normalised Legendre’s function, $\vartheta$ is the geocentric colatitude and $\theta$ is the geodetic colatitude. Each of these can be transformed to one another.

$$\tan \eta = \tan (\vartheta - \theta) = \frac{e^2 \sin \vartheta \cos \theta}{1 - e^2 \cos^2 \theta} \tag{1.5}$$

A spectral Butterworth filter $w(n)$ of degree ten was used to weight the spherical harmonic coefficients. The spherical version of the “low-pass” Butterworth filter is

$$w(n) = \frac{1}{1 + \left( \frac{n}{n_0} \right)^{2k}} \tag{1.6}$$

where $n_0$ is cut-off spherical harmonic degree, $k$ is the order of Butterworth filter and $k = 5$ is used in this study.

This filter changes smoothly but rapidly from unity for $n < n_0$ to almost zero for $n > n_0$.

### 1.5 The Ethiopian Airborne Gravity Survey

Space-based satellite gravity field observations can map the Earth’s gravity field in a homogeneous way but only with a lower resolution due to the altitude of the satellite’s orbit. Now, due to the recent advance of GNSS observations, airborne gravimetry is becoming a standard tool to provide homogenous coverage of gravity data that challenges conventional land gravimeter measurement. Gravity information is obtained from differencing the accelerations observed by the airborne gravimeter (which measures gravity plus aircraft accelerations) and the non-gravitational platform accelerations derived from GNSS. Airborne gravity surveying is the fastest
and most efficient technique for providing medium to shorter wavelength components of the gravity field.

The Ethiopian airborne gravity survey was done from 2006 to 2008. On any one survey line segment the aircraft flew at constant barometric altitude. The flight altitude has been kept as close as possible to the terrain in order to recover short wavelength gravity components caused by local topography. Measurement height varied between 1480 m and 5000 m, because of the contrast in elevation between the Ethiopian highland and low-lying regions like Afar. The survey was made at a speed of about 270 km/hr with 1 second (75 m) data recording, giving point values after filtering with 100 - 150 seconds. The along-track resolution is between 750 to 1,125 m. The track spacing was 18 km.

The overall data acquisition and processing was done by Arne Olesen and Rene Forsberg from the Danish National Space Centre (DNSC) and Addisu Hunegnaw from University of Edinburgh. 92,433 usable along-track point gravity disturbances, corresponding to 90,000 km flight distance, were synthesised. A cross-over analysis indicated a 2.6 mgal noise level on the gravity disturbances, assuming the noise to be uncorrelated from track to track (Olesen & Forsberg, 2007).

The integrated airborne gravimetry data acquisition system consists of two main subsystems, the gravity sensor and the navigation system. The gravity sensor used for the Ethiopian airborne gravity survey is a LaCoste and Romberg air-sea gravimeter (S-99) from University of Bergen, Norway mounted on a two-axis gyro-stabilized platform. The gravimeter was mounted at the centre of gravity of a Cessna Caravan aircraft owned by Abyssinian Flight Service. Three Global Positioning System (GPS) navigation receivers were used in differential mode. One reference GPS receiver was stationed at an airport near to the survey place. The airborne GPS receiver was compared with two ground-based reference receivers. The other reference receiver was stationed at Addis Ababa airport and used continuously throughout the survey. The aircraft trajectory was determined with the software package GPSurvey from Trimble using precise orbit/clocks and connected to an International GNSS Service (IGS) stations near Ethiopia. Many combinations of reference and aircraft receiver were produced for each flight and the best performing solution chosen for the final iteration of the gravity processing.
Airborne gravity surveying is done as a relative measurement. A second LaCoste and Romberg gravity meter is used to construct a network of reference gravity stations at the airports used in the survey. The network datum was fixed by the 1971 International Gravity Standardization Net (IGSN71) reference station at Addis Ababa Geophysical Observatory and other international IGSN71 sites. Then airborne measurements are taken relative to the reference gravity value at the airport station. The reference gravity readings at the airport were taken before and after each flight to minimise the gravity sensor drift effect. Because the gravimeter is assumed to be virtually drift-free during the short time span of a flight (~7 hours) (Olesen, 2003), biases due to the platform off-level can be optimally corrected. Using gravity reference stations at different airports, rather than using the airborne survey to connect to the IGSN71 reference station at Addis Ababa, errors in gravity due to drift accumulations minimises the contribution of long period drift, and hence long wavelength.

The airborne gravimetry observables and GPS-derived velocity and accelerations of the aircraft dynamics are low pass filtered using a Butterworth filter. The gravity disturbances at aircraft level are determined by the following formula.

\[
\delta g(h) = f_z - f_{z0} - h" + \delta g_{eotvos} + \delta g_{tilt} + g_0 - \gamma(h)
\]

where \( f_z \) is the gravimeter observation, \( f_{z0} \) the apron base reading, \( h" \) the GPS vertical acceleration, \( \delta g_{eotvos} \) the Eötvös correction, \( \delta g_{tilt} \) is gravity correction due to platform off-level caused by aerodynamics and \( g_0 \) the apron gravity value which is the reference absolute gravity value fixed at the airport, \( \gamma(h) \) is normal gravity evaluated at aircraft height.

This research used these airborne gravity disturbances (Figure 1.3) for the computation of geopotential number. Figure 1.4 shows the distribution of the airborne gravity measurements.
Figure 1.3. Airborne free air gravity disturbance (mgal) interpolated on to a regular 2-D grid of horizontal coordinates but at irregular flight altitudes.
Figure 1.4. Distribution of measurement points for the Ethiopian airborne gravity survey. Gravity disturbance in mgal.
1.6 **Global Navigation Satellite System measurement**

The Global Navigation Satellite System (GNSS) is a network of ground-station controlled satellites, which emit low power radio signals to provide precise geocentric position coordinates \((x, y, z)\) of a point on or above the Earth’s surface for navigation, surveying, geodesy and other position/location sensitive disciplines. The widely used Global Positioning System (GPS) consists of a constellation of 24 to 28 active satellites in six semi-synchronous circular orbital planes with each orbit containing four evenly spaced satellites at about 20,200 km altitude above the Earth’s surface (Keith, 2002). The geometrical configuration of the satellites is designed in a way that at least 4 satellites can be available anywhere in the world. Each GPS satellite broadcasts radio signals containing synchronised orbital information and timing signals at two frequencies, designated by \(L_1\) and \(L_2\) (Hofmann-Wellenhof & Moritz, 2006, Sect. 5.3). \(L_1\) is the principal GPS carrier signals, at a frequency of 1575.42 MHz, equivalent to a wavelength of about 19 cm. The GPS \(L_2\) signal, transmitted by the satellite at 1227.60 MHz (~24 cm wavelength), was established to provide GPS user receivers with a second frequency for ionospheric delay corrections. The combination of \(L_1\) and \(L_2\) frequencies provides a real-time technique for determining the ionospheric delay effects on the GPS signal paths caused by the free electron content in the ionosphere (Gao & Liu, 2002; Liao, 2000). This provides a unique opportunity to determine a real-time precise positioning as the satellites’ radio signal travels approximately at the speed of light, \(299,792,458 \, m/s\).

The position of a point on or above the Earth’s surface is determined by trilateration – by measuring distance (range) between a GPS receiver located at the desired point and a group of satellites in the space (Witte & Wilson, 2004). The distance from each satellite to the GPS receiver is obtained simply by multiplying the speed of light by the propagation time taken by the radio signal to reach at the receiver. Range measurement does not contain corrections for synchronization errors between clock of the GPS satellite transmitter and receiver. The geocentric position of the GPS receiver is determined by precisely calculating the intersection of the observed ranges to the satellites – in principle, this involves solving three unknown
parameters: the three Cartesian coordinates \((x, y, z)\) or alternatively latitude, longitude and ellipsoidal height. However, the GPS receiver clock error must also be determined so that, in practice, measurements to a minimum of four satellites are required to resolve the four unknowns.

In order to accurately determine the position of a receiver on or above the earth’s surface based on satellite positions, all the satellites and the receiver must adopt the same geodetic coordinate system. The datum used for GPS refers to the World Geodetic System (WGS84) ellipsoid (NIMA, 2000). At present, the GNSS technology can provide sub-centimetric accuracy in a static mode measurement with a longer observation period. The accuracy of the three dimensional coordinate will also increase as we get a redundant measurements from other upcoming satellite constellation such as Galileo and Compass.

GPS data were used to locate the aircraft and determine its vertical acceleration, during the airborne gravity survey, as described in the previous section. They were also used to locate the position along the line of geodetic levelling (see, Sec.5.10).
Chapter 2

Modelling Gravity at Aircraft Height

2.1 Introduction

The computation of a gravitational potential involves a surface integral of gravity disturbances or anomalies on a smooth regular surface. Gravity data must be reliable and representative. In reality, gravity is measured at discrete and variable heights either on or above the Earth’s surface. Attempting to use free air gravity disturbances or free air gravity anomalies is error prone because they may be neither representative nor reliable if topographic masses contribute short wavelength effects. To synthesize a more reliable and representative version of the measured gravity data, smoothing techniques are usually applied, but the complete field must be recoverable from them. This can be achieved by subtracting from the measured gravity a model for the attraction of SRTM topography as well as that of the reference ellipsoid and a global gravity model. The resulting residual “gravity” can then be legitimately interpolated and integrated to find residual potential. The effect of the subtracted gravity model is restored later as their potential. This is the rationale of the “remove-compute-restore” method.

Subtracting the normal gravity of the reference ellipsoid from the measured gravity gives gravity disturbances. They also contain very long wavelengths potentially larger than the size of the local region of surface gravity data. Their contribution to the potential could not be recovered from a local surface integral. In order to make the integration simpler, long to medium wavelength components of the gravity residuals must be removed. In practice, a global gravity model that is well controlled by satellite observations at long wavelengths is subtracted from the measured gravity disturbance to eliminate long wavelengths and trends. This gives gravity residuals that are mainly influenced by high frequency components caused by topographic irregularities and they must be smoothed before performing geodetic calculations.
The contribution of the topographic mass attraction has to be precisely removed from the gravity disturbance to synthesise a more reliable and representative grid values of band-limited residual gravity disturbance. In addition to suppressing the high frequency components from the residual gravity disturbance, the topographic modelling defines the resulting gravity disturbance to be harmonic field in the mass-free region outside the reference ellipsoid so that the Laplace’s equation can be used to simplify the downward continuation and transformation of gravity to potential.

The task of modelling the short wavelength components of the topography requires subtracting a geometrical model for the \textit{in situ} topography and adding back the attraction of identical masses condensed as a single layer on or below the reference ellipsoid or geoid. The gravity and potential models of the condensed topography can be precisely computed at measurement points. However, the calculation of the geometrical gravity and potential models for the SRTM topography is one of the challenging problems in regional gravity field modelling and geopotential computation, both in terms of its computation time and the accuracy of the result it provides. Particularly, the determination of high resolution precise geopotential model needs the effect of the removed topographic mass attraction to be precisely restored as its potential otherwise the “remove-restore” process introduces an error.

Traditionally, the Bouguer plate model was used together with a terrain correction to remove the gravitational attraction of the topographic masses from free-air gravity anomalies prior to geophysical interpretation. However, an infinitely extended plate poorly approximates the shape of the Earth and thus gravitational attraction of the Earth (cf. Qureshi, 1976). Consequently, others have followed Bullard (1936) and devised a rigorous closed-formula for computing Bullard’s Earth’s curvature correction to convert an infinite Bouguer slab to a spherical cap whose thickness is the elevation of the station and whose radius (arc length) from the station is the outer radius of the Hayford-Bowie near zone terrain system, 166.735 km (LaFehr, 1991; Smith et al., 2001; Vaniček et al., 2001). However, the radius of the spherical cap assumes different mass distributions as the calculation routine moves from one station to the next station. Unless the “far zone” corrections, which
include the rest of the world, are included this technique is disqualified for use in
geodesy where the effect of the same topographic mass distribution is needed at
every gravity measurement point; otherwise, the residual is not a harmonic function.

Due to the limiting application of the spherical cap for most geodetic works,
geodesists have adopted better topographic models having a simple geometrical
shape and a closed analytical or numerical formula for the gravity potential and its
first derivative. The rectangular prism handles the complexity of the topographic
shape well, but its evaluation is time-consuming owing to the many logarithmic and
arctan functions involved (Forsberg, 1984; MacMillan, 1930; Nagy, 1966; Nagy et
al., 2000, 2002; Smith, 2000, 2002; Tsoulis, 1999). Others have used the Fast Fourier
Transform (FFT) algorithm for terrain correction (Forsberg, 1985; Kirby &
Featherstone, 2002; Sideris, 1985) to reduce the computation time. But frequency-
domain techniques are usually more prone to errors due to surface roughness of the
in situ topography. For instance, Tsoulis (2001) showed the unreliability of the
planar FFT results in areas with rugged topography.

Others have represented the topography by surface harmonic functions
(Garmier & Barriot, 2001). Wieczorek (2007) has constructed a spherical harmonic
model of the Earth’s shape complete to degree and order 2160 using the method of
Driscoll and Healy (1994) from which gravity and potential can be calculated. Novák
& Grafarend (2005) have computed the effect of topographic isostatically
compensated mass on the satellite-to-satellite tracking data using high degree
spherical harmonics complete to 2700. However, spherical harmonics complete to
ultra-high degree and order of 445,000 would be required to maintain full resolution
of the 3 arc-seconds SRTM data. Computationally, this approach is too slow and too
computationally demanding to be of practical importance. Therefore, only space-
domain techniques can properly handle the local variation in the shape of the
topography, however, they are usually too slow unless topographic models with
simplified geometry and formulae are used.

Much active research has been carried out to develop more efficient
algorithms for the computation of the topographic effects. One example is the
spherical tesseroid method (Heck & Seitz, 2007). Adopting models with simple
gometry and analytical formulae for their gravity and potential is the central issue
for handling the high resolution topographic data with a reasonably fast computational speed. Another problem that degrades the efficiency of the models is their instability in the vicinity of the computation point. In most cases, stable models are usually slower and, when used for computation of the inner zones, they decrease the efficiency of the fast models. Heck & Seitz (2007) model excludes the 4 nearest points and they have used the prism formula in the direct vicinity of the computation point where the tesseroid formula provides unacceptable results. Similarly, Tsoulis (2001) has used the closed formula for an arbitrary polyhedron (Petrović, 1996; Tsoulis, 1999) for the inner zone and FFT for the outer zone.

This chapter discusses four gravity models: normal gravity, global gravity model and geometrical and condensed gravity models of the SRTM topography to smooth and remove trends from gravity data measured over the whole region of Ethiopia. It applies them to the Ethiopian airborne gravity survey. Most emphasis is given to modelling the geometrical gravity and potential models of the 3 arc-seconds SRTM data. It is a space-domain technique for computing the topographic effect. For most of the region it is based on representing each prism of the SRTM topography by a vertical line mass – making the Earth’s topography a structure analogous to the spines of a hedgehog or porcupine, hence the algorithm’s name ‘hedgehog’. In the near zone, it is the representation of the prism by a vertical array of point masses and hence it is called ‘multipoint’. These two models have simple analytical expressions for gravity and potential.

Section 2.2 presents theoretical formulae for the normal gravity and normal potential of the ellipsoidal Earth model. Section 2.3 explores how to use the EGM08 in combination with the local gravity data. This section also discusses two methods of computing the EGM08 derived gravity disturbance at irregular heights and horizontal positions of the airborne gravity observation points. Section 2.4 introduces stable and fast models for the gravity effect of a prism: the vertical line mass element, multipoint and sector for computing the gravity and gravitational potential of the geometrical topography. The convergence of the three models in the vicinity of the source mass and their computational speed has been investigated. The algorithms were also applied to compute the effects of the SRTM topographic masses above the reference ellipsoid within the region defined by latitudes of 6°S to
and longitudes of (24°E to 52°E) at 92,433 airborne gravity observation points. Similarly, section 2.5 evaluates the gravity and potential formulae for the topographic mass distribution represented as a single layer surface density on a sphere to compute the gravity and potential effects of topography. Section 2.6 first compares the EGM08 derived gravity disturbance and then compares the computed effects of in situ topography with that of condensed topography. Section 2.7 describes the benefits of using the global gravity model and SRTM data and emphasises on the accuracy that has to be made when using them in the remove-restore technique.

### 2.2 Normal Gravity

Normal gravity is subtracted from the measured gravity to provide reasonably small magnitudes of the anomalous gravity field. The residual ‘gravity’ is four or five order of magnitude smaller than the observed gravity. This research uses gravity disturbances, rather than gravity anomalies, so can be transformed to its equivalent disturbing potential on a fixed boundary surface. Normal gravity represents the main components of the real Earth’s gravity field that varies due to latitude, flattening and rotation of the Earth. It can be determined from the Pizzetti-Somigliana (P-S) theory of a level ellipsoid. The P-S ellipsoid is the equipotential surface (level surface) produced by a body with the same mass, rotational rate, equatorial radius and flattening as the real Earth, but no topography or lateral subsurface changes in density. The normal potential and its derivatives can be analytically determined on the surface of the P-S type level ellipsoid using four geodetic observable parameters.

\[ U = U_0 \equiv U_0(GM, a, J_2, \omega) \]  

(2.1)

where \( GM \) is the geocentric gravitational constant; \( a \) is the semi-major axis of the reference ellipsoid; \( J_2 \) takes the role of the flattening \( f \). It is the Earth’s gravitational flattening: a measure of the second zonal harmonic of the geopotential; and \( \omega \) is the Earth’s rotation rate.

The Earth’s rotation rate can be determined with sufficient accuracy needed by applications to physical geodesy. Now, advances in the space geodetic satellite observations gives the other three parameters with high accuracy on an epoch basis. The value of the semi-major axis is chosen by convention – in a sense that it
generates a level ellipsoid of revolution best-fitting to the geoid. The current values of these four geodetic parameters defining the WGS 84 reference ellipsoid are:

\[ a = 6378137 \text{ m} ; \quad GM = 3986004.418 \times 10^8 \text{ m}^3 \text{ s}^{-2} ; \quad J_2 = 108263 \times 10^{-8} ; \quad \omega = 7.292115 \times 10^{-11} \text{ rad s}^{-1} \]  \tag{2.2}

The value of the normal potential on the ellipsoid corresponding to these values is

\[ U_0 = 62636851.7146 \text{ m}^2 \text{ s}^{-2} \]  \tag{2.3}

Since the reference ellipsoid is defined to have the same mass and angular velocity as the real Earth, its gravitational part of the potential can be expanded into ellipsoidal harmonics and the normal potential is determined by adding the rotational potential (Heiskanen & Moritz, 1967, p. 67).

\[ U(u, \beta) = \frac{GM}{E} \tan^{-1} \frac{E}{u} + \frac{1}{2} \omega^2 a^2 \frac{q}{q_0} \left( \sin^2 \beta \frac{1}{3} + \frac{1}{2} \omega^2 (u^2 + E^2) \cos^2 \beta \right) \]  \tag{2.4}

where

\[ q = \frac{1}{2} \left[ \left( 1 + 3 \frac{u^2}{E^2} \right) \tan^{-1} \frac{E}{u} - 3 \frac{u}{E} \right] \]  \tag{2.5}

\[ q_0 = \frac{1}{2} \left[ \left( 1 + 3 \frac{b^2}{E^2} \right) \tan^{-1} \frac{E}{b} - 3 \frac{b}{E} \right] \]  \tag{2.6}

and \( u, \beta \) are the confocal ellipsoidal radius and reduced latitude respectively, \( E \) is the linear eccentricity and \( b \) is the semi-minor axis of the reference ellipsoid.

The reference ellipsoid best approximating the shape of the Earth is defined by \( u = b \) so its potential is

\[ U(b, \beta) = U_0 \]  \tag{2.7}

The formula for the normal gravity on the surface of the reference ellipsoid is found by taking the derivative of the normal potential along the ellipsoidal-harmonic coordinate \( u \) (Heiskanen & Moritz, 1967, p. 70).
\[
\gamma(\beta) = \frac{a \gamma_b \sin^2 \beta + b \gamma_a \cos^2 \beta}{\sqrt{a^2 \sin^2 \beta + b^2 \cos^2 \beta}} \tag{2.8}
\]

Where \( \gamma_a, \gamma_b \) are normal gravity at the equator and the pole and they are given by

\[
\gamma_a = \frac{GM}{ab} \left(1 - m - \frac{m' q'_0}{6} \right) \tag{2.9}
\]

\[
\gamma_b = \frac{GM}{a^2} \left(1 + \frac{m' q'_0}{3} \right) \tag{2.10}
\]

and

\[
q'_0 = 3 \left(1 + \frac{u^2}{E^2} \right) \left(1 - \frac{u}{E} \tan^{-1} \frac{E}{u} \right) - 1 \; ; \; m = \frac{\omega^2 a^2 b}{GM} \tag{2.11}
\]

\[
\epsilon' = \frac{E}{b} = \frac{\sqrt{a^2 - b^2}}{b}
\]

is the second eccentricity.

Analytical solutions for the normal potential and its derivatives are exactly known on the surface of the reference ellipsoid by equations (2.4) and (2.8), respectively. Now, with the available GNSS positioning, the task of computing gravity disturbances and geopotential number requires the normal potential and its higher order derivatives to be computed so that \( U \) and \( \gamma \) can be found at measurement positions using a Taylor series expansion.
2.2.1 Normal potential and its derivatives above the ellipsoid

The normal potential and normal gravity can be determined at any point above the reference ellipsoid from the Taylor series expansion of their values and their derivatives on the ellipsoid.

\[
U(R+h) = U(R) + (\mathbf{h} \cdot \nabla) U(R) + \frac{1}{2!} (\mathbf{h} \cdot \nabla)^2 U(R) + \frac{1}{3!} (\mathbf{h} \cdot \nabla)^3 U(R) + O(h^4) \\
= U(R) - h \gamma(R) + \frac{1}{2!} h (\mathbf{h} \cdot \nabla) \gamma(R) - \frac{1}{3!} h (\mathbf{h} \cdot \nabla)^2 \gamma(R) + O(h^4)
\]

(2.12)

\[
\gamma(R+h) = -\nabla U
\]

(2.13)

where \( \mathbf{h} \cdot \nabla = \frac{\partial}{\partial h} \) is gradient operator in the direction of ellipsoidal normal and \( R \) is the radius of the reference ellipsoid.

Heiskanen & Moritz (1967, p. 70) give the closed formula due to Bruns for the first derivative of the normal gravity along the ellipsoidal normal.

\[
\frac{\partial \gamma}{\partial h} = -\gamma \left( \frac{1}{M} + \frac{1}{N} \right) - 2\omega^2
\]

(2.14)

\( M \) and \( N \) are the principal radii of curvature: \( M \) is the radius in the direction of meridian, and \( N \) is the normal radius of curvature taken in the direction of prime vertical.

Equation (2.14) can be described in terms of geodetic latitude, \( \phi \), (Heiskanen & Moritz, 1967, Eq. 2-121)

\[
\frac{\partial \gamma}{\partial h} = \frac{-\gamma}{a(1-e^2)^\frac{1}{2}} \left( 1-e^2 \sin^2 \phi \right)^\frac{1}{2} (2 - e^2 - e^4 \sin^2 \phi) - 2\omega^2
\]

(2.15)

where \( e = \frac{\sqrt{a^2-b^2}}{a} \) is the first eccentricity.

The calculation of the higher order derivatives generally involves approximations but here a closed formula derived by Hipkin & Steinberger (1989) was used. They recasted the Bruns’ formula in ellipsoidal polar coordinates and then partially differentiated with respect to the minor axis.
\[
\frac{\partial^2 \gamma}{\partial h^2} = \frac{1}{\gamma} \left( \frac{\partial \gamma}{\partial h} + 2 \omega^2 \right) \frac{\partial \gamma}{\partial h} - \gamma \frac{\partial}{\partial h} \left[ \frac{1}{\gamma} \left( \frac{\partial \gamma}{\partial h} + 2 \omega^2 \right) \right] \tag{2.16}
\]

The first term is obtained from Bruns’ formula and the second term was derived in an exact expression by the authors as

\[
\gamma \frac{\partial}{\partial h} \left[ \frac{1}{\gamma} \left( \frac{\partial \gamma}{\partial h} + 2 \omega^2 \right) \right] = \frac{\gamma}{a^2 (1 - e^2)^2} \left[ (2 - e^2 + e^4) \right.
\]
\[-(7 - e^2 + e^4) \sin^2 \phi + 8 e^4 \sin^4 \phi
\] + \left. (1 - 2e^2 + 2e^4) \sin^6 \phi \right] \tag{2.17}

Normal gravity at a height \( h \) can now be found from

\[
\gamma(h) = \gamma(0) + \frac{\partial \gamma(0)}{\partial h} h + \frac{\partial^2 \gamma(0)}{\partial h^2} \frac{h^2}{2} + \frac{\partial^3 \gamma(0)}{\partial h^3} \frac{h^3}{6} \tag{2.18}
\]

It is sufficient to use a spherical approximation for the third derivative

\[
\frac{\partial^3 \gamma(0)}{\partial h^3} \approx -\frac{24GM}{a^3} \tag{2.19}
\]

### 2.3 EGM08 Gravity Disturbance

The computation of gravitational potential from local gravity data involves the use of the Earth Gravity Model (EGM) in a remove-compute-restore process to recover the longer wavelength components of the Earth’s gravitational potential. This study used the EGM08 model (Pavlis et al., 2008) synthesized from the combination of the satellite orbital perturbation, land and airborne gravimetry data and sea surface radar altimetry over the oceans. EGM08 is an observational model representing the Earth’s gravity field at high accuracy globally at long to medium wavelengths. It also provides an accurate representation of the regional gravity field at long to short wavelengths in regions where well distributed terrestrial gravimetry data were incorporated in the model. Our study applies to the whole region of Ethiopia where most of the airborne gravity data has been included in the EGM08. In this area, the EGM08 derived gravity disturbance can essentially remove the main features of the airborne acquired gravity disturbance so that resulting residuals are small. The EGM08 gravity disturbance was subtracted from airborne acquired gravity...
disturbance at along track aircraft positions providing residuals mainly containing high frequency components generated by topographic effect.

In practice, the long wavelength components of EGM08 are more accurate than those derived from local data so only medium to shorter wavelengths are replaced by the results of local gravity. A low pass Butterworth filter was used to synthesise the EGM08 gravity disturbance and its derivatives (for degree $n \leq 200$), so that its combination with local gravity data would produce a better resolution of potential model. As there is no direct forward criterion for the selection of the optimal filter length used for the combination of the local data with the EGM08, a test experiment was conducted by comparing the geopotential numbers computed from gravity with the same model computed from precise geodetic levelling data (see, Section 5.10).

Since the spherical harmonic coefficients of the EGM08 are given on the WGS84 reference ellipsoid, the harmonic disturbing potential and its derivatives can be analytically computed on or above the ellipsoid. The determination of disturbing potential and its derivatives involves the computation of Laplace’s solutions on the geodetic boundary surfaces. Because of the distortion caused by rotation, the actual shape of the Earth is better approximated by a rotationally symmetric ellipsoid than a sphere. Its gravity disturbance could then be expanded in a series of ellipsoidal harmonics. However, they are somewhat complex mathematically so that, for simplicity, it is common to expand the gravity disturbance in a series of spherical harmonics but synthesise gravity on the ellipsoid. The formula for the gravity disturbance and its derivatives was derived on the WGS84 ellipsoid using equation (1.4).

This study used both Taylor series expansion and analytical continuation methods to compute the EGM08 gravity disturbance at airborne gravity measurement points and also compared their results. The Taylor series expansion method predicts disturbing potential or gravity disturbance at aircraft height using the disturbing potential and its derivatives determined on the ellipsoid. In practice, this approach only needs a 2-D interpolator to predict the values determined on the ellipsoid at the horizontal coordinates of the measurement positions before applying Taylor expansion.
\[ T(R+h) = T(R) + (h \bullet \nabla)T(R) + \frac{1}{2!} (h \bullet \nabla)^2 T(R) \]
\[ + \frac{1}{3!} (h \bullet \nabla)^3 T(R) + O(h^3) \] (2.20)

\[ \delta_g(R+h) = \delta_g(R) + (h \bullet \nabla)\delta_g(R) + \frac{1}{2!} (h \bullet \nabla)^2 \delta_g(R) + O(h^3) \] (2.21)

Note that this approximation does not account for the curvature of the normal plumbline.

In practice, the synthesis of a high degree spherical harmonic series is only efficient if evaluated on a regular grid of latitude and longitude so that, before applying the Taylor expansion, this approach needs a 2-D interpolator to predict the values at the irregular horizontal coordinates of the point on the ellipsoid below the aircraft measurement. Alternatively, spherical harmonic synthesis can provide a regular grid of disturbing potential or its derivatives on a fixed ellipsoidal surfaces defined by geocentric radii \( r_k \), but not at irregular heights. That means, the geopotential coefficients given on WGS84 can be upward continued at multiple ellipsoids defined by \( r = R + k \Delta h \) using the analytical continuation terms \( \left( \frac{a}{r} \right)^{n+1} \) and \( (n+1) \left( \frac{a}{r} \right)^{n+2} \) for potential and gravity, respectively. The formulae for disturbing potential and gravity disturbance are given by

\[ T(\vartheta, \lambda, r) = \frac{GM}{a} \sum_{n=0}^{N} w(n) \left( \frac{a}{r} \right)^{n+1} \sum_{m=0}^{n} \left( C_n^m \cos m\lambda + S_n^m \sin m\lambda \right) P_n^m (\cos \vartheta) \] (2.22)

\[ \delta_g(\vartheta, \lambda, r) = \frac{GM}{a^2} \sum_{n=0}^{N} w(n) \left( \frac{a}{r} \right)^{n+2} \sum_{m=0}^{n} \left( C_n^m \cos m\lambda + S_n^m \sin m\lambda \right) \]
\[ \times \left[ (n+1) \cos \eta \overline{P}_n^m (\cos \vartheta) + \sin \eta \frac{\partial \overline{P}_n^m (\cos \vartheta)}{\partial \vartheta} \right] \] (2.23)

where \( T \) is the disturbing potential.

Practical calculation of gravity disturbance or disturbing potential at irregular observation points first requires the computation of regular grids of gravity disturbances or disturbing potential on multiple layers vertically whose radii...
encompass all ellipsoidal heights of the measured gravity data. They are then interpolated vertically to the irregular heights of the aircraft measurements. Therefore, this analytical continuation method requires the use of a precise 3-D interpolation algorithm. In this study, regular grids of gravity disturbances were calculated at (0, 500, ..., 6000) m ellipsoidal heights and a 3-D cubic interpolator was used to predict EGM08 disturbance at aircraft height. The next section discusses the 3-D cubic interpolation software used.

### 2.3.1 3-D Cubic Interpolation

The 3-D interpolation software is the key element for implementing the analytical continuation method to predict the gravity disturbance or disturbing potential at irregular measurement points using geopotential coefficients given on the reference ellipsoid. One such tool is the numerical methods used for precise prediction of the gravity value at the desired positions from regular grid of gravity data. Numerical methods are used to accurately determine the values of the function at the required interior positions within the range of the data distribution. Numerical methods compute the numerical derivatives up to the desired order in order to estimate the unknown function from the surrounding data points.

Numerical differentiation formulae are usually classified as forward, backward and central difference formulae, based on the position of the point to be predicted with respect to the pattern of the samples used in the calculation. Forward difference uses the samples at a mesh point and the next forward equally spaced points to calculate the derivative at the mesh point. Backward difference uses the samples at a mesh point and the previous equally sampled points whereas the central difference uses both forward and backward samples to calculate the derivative at the specified mesh point. The accuracy of numerical interpolation depends on the number of equally sampled data points used in the numerical differences. The order of numerical differentiation is one less than the number of data used. A numerical expression for a finite difference of order \( N \) is generally obtained by solving \( N \) equations directly from the Taylor series expansion or by the method of indeterminate coefficients (Khan & Ohba, 2000). Alternatively, equivalent numerical solutions have been devised based on interpolating polynomials such as Aitken, Bessel, Everett, Gauss, Lagrangian, Newton-Gregory, and Stirling.
Here, numerical differentiation methods that are a class of Newton-Gregory’s forward and central differences are used to compute a continuous piecewise cubic polynomial approximation to values of the gravity data on a regular discrete mesh. In principle, these numerical interpolation methods determine a unique polynomial function at the desired central point that best fits the known values of the discrete function to acceptable accuracy.

The gravity vector \( g(x, y, z) \) is a continuously differentiable function with respect to \( x, y \) and \( z \) real variables in the mass-free space

\[
g(x, y, z) = \{ g_x(x, y, z), g_y(x, y, z), g_z(x, y, z) \}
\]

(2.24)

where \( g_x, g_y, g_z \) are the orthogonal gravity components.

Since in reality measurements are made at discrete points or computations are made in a grid form, the prediction of gravity at \((x, y, z)\) in the space from the measured gravity vector/computed gravity disturbance needs numerical interpolation to be carried out in the three orthogonal directions. A fast 3-D interpolation method can be established by first performing horizontal interpolation at variable geocentric radii and then vertically interpolating at the desired positions.

Newton’s cubic polynomial interpolation method was used to precisely predict gravity disturbance at the desired point in the space from a regular 3-D discrete mesh of gravity disturbance pre-computed by analytical continuation formula showed in equation (2.23). The strategy is first to accurately predict gravity disturbance on the \(x-y\) plane at any central point from a regular 2-D discrete mesh of gravity disturbance pre-synthesised at different geocentric radii and then vertically interpolate to estimate its value at any \((x, y, z)\) interior positions. We have used 4x4 equally spaced 2-D discrete values of the residual gravity disturbance to compute an accurate prediction of its values at any points within the predefined rectangular mesh using Newton-Gregory’s central difference polynomial approximation. The determination of the numerical solutions on a two-dimensional \(x-y\) plane is a two dimensional boundary value problem. The prediction of gravity only depends on the discrete values of the known gravity disturbances which are assumed to vary as functions of two independent variables: \( x \) and \( y \). The gravity is predicted from the higher order terms of numerical derivatives defined over the region defined by \( n'\Delta x \)
length in the $\hat{x}$ direction and $m'\Delta y$ in the $\hat{y}$ direction. Within the area, a grid cell is defined by

$$x_0 + n'\Delta x \leq x \leq x_0 + (n'+1)\Delta x$$  \hspace{1cm} (2.25)

$$y_0 + m'\Delta y \leq y \leq y_0 + (m'+1)\Delta y$$  \hspace{1cm} (2.26)

where $\Delta x, \Delta y$ denotes the discretization grid sizes of the interpolation region while $n'$ and $m'$ denotes the number of discrete data points in $x$- and $y$- directions

Newton’s central difference polynomial approximation is first carried out either in the easting ($x$) or northing ($y$) direction to precisely predict gravity at an absolute 1-D coordinate of the central computation point in the chosen direction from the high order numerical derivatives generated by the known input data. Then, the gravity effects of these pre-computed gravity values geo-located at different locations with respect to the other direction are numerically computed to add additional adjustment corrections for prediction to be made precisely at the $(x,y)$ coordinates of the desired point for a fixed value of $z$. The final result of the prediction is the solution to the polynomial function that best fits all the known data points. For instance, the gravity data on the faces between cells $(i, j)$ and $(i+n', j)$ can be expanded into a Taylor series about the centre of the face by performing numerical differentiations in the $x$-direction along a constant $y$-band. In a simple expression, a two point forward difference can be computed in the $x$-direction approximating the first gradient of the gravity at $x$.

$$\frac{\partial}{\partial x} \delta g(x, y, z_0) = \frac{\delta g(x + \Delta x, y, z_0) - \delta g(x, y, z_0)}{\Delta x}$$  \hspace{1cm} (2.27)

where $\Delta x$ is the grid spacing between two consecutive points

Doing so, the higher order numerical differentiations are used to uniquely determine the $n^{th}$ degree polynomial approximation that passes through the $n'$ known data points to accurately predict gravity disturbance at the unknown central point $x$. In the three dimensional mesh representations, the gravity disturbance is assumed to be cell registered. The gravity disturbance of each cell is identified by the indices of the cell in terms of $i, j, k$ allowing it to be uniquely read during numerical differentiation. The numerical differentiation approximates the values of the derivatives of the
function at the centre of the cell faces. The 2-D interpolation on the x-y plane at different geocentric radii is done by the Newton’s central cubic polynomial approximation.

The central difference method uses both a forward and backward differentiation scheme. It gives more accurate prediction at any central point. The typical example below shows numerical prediction in the x direction computed from the gradient effects of the gravity data geo-referenced at cells between \((i, j)\) and \((i+3, j)\). The derivatives of gravity are computed at the centre of the faces between the nodes. The formula for the central cubic interpolation can be simply derived based on the Taylor series expansion.

\[
g_x(x, y_j, z_k) = g_x(x, y_j, z_k) + p(g_x(x_{i+1}, y_j, z_k) - g_x(x, y_j, z_k)) \\
+ \frac{1}{2!} p(p-1)(g_x(x_{i+1}, y_j, z_k) - 2g_x(x, y_j, z_k) + g_x(x_{i-1}, y_j, z_k)) \\
+ \frac{1}{3!} p(p+1)(p-1)(g_x(x_{i+2}, y_j, z_k) - 3g_x(x, y_j, z_k) + g_x(x_{i-2}, y_j, z_k)) \\
+ 3g_x(x, y_j, z_k) - g_x(x_{i-1}, y_j, z_k)
\]  (2.28)

where \( p = \frac{x - x_i}{\Delta x} \) is the proportion distance to the point of computation

This prediction yields four estimate values of gravity at an absolute x coordinate of the computation point, but positioned at different coordinates \((x, y_j, z_k)\) relative to the y direction for a particular value of z. The next stage repeats the prediction process in the y direction using the pre-estimated values, \( g_x(x, y_j, z_k) \) to determine the gravity value accurately at the desired \((x,y,z_k)\) coordinates.

\[
g_y(x, y, z_k) = g_y(x, y, z_k) + p(g_y(x_{y_{j+1}}, z_k) - g_y(x, y, z_k)) \\
+ \frac{1}{2!} p(p-1)(g_y(x_{y_{j+1}}, z_k) - 2g_y(x, y, z_k) + g_y(x_{y_{j-1}}, z_k)) \\
+ \frac{1}{3!} p(p+1)(p-1)(g_y(x_{y_{j+2}}, z_k) - 3g_y(x, y, z_k) + g_y(x_{y_{j-2}}, z_k)) \\
+ 3g_y(x, y, z_k) - g_y(x_{y_{j-1}}, z_k)
\]  (2.29)

Equation (2.29) is used to predict gravity at the easting and northing coordinates of the central point from regularly spaced discrete gravity data. Now, gravity data are estimated at \((x,y,z_k)\) coordinates of the measurement positions \((x,y,z)\) at regularly
spaced different ellipsoidal heights. Thus, the computation of gravity at any central coordinate \((x,y,z)\) involves vertical interpolation from gravity values known at different radii.

Vertical prediction involves both forward and central cubic interpolation methods depending on the position of the computation point. The forward formula provides an accurate result if the point of interpolation is close to the top or in the middle of the data points used. Here, it has been used for height ranges between zero to 500 m while central difference was used for the other height ranges. In conclusion, the complete formula for the 3-D gravity prediction can be summarized by showing an independent solution for both forward and central difference.

The Newton’s forward difference formula describing the vertical component of the Earth’s gravity field at \(z (z_k < z < z_{k+1})\) due to the variation of gravity as the function of space is given by

\[
g_{xyz}(x,y,z) = g_{xyz}(x,y,z_k) + p(g_{xyz}(x,y,z_{k+1}) - g_{xyz}(x,y,z_k))
\]

\[
+ \frac{1}{2!} p(p-1)(g_{xyz}(x,y,z_{k+2}) - 2g_{xyz}(x,y,z_{k+1}) + g_{xyz}(x,y,z_k))
\]

\[
+ \frac{1}{3!} p(p-1)(p-2)(g_{xyz}(x,y,z_{k+3}) - 3g_{xyz}(x,y,z_{k+2}) + 3g_{xyz}(x,y,z_{k+1}) - g_{xyz}(x,y,z_{k}))
\]

(2.30)

and the central formula is

\[
g_{xyz}(x,y,z) = g_{xyz}(x,y,z_k) + p(g_{xyz}(x,y,z_{k+1}) - g_{xyz}(x,y,z_k))
\]

\[
+ \frac{1}{2!} p(p-1)(g_{xyz}(x,y,z_{k+2}) - 2g_{xyz}(x,y,z_{k+1}) + g_{xyz}(x,y,z_{k}))
\]

\[
+ \frac{1}{3!} p(p+1)(p-1)(g_{xyz}(x,y,z_{k+3}) - 3g_{xyz}(x,y,z_{k+2}) + 3g_{xyz}(x,y,z_{k+1}) - g_{xyz}(x,y,z_{k-1}))
\]

(2.31)

To speed up the computation of the interpolation process, we have used the direct access binary file format.

### 2.3.2 Accuracy of the cubic interpolator

The numerical accuracy of the central cubic interpolation method has been verified by evaluating its relative accuracy with respect to the Generic Mapping Tools (GMT) “grdtrack” software (Wessel & Smith, 1998) in 2-D space. First, the GMT “surface”
interpolation software was used to convert the residual gravity disturbances known at irregular horizontal positions into a regular grid, in a context of horizontal coordinates. Then, the numerical accuracy of the cubic and “grdtrack” softwares were assessed by predicting a new estimates at measurement positions from a regular grid pre-synthesised by “surface” software. The cubic interpolation is applied to the Edinburgh University binary format version of the GMT grid data while “grdtrack” directly uses the GMT default grid format. The results of the prediction made by the two interpolation softwares are in a very good numerical agreement with the measured residual gravity data (Table 2.1). Conceptually, the misfit between the measured data and predicted data is significantly affected by the cumulative gridding and interpolation errors. Therefore, the difference between the measured and predicted data does not explicitly show the error in the cubic and “grdtrack” interpolation methods, rather it does show the accuracy level of the overall numerical predictions. But the accuracy of these interpolation methods can be showed by comparing the relative accuracy between the two interpolation methods. The two interpolation methods are in relative agreement of 0.0002 mgal (mean) and 0.052 mgal (stdev) in the point wise variation of the residual gravity disturbances. These results suggest that our cubic interpolation method can be applied for high accuracy numerical prediction purposes. Note that, a noticeably larger difference between the actual value and the predicted value as given in Table 2.1 shows that much of the interpolation error is introduced by “surface” software.

<table>
<thead>
<tr>
<th>Data points: 92433</th>
<th>Mean</th>
<th>Stdev</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed – cubic</td>
<td>0.0541</td>
<td>0.799</td>
<td>-18.026</td>
<td>19.742</td>
</tr>
<tr>
<td>Observed – grdtrack</td>
<td>0.0539</td>
<td>0.788</td>
<td>-18.425</td>
<td>18.985</td>
</tr>
<tr>
<td>Cubic - grdtrack</td>
<td>0.0002</td>
<td>0.052</td>
<td>-1.078</td>
<td>0.930</td>
</tr>
</tbody>
</table>

Table 2.1. Comparison between the residual gravity values predicted at observation points from a regular grid data using cubic and GMT grdtrack interpolation software. Units in mgal.
2.3.3 Taylor Series compared with analytical continuation

For the purpose of numerical comparison, the EGM08 gravity disturbance was calculated at airborne gravity measurement points using both a Taylor series of second order (see, Eq. 2.21) and analytical continuation methods. Table 2.2 shows that analytical continuation provides residual gravity with small standard deviation when subtracting the predicted EGM08 gravity disturbance from the measured gravity disturbance compared to the residuals derived from the Taylor method, but with a larger mean value. Therefore, the Taylor series is needed to be expanded to higher order to capture very high frequency components so that the resulting residuals will be much smoother. In this study we have used the analytical continuation method in the remove-restore process to smooth the external gravity field with the global gravity model and condensed topography.

<table>
<thead>
<tr>
<th>Observed – EGM08</th>
<th>Mean</th>
<th>Stdev</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taylor series</td>
<td>0.19</td>
<td>8.58</td>
<td>-104.3</td>
<td>80.7</td>
</tr>
<tr>
<td>Analytical continuation</td>
<td>0.26</td>
<td>4.45</td>
<td>-33.1</td>
<td>37.9</td>
</tr>
</tbody>
</table>

Table 2.2. Comparisons of the band-limited residual gravity disturbance obtained by subtracting the EGM08 derived gravity disturbance at aircraft height using Taylor series expansion and analytical continuation (Units in mgal).

2.4 Geometrical Gravity Models of *in situ* SRTM

The role of topographic computation for regional gravity field modelling or gravitational potential computation is based on the fact that much of the short wavelength components of the measured gravity field are due to the attraction of the topography. Both geometrical and spherical harmonic representations of the topographic gravity and potential are needed. This section deals with geometrical models of *in situ* topography. Their gravity effect needs to be removed to suppress the shorter wavelength components of the measured gravity field in order, first, to improve the reliability of interpolation and integration, and, secondly, to create mass-free space for Laplace’s equation to be used for solving geodetic boundary value problems. These two models are based on the Shuttle Radar Topography Mission (SRTM) in the form of a 3” nearly global grid. Since these models are first removed and then later restored, the final product is not degraded by any inaccuracies in them.
Note that, the SRTM data comes from measurements of ellipsoidal height but published after subtracting the EGM96 geoid in order to simulate orthometric heights. Here the EGM96 geoid is restored and we use ellipsoidal heights.

This section deals with modelling the gravity and potential effects of \textit{in situ} topography. The new method was applied to 92,433 points of airborne gravity data acquired over the whole region of Ethiopia. Computing the topographic gravity at a single observation point requires the calculations of 1.5 billion SRTM vertical prisms representing Ethiopia and its immediate surroundings. The inclusion of the surrounding region is computationally needed to avoid the edge effect. Removing the attraction of the topography from the measured gravity disturbance creates a complete Bouguer disturbance. The Bouguer disturbance has the required smoothness but is neither small nor trend-free. To overcome these deficiencies, a smoother representation of the topography restricted to longer wavelength components has to be put back to create a residual that is trend-free but still has a very small amplitude at the shortest wavelengths. This second topographic model involves a spherical harmonic representation of the same SRTM topography condensed to a surface density on the ellipsoid. This will be discussed in more detailed in section 2.5.

\textbf{2.4.1 The hedgehog model}

Analytical solutions for the gravity and gravitational potential of a vertical line-mass element (see, Figure 2.1) representing a homogenous mass-density distribution can be derived using Newton’s integral formula.

\[
\delta g_T = G \iiint \frac{dm}{r^2} = G \iiint \frac{\rho}{r^2} \, dV
\]  

(2.32)

and

\[
\delta \nu_T = G \iiint \frac{dm}{\ell} = G \iiint \frac{\rho}{\ell} \, dV
\]  

(2.33)

Where \( \delta g_T \) and \( \delta \nu_T \) denote gravity and potential models of the topography, respectively, \( dm \) and \( V \) are the mass and the volume of the vertical line mass
element, $\ell$ is the distance between the source mass and the computation point, $\rho$ is the rock density.

**Figure 2.1.** Geometry of a mass line element and its gravity attraction relative to radius vector $R_f$.

For a simplified case of line-mass element distribution, where all elements are directed radially outward along the radius vector, the vertical line-mass element model has solutions in spherical coordinates $(\theta, \lambda, h)$. In this model the ellipsoidal height of the topography is $h$, along the direction of the ellipsoidal normal, that is, the direction of normal gravity. Thus, the magnitude and orientation of the topographic element adopted by the ‘hedgehog’ model is consistent with the ellipsoidal height computed from GPS, e.g. for the airborne gravity measurement. Hence, the coordinates used for gravity and potential solutions of the vertical line mass element and for measured gravity are consistent.

The Newton integral expressions for gravity and potential of the vertical line mass element as defined in equations (2.32) and (2.33) can be grossly simplified to 1-D line integrals. To make the problem tractable, the computation uses spherical rather than ellipsoidal geometry, but sets the spherical radius $R$ equal to the mean
radius of the curvature of the ellipsoid at a central point with latitude \( \varphi_0 \) in the region of computation.

\[
R(\varphi_0) = a \sqrt{\frac{1 - (2 - e^2)(\sin \varphi_0)^2}{1 - (\sin \varphi_0)^2}}
\]

(2.34)

\[
\delta g_T = 2G \iint \frac{-\xi dm}{z \sqrt{1 + \xi^2}} = 2G \int \frac{-\xi dz}{z \sqrt{1 + \xi^2}}
\]

(2.35)

\[
\delta v_T = 2G \iint \frac{-\xi dm}{z \sqrt{1 + \xi^2}} = 2G \int \frac{-\xi dz}{z \sqrt{1 + \xi^2}}
\]

(2.36)

where

\[
\zeta = \rho R^2 d\varphi d\lambda \cos \varphi
\]

is mass per unit length, and it is equal to topographic density times the grid cell area, \( R^2 d\varphi d\lambda \cos \varphi \) ignoring the change in area element with height. A constant mean rock density of \( \rho = 2700 \text{ kg m}^{-3} \) has been used for topographic computation. \( d\varphi \) and \( d\lambda \) are the angular size of the SRTM grid cell in the north and east directions over which the line integral is performed.

Analytical formulae for the vertical gravity attraction at the required points can be easily derived from equation (2.35) using two step vector decompositions. First, the Newtonian gravity attraction of the line-mass element is decomposed into its local-horizontal and local-vertical components (see, Figure 2.2) using cylindrical polar co-ordinates. Then, the two components will be separately projected onto the direction of ellipsoidal normal to give the net vertical attraction at the computation point. Analytical solution of the vertical line mass element was derived based on the spherical approximation.
**Figure 2.2.** Vertical and horizontal gravity components of the vertical line-mass element relative to the direction of normal gravity at the mass element

- $X_0$ is perpendicular distance from the computation point to the line element. $z$ is the distance along the line element, measured from the point where it intersects $X_0$.
- Similarly, $z_1$ and $z_2$ represents an algebraic relative positions of the mean radius $R$ and the topographic line mass element height, respectively. The angle $\hat{\Lambda}$ is used to calculate the vertical and horizontal components of topographic gravity attraction $\delta g_I^z$ and $\delta g_I^h$ relative to the direction of normal gravity at the body point, respectively. They are expressed by

$$\delta g_I^z = G \int_{z_1}^{z_2} \frac{\ell \xi \, dz}{|\ell|^3} \cos \hat{\Lambda} = G \int_{z_1}^{z_2} \frac{\ell \xi \cos \hat{\Lambda}}{|\ell|^3} \, dz$$

$$\delta g_I^h = G \int_{z_1}^{z_2} \frac{\ell \xi \, dz}{|\ell|^3} \sin \hat{\Lambda} = G \int_{z_1}^{z_2} \frac{\ell \xi \sin \hat{\Lambda}}{|\ell|^3} \, dz$$

where

$$z = X_0 \tan \hat{\Lambda}, \quad dz = X_0 \sec^2 \hat{\Lambda} \, d\hat{\Lambda}, \quad \ell = X_0 \sec \hat{\Lambda}$$

(2.39)
Substituting equation (2.39) into equations (2.37) and (2.38), the local horizontal and local vertical components of the vertical line mass element gravity attraction become

\[
\mathbf{\delta g}_T^h = G \int_{z_1}^{z_2} \xi X_0 \sec^2 \hat{A} \cos \hat{A} \ d\hat{A} = \frac{G \xi}{X_0} \left( \sin \hat{A}_2 - \sin \hat{A}_1 \right)
\]  

(2.40)

and

\[
-\mathbf{\delta g}_T^v = G \int_{z_1}^{z_2} \xi X_0 \sec^2 \hat{A} \sin \hat{A} \ d\hat{A} = -\frac{G \xi}{X_0} \left( \cos \hat{A}_2 - \cos \hat{A}_1 \right)
\]  

(2.41)

where

\[
X_0 = (R + h_f) \sin \psi, \quad \cos \hat{A} = \frac{X_0}{\sqrt{z^2 + X_0^2}}, \quad \sin \hat{A} = \frac{z}{\sqrt{z^2 + X_0^2}}
\]  

(2.42)

The gravity components \(\mathbf{\delta g}_T^v\) and \(\mathbf{\delta g}_T^h\) are computed at the field point in the direction of the normal gravity and in the direction perpendicular to the normal gravity at the body point respectively. Their contribution in the direction of the ellipsoidal normal at the field point should be known for the proper reduction of the topographic effect. Combining equations (2.40) and (2.41) using vector projection leads to an analytical formula for the Newtonian vertical gravity attraction, \(\mathbf{\delta g}_T^v\), of the vertical line mass element at every desired point on or above the Earth’s surface.

\[
\mathbf{\delta g}_T^v = \frac{G \xi}{R + h_f} \left\{ \frac{R + h_b}{\ell(h_f, h_b)} - \frac{R}{\ell(h_f, 0)} \right\}
\]  

(2.43)

where

\[
\ell(h_f, h_b) = \sqrt{(R + h_f)^2 + (R + h_b)^2 - 2(R + h_f)(R + h_b) \cos \psi}
\]  

(2.44)
denotes the spatial distance between the computation point and the line-mass element and its corresponding spherical distance between the field point and the body point is given by

$$\cos \psi = \sin \phi_b \sin \phi_f + \cos \phi_b \cos \phi_f \cos (\lambda_f - \lambda_b)$$  \hspace{1cm} (2.45)

Here, $\phi_f, \lambda_f$ are geodetic latitude and longitude of a field point, $\phi_b, \lambda_b$ are latitude and longitude of a running point.

Similarly, the potential of the vertical line-mass element can be derived from equation (2.36)

$$\delta v_T = G \int_{Z_i}^{Z_2} \frac{\xi dZ}{\sqrt{X_0^2 + Z^2}} = G \xi \ln \left[ \frac{Z_2 + \sqrt{X_0^2 + Z_2^2}}{Z_1 + \sqrt{X_0^2 + Z_1^2}} \right]$$  \hspace{1cm} (2.46)

An analytical formula for the potential can be obtained by substituting for terms in the logarithmic function using equation (A2) derived in appendix A.

$$\delta v_T = G \xi \ln \left[ \frac{R + h_b - (R + h_f) \cos \psi + \ell(h_f, h_b)}{R - (R + h_f) \cos \psi + \ell(h_f, 0)} \right]$$  \hspace{1cm} (2.47)

Equations (2.43) and (2.47) are analytical semi-ellipsoidal solutions. The computation of the gravity attraction of every running point along constant latitude requires two square root functions $\ell(h_f, h_b)$ and $\ell(h_f, 0)$.

### 2.4.2 Multipoint expansion

The multipoint model has the same geometry and mass distribution as the vertical line-mass element. In this model the topographic vertical line-mass element is subdivided into a number of smaller cuboids. The mass of each discrete cell is assumed to be concentrated at its centre of mass defining a number of point masses in a vertical direction and hence the term multipoint is used. The best approximation of the attraction of a cuboid to that of a point mass comes when it is a cube. Thus the vertical distance $\Delta z_i$ (Figure 2.3) is chosen to be equal to the square root of the cross-sectional area. The division starts at the top, so that the residual cube with
unequal dimensions comes at the base, where the point mass approximation is less good, furthest from the computational point.

**Figure 2.3.** Approximating a vertical line mass element by a vertical array of point masses

The effect of the vertical prism on gravity and potential is approximated by the cumulative effect of all point masses synthesized from it. This technique is similar to the point mass approximation of the prism’s shape adopted by Tsoulis (2001). The gravity and potential of each of these discretized point masses can be calculated using the basic geometrical relations shown in Figure 2.1 but the vertical line mass element is now assumed to be replaced by a number of discrete point masses aligned in the vertical direction. The vertical position of each point mass can be analytically calculated along the ellipsoidal normal relative to the Earth’s centre of mass. The vertical position of this point mass together with its horizontal geodetic coordinates extracted from the mean values of the SRTM grid cell defines the coordinates of its centre of mass to be used for analytical calculations. For example, analytical solutions for gravity and potential of a single point mass positioned at a radial distance, $z_i$, relative to the Earth’s centre of mass can be derived as
\[ \delta g_i^T = \frac{G \rho \Delta V_i}{\ell^3(h_f, z_i)} \cos \left(90^\circ - \psi + \hat{\lambda}_i\right) \quad (2.48) \]

where \( z_i \) is used in terms of \( R + h_b \) as adopted by the vertical line mass element for defining the position of the body point relative to the Earth’s centre of mass.

The term \( G \rho \Delta V_i \ell^{-2} \) in equation (2.48) shows the gravity effect of a point mass at any point in space. The whole expression denotes its magnitude in the direction of normal gravity, \( \gamma \) at the field point. The cosine term represents the magnitude of the *dot product* between the unit vectors oriented from the computation point to a point mass and the unit vector in the direction of normal gravity at the field point.

\[
\cos(90^\circ - \psi + \hat{\lambda}_i) = \sin \psi \cos \hat{\lambda}_i - \cos \psi \sin \hat{\lambda}_i
\]

\[
= \sin \psi \frac{X_0}{\ell(h_f, z_i)} - \cos \psi \frac{z_i - (R + h_f) \cos \psi}{\ell(h_f, z_i)}
\]

\[
= \sin^2 \psi \frac{R + h_f}{\ell(h_f, z_i)} - \cos \psi \frac{z_i - (R + h_f) \cos \psi}{\ell(h_f, z_i)}
\]

The volume of a point mass is equivalent to the volume of a small cube whose mass is assumed to be concentrated at its centre of mass

\[ \Delta V_i = R^2 \partial \varphi \partial \lambda \Delta z_i \quad (2.50) \]

where \( \Delta z_i \) is discretization size, and \( \partial \varphi \) and \( \partial \lambda \) are the northing and easting size of the SRTM grid cell.

An analytical solution for the vertical gravity attraction of any point mass can be obtained by substituting equation (2.49) into equation (2.48).

\[ \delta g_i^T = \frac{G \rho \Delta V_i}{\ell^3(h_f, z_i)} \left[(R + h_f) \sin^2 \psi - \cos \psi (z_i - (R + h_f) \cos \psi)\right] \quad (2.51) \]

In a similar way the potential of the point mass is given by

\[ \delta \psi_i^T = \frac{G \rho \Delta V_i}{\ell(h_f, z_i)} \quad (2.52) \]

Equations (2.51) and (2.52) are analytical solutions for gravity and potential of the \( i^{th} \) point mass. Thus, the total topographic effect of a single SRTM vertical line mass element is equal to the sum over the contribution of all the point masses formed
by the discretization process. The *geocentric* vertical position of each point mass is given by

\[ z_i = R + h_b - \left( s - \frac{1}{2} \right) \Delta z_i ; \quad s = 1, \ldots, c \]  

(2.53)

where *c* is the number of point masses (cubes) formed from the discretization process and it is given by:

\[ c = INT \left( \frac{h_{SRTM}}{\Delta z_i} \right) \]  

(2.54)

where *INT* denotes the ‘integer value’

The effect of the residual mass left-over from the discretization process is also taken into account by changing the values of the volume quantity and \( z_i \) in the gravity and potential solutions. For clarity, we have used a different symbolic representation for its volume and position coordinate.

\[ \Delta V_i \rightarrow \Delta V_{residual} \]
\[ z_i \rightarrow z_{residual} \]  

(2.55)

\[ \Delta V_{residual} = R^2 \frac{\partial \phi \partial \lambda}{\partial h_{residual}} \]
\[ z_{residual} = R + \frac{1}{2} \partial h_{residual} \]  

(2.56)

where

\[ \partial h_{residual} = h_{SRTM} - c \times \Delta z_i \]  

(2.57)

The gravity and potential contribution of the residual component is calculated by substituting equation (2.56) into equations (2.51) and (2.52), respectively. The analytical solutions for gravity and potential of the point mass approximation of the vertical line mass element have one square root function \( \ell(h, z_i) \). Except a common overhead square root function for mean spherical Earth radius, much of its computation time is primarily taken by the loop operating over the cumulative effects of the discrete point masses.
2.4.3 Sector method for topographic computation

The sector method used circular vertical cylinder formulae (Heiskanen & Moritz, 1967, sect. 3.2) to compute gravity and potential models of the topography. The vertical circular cylinder only has analytical solutions for points on its axis. It has no simple analytical solutions at an off-axis point. Elsewhere we used the gravity effect of a sector cut from cylindrical shell, the method used by Hammer (1939) for computing terrain corrections.

![Geometry of the cylindrical sector used to approximate a vertical prism.](image)

Figure 2.4. Geometry of the cylindrical sector used to approximate a vertical prism.

In this method, the geometrical radius of the vertical circular cylinder is computed based on the mass equivalence principle. That is, the vertical circular cylinder should contain the same mass as the SRTM grid cell.

\[ \delta M_{\text{cylinder}} = \delta M_{\text{SRTM}} \]  

(2.58)

Theoretically, both models: the vertical cylinder and the SRTM topography are assumed to have same topographic height and density. Therefore, these conditions will lead to the equivalence between their surface areas

\[ \delta A_{\text{cylinder}} = \delta A_{\text{SRTM}} \]  

(2.59)

\[ \pi R_0^2 = R^2 \partial \phi \partial \lambda \cos \phi \]
\[
R_0 = \sqrt{\frac{R^2 \partial \varphi \partial \lambda \cos \varphi}{\pi}}
\]  (2.60)

where \( R_0 \) is radius of the vertical cylinder that conserves the mass of the rectangular SRTM grid cell.

This method adopts two separate steps for the computation of topographic effects on gravity and potential. The first step directly adopts the vertical cylinder formula for calculating the effect of the innermost zone, where the field point and the running point are coincident or, as a good approximation, where the field point is at an off-axis position within the SRTM grid cell. The small misfit between the actual gravity reduction point and the topographic computation point will not destroy the smoothing effect and generates no theoretical error because it is part of a remove-restore process.

The second step uses two coaxial vertical cylinders for calculating the effects of each SRTM grid mass positioned at a geodesic distances greater than or equal to 90 m. The topographic effect of the annulus between the two coaxial cylinders can be calculated as the difference between the contributions of the outer cylinder and the inner cylinder with radii \( R_e \) and \( R_i \), respectively. The heights of the two cylinders are equal to the height of the SRTM source mass and the radius of the outer cylinder is always larger than the radius of the inner cylinder by the sampling distance of the running points. The potential and gravity of the annulus region defined by two coaxial-cylinders is given by

\[
\begin{align*}
\delta \varphi_T &= \delta \varphi_T (R_e) - \delta \varphi_T (R_i) \\
\delta g_T &= \delta g_T (R_e) - \delta g_T (R_i)
\end{align*}
\]  (2.61)

where

\[
R_e = X_0 + R_0 \quad ; \quad R_i = X_0 - R_0
\]  (2.62)

\( X_0 = (R_f + h_f) \sin \psi \) is the perpendicular distance from the field point to the topographic mass element at the running point (see, Fig. 2.1 or Fig. 2.2).

In order to determine the gravity and potential formulae of a single SRTM prism mass as its representation by a cylindrical sector mass element, the annulus is
divided into a number of sectors having mass or area equal to that of the SRTM grid cell. The gravity and potential formulae for the cylindrical sector mass model can be determined by dividing the gravity and potential values of the annulus by the number of sectors which have the same mass as the SRTM grid cell.

\[
\Gamma = \frac{\delta g_T(R_e) - \delta g_T(R_i)}{\Gamma} \cdot \hat{h}_b; \quad \delta v_T = \frac{\delta v_T(R_e) - \delta v_T(R_i)}{\Gamma}
\]

(2.63)

where \( \Gamma \) is the number of sectors, each conserving the same mass as the SRTM grid cell. Note that, in this process, the \( \cos \phi \) term is used to predict the variation of the SRTM grid cell with latitude, \( \hat{h}_b \) is a unit vector in the direction of normal gravity at the body or running point.

The number of sectors can be directly computed from the mass conservation principle between the annulus zone and the actual SRTM grid cell or equivalently from the geometrical property of the sector model (see, Eqs. 2.64 & 2.65).

\[
\Gamma = \frac{\delta M_{SRTM}}{\delta M_{annulus}} = \frac{\pi (R_e^2 - R_i^2)}{R^2 \partial \phi \partial \lambda \cos \phi} = \frac{2\pi}{\phi}
\]

(2.64)

and

\[
\phi = 2 \tan^{-1} \left( \frac{R_u}{X_0} \right)
\]

(2.65)

where \( \phi \) is the angle subtended by the sector (see, Fig. 2.4)

Note that analytical solution for the gravity attraction of the sector mass element as given in Eq. (2.63) is determined at the computation point in the direction of the normal gravity at the body point. For practical applications the vertical component of the topographic attraction must be calculated at the computation point. This can be achieved by projecting the gravity given in equation (2.63) on to the direction of the normal gravity at the computation point. The gravity is then obtained by multiplying equation (2.63) by the cosine value of the spherical distance between the computation point and body point, see, Eq. (2.66). At this stage it should be clear that the potential is a scalar quantity and it does not depend on direction.
Here, the effect of a single sector or SRTM block mass is equal to the ratio of the SRTM grid cell area to the total area of the ring zone multiplied by the gravity or potential estimates of the annulus zone. In this fashion the total topographic effect is calculated by expanding the radius of the coaxial cylinders as desired.

2.4.4 Numerical Investigations

Precise calculation of the topographic gravity and potential models is very challenging due to large contributions from the topographic masses in the immediate locality of the computation point. The near zone topographic mass distribution recovers very high frequency components up to the shortest half-wavelength equivalent to the SRTM grid size. The contribution of the near zone is larger than the far zone effect because the number of constant volume elements at a distance \( r \) from the computation point grows like \( r \) but the gravity effect decreases by \( r^{-2} \). Many authors (Heck & Seitz, 2007; Kiamehr, 2006; Makhloof & Ilk, 2008) compute the gravity effect out to a fixed radius centred on the computation point. As a result most of the topographic models converge slowly near the source mass. However, the remove-compute-restore approach of geopotential computation requires a precise topographic modelling so that the removed gravity is restored as potential without altering the harmonic property of the Earth’s external potential. In contrast to gravity, the potential of a mass element decreases like \( r^{-1} \), so all annuli with the same size of volume element will produce the same effect, independently of their radius: the potential does not converge as the size of the region increases. Consequently, the task must be made definitive by giving fixed geographic limits to the area of topography. To keep the remove-restore process consistent, both the gravity and potential effect of every mass element in this region must be computed.

The task of evaluating the model’s convergence in the locality of the computation point is the key issue in the remove-restore technique of computing the geopotential number. Many authors have devised methods of handling the effect of the innermost zones. Heck & Seitz (2007) used the prism formula to calculate gravity and potential in the direct vicinity of the computation points where the

\[
\delta g_i = \frac{\delta g_T(R_i) - \delta g_T(R_e)}{\Gamma} \cos \psi
\]
tesseroid formula is inefficient. Others have used the rectangular prism formula for the computation of the innermost zone (Forsberg, 1984; Nagy, 1966; Nagy et al., 2000, 2002; Tsoulis, 1999).

To insure quality control in the remove-compute-restore technique of topographic modelling this study investigates the stability of the three models, the vertical line mass element, multipoint and cylindrical sector in the vicinity of the source mass. A computational test used a topographic element with a 5000 m ellipsoidal height and 90 m squared cross-section to estimate gravity and potential for the three models as a function of horizontal and vertical distances. Evaluating the attraction and potential of a single block of mass on a 2-D plane surface centred on the top of the object itself will give information about the stability of the models as horizontal distance changes.

The gravity and potential formulae of the vertical line mass element model diverges to infinity at the coincident point because the term $\ell^{-1}(h_f, h_b)$ is singular at the coincidence points $(h_f = h_b)$. For more detail see equations (2.43) and (2.47). In the vicinity of the computation point, the gravity and potential estimates of the three models behave differently due to the models’ inherent geometrical effect. The residual gravity and residual potential between the vertical line mass element and any of the other two models reveal better convergence trend as a function of variation in horizontal distance. The multipoint attraction is 0.172 mgal larger than the cylindrical sector attraction at the coincident point and it is also larger than the vertical line mass element attraction at the horizontal distance of 90 metre from the source mass by 0.169 mgal and at larger distances, their results converge to the same value with acceptable accuracy (see, Figures. 2.5a and 2.5b). The line mass element attraction is 0.3 μgal larger than the cylindrical sector at 90 m away from the source mass.
Figure 2.5. Comparing the accuracy of gravity and potential models of a vertical line mass element and multipoint with the corresponding models derived from a cylindrical sector as a function of horizontal distance. (a) Gravity difference (mGal). (b) Potential difference \((m^2s^{-2})\).
Since the actual topography undulates in a three-dimensional fashion it is also interesting to evaluate the accuracy of the models as a function of variation in vertical distance relative to the elevation of the computation point. This can be investigated by computing gravity and potential of the three models at variable heights and a fixed horizontal distance relative to the elevation of the source mass. For clarity, field points above the height of the body point are given positive elevation while field points below the body points are assigned negative elevation. Figure 2.6a-c shows the effect on gravity and potential of the synthetic test mass near its locality calculated at variable elevations relative to the elevation of the test mass. Varying elevation of the computation point from -70 m to zero metre relative to elevation of the source mass located at a constant horizontal distance of 90 m causes a change in the gravity difference obtained from the vertical line mass element and multipoint by 0.25 mgal (Figure 2.6b) and the gravity difference between the line mass element and cylindrical sector mass models converges from about 0.06 mgal to 0.02 μgal. Similarly, the potential difference is less than $0.0001 m^2 s^{-2}$ in the same elevation range (Figure 2.6c).
Figure 2.6a-c. Comparison of gravity and gravity potential of the vertical line mass element, multipoint and sector calculated at the vicinity of the source mass – at variable elevation and a constant horizontal distance of 90 m. (a) gravity (mgal). (b) gravity difference (mgal). (c) potential difference \((m^2s^{-2})\).
It is now verified that the vertical line mass element model is not accurate enough when the computation point falls within the grid cell of the SRTM source mass or in its immediate locality. Both the multipoint and sector models are convergent in the vicinity of the source mass but for field points lying within the grid of the source point only the multipoint has an analytical solution.

The numerical differences between the vertical line mass element and the multipoint model have also been investigated by computing the effects of the test mass i.e a 5000 m high prism at horizontal distances larger than 1000 m on a horizontal plane centred on the top of a test mass. Figure 2.7a-b shows precise numerical agreement between the model’s gravity and potential at far zone distances.

It has therefore been concluded that the vertical line mass element model can be used for precise topographic modelling except for SRTM masses closer than 1000 m (Figure 2.5a-b). The next section chooses a way of combining the vertical line mass element and the other two models based on their computational speed.
Figure 2.7a-b. Comparing the accuracy of a vertical line mass element relative to a multipoint due to the effect of the test mass on gravity and potential at longer distances. (a) Difference in gravity (b) Difference in potential.
2.4.5 A Combined Algorithm for SRTM Gravity Modelling

This section emphasises the advantage of using the hedgehog algorithm as compared to the multipoint and the sector methods. The three models were compared based on their computation time and the accuracy of the result they provide. The geographic region defined by longitudes of $37^\circ E - 38^\circ E$ and latitudes of $9^\circ N - 10^\circ N$ is used to study gravity and potential effects of the topography as a realistic computational test. The three arc seconds resolution SRTM data were used to calculate gravity and potential at 90 m resolution. At every computation point the net effect of the SRTM data on gravity and potential requires $1200 \times 1200$ points to be included in the calculation for each of the three models.

The relative computation time taken by the three models for the computation of gravity and potential of the defined region is shown in Figure 2.8. The vertical line mass element algorithm is 4 times faster than the multipoint for gravity and twice as fast for potential. The vertical line mass element and sector methods have nearly the same computational speed for computing the gravity effect, but for the potential, the former is twice as fast as the latter.

Even though the cylindrical sector formula is as fast as the vertical line mass element for computing the gravity effect, the latter has been used except for the innermost zone where it fails to converge because the vertical line mass element is faster in restoring the effect of the removed topographic masses as its potential.

In conclusion, despite its lower computational speed the multipoint was used for calculating the effect of the innermost zone within 1000 m radius while the vertical line mass element is used for the outer zone effect. This combined algorithm has been applied to compute the gravity and potential models of the topographic effects representing the region of Ethiopia and its immediate surroundings at 92,433 point airborne gravity disturbance (Figure 2.9a-b). Subtracting the topographic gravity model from gravity disturbance gives the Bouguer disturbance (Figure 2.10).
Figure 2.8. Comparison between the computational speed of the vertical line mass element, multipoint and sector needed to calculate the effects of topographic masses on gravity and potential. Percentages are relative to the multipoint computational speed.
Figure 2.9a-b. Gravity and potential of the *in situ* topography evaluated at airborne gravity measurement points from SRTM data representing Ethiopia and its immediate surroundings using multipoint for the inner zone within 1 km radius and hedgehog for larger radius. (a) Gravity (mgal). (b) Potential ($m^2s^{-2}$).
Figure 2.10. The Bouguer disturbance map of Ethiopia derived from airborne gravity disturbance after removing the effects of topographic attraction (mgal).

2.5 The Condensed Topography

The Bouguer disturbance has the required smoothness but is neither small nor trend-free and so it is not suited to implement simplified approximations for downward continuation and transformation of gravity to potential. Any transformation of gravity to potential must have trend-free residual gravity disturbances and will be more precise if they are small, smooth and localised. To eliminate the long wavelength components from the Bouguer disturbance, a smoother representation restricted to long wavelength components of the topography has to be added back. This will synthesise a residual that is trend-free and but still has localised small amplitudes at short wavelengths. The task of restricting the topographic gravity and potential models to longer wavelengths involves their expansion into spherical harmonic or ellipsoidal harmonic functions. Evaluating the effect of the same mass condensed onto the ellipsoid, rather than \textit{in situ}, simplifies the calculation and also leaves the space above the ellipsoid mass-free.
In this research, the topographic masses are condensed on the reference ellipsoid as a single layer preserving the mass of the 3-arc seconds SRTM elevation data and expanded into surface spherical harmonic functions (Ellmann & Vanicek, 2007; Heck, 2003; Vaníček et al., 2001). Surface spherical harmonic coefficients up to degree and order 2159 were computed from the SRTM data by direct numerical integration and the topographic gravity effect was added back to the smoothed Bouguer gravity disturbance at aircraft positions. This rigorous smoothing technique provides band-limited residual gravity disturbance maintaining small amplitude and smoothly varying high frequency components so that a simple planar Fast Fourier Transform can be efficiently used to solve huge computational task with high numerical accuracy.

The potential at \((r_2, \vartheta_2, \lambda_2)\) due to a surface density \(\sigma(\vartheta_1, \lambda_1)\) on a sphere of radius \(r_1\) is given by

\[
\delta v_c(r_2, \vartheta_2, \lambda_2) = G \int_S \frac{\sigma(r_1, \vartheta_1, \lambda_1)}{|r_1 - r_2|} r_1^2 \sin \vartheta_1 d\vartheta_1 d\lambda_1
\]

The condensation technique is based on the principle of mass conservations that converts the topographic mass of an elementary ‘tesseraid’ into an equivalent surface density on a sphere. The spherical solution of the surface density is obtained by integrating the condensed layer over a differential angular element \(\delta \Omega\) containing an equivalent mass of the topography. It is derived as:

\[
\delta n = \rho \int_R^{R+h} r^2 d\Omega = \rho \left( \frac{(R+h)^3 - R^3}{3} \right) \delta \Omega = \rho h \left( 1 + \frac{h}{R} + \frac{1}{3} \left( \frac{h}{R} \right)^2 \right) R^2 \delta \Omega
\]

and the spherical analytical solution of the surface density of the condensed topography corresponding to its equivalent geometrical topography is given by:

\[
\sigma = \left[ \rho h \left( 1 + \frac{h}{R} + \frac{1}{3} \left( \frac{h}{R} \right)^2 \right) \right]
\]

The non-linear terms in equation (2.69) with the topographic height can never contribute more than a few parts in \(10^5\) of the linear term. Practical conversion of equation (2.69) into surface harmonics is based on the fact that the surface varies laterally as a function of latitude and longitude and so can be expanded into a series of sinusoidal surface spherical harmonic functions.
\[ \sigma(\vartheta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m(\cos \vartheta) \left( \sigma_{n,c}^m \cos(m\lambda) + \sigma_{n,s}^m \sin(m\lambda) \right) \] (2.70)

Where \( \sigma_{n,c}^m \) & \( \sigma_{n,s}^m \) are the Schmidt semi-normalised coefficients; \( P_n^m \) is the Schmidt semi-normalised Legendre’s function.

Now, due to the availability of high resolution and accurate model of the Earth’s topography \( h(\vartheta, \lambda) \) from lidar and microwave remote sensing satellites, the dimensionless spherical harmonic coefficients of the surface density of the condensed topography can be determined from these observables using equation (2.69) as.

\[
\begin{align*}
\begin{bmatrix} \sigma_{n,c}^m \\ \sigma_{n,s}^m \end{bmatrix} &= \int \int \int \sigma(\vartheta, \lambda) P_n^m(\cos \vartheta) \begin{bmatrix} \cos(m\lambda) \\ \sin(m\lambda) \end{bmatrix} \sin \vartheta \, d\vartheta \, d\lambda \\
&= \frac{(2n+1)}{4\pi} \int \int \sigma(\vartheta, \lambda) P_n^m(\cos \vartheta) \begin{bmatrix} \cos(m\lambda) \\ \sin(m\lambda) \end{bmatrix} \sin \vartheta \, d\vartheta \, d\lambda 
\end{align*}
\] (2.71)

The inverse distance in equation (2.67) can be expanded into functions of Legendre’s polynomials

\[
\frac{1}{|r_1 - r_2|} = \frac{1}{r_2} \sum_{n=0}^{\infty} \left( \frac{r_1}{r_2} \right)^n P_n(\cos \psi)
\] (2.72)

where

\[
\cos \psi = \frac{r_1 \cdot r_2}{|r_1| |r_2|} \quad \text{and} \quad |r_2| > |r_1|
\] (2.73)

\[
\psi = \cos \vartheta_1 \cos \vartheta_2 + \sin \vartheta_1 \sin \vartheta_2 \cos(\lambda_2 - \lambda_1)
\] (2.74)

And the Legendre’s polynomial \( P_n(\cos \psi) \) can be expanded in terms of latitudes and longitudes of the running point \( r_1 \) and computation point \( r_2 \).

\[
P_n(\cos \psi) = \sum_{m=0}^{n} P_n^m(\cos \vartheta_1) P_n^m(\cos \vartheta_2) \cos[m(\lambda_1 - \lambda_2)]
\] (2.75)

An alternative expression can be also obtained by expressing the semi-normalised harmonics in terms of fully normalised harmonics.
\[ P_n^m(\cos \vartheta) = \frac{1}{\sqrt{2n+1}} P_n^m(\cos \vartheta) \quad (2.76) \]

Substituting equation (2.70) for \( \sigma \) and equations (2.72) and (2.75) for the inverse distance into equation (2.67) and using orthogonality principle, we get

\[
\delta_{r_2}(r_2, \vartheta_2, \lambda_2) = \frac{G}{r_2} \sum_n \left\{ \sum_m \left[ \left( \frac{r_1}{r_2} \right)^n \sum_{m=0}^{\infty} \left( \cos(m\lambda_1) \cos(m\lambda_2) + \sin(m\lambda_1) \sin(m\lambda_2) \right) \right] \right. \\
\times \sum_p \sum_q \left\{ P_p^q(\cos \vartheta_1) \left[ \sigma_{p,c}^q \cos(q\lambda_1) + \sigma_{p,s}^q \sin(q\lambda_1) \right] \right\} \left( \frac{r_1^2}{r_2} \right)^{n+1} \sin \vartheta_1 d\vartheta_1 d\lambda_1 \\
\left. = 4\pi G \sum_n \sum_m \frac{r_1}{2n+1} \left( \frac{r_1}{r_2} \right)^{n+1} \sum_{m=0}^{\infty} \sigma_{n,c,s}^m(r_1) \right\} \left( \frac{r_1}{r_2} \right)^{n+1} \quad (2.77) \]

Equation (2.77) is the fundamental formula for computing the gravitational potential of a single layer condensed mass at every geocentric radius \( r_2 \) on or above the reference surface of condensation. The solution can be alternatively expressed in terms of the cosine and sine harmonic coefficients of the surface density described in equation (2.69) as shown below.

\[
\delta_{r_2}(r_2) = 4\pi G \sum_{n=0}^{\infty} \frac{r_1}{2n+1} \left( \frac{r_1}{r_2} \right)^{n+1} \sum_{m=0}^{\infty} \sigma_{n,c,s}^m(r_1) \quad (2.78) \]

The gravitational attraction of the condensed layer can be simply derived by differentiating equation (2.78) with respect to \( r_2 \).

\[
\delta g_{r_2}(r_2) = -\frac{\partial \delta_{r_2}(r_2)}{\partial r_2} = 4\pi G \sum_{n=0}^{\infty} \frac{n+1}{2n+1} \left( \frac{r_1}{r_2} \right)^{n+2} \sum_{m=0}^{\infty} \sigma_{n,c,s}^m(r_1) \quad (2.79) \]

The same SRTM topographic mass elements used in the hedgehog algorithm were condensed on the reference ellipsoid to compute its gravitational potential and attraction. Equation (2.79) truncated to \( n = 2159 \) or, experimentally to lower degree, was used to calculate the longer wavelength components of gravity model of the topography (Figure 2.11). The attraction of the condensed topography was predicted at aircraft variable height using analytical continuation formula applied for EGM08 disturbance. The predicted gravity (Figure 2.11) was added back to the Bouguer
disturbance (Figure 2.10) to synthesise a band-limited gravity disturbance. Figure 2.12 shows smallness and smoothness of the residual gravity disturbance over Ethiopia acquired by de-trending the Bouguer disturbance. The residual gravity disturbance has a mean value of 1.2 mgal with a standard deviation of 5.2 mgal.

Figure 2.11. Gravity model of the condensed topography computed at airborne gravity observations from the SRTM data representing Ethiopia and its immediate surroundings (series complete to $n = 2159$)
2.6 Verification of the Models

Many aspects of data need verification. Is any adjustment to the smoothing that suppresses the aircraft acceleration needed for those lines that suffered unusual turbulence? How well does our version of measured gravity compared with that predicted from EGM08? How well does the synthetic topography effect predicted from spherical harmonic coefficients compare with the hedgehog computation? Although these questions need quantitative statistical answers, these can easily obscure the nature of misfits so results for every line can be inspected by plotting. Figure 2.13 is an example for one flight line. This line is more than 600 km long and flew at an approximately constant height over extreme topography – the elevation ranged from 170 m to 3450 m.
Figure 2.13. Verification plot for airborne gravity along one flight line.

The lower box of Figure 2.13 compares the measured gravity disturbance (red) with the synthetic value from EGM08 (blue). The lower resolution of EGM08 is clear. The middle box compares SRTM ground height (green) with a spherical harmonic synthesis (red) derived from the same information, again illustrating the lower resolution for spherical harmonic synthesis. (Note the scale for topography is in units of 10 m and is offset by 2 km.) The box also shows the gravity effect of this topography calculated with the hedgehog algorithm (blue) and synthesised from the spherical harmonic representation of the topography (purple). The top box shows the
The final residual – the net information in the gravity measurements additional to the EGM08 gravity model and the SRTM topographic model.

Note that, although of lower resolutions, the EGM08 and SRTM spherical harmonic models do represent the phenomena faithfully even in a region of extreme topographic relief as shown in Figure 2.13.

All graphs have along track distance in km along the bottom and gravity effects in mgal plotted vertically (except for the red and green curves in the middle box where the units are 10 m offset by 2 km for a better visualization).

Although a quantitative statistical analysis obscures the level of agreements between the models at a localised scale, it gives valuable information on how well the models fit each other at regional scale. Statistical analysis was carried out to show how well the models predict each other on the basis of point-wise comparison. Table 2.3 shows this for the whole region of Ethiopia. The models are generally in good agreement. The EGM08 predicts the observed gravity with a mean error and standard deviation of 0.26 mgal and 4.56 mgal respectively. The topographic models are in agreement with a mean of -0.96 mgal and standard deviation of 5.20 mgal. The larger differences in extrema reflect their different resolutions.

<table>
<thead>
<tr>
<th></th>
<th>Measured – EGM08</th>
<th>SRTM_g_hedgehog – SRTM_g_sph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.26</td>
<td>-0.96</td>
</tr>
<tr>
<td>Stdev.</td>
<td>4.56</td>
<td>5.20</td>
</tr>
<tr>
<td>Max</td>
<td>37.90</td>
<td>61.30</td>
</tr>
<tr>
<td>Min</td>
<td>-33.10</td>
<td>-39.10</td>
</tr>
</tbody>
</table>

Table 2.3. Comparison between measured, EGM08 and space-domain and spectral modelling of the topographic effects evaluated at 92433 points (all units are in mgal).

The final residual gravity disturbances synthesised from these three gravity models (EGM08, airborne and topographic models) are now essentially very small with 1.2 mgal and 5.2 mgal mean and standard deviation, respectively (see, Table 2.4). With the up coming Gravity field and steady-state Ocean Circulation Explorer (GOCE) gravity data, they will become even smaller.


2.6.1 Reliability and resolution of the residual gravity

The novelty of using the normal gravity, global gravity model and the geometrical and condensed topographic models to smoothing the measured gravity data can be numerically evaluated by quantifying the magnitude of the interpolation errors that can be introduced while synthesising a regular grid of residual “gravity” for some gravity processing techniques, for example using a planar FFT. Most geodetic computations such as downward continuation and Stokes-like integral transform involve surface integral of gravity data. The accuracy of the integration operation relies on the smoothness of the residuals. Also the stability and quality of the downward continued solution depends on the local residual gravity disturbance. In particular, the FFT method requires a regular grid of gravity data with sufficient smoothness and smallness. However, the prediction of a regular grid from the observed data can introduce errors in the mean value of the grid block size. This error usually causes oscillating noise in the downward continuation process which eventually produces spurious signal in the transformed potential model. The extent of the gridding effect usually depends on the smoothness of the residual gravity disturbance and the grid sampling interval used. Its effect can be statistically quantified by computing the misfit between the original observed data and the values predicted from gridded version of the data. This was a two-stage process: the original, irregularly distributed data were interpolated onto a regular grid using the Generic Mapping Tool (GMT) software called surface. The gridded values were then re-interpolated back to the measurement positions, using the Newton central difference cubic interpolator. For the purpose of making comparisons, the GMT surface interpolation software was used to produce regular grids at different grid spacing. The cubic interpolator automatically uses the grid spacing of input grid to calculate the misfit between the observed and predicted values in order to identify an optimal grid size for which the overall interpolation error is minimum (Table 2.4). The difference between the observed and the predicted values is indicative of the errors on grid values. The error on the gridded values should not be significantly bigger than the Root Mean Square (RMS) misfit of the re-interpolated values. The error on the gridded values is also smallest for the grid spacing for which the misfit is smallest, 2 km for this test. Most of the gridding errors as shown in Table 2.4 are
introduced by GMT *surface* software. The cubic interpolator causes only small errors, see section 2.3.2.

<table>
<thead>
<tr>
<th></th>
<th>observed (mgal)</th>
<th>1 km</th>
<th>2 km</th>
<th>3 km</th>
<th>4 km</th>
<th>5 km</th>
<th>6 km</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mean</strong></td>
<td>1.217</td>
<td>-0.004</td>
<td>8.4e-5</td>
<td>0.003</td>
<td>0.007</td>
<td>0.005</td>
<td>0.003</td>
</tr>
<tr>
<td><strong>Stdev.</strong></td>
<td>5.203</td>
<td>1.347</td>
<td>0.787</td>
<td>1.148</td>
<td>1.138</td>
<td>1.158</td>
<td>1.226</td>
</tr>
<tr>
<td><strong>Max</strong></td>
<td>38.738</td>
<td>24.424</td>
<td>19.256</td>
<td>22.978</td>
<td>17.798</td>
<td>17.805</td>
<td>17.704</td>
</tr>
</tbody>
</table>

*Table 2.4.* Misfit between the observed values and values predicted by the GMT *surface* software and cubic interpolation methods for variable grid spacing.

Our residual gravity disturbance is concentrated in the short wavelength end of the regional gravity spectrum. The gridding effects are generally small, ranging from -0.004 to 0.007 mgal and 0.787 to 1.347 mgal in the mean and standard deviation, respectively. The gridding effect at 1km is larger and it shows the degradation in the resolution of the residual gravity disturbance. The smallness of the gridding effect at 2km suggests that this grid spacing be used for the continuation and transformation processes. This is equivalent to recovering the gravity field up to wavelengths of 4 km.

### 2.7 Summary

The modern remove-compute-restore technique of geopotential computation uses the measured gravity disturbance, a global gravity model and a topographic model. The difference between the measured gravity disturbance and the global gravity model-derived gravity disturbance gives a band-limited small and smoothly varying residual gravity so that the interpolation and integration involved is reliable. But this is not by itself sufficient, because the computation also needs the residual gravity to be predicted on a smooth surface external to source masses. The transformation of surface gravity data to potential on this subterranean smooth surface requires the use of Laplace’s equation to evaluate the function in three dimensions from observations over a two-dimensional surface and to determine the function from its derivatives. Laplace’s equation is restricted to essentially mass-free space and so involves removing the topographic effects on gravity and potential using different mass models. The topographic modelling is also equally used to suppress the high
frequency components of the measured gravity, because the measured gravity disturbance is dominantly influenced by the geometry of the topography as compared to lateral subsurface density variations.

Subtracting the contribution of the short wavelength components of the *in situ* topographic effects, $\delta g(h)$ from the measured gravity disturbance gives a smooth version of gravity disturbance usually called Bouguer disturbance. But the gravity model of the condensed topography has to be put back to remove trends from Bouguer disturbance so that a simple planar FFT algorithm can be successfully applied for the downward continuation and transformation process. Hence, precise topographic computation must be implemented for use with the measured gravity disturbance and the global model (gravity and potential) if a vertical reference system with centimetre accuracy is required.

However, the computation of the topographic effect maintaining the full resolution of the SRTM data is still a time-consuming operation even if models with a grossly simplified geometry are used. This study shows that the computation time of the vertical line mass element is generally better than the multipoint and sector solutions. The multipoint algorithm gives reliable result for the coincident point and very nearby points while the cylinder has no analytical solution when computation point is located at an off-axis position within the dimension of the source mass. Consequently the multipoint was used to calculate the effects of the topographic mass attraction in the immediate vicinity of the computation point. In terms of computational speed the vertical line mass element is faster by a factor of 4 and 2 for gravity and potential, respectively as compared to multipoint.

This study computes the topographic effects representing the region of Ethiopia and its immediate surroundings at 92,433 point airborne gravity disturbance (Figure 2.10a-b) and the resulting topographic attraction has a mean value of 151.61 mgal with a standard deviation of 77.84 mgal while the gravity disturbance has 4.74 mgal mean value with a standard deviation of 34.08 mgal. The contribution of the condensed topography is similar in size as that of the *in situ* topographic effect. Due to the larger contribution of the topographic effect on the geopotential, it must be done with great precision so that the removed gravity and the restored potential represent the same mass distribution without conceptual error.
The creation of small, smoothly-varying trend-free residual anomalies evaluated at the along-track aircraft measurement points involves only space-domain operations. The next stage deals with their representation in three-dimensions so that they can be evaluated on the ellipsoid and then transformation from residual gravity to residual potential requires transformation to the spectral domain, here involving Fourier analysis.
Chapter 3

Spectral Domain Gravity Analysis

3.1 Introduction

This chapter explores how to predict gravity on a subterranean reference ellipsoid from observations taken at irregular heights on or above the Earth’s surface. Some operations equivalent to Poisson’s integral (see, e.g. Heiskanen & Moritz, 1967, Chap. 8; Hofmann-Wellenhof & Moritz, 2006, Chap. 6; Hunang & Véronneau, 2005; Hunegnaw, 2001; Martinec, 1996; Martinec & Vaníček, 1994) remains the basis for downward continuing observations of gravity. Also most space domain processing like Taylor-series (Moritz, 1980; Wang, 1989) and Least-Square-Collocation (Tscherning & Forsberg, 1992) are prohibitively slow operations and they are limited to small number of observations. The covariance matrix in the Least-Square-Collocation and discrete form of the Poisson’s integral involves the computation of the system of linear algebraic equations necessitating substantial memory requirement and much computation time when one deals with large datasets. This study presents an alternative spectral domain analysis using Fast Fourier Transform to compute the downward continuation and transformation of gravity to potential (Hipkin, 1988; Sideris & Schwarz, 1986). The Fast Fourier Transform is very effective over large set of gridded data and computations can be carried out for a group of countries in a single calculation.

In reality, we observe gravity at irregular positions and so some form of interpolation is required to implement Fast Fourier Transform efficiently. A more precise harmonic downward continuation also requires the solutions of the Fast Fourier Transform in the three-dimensions, so that, the effects of the topographic masses above the reference ellipsoid have to be removed to satisfy Laplace’s condition.

Section 3.2 shows how Laplace’s equation and Fourier Transforms can be used for computing vertical gradients and integrals of the data available on a fixed
boundary surface. Section 3.3 shows the validity of using the Fourier transform on a map projection coordinates. Section 3.4 presents an iterative spectral domain approach for continuing gravity data from variable heights and horizontal positions on to the surface of the reference ellipsoid using a planar Fast Fourier Transform and a 3-D cubic interpolation technique. Section 3.5 discusses how to transform gravity to potential on a map projected coordinates using Fast Fourier Transform operation. Section 3.6 discusses the Singleton’s mixed radix Fast Fourier Transform, padding and edge effects. Section 3.7 explores the computational accuracy of the iterative Fast Fourier Transform algorithm by using two different test experiments. The first test compares the gravity anomalies reproduced at the observation point after three iterations with the original residual anomalies. The second test uses purely an independent approach – it downward continues the band-limited high frequency components of the EGM08 derived gravity disturbance from 2000 m ellipsoidal height on to the surface of the ellipsoid using the FFT and compares the continued residuals with the residuals directly computed from the spherical harmonics.

3.2 Laplace’s Equation and Fourier Transforms

The Laplace’s equation is widely used for solving boundary value problems of the potential field. It analytically determines variations normal to the surface from variations of observables or a priori data known over a coordinate of a fixed boundary surface. For data on a plane of constant \( z \),

\[
\frac{\partial^2 f}{\partial z^2} = -\left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \tag{3.1}
\]

where \( f = f(x, y, z) \) is harmonic function.

The right hand side of equation (3.1) is determined by measurable gradients on the observational plane, so variations with distance away from the plane become known. The general solution to equation (3.1) for sources above the plane \( z = 0 \) is
\[ f(x, y, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k_x, k_y) e^{-\kappa z} \exp\left[j(k_x x + k_y y)\right] dk_x dk_y \]  \hspace{1cm} (3. 2)

where \( F(k_x, k_y) \) is the Fourier Transform of \( f(x, y, z) \) evaluated on a plane at \( z = 0 \)
that is \( F(k_x, k_y) = FT[f(x, y, 0)] \)

\[ k_x = \frac{2\pi}{\lambda_x}, \quad k_y = \frac{2\pi}{\lambda_y} \]  \hspace{1cm} (3. 3)

Where \( k_x, k_y \) are the wavenumbers in the \( x \) and \( y \) directions, respectively; \( j \) is an
imaginary number, \( \sqrt{-1} \).

\( F(k_x, k_y) \) is formed from the inverse Fourier Transform

\[ FT[f(x, y, 0)] = F(k_x, k_y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, 0) \exp[-j(k_x x + k_y y)] \, dx \, dy \]  \hspace{1cm} (3. 4)

So the function can be evaluated in the whole of mass-free space from

\[ FT[f(x, y, z)] = e^{-\kappa z} FT[f(x, y, 0)] \]  \hspace{1cm} (3. 5)

The coefficient of \( z \) in the general solution, \( \kappa \) is related to the in-plane wavenumbers by

\[ \kappa = \sqrt{k_x^2 + k_y^2} \]  \hspace{1cm} (3. 6)

Similarly, a harmonic function can be transformed into its derivatives or
integrals, for example

\[ FT\left[\frac{\partial f}{\partial z}\right] = -\kappa FT[f] \]  \hspace{1cm} (3. 7)

This gives a novel opportunity for measuring gravity and calculating its potential.

The task of computing gravitational potential at any point on or above the
Earth’s surface from gravity observations taken at discrete irregular heights, and
horizontal positions involves the remove-compute-restore process. The remove-
restore process mathematically removes the effects of the topographic masses and the
normal gravity so that the region on and above the boundary surface should be
essentially mass-free and non-rotating to satisfy Laplace’s condition. Note that, the
boundary surface can be any smooth surface (e.g. plane, reference ellipsoid, sphere),
not strictly a level surface used in a classical approach. Matching data to solutions of Laplace’s equation involves spectral analysis (e.g., Fourier Transforms or analysis into surface spherical harmonics) which requires integration of the surface where observations are made. This chapter deals with the spectral analysis of gravity data using FFT for downward continuation and transformation of gravity to potential.

The Fourier Transform performs a spectral representation of reasonably well-behaved non-periodic function into different spectral components, of amplitude and phase or real and imaginary part, of the sinusoidal function varying with spatial frequencies or wavenumbers. The Fourier Transform creates an opportunity to modify some part of the frequency spectrum of the function and thereby change the shape of the function in the spatial domain. The inverse Fourier Transform is usually carried out after some calculation is done in the frequency or wavenumber domain to determine the correct spatial domain form of the modified function. For example, the upward continuation of a continuously available gravity data can be performed by multiplying the Fourier transform of the gravity by \( \exp(-\kappa z) \) before performing the inverse Fourier transform operation.

Equations (3.2) and (3.4) are applicable to a continuously available function. Practical applications use point gravity observations that are irregularly distributed and so Fast Fourier Transform operates on discrete finite arrays of observed data. Some interpolation method is required to prepare a regular grid to maintain the computational efficiency of the FFT processing. The Fourier Transform of the finite observations available as regular array with \( M \) and \( N \) number of points in the \( x \) and \( y \) directions, respectively is given as:

\[
F(m\Delta k_x, n\Delta k_y) = \Delta x \Delta y \sum_{p=0}^{M-1} \sum_{l=0}^{N-1} f(p\Delta x, l\Delta y) \exp \left[ -j \left( \frac{mp}{M} + \frac{nl}{N} \right) \right]
\]  

(3.8)

and the inverse Fourier transform is

\[
f(p\Delta x, l\Delta y) = \Delta k_x \Delta k_y \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} F(m\Delta k_x, n\Delta k_y) \exp \left[ j \left( \frac{mp}{M} + \frac{nl}{N} \right) \right]
\]  

(3.9)

where spatial discrete wavenumbers are given by
\[ \Delta k_x = \frac{2\pi}{M\Delta x} \]  \hspace{1cm} (3.10)

\[ \Delta k_y = \frac{2\pi}{N\Delta y} \]  \hspace{1cm} (3.11)

Fast Fourier Transform analysis, however, decomposes a single signal into finite numbers of wavenumbers and it restricts the highest wavenumber to Nyquist wavenumber. The Nyquist wavenumber is given by

\[ k_{xN} = \pm \frac{2\pi}{2\Delta x} \hspace{1cm} k_{yN} = \pm \frac{2\pi}{2\Delta y} \]  \hspace{1cm} (3.12)

corresponding to a minimum resolvable wavelength of twice the data spacing.

A simple planar Fast Fourier Transform can accurately resolve short wavelengths smaller than the dimension of the data region. For the purpose of high numerical calculation the measured gravity has to be rigorously smoothed with global gravity model and topographic information to produce smoothly varying residuals with very small amplitude at the shortest wavelengths (see, Chapter 2). The smoother and the smaller the residual gravity the better the spectral information is captured. The next section shows the numerical accuracy of a planar Fast Fourier Transform algorithm operated on the Lambert’s azimuthal projection of a sphere on to a tangent plane. In this study we used an extension of the Fast Fourier Transform algorithm developed by Hipkin & Hussain (1983) and Hipkin (1988).

### 3.3 Validating the accuracy of Fourier Transforms

#### 3.3.1 Fourier transforms using map projection coordinates

The kinds of map projection used in cartography aim to minimise distortion over a specific region. Where different projections have a common region of small distortion, the maps produced are very similar. The analysis of Fourier transforms used Lambert’s conformal conical projection to project points from the WGS84 ellipsoid on to a cone (Hooijberg, 2008, pp. 183-193). A cone whose axis lies on the rotation axis of the Earth could sit directly on the ellipsoid, touching it along one parallel of latitude. Practical projections achieve a larger region of small distortion by lowering the cone and making it intersect the ellipsoid along two parallel, and pass
inside the Earth between them. The two standard parallels $\phi_1$ and $\phi_2$ used for the projection here were $15^\circ$ and $5^\circ$ N. In general, the accuracy of this map projection deteriorates as a function of distances away from the standard parallels where the cone intersects the ellipsoid. On the two selected parallels, arcs of longitude are represented to a scale factor unity or in their true lengths. Between these standard parallels the scale is less than unity and outside them the scale factor is greater than unity. By having two rather than one standard parallel, the region of small distortion is increased.

Meridians are represented by straight lines that meet at the apex of a cone which is usually located outside the limits of the map projection. In the projection, parallels of latitude are sections of concentric circles centred at the apex where meridians intersect. Meridians and parallels intersect at right angles so that the angles formed by any two lines on the earth’s surface are correctly represented on this projection.

The origin for this map projection coordinates can be chosen arbitrarily.

A 2-D map projection provides an opportunity to transform a large ellipsoidal surface to a plane on which position defined by northing and easting coordinates.

With the advent of gravity models involving very high degree spherical harmonic analysis, the proposition that gravity can be transformed to potential using planar Fourier transforms and a map projection can be tested.

The gravity disturbance can be evaluated on the surface of the ellipsoid from a series of spherical harmonics coefficients using equation (1.4) with a “high-pass” filter. Here we use those of the Earth gravity model, EGM08. The gravity disturbance expressed in equation (1.4) corresponds exactly to the potential

$$T(\vartheta, \lambda) = \frac{GM}{a} \sum_{n=0}^{N} w(n) \left( \frac{a}{R(\vartheta)} \right)^{n+1} \sum_{m=0}^{n} \left( \tilde{C}_n^m \cos m\lambda + \tilde{S}_n^m \sin m\lambda \right) \tilde{F}_n^m (\cos \vartheta)$$

(3.13)

The filter $w(n)$ is used to create a band-limited signal, eliminating wavelengths too long to be characterized by local data. The filter needs to exist in an equivalent form in both the spherical harmonic and Fourier domains and needs to have effectively total suppression of long wavelength components. We use the Butterworth filter with degree ten or order $k = 5$. The spherical version of the “high-pass” filter is
\[ w(n) = 1 - \frac{1}{1 + \left(\frac{n}{n_0}\right)^{2\kappa}} \]  
\[ w(\kappa) = 1 - \frac{1}{1 + \left(\frac{\kappa R}{n_0}\right)^{2\kappa}} \]  
(3.14)  
(3.15)

For which the corresponding Fourier domain filter is

Where \( \kappa = \frac{2\pi}{\lambda} \) is the wavenumber; \( R \) is mean radius of the Earth.

The test used different values of the cut-off parameter \( n_0 \) to synthesise both the gravity disturbance and its corresponding potential on a grid of latitude and longitude with a spacing of 0.02° using the spherical harmonic series complete to degree and order, \( n = 2180 \). The resulting gravity disturbance and potential were then interpolated on to a 2 km grid of eastings and northings, \( \delta g(x, y) \) and \( T(x, y) \), derived using Lambert’s formulae for the conformal projection of an ellipsoid onto a cone. The resulting region approximately 3000 km square was extended with a 100% margin of zero-padding and transformed to the equivalent band-limited potential using the standard formula relating a normal force component to its potential planar Fourier transform.

\[ T_{FT}(x, y) = FT^{-1}\left[ \frac{1}{\kappa} FT[\delta g(x, y)] \right] \]  
(3.16)

Table 3.1 shows the statistics for the residuals between the potential computed directly from the spherical harmonic synthesis and the version derived from the disturbance using the Fourier transform. It shows the result for different values of the cut-off degree \( n_0 \). It is clear that a planar Fourier transform operating on map projection coordinates is entirely adequate in carrying out the equivalent of Stokes’ integral on an ellipsoidal surface.
<table>
<thead>
<tr>
<th>Filter Cut-off $n_0 =$</th>
<th>disturbance</th>
<th>anomalous potential</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100 200 400</td>
<td>100 200 400</td>
</tr>
<tr>
<td>$\lambda_0 (km) =$</td>
<td></td>
<td>400.75 200.37 100.19</td>
</tr>
<tr>
<td>Mean</td>
<td>-0.05 -0.04 -0.003</td>
<td>-6.5 -4.6 -0.9</td>
</tr>
<tr>
<td>Stdev.</td>
<td>24.7 21.6 16.2</td>
<td>4.3 2.3 0.8</td>
</tr>
<tr>
<td>Max</td>
<td>560.3 532.0 457.7</td>
<td>3.0 6.5 5.3</td>
</tr>
<tr>
<td>Min</td>
<td>-176.0 -173.9 -174.8</td>
<td>-17.3 -12.1 -4.7</td>
</tr>
</tbody>
</table>

Table 3.1. Statistics of the residuals between the band-limited anomalous potential computed directly from EGM08 spherical harmonic coefficients and that derived from the planar Fourier transform of the EGM08 gravity disturbance. (Units are $10^{-2}$ m$^2$s$^{-2}$, approximately equivalent to 1 mm of height). For comparison, the second columns give corresponding statistics for the gravity disturbance (mgal).

### 3.4 An Iterative Fourier Transform Continuation Algorithm

The numerical accuracy of the planar Fourier Transform (FT) has shown to be accurate enough for transforming gravity to potential and so this section uses Fast Fourier Transform algorithm to calculate analytical continuation of irregularly distributed residual gravity data as a regular grid on the reference ellipsoid. In map projection coordinates, downward continuation involves the prediction of gravity onto a plane below the observation point.

With the availability of high resolution global gravity and topographic models, and precise gravity observations, the resulting residual gravity disturbances or anomalies can be synthesized with the required smallness. Due to the irregular distribution of the residual gravity and its short wavelength nature, the downward continuation process will be destabilised by noise associated with high frequency components.

\[
\delta g(x, y, z) = FT^{-1}[e^{\kappa z} FT[\delta g(x, y, z)]]
\]  

(3.17)

Alternatively, we used an upward continuation algorithm (see, Eq. 3.18) to minimise the amplification of short wavelengths, but the algorithm has to be devised
in an iterative approach to project residual gravity at aircraft height on to the ellipsoidal surface.

SRTM topography is available on a ~ 90 m grid. Airborne gravity corrections are also sampled at comparable along-track distance (~ 80 m) but have to be smoothed and then sub-sampled at ~ 800 m in order to suppress dynamic noise. Consequently removing the shortest wavelengths of SRTM gravity may increase short wavelength noise. As a result, a second Butterworth filter is introduced to suppress very high frequency effects in residual gravity, a ‘band-pass’ Butterworth filter is used. In practice, this produces a “band-pass” filter – very long wavelengths are removed to make the Fourier method valid and very short wavelengths to stabilize downward continuation. We used a spectral version of the filter $w(\kappa)$:

$$\delta g(x, y, z) = FT^{-1}[w(\kappa)e^{-\kappa z}FT(\delta g(x, y, 0))]$$  \hspace{1cm} (3. 18)

$$w(\kappa) = \frac{1}{1 + \left(\frac{\kappa}{\kappa_{0L}}\right)^{2k}} \left[1 - \frac{1}{1 + \left(\frac{\kappa}{\kappa_{0H}}\right)^{2k}}\right]$$  \hspace{1cm} (3. 19)

Where $\kappa_{0L}$ is the “low-pass” cut-off wavenumber; $\kappa_{0H}$ is the “high-pass” cut-off wavenumber.

The condition $\kappa_{0L} > \kappa_{0H}$ creates a band-pass filter that changes smoothly but rapidly from zero for $\kappa < \kappa_{0H}$ & $\kappa > \kappa_{0L}$ to unity for $\kappa > \kappa_{0H}$ & $\kappa < \kappa_{0L}$. The completeness and rapidity of this change is necessary to effectively suppress the long and very short wavelength components. In our calculation we have used the cut-off wavenumbers $\kappa_{0H} = 0.0314 km^{-1}$ and $\kappa_{0L} = 0.628 km^{-1}$, corresponding to cut-off wavelengths 200.137 km and 10 km, respectively in the spatial domain.

Analytical Fourier transform operations can determine gravity at height $z$ from a priori gravity data given on the plane projection of the reference ellipsoid. This involves the operator $e^{-\kappa z}$. Here, the height $z$ cannot be a function of the horizontal coordinates, so a different transformation is needed for each different height. To reduce the computation time, separate transforms compute gravity upward continued to height of 500 m, 1000 m, ..., 6000 m to provide values on planes
bracketing the height range of airborne survey of Ethiopia. A three-dimensional cubic interpolation then gives the value at each measurement point. Note that, the continuation with 250 m interval also gave the same result as the 500 m interval.

However, the problem is to deduce gravity on the plane \( z = 0 \) from observations at irregular heights not vice versa. The scheme used to solve this problem is iterative. A ‘first guess’ grid for gravity on the ellipsoid is upward continued to the observational data. The mismatch between the observations and the upward continued ‘first guess’ is added to the zero height data and the process is iterated until there is no significant improvement between the measured and predicted values at aircraft height (see, Figure 3.1). The ‘first guess’ was found by assuming all airborne data were observed at the same height. In reality, measurements are taken on or above the Earth’s surface and so practical geodetic calculations on subterranean reference ellipsoid or sphere necessitate the formulation of an iterative downward continuation algorithm. The iterative algorithm starts with an initial guess for \( \delta g (x, y, 0) \) that is modified iteratively to determine the total downward continuation effect and thence the downward continued gravity disturbance. The contribution of the downward continuation effect can be iteratively determined by:

\[
\delta g_{DC}^{(i+1)} = \delta g(x, y, z) - FT^{-1}\left[w(\kappa)e^{-\kappa z}FT\left\{\delta g(x, y, 0)_{(i)}\right\}\right]
\]  \hspace{1cm} (3.20)

This process usually starts with an initial value given by

\[
[\delta g(x, y, 0)]_{(0)} = \delta g(x, y, z)
\]  \hspace{1cm} (3.21)

The downward continuation of the actual measurements can be rigorously determined at an acceptable accuracy after a number of finite iterations.

\[
[\delta g(x, y, 0)]_{(i)} = \delta g(x, y, z) + \sum_{m=1}^{i} \delta g_{DC}^{(m)}
\]  \hspace{1cm} (3.22)

The second term in the right-hand side of equation (3.22) defines the cumulative downward continuation effect on gravity and it is given by

\[
\delta g_{DC}^{(i+1)} = \delta g(x, y, z) - FT^{-1}\left[w(\kappa)e^{-\kappa z}FT\left\{\sum_{m=0}^{i} \delta g(x, y, 0)_{(m)}\right\}\right]
\]  \hspace{1cm} (3.23)
In general, for all $i \geq 1$, $\delta g_{DC}^{(i+1)} < \delta g_{DC}^{(i)}$

Iterations continue until the differences between two consecutive values are sufficiently small otherwise the iterative process may add unnecessary short wavelength information causing instability in the continuation process. The stability and convergence of our algorithm has been controlled by comparing the contribution of the downward continuation effect for ten successive iterations. Figure 3.2a-b shows that the continuation algorithm converges rapidly after a few iteration.
Figure 3.1. Downward continuation scheme.

\[ \delta g(x, y, z) = \delta g_{\text{obs}}(x, y, z) - \delta g_{\text{EGM08}}(x, y, z) - \delta g_I(x, y, z) + \delta g_{RI}(x, y, z) \]

\[ \delta g(x, y, 0) \]

Current estimate of the residual gravity disturbance on the Lambert plane tangent to the reference ellipsoid

\[ \delta g(x, y, z) = FT^{-1} \left[ w(k)e^{-kz} FT\{\delta g(x, y, 0)\} \right] \]

Continuing residual gravity at different heights (0, 500, …, 6000) metres

3-D cubic interpolation

\[ \delta g^{i+1}(x, y, 0) = \delta g^{i}(x, y, 0) + \left[ \delta g(x, y, z) - FT^{-1} \left[ w(k)e^{-kz} FT\{\delta g^{i}(x, y, 0)\} \right] \right] \]

Downward continuation correction at zero height

Convergence ?

\[ i = i + 1 \]

No

Yes

Stop
Figure 3.2a-b. Convergence of the iterative FT algorithm for the downward continuation correction. a) Root Mean Square (RMS) mismatch between the residual gravity at aircraft height and the values predicted at heights for each iteration. b) RMS correction to the ‘first guess’ grid for gravity at zero height and the downward continuation corrected grid at zero height for each iteration.
3.5 Transforming Gravity to Potential

The classical approach to computing gravitational potential from surface gravity data involves two separate compute stages. The first compute stage performs downward continuation (e.g. using Poisson’s integral) and the second stage transforms gravity to potential using Stokes-like integral. With Fourier analysis the task of computing the gravitational potential from residual gravity disturbances irregularly distributed in position and height can be considered as a single compute stage. This involves downward continuation of the residual gravity and its transformation to potential on the reference ellipsoid. The Fourier Transform of the disturbing potential is simply found by dividing the Fourier Transform of the band-limited gravity disturbance by wavenumber $\kappa$.

$$\text{FT}[T(x,y,0)] = \frac{1}{\kappa} \text{FT}[\delta g(x,y,0)]$$  \hspace{1cm} (3.24)

The Fourier transform of residual gravity at zero height is the outcome of the downward continuation algorithm; dividing it by the wavenumber to get the potential is a trivial last step. The equivalent form of the disturbing potential in the spatial domain is directly computed by performing an inverse Fast Fourier Transform operation.

$$T(x,y,0) = \text{FT}^{-1} \left[ \frac{1}{\kappa} \text{FT} [\delta g(x,y,0)] \right]$$  \hspace{1cm} (3.25)

It is concluded that a planar Fourier Transform analysis on map projection coordinates is adequate in carrying out the equivalent of the Stokes-like integral transform on an ellipsoidal surface (see, Section 3.3). The transformation of gravity to potential can be carried out on any smooth surfaces (e.g. plane, sphere, ellipsoid), not necessarily a level surface used in a classical approach.

3.6 Improving Fourier Transform Analysis

Once trend-free residual gravity disturbance is produced at sufficient degree of smoothness and smallness, the stability of the continuation may be affected by the gridding and interpolation operations. But, these effects are unavoidable in the Fourier Transform analysis if a significant reduction in the computation time is
needed. Because, computing the Fourier transform of large dimension of irregularly distributed gravity disturbance involves the inversion of the full matrix of data, the irregular Discrete Fourier Transform (DFT) operation is limited to relatively small number of observations. Alternatively, if the data is first interpolated to a regular grid, then the Fourier transform can be efficiently used over large datasets. Moreover, our algorithm uses an extension of the Fourier transform based on the mixed-radix "Singleton" algorithm which works on a regularly spaced data in the form of linear dimension of array factorizable into an even power of any prime numbers to further increase the computational speed of the Fourier transform by fitting the observations more economically into the available data sizes.

The first stage of Fourier transform implementation involves the interpolation of irregularly spaced data into a regular grid which introduces non-zero estimates of gravity at points outside the measurement area. The transform of this grid data introduces spurious results due to the contribution of the interpolated gravity data residing in the region outside the measurement area. The application of Fourier Transform (FT) to irregularly shaped areas and rectangles whose dimensions are not consistent with the FT array size need to be padded out to produce an array of the required size. Because the mean value of real data should be zero, zero padding is commonly used, but the sharp transition or discontinuity in the data at the boundary of the irregularly shaped measurement region will also introduce spectral noise associated with high frequencies which may appear as edge effect in the space domain. An efficient padding approach has been carried out to minimise the edge effect by creating a smooth transition of the data across the boundary. The data outside the measurement area has been suppressed by a mask whose transition is smoothed with a 20 km Cosine Bell filter. This masking procedure will produce a smooth transition across the boundary thereby minimizing the marginal leakage from adjacent periodic reproductions of the data. Also the use of precise 3-D interpolation method is the key component to maintain the computational accuracy and speed of the FT algorithm to solving downward continuation and integration.
3.7 Computational Accuracy

3.7.1 Continuation effect

The downward continuation process enhances the local gravity anomaly particularly in areas having a large horizontal gravity gradient. Despite its theoretical importance, its contribution has been neglected in the classical computations. Many authors have investigated the contribution of the continuation effect to geoid height (Hipkin, 1988; Martinec, 1996; Tscherning & Forsberg, 1992). For instance, Hipkin (1988) reported 4.6 cm contribution to the Bouguer co-geoid height from the continuation effect computed over dissected topography in the Grampian Highlands of Scotland. Vanicek et al., (1996) has showed up to 100 cm contribution to the free air co-geoid height from the continuation effect computed over highly rugged Canadian Rocky Mountain. Conceptually, the effect varies depending on the geology and topography of the source mass. It might be significant in areas having larger complexity both in subsurface density distribution and topography. So, much attention has to be given to the downward continuation effect towards achieving centimetre accuracy of gravity derived height system.

A test experiment was carried out in order to quantify the downward continuation effect on both gravity and potential. The FT iterative algorithm was used to downward continue gravity on to the plane of a Lambert’s conic projected on the reference ellipsoid. The contribution of the continuation effect has been representatively shown for ten successive iterations (see, Figure 3.2a-b). The convergence of the continuation process was checked for each iteration process by upward continuing the current estimates of the downward continued gravity data back to sub-aircraft level otherwise, the iterative process will add unnecessary shorter wavelengths signal thereby destabilising the downward continuation process. The difference between the observed values and the values predicted at the observation points were determined with a standard deviation of 2.57 mgal after ten successive iterations (see, Table 3.2). This level of accuracy is sufficient for most geodetic applications because the prediction error is in the order of the accuracy of the measurements.
<table>
<thead>
<tr>
<th></th>
<th>Residual gravity disturbance (mgal)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Observed</td>
</tr>
<tr>
<td>Mean</td>
<td>1.217</td>
</tr>
<tr>
<td>Stdev.</td>
<td>5.20</td>
</tr>
<tr>
<td>Max</td>
<td>38.74</td>
</tr>
<tr>
<td>Min</td>
<td>-50.40</td>
</tr>
</tbody>
</table>

**Table 3.2.** Comparison between the actual observation and the value predicted at observation points using the iterative FFT algorithm.

In addition, the computational accuracy of the iterative FT downward continuation algorithm has been investigated by developing an independent closure test experiment. A band-limited gravity disturbance consisting of only high frequency components of the Earth Gravity Model EGM08 was computed on the surface of WGS84 and at 2000 m ellipsoidal height above it from a series of spherical harmonics coefficients using equation (1.4). This is a similar test to the one (see, Section 3.3) testing the FT algorithm itself.

The numerical test used different values of the cut-off parameter $n_0$ to synthesise band-limited gravity disturbances on the surface of the reference ellipsoid and at 2000 m ellipsoidal height above it on a grid of latitude and longitude with a spacing of 0.02° using the spherical harmonic series complete to degree and order 2180. These two band-limited gravity disturbances are projected on to a 2 km grid of eastings and northings, $\delta_{y_0}(x, y, 0)$ at $h = 0$ and $\delta_{y_0}(x, y, h)$ at $h = 2000$ m, using Lambert’s formulae for the conformal projection of an ellipsoid onto a cone. The resulting region approximately 3000 km square was extended with a 100% margin of zero-padding and the ultra-high frequency component of the EGM08 gravity disturbance computed at 2000 m ellipsoidal height was iteratively downward continued to the WGS84 ellipsoid.
\[ \delta g_h(x, y, h) \xrightarrow{FT} \delta g_{DC}(x, y, 0) \]  

\( \delta g_{DC}(x, y, 0) \) - is the downward continued gravity disturbance using Fourier Transform

The downward continued result was compared with the values directly computed on the same reference surface. The difference between the computed and predict values corresponds to the cumulative errors introduced by the continuation algorithm. Table 3.3 shows the statistics for the residuals between the gravity disturbances computed directly from the spherical harmonic synthesis and the downward continued gravity disturbance using the Fourier transform for different values of the cut-off degree \( n_0 \).

<table>
<thead>
<tr>
<th>Cut-off ( n_0 ) =</th>
<th>residual disturbance (mgal) at ( h = 2000 ) m</th>
<th>anomalous gravity misfit (mgal) on WGSR84 ellipsoid</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>300</td>
<td>200</td>
</tr>
<tr>
<td>Mean</td>
<td>-0.037</td>
<td>-0.006</td>
</tr>
<tr>
<td>Stdev.</td>
<td>17.95</td>
<td>15.0</td>
</tr>
<tr>
<td>Max</td>
<td>400.90</td>
<td>379.50</td>
</tr>
<tr>
<td>Min</td>
<td>-134.97</td>
<td>-133.70</td>
</tr>
</tbody>
</table>

**Table 3.3.** Statistics of the residuals between the band-limited gravity disturbances computed directly from EGM08 spherical harmonic coefficients over a 3000 km squared region centred on Ethiopia and the band-limited gravity disturbance downward continued from 2000 m ellipsoidal height using an iterative planar Fourier transform algorithm. The second columns give the statistics for the residual gravity disturbance computed at 2000 m ellipsoidal height.

It is clear that an iterative Fourier transform operating on map projection coordinates is entirely adequate in carrying out analytical continuation of free air gravity disturbances that are not corrected for topographic effects to an ellipsoidal surface at accuracies below 1 mgal. This level of computational accuracy is remarkably good as compared to the present day airborne gravimetric observational errors, (\( \sim 1 - 3 \)) mgal (Forsberg & Olesen, 2006).
3.8 Summary

The FT recursive algorithm has been applied to downward continue the gravity data using the flat-earth approximation. The FT analysis was carried out on the Lambert local cartesian coordinates. It predicts a regular grid of gravity disturbances onto the reference ellipsoid from residuals calculated at irregular measurement points. The iterative FT algorithm determines the vertical gradient of gravity from a priori ‘first guess’ grid for gravity on the reference ellipsoid and adds this back to the residuals at observation point to determine its estimates at zero-height. The FT analysis is very efficient in terms of computational speed and memory allocation compared to the space domain operations, example Poisson integral.

The convergence of the downward continuation algorithm has been tested using residual gravity disturbances acquired over the whole region of Ethiopia from the airborne survey. The algorithm converges rapidly after three iterations (Figure 3.2a-b). Besides, the accuracy of the downward continuation algorithm has been evaluated using a closure experiment based on the high frequency components of the EGM08 derived gravity disturbances. Band-limited gravity disturbances were calculated at 2000 m ellipsoidal height and analytically downward continued on to the reference ellipsoid using equation (3.22). The residuals between the continued and those directly obtained from spherical harmonic coefficients differ only with a mean value of 0.017 mgal and standard deviation of 0.85 mgal for 200 cut-off degree. For 300 degree cut-off, the mean is 0.015 mgal while the standard deviation is 0.8 mgal.

An iterative FT algorithm is computationally very efficient for continuing potential fields and it can also adequately transform gravity to potential but only restricted to the high frequency components. A rigorous gravity smoothing approach in conjunction with a precise 3-D interpolation algorithm is the basis for solving analytical continuation and Stokes-like integration as a single ‘compute’ stage using a planar Fourier transform.
Chapter 4

Stokes’ Integral and FFT

This chapter is based on my contribution to a paper submitted to Journal of Geodesy entitled: Stokes’ Integral and Fourier Transform

by
Roger Hipkin, Addisu Hunegnaw and Tulu Bedada

4.1 Introduction

The task of calculating precisely the gravitational potential at a point on or above the Earth’s surface from surface gravity data is one of the challenging problems in physical geodesy. Now, due to the availability of a high resolution and accurate global geopotential model (EGM08, Pavlis et al., 2008), all but the shortest wavelength components ($\lambda \leq 18 \text{ km}$) of the gravitational potential can be directly computed from the spherical harmonic coefficients on or above the Earth’s surface in a truly representative way. However, its resolution is not sufficient for precise geodetic applications at regional scale unless it is combined with local gravity and the global model on its own is only accurate enough for wavelength greater than about 400 km. The computation of high resolution and precise gravitational potential requires the Stokes-like integral transform of residual local gravity data to potential.

Usually, we measure the magnitude of gravity vector but this is very well approximated by the derivative of the gravity potential in the direction of the normal to an ellipsoid approximating the Earth’s shape. This approximation makes the transformation problem linear so that both gravity and the kernel can be separated into parts and each component integrated separately. In practice, models for some parts of the gravity field (e.g. reference ellipsoid, global gravity field and SRTM) are subtracted from observations and Stokes’ integral is applied only to small and smoothly varying residuals so that the interpolation involved is reliable. Later, the removed models can be exactly restored as potential without introducing any
conceptual error. However, a problem arises due to the fact that a dense enough coverage of precise gravity data are only available for some parts of the globe whereas the Stokes-like integrals relate the anomalous geopotential $T$ at point $D$ (the origin) to a global integral of gravity anomalies, $\Delta g$, or gravity disturbances $\delta g$. They are related as:

$$T_D = \frac{R}{4\pi} \int_{\psi = 0}^{\pi} \left( \int_{\alpha = 0}^{2\pi} \Delta g(\psi, \alpha) d\alpha \right) S(\psi) \sin \psi d\psi$$

or

$$T_D = \frac{R}{4\pi} \int_{\psi = 0}^{\pi} \left( \int_{\alpha = 0}^{2\pi} \delta g(\psi, \alpha) d\alpha \right) K(\psi) \sin \psi d\psi$$

$S(\psi)$ is the Stokes’ kernel and $K(\psi)$ is Hotine-Koch kernel.

The term in the brackets, to be denoted by $\Delta g$ or $\delta g$, defines the average value of the gravity anomaly or gravity disturbance around an annulus whose radius subtends the geocentric angle $\psi$.

The Stokes’ integral transform involves an integral of gravity anomalies over the whole globe, $0 \leq \psi \leq \pi$, whereas, in practice, detailed data are only available for a local region around the computation point, exemplified by a spherical cap $0 \leq \psi \leq \psi_0$. At present, most geopotential computation methods suppose that Stokes-like integral can be legitimately restricted to a local region. This solution was examined by Wong & Gore (1969), who sought a solution via kernel modification. Their approach has been developed further by many authors since (e.g. Featherstone et al., 1998; Jekeli, 1980; Meissl, 1971; Sjöberg, 1984; Vaníček & Kleusberg, 1987; Vaníček & Sjöberg, 1991). A key part of this thesis explores an alternative approach: we justify limiting the integral to a spherical cap from stochastic properties of the local gravity field, rather than any property of the kernel.

Section 4.2 explores how to compute gravitational potential on or above the Earth’s surface from surface gravity data using the Stokes-like integral. Section 4.3 presents how to localise Stokes-like global integral needed to transform gravity to
potential in using a deterministic approach and section 4.4 revised the properties of Stokes-like kernels. Although the deterministic approach of kernel modification is the most widely used technique, section 4.5 deals with a stochastic truncation – the localisation must be justified by the properties of the Earth’s gravity field, kernel modification is not needed. To do so, it develops a stochastic model of the global gravity field and shows that it is successful in predicting the behaviour of the gravity residuals in a local region. The stochastic prediction treats the gravity field as a random stationary variable, but one characterised by globally consistent correlation lengths. Section 4.6 shows that a stochastic model of the ring-averaged gravity, \( \Delta g \) or \( g_{\delta} \), becomes negligibly small when the ring radius reaches a few degrees. This stochastic model verified that the Stokes’ integral can be legitimately localised irrespective of the properties of the kernel. There are no short wavelengths errors of omission in the global gravity field when localising the Stokes’ integral. Section 4.7 shows how to compute the Fourier Transforms of geodetic kernels over finite spherical cap and justified the success of FFT to simplifying Stokes-like integral transform.

### 4.2 Stokes-like Integral Transform

The evaluation of the Stokes-like integral relates the disturbing potential at the computation point to a global integral of gravity anomalies \( \Delta g \) or gravity disturbance \( \delta g \). The transformation formulae have been shown in equations (4.1) and (4.2). Stokes’ integral involves the regularisation of a global topographic model and an integral transform of the residual gravity anomalies over the whole globe, \( 0 \leq \psi \leq \pi \), to compute the anomalous geopotential. In practice, however, detailed data are only available for a local region around the computation point, exemplified by a spherical cap \( 0 \leq \psi \leq \psi_0 \). The original Stokes’ approach starts with a spherical Earth model and subtracts in situ topographic mass, adding it back as a coating on the surface of zero height. The gravity field above this surface then satisfies Laplace’s condition. Recent advances in satellite geodesy have determined gravity sufficiently and accurately for elliptical corrections to the Stokes’ spherical approximation to be needed (Fei & Sideris, 2000; Huang et al., 2003; Sjöberg,
(2003b). Hipkin (2004) has derived an ellipsoidal geoid solution for spherical harmonics. Martinec & Grafarend (1997) have derived solutions in ellipsoidal boundary-value problem. With the use of global gravity model, some authors have divided the Stokes’ global integral into two parts: the near zone and the far zone (Heiskanen & Moritz, 1967, p. 121). For a particular computation point, the near zone integration uses the local gravity data within a spherical cap radius of $0 \leq \psi \leq \psi_0$ while the far zone integration uses the global gravity model. The evaluation of the Stokes’ integral on a global scale is not reliable due to contamination of the resulting potential solutions caused by the cumulative errors of the medium wavelengths of gravity anomalies from the far zone area. The integration is not simple due to large magnitudes of the gravity anomalies and also it requires much computation time. Compared with the integration of the whole gravity anomaly or disturbance, the integral of band-limited residual gravity disturbances provides more reliable solutions.

The modern ‘Remove-Compute-Restore’ technique computes the anomalous potential or residual geoid height rather than computing the whole components. This approach, in which observed gravity anomalies were transformed to small residuals by subtracting the global model EGM96 and the difference in the gravity effect of in situ and condensed topography, and these residuals further reduced by tailoring them to a spherical harmonic model, has been successfully implemented by Hipkin et al., (2004), who has computed the EDIN2000 geoid for the British Isles. The error due to truncating Stokes’ kernel by planar FFT was considered negligible but is examined further here. According to this modern approach the Stokes-like integral operates on the residual gravity anomalies. The total potential field is then obtained from direct evaluation of the low degree spherical harmonic coefficients of the global gravity model plus contributions of the residuals from the Stokes’ integral. This is written as:

$$T_P = T_{\text{EGM96}} + \frac{R}{4\pi} \left( \int_{\psi_0}^{\psi} \left( \int_{\alpha=0}^{2\pi} \Delta g (\psi, \alpha) d\alpha \right) S(\psi) \sin \psi \, d\psi \right)$$

$$(4.3)$$

$$+ \frac{R}{4\pi} \left( \int_{\psi_0}^{\psi} \left( \int_{\alpha=0}^{2\pi} \Delta g (\psi, \alpha) d\alpha \right) S(\psi) \sin \psi \, d\psi \right)$$
Where the first term in equation (4.3) represents the contribution from the long wavelength components of the global geopotential model complete to spherical harmonic degree and order \( n \), the other two integrals are contributions from the residual gravity anomalies.

Practical, evaluation of equation (4.3) provides reliable solutions as compared with the original Stokes integral which deals with large magnitude of gravity anomalies. But, the solution is biased by the truncation error (the third term in Eq. 4.3), if the integration over the residual gravity anomalies is not properly handled. Much active research has been conducted to seek an equivalent solution to the Stokes’ global integral by truncating it to a limited size of spherical cap around the computation point in a way that the contribution from third term is essentially negligible. The next sections discuss the deterministic and stochastic approaches of localising the Stokes-like global integral into a limited spherical cap.

### 4.3 Deterministic Approach

The truncation of the Stokes’ global integral into a limited spherical cap radius has been practised by modifying the Stokes’ kernel via deterministic approach. The strategy of the deterministic approach is to reduce the effect of the far-zone gravity field while truncating the Stokes-like global integral to a limited spherical cap. This issue was examined by Wong & Gore (1969) to deal with the transformation of the residual from which the global gravity model has been removed. They modified the Stokes’ kernel by removing the low-degree terms of the Legendre polynomials (\( 2 \leq n \leq p \)) from the original kernel.

\[
S_{WG}(\psi) = S(\psi) - \sum_{n=2}^{p} \frac{2n+1}{n-1} P_n(\cos \psi)
\]

(4.4)

For \( 0 \leq \psi \leq \pi \)

where, \( p \) corresponds to degree of modification.

They defined the error kernel by

\[
\bar{S}_{WG}(\psi) = \begin{cases} 
0 & \text{for } 0 \leq \psi \leq \psi_0 \\
S_{WG} & \text{for } \psi_0 \leq \psi \leq \pi
\end{cases}
\]

(4.5)
This error kernel is used to evaluate the third term of equation (4.3). The modified kernel decreases as $\psi$ approaches $\psi_0$ from outside and it converges to zero at $\psi = \psi_0$, so that, it could minimise the distortion in the long wavelength contribution of the remote zone.

Following their work, many authors have conducted active research in how best to modify the Stokes’ kernel (e.g. Featherstone et al., 1998; Jekeli, 1980; Meissl, 1971; Sjöberg, 1984; Vaníček & Kleusberg, 1987; Vaníček & Sjöberg, 1991). In general, all kernel modification approaches are related to each other by making some changes, the original being the Wong & Gore (1969). In practice, the shape of the Stokes’ kernel is altered so as to reduce the contribution of the residual gravity in the truncated region, $\psi_0 < \psi \leq \pi$ (called truncation error) to the solution of the Stokes’ integral evaluated within the spherical cap of radius $\psi_0$. The key reasoning for modifying the Stokes’ kernel is to avoid the distortion in the long wavelength components of the gravity field when the integration is performed over a limited spherical cap. Distortion in the long wavelengths only occurs due to breaking down of the orthogonality of the low degree spherical harmonic coefficients which are purely determined from perturbations of the Earth’s gravity field. In principle, the Stokes’ kernel is just a weighting function; it is gravity field that varies randomly over the surface of the Earth. Do the un-modified Stokes kernel properties suggest essential implications of truncating the Stokes global integral into a local spherical capsize?

### 4.4 Kernel Properties

Visualising the properties of the kernel on the surface of the plane Cartesian coordinate relating the physical distance on the Earth’s surface to the geocentric angle is very important to understand the level of approximation that can be made to the Stokes’ integral, for example using Fourier Transforms. The argument of Stokes kernel depends on the geocentric angle $\psi$. A new variable,

$$\xi = \sin \frac{\psi}{2} = \frac{\rho}{2R}$$

(4.6)
is convenient because $\rho$ is the length of the chord joining the two surface points. This distance property will be used in section 4.7 for a solution using Fourier Transforms. Here, the analysis uses the variable $\xi$. Following the method originally implemented by Lambert & Darling (1936) and later developed by Kearsley (1986), the gravity anomaly is averaged over the azimuth $\alpha$ in terms of the new variable, $\xi$, such that the Stokes and Hotine-Koch integral take the new form.

$$ T = 2R \int_{\xi=0}^{1} \overline{\Delta g}(\xi) F^*(\xi) \, d\xi $$

$$ T = 2R \int_{\xi=0}^{1} \overline{\Delta g}(\xi) H^*(\xi) \, d\xi $$

The modified Stokes-Lambert and Hotine-Koch kernels are

$$ F^*(\xi) = 1 - 4\xi - 6\xi^2 + 10\xi^3 - 3(1 - 2\xi^2)\xi \ln(1 + \xi) - 3(1 - 2\xi^2)\xi \ln(\xi) $$

$$ H^*(\xi) = 1 - \xi \ln(1 + \xi) + \xi \ln(\xi) $$

Figure 4.1 shows the shape of the kernels $F^*(\xi)$ and $H^*(\xi)$ for the range $0 \leq \xi \leq 1$, but, for easy of comparison the axis shows $0 \leq \psi \leq 180^\circ$.

**Figure 4.1.** The modified Lambert and Hotine-Koch kernel functions, $F^*$ and $H^*$
Neither kernel is strongly peaked around the origin so neither gives any support to limiting the range of integration to a local region. In fact, \( F \) reaches its maximum of 3.08 at the antipodes, not at the computation point. There is a local maximum of 1.2337 at \( \psi = 7.46^\circ \) and a minimum of -1.4131 at \( \psi = 79.3^\circ \).

Note that neither kernel is an analytical function, so cannot be approximated by a Taylor series. The last logarithmic term in equations (4.9) and (4.10) has an infinite first derivative. Lambert’s kernel has a downward pointing cusp at the origin while the cusp for Hotine-Koch kernel point upwards. What difficulty there is with the Stokes-like kernels comes from the \( \xi \ln(\xi) \) term, not their leading term \( 1/\xi \).

Therefore, if the Stokes’ integral is to be legitimately truncated to a spherical cap, this has to be justified by the property of the gravity field \( \Delta g(\psi) \), not by the kernel. This study explores an alternative approach of limiting the integral to a spherical cap using the stochastic properties of the local gravity field.

### 4.5 Stochastic Approach

The evaluation of the Stokes integral uses the Stokes’ kernel as a weighting function for the gravity anomalies to determine their contribution at the computation point. Here it is supposed that gravity anomalies vary randomly over the Earth’s surface. Different parts of the Earth have different geological mass distributions and varying density and topography thereby producing different wavelengths of gravity anomalies. This section examines the consequences of treating these variations as random. If Stokes’ integral is to be legitimately truncated to a spherical cap, the gravity anomalies need to be correctly isolated at different wavelengths so that there are no far field omission errors by truncating the integral to a spherical cap. This study shows that, the truncation of the Stokes’ integral to a spherical cap can be justified by a property of the gravity field \( \Delta g(\psi) \), rather than the kernel.

Several processing methods suppose that gravity anomalies can be represented by a function varying ‘randomly’ over the Earth’s surface but with a finite correlation length. In this case, the average product of gravity at two points a distance \( \rho \) apart will become small if \( \rho \) is much larger than the correlation length.
and it will tend to a locally representative value if the distance is much smaller than the correlation length.

A stochastic gravity model provides a rational way to estimate what size of spherical cap is required in an approximation that integrates over a local region rather than the whole globe: if the circumference of the annulus involved in the integral over azimuth is much larger than the correlation length of the gravity anomalies, $\Delta g(\xi)$ will tend to zero and the contribution of Stokes’ integral from all annuli with larger $\xi$ may be neglected. Except for the requirement that the kernel function is reasonably smooth and does not itself tend to zero or large values as distance increases, the appropriate size of the cap will be independent of the kernel function.

Hipkin (2001) used a ‘pink noise’ model to show that the power spectrum of the global model EGM96 represented 5 independent components, each corresponding to a random ‘white-noise’ source distribution over an internal geocentric sphere with a different radius. This section uses the pink noise model to make a stochastic prediction to estimate the correlation length of the different components now derived from EGM08. Their correlation only will determine which component of the Earth’s gravity field can be used in Stokes’ integral if it is to be restricted legitimately to a local cap.

**4.5.1 The Gravity Power Spectrum and Autocorrelation Function**

A model of a covariance function can be used to characterise the properties of the gravity at both local and global scales. Many researchers have developed different models of the covariance function to determine the spatial correlation of gravity anomalies by fitting their model to spectral variance of the Earth gravity field represented by a series of spherical harmonics (Heiskanen & Moritz, 1967; Hipkin, 2001; Tscherning & Rapp, 1974).

Hipkin (2001) discussed the statistics of spatially white noise distributed over the surface of an internal sphere with radius $r_B$, which was then attenuated by upward continuation to the surface with radius $a$. The internal white noise surface is analogous to the Bjerhammer sphere. Hipkin (2001) has also corrected the definition
of the power spectrum expressed in terms of spherical harmonic coefficients and showed how it is related to the autocorrelation function.

\[ C_{gg}(\psi) = \sum_n (2n + 1) P_n \, P_n(\cos\psi) \]  

(4.11)

According to (Hipkin, 2001) definition, the power spectrum \( P_n \) of a global gravity model described by spherical harmonics whose fully normalised dimensionless coefficients are \( \{ C^m_n, S^m_n \} \) is given by

\[ P_n = \left( \frac{GM}{a^2} \right)^2 (n+1) \left\{ \frac{C^0_n^2}{n^2} + \sum_{m=1}^n \left[ \frac{C^m_n^2 + S^m_n^2}{n^2} \right] \right\} \]  

(4.12)

Note that equation (4.12) is neither the ‘degree variance’, for which some geodesist use the term gravity spectrum, nor the Lowes-Mauersburger ‘power spectrum’ used in geomagnetism (Hipkin, 2001).

If the power spectrum describes a white noise source attenuated by upward continuation, it will have the following form:

\[ P_n = A \, \beta^n ; \quad \left( \beta = \left( \frac{r_B}{a} \right)^2 < 1 \right) \]  

(4.13)

A ‘white noise’ signal is one for which the average power is the same for every degree of freedom. With the correct definition of the power spectrum used here, the gravity field can be described by ‘white noise’ signal defined on the inner Bjerhammer sphere of radius \( r_B \), over which the power spectrum averages to a constant for every independent spherical harmonic. In practice, the power spectrum of the global gravity model can be computed on a sphere of radius \( a \) so that the depth of its white noise corresponds to downward continuing the power spectrum until it no longer decreases as degree \( n \) increases. A plot of the power spectrum against spherical harmonic degree \( n \) defines a straight line whose slope and intercept are \( \ln \beta \) and \( \ln A \) respectively. In this way the power spectrum is used to determine the covariance model parameter \( \beta \) from which the angular correlation length \( \xi_{1/2} \) or its equivalent geometrical map projection distance \( \rho_{1/2} \) can be determined.
For rotationally invariant or isotropic statistics, the autocorrelation function \( \phi_{\rho \rho} \) is the same as the covariance function \( C_{\rho \rho} \) and has the form (see, Hipkin, 2001)

\[
C_{\rho \rho}(\psi) \equiv \phi_{\rho \rho}(\psi) \equiv \phi_{\rho \rho}(\xi) = \sigma^2 \left[ 1 + \frac{4\beta}{(1-\beta)^2} \xi^2 \right]^{3/2}
\] (4.14)

The angular correlation length \( \xi_{1/2} \) is then easily determined from the power spectrum (Hipkin, 2001): \( \xi_{1/2} \) is defined here as the argument for which the autocorrelation function drops to half its peak value:

\[
\xi_{1/2} = \sin^{-1} \left( \frac{\sqrt{2/3} - 1}{2\sqrt{\beta}} (1-\beta) \right)
\] (4.15)

\( \rho_{1/2} \) is the equivalent quantity given as a map projection distance.

To model uncorrelated random sources distributed over several spheres, Hipkin (2001) fitted the power spectrum of the global gravity model EGM96 to a series of independent pink noise terms.

\[
\ln P_n = \ln \sum_i A_i \beta_i^n
\] (4.16)

By varying the number of terms as well as the parameters \( \{A_i, \beta_i\} \) a non-linear least-squares algorithm found a minimum of \( \chi^2 \) as a way of estimating the optimal model for EGM96 (Lemoine et al., 1998): the result had 4 pink noise terms with a very small components (~ 14 \( \mu \)Gal) of random surface noise. The analysis was repeated with EGM08 (Pavlis et al., 2008), for which two further pink noise terms and no surface noise were needed.

The stochastic properties of the Earth’s gravity field are given in Table 4.1 for the two global gravity models. The analysis of the two Earth gravity models gives remarkably similar results. For both, the first three components have such long correlation lengths, \( \rho_{1/2} \), that a ring-averaged gravity for an annulus within a local cap will not become small. At present, these first three components of the Earth’s gravity are determined form satellite orbital perturbations and they have to be effectively removed from observations in order to localise the Stokes integral to a small part of the globe. In contrast, components (v) and (vi) have small correlation
lengths and so are appropriate for a computation within a small spherical cap. Note that, the solution with components (iv) with EGM96 and EGM08 are very close, with correlation lengths of 143 km and 154 km and zero-lag amplitudes of 345 mGal$^2$ and 320 mGal$^2$.

Section 4.5 discussed how to determine the spatial and angular correlation lengths of different components of the Earth gravity field. Now, what we need to solve the Stokes’ integral is to know the relative contribution of gravity anomalies as a function of their distance relative to the computation point.
Table 4.1. Parameters of the covariance functions derived from EGM96 and EGM08 estimating spatial correlation length for different components of global gravity model.

<table>
<thead>
<tr>
<th>Gravity dominating Component degree range</th>
<th>lnA</th>
<th>lnB</th>
<th>( C_{\text{gg}}(0) ) (mgal)(^2)</th>
<th>( \rho_{1/2} ) (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) 2 ( \leq n \leq 4 )</td>
<td>-19.9198</td>
<td>-19.9487</td>
<td>-1.8024</td>
<td>-1.7352</td>
</tr>
<tr>
<td>(ii) 4 ( \leq n \leq 12 )</td>
<td>-23.3153</td>
<td>-23.4237</td>
<td>-0.4450</td>
<td>-0.4260</td>
</tr>
<tr>
<td>(iii) 14 ( \leq n \leq 50 )</td>
<td>-28.3397</td>
<td>-28.9644</td>
<td>-0.10089</td>
<td>-0.07322</td>
</tr>
<tr>
<td>(iv) 70 ( \leq n \leq 300 )</td>
<td>-19.5522</td>
<td>-29.6583</td>
<td>-0.01544</td>
<td>-0.01575</td>
</tr>
<tr>
<td>(v) 500 ( \leq n \leq 1000 )</td>
<td>-32.8611</td>
<td>-0.00531</td>
<td>191</td>
<td>191</td>
</tr>
<tr>
<td>(vi) 1500 ( \leq n \leq 2000 )</td>
<td>-36.8886</td>
<td>-0.00156</td>
<td>87</td>
<td>87</td>
</tr>
</tbody>
</table>
4.6 Stochastic Gravity Model and Stokes’ Integral

4.6.1 The expected value of ring-averaged gravity $\Delta g(\psi)$

The relative contributions of the gravity anomalies to the Stokes’ integral can be estimated from the knowledge of the covariance function. It is assumed that on average, a gravity anomaly $\Delta g(\xi)$ at angular distance $\xi$ away from the computation point (origin) is linearly predictable from the gravity anomaly at the origin $\Delta g(0)$.

$$\Delta g(\xi) = Z(\xi)\Delta g(0) \quad (4.17)$$

Multiplying through by $\Delta g(0)$ and averaging gives

$$Z(\xi) = \frac{\Delta g(\xi)\Delta g(0)}{\Delta g(0)^2} = \frac{\phi_{gg}(\xi)}{\sigma^2} = \frac{1}{[1 + a^2\xi^2]^{3/2}} \quad (4.18)$$

$$a^2 = \frac{4\beta}{(1 - \beta)^2} \quad (4.19)$$

For residual gravity anomalies distributed around a ring whose angular radius is $\xi$, the expected value of ring averaged gravity is evaluated by first choosing gravity at two different points on the ring, separated by the azimuth $\alpha$. Integrating with respect to azimuth gives the expected value of the ring averaged with respect to an arbitrary point on the ring. Then apply the formula again to relate the expected value at one point on the ring to that at the origin. The expected value of the ring averaged gravity is given by the following formula.

$$\frac{\Delta g(\xi)}{\Delta g(0)} = \frac{1}{[1 + a^2\xi^2]^{3/2} [1 + 4a^2[1 - \xi^2/\xi_0^2]]^{3/2}} \quad (4.20)$$

A derivation Eq. (4.20) is given in Hipkin et al., (2010)

Now Stokes’ integral can be evaluated numerically over a spherical cap radius of $\xi_0$ by multiplying the expected value of the ring-averaged gravity by the kernel.
4.6.2 Different sizes of spherical cap

The contribution to the geopotential or geoid from the Stokes’ integral evaluated over different cap sizes relative to the contribution from the global integral can determine how large the cap must be for each components of the gravity field (i) to (v). This is done by evaluating equation (4.21).

\[
\Delta N(\xi) = \frac{R}{\gamma} \int_{0}^{\xi} \frac{\Delta g(\xi)}{\Delta g(\xi)} F^*(\xi) d\xi
\]  

(4.21)

Within the integrand, the expected value of ring-averaged gravity is given by equation (4.20) and the kernel functions by equations (4.9) or (4.10); both are now functions of \(\xi\) described by closed formulae. Figure 4.2 shows the effect of dealing separately with each of the sources (i) to (v), evaluated for a unit kernel and for the Lambert and Hotine-Koch kernels. The left-hand column of Figure 4.2 shows how gravity at different distances contributes to the potential computed at the origin. It displays the effect of multiplying ring-averaged gravity by the Lambert or Hotine-Koch kernels, for each of the gravity components (i) to (v). It also shows the result of approximating the kernels by unity.

The right-hand column of Figure 4.2 shows that the difference between the ‘Stokes’ integral’ with different kernels, including the unit kernel, decreases from component (i) to (v). It is barely distinguishable for the latter. Both columns show that the response of components (i) and (ii) with a Lambert kernel is qualitatively different from components (iii) and (iv) and from all components with the Hotine-Koch kernel. For all of the latter, the integral approaches the result for a global integral via a monotonically asymptotic route. For these cases, Table 4.2 gives the cap sizes – in terms of the geocentric angle \(\psi\) – at which the integrals (equation 4.21) reaches 99% of their global value. A localised integral with the Lambert kernel for components (i) and (ii) would generate gross errors, even giving an effect with the wrong sign for component (i). Noting that the region where the geopotential or geoid is to be computed should be surrounded by a marginal zone of data whose width needs to be equal to the cap radius, only components (iv) and (v) are suitable for localised integration. For successful geopotential computation over a local cap, the primary task must be to eliminate components (i), (ii) and (iii) from the gravity
field, so that integration is restricted to components (iv) and (v). This understanding

gives new insight into the need to remove a global gravity model. However, if this
separation can be achieved, Table 4.2 shows that a cap whose radius is less than 2°
will capture 99% of the contribution of a global integral for components (iv) and (v).
For these components, there is no further ‘omission error’ from gravity data outside
the cap.

<table>
<thead>
<tr>
<th>Component</th>
<th>Radius $\psi_0$ for 99% of Lambert</th>
<th>Radius $\psi_0$ for 99% of Hotine-Koch</th>
<th>Contribution from unit kernel WrLambert</th>
<th>Contribution from unit kernel WrLambert</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>n/a</td>
<td>160.4</td>
<td>-175%</td>
<td>172%</td>
</tr>
<tr>
<td>(ii)</td>
<td>n/a</td>
<td>90.6</td>
<td>566%</td>
<td>124%</td>
</tr>
<tr>
<td>(iii)</td>
<td>5.61</td>
<td>10.4</td>
<td>135%</td>
<td>106.4%</td>
</tr>
<tr>
<td>(iv)</td>
<td>1.53</td>
<td>1.66</td>
<td>105.7%</td>
<td>101.31%</td>
</tr>
<tr>
<td>(v)</td>
<td>0.505</td>
<td>0.521</td>
<td>101.87%</td>
<td>100.16%</td>
</tr>
</tbody>
</table>

Table 4.2. Cap size needed to get a given precision for the difference gravity field
components deduced from EGM96 characterised by their different correlation
lengths (See, Table 4.1). The last two columns show the result of replacing the kernel
by unity, normalised to the result for the full kernel integrated over the whole globe
(Hipkin, 2008).
Figure 4.2. Left column: normalised contribution of gravity at different distances $\psi^2$ from the computation point; right column: ‘Stokes’ integral over different cap sizes, relative to the global integral; also showing the effect of replacing the Lambert or Hotine-Koch kernel by unity.

Rows show the result for components (i) (top) to (v) (bottom), Hipkin (2008).
Section 4.6.2 looked at how much of the global contribution to Stokes’ integral was recovered for different cap sizes. This section uses the same approach to assess the relative contribution to the result from different terms in the kernel; in particular it looks how well $F^*(\xi)$ and $H^*(\xi)$ are approximated by their leading term, unity. This corresponds to the term $1/\sin(\psi/2)$ in Stokes kernel. The right-hand column of Figure 4.2 shows the effect of replacing $F^*(\xi)$ and $H^*(\xi)$ by 1, and also shows the systematic error in doing so. Based on the stochastic model for component (iv), a unit kernel overestimates the true result for a Lambert kernel by 5.7% and 1.3% for a Hotine-Koch kernel. In many cases residual gravity is already so small that this error would be negligible.

The next section looks at a deterministic rather than a stochastic approach to the problem of Stokes integral.

### 4.7 The Fourier transform of geodetic kernels computed over a finite region

Because the computation point is always chosen to be the origin, the fact Stokes integral is a two-dimensional convolution integral is less evident. Classically, the kernel argument is the geocentric angle between its gravity field point and computation point $\psi$, which can be transformed into the variable $\xi = \sin(\psi/2)$ or into the chord length $\rho = 2R\xi$. If we choose a special map projection in which radial distance on the projected plane is equal to the chord length between corresponding field and computation points on the sphere Stokes integral maps exactly to a plane cartesian. Eq. (4.2) then becomes

$$T(0,0) = \frac{1}{4\pi} \int_{\rho=0}^{R} \int_{\alpha=0}^{2\pi} \delta_k(\rho, \alpha) K(\rho) \, d\rho \, d\alpha \quad (4.22)$$

Transforming from polar to cartesian coordinates and denoting the computation point by $(x', y')$ gives
\[ T(x', y') = \frac{1}{4\pi} \iint_{xy} \delta(x, y) K(\rho(x-x', y-y')) \frac{dx dy}{\rho(x-x', y-y')} \]
\[ = \frac{1}{4\pi} \iint_{xy} \delta(x, y) H^{*}(x-x', y-y') dx dy \] (4.23)

This is now more obviously a convolution integral and the theorem that space-domain convolution transforms to multiplication in the Fourier domain gives
\[ FT [t] = \frac{1}{4\pi} FT [\delta] FT [H^{*}] \] (4.24)

Equivalently
\[ FT [t] = \frac{1}{4\pi} FT [\Delta g] FT [F^{*}] \] (4.25)

Fourier transform have become widely used for geoid computation but most popularly the 1-D FFT of Haagmans et al., (1993) or variants of planar FFT’s reviewed, for example by Schwarz et al., (1990). Like the earlier part of this chapter, this section looks at the effect of restricting the Fourier transform to a disk on the map-projection plane, corresponding to an integral over a spherical cap, as well as the effects of replacing the kernels by their leading term, unity. Both tasks need to compute the Fourier Transform of the kernel.

When dealing with the Fourier transform of the kernel, there are some simplifications: first, the function is described by a closed formula and so continuously available and, secondly, kernel functions are rotationally symmetrical about the origin. Note that, by using Lambert’s azimuthal projection of a sphere onto a tangent plane, the radial distances in the projection is exactly given by the chord distance \( \rho \). For this special map projection, the algebraic form of the kernel as a function of the distance \( \rho \) in the projection has already included all curvature effects and the relation between potential and its radial derivative: there is no ‘spherical to planar’ approximation. Lambert’s azimuthal tangential projection is not one in common use nor is it the one used for subsequent practical computation. It is used here because its use allows the Fourier transform of kernel functions to be evaluated without approximation. This section uses it to compare a numerically computed Fourier transform of the kernel over a spherical cap with analytical integration over an infinite plane. Later it is argued that the conclusion will remain valid for any map projection with small distortion in the region of integration.
For a function that is rotationally symmetrical about the origin, the integral defining the Fourier transform in plane Cartesian coordinates \( \{x, y\} \) can be transformed to one involving plane polar coordinates \( \{\rho, \alpha\} \), where \( \alpha \) is the azimuth and \( \rho \) is the radial distance (\( \rho = 2R \xi \)):

\[
\begin{align*}
FT[f(x, y)] &= FT[f(\rho, \alpha)] \\
&= F(k, \varphi) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \exp[j(k_x x + k_y y)] dx dy \\
&= \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{\infty} f(\rho, \alpha) \exp[j(k \rho \cos(\varphi - \alpha))] \rho d\rho d\alpha
\end{align*}
\]

(4.26)

For the full Stokes-Lambert kernel, Eq. (4.9)

\[ f_{\text{Lambert}}(\rho, \alpha) = \frac{F(\frac{\rho}{2R})}{\rho} \]

(4.27)

and for the Hotine-Koch kernel, Eq. (4.10)

\[ f_{\text{Hotine}}(\rho, \alpha) = \frac{H(\frac{\rho}{2R})}{\rho} \]

(4.28)

but no analytical solution to equation (4.26) is available for an infinite plane unless we make the approximation \( f = 1 \). Otherwise only numerical evaluation is available.

Before abandoning an analytical approach, we explore the effect of replacing the infinite plane by a spherical cap using the ‘Newton’ kernel, corresponding to the leading term in the other two cases

\[ f(\rho, \alpha) = \frac{1}{\rho} \]

(4.29)

The Fourier transform over an infinite plane is

\[ F(k, \varphi) = \int_{0}^{2\pi} \int_{0}^{\infty} \exp[j(k \rho \cos(\varphi - \alpha))] \rho d\rho d\alpha = \int_{0}^{\infty} J_0(k \rho) \rho d\rho = \frac{1}{k} \]

(4.30)

Equations (4.26) – (4.30) have expressed the wavenumber \( \{k_x, k_y\} \) in radial and azimuthal polar coordinates \( \{k, \varphi\} \).

The Fourier transform over all or part of the curved surface of the Earth can be seen by replacing the infinite limit by \( \rho_{\text{max}} \leq 2R \). The analytical result for the infinite integral (Eq. 4.30) then has a correction term involving products of Struve functions.
and Bessel function of degrees 0 and 1 (Gradshteyn & Ryzhik, 1980, sect. 6.551.8). However, the spherical cap integral is best evaluated numerically and then illustrated by plotting

\[ k F(k; \rho_{\text{max}}) = k \int_0^\rho_{\text{max}} J_0(k \rho) \, d\rho \]  

(4.31)

whose value is unity when \( \rho_{\text{max}} \to \infty \).

Figure 4.3a shows the result for a cap radius of 100 km. The Fourier transform of the kernel oscillates about unity, the result for an infinite plane. The period and the amplitude of the oscillation decrease for larger discs and the amplitude decreases for larger wavenumbers. The main discrepancy comes from integration over a cap of finite size, with the oscillation only dropping below 20% for wavelengths smaller than 200 km.

In contrast, the differences between a unit kernel and the Stokes-Lambert and Hotine-Koch kernels (see, Fig. 4.3b) are much smaller, decreasing from 3.5% and 2% respectively at this wavelength. This discussion has used a special map projection in which chord length on the sphere maps to radial distance on the projected plane. For a region near the centre of any practically useful map projection, scale and shape distortion will be small. We therefore infer that the results deduced for Lamberts azimuthal projection would also apply with adequate precision for most other projections used for national cartography. The numerical verification of this was described in section 3.3.
Figure 4.3. The Fourier transform of kernel functions evaluated over a spherical cap of radius 400 km. Each has been multiplied by the wavenumber, so that the result for integrating over an infinite plane would be unity. The maximum of the wavenumber ordinate corresponds to a wavelength of 10.5 km. (Hipkin, 2008)

(a) the general behaviour – small oscillations about the infinite plane solution.

(b) Differences between the Fourier transforms of the Hotine-Koch and Stokes-Lambert kernels and a unit kernel.
4.8 Summary

This chapter presented a new insight into how best to restrict Stokes’ global integral to a local region where precise gravity data are available. It erects a stochastic model for the Earth’s gravity field based on the global gravity models EGM08 and EGM96. With the use of the correct definition of the gravity field power spectrum adopted by Hipkin (2001, Eq. 7), the concept of white noise finds the parameters of the stochastic model of the global gravity field that are successfully used to define autocorrelation functions and hence correlation length for different spectral parts of the Earth’s gravity field. It showed how a stochastic ring-averaged gravity field varies with distance from the computation point, thereby providing a rational way of predicting the size of the spherical cap needed to solve Stokes’ integral. All the residual gravity anomalies associated with Bjerhammer sphere with depths less than 50 km, the global integral can be appropriately restricted to a spherical cap size less than 2° causing no far field omission error. This justifies that the deterministic approach of kernel modification which depends on the properties of the gravity field is not needed. For these short wavelength components of the Earth’s gravity field, the Stokes’ spherical solution can be adequately computed by a simple planner FFT. Approximating the Stokes-Lambert or Hotine-Koch kernels by unity kernel does not affect the accuracy of geoid computation (Hipkin et al., 2010).
5.1 Introduction

Traditionally, a vertical reference system is established by geodetic levelling. In practice, the geodetic levelling connects the continental topography to the mean sea level through one or more tide gauge stations. Mean sea level is used as a datum for a vertical reference system, although this prospective has challengers. Hipkin (2002, p.376) this is the “… nineteenth century approach to establish a global vertical datum supposed that mean sea level could bridge regions not connectable by levelling”. Now, due to the emerging high precision observations of sea surface topography as acquired from satellite altimetry and a geoid model computed from the gravity field quantify the significant distinction between the mean sea level and the geoid. There is a permanent sea surface topography with a range of about 2 m due to non-random wind stress and water density variations. In addition, the mean level of the sea surface appears to be changing with time because of thermal expansion of the oceans due to climate changes, which contributes up to 1-2 mm/year of mean sea surface rise with little corresponding displacement of the equipotential surface (Andersen & Knudsen, 2000). Although the performance of satellite altimetry is very good over open seas (i.e 2 to 4 cm for TOPEX/POSEIDON), the measurement accuracy significantly deteriorates along the coast, over the shelf (which varies from tens to hundreds of kilometres off the coasts), in shallow depths, and fresh water inflows; where most tide gauges used for a local vertical datum are located. Now, it has been proved that, the mean sea level is not an equipotential surface: different tide gauges are theoretically at different equipotential surfaces. Therefore, the unification of levelling data derived from different tide gauges will introduce distortions in the vertical control network due to the inconsistence among the datum
points. This will give rise to offsets among height systems on a global scale (Burša et al., 2004; Rapp, 1994). The results of remeasurement campaigns in France (Kasser, 1989), Great Britain (Kelsey, 1972) and United States (Fischer, 1975) make the validity of the apparent slopes in mean sea level found by long distance levelling from the tide gauges questionable. The rationale of establishing a vertical reference system from levelling tied to tide gauges is even in doubt at local scale, never mind an extension of the process world-wide!

At present, when mean sea level is known to be non-equipotential surface, many authors (Amos & Featherstone, 2009; Ardalan et al., 2002; Ardalan et al., 2010; Ardalan & Safari, 2005; Burša et al., 2007; Ihde & Sánchez, 2005; Olsen & Gelderen, 1998) prefer to adopt the classical approach by making some corrections (e.g. sea surface topography or change in potential) on an epoch basis. The main target of these groups of researchers is to unify all globally available tide gauge stations in order to establish a unified global vertical reference system. Different height systems are adjusted to a level surface chosen conventionally. The mean sea level is changing both at local and global scales. Making continuous adjustment to the observable sea phenomenon confuses the crucial role geodesy can play in monitoring sea level rise, climate change and ocean circulation (Hipkin, 2002). This approach does not really solve the key problem: “Can we compute a globally absolute physical height on or above the Earth’s surface without referring to any local height networks?”

Our approach to determining a global vertical reference system computes the potential directly on or above the Earth’s surface from gravity observables. It uses band-limited gravity disturbance synthesised by combining local gravity data, global gravity model EGM08 and Shuttle Radar Topographic Mission (SRTM) topographic gravity models both in geometrical and spherical harmonic representations to compute gravity potential \( W \). Section 5.2 discusses the principle of geopotential computation and section 5.3 presents the conventional approach of computing geopotential number using geodetic spirit levelling. Section 5.4 explores how to measure a true height system based on the Earth’s gravity field and suggests to adopt the two fundamental properties of a true height system defined by (Hipkin, 2002) rather than sticking to the third less fundamental property which can be sacrificed.
Section 5.5 discusses the role of the physical height system and section 5.6 introduces an alternative approach of tracing the gravitational potential level surface in space through the GNSS provided geometrical coordinates. Section 5.7 discusses the criterion that should be considered to establish a vertical reference system which is absolute in a global sense. Section 5.8 presents the computational procedures involved to define the new geopotential vertical reference system which can be calculated for a single isolated point without connecting to any local height networks. Section 5.9 explores possible scenarios of improving the resolution of the EGM08 by using the airborne gravity data and SRTM models. Section 5.10 compares the geopotential numbers calculated from EGM08 and the new model derived from global gravity model, airborne gravity and SRTM data with an equivalent model independently computed from precise geodetic levelling data for Ethiopia.

5.2 Principle of geopotential computation

The determination of the changes in the Earth’s gravitational potential energy with respect to the potential of the geoid \( W_0 \) at points on or above the topography is what physical geodesy seeks for defining a level surface in space. Knowing the potential difference between two points involves the integral of gravity along a path between the two points. For instance, the potential difference between point \( A \) and point \( D \) is described by

\[
\int_A^D g \cdot d\ell = -\int_A^D \nabla W \cdot d\ell = W_A - W_D
\] (5.1)

If point \( A \) is chosen as a datum and assigned a potential \( W_0 \), then the geopotential number at point \( D \) is defined by (Heiskanen & Moritz, 1967, p. 162).

\[
c_D (\varphi, \lambda, h) = W_0 - W_D (\varphi, \lambda, h)
\] (5.2)

The geopotential number is used to determine the change in the gravitational energy of that point with respect to the datum potential \( W_0 \). It could answer whether two points lie on a level surface and if not how much potential difference or gravity is there. It is the only geodetic physical parameter used to uniquely characterise level surface on or above the Earth’s surface. Finding a geopotential number will help to
determine the shape of a level surface required to solve engineering problems related to energetic mass transports. Since 1951, the geopotential number has been internationally adopted as the standard way of specifying heights for scientific purposes.

At present, finding the geopotential number involves two different approaches. The first method uses geodetic spirit levelling while the second approach involves a Stokes-like integral of surface gravity data. This study first reviews how to compute the geopotential number from spirit levelling and then it presents a new rational way of computing the classical Stokes’ integral at accuracy good enough to replace the conventional spirit levelling method.

5.3 Spirit levelling

Geopotential spirit levelling is used to determine the shape of horizontal (level) surfaces and to measure the vertical distance separating them. The instrumentation and the principles of spirit levelling measurements are governed by the laws of physics defining the level surface. A level surface is one over which a sphere would have no tendency to roll away: the direction of gravity defines the \textit{vertical}, so must be right angles to a level surface – any component of gravity tangential to the surface would cause the sphere to roll away. Level surfaces are therefore \textit{horizontal}. Thus, the orientation of both the vertical and the horizontal are determined by the Earth’s gravity field.

Gravity is defined as the gradient of the gravitational potential. The gradient of the potential is at right angles to a surface of constant potential. Therefore, level surfaces are equipotential surfaces – surfaces where the potential is a constant.

Spirit levelling is used for a precise measurement of height differences between two nearby points occupied by a pair of levelling rods. These elevation differences are path dependent unless combined with gravity data.
Figure 5.1. The relation between levelling and gravity. Note: this diagram greatly exaggerates the curvature of level surfaces.

The incremental change in the gravitational potential is the product of the gravity and the increment of the path length projected in the direction of gravity field. Finding the potential difference between two points involves adding these increments of potential together along the complete path. Evaluating this line integral

$$ W_A - W_D = \int_A^D g \cdot d\ell $$

along a path that follows the sight lines between levelling staves and then goes up or down the stave, only the path increments where $g$ and $d\ell$ are parallel (up and down the levelling staff) contribute to the integral; other increments, where the path is along a level sight-line so that $g$ and $d\ell$ are perpendicular, do not contribute to the integral.

If the equipotential surface through the point $A$ in Figure 5.1 is chosen as a datum and given the value $W_0$ then the potential at point $D$ will be
The quantity

\[ c_D = W_0 - W_D = \sum g_i \Delta h_i \tag{5.5} \]

is called the geopotential number at point D.

It is clear that \( \sum \Delta h_i \) and \( \sum g_i \Delta h_i \) cannot both be a unique function of position unless gravity is a constant. In particular, \( \sum \Delta h_i \) gives different results if measurements are made along different routes between the datum and point D. In contrast, the combination of levelling and gravity is a unique way to characterise height and is independent of which route is taken. Knowledge of gravity is essential for converting levelling increments into geopotential number or a physical height system.

Conventionally, a geopotential vertical reference system is established by connecting the actual topography to one or more mean sea level tide gauge stations. According to this classical approach, mean sea level was considered as a gravitational equipotential surface of a fluid in hydrostatic equilibrium and it is realised as a vertical datum - a level surface of zero geopotential number. A vertical reference network is extended over each country using a co-located measurement of gravity and spirit levelling.

The new approach uses the middle term in equation (5.5) to determine \( W_D \) from surface gravity measurement. So, a convenient global value of the datum potential \( W_0 \) can be chosen conventionally to provide an absolute geopotential number in a global sense. This will be discussed in the next sections.

### 5.4 The new height system

The new geodesy provides a new definition for the realisation of a true height measurement made between two points experiencing the Earth’s gravity attraction.
Height is measured as a gravitational potential gained by a unit mass falling inside a gravitational field along measurement trajectories. The trajectories are controlled by the change in the increments of the plumb lines as the changes in the gravity field are measured. The variations of the gravity field strength along different trajectories do not alone affect this sort of distance measurements. Consider the measurement of gravity \( g = -\nabla W \) and distance increment measured along a path from point A to D on or above the Earth’s surface.

\[
\int_{A}^{D} g \cdot d\ell = -\int_{A}^{D} \nabla W \cdot d\ell = W_{A} - W_{D}
\]

(5. 6)

If the point mass eventually returns to its initial position, it acquires its initial gravitational potential \( W_{A} \), then the total integral must be zero irrespective of the length and shape of the trajectory.

\[
\oint g \cdot d\ell = 0
\]

(5. 7)

This condition confirms that \( W \) is a function of position only: to every point there corresponds a unique value \( W \). In general, this has been stated by the Stokes’ Theorem: the integral of any vector \( g \) derived from a scalar potential is path independent. Hence \( W \) is single valued.

So far we have discussed how to measure a true height between two points under the influence of the Earth’s gravity field. Now, we deal with the fundamental properties of a true measure of height system. Hipkin (2002) suggested two necessary conditions:

(i) Height must be single-valued
(ii) A surface of constant height must be a level (equipotential) surface

He also sets the third condition which would be convenient but is much less fundamental and might be sacrificed if necessary:

(iii) Height should be a geometrical distance and therefore measured in metres

With these properties of height system, a true measure of height (potential difference between two points) on or above the Earth’s surface can be converted to a globally absolute height by transforming surface gravity data to gravitational potential using Stokes-like integral. Then, the computation of geopotential number requires the \( W_{o} \) value to be chosen conventionally.
This geopotential vertical reference system would be global and absolute if $W_0$ is determined from a physical property of the Earth and the computation of $W_A$ from gravity data is consistent with this value. The geopotential number has all the necessary properties of a height system: it can identify whether two points lie on the same level surface and, if not, what gravity is there. Each point on or above the earth’s surface has a unique value of geopotential number, however, it is not measured in metres.

Any of the current height systems in use do not satisfy at least the two necessary conditions, because the third condition was given priority as compared to the two necessary properties of a true height system. This rather leads to a misunderstanding of how a true height is measured within the gravitational field. In practice, all the present physical height systems are directly measured as a potential difference. For instance, spirit levelling gives geopotential difference while the classical geodetic boundary-value problem provides potential on the ellipsoid from surface measurements of gravity data. In principle, any conversion of geopotential number to geometrical height forms a derivative rather than a definitive part of a vertical reference system. Only dynamic height has all the properties of the height system (Heiskanen & Moritz, 1967, Sect. 4.2). But, due to historical reasons dynamic height is considered as an obsolete. The present height systems: ‘orthometric’ height and ‘normal’ height do not satisfy the two basic conditions. This study presents a modern approach of computing a geopotential vertical reference system and some criteria that has to be considered to establish a global geopotential vertical reference system.

5.5 Why do we need a physical height system?

Nowadays satellite-based positioning systems have provided 3-D coordinates in an absolute geocentric reference system with an accuracy of a few centimetres during a real time surveying, so why do we need different vertical reference system for height. GNSS directly provides the geocentric radius $r$ or equivalently ellipsoidal height for most positioning applications: we no longer need conventional height to tell us where we are (Hipkin, 2002). However, some residual sets of tasks, mostly related to
hydrodynamics and energetics of transporting mass still requires another height to map level surfaces on or above the Earth’s surface. This sort of height must exactly determine the shape of a level surface at the point of interest in terms of the coordinates given by GNSS. It should determine whether the potential energy at point A is more or less than point D and, if so, by how much. Note that tying the physical geopotential height with the geometrical coordinates will potentially utilise the positioning application of GNSS technology.

We argue that any definition and realisation of the physical height system should arise from the fact that it has to map level surfaces in space. This is why we need them: the realisation should answer their objectives. With these new concepts of height, the realisation of a physical height system must at least satisfy the two necessary conditions of a true height system (Hipkin, 2002). Do the present height systems satisfy these two basic criteria? The answer to this is no. Meyer et al., (2006) “there is, in fact, no single height system that is both geometric and honours level surfaces simultaneously because these two concepts are physically incompatible due to non-parallelism of the equipotential surfaces of the Earth’s gravity field.” The two necessary conditions form the basis for the definition and realisation of the physical height systems. With the objective of defining level surfaces in space the third condition should be sacrificed, so that, the physical height is a scaled geopotential number. Any present physical height system such as orthometric and normal heights is excluded.

5.6 How can we map level surfaces in space?

The problem of mapping level surfaces in space involves the technology that can deliver accurate measurements of gravitational potential energy of points in space. This new approach should also determine whether potential energy at point A is more or less than point D and, if so, by how much. What we really need is to devise a procedure that would enable us to compute the gravitational potential energy of points on or above the Earth’s surface during real time practical field surveying. This provides what we need to define a global vertical reference system. The strategy is to compute the gravity potential energy on or above the Earth’s surface through the geometrical coordinates provided by the GNSS technology. GNSS provides absolute
geocentric coordinates \( \{x, y, z\} \) for any point \( A \) on or above the Earth’s surface. Then, the global vertical reference system is defined by computing \( W_A = W(x_A, y_A, z_A) \) as the output when the GNSS provided position coordinates are the inputs. Using the alternative geometric coordinates \( \{\lambda, \varphi, h\} \), equivalent to \( \{x, y, z\} \), we can trace the ellipsoidal height of a level surface through point \( A \) by solving the equation \( W(h, \varphi, \lambda) = W_A \) for \( h(\varphi, \lambda) \). Afterwards the computation of the potential energy differences between two points in the space becomes trivial. What we need is only their geometrical coordinates from GNSS. For instance, the potential difference between points \( A \) and \( D \) can be computed as:

\[
W_A - W_D = W_A(x_A, y_A, z_A) - W(x_D, y_D, z_D)
\]

Relative potential difference can be used for specific set of tasks like those needed by classical spirit levelling techniques but it does not give an absolute value as the total potential \( W \). The equipotential lines roughly follow the topographic shape of the Earth. They bow up over mountains and dip down into valleys. To better visualise the gravitational potential energy surface undulations on or above the Earth’s surface in an absolute sense, it is elegant to deal with the geopotential numbers rather than the total potential. The magnitude of the total potential is so large that it masks the local details – because it tells you the potential height relative to the Earth’s centre of mass. Alternatively, geopotential number is defined conventionally in such a way that its magnitude is small, making it easily manageable. The geopotential number at any point \( A \) on or above the Earth’s surface can be uniquely realised by the change in the gravitational potential of the point relative to the potential of the zero level vertical reference surface.

\[
c_A = c_A(x_A, y_A, z_A) = W_0 - W_A(x_A, y_A, z_A)
\]

where the potential of the geoid \( W_0 \) is defined by normal potential \( U_0 \).

Since both \( W_0 \) and the GNSS position coordinates, \( \{\lambda, \varphi, h\} \) are used for the real time computation of geopotential number, the potential of the WGS84 ellipsoid is preferred for the sake of consistency. The next section discusses the overall computational procedures needed to get an absolute geopotential number during a real time GNSS surveying.
5.7 Standards for unifying a vertical reference system

The unification of a geopotential vertical reference system requires a consistent usage of four parameters in a global sense:

(i) Geopotential model
(ii) Reference ellipsoid
(iii) The geoid’s potential value
(iv) Tide system

The same Earth gravity model (geopotential model) must be used for the realisation of the world height system unification. As our knowledge of the Earth’s gravity field will be improved in the near future with the coming new gravity mission satellites and from expeditions in the airborne gravimetry campaigns, the geopotential vertical reference system has to be updated with the release of a new Earth’s gravity field model. For instance, the newly launched European Space Agency’s (ESA’s) gravity mission satellite, Gravity field and steady-sate Ocean Circulation Explorer (GOCE) will bring unprecedented resolution and accuracy of the Earth’s gravity field thereby revolutionising the realisation of the vertical reference system (ESA, 1999). In a strict sense, the International Association of Geodesy (IAG) has to be responsible for standardisation of the geopotential model.

The reference ellipsoid can be the World Geodetic System WGS84 – which is used as the reference level for the GNSS positioning. This ties the geometrical reference system and the geopotential vertical reference system together. Its potential and gravity field are defined by four geodetic parameters \((GM, a, J_2, \omega)\) which are directly determined from space geodetic observations. In addition, the values of \((a, J_2)\) parameters and the ellipsoidal height of the reference ellipsoid must be reduced to the same tide system adopted to generate the Earth gravity model and local gravity data.

The geoid potential \(W_0\) should be determined from the property of the Earth’s gravity field. At present, there is no agreement as to which value for \(W_0\) should be chosen as a global datum (Hipkin, 2002). Smith (1998) suggested two ways of calculating \(W_0\): choose a “reasonable” value or adopt a so-called “best fitting ellipsoid”. Hipkin (2002) has argued for the first approach: “To me it seems
inevitable that, in the near future, we shall adopt a vertical reference system based on adopting a gravity model and one that incorporates \( W = W_0 = U_0 \) to define its datum” . Approximating \( W_0 \) by \( U_0 \) does not introduce any systematic errors; it only causes a vertical shift in the reference equipotential surface - no change in the shape of the geopotential surface. The practical importance of this philosophy is much more significant. It ties the physical reference system to a uniquely realised geometrical reference system defined by equipotential surface of constant gravity potential. For instance, the use of the normal potential of the World Geodetic Reference System WGS84 as a datum point could give direct access to real-time processing of the GNSS signal to obtain real-time geopotential number.

### 5.7.1 Tide Systems

In practice, geodetic measurements are affected by the non-zero time-average of the tide-generating potentials of the \textit{Moon} and the \textit{Sun} and the permanent deformation of the Earth they cause. The Earth permanently exhibits deformation due to the permanent existence of the \textit{Sun} and the \textit{Moon}. The first part is usually called the “direct” component of the permanent Earth tide while the second is called the “indirect effect”. The tide-generating potentials cause periodic and time-independent perturbations in the Earth’s potential, and therefore in the shape of its equipotential surface, due to the orbital motions of the \textit{Moon} and the \textit{Sun} with respect to the Earth. At the surface of the Earth its magnitude varies from place to place, parts of the Earth closer to the \textit{Moon} and the \textit{Sun} experiences larger attraction than the other parts on the opposite side. The magnitude and ranges of the tidal potential reaches a few parts of \( 10^{-8} \) of the potential of the Earth (Mäkinen & Ihde, 2009) at the surface of the Earth.

The tidal-generating potential varies on the Earth’s surface spatially as functions of geocentric radius and geodetic latitude and longitude \((r, \varphi, \lambda)\) and can be precisely approximated by the second degree harmonic term (Poutanen et al., 1996).
\[ W_z(r, \phi, \lambda) = \frac{3}{4} \frac{GM_e r^2}{d^3} \times \left[ \cos^2 \phi \cos^2 \delta \cos 2h + \sin 2\phi \sin 2\delta \cos h + (3\sin^2 \phi - 1)(\sin^2 \delta - \frac{1}{3}) \right] \]  

(5.11)

where \( \delta, h \) are declination and hour angle of the celestial body, respectively, \( M_e \) is the mass of the celestial body (Moon and Sun), \( d \) is the geocentric distance of the measurement point with respect to the centre of celestial body.

Note that, ignoring the higher order terms would only result in an error of 0.03% (Vaníček & Krakíwsky, 1986). The first, second and third terms in equation (5.11) are equivalent to tesseral, sectorial and zonal harmonics, respectively. The tesseral harmonics varies in both latitude and longitude (time) causing semi-diurnal tide. The sectorial harmonic does not change sign within certain longitude intervals, it gives rise to the two high/low tides each day. The time average of the tesseral and sectorial over the whole sphere is zero. The zonal part does not vary in longitude (time) and it produces permanent or long period tide which varies in latitude, not in longitude. The expression for the permanent tide generating potential is given by:

\[ W_z(r, \phi, \lambda) = \frac{3}{4} \frac{GM_e a^2}{d^3} r \left[ (3\sin^2 \phi - 1)(\sin^2 \delta - \frac{1}{3}) \right] \]  

(5.12)

The nearly constant term varying over the Earth’s surface:

\[ D = \frac{3}{4} \frac{GM_e r^2}{d^3} \]  

(5.13)

is called the Doodson’s tidal constant and its value was numerically determined to be \( 2.6248 \times 10^5 \) m.mgal for the Moon and \( 1.2054 \times 10^5 \) m.mgal for the Sun (Xi, 1982).

The direct tide-generated gravity effect is the first radial derivative of the tide-generating potential \( W_z(r, \phi, \lambda) \).

\[ g_z(r, \phi, \lambda) = -\frac{\partial W_z(r, \phi, \lambda)}{\partial r} = \frac{2}{r} W_z(r, \phi, \lambda) \]  

(5.14)

If the tide generating force interacts with a perfectly elastic solid Earth, the Earth distorts linearly. Love’s numbers \( \{ h, k, l \} \) are the constants of proportionality
for this linear interaction: \( h \) for vertical displacement, \( l \) for the horizontal displacement and \( k \) for the potential generated by the tidally redistributed mass. The distortion has two effects: a gravimeter sitting on the Earth’s surface is raised through the general Earth’s gravity field by an amount

\[
\delta \xi = \frac{h}{g} W_2
\]  

(5.15)

There is a consequent change in potential

\[
\delta W_2 = g \delta \xi = hW_2
\]  

(5.16)

and a gravity effect of

\[
\delta g = \frac{2}{r} hW_2
\]  

(5.17)

In addition, the tidally redistributed mass creates a new internal potential

\[
\delta W_2 = kW_2
\]  

(5.18)

So its gravity effect is

\[
\delta g = -\frac{3}{r} kW_2
\]  

(5.19)

Thus the net effect on a gravimeter is

\[
\delta g = \left[ 1 + h - \frac{3}{2} k \right] \delta \delta g_2
\]  

(5.20)

and \( \delta \) defines the gravimetric factor.

Note that the gravimetric factor is evaluated using the results from seismology, where the Earth response is measured for stresses with typical seismic frequency band, say \( 10^3 \text{ Hz} \) to \( 10^5 \text{ Hz} \), extended to \( 10^6 \text{ Hz} \) by Earth-tide observations, the solid Earth fits Loves elastic theory and gives a precisely determined value of \( k \approx 0.30 \). The response to very long period stress – in principle up to static conditions – will be quite different so the elastic gravimetric factor will not apply to the permanent tide. We do not know what value to give \( k \) for permanent tide. The seismic value \( k \approx 0.30 \) is wrong and the \( k_{fluid} \approx 0.96 \) value for the ‘solid’ Earth behaving as a perfect fluid is uncertain (Hipkin, 2002). The size of the
distortion of the Earth to the permanent tide is unknown and will in any case not be distinguished for the net term described by the parameter $J_2$, so multiplying a closed expression for the whole tide by the elastic gravimetric factor will be incorrect - only the direct effect of the tide-generating potential should be removed, that is the 1 in the gravimetric factor (Hipkin, 2002), see equation (5.20) above.

The permanent tide-generating potential affects all space and terrestrial geodetic measurements (gravity, GNSS solutions, altimetry, etc) and other geodetic parameters (the semi-major axis and gravity flattening). Particularly, the second degree zonal term $C_{20}$ or gravity flattening $J_2$ and the semi-major axis of the reference ellipsoid are used for most geodetic computations (gravity anomalies, normal gravity, geoid, potential, etc). Therefore, they must be consistently used with tidal corrections applied to the other quantities involved in the computations. Currently, regarding the consistence in tidal correction among geodetic measurements, there are three tide systems in use (Ekman, 1989).

**Tide-free (non-tidal) system** – is when the effect of permanent tide is eliminated from observations including “direct” tide and “indirect effect”. It is equivalent to removing the Sun and the Moon to infinity. The permanent Earth deformation is computed using the Love numbers (h and k) to define a conventional tide-free system, instead of estimating the secular (fluid) Love numbers. Conventional tide-free love numbers are h=0.6, k=0.3, $\delta = 1.15$ and their values in the fluid tide-free are h=1.93, k=0.93, $\delta = 1.535$.

**Mean tide system** – is when the permanent tidal effect is included in the measurements. Thus the shape of the gravity and potential corresponds to the long-time average under the tidal forces. It includes the tide-generating potential which is not even related to the mass of the Earth.

**Zero tide system** – assumes the removal of the tide-generating potential but retains its indirect effect.

Present practice showed that different geodetic measurements are reduced to different tide systems. Currently zero tide system is used for gravity and tide-free system for GNSS positioning. The spherical harmonic coefficients of the global gravity model use the International Earth Rotation Service (IERS) convention (free-tide system). Now EGM08 model is also available in zero-tide system. It is $C_{20}$ or $J_2$.
term of the zonal harmonic coefficient that is affected by tidal forces. The transformation of the global gravity or potential from one tide system to another involves the modification of the second degree zonal harmonic coefficient or if the ultimate goal is to compute gravitational potential or geoid using a conventional $U_0$ value of the reference ellipsoid, a zonally uniform correction has to be added (Rapp, 1989). To convert between different tide systems we used (EGG-C, 2010).

\[
\bar{C}^{(\text{mean tide})}_{2,0} - \bar{C}^{(\text{zero tide})}_{2,0} = \langle \Delta \bar{C}_{20} \rangle 
\]

\[
\bar{C}^{(\text{zero tide})}_{2,0} - \bar{C}^{(\text{free tide})}_{2,0} = k_2 \langle \Delta \bar{C}_{20} \rangle 
\]

\[
\bar{C}^{(\text{mean tide})}_{2,0} - \bar{C}^{(\text{free tide})}_{2,0} = (1 + k_2) \langle \Delta \bar{C}_{20} \rangle
\]

where

\[
\langle \Delta \bar{C}_{20} \rangle = \frac{(-0.198 m \times r^3 g)}{a^2 GM \sqrt{5}} = -1.39142 \times 10^{-8} ; \quad k_2 = 0.3019
\]

$\langle \Delta \bar{C}_{20} \rangle$ is the value of the second degree time-independent gravity tide for Sun and Moon, $k_2$ is Love number for second degree zonal coefficient, $g$ is the mean gravity.

Similarly, gravity observations can be converted from one tide to another tide system by using the formulas and constants derived by Ekman, (1989).

\[
g^{(\text{mean tide})} - g^{(\text{zero tide})} = (30.4 + 91.2 \sin^2 \varphi) \times 10^{-8} \text{ mgal} \quad (5.25)
\]

\[
g^{(\text{zero tide})} - g^{(\text{free tide})} = 0.16(30.4 + 91.2 \sin^2 \varphi) \times 10^{-8} \text{ mgal} \quad (5.26)
\]

\[
g^{(\text{mean tide})} - g^{(\text{free tide})} = 1.16(30.4 + 91.2 \sin^2 \varphi) \times 10^{-8} \text{ mgal} \quad (5.27)
\]

Computing an absolute geopotential number in a global sense requires the use of the same Earth gravity model and consistent tidal conventions between the local gravity observations and the global gravity model. The observation of gravity in a static or kinematic mode on and above the Earth’s surface is influenced by permanent tide generating potential due to the external masses (the Sun and Moon), varying as $r^2$ not $r^{-3}$, and so must be removed. The removal of the Moon’s and Sun’s mass attractions has rational property of being connected by the Stokes’ formula, as it refers to a situation in which the external masses have to be computationally removed. The removed tidal gravity can be restored in the form of potential without introducing any conceptual error as discussed above. In this way, consistent values of geopotential numbers at the boundaries between different countries can be achieved.
preferably in a zero tide system as adopted by the IAG resolution 16, 1983. More on the treatment of permanent tide in gravimetry, height system and GNSS position solutions can be obtained (Ekman, 1989; Hipkin, 2002; Mäkinen & Ihde, 2009; Poutanen et al., 1996; Rapp, 1989).

5.8 Computing the Geopotential Model

The modern approach of mapping a level surface in space requires the potential to be evaluated on or above the Earth’s surface where the field surveyor needs it. In practice, we usually measure the magnitude of the gravity vector, but some operation equivalent to the Stokes’ integral is needed to transform observations of surface gravity data to geopotential number, see section 3.3. At present, the coverage of global gravity model (e.g. EGM08), topographic models (SRTM) and local gravity data is essentially complete and their quality so high that Stokes-like integral is applied only to the small residuals. The transformation process is carried out on the surface of the reference WGS84 ellipsoid and this procedure involves three different stages.

5.8.1 The first stage: Gravity Field Modelling

The first stage involves the removal of a gravity model from the measured gravity in order to synthesise band-limited high frequency components of gravity disturbance to make the integration reliable. This has been discussed in chapter 2. In practice, models for some parts of the gravity field are subtracted from observations. The removed gravity model must be restored as potential using a closed formula to describe exactly the same model. These models include a representation of a global ellipsoid, long wavelength components of the global gravity model developed as spherical harmonic coefficients complete to high degree and order and the gravity effects of the topography. Subtracting the normal gravity of the global reference ellipsoid from the measured gravity gives the observed gravity disturbance at measurement positions. After the gravity disturbance computed from the EGM08 is subtracted from the observed gravity disturbance, most of the remaining high frequency component in the residual comes from the topographic influence. Usually, this residual has the required smallness but it is not sufficiently smooth. Therefore, the remaining high frequency components have to be eliminated by removing the
gravity model of the geometrical SRTM topography and by adding back its spherical harmonic representation. In this way trend-free, small and smooth gravity disturbance can be synthesised so that the compute stage involves simplifying approximations as the band-limited residual gravity disturbances are not affected by long wavelengths. Note that, since the effects of the mass models are large, the removed gravity and restored potential must be done with high precision and must involve exactly the same quantity and distribution of mass.

5.8.2 The second stage: downward continuation and transformation

The second stage generally involves two separate tasks as described in chapter 3. The first task involves the downward continuation of the residual gravity disturbance from the measurement positions on to the geodetic boundary surface (e.g. reference ellipsoid, sphere). The problem is to compute \( \delta g(x,y,0) \) from the known values of \( \delta g(x,y,h) \). The development of more efficient algorithm in terms of large data handling, stability and convergence are the key factors for achieving high computational accuracy. In this study, a 2-D planar FFT algorithm was used to continue the residual gravity from along track airborne measurement positions onto the reference ellipsoid (see, Chapter 3). We used a band-pass Butterworth filter to eliminate the leakage of the long wavelengths and to suppress the amplification of noise from the high frequency end of the spectrum (see, Section 3.4). The analysis of Fourier transforms on Lambert plane has yielded the band-limited gravity disturbance on to a 2 km grid of eastings and northings, derived using Lambert’s formulae for the conformal projection of an ellipsoid onto a cone.

The second task computes the disturbing potential from the downward continued residual gravity: \( \delta g(x,y,0) \Rightarrow \delta T(x,y,0) \). Different forms of Stokes-like integral transform algorithms could be used (spherical Stokes-like integral in both space and frequency domain, Least-Square-Collocation etc). Here, the process of downward continuation has already delivered a spectral domain version of residual gravity so its conversion to residual potential is trivial \( FT[r] = \frac{1}{\kappa} \bar{FT}[\delta g] \). The validity
of this planar operation was demonstrated in section 3.3. This stage adds geopotential components in the wavelengths range $4\ km < \lambda < 400\ km$ (see, Table 5.6 & Table 5.7).

5.8.3 The third stage: upward continue and restore the potential models

The third stage computes the total geopotential number of a point on or above the Earth’s surface. It first continues the residual potential up to the desired field point whose coordinate is provided by GNSS receiver. Note that, the upward continuation algorithm should perform opposite operation as compared to the second stage in a way that is consistent. Then the exact value of the geopotential number is computed by restoring the potential of the reference ellipsoidal, SRTM geometrical model, long wavelength components of the Earth Gravity Model (EGM08) and by removing the potential of the condensed topography. In practice, only some parts of this third stage need to be done after GNSS observations. A globally absolute geopotential number of a point on or above the Earth’s surface should be determined by interpolating from the pre-synthesised digital grids of potential carried in a chip for use with a GNSS receiver in the field. Only the effect of high resolution local topography needs to be computed from scratch in the field. This now gives a geopotential height at any isolated point where there is a GNSS measurement without reference to a network or regional datum point. Figure 1.2 gives a summary of the computational procedures used in stage one to stage three to get a geodetic quality geopotential number.

5.9 EGM08 compared with airborne gravity and SRTM corrections

The release of the EGM08 model has attracted many geodesists to investigate its actual accuracy with different validation techniques and independent data sets from different parts of the globe (e.g. Ellmann, 2010; Kotsakis et al., 2008; Łyszkowicz, 2009) This section first investigates the performance of the three gravity models based on the contribution that they can make to improve the resolution of the local residual potential and also the level of accuracy that can be achieved.
<table>
<thead>
<tr>
<th>Gravity models</th>
<th>Components</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model1</td>
<td>EGM08</td>
</tr>
<tr>
<td>Model2</td>
<td>EGM08 for $\lambda &gt; \lambda_0$ &amp; airborne gravity for $\lambda \leq \lambda_0$</td>
</tr>
<tr>
<td>Model3</td>
<td>EGM08 for $\lambda &gt; \lambda_0$ &amp; airborne gravity and SRTM for $\lambda \leq \lambda_0$</td>
</tr>
</tbody>
</table>

Table 5.1. The three gravity models ($\lambda_0$ is the cut-off wavelength)

The first is the EGM08 gravity model – dominated by airborne gravity data locally but by satellite gravity for wavelengths greater than about 400 km. The second gravity model replaces EGM08 gravity by the airborne gravity data for wavelengths shorter than 400 km and the local data will be down-weighted for wavelengths longer than 400 km. The third gravity model modifies the second by removing high resolution topographic effects (hedgehog) and adding lower resolution spherical harmonic representation of the same topographic masses condensed to some coating on or below the ellipsoid.

The first test explores the improvement that can be achieved by replacing EGM08 with airborne gravity data. The best approach of validating any further enhancement of the local disturbing potential field and hence improvement in its resolution is to compute the contribution from the difference between these two gravity models, model two minus model one (EGM08 only). Figure 5.2 shows the residual potential calculated from gravity model two minus model one on the surface of the ellipsoid using FFT technique. This indeed makes significant contribution, with a mean of 4.4 cm and standard deviation of 7.9 cm (Table 5.2). The resulting residual potential has local features with amplitude of about a decimetre that actually correlates with topography and so does represent real improvement in the EGM08 because of increased resolution of the local gravity data. Figure 5.2 also shows some features, in particular, the ~ 35 cm high near (35°E, 10°N) reaching a few decimetres associated with long wavelengths of up to 500 km. It may result from EGM08 including commercial land gravity with a poorly constrained datum and needs to be further investigated by the coming GOCE gravity data.
Anomalous potential computed from observed minus EGM08 gravity disturbance
residual gravity (mgal)       disturbing potential
(Units are $10^{-1} \text{ m}^2\text{s}^{-2} \sim 1 \text{ cm of height}$)

<table>
<thead>
<tr>
<th></th>
<th>Anomalous potential computed from observed minus EGM08 gravity disturbance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>residual gravity (mgal)</td>
</tr>
<tr>
<td>mean</td>
<td>0.26</td>
</tr>
<tr>
<td>Stdev</td>
<td>4.56</td>
</tr>
<tr>
<td>Max</td>
<td>37.90</td>
</tr>
<tr>
<td>Min</td>
<td>-33.10</td>
</tr>
</tbody>
</table>

*Table 5.2.* Disturbing potential obtained by replacing EGM08 with airborne gravity.
Figure 5.2. Effect of replacing EGM08 gravity by the airborne gravity on disturbing potential (Units are $10 \text{ m}^2\text{s}^{-2}$, approximately equivalent to 1 m of height).
The second test examines the contribution of model three compared to model one. It first shows the contribution of the in situ topography to the residual potential and then illustrates the effects of replacing the EGM08 by local gravity data plus topographic models for medium to shorter wavelengths.

The contribution of the difference between the gravity models for the in situ topography (hedgehog) and its spherical representation usually provides small effect but with unprecedented resolution and expected behaviour of correlation with topography. It recovers only shorter wavelength features and they correspond to regions of high and dissected topography. These small features reach up to an order of -16.4 cm in a mean value with a standard deviation of 22.5 cm in the residual potential (see, Table 5.3). Note that, these figures do not represent the overall contribution of the topographic effect because they will be minimised when restoring the effects of the removed topographic gravity models in the form of potentials.

<table>
<thead>
<tr>
<th>Anomalous gravity and potential computed from in situ topography minus condensed topography</th>
<th>residual gravity</th>
<th>disturbing potential</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mgal</td>
<td>$10^{-1} \text{m}^2\text{s}^{-2}$ (~ 1 cm of height)</td>
</tr>
<tr>
<td>mean</td>
<td>-0.96</td>
<td>-16.4</td>
</tr>
<tr>
<td>Stdev</td>
<td>5.2</td>
<td>22.5</td>
</tr>
<tr>
<td>Max</td>
<td>61.3</td>
<td>36.5</td>
</tr>
<tr>
<td>Min</td>
<td>-39.1</td>
<td>-96.5</td>
</tr>
</tbody>
</table>

**Table 5.3.** Effect of the topographic residual gravity on disturbing potential

In reality, these very short wavelengths topographic generated gravity field cannot be fully recovered from airborne gravimetric observations. For instance, the dipolar anomaly near (39°E,14°N) suggests that the same feature also seen in Figure 5.2 is real signal and may be caused due to a lack of resolution in spherical
harmonic analysis rather than a data artefact, because Figure 5.3 depends only on SRTM topography not on any gravity measurement. The need for the topographic correction to improve the resolution of widely spaced local gravity data and global gravity model is now justified by using the third gravity model.

Figure 5.4 showed a better improvement in the EGM08 and it suggests that the combined use of local gravity data, global gravity model and SRTM in a way that they could provide complementary spectrum of the Earth’s gravity field is the key element to achieve a high resolution geopotential model. The effect of gravity model three on the disturbing potential had a mean value of 20.8 cm and a standard deviation of 28.8 cm (see, Table 5.4). Now, we showed how best to improve the resolution of the EGM08 over Ethiopia and the next section tests its accuracy against precise levelling data.

**Figure 5.3.** Effect on disturbing potential of the difference between hedgehog and a spherical harmonic model of SRTM topography (Units are $10 \text{ m}^2\text{s}^{-2}$, approximately equivalent to 1 m of height).
Figure 5.4. Effect of combining topographic models and airborne gravity on disturbing potential compared to EGM08 (Units are 10 m$^2$s$^{-2}$, approximately equivalent to 1 m of height).

Table 5.4. Effect of gravity model three on disturbing potential
5.10 Comparison with Levelling

Precise levelling data and GPS control position coordinates were made available by the Ethiopian Mapping Agency to validate the accuracy of the geopotential numbers computed from the three gravity models: model one, model two and model three, see section 5.9. For each levelling stations, the information is arranged as benchmark name, geographic location (longitude, latitude and ellipsoidal height), levelling height and levelling increments (or foresight and backsight measurements), see Table 5.5. The levelling data were collected both in the fore and backward directions between two successive levelling stations and the data with high accuracy was used. About 10 to 90 foresight and backsight levelling measurements are available from each fore and back levelling network between two consecutive levelling stations. Geodetic levelling and precise GNSS surveying were carried out across 100 km long baseline in the north-south direction from Addis Ababa to Nazret. Figure 5.5 shows the distribution of 16 levelling stations with high quality levelling data and GPS coordinates used for comparison against geopotential numbers derived from gravity models.

In order to validate the accuracy of the gravity derived geopotential numbers, I converted the levelling increments to geopotential increments using equation (5.5). This requires precise prediction of gravity at every levelling point from the EGM08 and normal gravity. The sum over the product of gravity and levelling increments between two stations gives an incremental change of geopotential number.
### Table 5.5. Geopotential numbers calculated from levelling increments and gravity using GPS position coordinates at each levelling sites.

<table>
<thead>
<tr>
<th>Levelling Site name</th>
<th>Observables</th>
<th>Corrections</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Longitude</td>
<td>latitude</td>
</tr>
<tr>
<td>BA202</td>
<td>38.76790</td>
<td>8.93009</td>
</tr>
<tr>
<td>BMQ25</td>
<td>38.73535</td>
<td>9.00952</td>
</tr>
<tr>
<td>BM904</td>
<td>38.79580</td>
<td>8.87297</td>
</tr>
<tr>
<td>BMDK1</td>
<td>38.75370</td>
<td>8.97675</td>
</tr>
<tr>
<td>BMDK3</td>
<td>38.81930</td>
<td>8.84309</td>
</tr>
<tr>
<td>BMDZ1</td>
<td>38.96010</td>
<td>8.75188</td>
</tr>
<tr>
<td>BMJ1a</td>
<td>38.98250</td>
<td>8.74828</td>
</tr>
<tr>
<td>BMJ3a</td>
<td>39.04560</td>
<td>8.67488</td>
</tr>
<tr>
<td>BMJ4a</td>
<td>39.06050</td>
<td>8.65589</td>
</tr>
<tr>
<td>BMJ5a</td>
<td>39.09500</td>
<td>8.62340</td>
</tr>
<tr>
<td>BMCaa</td>
<td>39.01570</td>
<td>8.71836</td>
</tr>
<tr>
<td>BMNZ1</td>
<td>39.13770</td>
<td>8.58011</td>
</tr>
</tbody>
</table>
Figure 5.5 Location of the geodetic quality levelling stations.

First, it should be emphasised that, the levelling data used for this test do not necessarily provide a geocentric geopotential number because the geopotential number of the national vertical datum is not accurately known as it was thought to be connected to tide gauge station located at the port of Alexandria through long distance levelling networks (~2500 km) carried down the Nile river basin. Rather, it gives relative geopotential difference between points that are connected by levelling measurements with high accuracy. Here, we carried out comparison over long baseline by calculating the difference in the geopotential numbers between all levelling stations with respect to one particular reference levelling station called BA202 (see, Table 5.5). Whereas, the three gravity models provide a geocentric values of geopotential numbers with respect to the potential of the geoid defined by \( W_0 = U_0 \) (Hipkin, 2002). The first test evaluates the geocentric geopotential number at each levelling benchmark stations for the three gravity models and then computes the relative values of each station with respect to a base station, BA202. However,
this requires precise GPS position coordinates. For this test, the original GPS coordinates were reprocessed by Hunegnaw (personal communication) using GAMIT software (King & Bock, 1987) and shown to have an accuracy of 2 - 3 cm.

Table 5.6 shows the difference between the levelling and gravity model one and model two derived geopotential numbers. The EGM08 predicts the levelling geopotential number with standard deviation of 5.3 cm compared with 4.6 cm for model two. Table 5.7 shows that gravity model three gives better numerical agreement with levelling geopotential number with a standard deviation of 3.9 cm. Figure 5.6 also shows the difference between the levelling data and three gravity models, at each levelling benchmark stations.

<table>
<thead>
<tr>
<th>Cut-off ( \lambda_0 (\text{km}) = )</th>
<th>Anomalous geopotential difference between levelling &amp;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>EGM08</td>
</tr>
<tr>
<td></td>
<td>400.75</td>
</tr>
<tr>
<td>Mean</td>
<td>-6.3</td>
</tr>
<tr>
<td>Stdev.</td>
<td>5.3</td>
</tr>
<tr>
<td>Max</td>
<td>3.5</td>
</tr>
<tr>
<td>Min</td>
<td>-16.9</td>
</tr>
</tbody>
</table>

Table 5.6. Comparison of model one and model two based geopotential heights against an equivalent model derived from geodetic levelling (Units are in \(10^{-1} \text{m}^2 \text{s}^{-2}\) equivalent to 1cm).

<table>
<thead>
<tr>
<th>Cut-off ( \lambda_0 (\text{km}) = )</th>
<th>Anomalous geopotential difference between levelling &amp;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>EGM08</td>
</tr>
<tr>
<td></td>
<td>400.75</td>
</tr>
<tr>
<td>Mean</td>
<td>-6.3</td>
</tr>
<tr>
<td>Stdev.</td>
<td>5.3</td>
</tr>
<tr>
<td>Max</td>
<td>3.5</td>
</tr>
<tr>
<td>Min</td>
<td>-16.9</td>
</tr>
</tbody>
</table>

Table 5.7. Statistics of the geopotential heights of the EGM08, airborne gravity and SRTM and their combination compared to the levelling derived geopotential heights. All units are in \(10^{-1} \text{m}^2 \text{s}^{-2}\) equivalent to 1cm.
Figure 5.6. Difference between geodetic levelling and gravity derived geopotential heights ($10^{-1} m^2 s^{-2}$ is approximately 1cm).
5.11 Conclusions

- A global vertical reference system should be defined by the Earth’s gravity field and it must trace level surfaces in space.

- A global vertical reference system should be based on a model of the Earth’s gravity field so that its realisation does not involve levelling networks between nations, geodetic/oceanographic levelling over the ocean surface or tide gauge geodetic benchmarks.

- A choice of a $W_0$ value is conventional, but deriving $U_0$ from the parameters of the global gravity model and then adopting $W_0 = U_0$ ties the physical and geometrical reference systems together inseparably and makes the system globally absolute.

- A unification of a geopotential height system in a global sense requires a consistent usage of tide systems (zero tide system).

- A globally absolute geopotential number can then be determined for any isolated field point without connecting to any local height network.

- The accuracy of the geopotential number depends on the accuracy of the gravity model. Table 5.7 shows that local gravity data do not corrupt the global datum determined from the global gravity model but their quality influences short wavelength undulations. The computational method could only introduce errors of a few millimetres.

- This rationale of computing a global geopotential number in a real time with GNNS receiver is pioneering work that can be applied on a global scale to resolve the real bottleneck of levelling.
Chapter 6

Discussion

6.1 Vertical Reference System

This thesis has determined an absolute geopotential height system for Ethiopia, making two fundamental contributions. First, it develops a new definition of what a vertical reference system is; secondly it shows that this novel approach is able to predict the results of classical geodetic levelling: the mismatch has a standard deviation of only a few centimetres.

The gravitational potential $W$ is central to defining a globally absolute vertical reference system. Chapter 5 summarized the problems in the definition and realization of present height systems. Currently, they involve (i) levelling and gravity to find a geometrical height $H$ above a levelling datum, and (ii) finding the geometrical height $N$ of this datum surface above the ellipsoid whose conventionally defined radius is $R$. In the pre-satellite era, the combination $r = H + N + R$ was the only way of finding absolute geocentric coordinates of a point on the Earth’s surface.

The advent of GNSS technology now determines geocentric positioning coordinates directly and with greater accuracy than the classical method. This has revolutionized the relation between geometrical and physical geodesy. Now a separate vertical reference system is not needed for positioning – it is only needed for sets of specialized problems related to the energetics of transporting mass that need to identify what is “uphill”. It must be able to determine gravitational potential energy differences rigorously. Its primary definition must therefore be the geopotential number $c = W_0 - W$.

The present definition of orthometric height as a potential energy difference scaled by a prediction of local gravity is convenient but flawed. It fails the primary objective of a vertical reference system – mapping a level surface in space – because
a surface of constant orthometric height is not level. The new system should only provide a geometrical height as a derivative for convenience – it cannot be definitive.

Being able to define and realise the new vertical reference system must therefore involve finding and adopting a model for the Earth’s geopotential as a function of the geometrical coordinates provided by GNSS, latitude, longitude and ellipsoidal height, $w(h, \varphi, \lambda)$.

Traditionally, the datum constant has been chosen to be related to mean sea level determined at a reference tide gauge. The datum was then a physical reference point and $w_0$, the value of $w$ at that point. Because of sea surface topography, mean sea level does not tie-in this local datum globally. For a land locked country like Ethiopia, where the reference tide gauge is 2500 km away, the connection of the practical datum to mean sea level is a large additional source of error.

In the new vertical reference system, $w_0$ is set equal to $U_0$, the value of the reference potential on the reference ellipsoid. It is then determined by adopted physical constants of the Earth. With this new approach, the unification of world height systems must use consistent usage of global gravity model, datum potential and the same tide system (Hipkin, 2002; Ihde & Sánchez, 2005; Mäkinen & Ihde, 2009). Present practice uses the zero-tide for gravimetry, station positions and other geodetic measurements as adopted at the XVIII General Assembly of the International Union of Geophysics and Geodesy (IUGG) in Hamburg in 1983 (Tcherning, 1984). The Earth’s gravitational flattening $J_2$ and semi-major axis of the reference ellipsoid $a$ must be determined in the same tide system used for gravimetry so that $w_0 = U_0$ defines the resulting geopotential number to be independent of the permanent tide. Effect of the permanent tide on GNSS position coordinates is also significant (Poutanen et al., 1996) and they are determined in a zero-tide system.

The main part of this thesis, chapter 2, 3 and 5 describe how a very precise model of the geopotential $w(h, \varphi, \lambda)$ was determined for Ethiopia by combining EGM08, the airborne gravity survey and the digital topographic model SRTM.
6.2 Computational overview

This thesis has developed a new approach to computing the geopotential, and hence height, from observations of gravity. The classical Stokes’ approach involves a surface integral of gravity using the normal gravity field and the in situ topography and topography condensed to surface density to make integration simpler and reliable. It was a linearised and spherical approximation. Molodensky developed an equivalent surface integral but now over the topographic surface and leaving the in situ topography. He also proposed an alternative method of computing the physical surface but based on levelling and astrogeodetic measurements alone.

However, the advent of satellite gravity and altimetry missions has changed the way the potential of a point on or above the Earth’s surface can be determined. Recent satellite missions put Stokes’ approach back into a modern ‘Remove-Compute-Restore’ method for computing the geopotential, geoid or geoid-like surfaces. This now gives a geocentric gravitational potential $W$ at points on or above the Earth’s surface using satellite-based gravity information, local gravity observations and topographic models. Finding the geopotential $W$ gives a geopotential number, which uniquely characterises level surfaces in space.

The problem of solving the Stokes’ integral is made tractable by developing a remove-restore approach to smoothing the external gravity field. Satellite gravity information determines a global gravity model with sufficient accuracy for a global vertical reference system with a resolution of about 500 km. Soon the very short baseline gradiometry of GOCE mission may reduce this resolution to 200-300 km (ESA, 1999). However, the determination of high resolution and accurate geopotential model still requires the contribution of local gravity data and the in situ topographic model via Stokes-like integration. With the availability of homogenous gravity and topographic data, the computation of high resolution geopotential perturbations at local scale needs to perform time consuming topographic computation before applying Stokes-like integration. For the first time in the geodetic history, the topographic elevation is available at unprecedented resolution (3-arcseconds or ~90 m) on a nearly global grid and the impact of the SRTM data challenges the computational speed and the accuracy of computers needed to integrate the effects of the topographic mass elements.
6.3 Space Domain Gravity Pre-processing

The second chapter explores gravity modelling techniques used to synthesise small and smoothly varying localised residual gravity disturbance from gravity measured at aircraft height. A gravity model of the reference ellipsoid, global gravity model (e.g. EGM08) and in situ and condensed models of the SRTM topographic attraction have been used in a remove-restore technique to produce residual “gravity” that can be reliably interpolated and integrated. Subtracting a gravity model of the reference ellipsoid from observed gravity gives gravity disturbance at the measurement point. However, the resulting gravity disturbances are neither small nor smooth due to their strong correlation with topography. A modern remove-restore process also subtracts a global gravity model from the gravity disturbance to remove long to medium wavelengths. The resulting residual gravity disturbances are small but still they are influenced by high frequency components of the topographic attraction. Stokes’ understanding that topographic models ought to be used to recover short wavelength gravity information from more widely spaced gravity observations remains the key component in the remove-restore technique to synthesise small, smooth and trend-free residual gravity. This research used two models of topographic attraction to suppress short wavelength components from the residual gravity disturbance calculated at aircraft height.

Detailed models of the Earth’s topography are now much more accurately determined using lidar and microwave remote sensing from satellites and aircraft. The first method presents a new algorithm for the computation of the SRTM topographic effect on gravity data and applies it to the Ethiopian airborne gravity survey. This algorithm was devised to recover the total attraction of the topographic masses (long to short wavelengths) at gravity observation points. The representation of this complex topographic shape by a simple geometrical model of vertical mass prisms distributed over the surface of ellipsoidal or spherical Earth model is computationally very demanding. The use of Bouguer plate (Heiskanen & Moritz, 1967, p. 130) or its spherical shell approximation (Smith et al., 2001; Vaníček et al., 2001) does not resolve the complex shape of the actual topography. Although many active researches have been conducted to evaluate the topographic effect using vertical prism (Nagy et al., 2000), polyhedron (Tsoulis, 2001) and tesseroid (Heck &
in a more effective way, still the issues of their computational speed, and the accuracy of the models in the immediate locality of the computation point except for the prism model are not satisfactorily answered. Hence, the demand for a model with a better performance is needed. The task of implementing a more efficient algorithm needs to create a model with a grossly simplified geometry but one that can preserve the shape of the actual topography accurately.

This research presents a novel algorithm that can maintain the full resolution of the SRTM data. The new model involves vertical line mass elements distributed over the surface of a sphere and directed radially outwards from the centre of the Earth. Each prism of the SRTM topography is then represented by a vertical line mass – making the Earth’s topography a structure analogous to the spines of a hedgehog. In the immediate locality of the computation point the vertical line mass element is replaced by more stable vertical array of point masses. For this case study, the computation of geopotential number requires calculating the gravity and potential of the array of 1.5 billion vertical prisms, needed to represent Ethiopia and its immediate surroundings at 92,433 point gravity observations.

The classical approach of working with “Bouguer” gravity, obtained by subtracting the topographic attraction from the gravity disturbances evaluated at observation points, does not provide the opportunity to make simplifying approximations to Stokes’ integral transform and the downward continuation process, for example using an FFT, because “Bouguer” gravity is neither small nor trend-free. In order to use the computational efficiency of the FFT, the modern approach uses a spherical/ellipsoidal harmonic gravity model of the same topography condensed as a surface density on a reference ellipsoid. Spectral analysis and a low-pass filter applied to the condensed topographic attraction was subtracted from the in situ gravity model calculated by the hedgehog algorithm to create a high-pass filtered version of topographic gravity. This gives a small band-limited, smooth and trend-free residual gravity disturbance that can be reliably interpolated and integrated using FFT.

### 6.3.1 Topographic Gravity Processing

The task of improving the resolution of the gravity and potential fields still involves precise computation of the gravity and potential due to geometrical models
of *in situ* topography. Since the removed gravity must be exactly restored as potential without introducing any computational error in recovering the real potential, the topographic computation should be done with high precision and must involve exactly the same mass distribution. The accuracy and computational speed of the hedgehog algorithm has been tested in comparison with its approximation by a vertical array of point masses (the ‘multipoint’) and a cylindrical sector in the vicinity of the computation point (see, Section 2.4). The analytical solutions of the gravitational potential and attraction of these three models have been investigated in the inner-zone of the computation point defined by spherical distance \(0.02^\circ \geq \psi \geq 0^\circ\). The line mass element approximation for the attraction of a vertical prism is no longer adequate at and very near to the computation point. There, it was replaced by the multipoint algorithm. A line mass element is good beyond a spherical distance of \(30^\circ\). The computer program used multipoint for all SRTM elevations within 1 km of the computation point and line mass element beyond. Compared to the multipoint, the hedgehog algorithm is computationally faster by a factor of 4 and 2 for gravity and potential, respectively. The computational efficiency of the hedgehog algorithm, together with the multipoint for the inner-zone, was successfully used to improve the resolution of about 92,433 usable point gravity data acquired over the whole region of Ethiopia. Even after code-parallelization, this remains near the limits of practical computing – it took about a fortnight on the Edinburgh University supercomputer.

In contrast, spherical harmonic analysis of the condensed topography, and then synthesising its gravity and potential effect could be completed to degree 2160 (grid spacing ~ 10 km) in the order of one day with conventional computer. However, this method is not able to achieve 90 m resolution.

### 6.3.2 Gravity models verification and validation

Computations of the SRTM topographic contribution and the global gravity model were followed by data verification. Now, since almost the complete coverage of the Ethiopian airborne gravity data had been included in the EGM08 model, this model could be used for controlling the quality of our representation of the airborne gravity data. EGM08 used least squares collocation to estimate gravity at zero height, followed by least-square estimation to evaluate global spherical harmonic
coefficients up to degree 2190, equivalent to a 6° (or ~ 10 km) grid. This thesis formed a 2 km grid optimal and so should have a better resolution but if both are otherwise equivalent, the mean value should be the same. Table 2.3 in chapter 2 shows that the residual between point airborne gravity measurements and EGM08 gravity evaluated at the same points had a mean value of 0.26 mgal and a standard deviation of 4.6 mgal. Olesen & Forsberg (internal report) gives point accuracy for the airborne gravity measurements of 2.7 mgal and 2.6 mgal from cross-over analysis for the two seasons. There is therefore a very satisfactory agreement in the mean values and we can have confidence that the absolute datum of EGM08 gravity will be retained when higher resolution residual gravity is added. The residual standard deviation is compatible with the inherent point accuracy of the airborne gravity and the lower resolution of EGM08. Apart from statistical analysis of the whole data set, airborne and EGM08 gravity were compared visually (Figure 2.13, chapter 2). For each flight line, the measured gravity disturbance was compared with that predicted from EGM08 in order to analyse whether any adjustment to the smoothing operation that suppresses the aircraft acceleration needed for those lines that suffered unusual turbulence.

The spherical harmonic representation of the SRTM topographic gravity effect was also compared with the effect of the same mass distribution computed using the hedgehog algorithm. The residual topographic gravity difference (in situ hedgehog minus spherical harmonic condensed) had a mean of -0.96 mgal and a standard deviation of 5.2 mgal (Table 2.3, chapter 2). These figures do not represent errors introduced to the overall computation because the in situ and condensed topography are individual models and each is removed and restored separately. However, their relatively small magnitudes noting their very different resolution (10 km and 90 m), gives some confidence that the algorithm are correct. The quality of the three gravity models has been verified and the residuals from these three gravity models are now essentially very small with 1.2 mgal and 5.2 mgal mean and standard deviation, respectively (Table 2.4, chapter 2 & Table 3.2, chapter 3). With the up coming GOCE gravity data, they will become even smaller.

The tests described so far are internal: they compare the results of different ways of treating the same data. An external test comparing predicted height with
geodetic levelling improved standard deviation from 5.3 cm with EGM08 to 4.6 cm and 3.9 cm when airborne gravity, and airborne gravity plus topographic models are included, respectively. These small figures in the standard deviation justifies that a new height system can be better fitted to a precise geodetic levelling height when precise local gravity information is added to the EGM08 model. A mismatch in the mean values of geopotential heights computed from levelling data and gravity models indicates the vertical offset between their vertical datums. The improvement in the mean values from -6.3 cm with EGM08 to -4.8 when airborne gravity and topographic models are included justifies that the absolute datum of EGM08 gravity can be determined at an accuracy of 1.5 cm when high resolution local gravity information is added. This validates the novelty of the computational methods used in this study to determining an absolute geopotential height system at centimetric accuracy.

6.4 Spectral Domain Gravity Processing

Analytical continuation from data at irregular elevations to determine values on a smooth zero height surface requires the data to be described in the spectral domain. Once this is available, transformation from gravity to potential is trivial and the two operations can be combined.

The third chapter of this thesis first explores how to predict a regular grid of residual gravity disturbances on the ellipsoidal surface from airborne gravity data measured at irregular heights and horizontal positions. Secondly, it transforms the residual gravity disturbances to disturbing potential. Having trend-free and smoothly varying residual gravity disturbance with small amplitude changes the way analytical continuation and integration has to be implemented with great precision and efficiency. In section 4.7 we showed that the Stokes-like convolution integral can be solved on a particular kilometric grid of a particular map projection say Lambert azimuthal projection, allows all the effects of Earth curvature and the non-linear parts of kernels to be correctly represented using a simple planar Fast Fourier Transform. This proposition was justified in section 3.3 by transforming gravity data interpolated on to a kilometric grid of Lambert’s conformal conical projection of an ellipsoid on to a cone to potential. It is supposed that this result would hold equally well for any
other map projection whose distortion was equivalently small over the region of computation.

This was tested numerically with a closure experiment. Very high resolution grids of gravity disturbance and potential were computed for a 3000 km squared region of the surface of the reference ellipsoid using the spherical harmonic coefficients of EGM08. Both were band limited by high-pass filters with cut off degrees between 100 (wavelength ~ 400 km) and 400 (wavelength ~ 100 km). The gravity disturbance was interpolated onto a 2 km grid of a conventional map projection, in this case Lambert’s conformal projection of the WGS84 ellipsoid on to a cone, and then transformed to potential using the simple 2-D planar FFT with 100% zero-padding. The result was re-interpolated on to the regular grid of latitude and longitude for comparison with the potential evaluated directly from EGM08 coefficients. The FFT technique was able to recover the disturbing potential with a mean error and standard deviation better than $10^{-5}$ m and $10^{-3}$ m for residuals with a high pass cut-off of 400 km and better than $10^{-2}$ m and $10^{-1}$ m for a cut-off wavelength of 100 km, respectively. $10^{-2}$ m is equivalent to about 1 mm of height. This experiment verifies that the FFT is able to transform band limited gravity given on an ellipsoidal surface to the equivalent potential. The experiment is independent of the quality of the gravity or of the potential used for comparison – both are ‘exact’; so is their dependence on each other. It therefore demonstrates that the FFT routine can be used to transform real gravity data to their potential instead of classical methods based on Stokes’ integral. No practical error is introduced by the algorithm. The validity of using simple planar FFT for geoid computation has been assumed intuitively and incorporated in a series of geodetic algorithms developed in Edinburgh since 1982 (Hipkin & Hussain, 1982). It is now verified.

We also showed that, an iterative FFT algorithm can be used effectively in conjunction with a precise 3-D interpolation technique to project residual gravity disturbances available at irregular heights and horizontal positions, on to a regular grid on surface of the ellipsoid. The overall numerical accuracy of the FFT algorithm and the cubic Newton’s finite difference interpolation has been examined. It was able to recover the gravity disturbance with a mean and standard deviation better than
0.017 mgal and 0.85 mgal with a high pass cut-off of 200 km, respectively. For a cut-off wavelength of 133 km, we get a mean and standard deviation better than 0.015 mgal and 0.8 mgal, respectively (see, Section 3.7).

The efficiency of the FFT routine allows many novel algorithms that need iterative solutions to Stokes’ problem. The Iterative Combination Method (ICM) (Hunegnaw et al., 2009; Kirby, 1995) generates a self-consistent combination of gravity potential information derived from sea surface altimetry and marine gravimetry, leading to ocean-wide gravity anomaly data sets with a standard deviation of only ~3mGal. In this thesis, the FFT has routinely been used to upward continue the Stokes-like integral derived anomalous geopotential to the field survey point in order to determine the gravitational potential \( W \) on or above the Earth’s surface.

### 6.5 Localization of Stokes’ Integral

The fourth chapter discusses using the planar FFT to solve Stokes’ integral. Now, when by far the largest component of the gravitational potential comes from the satellite derived global gravity model, Stokes-like integration is restricted to adding small local details to a global model (Hipkin, 2004). These global model (EGM08) contributions are about \( 34 \times 10^2 \) m\(^2\) s\(^{-2}\) standard deviation of geopotential number \( (10^2 \) m\(^2\) s\(^{-2}\) corresponds to approximately 1 m of elevation). Residual gravity for the 3000 km square region centred on Ethiopia in the wavenumber band not adequately controlled by satellite data, has a standard deviation of only \( 0.33 \times 10^2 \) m\(^2\) s\(^{-2}\). Having an integrand which is two orders of magnitude smaller and confined to a known spectral band changes the precision needed for integration and thence the strategy for its implementation.

It develops a new way of showing why Stokes’ global integral can be limited to a local spherical cap. This uses a stochastic model to predict how the average of gravity around an annulus decreases with its radius. Section 4.5 explores a stochastic model for the Earth’s gravity field based on the global gravity model EGM08. Bjerhammer (1962) introduced a fundamental concept in geodetic theory: the identification of the minimum radius of sphere inside which no source distribution can account for a harmonic representation of the Earth’s external gravity field. This
Bjerhammer sphere is closely linked to the concept of white noise in communications theory: a random surface density distribution on the Bjerhammer sphere has a constant (white) spectrum but is attenuated and reddened by upward continuation to the observable regions on or above the Earth’s surface. Hipkin (2001) used the term ‘pink noise’ for an observed gravity spectrum reducible by downward continuation to a particular spherical surface where it becomes ‘white noise’.

The radius of the Bjerhammer sphere is deduced from the gravity power spectrum but requires a physically correct definition of the spherical power spectrum. The versions developed by Mauersberger (1956) and Lowes (1974) for geomagnetism and by Kaula (1972) for geodesy were different from each other and both were incorrect. A different picture of the power spectrum of the Earth emerged when the correct definition was derived (Hipkin, 2001): a non-linear, minimum variance search of the power spectrum of EGM96 found four ‘Bjerhammer spheres’ at depths ranging from about 3000 km to 50 km. Extending this search to degree 2000 for the EGM08 model recovered similar values for the radii of the first four spheres but needed two more at shallower depth. Now we have a twelve parameter stochastic model fitting some eight million independent coefficients.

The stochastic model of the global field defines covariance function for the component of the global gravity field associated with each Bjerhammer sphere. The global model parameters characterize the correlation lengths for each component. The covariance function predicts how ring-averaged residual gravity decreases with distance, thereby providing a rational way of predicting how large a spherical cap is needed for ‘Stokes’ integral of residual gravity to capture the whole of the disturbing potential. Figures “4.2” show how ring-averaged gravity is predicted to decrease with ring radius and how much of the potential is generated by anomalies at different distances. Provided that the residual gravity anomalies are restricted to the components associated with the three shallowest Bjerhammer spheres, where depths are less than 50 km, the integral is complete for cap radii less than 2°. There are then no ‘far field’ corrections due to these three components. A simple Butterworth filter is used to eliminate contributions from sources with Bjerhammer spheres deeper than 50 km. The choice of filter is arbitrary but must cut out long wavelength terms completely while maintaining smoothness near the cut-off wavelength. The
Butterworth filter is also convenient because it has simple forms in both the Fourier and spherical harmonic spectra.

**6.6 Future work**

This research has verified how it is possible to determine a vertical reference system that honours the properties to map level surfaces in space from gravity data using the Stokes’ approach of geodetic boundary value problem. Now, the modern ‘Remove-Compute-Restore’ approach of geopotential number computation involves the process of combining the global gravity model, local gravity data and SRTM elevation data in a way that they give complementary solutions.

This work has identified four areas that need new further theoretical development or improved data.

- The representation of topography condensed onto the ellipsoid as the spherical harmonic series worked well enough for Ethiopia where the ellipsoid is close to the circumscribed sphere but gives a divergent solution at higher latitudes. A proper representation in ellipsoidal harmonics is being developed. So, a purely ellipsoidal harmonic representation of the condensed topographic mass elements has to be dealt with in the future to get exact solution on the ellipsoid at a global scale.

- The accuracy of our computational approach has been validated by comparing the geopotential numbers with equivalent solutions derived from an independent precise levelling data collected over 100 km baseline. This form of quality control does not yet cover a large enough region to verify the accuracy of our model at national scale – further precise geodetic levelling is planned for verification to the north.

- The closure experiment testing how well the potential was recovered when EGM08 was truncated at degrees 100, 200 and 400 showed that all gave an adequate representation although the \( n = 400 \) was best. It is anticipated that the new gravity satellite GOCE will provide an independent satellite-only model up to degree 200 (or 100 km spatial resolution) rather than the limit of less than 100 for the satellite-only component of EGM08. The new GOCE model will be able to test how well the model developed for Ethiopia has determined these longer wavelengths.
The quality of the SRTM topographic data used in this study is limited by the accuracy of the model caused by the re-tracking problems over a region of rugged mountains, forests and inland water and so using a more refined brand new Altimeter Corrected Elevations (ACE2) topographic model released by Berry et al., (2007) is necessary to accurately determine the topographic gravity and potential models. ACE2 is synthesised from the combination of the SRTM dataset with altimeter heights derived from the ERS-1, ERS-2, Envisat and Jason Missions. The new topographic model is available at nearly global coverage maintaining full resolution of the SRTM data (3 arc seconds) with much better accuracy.
Appendix A

Gravity model of the geometrical topography

This appendix presents the derivation of the Newtonian vertical gravity attraction for the vertical line mass element representation of the topography by the “hedgehog” model.

The topographic gravity at the field point is known in the direction of the normal gravity at the body point and in the direction perpendicular to the normal gravity at the body point. But the computation of the vertical gravity attraction at the field point can be calculated using vector decomposition principle.

\[
\delta G_z = \delta G_y \sin \psi + \left(-\delta G_x\right) \cos \psi
\]

\[
= \frac{G \xi}{X_0} \left( \frac{Z_2}{\sqrt{Z_2^2 + X_0^2}} - \frac{Z_1}{\sqrt{Z_1^2 + X_0^2}} \right) \sin \psi + \left( \frac{X_0}{\sqrt{Z_2^2 + X_0^2}} - \frac{X_0}{\sqrt{Z_1^2 + X_0^2}} \right) \cos \psi
\]  \hfill (A 1)

where

\[
Z_1 = R_b - (R_f + h_f) \cos \psi, \quad Z_2 = R_b + h_b - (R_f + h_f) \cos \psi,
\]

\[
\ell(h_f, 0) = \sqrt{Z_1^2 + X_0^2}, \quad \ell(h_f, h_b) = \sqrt{Z_2^2 + X_0^2}, \quad X_0 = (R_f + h_f) \sin \psi
\]  \hfill (A 2)

After substituting equation (A2) into equation (A1), the solutions for the vertical line mass element attraction becomes

\[
\delta G_z^v = \frac{G \xi}{R_f + h_f} \left( \frac{R_b + h_b}{\ell(h_f, h_b)} - \frac{R_b}{\ell(h_f, 0)} \right)
\]  \hfill (A 3)
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