Prime Ideals in Quantum Algebras

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Ewan Russell*)

To my family

Abstract

The central objects of study in this thesis are quantized coordinate algebras. These algebras originated in the 1980s in the work of Drinfeld and Jumbo and are noncommutative analogues of coordinate rings of algebraic varieties. The organic nature by which these algebras arose is of great interest to algebraists. In particular, investigating ring theoretic properties of these noncommutative algebras in comparison to the properties already known about their classical (commutative) counterparts proves to be a fruitful process.

The prime spectrum of an algebra has always been seen as an important key to understanding its fundamental structure. The search for prime spectra is a central focus of this thesis. Our focus is mainly on Quantum Grassmannian subalgebras of quantized coordinate rings of Matrices of size $m \times n$ (denoted $\mathcal{O}_q(M_{m,n})$). Quantum Grassmannians of size $m \times n$ are denoted $G_q(m,n)$ and are the subalgebras generated by the maximal quantum minors of $\mathcal{O}_q(M_{m,n})$. In Chapter 2 we look at the simplest interesting case, namely the 2×4 Quantum Grassmannian ($G_q(2,4)$), and we identify the H-primes and automorphism group of this algebra. Chapter 3 begins with a very important result concerning the dehomogenisation isomorphism linking $G_q(m,n)$ and $\mathcal{O}_q(M_{m,n-m})$. This result is applied to help to identify H-prime spectra of Quantum Grassmannians.

Chapter 4 focuses on identifying the number of H-prime ideals in the $2 \times n$ Quantum Grassmannian. We show the link between Cauchon fillings of subpartitions and H-prime ideals. In Chapter 5, we look at methods of ordering the generating elements of Quantum Grassmannians and prove the result that Quantum Grassmannians are Quantum Graded Algebras with a Straightening Law is maintained on using one of these alternative orderings.

Chapter 6 looks at the Poisson structure on the commutative coordinate ring, $G(2,4)$ encoded by the noncommutative quantized algebra $G_q(2,4)$. We describe the symplectic ideals of $G(2,4)$ based on this structure. Finally in Chapter 7, we present an analysis of the 2×2 Reflection Equation Algebra and its primes. This algebra is obtained from the quantized coordinate ring of 2×2 matrices, $\mathcal{O}_q(M_{2,2})$.

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Contents

Chapter 1

Preliminaries

Tools for Noncommutative Algebras

Throughout, k will denote a field. In this initial chapter we will introduce some of the crucial ideas and notions from noncommutative ring theory which will be utilised greatly in subsequent chapters. We will also introduce some of the key algebras whose properties we will be studying throughout the thesis.

1.1 Localization

In the commutative setting, localizations of a ring with respect to a subset and the construction of a field of fractions is well known. In the situation where our ring R is not necessarily commutative, greater care must be taken and extra considerations must be made.

Definition 1.1.1. An element $x \in R$ is called *normal* if $xR = Rx$.

Definition 1.1.2. An element $x \in R$ is called a *regular* element if it is not a divisor of zero. Hence x is regular if

$$\text{ann}_r(x) = \text{ann}_l(x) = 0$$

where $\text{ann}_r(x)$ and $\text{ann}_l(x)$ are the standard right and left annihilator ideals of x respectively. Explicitly, these are the sets

$$\text{ann}_r(x) = \{r \in R : xr = 0\}$$

and

$$\text{ann}_l(x) = \{r \in R : rx = 0\}.$$

Definition 1.1.3. A *multiplicative set* X in a ring R is a subset such that $1 \in X$ and for all $x, y \in X$, $xy \in X$.

Definition 1.1.4. Let R be a ring and let X be a multiplicative set of regular elements of R. We define a *right ring of fractions* for R with respect to the set X to be an overring S of R such that
(i) Every element of X has an inverse in S.
(ii) Each element of S can be expressed in the form ax^{-1} for some $a \in R$ and some $x \in X$.

Definition 1.1.5. Let X be a multiplicative set in a ring R. Then X is a *right Ore set* if, for each $x \in X$ and $r \in R$

$$xR \cap rX \neq \phi.$$

Theorem 1.1.6. *Let R be a ring and let $X \subseteq R$ be a multiplicative set of regular elements. Then a right ring of fractions for R with respect to X exists if and only if X is a right Ore set.*

Proof. See [11], Chapter 6. □

Any such right ring of fractions for a ring R with respect to a right Ore set X of regular elements will be denoted RX^{-1}. Note the special case however where the set X consists solely of powers of a single element x. In this case, we simply denote the localization as $R[x^{-1}]$.

Lemma 1.1.7. *Let x be a regular normal element in a ring R. Then the set*

$$X := \{x^n : n \geq 0\}$$

is a right Ore set in R.

Proof. Let $x^n \in X$ and $r \in R$. Then since x is normal,

$$x^n r = x^{n-1} r_1 x = x^{n-2} r_2 x^2 = ... = r_n x^n$$

for some $r_1, r_2, .., r_n \in R$. Hence X is a right Ore set. □

Now, we define a product ST of two subsets of a ring as

$$ST := \{st : s \in S,\ t \in T\}$$

We take note now of that fact that, if X is a right Ore set in a k-algebra A, then k^*X is also a right Ore set in A.

Proposition 1.1.8. *Let S and T be right Ore sets in a ring R such that $ST = TS$. Then ST is a right Ore set in R.*

Proof. We begin by checking that ST is a multiplicative set. Let $s_1, s_2 \in S$ and $t_1, t_2 \in T$. Then

$$s_1 t_1 s_2 t_2 = s_1 s' t' t_2$$

for some $s' \in S$ and some $t' \in T$. Hence we have that

$$s_1 s' t' t_2 \in ST$$

since S and T are multiplicative sets. Now let $st \in ST$ and let $r \in R$. S is right Ore so there exists $s_1 \in S$ and $r_1 \in R$ such that

$$sr_1 = rs_1$$

Now by the right Ore condition on T applied to $t \in T$ and $r_1 \in R$, there exists $t_1 \in T$ and $r_2 \in R$ such that

$$tr_2 = r_1 t_1$$

So since $sr_1 = rs_1$ we get that

$$(st)r_2 = s(tr_2) = sr_1 t_1 = (sr_1)t_1 = r(s_1 t_1)$$

Hence ST is right Ore. □

The theory of localization will have a very useful purpose later in our analysis of the prime spectra of various algebras.

1.2 Skew Polynomial Extensions

Many quantized coordinate rings can be expressed best in algebraic terms using generators and a list of commutation relations for these generators. Here we outline a construction which can often be applied in such circumstances. Showing that commutation relations between generators can be used to exhibit a given algebra as a *skew polynomial ring* allows us to deduce many ring theoretic properties thanks to the general theory of these extension rings.

Definition 1.2.1. Let R be a ring and let $\sigma : R \longrightarrow R$ be an automorphism of R. We define a *σ-derivation* of R to be a map

$$\delta : R \longrightarrow R$$

which has the properties that, for all $r, s \in R$,

$$\delta(r + s) = \delta(r) + \delta(s)$$
$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s.$$

Specifically in this case, we should say that δ is a *left* σ-derivation of R. Note that there is an obvious notion of right σ-derivation. It should also be noted that when σ is the identity automorphism, the resulting δ is just a regular derivation.

Once we have an automorphism, σ of R and a σ-derivation, δ of R we may define a new extension ring of R.

Definition 1.2.2. With the notation above, we define a *skew polynomial extension* of R to be a ring

$$S := R[x;\sigma,\delta]$$

in which the elements are polynomials in x. Hence a typical element $f \in S$ has the form

$$f = \sum_i r_i x^i.$$

The rule for multiplication of ring elements with the indeterminate x is

$$xr := \sigma(r)x + \delta(r).$$

When $\sigma =$id we will shorten the notation to $S = R[x,\delta]$. Similarly, when $\delta = 0$ we will shorten to $S = R[x;\sigma]$. We will often use the phrase *Ore extension* instead of skew polynomial extension. The details of the construction of these skew extensions is given in [11].

Theorem 1.2.3. *Let σ be an automorphism of R and let δ be a σ-derivaton. Set $T = R[x;\sigma,\delta]$.*
(a) If R is right (left) noetherian then so is T,
(b) If R is a domain then T is a domain.

Proof. See [11], Chapter 2. □

1.3 Gelfand Kirillov Dimension

The algebras which we will be studying are quantized coordinate rings of affine varieties and hence we may express them in terms of a finite number of generating elements, subject to certain commutation relations. In this section we explain the concept of Gelfand Kirillov Dimension for affine k-algebras.

Definition 1.3.1. Let A be an affine k-algebra with finite generating set $\{a_1, ..., a_n\}$. Let V be a finite dimensional subspace of A. We say that V is a *finite dimensional generating subspace* for A if we can express every element of A as a linear combination of monomials formed by elements of V.

An example is the case where V is the subspace of A spanned by the generators $a_1, ..., a_n$. Then if we set

$$V^0 := k$$

and

$$V^n := \text{the subspace spanned by monomials of the form } a_{i_1}^{s_1} a_{i_2}^{s_2} ... a_{i_m}^{s_m}$$

where $a_{i_j} \in \{a_1, ..., a_m\}$ and $\sum s_i = n$. We have

$$A_n = \sum_{i=0}^{n} V^i$$

and $A = \cup_{n=0}^{\infty} A_n$.
Define $d_V(n) := \dim_k(A_n)$

Definition 1.3.2. The *Gelfand Kirillov Dimension* of A is

$$\mathrm{GKdim}(A) := \overline{\lim}\left(\frac{\log d_V(n)}{\log(n)}\right)$$

for V a finite dimensional generating subspace of A.

The Gelfand Kirillov Dimension of the algebra A is independent of the choice of V and useful formulae to aid in calculating Gelfand Kirillov Dimension are given in [13], Chapter 1.

Proposition 1.3.3. *Let A be an affine k-algebra and let B be a k-subalgebra of A. Then*

$$\mathrm{GKdim}(B) \leq \mathrm{GKdim}(A)$$

Proof. See [16], Lemma 3.1. □

Proposition 1.3.4. *Let A be an affine k-algebra and let $B = A[x;\sigma,\delta]$ be an Ore Extension of A. Then*

$$\mathrm{GKdim}(B) \geq \mathrm{GKdim}(A) + 1.$$

Proof. See [13], Lemma 1.3.4. □

Proposition 1.3.5. *Let A be an affine k-algebra and let $B = A[x;\sigma,\delta]$ be an Ore Extension of A. Assume that A has a finite dimensional generating subspace V with $1_A \in V$ and $\sigma(V) \subseteq V$. Then*

$$\mathrm{GKdim}(B) = \mathrm{GKdim}(A) + 1.$$

Proof. See [13], Lemma 1.3.5. □

1.4 Graded Rings

Let G be an additive group and R be a ring.

Definition 1.4.1. We say that R is (G-)graded if there exists a family $\{R_g : g \in G\}$ of additive subgroups such that

$$R = \bigoplus_{g\in G} R_g$$

and

$$R_g R_h \subseteq R_{g+h}$$

for all $g, h \in G$.
An intuitive definition exists for graded R-Modules.

Definition 1.4.2. Let M be a right R-Module. We say that M is (G-)graded if there exists a family $\{M_g : g \in G\}$ of additive subgroups of M such that

$$M = \bigoplus_{g \in G} M_g$$

and

$$M_g R_h \subseteq M_{g+h}$$

for all $g, h \in G$.
We may express any $m \in M$ uniquely as

$$m = \sum_{g \in G} m_g$$

where all but finitely of the m_gs are nonzero. The m_gs are called the *graded components* of m and when $m = m_g$ for some $g \in G$, we say that m is homogeneous of degree g.

1.5 Quantum Matrices and Quantum Grassmannians

The central algebras of study in this thesis will be Quantum Grassmannians. These algebras are subalgebras of Quantum Matrices.

Definition 1.5.1. Let $m, n \in \mathbb{N}$ and let $q \in k$ be a nonzero scalar which we will assume is not a root of unity for the remainder of the thesis. We define the *quantized coordinate ring of $m \times n$ matrices* (or simply $m \times n$ quantum matrices for short) to be the k-algebra generated by mn indeterminates, x_{ij} subject to the commutation relations

$$\begin{aligned} x_{ij}x_{il} &= qx_{il}x_{ij} \\ x_{ij}x_{kj} &= qx_{kj}x_{ij} \\ x_{il}x_{kj} &= x_{kj}x_{il} \\ x_{ij}x_{kl} - x_{kl}x_{ij} &= (q - q^{-1})x_{il}x_{kj} \end{aligned}$$

for $1 \leq i < k \leq m$ and $1 \leq j < l \leq n$.
In terms of matrix entries, the relations tell us that commutations between coordinates on the same row or column come with the production of a factor of q or q^{-1}, commutations between anti-diagonal coordinates are regular commutations, and commutations between coordinates diagonal to one another are messier.

Example 1.5.2. One of the standard examples of quantum matrices is the 2×2 case. This is the k-algebra

$$\mathcal{O}_q(M_{2,2}) = k \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

subject to the 6 relations

$$\begin{aligned}
&x_{11}x_{12} = qx_{12}x_{11}, \quad x_{11}x_{21} = qx_{21}x_{11}, \quad x_{12}x_{21} = x_{21}x_{12} \\
&x_{12}x_{22} = qx_{22}x_{12}, \quad x_{21}x_{22} = qx_{22}x_{21} \\
&x_{11}x_{22} = x_{22}x_{11} + (q - q^{-1})x_{12}x_{21}
\end{aligned}$$

There is a special element in $\mathcal{O}_q(M_{2,2})$ called the *quantum determinant.* This is the element

$$D_q := x_{11}x_{22} - qx_{12}x_{21}$$

The element D_q is central in the algebra. $\mathcal{O}_q(M_{2,2})$

The quantum determinant exists in arbitrary $n \times n$ quantum matrices and it is a central element. It is defined as

$$D_q := \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} x_{1,\sigma(1)}x_{2,\sigma(2)}...x_{n,\sigma(n)}$$

where $\ell(\sigma)$ denotes the length of the permutation σ.

Proposition 1.5.3.

$$\mathcal{O}_q(M_{m,n}) = k[x_{11}][x_{12}; \sigma_{12}, \delta_{12}]...[x_{1n}; \sigma_{1n}, \delta_{1n}][x_{21}; \sigma_{21}, \delta_{21}]...[x_{mn}; \sigma_{mn}, \delta_{mn}]$$

Hence we may express the algebra $\mathcal{O}_q(M_{m,n})$ as an iterated Ore extension on adding the generators in lexicographic order.

Proof. (sketch) We define the automorphisms as

$$\sigma_{st}(x_{ij}) := \begin{cases} q^{-1}x_{ij} & \text{for } i = s, j < t \\ q^{-1}x_{ij} & \text{for } i < s, j = t \\ x_{ij} & \text{for } i < s, j \neq t \end{cases}$$

and we define the σ-derivations as

$$\delta_{st}(x_{ij}) := \begin{cases} 0 & \text{for } i = s, j < t \\ 0 & \text{for } i < s, j \geq t \\ (q^{-1} - q)x_{it}x_{sj} & \text{for } i < s, j < t \end{cases}$$

□

Example 1.5.4.

$$\mathcal{O}_q(M_{2,2}) = k[x_{11}][x_{12}; \sigma_{12}][x_{21}; \sigma_{21}][x_{22}; \sigma_{22}, \delta_{22}]$$

Hence for the 2×2 case a non-trivial derivation is only required on adding the generator x_{22} to the extension.

Corollary 1.5.5. *The algebra $\mathcal{O}_q(M_{m,n})$ is a noetherian domain.*

Proof. This follows directly from the fact that $\mathcal{O}_q(M_{m,n})$ can be expressed as an iterated Ore extension, applying 1.2.3. □

Corollary 1.5.6. *The algebra $\mathcal{O}_q(M_{m,n})$ has Gelfand Kirillov dimenson mn.*

Proof. This follows from 1.3.5. □

For the majority of this thesis we will be interested in a special k-subalgebra of $\mathcal{O}_q(M_{m,n})$.

Definition 1.5.7. Within the quantum matrix $\mathcal{O}_q(M_{m,n})$, let

$$I \subseteq \{1, ..., m\}$$

and

$$J \subseteq \{1, ..., n\}$$

such that $|I| = |J|$. We may use any such pair of subsets to specify a *quamtum minor* within the quantum matrix. Explicitly the quantum minor denoted $[I|J]$ is the determinant of the quantum submatrix with rows indexed by the elements of I and columns indexed by the elements of J.

Definition 1.5.8. Let $m, n \in \mathbb{N}$ with $m \leq n$. We define the *quantized coordinate ring of the $m \times n$ Grassmannian* (or $m \times n$ *Quantum Grassmannian* for short) to be the k-subalgebra of $\mathcal{O}_q(M_{m,n})$ which is generated by the maximal $m \times m$ quantum minors. This subalgebra is denoted $G_q(m, n)$.
The commutation relations for $G_q(m, n)$ can be calculated from the relations for quantum matrices.

Example 1.5.9. $G_q(2, 4)$ is generated by the six minors $[12], [13], [14], [23], [24]$ and $[34]$ subject to the relations :

$$[12][13] = q[13][12], \;\; [12][14] = q[14][12], \;\; [12][23] = q[23][12],$$

$$[12][24] = q[24][12], \;\; [12][34] = q^2[34][12], \;\; [13][14] = q[14][13],$$

$$[13][23] = q[23][13], \;\; [13][24] = [24][13] + (q - q^{-1})[14][23],$$

$$[13][34] = q[34][13], \;\; [14][23] = [23][14], \;\; [14][24] = q[24][14],$$

$$[14][34] = q[34][14], \;\; [23][24] = q[24][23], \;\; [23][34] = q[34][23],$$

$$[24][34] = q[34][24].$$

Together with the Quantum Plücker relation

$$[12][34] - q[13][24] + q^2[14][23] = 0.$$

This can be rewritten in the form

$$[34][12] - q^{-1}[24][13] + q^{-2}[23][14] = 0.$$

Quantum Plücker relations exist in the more general setting and the explicit formulae for these have been calculated.

Proposition 1.5.10 (Generalised Quantum Plücker Relations)**.** *Let* $J_1, J_2, K \subseteq \{1, 2, ..., n\}$ *be such that* $|J_1|, |J_2| \leq m$ *and* $|K| = 2m - |J_1| - |J_2| > m$. *Then*

$$\sum_{K' \sqcup K'' = K} (-q)^{\ell(J_1;K') + \ell(K';K'') + \ell(K'',J_2)} [J_1 \sqcup K'][K'' \sqcup J_2] = 0$$

where $\ell(I; J) := |\{(i, j) \in I \times J : i > j\}|$.

Proof. See [14] p.16, Theorem 2.1. □

The commutation relations for the k-algebra $G_q(2, n)$ for arbitrary $n \in \mathbb{N}$ may be deduced easily from the relations ordering on minors for $G_q(2, 4)$. We have

$$[ab][cd] = q[cd][ab] \text{ if } |\{a, b\} \cap \{c, d\}| = 1 \text{ and } a < b \text{ or } c < d$$
$$[ab][cd] = q^2[cd][ab] \text{ if } a < b < c < d$$
$$[ab][cd] = [cd][ab] \text{ if } a < c < d < b$$
$$[ab][cd] = [cd][ab] + (q - q^{-1})[cb][ad] \text{ if } a < c < b < d$$

Commutation relations in the Quantum Grassmannian $G_q(m, n)$ can get extremely complicated. However, we may appeal to Muir's Law in certain cases.

Proposition 1.5.11 (Muir's Law)**.** *Let* P *and* Q *be subsets of* $\{1, ..., n\}$ *with the same cardinality. Set* $\bar{P} := \{1, ..., n\} \backslash P$ *and* $\bar{Q} := \{1, ..., n\} \backslash Q$. *Consider* $d \in \mathbb{N}^*$ *and, for* $1 \leq s \leq d$, *elements* $c_s \in k$ *and subsets* $I_s, K_s \subseteq P$ *and* $J_s, L_s \subseteq Q$ *such that* $|I_s| = |J_s|$ *and* $|K_s| = |L_s|$. *If the relation*

$$\sum_{s=1}^{d} c_s [I_s|J_s][K_s|L_s] = 0$$

holds in $\mathcal{O}_q(M_n)$, *then the relation*

$$\sum_{s=1}^{d} c_s [I_s \cup \bar{P}|J_s \cup \bar{Q}][K_s \cup \bar{P}|L_s \cup \bar{Q}] = 0$$

holds in $\mathcal{O}_q(M_n)$.

Proof. See [23] Proposition 1.3. □

Muir's Law often allows us to simplify commutation calculations within $G_q(m, n)$ by effectively disregarding any common column indices between the minors involved. In this case, Muir's Law tells us that the commutation will be the same as the commutation between the remaining column indices.

Example 1.5.12. Within the algebra $G_q(5,9)$, if we want to find the commutation relation between the minors

$$[23689][35789]$$

we simply need to set

$$L := \{3, 8, 9\}$$

and observe that the commutation we have is

$$[26\ \ L][57\ \ L]$$

and hence applying Muir's Law, we have

$$[26\ \ L][57\ \ L] = [27\ \ L][56\ \ L] + (q - q^{-1})[27\ \ L][56\ \ L]$$

Hence

$$[35689][23789] = [23789][35689] + (q - q^{-1})[23789][35689].$$

Theorem 1.5.13. *(Kelly) $G_q(m,n)$ is a noetherian domain.*

Proof. See [13] Theorem 2.2.7. □

Theorem 1.5.14. $\mathrm{GKdim}(G_q(m,n)) = m(n-m) + 1$.

Proof. See [13] Proposition 2.3.14. □

1.6 Prime Ideals

It is well understood that the prime ideals are the building blocks of a ring. In the non-commutative case, identifying all the prime ideals of a given ring can be a difficult task.

Definition 1.6.1. Let R be a ring. We say that an ideal P of R is a *prime ideal* if, whenever I and J are ideals of R such that

$$IJ \subseteq P$$

then $I \subseteq P$ or $J \subseteq P$.
A *prime ring* is one in which 0 is a prime ideal.

Definition 1.6.2. An ideal I of a ring R is called *semiprime* if

$$I = \bigcap P_\lambda$$

where $\{P_\lambda\}$ is a collection of prime ideals of R.

Proposition 1.6.3. *Let R be a ring and let P be a proper ideal of R. The follwing are equivalent:*

(a) P is prime,
(b) R/P is a prime ring,
(c) For all $x, y \in R, xRy \subseteq P \Longrightarrow x \in P$ or $y \in P$.

Proof. See [11], Chapter 3. □

Definition 1.6.4. Let $R = \bigoplus_{g \in G} R_g$ be a graded ring. An ideal $I \trianglelefteq R$ is called a *graded ideal* of R if

$$I = \bigoplus_{g \in G} I \cap R_g$$

A *graded prime ideal* of R is a proper graded ideal, $P \triangleleft R$, such that whenever A and B are graded ideals of R

$$AB \subseteq P \Longrightarrow A \subseteq P \quad \text{or} \quad B \subseteq P.$$

In a ring R we will use the notation $\mathrm{Spec}(R)$ to denote the set of all prime ideals of R. We also make note here of the Zariski Topology which we have on the set $\mathrm{Spec}(R)$. If we let I be an ideal of R and set

$$V(I) := \{P \in \mathrm{Spec}(R) : I \subseteq P\}$$

and

$$W(I) := \{P \in \mathrm{Spec}(R) : I \nsubseteq P\}.$$

Then the collection

$$\mathcal{W} := \{W(I) : I \trianglelefteq R\}$$

is the family of open sets for the Zariski Topology. The closed sets are exactly the $V(I)$s.
When working with Quantum Algebras, identifying a group acting via automorphisms often proves extremely useful in identifying the prime spectrum of the algebra. The basis behind this is given in the theory of Stratification given by Goodearl and Letzter.
Let H be a group acting by automorphisms on a ring R. By this we mean that we have specified a homomorphism of groups

$$\alpha : H \longrightarrow \mathrm{Aut}(R).$$

To simplify our calculations we will omit the symbol α and we will write $h.a$ instead of $\alpha(h)(a)$ when the group acts on the element $a \in R$.
An *H-ideal* of R is any ideal I of R such that $h(I) = I$ for all $h \in H$.
An *H-Prime ring* is any ring in which any finite product of nonzero H-ideals is nonzero. Hence we define an *H-Prime ideal* of R as an ideal I of R such that the factor ring R/I is an H-Prime ring.
Given any ideal I of R, we set

$$(I : H) := \bigcap_{h \in H} h(I).$$

This is the largest H-ideal of R contained in I.

Proposition 1.6.5. *Let P be a prime ideal of R. Then $(P : H)$ is an H-Prime ideal of R.*

Proof. Let I and J be H-ideals of R such that

$$IJ \subseteq (P : H) \subseteq P.$$

Then since P is a prime ideal, we must have $I \subseteq P$ or $J \subseteq P$. Hence, for all $h \in H$, we must have

$$I = h(I) \subseteq h(P)$$

i.e.

$$I \subseteq \bigcap_{h \in H} h(P) = (P : H)$$

or

$$J = h(J) \subseteq h(P)$$

i.e.

$$J \subseteq (P : H).$$

Hence $(P : H)$ is H-prime as claimed. □

Armed with this fact, we define an equivalence on Spec(R) by declaring that

primes P and Q of R are equivalent if and only if $(P : H) = (Q : H)$.

This equivalence splits Spec(R) up into disjoint H-strata. For any H-Prime ideal J of R, the *H-stratum* in spec$R$ corresponding to J is

$$\mathrm{Spec}_J(R) = \{P \in \mathrm{Spec}(R : (P : H) = J\}.$$

Then

$$\mathrm{Spec}(R) = \bigsqcup \mathrm{Spec}_J(R)$$

as J varies through $H - \mathrm{Spec}(R)$.
Hence we may analyse Spec(R) by identifying $H - \mathrm{Spec}(R)$ and then describing each individual H-stratum.
The notion of stratification works very nicely for a variety of quantized function algebras where an intuitive action by automorphisms by an algebraic torus can often be used. The generators are H-eigenvectors.

Example 1.6.6. With $\mathcal{O}_q(M_{2,2})$, the torus $(k^*)^4$ acts on each of the four generators as

$$(\alpha_1, \alpha_2, \beta_1, \beta_2).x_{11} = \alpha_1\beta_1 x_{11}, \quad (\alpha_1, \alpha_2, \beta_1\beta_2).x_{12} = \alpha_1\beta_2 x_{12},$$

$$(\alpha_1, \alpha_2, \beta_1, \beta_2).x_{21} = \alpha_2\beta_1 x_{21}, \quad (\alpha_1, \alpha_2, \beta_1, \beta_2).x_{22} = \alpha_2\beta_2 x_{22}.$$

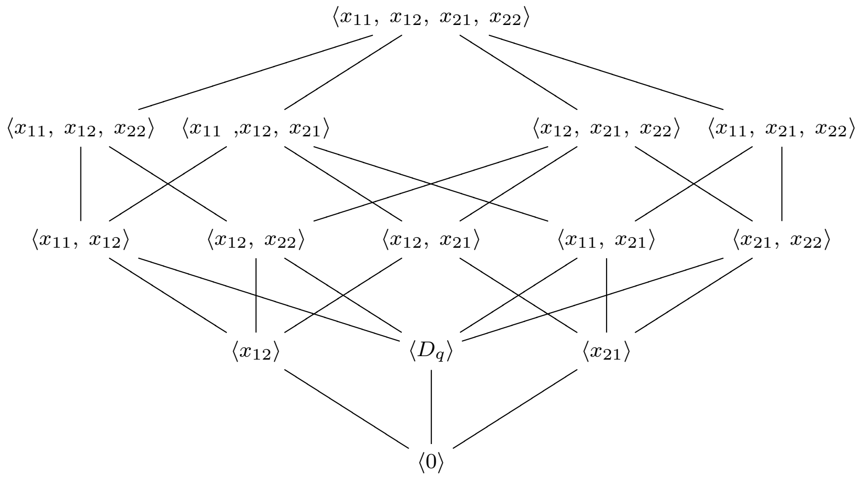


Figure 1.1: H-Spec($\mathcal{O}_q(M_{2,2})$)

There are 14 different H-primes under this action and they are displayed in Figure 1.1. It has been proved by Goodearl and others that there are exactly 14 H-primes thus allowing a complete description of the prime spectrum of $\mathcal{O}_q(M_{2,2})$. For a detailed account of this see [8].

1.7 Dehomogenisation

Definition 1.7.1. Let R be a commutative graded algebra and $x \in R$ is a regular homogeneous element of degree 1. We define the *dehomogenisation* of R at x to be the factor algebra

$$\text{Dhom}(R, x) := R/ < x - 1 > .$$

It often proves to be fruitful to study the properties of the algebra $\text{Dhom}(R, x)$ in order to uncover properties of R. Unfortunately this definition of dehomogenisation is unsuitable as a general definition in the noncommutative world. However, there is an alternative definition for the commutative case from which we can make a definition of noncommutative dehomogenisation.

Proposition 1.7.2. *With notation as above, let $S = R[x^{-1}]$ be the localization of R at x. Then S has a natural $\mathbb{Z}$-grading and we have*

$$\text{Dhom}(R, x) \cong S_0$$

Proof. See [13] chapter 3.0.2. □

In the noncommutative setting we need to be slightly careful with the setup. Firstly, we should take a $\mathbb{N}$-graded ring R and let $x \in R$ be regular, homogeneous of degree one and normal. The normality here guarantees that we are able to localize with respect to the set

$$X := \{x^i : i \geq 0\}$$

(see Section 1.1). As usual we will denote this localization by $S = R[x^{-1}]$. For $i \in \mathbb{Z}$ we now define

$$S_i := \sum_{l=0}^{\infty} R_{i+l} x^{-l}.$$

Kelly shows in [13] that the additive family of subgroups $\{S_i : i \in \mathbb{Z}\}$ forms a $\mathbb{Z}$-grading on the localization S.

Definition 1.7.3. In the setup described above, the degree zero algebra in the $\mathbb{Z}$-grading of S is defined to be the dehomogenisation of R at the element x and we shall write

$$\mathrm{Dhom}(R, x) := S_0.$$

Proposition 1.7.4. *Let R be an $\mathbb{N}$-graded algebra and let $x \in R$ be a regular normal homogeneous element of degree 1. Then there is an isomorphism*

$$\theta : \mathrm{Dhom}(R, x)[y, y^{-1}; \sigma] \longrightarrow R[x^{-1}]$$

which is the identity on $\mathrm{Dhom}(R, x)$ *and maps y to x.*

Proof. See [14, Lemma 3.2]. □

The next three results give important noncommutative properties of the dehomogenisation. For proofs of these results see [13] p. 62-63

Proposition 1.7.5. *If* $\mathrm{Dhom}(R, x)$ *is a domain, then R is a domain.*

Proposition 1.7.6. *Let R be a noetherian ring, then $S := R[x^{-1}]$ and* $\mathrm{Dhom}(R, x)$ *are also noetherian.*

Proposition 1.7.7. *Let R be a finitely generated, $\mathbb{N}$-graded algebra with a regular normal homogeneous element x of degree 1. Let σ be the automorphism of R obtained by conjugating by x. Suppose that R has a finite dimensional generating subspace V with $1 \in V$ such that $\sigma(V) = V$. Then* $\mathrm{GKdim}(S) = \mathrm{GKdim}(R)$ *and* $\mathrm{GKdim}(\mathrm{Dhom}(R, x)) = \mathrm{GKdim}(R) - 1$.

In the classical situation the dehomogenisation of the coordinate ring of the $m \times n$ Grassmannian at the rightmost minor is isomorphic to the coordinate ring of $m \times (n - m)$ matrices. Hence

$$\mathcal{O}(G(m, n)) \big/ < [n - m + 1, ..., n] > \cong \mathcal{O}(M_{m,n-m}).$$

The search for a notion of noncommutative dehomogenisation and the noncommutatve analogue of this classical result was resolved by Kelly.

Theorem 1.7.8 (Kelly)**.**

$$\mathcal{O}_q(M_{m,n-m}) \cong \mathrm{Dhom}(G_q(m,n), [n-m+1, ..., n]).$$

Proof. See [14] Theorem 4.3. □

Chapter 2

The Algebra $G_q(2,4)$

In this chapter we will investigate properties of the 2×4 Quantum Grassmannian, $G_q(2,4)$. This is the simplest non-trivial example of the Quantum Grassmannian algebras. We study the H-prime ideals of this algebra and identify them explicitly using a successive factorisation and dehomogenisation procedure. We use the H-invariant primes at height one to find the automorphism group of $G_q(2,4)$. Set $G := G_q(2,4)$ throughout this chapter.

2.1 Partial Order and the Preferred Basis

We take note in this section of the standard partial order on the algebra $G_q(m,n)$.

Definition 2.1.1. Let $m, n \in \mathbb{Z}$ with $m \leq n$ and let $1 \leq t \leq m$. An *index pair* (I, J) of cardinality t is a pair of subsets $I \subseteq \{1, ..., m\}$ and $J \subseteq \{1, ..., n\}$ such that $|I| = |J| = t$. If $I = \{i_1, ..., i_t\}$ and $J = \{j_1, ..., j_t\}$ with $1 \leq i_1 < ... < i_t \leq m$ and $1 \leq j_1 < ... < j_t \leq n$ then we will sometimes write

$$I = \{i_1 < ... < i_t\}$$

and

$$J = \{j_1 < ... < j_t\}.$$

Generally we will denote the set of all index pairs by $\Delta_{m,n}$. To any index pair $(I, J) \in \Delta_{m,n}$ of cardinality t we associate the $t\times t$ quantum minor $[I|J]$. Similarly we define an *index set* to be a subset J of m pairwise distinct elements of $\{1, ..., n\}$. The set of all index sets is denoted by $\Pi_{m,n}$.

The set $\Pi = \Pi_{m,n}$ of index sets (equivalently, of generating quantum minors of $G_q(m,n)$) carries a natural partial order defined in the following way. Let $I = \{i_1 < \cdots < i_m\}$ and $J = \{j_1 < \cdots < j_m\}$ be two index sets, then

$$I \leq_{\text{st}} J \iff i_k \leq j_k \quad \text{for} \quad 1 \leq k \leq m.$$

The partial order on $G_q(2,4)$ is displayed in Figure 2.1. In Theorem 1.5.14, we

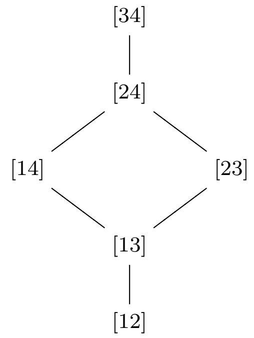


Figure 2.1: The poset Π_1 on $G_q(2,4)$.

saw that $\text{GKdim}(G_q(m,n)) = m(n-m)+1$. In this section, we will make use of the *preferred basis* of $G_q(m,n)$ which we describe here in the 2×4 case.

Definition 2.1.2. We will declare that the products

$$[T_{\mathbf{a}}] = [12]^{a_1}[13]^{a_2}[14]^{a_3}[24]^{a_4}[34]^{a_5}$$

and

$$[S_{\mathbf{b}}] = [12]^{b_1}[13]^{b_2}[23]^{b_3}[24]^{b_4}[34]^{b_5}$$

where a_i, b_i are non-negative integers, $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$ and $\mathbf{b} = (b_1, b_2, b_3, b_4, b_5)$ are *preferred products* in G.

Proposition 2.1.3. *The set of preferred products $\{[T_a], [S_b]\}$ forms a basis for $G_q(2,4)$.*

Proof. See [14, Proposition 2.2]. □

In Chapter 2 of [13], Kelly discusses preferred products in $G_q(m,n)$ and proves that these form a basis of $G_q(m,n)$. The technique is by using Young Tableaux. For details of this see [13].

The elements $[12], [23], [34]$ and $[14]$ are all normal elements in G and hence the localization of G at each of these can be formed. Although $[13]$ is not a normal element of G, we can still show that the non-negative powers of this quantum minor forms a right Ore set.

Lemma 2.1.4. *For all $i \geq 2$ we have*

$$[24][13]^i = [13]^{i-1}([13][24] - (q-q^{-1})\sum_{k=0}^{i-1} q^{-2k}[14][23])$$

in $G_q(2,4)$.

Proof. Firstly note that we have

$$[24][13] = [13][24] - (q-q^{-1})[14][23]$$

From this relation we deduce easily that

$$[24][13]^2 = [13]([13][24] - (q - q^{-1})(1 + q^{-2})[14][23])$$

which establishes the case $i = 2$. Now assume that the result holds for $i = t$. Then we have

$$\begin{aligned}
&[24][13]^{t+1} \\
&= [24][13]^t[13] \\
&= [13]^{t-1}([13][24] - (q - q^{-1}) \sum_{k=0}^{t-1} q^{-2k}[14][23])[13] \\
&= [13]^{t-1}([13][24][13] - (q - q^{-1}) \sum_{k=0}^{t-1} q^{-2k}q^{-2}[13][14][23]) \\
&= [13]^{t-1}([13]^2[24] - (q - q^{-1})[13][14][23] - (q - q^{-1}) \sum_{k=0}^{t-1} q^{-2(k+1)}[13][14][23]) \\
&= [13]^t([13][24] - (q - q^{-1}) \sum_{k=0}^{t} q^{-2k}[14][23]).
\end{aligned}$$

□

Corollary 2.1.5. *The set*

$$X := \{[13]^t : t \geq 0\}$$

is a right Ore set in $G_q(2, 4)$.

Proof. Let $x = [13]^t \in X$ and let $\alpha \in G_q(2, 4)$. Then we may take

$$\alpha = \sum_{\mathbf{a}} \alpha_{\mathbf{a}}[12]^{a_1}[13]^{a_2}[14]^{a_3}[24]^{a_4}[34]^{a_5} + \sum_{\mathbf{b}} \alpha_{\mathbf{b}}[12]^{b_1}[13]^{b_2}[23]^{b_3}[24]^{b_4}[34]^{b_5}$$

with $a_k, b_k \geq 0$, $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$, $\mathbf{b} = (b_1, b_2, b_3, b_4, b_5)$, and $\alpha_{\mathbf{a}}, \alpha_{\mathbf{b}} \in k$. Now we need to show that there exist $y \in X$ and $\beta \in G_q(2, 4)$ such that

$$x\beta = \alpha y$$

Thanks to the preceding lemma we can satisfy the condition by taking

$$y = [13]^{t+\max(a_4, b_4)}.$$

□

2.2 H Action On G

There is a natural action of the group $H = (k^*)^4$ on G where

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4).x_{ij} = \alpha_j x_{ij}.$$

We have the result that

$$\text{Dhom}(G, [34]) \cong \mathcal{O}_q(M_{2,2})$$

and in the more general setting,

$$\text{Dhom}(G_q(m,n), [n-m+1, ..., n]) \cong \mathcal{O}_q(M_{m,n-m}).$$

Hence we need to find out how the natural action on G is related to the standard H action on $\mathcal{O}_q(M_{2,2})$. The standard torus action of $H := (k^*)^m \times (k^*)^{n-m}$ on $\mathcal{O}_q(M_{m,n-m})$ is defined in such a way that the element $\alpha := (\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_{n-m}) \in H$ has the effect that

$$\alpha.x_{ij} = \alpha_i \beta_j x_{ij}$$

for $1 \leq i \leq m,\ 1 \leq j \leq n-m$.

Notation 2.2.1. Let $1 \leq s \leq m$, then the element $[i_1 ... \widehat{i_s}, ..., i_m]$ is the minor obtained from the columns indexed by the set $\{i_1, ..., i_m\} \backslash \{i_s\}$.

Proposition 2.2.2. *Let $\{-\} := [-][n-m+1, ..., n]^{-1}$ in Dhom$(G_q(m,n), [n-m+1, ..., n])$.*
$\alpha := (\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_{n-m})$ acts on the general $m \times n$ quantum grassmannian, $G_q(m,n)$ via :

$$\alpha.x_{ij} = \beta_j x_{ij}, \quad (j = 1, ..., n-m)$$
$$\alpha.x_{ij} = \alpha_{n+1-j}^{-1} x_{ij}, \quad (j > n-m)$$

such that, if

$$\theta : \mathcal{O}_q(M_{m,n-m}) \longrightarrow \text{Dhom}(G_q(m,n), [n-m+1, ..., n])$$

is the isomorphism defined by

$$\theta(x_{ij}) = \{j,\ n-m+1, ..., \widehat{n-i+1}, ..., n\}$$

$1 \leq i \leq m,\ \ 1 \leq j \leq n-m,$
then

$$\theta^{-1} \alpha \theta = \alpha.$$

Proof. We have

$$\theta^{-1}\alpha\theta(x_{ij}) = \theta^{-1}\alpha\{j,\ n-m+1, .., \widehat{n-i+1}, .., n\}$$

$$= \theta^{-1}\alpha([j,\ n-m+1, .., \widehat{n-i+1}, .., n])\alpha([n-m+1, ..., n]^{-1})$$

$$= \theta^{-1}(\beta_j \alpha_m^{-1}\alpha_{m-1}^{-1}...\alpha_{i+1}^{-1}\alpha_{i-1}^{-1}...\alpha_1^{-1}\alpha_m\alpha_{m-1}...\alpha_{i+1}\alpha_i\alpha_{i-1}...\alpha_1\{j, n-m+1, .., \widehat{n-i+1}, .., n\})$$
$$= \alpha_i\beta_j x_{ij}.$$

□

A feature of the torus actions on the algebras $\mathcal{O}_q(M_{2,2})$ and $G_q(2,4)$ which we take note of here is the fact that the action of $H = (k^*)^4$ on $G_q(2,4)$ is not faithful. To see that the action of H on $G_q(2,4)$ is not faithful, it is easily checked that the element $h := (-1,-1,-1,-1) \in H$ has $h([ij]) = [ij]$ for all generating minors $[ij]$ of $G_q(2,4)$.

Lemma 2.2.3. *Suppose that $h \in (k^*)^4$ has $h([ij]) = [ij]$ for all generating minors $[ij]$ of $G_q(2,4)$. Then either $h = (-1,-1,-1,-1)$ or $h = (1,1,1,1)$, the identity element of the torus.*

Proof. Let $h = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in H$ be such that

$$h([ij]) = [ij] \tag{2.1}$$

for all generating minors $[ij] \in G_q(2,4)$. Now by the definition of the H-action, we have that

$$h([ij]) = \alpha_i\alpha_j[ij].$$

By condition 2.1 we have that

$$\alpha_1\alpha_2 = 1,$$
$$\alpha_1\alpha_3 = 1,$$
$$\alpha_1\alpha_4 = 1,$$
$$\alpha_2\alpha_3 = 1,$$
$$\alpha_2\alpha_4 = 1,$$
$$\alpha_3\alpha_4 = 1.$$

Hence we must have

$$\alpha_1\alpha_2 = \alpha_1\alpha_3 = \alpha_1\alpha_4$$

and so $\alpha_2 = \alpha_3 = \alpha_4$. Also

$$\alpha_1\alpha_2 = \alpha_2\alpha_3 = \alpha_2\alpha_4$$

implying that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$. Hence it is easy to see that, for all $i \in \{1,2,3,4\}$ we have

$$\alpha_i^2 = 1 \Longrightarrow \alpha_i = \pm 1.$$

Hence

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$$

or

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -1.$$

□

Now let R be an $\mathbb{N}$-graded ring and let $x \in R$ be a regular, normal homogeneous element of degree one. Set $S := R[x^{-1}]$. Then $\mathrm{Dhom}(R, x) := S_0$.
Assume that there is a group H acting on R by automorphisms and that, under this action, the element $x \in R$ is an H-eigenvector. Then the H action on R extends to the localization S and hence there is also an induced H action on the dehomogenisation S_0.

Lemma 2.2.4. *Let Y be the set of all graded H-Primes in R modulo which x is regular. Let Z denote the set of proper H-Primes of S_0. Then the map*

$$\Gamma_x : Y \longrightarrow Z$$

given by the chain

$$I \mapsto IS \mapsto IS \cap S_0$$

is
(1) Inclusion Preserving,
(2) Injective,
(3) Surjective

Proof. Most of this is checked by Kelly in [13], Chapter 3. We are left to verify firstly that $IS \cap S_0$ is indeed a proper H-Prime of S_0.
We have that IS is an H-ideal of S since for all $h \in H$

$$h(IS) \subseteq h(I)h(S) \subseteq Ih(S) \subseteq IS$$

Now IS is an H-Prime of S since, if A and B are H-ideals of S such that

$$AB \subseteq IS$$

then

$$(A \cap R)(B \cap R) \subseteq AB \cap R \subseteq IS \cap R = I$$

Now I is an H-Prime of R and so

$$A \cap R \subseteq I \text{ or } B \cap R \subseteq I$$

Hence

$$(A \cap R)S \subseteq IS \text{ or } (B \cap R)S \subseteq IS$$

i.e.

$$A \subseteq IS \text{ or } B \subseteq IS$$

Hence IS is an H-Prime of S. Now the action of H on S restricts to an action of H on S_0. Hence $IS \cap S_0$ is a proper H-Prime of S_0.

Properties (1) and (2) are proved by Kelly in [13], Chapter 3.. To see that

Γ_x is onto, let T be a proper H-Prime of S_0. Then TS is an H-Prime of S by an argument similar to that above. Now $TS \cap R$ is an H-Prime of R and x is regular modulo $TS \cap R$. Finally,

$$\Gamma_x(TS \cap R) = (TS \cap R)S \cap S_0 = TS \cap S_0 = T.$$

□

From the partial ordering on $G_q(2,4)$ and referring to [13, Chapter 4], we see that if P is an H-Prime of $G_q(2,4) := G$, then there are seven possible cases to consider :

Case 1 : $[34] \notin P$,
Case 2 : $[34] \in P$, $[24] \notin P$,
Case 3 : $[34], [24] \in P$, $[23] \notin P$,
Case 4 : $[34], [24], [23] \in P$, $[14] \notin P$,
Case 5 : $[34], [24], [23], [14] \in P$, $[13] \notin P$,
Case 6 : $[34], [24], [23], [13], [14] \in P$, $[12] \notin P$.
Case 7 : $[34], [24], [23], [13], [14], [12] \in P$.

2.3 An Analysis Of The 7 Cases

We now give an analysis of the seven cases detailed in section 2.2. Note that in what follows, the notation $[ij]$ for a minor in $G_q(2,4)$ will also be used to denote the image of the relevant minor in the various factor rings which we will construct.
CASE 1
Let $S = G[[34]^{-1}]$. Then S_0 =Dhom$(G, [34])$ and we have the result that

$$\text{Dhom}(G, [34]) \cong \mathcal{O}_q(M_{2,2})$$

via the map

$$\phi : \mathcal{O}_q(M_{2,2}) \longrightarrow S_0$$

such that

$$a \mapsto [\bar{13}], \;\; b \mapsto [\bar{23}]$$
$$c \mapsto [\bar{14}], \;\; d \mapsto [\bar{24}]$$
$$D_q \mapsto [\bar{12}]$$

where $[\bar{-}] := [-][34]^{-1}$. Hence, using the map $\Gamma_{[34]}^{-1}$, and the fact that the H-Prime spectrum of $\mathcal{O}_q(M_{2,2})$ has been identified, we can identify those H-Primes of G which do not contain $[34]$.

Proposition 2.3.1. *There are 14 H-Primes of G which do not contain* $[34]$.

These are displayed in the figure below.

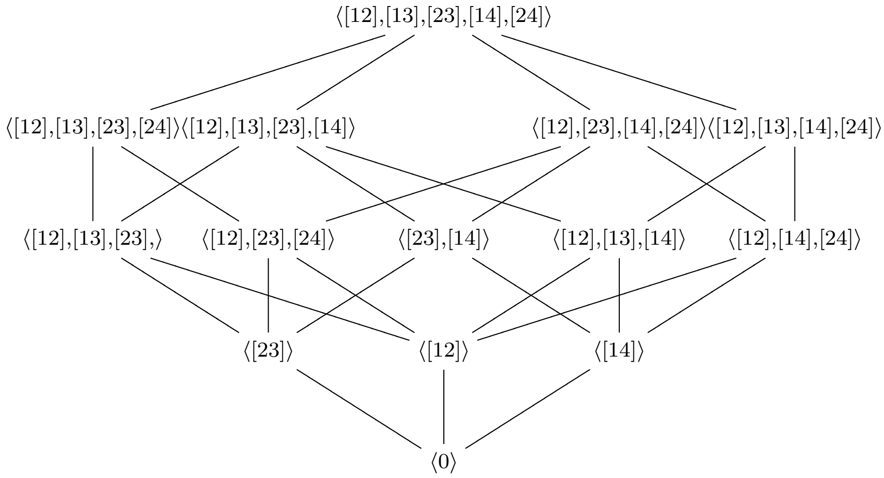


Figure 2.2: The Case 1 H-Primes

CASE 2

Set $G_1 := G \diagup \langle [34] \rangle$. Then Kelly has proved in Chapter 4 that [24] is regular and normal in G_1. Thus we can form the localization $T := G_1[[24]^{-1}]$. It follows that

$$T_0 = \text{Dhom}(G_1, [24]).$$

It is then easy to verify that T_0 is generated by

$$\widetilde{[12]},\ \widetilde{[13]},\ \widetilde{[14]},\ \widetilde{[23]}$$

where $\widetilde{[-]} := [-][24]^{-1}$.

Now Kelly observes that, upon considering the Quantum Plucker relation modulo [34], a generating set for T_0 is in fact

$$\widetilde{[12]},\ \widetilde{[14]},\ \widetilde{[23]},$$

Kelly also shows that

$$T_0 \cong R$$

where R is the k-algebra generated by three elements X, Y, Z subject to the commutation rules :

$$XY = qYX,\ \ XZ = qZX,\ \ YZ = ZY.$$

Explicitly, this isomorphism is defined on the generators by sending

$$\widetilde{[12]} \mapsto X,\ \ \widetilde{[14]} \mapsto Y,\ \ \widetilde{[23]} \mapsto Z.$$

Hence we can identify T_0 as a quantum affine 3-space. We now recall a result from Goodearl and Brown which tells us about the H-Primes in such spaces.

Proposition 2.3.2. *([2] p.157) Let* $A = \mathcal{O}_q(k^n) = k\langle x_1, ..., x_n : x_i x_j = q_{ij} x_j x_i, i > j\rangle$ *have the standard torus action of* $H = (k^*)^n$ *with* k *infinite. For* $\omega \subseteq \{1, ..., n\}$, *set*

$$J_\omega := \langle x_i : i \in \omega \rangle .$$

Then the ideals J_ω *are precisely the* H*-Primes of* A.

Proposition 2.3.3. *The Case 2* H*-Primes of* G *are the ideals*

$$\{\langle [34] \rangle, \langle [12], [34] \rangle, \langle [34], [14], [13] \rangle, \langle [34], [23], [13] \rangle, \langle [34], [14], [12], [13] \rangle,$$

$$\langle [34], [14], [23], [13] \rangle, \langle [34], [23], [12], [13] \rangle, \langle [34], [14], [12], [23], [13] \rangle\}.$$

CASE 3

Kelly shows that, if P is one of the Case 3 H-Primes, then as well as having $[34] \in P$ and $[24] \in P$, we must also have $[14] \in P$. This follows from the Quantum Plucker relation.

Set $G_2 := G/\langle [34], [24], [14] \rangle$. It is shown by Kelly that the image of the element $[23]$ is normal and regular in G_2. Hence we can form the localization $U = G_2[[23]^{-1}]$ and we have that

$$U_0 = \text{Dhom}(G_2, [23]).$$

Now U_0 is generated by the 2 elements

$$\widehat{[12]}, \ \widehat{[13]}$$

where $\widehat{[-]} := [-][23]^{-1}$. We have the commutation rule

$$\widehat{[12]}\widehat{[13]} = q\widehat{[13]}\widehat{[12]}$$

from which it follows easily that

$$U_0 \cong k_q[x, y]$$

Hence using the isomorphism above and the map $\Gamma^{-1}_{[23]}$, we can calculate easily the Case 3 H-Primes of G.

Proposition 2.3.4. *The Case 3* H*-Primes of* G *are :*

$$\{\langle [34], [24], [14] \rangle, \langle [34], [24], [14], [12] \rangle, \langle [34], [24], [14], [13] \rangle, \langle [34], [24], [14], [12], [13] \rangle\}.$$

CASE 4

Set $G_3 = G/\langle [34], [24], [23] \rangle$. Then, as before, $[14]$ is a regular, normal element in G_3 and so we may form $V = G_3[[14]^{-1}]$. Hence $V_0 = \text{Dhom}(G_3, [14])$ and V_0 is generated by

$$\dot{[12]}, \ \dot{[13]},$$

where $\dot{[-]} := [-][14]^{-1}$. We have the relation

$$[\dot{12}][\dot{13}] = q[\dot{13}][\dot{12}]$$

Hence, as in the previous case, it follows that

$$V_0 \cong k_q[x, y].$$

Thus we know the Case 4 H-Primes.

Proposition 2.3.5. *The Case 4 H-Primes are :*

$\{\langle[34],[24],[23]\rangle, \langle[34],[24],[23],[12]\rangle, \langle[34],[24],[23],[13]\rangle, \langle[34],[24],[23],[12],[13]\rangle\}.$

CASE 5
Set $G_4 = G\diagup\langle[34],[24],[23],[14]\rangle$. Then [13] is a normal, regular element in G_4 and so we may localize to obtain $W = G_4[[13]^{-1}]$. Then we have

$$W_0 = \mathrm{Dhom}(G_4, [13])$$

and W_0 is generated by $[12][13]^{-1}$. Further, Kelly shows that

$$W_0 \cong k[x]$$

under tha map where $x \mapsto [12][13]^{-1}$.
Hence, using this isomorphism together with the map $\Gamma^{-1}_{[13]}$, we obtain the Case 5 H-Primes of G.

Proposition 2.3.6. *The Case 5 H-Primes of G are :*

$$\langle[34],[24],[23],[14]\rangle, \langle[34],[24],[23],[14],[12]\rangle\}.$$

CASE 6
Set $G_5 = G\diagup\langle[34],[24],[23],[14],[13]\rangle$. [12] is normal and regular in G_5 and so we can form the localization $Y = G_5[[12]^{-1}]$. We then have that

$$Y_0 = \mathrm{Dhom}(G_5, [12])$$

and Y_0 is generated by $[12][12]^{-1} = 1$. Hence

$$Y_0 \cong k.$$

Proposition 2.3.7. *The H-Primes of G arising in Case 6 are*

$$\langle[34],[24],[23],[14],[13]\rangle, \quad \langle[34],[24],[23],[14],[13],[12]\rangle.$$

Summary
Combining the 7 possible cases of occurrences of H-Primes in G, we can see that

there are precisely 34 H-Primes in this algebra. These are displayed in Figure 2.3.

2.4 The Automorphism Group of $G_q(2,4)$

In this section, we identify the automorphism group of the 2×4 quantum grassmannian. The methods employed in the approach will be similar to those utilised by Lenagan and Launois in [17]. We will first attempt to identify the height one primes of $G_q(2,4)$ and then use the fact that any automorphism must permute these primes. Assume for the remainder of this chapter that our base field k is algebraically closed. Throughout the remainder of the chapter we will work with $H = (k^*)^4 / \langle C_2 \rangle$ where C_2 denotes the cyclic subgroup of $(k^*)^4$ of order 2 generated by the element $(-1,-1,-1,-1)$.

Alev and Chamarie have found the automorphism group of the algebra $\mathcal{O}_q(M_{2,2})$ by a direct computation

Theorem 2.4.1. *(Alev, Chamarie)*

$$\mathrm{Aut}(\mathcal{O}_q(M_{2,2}) \cong (k^*)^3 \times \langle \mu \rangle$$

where μ denotes the automorphism of $\mathcal{O}_q(M_{2,2})$ such that

$$\begin{aligned} x_{11} &\mapsto x_{11}, \\ x_{12} &\mapsto x_{21}, \\ x_{21} &\mapsto x_{12}, \\ x_{22} &\mapsto x_{22}. \end{aligned}$$

Proof. See [1], Theorem 2.3. □

Extensive work has been done on the automorphism group of arbitrary $\mathcal{O}_q(M_{m,n})$ by Lenagan and Launois in both the square and non-square cases. For details of this see [17].

2.4.1 Height One Primes Of $G_q(2,4)$

Proposition 2.4.2 ([2] p.141)**.** *Let R be a noetherian ring and let $H = (k^*)^r$ be a torus acting on R by automorphisms. If k is algebraically closed, then all H-primes of R are prime.*

From our analysis of the H-prime spectrum of $G_q(2,4)$ in the previous section, we can immediately identify the set of height one primes of $G_q(2,4)$ which are H-invariant.

Proposition 2.4.3. *The H-invariant height one primes of $G_q(2,4)$ are the ideals*

$$\langle [12] \rangle\,,\ \ \langle [14] \rangle\,,\ \ \langle [23] \rangle\,,\ \ \langle [34] \rangle\,.$$

We now need to identify any other height one primes of $G_q(2,4)$. Note first that, if P is a height one prime which is not H-invariant, then $(P : H)$ is an H-prime where

$$(P : H) := \bigcap_{h \in H} h(P)$$

We observe that $(P : H) \subseteq P$ and hence, since P has height one, we must have that $(P : H) = P$ or $(P : H) = 0$. The former cannot occur since we assume that P is not H-invariant and so we must have

$$(P : H) = 0.$$

Hence P must be in the $\langle 0 \rangle$ stratum of $\mathrm{Spec}(G_q(2,4))$.
Now each of the elements $[12], [14], [23], [34]$ are normal in $G_q(2,4)$ and q-commute with each other and so the set consisting of products of powers of these four elements is a right Ore set in $G_q(2,4)$. Thus, we may form the localization (S, say) of $G_q(2,4)$ with respect to this right Ore set. We then have a sequence of homeomorphisms

$$\mathrm{Spec}_{\langle 0 \rangle} G \longrightarrow \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(Z(S))$$

as in [17], where $Z(S)$ denotes the centre of S and $G = G_q(2,4)$. These homeomorphisms are given by extension and contraction. Namely

$$P \mapsto PS \mapsto PS \cap Z(S).$$

Hence, we are interested in the centre of the localization S. Upon examining the defining relations for G once again, we see that the elements [14] and [23] commute with each other and commute in the same way with all other generators of G. Namely, if

$$[14][-] = q^t[-][14]$$

then

$$[23][-] = q^t[-][23]$$

for all generating quantum minors $[-] \in G$ and where $t \in \mathbb{Z}$.

Proposition 2.4.4. *The element $[14][23]^{-1}$ is central in the localization S.*

Proof. It suffices to check the commutation with the remaining 4 minors of G.

We have

$$
\begin{aligned}
[12][14][23]^{-1} &= q[14][12][23]^{-1} = [14][23]^{-1}[12],\\
[13][14][23]^{-1} &= q[14][13][23]^{-1} = [14][23]^{-1}[13],\\
[24][14][23]^{-1} &= q^{-1}[14][24][23]^{-1} = [14][23]^{-1}[24],\\
[34][14][23]^{-1} &= q^{-1}[14][34][23]^{-1} = [14][23]^{-1}[34].
\end{aligned}
$$

□

Proposition 2.4.5. *$Z(S) = k[u^{\pm 1}]$ where $u = [14][23]^{-1}$.*

Proof. We have that

$$k[u^{\pm 1}] \subseteq Z(S)$$

by 2.4.4. By Proposition 1.7.4 there is an algebra isomorphism

$$\theta : \mathcal{O}_q(M_{2,2})[y, y^{-1}; \phi] \longrightarrow G_q(2,4)[[34]^{-1}]$$

such that $\phi(x_{ij}) = q^{-1}x_{ij}$.
Now let S be the localization of $G_q(2,4)$ with respect to the right Ore set generated by the elements $[12], [23], [34]$ and $[14]$. Note that these are the generating quantum minors of $G_q(2,4)$ which are normal elements and so it is possible to localize at this particular set. Note that

$$G_q(2,4)[[34]^{-1}] \subseteq S$$

and that S can be thought of as the localization of $G_q(2,4)[[34]^{-1}]$ at the right Ore set generated by $[12][34]^{-1}, [23][34]^{-1}$ and $[14][34]^{-1}$.
Now we have that D_q is central in $\mathcal{O}_q(M_{2,2})[y, y^{-1}; \phi]$ and x_{12} and x_{21} are both normal elements in the algebra so we may localize at the right Ore set generated by the three elements D_q, x_{12} and x_{21}. Let T denote this localization. Now we have

$$
\begin{aligned}
\theta(D_q) &= [12][34]^{-1},\\
\theta(x_{12}) &= [23][34]^{-1},\\
\theta(x_{21}) &= [14][34]^{-1},
\end{aligned}
$$

so θ extends to an isomorphism from T to S. Hence we may calculate the centre of S by calculating the centre of T and applying θ.
Denote by B the localization of $\mathcal{O}_q(M_{2,2})$ at the right Ore set generated by the elements D_q, x_{12} and x_{21}. θ extends to B and to $T = B[y, y^{-1}; \phi]$. Hence a typical element, α, in the algebra T may be written uniquely in the form $\alpha = \sum_m a_m y^m$ for some indices $m \in \mathbb{Z}$ and $a_m \in B$. Suppose that $\alpha \in Z(T)$. Then we have

$$\alpha x_{ij} = x_{ij}\alpha = \sum_m x_{ij} a_m y^m.$$

In addition, we have

$$\alpha x_{ij} = (\sum_m \alpha_m y^m) x_{ij} = \sum_m q^{-m} \alpha_m x_{ij} y^m.$$

Hence

$$\sum_m x_{ij} \alpha_m y^m = \sum_m q^{-m} \alpha_m x_{ij} y^m.$$

Thus

$$\sum_m (x_{ij}\alpha_m - q^{-m}\alpha_m x_{ij}) y^m = 0.$$

This implies that, for each m and for each x_{ij}, we have

$$x_{ij}\alpha_m - q^{-m}\alpha_m x_{ij} = 0. \tag{2.2}$$

This gives us $\alpha_0 \in Z(B)$. Now assume that $m \neq 0$. Then using equation 2.2, we have

$$D_q \alpha_m = q^{-2m}\alpha_m.$$

However, we know that D_q is central so we must have

$$D_q \alpha_m = \alpha_m D_q.$$

From these two equations we obtain the condition

$$\alpha_m = q^{-2m}\alpha_m$$

i.e.

$$(1 - q^{-2m})\alpha_m = 0.$$

Since we assume that q is not a root of unity, we must have $a_m = 0$. Thus we see that if $\alpha \in Z(T)$, then $\alpha \in Z(B)$. We must now investigate how elements of $Z(B)$ commute with the element y. Let $z \in Z(B)$. Then by [17, Theorem 2.4] we have

$$z = \sum_{i,j} \lambda_{ij} D_q^i (x_{12} x_{21}^{-1})^j.$$

Now by the fact that z is central, we must have

$$zy = yz = y(\sum_{i,j} \lambda_{ij} D_q^i (x_{12} x_{21}^{-1})^j).$$

Alternatively, we may carry out this calculation using the commutation rules given for ϕ. In this case, we get

$$\begin{aligned}
& zy \\
& = (\sum_{i,j} \lambda_{ij} D_q^i (x_{12}x_{21}^{-1})^j)y \\
& = \sum_{i,j} y\lambda_{ij} q^{-2i} D_q^i (x_{12}x_{21}^{-1})^j \\
& = y(\sum_{i,j} \lambda_{ij} q^{-2i} D_q^i (x_{12}x_{21}^{-1})^j).
\end{aligned}$$

Hence we must have

$$\sum_{i,j} \lambda_{ij} D_q^i (x_{12}x_{21}^{-1})^j = \sum_{i,j} \lambda_{ij} q^{-2i} D_q^i (x_{12}x_{21}^{-1})^j.$$

Now since q is not a root of unity, we must have $\lambda_{ij} = 0$ for $i \neq 0$. Hence z must be of the form

$$z = \sum_j \lambda_j (x_{12}x_{21}^{-1})^j.$$

Hence

$$Z(T) = k[(x_{12}x_{21}^{-1})^{\pm 1}].$$

Finally we have

$$\theta(x_{12}x_{21}^{-1}) = [23][34]^{-1}([14][34]^{-1})^{-1} = [23][14]^{-1}$$

It follows that

$$Z(S) = k[([23][14]^{-1})^{\pm 1}]$$

as required. □

Since k is algebraically closed, the nonzero primes of $Z(S)$ are the ideals

$$\left\langle [14][23]^{-1} - \lambda \right\rangle$$

where $0 \neq \lambda \in k$.
It follows that, on tracing back to G, that the primes in $\mathrm{Spec}_{\langle 0 \rangle} G$ are the ideals

$$\langle [14] - \lambda[23] \rangle$$

where $0 \neq \lambda \in k$.
We now have a complete list of the height one primes of G. These are the ideals $\langle [12] \rangle$ and $\langle [34] \rangle$, along with the ideals

$$\langle \alpha[14] + \beta[23] \rangle$$

where $(\alpha, \beta) \in k^2$ with $(\alpha, \beta) \neq (0, 0)$.

2.5 Automorphisms Of G

Now that we have identified the height one primes of the algebra G, we are interested in how an automorphism σ permutes these primes.

From the relation

$$[12][34] = q^2[34][12]$$

we see that, for any automorphism σ of G, we must have

$$\sigma([12])\sigma([34]) = q^2\sigma([34])\sigma([12]).$$

But [12] and [34] must be generators of height one primes. Hence we see that $\langle[12]\rangle$ and $\langle[34]\rangle$ must be fixed i.e.

$$\sigma(\langle[12]\rangle) = \langle[12]\rangle\,,\ \ \sigma(\langle[34]\rangle) = \langle[34]\rangle$$

since [12] and [34] are the only two elements of G generating height one primes which q^2-commute with each other.

Proposition 2.5.1. *Let R be a ring with group of units k^* and let u and v be nonzero normal elements of a ring R such that $\langle u\rangle = \langle v\rangle$. Then there exist $\lambda, \mu \in k^*$ such that $u = \lambda v$ and $v = \mu u$.*

Proof. See [17] Lemma 3.1. □

Corollary 2.5.2. *There exist $\alpha_{12}, \alpha_{34} \in k^*$ such that*

$$\sigma([12]) = \alpha_{12}[12],\ \ \sigma([34]) = \alpha_{34}[34].$$

Proof. Using the fact above that

$$\sigma(\langle[12]\rangle) = \langle[12]\rangle\,,\ \ \sigma(\langle[34]\rangle) = \langle[34]\rangle\,,$$

together with the facts that the elements [12],[34], $\sigma([12])$ and $\sigma([34])$ are normal in G, the result follows immediately from the previous proposition. □

As previously noted, the elements [14] and [23] commute in exactly the same way with all elements of G. Hence, an obvious automorphism which will not fix the ideals generated by these elements is the automorphism

$$\tau : G \longrightarrow G$$

defined by $\tau([14]) = [23]$, $\tau([23]) = [14]$ and fixing the other four generators.

We proceed to show that the generators $[13][14], [23]$ and [24] are fixed up to a scalar by automorphisms of G. First, we take note of an important trick which we will use to simplify some calculations. Note that the torus element $h = (\alpha_{12}^{-1}, 1, \alpha_{34}^{-1}, 1) \in H$ has

$$h \circ \sigma([12]) = h(\alpha_{12}[12]) = \alpha_{12}^{-1}\alpha_{12}[12] = [12].$$

Similarly $h \circ \sigma([34]) = [34]$. Hence given the automorphism σ, we may adjust by the torus element h to simplify our calculations.

Proposition 2.5.3. *Let σ be an automorphism of G. Then*

$$\sigma([13]) = \alpha_{13}[13]$$

for some $\alpha_{13} \in k^*$.

Proof. Using the preferred basis in G, we may take

$$\begin{aligned}
&\sigma([13]) \\
&= \sum_{\mathbf{a}} \lambda_{\mathbf{a}}[12]^{a_1}[13]^{a_2}[14]^{a_3}[24]^{a_4}[34]^{a_5} + \sum_{\mathbf{b}} \mu_{\mathbf{b}}[12]^{b_1}[13]^{b_2}[23]^{b_3}[24]^{b_4}[34]^{b_5} \\
&= \sum_{\mathbf{a}} \lambda_{\mathbf{a}}[T_{\mathbf{a}}] + \sum_{\mathbf{b}} \mu_{\mathbf{b}}[S_{\mathbf{b}}].
\end{aligned}$$

where $a_k, b_k \geq 0$, $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$, $\mathbf{b} = (b_1, b_2, b_3, b_4, b_5)$ and $\lambda_{\mathbf{a}}, \mu_{\mathbf{b}} \in k$. Now we have the commutation relation

$$\sigma([13])\sigma([12]) = q^{-1}\sigma([12])\sigma([13])$$

Thus we have that

$$\sigma([13])[12] = q^{-1}[12]\sigma([13])$$

up to a scalar. In addition we have the equation

$$\sigma([13])[12] = [12](q^{-2a_5-a_4-a_3-a_2}\lambda_{\mathbf{a}}[T_{\mathbf{a}}] + q^{-2b_5-b_4-b_3-b_2}\mu_{\mathbf{b}}[S_{\mathbf{b}}]).$$

Hence, whenever $\lambda_{\mathbf{a}} \neq 0$, we have

$$2a_5 + a_4 + a_3 + a_2 = 1.$$

Similarly, whenever $\mu_{\mathbf{b}} \neq 0$, we have

$$2b_5 + b_4 + b_3 + b_2 = 1.$$

Since a_k, b_k are non-negative integers we must have $a_5 = b_5 = 0$.
Now assume that $[T_{\mathbf{a}}]$ occurs in $\sigma([13])$.
Up to a scalar we have the relation

$$\sigma([13])[34] = q[34]\sigma([13]).$$

Now

$$[T_{\mathbf{a}}][34] = q^{a_4+a_3+a_2+2a_1}[34][T_{\mathbf{a}}].$$

Hence we obtain the equation

$$a_4 + a_3 + a_2 + 2a_1 = 1.$$

As above, since $a_k \geq 0$ is a non-negative integer, we get that $a_1 = 0$.
Notice now that

$$\sigma([14]) = \alpha[14] + \beta[23]$$

where $(\alpha, \beta) \neq (0, 0)$. Hence $\sigma([14])$ commutes with all generating minors $[ij]$ in the same way as [14]. Thus we have the relation,

$$\sigma([13])\sigma([14]) = q\sigma([14])\sigma([13])$$

We can analyse the LHS using the commutation

$$[T_{\mathbf{a}}][14] = q^{-a_5 - a_4 + a_2 + a_1}[14][T_{\mathbf{a}}].$$

Hence we obtain a third equation

$$a_1 + a_2 - a_4 - a_5 = 1.$$

Now since $a_1 = a_5 = 0$, we have

$$a_2 - a_4 = 1.$$

Hence $a_2 \neq 0$. The same three equations hold among the positive integers b_i when $\mu_{\mathbf{b}} \neq 0$ by easy direct calculation since [14]and [23] commute in the same way with all elements of G. Hence the equations $a_2 - a_4 = 1$ and $b_2 - b_4 = 1$, combined with the facts that $a_i, b_i \geq 0$ and the products $[T_{\mathbf{a}}]$ and $[S_{\mathbf{b}}]$ form a preferred basis for G show that [13] occurs in $\sigma([13])$.
This gives us the result that $\sigma([13]) \in [13]G$. Hence $\sigma([13]) = [13]r$ for some $r \in G$. In a similar way, we can see that $\sigma^{-1}([13]) \in [13]G$ and so $\sigma^{-1}([13]) = [13]s$ for some $s \in G$. Thus

$$\begin{aligned}&[13]\\ &= \sigma(\sigma^{-1}([13]))\\ &= \sigma([13]s) = \sigma([13])\sigma(s)\\ &= [13]r\sigma(s).\end{aligned}$$

Hence $r\sigma(s) = 1$ and so r is invertible, i.e. $r \in k$. This gives us the desired result that

$$\sigma([13]) = \alpha_{13}[13]$$

for some $\alpha_{13} \in k^*$. □

Lemma 2.5.4. *Let σ be an automorphism of G. Then*

$$\sigma([24]) = \alpha_{24}[24]$$

for some $\alpha_{24} \in k^*$.

Proof. This proof is very similar to the the previous proof, concerning the effect of σ on [13]. Once again we use the preferred basis to write

$$\sigma([24]) = \sum_{\mathbf{a}} \lambda_{\mathbf{a}}[T_{\mathbf{a}}] + \sum_{\mathbf{b}} \mu_{\mathbf{b}}[S_{\mathbf{b}}]$$

and we obtain the same two equations from the commutations of $\sigma([24])$ with [12] and [34] respectively as we did for $\sigma([13])$, thus giving us the result that $a_1 = a_5 = b_1 = b_5 = 0$.
Once again, we observe that $\sigma([14])$ commutes like [14] and so we have

$$\sigma([24])\sigma([14]) = q^{-1}\sigma([14])\sigma([24]).$$

Now we have the explicit commutation

$$[T_{\mathbf{a}}][14] = q^{-a_5-a_4+a_2+a_1}[14][T_{\mathbf{a}}].$$

Since $a_1 = a_5 = 0$, we get that $a_2 - a_4 = -1$ and similarly $b_2 - b_4 = -1$. Hence $a_4 - a_2 = 1$ and $b_4 - b_2 = 1$. Again, since $a_k, b_k \geq 0$, we must have that [24] appears in $\sigma([24])$. The same argument as for $\sigma([13]$ now establishes the result. □

We now seek to show that the elements [14] and [23] remain fixed or are interchanged (up to a scalar) under the action of an automorphism σ. Once again, we may apply an adjustment by a torus element to simplify the calculation. We have an automorphism σ such that $\sigma([ij]) = \alpha_{ij}[ij]$ for $[ij] = [12], [13], [24], [34]$. We have the relation

$$[13][24] - q^2[24][13] = (q^{-1} - q)[12][34].$$

Applying σ we obtain

$$\alpha_{13}\alpha_{24}([13][24] - q^2[24][13]) = \alpha_{12}\alpha_{34}(q^{-1} - q)[12][34].$$

Multiplying through the original equation by $\alpha_{13}\alpha_{24}$ gives

$$\alpha_{13}\alpha_{24}([13][24] - q^2[24][13]) = \alpha_{13}\alpha_{24}(q^{-1} - q)[12][34].$$

Comparing the two equations and recalling that G is a domain we have

$$\alpha_{13}\alpha_{24} = \alpha_{12}\alpha_{34}. \tag{2.3}$$

Utilising this fact we aim to show that there exists an element $h = (u, v, w, x) \in H$ such that $h \circ \sigma([ij]) = [ij]$ for $[ij] = [12], [13], [24], [34]$. Hence we require that

$$\begin{aligned} uv\alpha_{12} &= 1, \\ uw\alpha_{13} &= 1, \\ vx\alpha_{24} &= 1, \\ wx\alpha_{34} &= 1. \end{aligned}$$

We see that such an h should be of the form $h = (u, \alpha_{12}^{-1}u^{-1}, \alpha_{13}^{-1}u^{-1}, \alpha_{24}^{-1}\alpha_{12}u)$. We have

$$\begin{aligned} h \circ \sigma([12]) &= u\alpha_{12}^{-1}u^{-1}\alpha_{12}[12] = [12], \\ h \circ \sigma([13]) &= u\alpha_{12}^{-1}u^{-1}\alpha_{13}[13] = [13], \\ h \circ \sigma([24]) &= \alpha_{12}^{-1}u^{-1}\alpha_{24}^{-1}\alpha_{12}u\alpha_{24}[24] = [24]. \end{aligned}$$

The final check is

$$h \circ \sigma([34]) = \alpha_{13}^{-1}u^{-1}\alpha_{24}^{-1}\alpha_{12}u\alpha_{34}[34] = [34]$$

by equation 2.3. Hence we can find a suitable h. Setting $u := \alpha_{13}^{-1}$, we can use

$$h = (\alpha_{13}^{-1}, \alpha_{13}\alpha_{12}^{-1}, 1, \alpha_{13}^{-1}\alpha_{12}\alpha_{24}^{-1}).$$

Then h has the effect that $h \circ \sigma([ij]) = [ij]$ for $[ij] = [12], [13], [24], [34]$.

Lemma 2.5.5. *Let σ be an automorphism of G. Then*

$$\sigma([14]) = \alpha_{14}[14]$$

and

$$\sigma([23]) = \alpha_{23}[23]$$

for some scalars $\alpha_{14}, \alpha_{23} \in k^*$.

Proof. We know that

$$\sigma([14]) = \alpha[14] + \beta[23]$$

with $(\alpha, \beta) \neq (0, 0)$. If $\beta \neq 0$ then

$$\tau \circ \sigma([14]) = \alpha[23] + \beta[14]$$

Hence adjusting by τ, we may assume that

$$\sigma([14]) = \alpha[14] + \beta[23]$$

with $\alpha \neq 0$. In this case we aim to show that $\beta = 0$.
Now we have

$$\sigma([23]) = \gamma[14] + \delta[23]$$

for some pair $(\gamma, \delta) \neq (0, 0)$. Now from the standard quantum plücker relation, we get

$$[14][23] = q^{-1}[13][24] - q^{-2}[12][34].$$

Applying σ to this equation gives

$$(\alpha[14] + \beta[23])(\gamma[14] + \delta[23]) = q^{-1}[13][24] - q^{-2}[12][34] = [14][23].$$

Now we expand the LHS to obtain the equation

$$\alpha\gamma[14]^2 + \alpha\delta[14][23] + \beta\gamma[23][14] + \beta\delta[23]^2 = [14][23].$$

Simplifying, we obtain

$$\alpha\gamma[14]^2 + (\alpha\delta + \beta\gamma - 1)[14][23] + \beta\delta[23]^2 = 0.$$

Rewriting the product [14][23] in terms of the preferred basis, we get

$$\alpha\gamma[14]^2 + (\alpha\delta + \beta\gamma - 1)q^{-1}[13][24] - (\alpha\delta + \beta\gamma - 1)q^{-2}[12][34] + \beta\delta[23]^2 = 0.$$

The minors involved in the above equation are in the preferred basis and so are linearly independent. Hence we get

$$\begin{aligned} &\alpha\gamma = 0, \\ &\alpha\delta + \beta\gamma = 1, \\ &\beta\delta = 0. \end{aligned}$$

Now since we chose $\alpha \neq 0$, we must have $\gamma = 0$. This implies that $\alpha\delta = 1$ Hence $\delta \neq 0$. This forces $\beta = 0$ in the final equation. Hence

$$\sigma([14]) = \alpha[14]$$

and

$$\sigma([23]) = \alpha^{-1}[23].$$

□

Now we know that given an automorphism σ of $G_q(2,4)$, we have that either there exist scalars $\alpha_{ij} \in k^*$ such that

$$\begin{aligned} [12] &\mapsto \alpha_{12}[12], \\ [13] &\mapsto \alpha_{13}[13], \\ [14] &\mapsto \alpha_{14}[14], \\ [23] &\mapsto \alpha_{23}[23], \\ [24] &\mapsto \alpha_{24}[24], \\ [34] &\mapsto \alpha_{34}[34], \end{aligned}$$

under σ, or $\sigma \circ \tau$ has this effect.
We now observe that certain restrictions are imposed upon the choice of the scalars α_{ij} by the relations in $G_q(2,4)$.
From the relation

$$[13][24] = [24][13] + (q - q^{-1})[14][23]$$

we see that applying σ provides us with the equation

$$\alpha_{13}\alpha_{24}[13][24] = \alpha_{24}\alpha_{13}[24][13] + \alpha_{14}\alpha_{23}(q - q^{-1})[14][23].$$

Now if we multiply through the original equation by $\alpha_{13}\alpha_{24}$, we obtain

$$\alpha_{13}\alpha_{24}[13][24] = \alpha_{13}\alpha_{24}[24][13] + \alpha_{13}\alpha_{24}(q - q^{-1})[14][23].$$

Comparing the two previous equations, and recalling that $G_q(2,4)$ is a domain, we get the relation

$$\alpha_{13}\alpha_{24} = \alpha_{14}\alpha_{23}.$$

Now recall the Quantum Plücker Relation

$$[12][34] - q[13][24] + q^2[14][23] = 0.$$

Applying σ to this equation gives

$$\alpha_{12}\alpha_{34}[12][34] - \alpha_{13}\alpha_{24}q[13][24] + \alpha_{14}\alpha_{23}q^2[14][23] = 0.$$

Multiplying the original Quantum Plücker Relation by $\alpha_{13}\alpha_{24}$ provides us with the equation

$$\alpha_{13}\alpha_{24}[12][34] - \alpha_{13}\alpha_{24}q[13][24] + \alpha_{13}\alpha_{24}q^2[14][23] = 0.$$

Now comparing the previous two equations and using the fact that $\alpha_{13}\alpha_{24} = \alpha_{14}\alpha_{23}$ we obtain the further relation

$$\alpha_{12}\alpha_{34} = \alpha_{13}\alpha_{24}.$$

Hence we have that

$$\alpha_{12}\alpha_{34} = \alpha_{13}\alpha_{24} = \alpha_{14}\alpha_{23}.$$

Theorem 2.5.6.

$$\mathrm{Aut}(G_q(2,4)) \cong ((k^*)^4 \diagup \langle C_2 \rangle) \times \langle \tau \rangle,$$

the semi-direct product of the factor group $(k^*)^4 \diagup \langle C_2 \rangle$ *and the subgroup generated by the automorphism* τ *which interchanges* [14] *and* [23] *but fixes all other generators of* G*. Here* C_2 *denotes the cyclic subgroup of the 4 torus of order 2 generated by the element* $(-1,-1,-1,-1)$.

Proof. We seek a torus automorphism $h = (u, v, w, x) \in H \times \langle \tau \rangle$ such that

$$h \circ \sigma([ij]) = [ij]$$

for each $[ij]$. In this case, $\sigma = h^{-1}$ and σ is in $H \times \langle \tau \rangle$.

If such an h exists, we require

$$\begin{aligned}
uv\alpha_{12} &= 1,\\
uw\alpha_{13} &= 1,\\
ux\alpha_{14} &= 1,\\
vw\alpha_{23} &= 1,\\
vx\alpha_{24} &= 1,\\
wx\alpha_{34} &= 1.
\end{aligned}$$

Hence we deduce that h should be of the form

$$h = (u, \alpha_{12}^{-1}u^{-1}, \alpha_{13}^{-1}u^{-1}, \alpha_{14}^{-1}u^{-1}).$$

We need $\alpha_{23}^{-1} = vw = \alpha_{12}^{-1}u^{-1}\alpha_{13}^{-1}u^{-1}$, leading to the condition

$$u^2 = \alpha_{12}^{-1}\alpha_{13}^{-1}\alpha_{23}.$$

Hence

$$u = \beta_{12}^{-1}\beta_{13}^{-1}\beta_{23}$$

where $\beta_{ij}^2 = \alpha_{ij}$. Note that since we choose k to be algebraically closed our introduction of the $\beta_{ij}s$ is justified.
Following this we see that h should be of the form

$$h = (\beta_{12}^{-1}\beta_{13}^{-1}\beta_{23},\quad \beta_{12}^{-1}\beta_{13}\beta_{23}^{-1},\quad \beta_{12}\beta_{13}^{-1}\beta_{23}^{-1},\quad \beta_{12}\beta_{13}\beta_{23}^{-1}\beta_{14}^{-2}).$$

It is easily verified that the composition $h \circ \sigma$ gives the desired effect on all generating minors of G. We show here the cases for [24] and [34] as they are the cases which invoke the equality $\alpha_{12}\alpha_{34} = \alpha_{13}\alpha_{24} = \alpha_{14}\alpha_{23}$ in the calculation.
We have

$$\begin{aligned}
h \circ \sigma([24])&\\
&= h(\alpha_{24}[24])\\
&= \beta_{12}^{-1}\beta_{13}\beta_{23}^{-1}\beta_{12}\beta_{13}\beta_{23}^{-1}\beta_{14}^{-2}\alpha_{24}[24]\\
&= \beta_{13}^{2}\beta_{23}^{-2}\beta_{14}^{-2}\alpha_{24}[24]\\
&= \alpha_{13}\alpha_{23}^{-1}\alpha_{14}^{-1}\alpha_{24}[24]\\
&= \alpha_{14}\alpha_{23}\alpha_{14}^{-1}\alpha_{23}^{-1}[24]\\
&= [24].
\end{aligned}$$

Similarly we have

$$
\begin{aligned}
h \circ \sigma([34]) & \\
&= h(\alpha_{34}[34]) \\
&= \beta_{12}\beta_{13}^{-1}\beta_{23}^{-1}\beta_{12}\beta_{13}\beta_{23}^{-1}\beta_{14}^{-2}\alpha_{34}[34] \\
&= \beta_{12}^{2}\beta_{23}^{-2}\beta_{14}^{-2}\alpha_{34}[34] \\
&= \alpha_{12}\alpha_{34}\alpha_{14}^{-1}\alpha_{23}^{-1}[34] \\
&= \alpha_{12}\alpha_{34}\alpha_{12}^{-1}\alpha_{34}^{-1}[34] \\
&= [34],
\end{aligned}
$$

as required. □

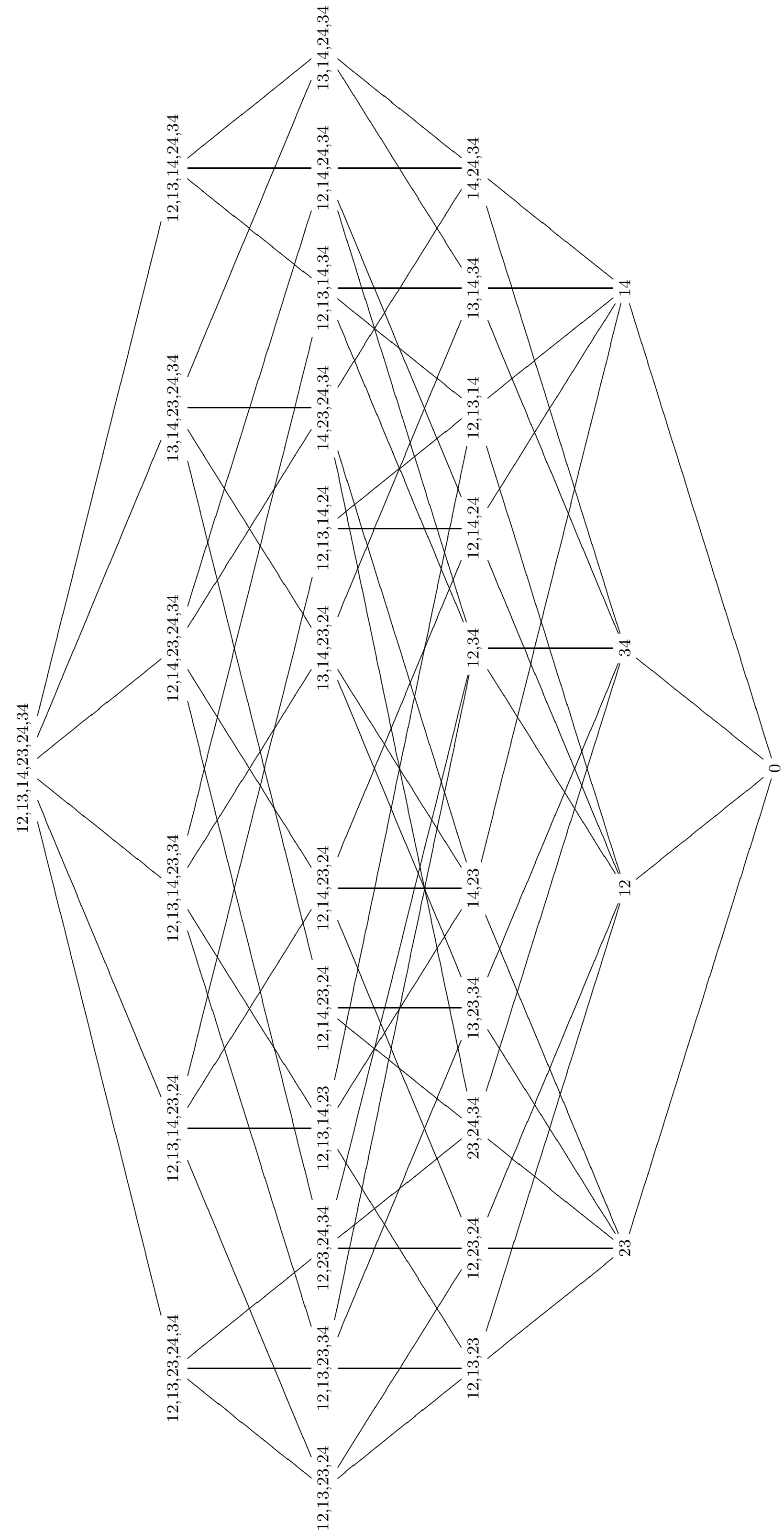


Figure 2.3: The complete H-prime spectrum of the quantum grassmannian $G_q(2,4)$.

Chapter 3

Consecutive Minors and Dehomogenisation

We have already mentioned in Chapter 1 the extremely important result of Kelly which showed that there does indeed exist a quantum analogue of the classical dehomogenisation isomorphism between the coordinate ring of $m \times (n-m)$ matrices and the dehomogenisation of the coordinate ring of the $m \times n$ Grassmannian at the rightmost minor (namely the element $[n-m+1, ..., n]$). In this chapter, we will discover that the elements of $G_q(m,n)$ which are *consecutive minors* are very special indeed. In fact, we prove the remarkable result that the dehomogenisation of the algebra $G_q(m,n)$ at any of these consecutive minors is isomorphic to $\mathcal{O}_q(M_{m,n-m})$.

3.1 Consecutive Minors

We already know that consecutive minors in $G_q(m,n)$ are normal, regular elements so we may localize at the Ore sets consisting of the non-negative powers of any such consecutive minor. From this localization, we may identify the degree zero component within the inherited $\mathbb{Z}$-grading. One of the main issues which we must address when performing calculations within these dehomogenisations will be congruence issues relating to the position where the consecutive minor is within the quantum matrix of $\mathcal{O}_q(M_{m,n})$. For example, within the algebra $\mathcal{O}_q(M_{2,4})$, the minor $[12|14]$ is considered to be consecutive. Hence we must introduce some notation to cope with this "wrap around" case.

Definition 3.1.1. Fix a positive integer $n \in \mathbb{N}$. For each $j \in \mathbb{N}$, we define $\tilde{j} \in \{1, ..., n\}$ to be the positive integer such that

$$\tilde{j} \equiv j \ \text{(modulo } n).$$

Definition 3.1.2. A *consecutive minor* in the algebra $G_q(m,n)$ is any minor of the form

$$I = [\widetilde{a}, \ , \widetilde{a+1}, ..., \widetilde{a+m}-1]$$

where $a \in \{1, ..., n\}$.

Notation 3.1.3. Let $1 \leq s \leq m$, then the element $[i_1, ..., \widehat{i_s}, ..., i_m]$ is the minor obtained from the columns indexed by the set $\{i_1, ..., i_m\} \backslash \{i_s\}$.

Note now that the commutation rules for $G_q(2,4)$ allow is to deduce the relations for $G_q(2,n)$ as given in section 1.5.

Lemma 3.1.4. *In $G_q(2,4)$ we have that*

$$q[24][13] - q^{-1}[13][24] = (1 - q^{-2})[12][34].$$

Proof. Using the reformulation of the Quantum Plucker relation, we have

$$q[24][13] - q^{-1}[13][24] = q(q[34][12] + q^{-1}[23][14]) - q^{-1}[13][24]$$

$$= [12][34] + (q^{-1}[13][24] - q^{-2}[12][34]) - q^{-1}[13][24]$$

$$= (1 - q^{-2})[12][34].$$

□

Corollary 3.1.5. *Let $i, j, k, l \in \{1, ..., n\}$ with $i < j < k < l$. Then in $G_q(2,n)$ we have the relation*

$$q[jl][ik] - q^{-1}[ik][jl] = (1 - q^{-2})[ij][kl].$$

Note that we also have a very simple relation in $G_q(2,4)$, namely that

$$[13][24] - [24][13] = (q - q^{-1})[14][23].$$

Lemma 3.1.6. *Let $i, j, k, l \in \{1, ..., n\}$ with $i < j < k < l$. Then in $G_q(2,n)$ we have the relation*

$$[ik][jl] - [jl][ik] = (q - q^{-1})[il][jk].$$

Lemma 3.1.7. *The k-algebra* $\mathrm{Dhom}(G_q(m,n), [\widetilde{a}, ..., \widetilde{a+m} - 1]) := S_0$ *is generated by the elements* $\{\{j, \widetilde{a}, ..., \widehat{i}, ..., \widetilde{a+m} - 1\}\} := [j, \widetilde{a}, ..., \widehat{i}, ..., \widetilde{a+m} - 1][\widetilde{a}, ..., \widetilde{a+m} - 1]^{-1}$ *where* $j \in \{1, ..., n\} \backslash \{\widetilde{a}, ..., \widetilde{a+m} - 1\}$ *and* $i \in \{\widetilde{a}, ..., \widetilde{a+m} - 1\}$.

Proof. S_0 is generated by the elements

$$\{\{I\}\} := [I][a, ..., a + m - 1]^{-1}$$

where $I \subseteq \{1, ..., n\}$ and $|I| = m$. We show that each such element may be expressed as a k-linear combination of products of elements of the form $\{\{j,\ a, ..., \widehat{i}, ..., a + m - 1\}\}$.
Denote by A the subalgebra of S_0 generated by the elements $\{\{j,\ a, ..., \widehat{i}, ..., a + m - 1\}\}$. Note that

$$|\{j, \widetilde{a}, ..., \widehat{i}, ..., \widetilde{a+m} - 1\} \cap \{\widetilde{a}, ..., \widetilde{a+m} - 1\}| = m - 1.$$

Assume that

$$I := \{i_1, ..., i_m\} \neq \{\widetilde{a}, ..., \widetilde{a+m-1}\}$$

is an ordered subset of $\{1, ..., n\}$ with

$$|I \cap \{\widetilde{a}, ..., \widetilde{a+m-1}\}| = m - t.$$

Say that

$$m - t = |I \cap \{\widetilde{a}, ..., \widetilde{a+m-1}\}| = |\{b_1, ..., b_{m-t}\}|.$$

We use induction on the cardinality of the set $I \cap \{\widetilde{a}, ..., \widetilde{a+m-1}\}$ to show that $\{\{I\}\} \in A$.

If $t = 1$, then I is of the form $\{j, \ \ a, ..., \widehat{i}, ..., a+m-1\}$. Hence $\{\{I\}\} \in A$. Now consider a fixed $t > 1$. Assume that the result is true for cardinality $t - 1$. Consider $[I] = [i_1 ... i_m]$ with

$$|I \cap \{\widetilde{a}, ..., \widetilde{a+m-1}\}| = m - t.$$

Choose $c \in I \backslash \{\widetilde{a}, \widetilde{a+1}, ..., \widetilde{a+m-1}\}$. We use the quantum Plücker relations to rewrite the product

$$[\widetilde{a}, ..., \widetilde{a+m-1}][i_1, ..., i_m].$$

Let $K = \{c\} \sqcup \{\widetilde{a}, ..., \widetilde{a+m-1}\}$, $J_1 = \phi$ and $J_2 = \{i_1, ..., i_m\} \backslash \{c\}$. Then

$$\sum_{K' \sqcup K'' = K} (-q)^{\ell(K';K'') + \ell(K'';J_2)} [K'][K'' \sqcup J_2] = 0$$

where either $K' = \{\widetilde{a}, ..., \widetilde{a+m-1}\}$ and $K'' = \{c\}$ or $K' = \{c\} \sqcup \{\widetilde{a}, ..., \widehat{l}, ..., \widetilde{a+m-1}\}$ and $K'' = \{l\}$, where $l \notin J_2$.

Let $S := \{\widetilde{a}, ..., \widetilde{a+m-1}\} \backslash J_2$. Rearranging the above, we get that

$$[\widetilde{a}, ..., \widetilde{a+m-1}][i_1, ..., i_m] = -\sum_{l \in S} (-q)^{\bullet} [c, \ \ \widetilde{a}, ..., \widehat{l}, ..., \widetilde{a+m-1}][l, \ \ J_2]$$

Multiplying through from the right by $[\widetilde{a}, ..., \widetilde{a+m-1}]^{-2}$ gives

$$\{\{i_1, ..., i_m\}\} = \sum_{l \in S} \pm (-q)^{\bullet} \{\{c, \ \ \widetilde{a}, ..., \widehat{l}, ..., \widetilde{a+m-1}\}\}\{\{l, \ \ J_2\}\}.$$

Now $|\{l, J_2\} \cap \{\widetilde{a}, ..., \widetilde{a+m-1}\}| = m - t + 1 = m - (t-1)$ and so by the inductive hypothesis

$$\{\{l, \ \ J_2\}\} \in A.$$

Clearly $\{\{c, \ \ a, ..., \widehat{l}, ..., a+m-1\}\} \in A$ and so $\{\{i_1, ..., i_m\}\} \in A$ and the result follows. □

Theorem 3.1.8. *Let* $1 \leq a \leq n$. *Then*

$$\mathcal{O}_q(M_{m,n-m}) \cong \mathrm{Dhom}(G_q(m,n), [\widetilde{a}, ..., \widetilde{a+m}-1])$$

via the isomorphism

$$\theta : \mathcal{O}_q(M_{m,n-m}) \longrightarrow \mathrm{Dhom}(G_q(m,n), [\widetilde{a}, ..., \widetilde{a+m}-1])$$

given explicitly by

$$x_{ij} \mapsto [j+\widetilde{a+m}-1,\ \widetilde{a}, ..., \widehat{a+m-}i, ..., \widetilde{a+m}-1][\widetilde{a}, ..., \widetilde{a+m}-1]^{-1}.$$

Proof. Firstly, we must check that θ is indeed a morphism of algebras. Hence, we must verify that the images of the generators of $\mathcal{O}_q(M_{m,n-m})$ still obey the commutation relations. There are four such relations to consider. Namely, for $1 \leq i < k \leq m$ and $1 \leq j < l \leq n$,

$$\theta(x_{ij})\theta(x_{il}) = q\theta(x_{il})\theta(x_{ij}),$$

$$\theta(x_{ij})\theta(x_{kj}) = q\theta(x_{kj})\theta(x_{ij}),$$
$$\theta(x_{il})\theta(x_{kj}) = \theta(x_{kj})\theta(x_{il}),$$
$$\theta(x_{ij})\theta(x_{kl}) - \theta(x_{kl})\theta(x_{ij}) = (q-q^{-1})\theta(x_{il})\theta(x_{kj}).$$

In our analysis of the 4 relations we will make extensive use of Muir's Law as this will greatly simplify the complexity of our calculations. Due to the generality with which we choose our consecutive minor at which we localize, we must also take note of the various possibilities for ordering on the indexing sets of the minors involved.

Relation 1, Case (a)

Firstly, note that

$$a+m-k < a+m-i < j+a+m-1 < l+a+m-1$$

since $i < k$ and $j < l$.

Set

$$I := \{\widetilde{a}, ..., \widetilde{a+m}-1\} \backslash \{a+m-k, a+m-i\}$$

For Relation 1, we should take notice of the fact the positive integer k is not involved and so we only have three possible orderings to consider. For our initial case, we assume that

$$\widetilde{a+m}-i < j+\widetilde{a+m}-1 < l+\widetilde{a+m}-1.$$

We have

$$
\begin{aligned}
&\theta(x_{ij})\theta(x_{il}) \\
&= [j+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-1}[l+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-1} \\
&= q^{-1}[j+\widetilde{a+m}-1,\ I][l+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-2} \\
&= [l+\widetilde{a+m}-1,\ I][j+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-2} \\
&= q[l+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-1}[j+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-1} \\
&= q\theta(x_{il})\theta(x_{ij}).
\end{aligned}
$$

Relation 1, Case (b)
Assume now that we have

$$l+\widetilde{a+m}-1 < \widetilde{a+m}-i < j+\widetilde{a+m}-1.$$

We have

$$
\begin{aligned}
&\theta(x_{ij})\theta(x_{il}) \\
&= [j+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-1}[l+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-1} \\
&= q[j+\widetilde{a+m}-1,\ I][l+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-2} \\
&= [l+\widetilde{a+m}-1,\ I][j+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-2} \\
&= q[l+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-1}[j+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-1} \\
&= q\theta(x_{il})\theta(x_{ij}).
\end{aligned}
$$

Relation 1, Case (c)
Assume that

$$j+\widetilde{a+m}-1 < l+\widetilde{a+m}-1 < \widetilde{a+m}-i.$$

We have

$$
\begin{aligned}
&\theta(x_{ij})\theta(x_{il}) \\
&= [j+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-1}[l+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-1} \\
&= q[j+\widetilde{a+m}-1,\ I][l+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-2} \\
&= q^2[l+\widetilde{a+m}-1,\ I][j+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-2} \\
&= q[l+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-1}[j+\widetilde{a+m}-1,\ I][\widetilde{a+m}-i,\ I]^{-1} \\
&= q\theta(x_{il})\theta(x_{ij}).
\end{aligned}
$$

Relation 2, Case (a)
Notice that, with Relation 2, we only have three possible cases to consider. Firstly assume that

$$\widetilde{a+m}-k < \widetilde{a+m}-i < j+\widetilde{a+m}-1.$$

Set

$$M := \{\widetilde{a+m}-k, \widetilde{a+m}-i, I\}.$$

Then we have

$$\begin{aligned}
&\theta(x_{ij})\theta(x_{kj}) \\
&= [j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1} \\
&= q^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-2} \\
&= [j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-2} \\
&= q[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1} \\
&= q\theta(x_{kj})\theta(x_{ij}).
\end{aligned}$$

Relation 2, Case (b)
Assume now that

$$j+\widetilde{a+m}-1 < \widetilde{a+m}-k < \widetilde{a+m}-i.$$

We have

$$\begin{aligned}
&\theta(x_{ij})\theta(x_{kj}) \\
&= [j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1} \\
&= q[j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-2} \\
&= q^{2}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-2} \\
&= q[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1} \\
&= q\theta(x_{kj})\theta(x_{ij}).
\end{aligned}$$

Relation 2, Case (c)
Finally, assume that

$$\widetilde{a+m}-i < j+\widetilde{a+m}-1 < \widetilde{a+m}-k.$$

Then we have

$$\begin{aligned}
&\theta(x_{ij})\theta(x_{kj}) \\
&= [j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1} \\
&= q[j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-2} \\
&= [j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-2} \\
&= q[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1} \\
&= q\theta(x_{kj})\theta(x_{ij}).
\end{aligned}$$

Relation 3, Case (a)
Fistly we must note that with relation 3 we are not so lucky with common index

set components of minors within our calculations and so we must split our analysis into four separate cases. Firstly, we assume that

$$\widetilde{a+m}-k < \widetilde{a+m}-i < j+\widetilde{a+m}-1 < l+\widetilde{a+m}-1.$$

Then we have

$$\begin{aligned}
&\theta(x_{il})\theta(x_{kj}) \\
&= [l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1} \\
&= q^{-1}[l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-2} \\
&= q^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-2} \\
&= [j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1}[l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1} \\
&= \theta(x_{kj})\theta(x_{il}).
\end{aligned}$$

Relation 3, Case (b)
Now assume that

$$l+\widetilde{a+m}-1 < \widetilde{a+m}-k < \widetilde{a+m}-i < j+\widetilde{a+m}-1.$$

We have

$$\begin{aligned}
&\theta(x_{il})\theta(x_{kj}) \\
&= [l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1} \\
&= q^{-1}[l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-2} \\
&= q[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-2} \\
&= [j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1}[l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1} \\
&= \theta(x_{kj})\theta(x_{il}).
\end{aligned}$$

Relation 3, Case (c)
Assume

$$j+\widetilde{a+m}-1 < l+\widetilde{a+m}-1 < \widetilde{a+m}-k < \widetilde{a+m}-i.$$

Then

$$\begin{aligned}
&\theta(x_{il})\theta(x_{kj}) \\
&= [l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1} \\
&= q[l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-2} \\
&= q[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-2} \\
&= [j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1}[l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1} \\
&= \theta(x_{kj})\theta(x_{il}).
\end{aligned}$$

Relation 3, Case (d)
Finally assume that

$$\widetilde{a+m}-i<j+\widetilde{a+m}-1<l+\widetilde{a+m}-1<\widetilde{a+m}-k.$$

Then we have

$$\begin{aligned}
&\theta(x_{il})\theta(x_{kj})\\
&=[l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1}\\
&=q[l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-2}\\
&=q^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-2}\\
&=[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1}[l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1}\\
&=\theta(x_{kj})\theta(x_{il}).
\end{aligned}$$

Relation 4, Case (a)
With Relation 4 there are once again four separate cases for consideration. In this initial case we assume that

$$\widetilde{a+m}-k<\widetilde{a+m}-i<j+\widetilde{a+m}-1<l+\widetilde{a+m}-1.$$

We have

$$\begin{aligned}
&\theta(x_{ij})\theta(x_{kl})-\theta(x_{kl})\theta(x_{ij})\\
&=[j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1}[l+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1}\\
&-[l+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1}\\
&=q^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][l+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-2}\\
&-q^{-1}[l+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-2}
\end{aligned}$$

Now we apply 3.1.6 and we see that

$$\begin{aligned}
&q^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][l+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-2}\\
&-q^{-1}[l+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-2}\\
&=q^{-1}(q-q^{-1})[l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-2}\\
&=(q-q^{-1})[l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1}\\
&=(q-q^{-1})\theta(x_{il})\theta(x_{kj}).
\end{aligned}$$

Relation 4, Case (b)
Assume now that

$$l+\widetilde{a+m}-1<\widetilde{a+m}-k<\widetilde{a+m}-i<j+\widetilde{a+m}-1.$$

We have

$$
\begin{aligned}
&\theta(x_{ij})\theta(x_{kl}) - \theta(x_{kl})\theta(x_{ij}) \\
&= [j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][M]^{-1}[l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][M]^{-1} \\
&- [l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][M]^{-1}[j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][M]^{-1} \\
&= q[j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][M]^{-2} \\
&- q^{-1}[l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][M]^{-2}
\end{aligned}
$$

Now we apply Corollary 3.1.5 and we see that

$$
\begin{aligned}
&q[j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][M]^{-2} \\
&- q^{-1}[l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][M]^{-2} \\
&= (1 - q^{-2})[l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][M]^{-2} \\
&= (q - q^{-1})[l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][M]^{-1}[j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][M]^{-1} \\
&= (q - q^{-1})\theta(x_{il})\theta(x_{kj}).
\end{aligned}
$$

Relation 4, Case (c)
Assume in this case that

$$
j + \widetilde{a+m} - 1 < l + \widetilde{a+m} - 1 < \widetilde{a+m} - k < \widetilde{a+m} - i.
$$

We have

$$
\begin{aligned}
&\theta(x_{ij})\theta(x_{kl}) - \theta(x_{kl})\theta(x_{ij}) \\
&= [j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][M]^{-1}[l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][M]^{-1} \\
&- [l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][M]^{-1}[j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][M]^{-1} \\
&= q[j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][M]^{-2} \\
&- q[l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][M]^{-2}
\end{aligned}
$$

Applying 3.1.6 once more, we see that

$$
\begin{aligned}
&q[j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][M]^{-2} \\
&- q[l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][M]^{-2} \\
&= q(q - q^{-1})[j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][M]^{-2} \\
&= q(q - q^{-1})[l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][M]^{-2} \\
&= (q - q^{-1})[l + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - k,\ \ I][M]^{-1}[j + \widetilde{a+m} - 1,\ \ \widetilde{a+m} - i,\ \ I][M]^{-1} \\
&= (q - q^{-1})\theta(x_{il})\theta(x_{kj}).
\end{aligned}
$$

Relation 4, Case (d)
Finally, we examine the case where

$$\widetilde{a+m}-i < j+\widetilde{a+m}-1 < l+\widetilde{a+m}-1 < \widetilde{a+m}-k.$$

We have

$$\begin{aligned}
&\theta(x_{ij})\theta(x_{kl}) - \theta(x_{kl})\theta(x_{ij}) \\
&= [j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1}[l+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1} \\
&- [l+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1} \\
&= q[j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][l+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-2} \\
&- q^{-1}[l+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-2}
\end{aligned}$$

Once more we use Corollary 1.3 and we get that

$$\begin{aligned}
&q[j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][l+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-2} \\
&- q^{-1}[l+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-2} \\
&= (1-q^{-2})[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-2} \\
&= (1-q^{-2})[l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-2} \\
&= (q-q^{-1})[l+\widetilde{a+m}-1,\ \widetilde{a+m}-k,\ I][M]^{-1}[j+\widetilde{a+m}-1,\ \widetilde{a+m}-i,\ I][M]^{-1} \\
&= (q-q^{-1})\theta(x_{il})\theta(x_{kj}).
\end{aligned}$$

Hence θ is indeed a morphism of algebras. By 3.1.7 θ is an epimorphism. Now assume that $\ker(\theta) \neq 0$. Then, since θ is onto, we have that

$$\mathrm{GKdim}(\mathcal{O}_q(M_{m,n-m})/\ker(\theta)) = \mathrm{GKdim}(\mathrm{Dhom}(G_q(m,n),[a,...,a+m-1])$$

Hence

$$\mathrm{GKdim}(\mathrm{Dhom}(G_q(m,n),[a...a+m-1]) < \mathrm{GKdim}(\mathcal{O}_q(M_{m,n-m})) = m(n-m)$$

However, from 1.7.7 we also know that

$$\mathrm{GKdim}(\mathrm{Dhom}(G_q(m,n),[a...a+m-1]) = \mathrm{GKdim}(G_q(m,n)) - 1$$

$$= m(n-m)+1-1 = m(n-m).$$

Thus, we obtain a contradiction and so we must have that $\ker(\theta) = 0$ and hence θ is a monomorphism. □

3.2 An Application

A Count Of The Number Of H-Primes In The 2×5 Quantum Grassmannian

As we have already seen, the torus $H := (k^*)^n$ acts on the quantum grassmannian $G_q(2, n)$ by automorphisms and the height one H-invariant prime ideals under this standard action are the ideals generated by the n elements

$$\mathcal{H}_1 := \{[i, \ \ i+1] : 1 \leq i \leq n-1\} \sqcup \{[1, \ \ n]\}.$$

Proposition 3.2.1. *The height one H-invariant primes of $G_q(2, n)$ are the ideals generated by the elements in $\mathcal{H}_1$.*

Proof. Note first that the ideal $\langle [n-1 \ \ n] \rangle$ is certainly H-invariant and, by results in Section 4 of [18], is prime. Now

$$\mathrm{Dhom}(G_q(2, n), [n-1 \ \ n]) \cong \mathcal{O}_q(M_{2,n-2})$$

and the height one H primes in $\mathcal{O}_q(M_{2,n-2})$ have been completely described by Lenagan and Launois in [17]. They are precisely the ideals generated by the elements b_i where

$$b_i := [1, ..., i | n-i-1, n-2]$$

if $1 \leq i \leq 2$,

$$b_i := [12 | n-i-1, ..., n-i]$$

if $2 < i \leq n-2$, and

$$b_i := [i-n+3, ..., 2 | 1, .., n-i]$$

if $n-2 < i \leq n-1$.

Explicitly, these are the ideals

$$\langle x_{1,n-2} \rangle \, , \ \ \langle [n-3, \ \ n-2] \rangle \, , \ \ \langle [n-4, \ \ n-3] \rangle \, , \ \ ..., \langle [12] \rangle \, , \langle x_{21} \rangle \, .$$

Tracing back to $\mathrm{Dhom}(G_q(2, n), [n-1 \, n])$ and then to $G_q(2, n)$ using the bijection $\Gamma_{[n-1 \ n]}$ defined in Chapter 2, we obtain the height one H-primes

$$\langle [n-2, \ \ n-1] \rangle \, , \ \ \langle n-3, \ \ n-2] \rangle \, , \ \ ..., \langle [12] \rangle \, , \langle [1, \ \ n] \rangle$$

□

We will now use the fact that dehomogenising the $2 \times n$ grassmannian at any of these n elements is isomorphic to the quantized coordinate ring of quantum matrices, $\mathcal{O}_q(M_{2,n-2})$.

The specific case which we will analyse is the case where $n = 5$. We note that in this case the dehomogenisation of $G := G_q(2, 5)$ at each of the 5 elements which generate height one H-prime ideals is isomorphic to $\mathcal{O}_q(M_{2,3})$. The H-spectrum of $\mathcal{O}_q(M_{2,3})$ has been studied in detail by Goodearl and Lenagan in [9].
In our analysis, we will make use of the inclusion-exclusion principle. Hence, we will split our count into 6 cases. Namely, if we take P to be an H-prime of G, we first consider the number of such primes which do not contain one of the elements which generate a height one H-prime, then we count the number of such primes which do not contain two of the aforementioned elements, and so on. Finally, we need to count the number of such primes which contain all of the elements which generate height one H-primes.

Theorem 3.2.2. *(Inclusion-Exclusion Principle) Suppose that a finite set S is the union of a family of subsets $A_1, ..., A_n$. For each subset $I \subseteq \{1, ..., n\}$, let*

$$A_I := \bigcap_{i \in I} A_i.$$

Then

$$|S| = \sum_{|I|=1} |A_I| - \sum_{|I|=2} |A_I| + \sum_{|I|=3} |A_I| - ... + (-1)^{n-1} \sum_{|I|=n} |A_I|.$$

The H-prime spectrum of the 2×3 quantum matrix algebra has been described in detail by Lenagan and Goodearl.

Notation 3.2.3. When describing the H-primes of $\mathcal{O}_q(M_{2,3})$, we will use bullets and circles as given in the following example :
The prime

$$\begin{pmatrix} \bullet\bullet\circ \\ \circ\circ\circ \end{pmatrix}$$

is the ideal generated by the elements x_{11} and x_{12} of $\mathcal{O}_q(M_{2,3})$. Similarly, the ideal

$$\left(\square \begin{smallmatrix} \bullet \\ \circ \end{smallmatrix} \right)$$

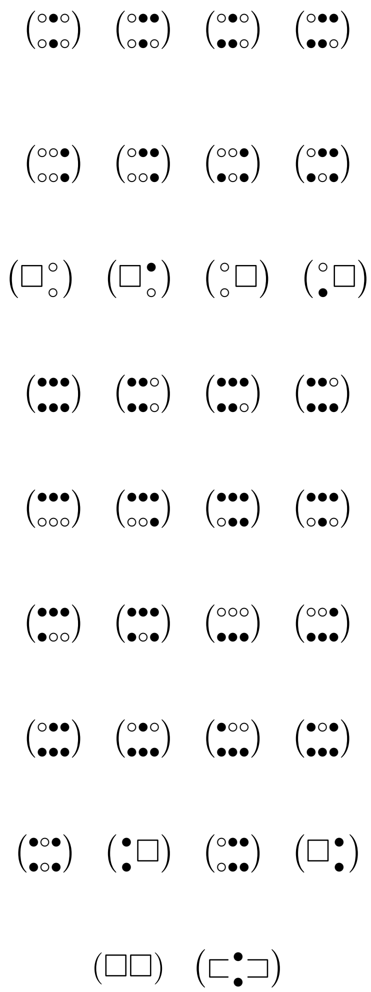
is the ideal generated by the 2×2 quantum minor obtained from the first and second columns and by the element $x_{13} \in \mathcal{O}_q(M_{2,3})$.

Proposition 3.2.4. *There are exactly 46 H-prime ideals in the algebra $\mathcal{O}_q(M_{2,3})$ and these are the ideals given below :*

$$\begin{pmatrix} \circ\circ\circ \\ \circ\circ\circ \end{pmatrix} \quad \begin{pmatrix} \circ\circ\bullet \\ \circ\circ\circ \end{pmatrix} \quad \begin{pmatrix} \circ\circ\circ \\ \bullet\circ\circ \end{pmatrix} \quad \begin{pmatrix} \circ\circ\bullet \\ \bullet\circ\circ \end{pmatrix}$$

$$\begin{pmatrix} \circ\bullet\bullet \\ \circ\circ\circ \end{pmatrix} \quad \begin{pmatrix} \circ\bullet\bullet \\ \bullet\circ\circ \end{pmatrix} \quad \begin{pmatrix} \circ\circ\circ \\ \bullet\bullet\circ \end{pmatrix} \quad \begin{pmatrix} \circ\circ\bullet \\ \bullet\bullet\circ \end{pmatrix}$$

$$\begin{pmatrix} \bullet\circ\circ \\ \bullet\circ\circ \end{pmatrix} \quad \begin{pmatrix} \bullet\circ\bullet \\ \bullet\circ\circ \end{pmatrix} \quad \begin{pmatrix} \bullet\circ\circ \\ \bullet\bullet\circ \end{pmatrix} \quad \begin{pmatrix} \bullet\circ\bullet \\ \bullet\bullet\circ \end{pmatrix}$$



Proof. See [9]. $\square$

The Exclusion Of 1 Element
We now recall that the five elements of G which generate height one H-primes are

$$[12],\ [23],\ [34],\ [45],\ [15].$$

The dehomogenisation of G at any of these 5 elements is isomorphic to $\mathcal{O}_q(M_{2,3})$. There are 46 H-primes in $\mathcal{O}_q(M_{2,3})$ and so counting the 5 possible dehomogenisations here, we count a total of 230 H-primes which do not contain a choice of one of the five elements which generate a height one H-prime ideal.
The Exclusion Of 2 Elements
If we now examine the H-invariant primes P which do not contain a choice of 2 of the height one H-prime generating elements of G, we find that there are 10 possible cases to consider. Namely,
Case 2.1 : $[12], [23] \notin P$.
We first note that $\mathcal{O}_q(M_{2,3}) \cong \mathrm{Dhom}(G, [12])$. By one of our previous results, the explicit map is given by

$$x_{11} \mapsto \widehat{[13]},$$

$$x_{12} \mapsto \widehat{[14]},$$
$$x_{13} \mapsto \widehat{[15]},$$
$$x_{21} \mapsto \widehat{[23]},$$
$$x_{22} \mapsto \widehat{[24]},$$
$$x_{23} \mapsto \widehat{[25]},$$

where $\widehat{[-]} := [-][12]^{-1}$.
Hence to count the number of H-primes occurring in Case 2.1, we simply need to count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain the generator x_{21}. There are 18 such H-primes.
Case 2.2 : $[23], [45] \notin P$.
As in Case 2.1, we have that $\mathrm{Dhom}(G, [45]) \cong \mathcal{O}_q(M_{2,3})$ In this case however, $\hat{[23]}$ corresponds to a 2×2 quantum minor in $\mathcal{O}_q(M_{2,3})$. To see this, we appeal to the Quantum Plücker relations. We have

$$[23][45] - q[24][35] + q^2[25][34] = 0$$

and hence multiplying through the equation on the right by $[45]^{-2}$ gives

$$\widehat{[23]} = q[24][35][45]^{-2} - q^2[25][34][45]^{-2} = \widehat{[24]}\widehat{[35]} - q\widehat{[25]}\widehat{[34]}.$$

So tracing back from the isomorphism

$$\theta : \mathcal{O}_q(M_{2,3}) \longrightarrow \mathrm{Dhom}(G, [45])$$

given by

$$x_{11} \mapsto \widehat{[14]},$$
$$x_{12} \mapsto \widehat{[24]},$$
$$x_{13} \mapsto \widehat{[34]},$$
$$x_{21} \mapsto \widehat{[15]},$$
$$x_{22} \mapsto \widehat{[25]},$$
$$x_{23} \mapsto \widehat{[35]},$$

we see that $\widehat{[23]}$ corresponds to the minor $[23] \in \mathcal{O}_q(M_{2,3})$. So to count the number of H-primes occurring in Case 2.2 we need to count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain the minor $[23]$. There are 12 such H-primes.
Case 2.3 : $[34], [45] \notin P$.
Once again, if we dehomogenise G at the element $[45]$, we have the result that $\mathcal{O}_q(M_{2,3}) \cong \mathrm{Dhom}(G, [45])$. Examining the defining map once more reveals that $\hat{[34]}$ corresponds to the generator $x_{13} \in \mathcal{O}_q(M_{2,3})$ and so to count the H-primes in case 2.3, we simply need to count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain the generator x_{13}. There are 18 such primes.

Case 2.4 : $[12], [45] \notin P$.
Still dehomogenising at the element [45], we use the aforementioned isomorphism

$$\theta : \mathcal{O}_q(M_{2,3}) \longrightarrow \mathrm{Dhom}(G, [45])$$

to identify the element $\widehat{[12]}$ in $\mathrm{Dhom}(G, [45])$. We have the Quantum Plucker relation

$$[12][45] - q[14][25] + q^2[15][24] = 0$$

and hence if we multiply through on the right by $[45]^{-2}$ we obtain

$$\widehat{[12]} = \widehat{[14]}\widehat{[25]} - q\widehat{[15]}\widehat{[24]}.$$

Hence $\widehat{[12]}$ corresponds to the minor [12] in $\mathcal{O}_q(M_{2,3})$. Thus in the count for Case 2.4, we need to look for the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain the minor [12]. There are 12 such H-invariant primes.
Case 2.5 : $[15], [45] \notin P$.
We follow the pattern above and see that $\widehat{[15]} \in \mathrm{Dhom}(G, [45])$ corresponds to the generator $x_{21} \in \mathcal{O}_q(M_{2,3})$. Hence we count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain the generator x_{21}. There are 18 such primes.

Case 2.6 : $[12], [34] \notin P$.
As in Case 2.1, we dehomogenise at [12] and by the same methods above, we see that the element $\widehat{[34]} \in \mathrm{Dhom}(G, [12])$ corresponds to the minor $[12] \in \mathcal{O}_q(M_{2,3})$. Hence we count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain [12]. There are 12 primes arising in this case.
Case 2.7 : $[12], [15] \notin P$.
Once again, we dehomogenise at [12] and we have that $\widehat{[15]}$ corresponds to the generator $x_{13} \in \mathcal{O}_q(M_{2,3})$. Thus the number of H-primes in this case is 18.
Case 2.8 : $[23], [34] \notin P$.
newline Dehomogenisation of G at the element [34] gives the isomorphism

$$\phi : \mathcal{O}_q(M_{2,3}) \longrightarrow \mathrm{Dhom}(G, [34])$$

via

$$x_{11} \mapsto \widehat{[35]},$$
$$x_{12} \mapsto \widehat{[13]},$$
$$x_{13} \mapsto \widehat{[23]},$$
$$x_{21} \mapsto \widehat{[45]},$$
$$x_{22} \mapsto \widehat{[14]},$$
$$x_{23} \mapsto \widehat{[24]}.$$

Hence to count the number of H-primes in this case, we need to count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain the generator x_{13}. As in previous cases, there are 18 such H-primes.

Case 2.9 : $[23], [15] \notin P$.
Now $\mathcal{O}_q(M_{2,3}) \cong \text{Dhom}(G, [15])$ via

$$x_{11} \mapsto \widehat{[25]},$$

$$x_{12} \mapsto \widehat{[35]},$$

$$x_{13} \mapsto \widehat{[45]},$$

$$x_{21} \mapsto \widehat{[12]},$$

$$x_{22} \mapsto \widehat{[13]},$$

$$x_{23} \mapsto \widehat{[14]}.$$

Now $\widehat{[23]}$ corresponds to the minor $[12] \in \mathcal{O}_q(M_{2,3})$. Hence we must count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain $[12]$. There are precisely 12 of these primes.
Case 2.10 : $[15], [34] \notin P$.
As with the previous case, we dehomogenise at the element $[15]$ and it is easily seen that $\widehat{[34]} \in \text{Dhom}(G, [15])$ corresponds to the 2×2 minor $[23] \in \mathcal{O}_q(M_{2,3})$. Hence there are 12 H-primes arising in this case.
The Exclusion Of 3 Elements
Now we aim to analyse each of the possible cases where three of the elements which generate a height one H-prime of G are excluded from the count. Once again, there are 10 possible cases for consideration.
Case 3.1 : $[12], [23], [15] \notin P$.
We have that $\mathcal{O}_q(M_{2,3}) \cong \text{Dhom}(G, [12])$ as detailed in the previous section. From the defining isomorphism given in the previous section, we can read off that $\widehat{[15]}$ corresponds to the generator $x_{13} \in \mathcal{O}_q(M_{2,3})$ and $\widehat{[23]}$ corresponds to the generator $x_{21} \in \mathcal{O}_q(M_{2,3})$. So we count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain x_{13} and which do not contain x_{21}. There are 6 of these H-primes.
Case 3.2 : $[34], [45], [15] \notin P$.
In this case, we dehomogenise at $[45]$ and use the isomorphism from section 2 to see that $\widehat{[34]}$ corresponds to $x_{13} \in \mathcal{O}_q(M_{2,3})$ and $\widehat{[15]}$ corresponds to $x_{21} \in \mathcal{O}_q(M_{2,3})$. Thus, as in the previous case, there are 6 H-primes arising here.
Case 3.3 : $[12], [23], [34] \notin P$.
Here, we use the isomorphism $\mathcal{O}_q(M_{2,3}) \cong \text{Dhom}(G, [12])$. Under this isomorphism, $x_{21} \mapsto \widehat{[23]}$ and $\widehat{[34]}$ corresponds to the 2×2 minor $[12] \in \mathcal{O}_q(M_{2,3})$. We count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain x_{21} and which do not contain the minor $[12]$. There are 6 such H-primes.
Case 3.4 : $[23], [45], [15] \notin P$.
Using $\mathcal{O}_q(M_{2,3}) \cong \text{Dhom}(G, [45])$, we have that $x_{21} \mapsto \widehat{[15]}$ and that $\widehat{[23]}$ corresponds to the minor $[23] \in \mathcal{O}_q(M_{2,3})$. We count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain x_{21} and which do not contain $[23]$. There are 4 H-primes in this case.
Case 3.5 : $[23], [34], [45] \notin P$.

Once again we use the isomorphsim $\mathcal{O}_q(M_{2,3}) \cong \text{Dhom}(G, [45])$. Under this isomorphism, we see that $x_{13} \mapsto \widehat{[34]}$. Now, we recall that $\widehat{[23]}$ corresponds to the minor $[23] \in \mathcal{O}_q(M_{2,3})$ and so we need to count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain x_{13} and which do not contain [23]. Hence there are 6 H-primes arising in this case.

Case 3.6 : $[12], [45], [15] \notin P$.

Using the same isomorphism as Case 3.5 we have that $x_{21} \mapsto \widehat{[15]}$ and that $\widehat{[12]}$ corresponds to the minor $[12] \in \mathcal{O}_q(M_{2,3})$. We count the H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain x_{21} and which do not contain the minor [12]. Once again, there are 6 such H-primes.

Case 3.7 : $[12], [34], [45] \notin P$.

Still using the same isomorphism as the previous case, we have that $x_{13} \mapsto \widehat{[34]}$ and $\widehat{[12]}$ corresponds to the minor [12]. Hence, we count the 4 H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain x_{13} and which do not contain [12].

Case 3.8 : $[12], [23], [45] \notin P$.

In this case, we appeal to the isomorphism $\mathcal{O}_q(M_{2,3}) \cong \text{Dhom}(G, [12])$ under which $x_{21} \mapsto \widehat{[23]}$ and the minor $[23] \mapsto \widehat{[45]}$. We count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain the generator x_{21} and which do not contain the minor [23]. Hence there are 4 H-primes in this case.

Case 3.9 : $[12], [34], [15] \notin P$.

As with the previous case, we dehomgenise at the element $[12] \in G$ and we use the fact that the dehomogenisation of G at [12] is isomorphic to $\mathcal{O}_q(M_{2,3})$ with $x_{13} \mapsto \widehat{[13]}$ and $[12] \mapsto \widehat{[34]}$. So we need to count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain x_{13} and which do not contain [12]. There are 4 such H-primes.

Case 3.10 : $[23], [34], [15] \notin P$.

In this case, we use the isomorphism

$$\phi : \mathcal{O}_q(M_{2,3}) \longrightarrow \text{Dhom}(G, [34])$$

under which we have that $x_{13} \mapsto \widehat{[23]}$ and $\widehat{[15]}$ corresponds to the 2×2 minor $[12] \in \mathcal{O}_q(M_{2,3})$. Thus, we must count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain the generator x_{13} and which do not contain the minor [12]. As recorded previously, there are 4 such H-primes.

The Exclusion Of 4 Elements

We noe turn our attention to the exclusion of a choice of four of the possible five elements which generate height one H-primes of G. There are thus $\binom{5}{4} = 5$ possible cases to consider.

Case 4.1 : $[12], [23], [34], [45] \notin P$.

In this case, we use the isomorphism

$$\theta : \mathcal{O}_q(M_{2,3}) \longrightarrow \text{Dhom}(G, [45])$$

Reading off the details from our explicit definition of θ in section 2, we see that $x_{13} \mapsto \widehat{[34]}$ under θ while $\widehat{[12]}$ and $\widehat{[23]}$ correspond to the minors [12] and [23] in $\mathcal{O}_q(M_{2,3})$ respectively. Hence, we must count the number of H-primes of $\mathcal{O}_q(M_{2,3})$

which do not contain the minors [12]and [23] and which do not contain the generator x_{13}. There are only 2 of these primes.
Case 4.2 : $[12], [34], [45], [15] \notin P$.
Once again using the isomorphism θ from the previous case, we have that $x_{13} \mapsto \widehat{[34]}$ and $x_{21} \mapsto \widehat{[15]}$ while $\widehat{[12]}$ corresponds to the 2×2 quantum minor $[12] \in \mathcal{O}_q(M_{2,3})$. Thus, we count the H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain either of the generators x_{13} or x_{21} and which do not contain the minor [12]. There are 2 of these H-primes.
Case 4.3 : $[12], [23], [45], [15] \notin P$.
Still using the isomorphism θ from the previous cases, we have that $x_{21} \mapsto \widehat{[15]}$ while $\widehat{[12]}$ and $\widehat{[23]}$ correspond to the minors [12] and [23] respectively in $\mathcal{O}_q(M_{2,3})$. Hence, we count the H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain either of the aforementioned minors and which do not contain the generator x_{21}. There are 2 such primes.
Case 4.4 : $[23], [34], [45], [15] \notin P$.
Once again we appeal to the isomorphism θ and we see that $x_{13} \mapsto \widehat{[34]}$ and $x_{21} \mapsto \widehat{[15]}$ while $\widehat{[23]}$ corresponds to the 2×2 minor $[23] \in \mathcal{O}_q(M_{2,3})$. We count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which do not contain either of the generators x_{13} and x_{21} and which do not contain the minor [23]. There are 2 of these primes.
Case 4.5 : $[12], [23], [34], [15] \notin P$.
For this case, we make use of the isomrphism $\mathcal{O}_q(M_{2,3}) \cong \mathrm{Dhom}(G, [12])$. Looking back at section 2 where the map is given explicitly, we see that $x_{13} \mapsto \widehat{[15]}$ and $x_{21} \mapsto \widehat{[23]}$ while $\widehat{[34]}$ corresponds to the minor $[12] \in \mathcal{O}_q(M_{2,3})$. Hence our count for this case is the same as that of the previous case and so 2 H-primes occur here.
We now need to consider the case where all five height one H primes are excluded. Using the isomorphism

$$\mathcal{O}_q(M_{2,3}) \cong \mathrm{Dhom}(G, [12])$$

we see that

$$\begin{aligned} x_{13} &\mapsto \widehat{[15]}, \\ x_{23} &\mapsto \widehat{[23]}, \\ [12] &\mapsto \widehat{[34]}, \\ [23] &\mapsto \widehat{[45]}. \end{aligned}$$

So we must count the number of H-primes of $\mathcal{O}_q(M_{2,3})$ which contain do not contain any of the four elements $x_{13}, x_{21}, [12], [23]$. There is clearly only one H-prime occurring here, namely the ideal $\langle 0 \rangle$.
The Inclusion Of All 5 Elements
Our final case for consideration is the H-primes, P, which include all of the 5

elements which generate height one H-primes. If we set

$$I := \langle [12], [23], [34], [45], [15] \rangle$$

then to analyse the H-primes occurring in this final case we should look at the factor algebra

$$T := G / I.$$

If we now set

$$[\tilde{-}] := [-] + I$$

in T, then we have the relations

$$[\tilde{13}][\tilde{14}] = q[\tilde{14}][\tilde{13}], \quad [\tilde{13}][\tilde{24}] = [\tilde{24}][\tilde{13}], \quad [\tilde{13}][\tilde{25}] = [\tilde{25}][\tilde{13}],$$

$$[\tilde{13}][\tilde{35}] = q[\tilde{35}][\tilde{13}], \quad [\tilde{14}][\tilde{24}] = q[\tilde{24}][\tilde{14}], \quad [\tilde{14}][\tilde{25}] = [\tilde{25}][\tilde{14}],$$

$$[\tilde{14}][\tilde{35}] = [\tilde{35}][\tilde{14}], \quad [\tilde{24}][\tilde{25}] = q[\tilde{25}][\tilde{24}], \quad [\tilde{24}][\tilde{35}] = [\tilde{35}][\tilde{24}],$$

$$[\tilde{25}][\tilde{35}] = q[\tilde{35}][\tilde{25}].$$

We also have the Quantum Plücker relations in $G_q(2,5)$

$$\begin{aligned}
[12][34] - q[13][24] + q^2[14][23] &= 0,\\
[12][35] - q[13][25] + q^2[15][23] &= 0,\\
[12][45] - q[14][25] + q^2[15][24] &= 0,\\
[13][45] - q[14][35] + q^2[15][34] &= 0,\\
[23][45] - q[24][35] + q^2[25][34] &= 0.
\end{aligned}$$

Now in T we get the five equations

$$\begin{aligned}
[\tilde{13}][\tilde{24}] &= 0,\\
[\tilde{13}][\tilde{25}] &= 0,\\
[\tilde{14}][\tilde{25}] &= 0,\\
[\tilde{14}][\tilde{35}] &= 0,\\
[\tilde{24}][\tilde{35}] &= 0.
\end{aligned}$$

Hence we may identify T with the quantum affine 5-space but subject to the 5 equations above. Note now that there are $2^5 = 32$ H-primes of a typical quantum affine 5-space and that these are simply given as those ideals given by the 32 possible combinations of subsets of the generators. In the special case of our algebra T however, there are conditions imposed upon us via the 5 equations above. For example, let P be an H-prime of T. Then

$$[\tilde{13}][\tilde{24}] = 0 \in P \Rightarrow [\tilde{13}] \in P \text{ or } [\tilde{24}] \in P.$$

Hence we obtain 5 conditions for an H-prime, P, of T

$$\begin{aligned}
\tilde{[13]} \in P &\text{ or } \tilde{[24]} \in P \\
\tilde{[13]} \in P &\text{ or } \tilde{[25]} \in P \\
\tilde{[14]} \in P &\text{ or } \tilde{[25]} \in P \\
\tilde{[14]} \in P &\text{ or } \tilde{[35]} \in P \\
\tilde{[24]} \in P &\text{ or } \tilde{[35]} \in P
\end{aligned}$$

It is clear from these 5 conditions that for P to satisfy all of these conditions, P must have height at least equal to 3. If T were a true quantum affine 5-space then there would be 10 valid height 3 H-primes given by the $\binom{5}{3} = 10$ possible combinations of any 3 of the 5 generators. It is easy to verify that only 5 of these 10 ideals satisfy the requirements of the 5 conditions. These five ideals are H-primes of the algebra T and these are the ideals

$$\begin{gathered}
\left\langle \tilde{[13]},\ \tilde{[14]},\ \tilde{[24]} \right\rangle \\
\left\langle \tilde{[13]},\ \tilde{[14]},\ \tilde{[35]} \right\rangle \\
\left\langle \tilde{[13]},\ \tilde{[25]},\ \tilde{[35]} \right\rangle \\
\left\langle \tilde{[14]},\ \tilde{[24]},\ \tilde{[25]} \right\rangle \\
\left\langle \tilde{[24]},\ \tilde{[25]},\ \tilde{[35]} \right\rangle
\end{gathered}$$

At height 4 in a standard quantum affine space there are $\binom{5}{4} = 5$ H-primes and we see that in the algebra T, each of the height four ideals satisfy the 5 conditions given above. The height 4 H-primes of T are the ideals

$$\begin{gathered}
\left\langle \tilde{[13]},\ \tilde{[14]},\ \tilde{[24]},\ \tilde{[25]} \right\rangle \\
\left\langle \tilde{[13]},\ \tilde{[24]},\ \tilde{[25]},\ \tilde{[35]} \right\rangle \\
\left\langle \tilde{[13]},\ \tilde{[14]},\ \tilde{[25]},\ \tilde{[35]} \right\rangle \\
\left\langle \tilde{[13]},\ \tilde{[24]},\ \tilde{[25]},\ \tilde{[35]} \right\rangle \\
\left\langle \tilde{[14]},\ \tilde{[24]},\ \tilde{[25]},\ \tilde{[35]} \right\rangle
\end{gathered}$$

Finally, there there is one ideal at height five in the standard quantum affine 5-space and it is easy to see that this ideal satisfies the required conditions. So we add

$$\left\langle \tilde{[13]},\ \tilde{[14]},\ \tilde{[24]},\ \tilde{[25]},\ \tilde{[35]} \right\rangle$$

to our list of the H-primes of T.
Now we have listed all the possible H-primes of T. There are therefore $5+5+1 = 11$ H-primes in the algebra T.

A Count Of The Number Of H-primes In G

Taking a total count of all the H-primes occurring in each of the above cases, we have 230 primes excluding one of the five elements, 150 excluding 2 elements, 50 excluding 3 elements, 10 excluding 4 elements and 1 excluding all 5 elements. Taking into account the 11 primes which contain all 5 elements, we have by the inclusion-exclusion principle that the total number of H-primes in G is

$$230 - 150 + 50 - 10 + 1 + 11 = 132.$$

Chapter 4

H-Primes in $G_q(2, n)$

In Chapter 4 we obtained a count of the number of H-prime ideals in the algebra $G_q(2, 5)$ through a very lengthy structured process where we exploited the Inclusion/Exclusion Principle and dehomogenisation of $G_q(2, 5)$ at consecutive minors. Where this provided a very interesting and useful application of Theorem 3.1.8, the process in general is unfeasible as a method for counting H-prime ideals in arbitrary size Quantum Grassmannian algebras. In this chapter we introduce a method for counting the number of H-prime ideals in $G_q(2, n)$ where $n \in \mathbb{N}$ such that $n \geq 2$. This method relates H-primes in $G_q(2, n)$ to Cauchon Fillings of Young Diagrams.

4.1 QASLs, Quantum Schubert Varieties and Ladder Rings

The material in this section is presented in greater detail in [21]. Here we summarise the main ideas which will be utilised for our purpose.

Definition 4.1.1. Let A be an algebra and let Π be a finite subset of elements of A with a partial ordering, $<_{st}$. We say that an element of A is a standard monomial if it is either 1 or of the form $a_1 a_2 ... a_s$ for some $s \geq 1$ and with $a_1, a_2, ..., a_s \in \Pi$ such that $a_1 \leq_{st} a_2 \leq_{st} ... \leq_{st} a_s$.

Definition 4.1.2. Let A be an $\mathbb{N}$-graded algebra and let Π be a finite subset of A equipped with a partial ordering, $\leq_{st}$. We say that A is a quantum graded algebra with a straightening law (or QASL) on the poset $(\Pi, \leq_{st})$ if the following are satisfied :
(1) The elements in Π are homogeneous of positive degree.
(2) A is generated as a k-algebra by the elements of Π.
(3) The set of standard monomials on Π is linearly independent.
(4) If $\alpha, \beta \in \Pi$ are not comparable for $<_{st}$, then $\alpha\beta$ is a linear combination of terms λ or $\lambda\mu$ where $\lambda, \mu \in \Pi$ with $\lambda \leq_{st} \mu$ and $\lambda <_{st} \alpha, \beta$.

(5) For all $\alpha, \beta \in \Pi$, there exists $c_{\alpha,\beta} \in k^*$ such that $\alpha\beta - c_{\alpha,\beta}\beta\alpha$ is a linear combination of terms λ or $\lambda\mu$ where $\lambda, \mu \in \Pi$ with $\lambda \leq_{st} \mu$ and $\lambda <_{st} \alpha, \beta$.

Lenagan and Rigal have proved in [21] that Quantum Grassmannians are QASLs.

Definition 4.1.3. Let $\gamma \in \Pi_{m,n}$ and set

$$\Pi_{m,n}^{\gamma} := \{\alpha \in \Pi_{m,n} : \alpha \ngeq \gamma\}.$$

The quantum Schubert variety associated to γ is

$$G_q(m,n)_\gamma := G_q(m,n) \big/ \left\langle \Pi_{m,n}^{\gamma} \right\rangle.$$

Definition 4.1.4. On the poset $\Pi_{m,n}$, take $\gamma = (\gamma_1, ..., \gamma_m)$ with $1 \leq \gamma_1 < ... < \gamma_m \leq n$. To any such γ we may associate the a subset $\mathcal{L}_\gamma \subseteq \{1, ..., m\} \times \{1, ..., n\}$ defined as follows :

$$\mathcal{L}_\gamma = \{(i,j) \in \{1, ..., m\} \times \{1, ..., n\} : j > \gamma_{m+1-i}, \quad j \neq \gamma_l, \quad 1 \leq l \leq m\}.$$

Note that for each $(i,j) \in \mathcal{L}_\gamma$, the set

$$(\{\gamma_1, ..., \gamma_m\} \backslash \{\gamma_{m+1-i}\}) \cup \{j\}$$

is a subset of $\{1, ..., n\}$ which contains m distinct elements. Hence, we may associate the maximal quantum minor

$$m_{ij} = |\{\gamma_1, ..., \gamma_m\} \backslash \{\gamma_{m+1-i}\} \cup \{j\}| \in \mathcal{O}_q(M_{m,n})$$

to any pair (i,j).
We define the quantum ladder matrix ring, denoted $\mathcal{O}_q(M_{m,n})_\gamma$, associated to any such γ to be the subalgebra of $\mathcal{O}_q(M_{m,n})$ generated by the elements x_{ij} where $(i,j) \in \mathcal{L}_\gamma$.

Theorem 4.1.5. *Let $\gamma \in \Pi_{m,n}$. There is a k-algebra isomorphism*

$$d_\gamma : \mathcal{O}_q(M_{m,n})_\gamma[Y^{\pm 1}; \sigma] \longrightarrow G_q(m,n)_\gamma[\overline{\gamma}^{-1}]$$

such that

$$x_{ij} \mapsto \overline{m_{ij}}$$

and

$$Y \mapsto \overline{\gamma}.$$

Proof. See [20, Theorem 3.1.6.]. □

4.2 Dehomogenisation and Ladder Rings

Now that we have the tools required, we may begin analysing the H-prime spectrum of the $2 \times n$ Quantum Grassmannian, $G_q(2,n)$. Recall first that dehomogenising $G_q(2,n)$ at any consecutive minor produces an algebra which is isomorphic to Quantum Matrices of size $2\times(n-2)$. To count the number of H-primes in $G_q(2,n)$, we will work our way through the partial ordering and make use of facts derived from the fact that $G_q(2,n)$ is a QASL as well as Dehomogenisation theory. For our purpose, it will prove extremely useful to link up the notions of partitions with Quantum minors.

Prime Ideals in Relation to Quantum Schubert Varieties

Thanks to the machinery of QASLs and Quantum Schubert varieties we may count the number of H-primes in $G_q(2,n)$.

Lemma 4.2.1. *Let $\gamma \in \Pi_{2,n}$. The image of γ is normal in the Quantum Schubert Variety $G_q(2,n)_\gamma$.*

Proof. See [21, Remark 3.1.5.]. □

Now, as is also noted in [21, Remark 3.1.5.], the image of γ is the unique minimal element in the set $\Pi_{2,n}\backslash\Pi_{2,n}^{\gamma}$. Hence one of our main concepts will be to associate to each minor $\gamma \in \Pi_{2,n}$ the collection of prime ideals $\{P_\gamma\}$ of the Quantum Schubert variety $G_q(2,n)_\gamma$.

Definition 4.2.2. *Let R be an $\mathbb{N}$-graded ring. The irrelevant ideal is the ideal generated by all graded elements in R of positive degree.*

Lemma 4.2.3. *Let P be a prime of $\Pi_{2,n}$ with P not equal to the irrelevant ideal. Then certainly there exists some $\gamma \in \Pi_{2,n}$ with $\gamma \notin P$. Now choose $\gamma \notin P$ such that*

$$\alpha < \gamma \Rightarrow \alpha \in P.$$

Then

$$\alpha \ngeq \gamma \Rightarrow \alpha \in P.$$

Further to this, γ is unique.

Proof. Assume that $\alpha \ngeq \gamma$. If we have $\alpha < \gamma$, then by our choice of γ we have $\alpha \in P$ as required. Otherwise α and γ are not comparable in $\Pi_{2,n}$. Hence by definition of a QASL,

$$\alpha\gamma = \sum \lambda\mu$$

where $\lambda \leq \mu$ and $\lambda < \alpha, \gamma$.
So each $\lambda \in P$. Therefore

$$\alpha\gamma \in P.$$

Now $\gamma \notin P$ and γ is normal in the Quantum Schubert Variety

$$G_q(2,n)_\gamma = G_q(2,n) \diagup \left\langle \Pi_{2,n}^{\gamma} \right\rangle$$

with

$$\{\alpha \in \Pi_{2,n} : \alpha < \gamma\} \subseteq P.$$

Hence γ is normal mod P and $\gamma \notin P$. Hence $\alpha \in P$.
Assume now that $\gamma_1, \gamma_2 \in \Pi_{2,n}$ and that

$$\gamma_1 \notin P \;\;\&\;\; \mu < \gamma_1 \Rightarrow \mu \in P.$$

Assume also that

$$\gamma_2 \notin P \;\;\&\;\; \mu < \gamma_2 \Rightarrow \mu \in P.$$

We claim that $\gamma_1 = \gamma_2$.
Suppose that $\gamma_1 \neq \gamma_2$. If γ_1 and γ_2 are comparable, without loss of generality we may assume that $\gamma_1 < \gamma_2$ which is a contradiction since $\gamma_1 \notin P$.
Suppose now that γ_1 and γ_2 are not comparable. Then $\gamma_2 \ngtr \gamma_1$ which is again a contradiction. □

Hence we may localize at the element $\overline{\gamma} \in G_q(2,n)_\gamma$ and we have that

$$\mathrm{Dhom}(G_q(2,n)_\gamma, \overline{\gamma}) \cong \mathcal{O}_q(M_{2,n})_\gamma.$$

Hence, to count the number of H-primes in $G_q(2,n)$ we need to work our way upwards in the partial ordering, dehomogenising and counting the number of H-primes occurring at each stage.
Assume from now on that
(a) $S = R[x;\sigma]$ is a skew polynomial algebra over a noetherian k-algebra R.
(b) H is a group acting on S by automorphisms.
(c) R is H-stable and x is an H-eigenvector.
(d) There exists $h_0 \in H$ such that $h_0|_R = \sigma$ and such that the h_0-eigenvalue of x is not a root of unity.

Lemma 4.2.4. *([2, Corollary II.5.10]) Assume the setup above and that R is H-simple but S is not. Then the only H-prime ideals of S are 0 and xS.*

Lemma 4.2.5. *([11, Ex 2ZA]) Let $S = R[x;\alpha,\delta]$ be a skew polynomial ring and I an ideal of R such that $\alpha(I) \subseteq I$ and $\delta(I) \subseteq I$. Let $\bar{\alpha}$ and $\bar{\delta}$ denote the ring automorphism and skew derivation on R/I induced by α and δ. Then IS is a 2-sided ideal of S such that $IS \cap R = I$. We also have that*

$$S/IS \cong (R/I)[\bar{x};\bar{\alpha},\bar{\delta}].$$

Theorem 4.2.6. *Assuming the setup above, let P be an H-prime of S. Then either*

$$P = (P \cap R)S$$

when $x \notin P$, or else

$$P = (P \cap R)S + xS$$

when $x \in P$.

Proof. We have a group H acting on S and there is an action on R via restriction.

Further, $Q = P \cap R$ is H-prime in R. Now R embeds in the skew extension S, which we may in turn map onto the factor algebra S/P. Hence

$$R/Q = R/P \cap R \hookrightarrow S/P.$$

Set

$$\bar{R} = R/Q, \quad \bar{S} = S/QS = S/(P \cap R)S.$$

Then $\bar{\sigma}$ acts on both $\bar{R}$ and $\bar{S}$ and by 4.2.5 we have that

$$\bar{S} \cong \bar{R}[\bar{x}; \bar{\sigma}].$$

Let $F = Frac(R)$ be the localization of R at the Ore set generated by all the normal regular elements of R. Assume that $P \cap R = 0$. Then $F^* = F \backslash \{0\}$ is Ore in S and we have $P \cap F^* = \phi$.Now $FS = F[x; \sigma]$ and so by 4.2.4,

$$xFS = PF.$$

Hence $x = pab^{-1}$ for some $a, b \in R$ and some $p \in P$. i.e.

$$xb = pa \in P.$$

Hence we must have that $x \in P$ or $b \in P$. Note that, in the latter case, we would have that $b \in P \cap R = 0$ and so we obtain a contradiction.

Assume then that $x \in P$. Then $xS \subseteq P$. Suppose that $p \in P$. Then there exist $r_i \in R$ such that

$$p = \sum_{i=0}^{n} r_i x^i$$

with $r_i x^i \in P$ for all $i \geq 1$.
Hence $r_0 \in P$ and so we have $r_0 \in P \cap R = 0$. So $p \in xS$. Hence $P = xS$. □

Assume now that
(a) $A = k[x_1][x_2; \sigma_2, \delta_2][x_3; \sigma_3, \delta_3]...[x_s; \sigma_s, \delta_s][x_{s+1}; \sigma_{s+1}]...[x_n; \sigma_n]$ is an iterated skew polynomial algebra over k.
(b) H is a group acting on A by k-algebra automorphisms.
(c) $x_1, ..., x_n$ are H-eigenvectors.
(d) There exist $h_1, ..., h_n \in H$ such that $h_i(x_j) = \sigma_i(x_j)$ for all $i > j$ and such that the h_i-eigenvalue of x_i is not a root of unity.

Corollary 4.2.7. *Let P be an H-prime of A and let $B = k[x_1][x_2; \sigma_2, \delta_2]...[x_s; \sigma_s, \delta_s]$ with $s < n$. Then*

$$P = (P \cap B)A + \sum_{j \in I} x_{ij} A$$

for some $I \subseteq \{s + 1, ..., n\}$.

4.3 Partitions Associated to Quantum Minors and Cauchon Diagrams

In this section we examine how we go about counting the H-primes of $G_q(2,n)$ in a concrete example. Our approach will be to begin at the lowest nonzero quantum minor in the partial ordering, namely the minor [12]. Via the dehomogenisation process we may then count the number of H-primes not containing the minor [12].

Definition 4.3.1. *An $m \times n$ Cauchon Diagram is an $m \times n$ grid consisting of mn boxes in which certain squares are coloured black. We require that a box can only be coloured black if :*
(1) Every box above is also coloured black.
(2) Every box to the left is also coloured black.

Cauchon shows in [4] that there is a bijection between the collection of H-prime ideals of $\mathcal{O}_q(M_{m,n})$ and the Cauchon Diagrams on an $m \times n$ grid. Recall that

$$\mathrm{Dhom}(G_q(2,n),[12]) \cong \mathcal{O}_q(M_{2,n-2}).$$

Hence, we count the number of Cauchon Fillings of a $2 \times (n-2)$ grid as detailed in [4].
Continuing in this manner, we can take an element, $\gamma \in \Pi_{2,n}$ in the partial ordering. Now

$$\mathrm{Dhom}(G_q(2,n)_\gamma,\overline{\gamma}) \cong \mathcal{O}_q(M_{2,n})_\gamma.$$

so the H-primes, P, which are associated with γ can be counted by counting the number of H-primes in the ladder ring $\mathcal{O}_q(M_{2,n})_\gamma$ in light of 4.2.1.

The Young Diagram Associated to a Quantum Minor

To each element $\gamma \in \Pi_{2,n}$, we may associate a Young Diagram.

Definition 4.3.2. Let $\gamma \in \mathbb{N}$ and recall that a partition of γ is a tuple $(\gamma_1, \gamma_2, ..., \gamma_t)$ such that

$$\gamma_1 \geq \gamma_2 \geq ... \geq \gamma_t$$

and

$$\sum_{i=1}^{t} \gamma_i = \gamma.$$

Definition 4.3.3. Let $\gamma = (\gamma_1, ..., \gamma_t)$ be a partition of $\gamma \in \mathbb{N}$. We say that a partition $\lambda = (\lambda_1, ..., \lambda_t)$ is a subpartition of γ if $\lambda_i \leq \gamma_i$ for all $i \in \{1, ..., t\}$.

Definition 4.3.4. A Young Diagram is a collection of boxes arranged in rows which are left-justified with a weakly decreasing number of boxes in each row.
To each partition $\gamma = (\gamma_1, ..., \gamma_t)$, we may associate a Young Diagram by declaring that row i of the Young Diagram should consist of γ_i boxes for $1 \leq i \leq t$. Sub-Young Diagrams are defined in the obvious way.

Definition 4.3.5. Let $[ij]$ be a quantum minor in $G_q(2,n)$. To $[ij]$, we associate the partition
$$\lambda_{ij} := (n-i-1, n-j).$$
Let $\lambda = (\lambda_1, \lambda_2)$ be a subpartition of $(n-2, n-2)$. To the partition λ we may associate the quantum minor
$$\gamma_\lambda := [n-1-\lambda_1, \quad n-\lambda_2] \in G_q(2,n).$$

Recalling the definition of the ladder ring, we see that the indexing set $\mathcal{L}_{\gamma_\lambda}$ is given as
$$\mathcal{L}_{\gamma_\lambda} = \{(i,j) \in \{1,2\} \times \{1,...,n\} : j > 3-i, \quad j \neq n-1-\lambda_1, \quad n-\lambda_2\}.$$
We see that, for $i=1$, this forces $j > n-\lambda_2$ and for $i=2$, this forces $j > n-1-\lambda_1$. Note that, under this construction, we always have at least as many $x_{2j}s$ in the corresponding matrix ladder ring, $\mathcal{O}_q(M_{2,n})_{\gamma_\lambda}$, as we do $x_{1j}s$.

Lemma 4.3.6. *There is an isomorphism of k-algebras*
$$\mathcal{O}_q(M_{m,n}) \longrightarrow \mathcal{O}_{q^{-1}}(M_{m,n})$$
such that
$$x_{ij} \mapsto x_{m+1-i\ n+1-j}.$$

Proof. See [9, Corollary 5.9]. □

Hence applying this isomorphism to the ladder ring twice, we can relabel our generators in the ladder ring in such a way that there are now at least as many $x_{1j}s$ as there are $x_{2j}s$.
Now we may view our ladder ring as a skew polynomial extension of a quantum matrix algebra. We have
$$\mathcal{O}_q(M_{2,n})_{\gamma_\lambda} = \mathcal{O}_q(M_{2,t})[x_{1\ t+1}, ..., x_{1\ t+p}\ ;\ \sigma_{t+1}, ..., \sigma_{t+p}]$$
with the automorphisms σ_i defined as
$$\sigma_i(x_{1j}) = q^{-1}x_{1j}$$
and
$$\sigma_i(x_{2j}) = x_{2j}$$
for all $j = 1, ..., t$ and for all $i = t+1, ..., t+p$.
Hence we see that the ladder ring $\mathcal{O}_q(M_{2,n})_{\gamma_\lambda}$ may be expressed in a such a way that 4.2.7 applies. Further, we know how to count the number of H-prime ideals in quantum matrix algebras. They are in bijection with the Cauchon fillings of the corresponding Young Diagram.
Now note that we have
(a) $\mathcal{O}_q(M_{2,n})_{\gamma_\lambda} = \mathcal{O}_q(M_{2,t})[x_{1\ t+1}, ..., x_{1\ t+p}\ ;\ \sigma_{t+1}, ..., \sigma_{t+p}]$.

(b) $H = (k^*)^{n+2}$ acts via automorphisms on $\mathcal{O}_q(M_{2,n})_{\gamma_\lambda}$.
(c) Each x_{ij} is an H-eigenvector for $i = 1, 2$, $j = 1, ..., t + p$.
(d) There exist $h_{t+1}, ..., h_{t+p} \in H$ such that $h_i(x_{jk}) = \sigma_i(x_{jk})$ for all $i > k$, and the h_i-eigenvalue of x_{ji} is not a root of unity for all i.
Note that these conditions are precisely the setup given in [2], II.5.1.

Theorem 4.3.7. *Let P be an H-prime of $S = \mathcal{O}_q(M_{2,n})_{\gamma_\lambda}$. Then*

$$P = (P \cap \mathcal{O}_q(M_{2,t}))S + \sum_{j \in I} x_{ij} S$$

for some subset $I \subseteq \{t + 1, ..., p\}$.

Proof. This follows immediately from 4.2.7. □

Now we can see the relation between the H-primes of S and the Cauchon Fillings of a Young Diagram derived from the corresponding partition. The H-primes coming from the $2 \times t$ quantum matrix algebra are in correspondence with the Cauchon Fillings of a $2 \times t$ array. The extension of this array by the addition of the remaining generators is incorporated into the H-prime count by colouring the box in the Young Diagram corresponding to the generator x_{ij} black if x_{ij} is contained in the H-prime P and white if $x_{ij} \notin P$.
We note now that a count of the number of Cauchon Fillings of all sub-Young Tableaux with associated partition of type $k \times n$ are given in [29, Section 2.4] under an alternative name. Reading from Theorem 2.4.1 in the paper and setting $q = 1$ in the definition of $A_{k,n}(q)$, we may read off the number of H-primes in $G_q(2, n)$

Theorem 4.3.8.

$$|H - \text{Spec}(G_q(2, n))| = 3^n - (2 + n)2^{n-1} + 1.$$

Note that the formula given for our particular case says that the number of Cauchon Fillings is $3^n - (2 + n)2^{n-1}$. However, since $G_q(2, n)$ is a projective variety, we lose the irrelevant ideal which is an H-prime. This accounts for the“+1” in the formula.

4.4 Illustrations of the main theorem

4.4.1 $G_q(2, 4)$

By Theorem (4.3.8) we have

$$|\text{H} - \text{Spec}(G_q(2, 4)| = 3^4 - (6 \times 2^3) + 1 = 34.$$

Using the technique we have introduced above, we look at all possible subpartitions of (2,2). These are displayed corresponding to the standard partial ordering in the Figure A.

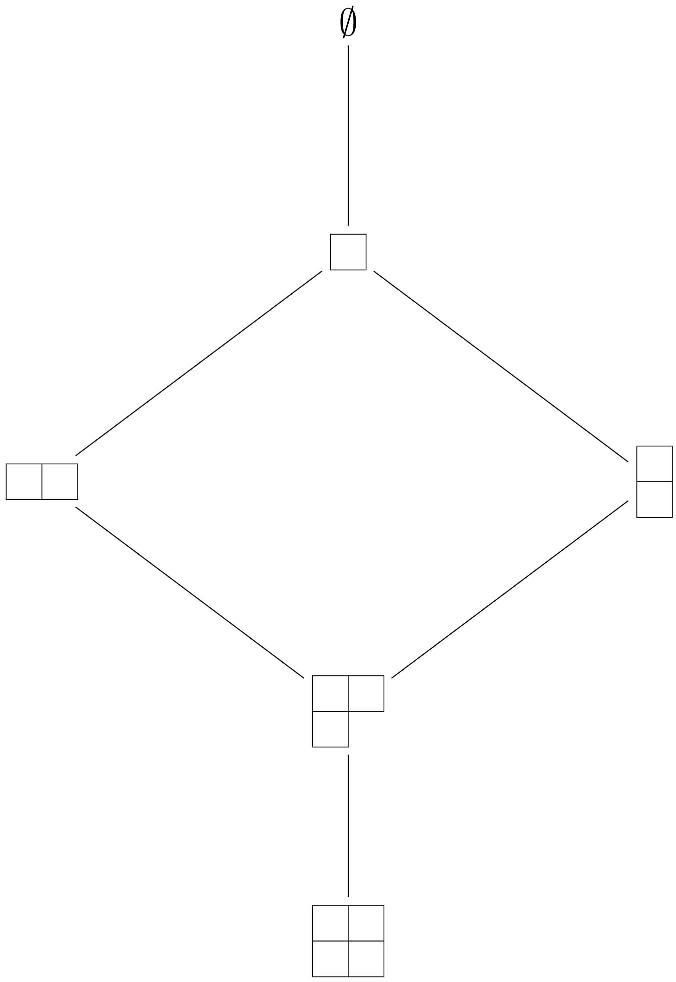


Figure A

We then count the number of Cauchon fillings of each of these Young Diagrams and we take the sum of these values to obtain the number of H-primes in $G_q(2,4)$. The total we obtain is 33. Taking into account the irrelevant ideal, we come to the answer 34 as we have shown earlier.

4.4.2 $G_q(2,5)$

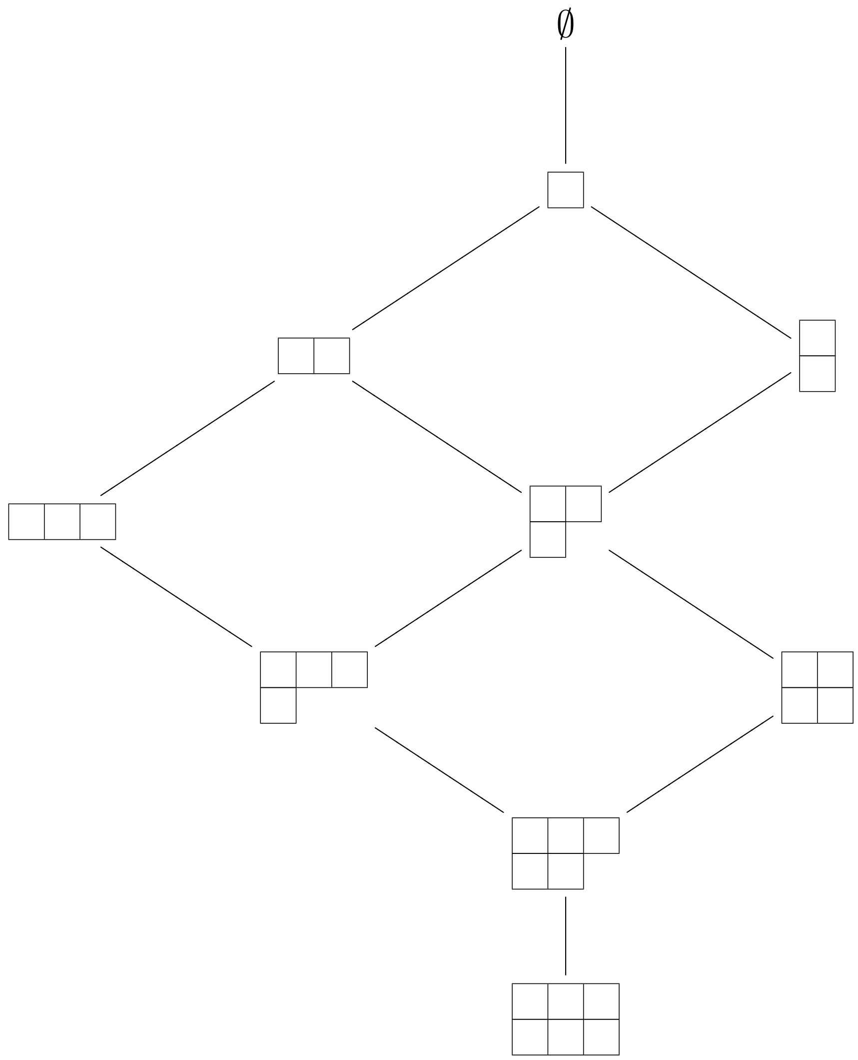


Figure B

Once again, we see that summing the number of Cauchon fillings of each of these Young Diagrams gives us a total of 131 and adding our irrelevant ideal we get the correct total of 132 as verified earlier.

Remark 4.4.1. The results of this chapter have recently been extended to count the number of H-prime ideals in the algebra $G_q(m,n)$. For details of this see [19].

Chapter 5

The Cyclic Ordering

Lenagan and Rigal introduced the notion of a Quantum Graded Algebra with a Straightening Law (QASL) in [21]. One of the main reasons that exhibiting an algebra as a QASL is that this fact guarantees that complex relations with generating elements can always be rewritten in a more structured way with respect to the ordering associated with the algebra.

In this chapter we look at an alternative ordering on generating minors of $G_q(m, n)$. We prove that this ordering is somewhat special in the sense that the QASL property of $G_q(m, n)$ is maintained when viewed with this ordering.

5.1 The cyclic order $<_s$

In order to study properties of the quantum grassmannian, the notion of a quantum graded algebra with a straightening law (on a partially ordered set Π) was introduced in [21].

By [21, Proposition 1.1.4], if A is a quantum graded algebra with a straightening law on the partially ordered set $(\Pi, <_{\mathrm{st}})$, then the set of standard monomials on Π forms a k-basis of A. Hence, in the presence of a standard monomial basis, the structure of a quantum graded algebra with a straightening law may be seen as providing more detailed information on the way standard monomials multiply and commute.

It is shown, in [21, Theorem 3.4.4], that $G_q(m, n)$ is a quantum graded algebra with a straightening law on $(\Pi_{m,n}, \leq_{\mathrm{st}})$.

The aim in this section is to show that there are other partial orderings that can be put on Π in such a way that $G_q(m, n)$ has the structure of a quantum graded algebra with a straightening law.

Consider the order $<_s$ defined by $s <_s s+1 <_s \ldots <_s n <_s 1 <_s \ldots <_s s-1$.

We use this ordering of the set $\{1, \ldots, n\}$ of column indices of $\mathcal{O}_q(M_{m,n})$ to induce a partial ordering $<_s$ on $\Pi = \Pi_{m,n}$: let $I = \{i_1 <_s \cdots <_s i_m\}$ and $J = \{j_1 <_s \cdots <_s j_m\}$ be two index sets, then

$$I \leq_s J \qquad \Longleftrightarrow \qquad i_k \leq_s j_k \quad \text{for each} \quad k \in \{1, \ldots, m\}.$$

When we are considering Π with this induced partial ordering, we will use the notation Π_s.

For example, Figure 5.1 shows the poset Π_2 in $G_q(2,4)$.

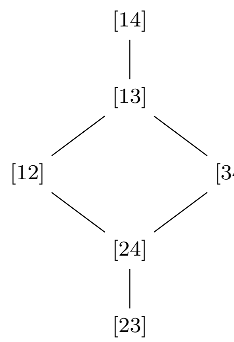


Figure 5.1: The poset Π_2 on $G_q(2,4)$.

The aim in this section is to show that $G_q(m,n)$ is a graded quantum algebra with a straightening law with respect to the poset Π_s.

Set $M = \{\widetilde{a}, \widetilde{a+1}, \ldots, \widetilde{a+m-1}\}$ for some $1 \leq a \leq n$. In Chapter 3, we have seen that the dehomogenisation of $G_q(m,n)$ at the quantum minor $[M]$ is isomorphic to $\mathcal{O}_q(M_{m,n-m})$. We will show that the usual standard partial order on the quantum minors of $\mathcal{O}_q(M_{m,n-m})$ is order isomorphic to the partial order Π_s on $G_q(m,n)$ when $a = s - m$. Once this is established, we use the fact that $\mathcal{O}_q(M_{m,n-m})$ is a graded quantum algebra with a straightening law to obtain the desired result.

In order to do this, we need to know how the quantum minors of $\mathcal{O}_q(M_{m,n-m})$ behave under the dehomogenisation isomorphism θ of Theorem 3.1.8.

Note that

$$\theta(x_{ij}) = \{\{j + a + m - 1,\ a, \ldots, \widehat{a+m-i}, \ldots, a + m - 1\}\},$$

for $1 \leq i \leq m$ and $1 \leq j \leq n - m$.

Consider the quantum minor $[I|J]$ of $\mathcal{O}_q(M_{m,n-m})$. Suppose that $I = \{i_1, \ldots, i_t\}$ and $J = \{j_1, \ldots, j_t\}$, for some $1 \leq t \leq m$, with $i_k \in \{1, \ldots, m\}$ and $j_k \in \{1, \ldots, n - m\}$. Define the maximal quantum minor $[Q(I,J)] \in \Pi_s$ to be the quantum minor with index set $Q(I,J)$ defined by

$$\begin{aligned} Q(I,J) &:= \{(j_1+\widetilde{a+m}-1),(j_2+\widetilde{a+m}-1)\dots,(j_t+\widetilde{a+m}-1)\} \\ &\bigsqcup\left(\{\widetilde{a},\widetilde{a+1},\dots,\widetilde{a+m}-1\}\backslash\{\widetilde{a+m}-i_1,\widetilde{a+m}-i_2,\dots,\widetilde{a+m}-i_t\}\right). \end{aligned}$$

In the special case where $I=\{i\}$ and $J=\{j\}$, we will write $Q(i,j)$ for $Q(I,J)$. Thus,

$$\theta(x_{ij}) = \{\{j+a+m-1,\ a,\dots,\widehat{a+m-i},\dots,a+m-1\}\} = [Q(i,j)][M]^{-1}.$$

Finally, define

$$\{\{Q(I,J)\}\} := [Q(I,J)][M]^{-1}.$$

The aim is to show that $\theta([I|J]) = \{\{Q(I,J)\}\}$. The main calculation is performed in the following preparatory lemma.

Set $\mathrm{sign}(i,j)=\ell(j,i)-\ell(i,j)$; so that $\mathrm{sign}(i,j)=\begin{cases} 1, & \text{if } i<j; \\ 0, & \text{if } i=j; \\ -1, & \text{if } i>j. \end{cases}$

Lemma 5.1.1. *Suppose that* $I=\{i_1,i_2,\dots,i_t\}$ *and* $J=\{j_1,j_2,\dots,j_t\}$ *with* $t\le\min\{m,n-m\}$. *Let* $M=\{\widetilde{a},\widetilde{a+1},\dots,\widetilde{a+m}-1\}$, *for some* $1\le a\le n$. *Then*

$$[Q(I,J)][M]+\sum_{k=1}^{t}(-q)^{(t-k)-\mathrm{sign}(\widetilde{a+m}-i_t,\,j_k+\widetilde{a+m}-1)}[Q(I\backslash\{i_t\},J\backslash\{j_k\})][Q(x_{i_tj_k})] = 0 \tag{5.1}$$

in $G_q(m,n)$.

Proof. **Special case:** We start by considering the special case where $t=m$ and $n=2m$. In this case, $I=J=\{1,\dots,m\}$. Thus, $Q(I,J)=\{\widetilde{a+m},\dots,\widetilde{a+2m}-1\}$ and $M=\{\widetilde{a},\dots,\widetilde{a+m}-1\}$.

Special case, subcase 1: First, consider the case where $m+1\le a\le 2m$, and write $a=m+1+b$, with $0\le b\le m-1$. Note that $k+\widetilde{a+m}-1=b+k$ and $\mathrm{sign}(a,k+\widetilde{a+m}-1)=\mathrm{sign}(a,k+b)=-1$, because $k+b<a$. Also, (5.1), which is what we need to prove, becomes

$$[Q(I,J)][M]+\sum_{k=1}^{m}(-q)^{m+1-k}[Q(I\backslash\{m\},J\backslash\{k\})]\times[k+b,\ a+1,\dots,2m,1,\dots,b]=0 \tag{5.2}$$

The proof uses the generalised quantum plücker relations with $J_1=\emptyset$. Thus,

$$\sum_{K'\sqcup K''=K}(-q)^{\ell(K';K'')+\ell(K'';J_2)}[K'][K''\sqcup J_2]=0, \tag{5.3}$$

and we set $K = \{b+1, \ldots, b+m\} \sqcup \{a\}$ and $J_2 = \{1 \ldots, b,\, a+1, \ldots, 2m\}$.

There are $m+1$ terms in this sum, corresponding to the choices $K'' = \{a\}$ and $K'' = \{b+k\}$ for $1 \le k \le m$.

When $K'' = \{a\}$ and $K' = \{b+1, \ldots, b+m\}$ we have

$$\begin{aligned}\ell(K';K'') + \ell(K'';J_2) &= \ell(\{b+1,\ldots,b+m\};\{a\}) + \ell(\{a\};\{a+1,\ldots,2m,1,\,\ldots,b\}) \\ &= 0 + b = b\end{aligned}$$

and so the corresponding term in the sum is $(-q)^b[Q(I,J)][M]$.

When $K'' = \{b+k\}$ and $K' = \{b+1,\ldots,b+m\}\backslash\{b+k\} \sqcup \{a\}$ we have

$$\begin{aligned}\ell(K';K'') + \ell(K'';J_2) &= \ell(\{b+1,\ldots,b+m\}\backslash\{b+k\} \sqcup \{a\};\{b+k\}) \\ &\quad +\ell(\{b+k\};\{a+1,\ldots,2m,1,\,\ldots,b\}) \\ &= (m+1-k) + b\end{aligned}$$

and so the corresponding term in the sum is $(-q)^{m+1-k+b}Q(I\backslash\{m\}, J\backslash\{k\})Q(m,k)$.

Thus,

$$(-q)^b[Q(I,J)][M] + \sum_{k=1}^{m}(-q)^{m+1-k+b}Q(I\backslash\{m\}, J\backslash\{k\})Q(m,k) = 0.$$

Cancelling $(-q)^b$ gives (5.2), the equality we need to finish this case.

Special case, subcase 2: Now, consider the case where $1 \le a \le m$. Note that $k+a+m-1 \le 2m$ when $k \le m-a+1$ while $k+a+m-1 > 2m$ when $k > a+m-1$. Thus, $k + \widetilde{a+m-1} = k+a+m-1$ for $k \le m-a+1$ and $k + \widetilde{a+m-1} = k+a-m-1$ for $k > a+m-1$. Set $\overline{k} = k + \widetilde{a+m-1}$ in each of these cases.

Now, $\mathrm{sign}(a, k+\widetilde{a+m-1}) = \mathrm{sign}(a,\overline{k}) = 1$ when $k \le m-a+1$ and, similarly, $\mathrm{sign}(a, k+\widetilde{a+m-1}) = -1$ when $k > m-a+1$.

Thus, in this case, (5.1), which is what we need to prove, becomes

$$\begin{aligned}&[Q(I,J)][M] \quad + \\ &\quad \sum_{k=1}^{m-a+1}(-q)^{m-1-k}[Q(I\backslash\{m\}, J\backslash\{k\})][k+a+m-1,\ a+1,\ldots,a+m-1] \\ &\quad -\Big(\sum_{k>m-a+1}^{m}(-q)^{m+1-k}[Q(I\backslash\{m\}, J\backslash\{k\})][k+a-m-1,\ a+1,\ldots,a+m-1]\Big) \\ &\quad = 0. \qquad (5.4)\end{aligned}$$

The proof again uses the generalised quantum Plücker relations with $J_1 = \emptyset$, but

$K = \{1, \ldots, a-1, a+m, \ldots, 2m\} \sqcup \{a\}$ and $J_2 = \{a+1, \ldots, a+m-1\}$. When $K'' = \{a\}$ and $K' = \{1, \ldots, a-1, a+m, \ldots, 2m\}$ we have

$$\begin{aligned} \ell(K'; K'') + \ell(K''; J_2) &= \ell(\{1, \ldots, a-1, a+m, \ldots, 2m\}; \{a\}) \\ &\quad + \ell(\{a\}; \{a+1, \ldots, a+m-1\}) \\ &= (m+1-a) + 0 = m+1-a \end{aligned}$$

and so the corresponding term in the sum is $(-q)^{m+1-a}[Q(I,J)][M]$.

Consider the case that $1 \leq k \leq m-a+1$. In this case, $\overline{k} = k+a+m-1$ and $a+m \leq \overline{k} \leq 2m$. When $K'' = \overline{k}$ and $K' = \{1, \ldots, a-1, a+m, \ldots, 2m\} \backslash \{\overline{k}\} \sqcup \{a\}$ we have

$$\begin{aligned} \ell(K'; K'') + \ell(K''; J_2) &= \ell(\{1, \ldots, a-1, a+m, \ldots, 2m\} \backslash \{\overline{k}\} \sqcup \{a\}; \{k+a+m-1\}) \\ &\quad + \ell(\{k+a+m-1\}; \{a+1, \ldots, a+m-1\}) \\ &= 2m - (k+m+a-1) + m - 1 = 2m - a - k \end{aligned}$$

and so the corresponding term in the sum is

$$(-q)^{2m-a-k}[Q(I \backslash \{m\}, J \backslash \{k\})][k+a+m-1,\ a+1, \ldots, a+m-1]).$$

Next, consider the case where $m-a+1 < k \leq m$. In this case, $\overline{k} = k+a-m-1$ and $1 \leq \overline{k} \leq a-1$. When $K'' = \overline{k}$ and $K' = \{1, \ldots, a-1, a+m, \ldots, 2m\} \backslash \{\overline{k}\} \sqcup \{a\}$ we have

$$\begin{aligned} \ell(K'; K'') + \ell(K''; J_2) &= \ell(\{1, \ldots, a-1, a+m, \ldots, 2m\} \backslash \{\overline{k}\} \sqcup \{a\}; \{k+a-m-1\}) \\ &\quad + \ell(\{k+a-m-1\}; \{a+1, \ldots, a+m-1\}) \\ &= m+1-\overline{k}+0 = m+1-(k-a-m-1) = 2(m+1)-k-a; \end{aligned}$$

and so the corresponding term in the sum is

$$(-q)^{2(m+1)-a-k}[Q(I \backslash \{m\}, J \backslash \{k\})][k+a+m-1,\ a+1, \ldots, a+m-1]).$$

Thus,

$$\begin{aligned} (-q)^{m+1-a}[Q(I,J)][M] &+ \sum_{k=1}^{m-a+1} (-q)^{2m-a-k}[Q(I \backslash \{m\}, J \backslash \{k\})][Q(x_{mk})] \\ &+ \sum_{k>m-a+1}^{m} (-q)^{2(m+1)-a-k}[Q(I \backslash \{m\}, J \backslash \{k\})][Q(x_{mk})] = 0. \end{aligned}$$

Cancelling $(-q)^{m+1-a}$ gives (5.4), the equality we need to prove to finish this case.

This establishes the special case.

General case: Now, consider the general case. Here, the proof is by induction. The base case of $G_q(1,2)$ is trivial to check. First, suppose that the result holds in $G_q(m',n')$ for all $m' \le n' < n$. Next, suppose that the result holds in all $G_q((m',n)$ for all $m' < m$. Finally, suppose that the result holds in $G_q(m,n)$ for all values of $t' < t$.

Suppose that $t < n - m$. Then $t + m < n$; and so there is an index c, say, with $c \notin M \sqcup \{j_1, \dots, j_t\}$. Note that the index c does not occur in any of the terms in (5.1). Thus, we may ignore the column c and work in $G_q(m, n-1)$ where the result holds by the inductive hypothesis.

Next, suppose that $t = n - m < m$. Choose an index $r \in \{1, \dots, m\} \backslash \{i_1, \dots, i_t\}$. The index $\widetilde{a+m} - r$ occurs in each of the quantum minors in (5.1). By the inductive hypothesis, the result (5.1) holds for the triple $I' := I \backslash \{\widetilde{a+m} - r\}$, $J' := J \backslash \{\widetilde{a+m} - r\}$, $M' := M \backslash \{\widetilde{a+m} - r\}$ in the copy of $G_q(m-1, n-1)$ that sits inside the copy of $\mathcal{O}_q(M_{m-1,n-1})$ obtained by removing the row r and the column $\widetilde{a+m} - r$: call the resulting equation (1′). We obtain the desired result by invoking the Quantum Muir Law, Proposition 1.5.11, to insert the index $\widetilde{a+m} - r$ in each quantum minor occuring in (1′).

It only remains to consider the case where $t = n - m = m$. However, this is the special case that was established in the first part of the proof. □

Proposition 5.1.2. $\theta([I|J]) = \{\{Q(I,J)\}\}$.

Proof. The proof is by induction on t. The case $t = 1$ is given in Theorem 3.1.8.

Suppose that $I = \{i_1, i_2, \dots, i_t\}$ and $J = \{j_1, j_2, \dots, j_t\}$, with $t \ge 2$. Expand $[I|J]$ along its final row, by using [26, Corollary 4.4.4], to obtain

$$[I|J] = \sum_{k=1}^{t} (-q)^{t-k} [I \backslash \{i_t\} | J \backslash \{j_k\}] x_{i_t j_k}.$$

Now apply θ to this expression, using the inductive hypothesis on the quantum minors $[I \backslash \{i_t\} | J \backslash \{j_k\}]$ to obtain

$$\theta([I|J]) \;=\; \sum_{k=1}^{t} (-q)^{t-k} [Q(I \backslash \{i_t\}, J \backslash \{j_k\})][M]^{-1} [Q(i_t, j_k)][M]^{-1}$$

Note that the index sets $Q(i_t, j_k) = \{(j_k + \widetilde{a+m} - 1),\ \widetilde{a}, \dots, \widehat{\widetilde{a+m} - i_t}, \dots, \widetilde{a+m} - 1\}$ and $M = \{\widetilde{a}, \widetilde{a+1}, \dots, \widetilde{a+m} - 1\}$ differ only in the indices $(j_k + \widetilde{a+m} - 1)$ and

$\widetilde{a+m-i_t}$; so that

$$[M]^{-1}[Q(i_t,j_k)] = q^{-\mathrm{sign}(\widetilde{a+m-i_t},j_k\widetilde{+a+m}-1)}[Q(i_t,j_k)][M]^{-1}.$$

Thus,

$$\theta([I|J]) \quad = \quad -(\sum_{k=1}^{t}(-q)^{(t-k)-\mathrm{sign}(\widetilde{a+m-i_t},j_k\widetilde{+a+m}-1)}[Q(I\backslash\{i_t\},J\backslash\{j_k\})][Q(x_{ij})])[M]^{-2}.$$

However,

$$-(\sum_{k=1}^{t}(-q)^{(t-k)-\mathrm{sign}(\widetilde{a+m-i_t},j_k\widetilde{+a+m}-1)}[Q(I\backslash\{i_t\},J\backslash\{j_k\})][Q(x_{ij})]) = [Q(I,J)][M]$$

by Lemma 5.1.1; so

$$\theta([I|J]) = [Q(I,J)][M][M]^{-2} = [Q(I,J)][M]^{-1} = \{\{Q(I,J)\}\}$$

as required.

□

Recall the definition of an index pair (I,J) and the corresponding quantum minor $[I \mid J]$ in a fixed quantum matrix algebra, say $\mathcal{O}_q(M_{m,n-m})$. Let Δ denote the set of index pairs (or quantum minors).

We put a partial order on Δ that we denote by $\leq_{\mathrm{st}}$. Let u, v be integers such that $1 \leq u \leq m$ and $1 \leq v \leq n-m$, and let (I,J) and (K,L) be index pairs with $I = \{i_1 < \cdots < i_u\}, K = \{k_1 < \cdots < k_v\} \subseteq \{1,\ldots,m\}$, and $J = \{j_1 < \cdots < j_u\}, L = \{l_1 < \cdots < l_v\} \subseteq \{1,\ldots,n-m\}$. We define $\leq_{\mathrm{st}}$ as follows:

$$(I,J) \leq_{\mathrm{st}} (K,L) \qquad \Longleftrightarrow \qquad \begin{cases} u \geq v, \\ i_s \leq k_s \quad \text{for} \quad 1 \leq s \leq v, \\ j_s \leq l_s \quad \text{for} \quad 1 \leq s \leq v. \end{cases}$$

In [21, Theorem 3.5.3] it is shown that quantum matrices form a graded algebra with a straightening law with respect to this order.

Let $[M] = [\widetilde{a}, \widetilde{a+1}, \ldots, \widetilde{a+m-1}]$. The previous proposition shows that for each quantum minor $[I \mid J]$ of $\mathcal{O}_q(M_{m,n-m})$ produces, in a natural way, a generating minor $[Q(I,J)] = \theta([I \mid J])M$ of $G_q(m,n)$. It is easy to check that every generating minor of $G_q(m,n)$, apart from $[M]$ itself, arises in this way. Thus, we can use the previous proposition to induce a partial order on Π, the set of generating minors of $G_q(m,n)$. The following combinatorial lemma identifies this partial order.

Proposition 5.1.3. *Let $1 \leq s \leq n$ and set $a = \widetilde{s}$. Then $[I|J] \leq_{\mathrm{st}} [K|L]$ if and only if $Q(I,J) <_s Q(K,L)$.*

Proof. This is similar to the proof of [3, Lemma 4.9]. □

Note that $[M] = [\widetilde{a}, \widetilde{a+1}, \dots, \widetilde{a+m-1}]$ is the maximal element in the partially ordered set Π_s. Figure 5.2 illustrates the previous result in $G_q(2,4)$ with $s = 2$.

$\leq_{\text{st}}$ on $\mathcal{O}_q(M_{2,2})$ $\qquad$ Π_2 on $G_q(2,4)$

Figure 5.2:

We use the previous results to transfer the graded algebra with a straightening law property from $\mathcal{O}_q(M_{m,n-m})$ to $G_q(m,n)$. The proof is essentially obtained by reversing the direction of the proof of [21, Theorem 3.5.3], and, for this reason, we merely sketch the proof.

Theorem 5.1.4. *The quantum grassmannian $G_q(m,n)$ is a graded quantum algebra with a straightening law on the poset Π_s for each $1 \leq s \leq n$.*

Proof. There are five conditions in the definition of a graded quantum algebra with a straightening law, see Definition 4.1.2. Conditions (1) and (2) are immediate; so we need to check (3), (4) and (5). We use Theorem 3.1.8 with $a = \widetilde{s-m}$.

The map θ of Theorem 3.1.8 extends to an isomorphism

$$\theta : \mathcal{O}_q(M_{m,n-m})[y, y^{-1}; \sigma] \longrightarrow \text{Dhom}(G_q(m,n), [\widetilde{a}, \widetilde{a+1}, \dots, \widetilde{a+m-1}])$$

with $\theta(y) = [M]$, cf. [14, Corollary 4.1]. Let ρ denote the inverse of this isomorphism. Note that y quasi-commutes with each of the quantum minors in $\mathcal{O}_q(M_{m,n-m})$.

Suppose that $[I_1]^{a_1}[I_2]^{a_2} \dots [I_t]^{a_t}[M]^a$ is a standard monomial with respect to the ordering $<_s$, and suppose that $I_t \neq M$. Let $\theta([K_i \mid L_i]) = [I_i][M]^{-1}$ for each $i = 1, \dots, t$. Then

$$\rho([I_1]^{a_1}[I_2]^{a_2} \dots [I_t]^{a_t}[M]^a) = (-q)^{\bullet}[K_1 \mid L_1]^{a_1}[K_2 \mid L_2]^{a_2} \dots [K_t \mid L_t]^{a_t} y^{a+\sum a_i}.$$

Note that this image is a non-zero scalar multiple of a term in the standard basis of $\mathcal{O}_q(M_{m,n-m})$ multiplied by a power of y. Note also that distinct $[I_1]^{a_1}[I_2]^{a_2} \dots [I_t]^{a_t}[M]^a$

produce distinct images. Thus, a linear combination of such terms is mapped to a linear combination of terms which are linearly independent, and so the standard monomials with respect to the ordering $<_s$ are linearly independent. This establishes (3).

Next, suppose that $[I], [J]$ are incomparable with respect to $<_s$. Note that neither $[I]$ nor $[J]$ is equal to $[M]$, since $[M]$ is the maximal element of the poset Π_s. Thus, there are quantum minors $[K \mid L], [U \mid V]$ with $\theta([K \mid L]) = [I][M]^{-1}$ and $\theta([U \mid V]) = [I][M]^{-1}$. Note that $[K \mid L]$ and $[U \mid V]$ are incomparable, by Proposition 5.1.3. As $\mathcal{O}_q(M_{m,n})$ is a graded quantum algebra with a straightening law, there is an equation

$$[K \mid L][U \mid V] = \sum \alpha_i [K_i \mid L_i][U_i \mid V_i]$$

with $\alpha_i \in k$ and $[K_i \mid L_i] <_{\text{st}} [U_i \mid V_i]$ while $[K_i \mid L_i] <_{\text{st}} [K \mid L], [U \mid V]$.

Apply θ to this equation, and cancel $[M]^{-2}$ from the resulting equation to obtain an equation

$$[I][J] = \sum \alpha_i (-q)^{\bullet} [I_i][J_i]$$

and note that $[I_i] <_s [J_i]$ and $[I_i] <_s [I], [J]$ for each i, by using Proposition 5.1.3. This establishes (4).

Finally, suppose that $[I], [J] \in \Pi_s$. If $[I] = [M]$ or $[J] = [M]$ then these quantum minors quasi-commute; and so (5) is established for this pair. Otherwise, assume as above that

$$[M] \neq [I], [J] \in \Pi_s.$$

There are quantum minors $[K \mid L], [U \mid V]$ such that $\theta([K \mid L]) = [I][M]^{-1}$ and $\theta([U \mid V]) = [J][M]^{-1}$. Now, since $\mathcal{O}_q(M_{m,n})$ is a graded quantum algebra with a straightening law, there is a unit $c \in k^*$ such that

$$[K \mid L][U \mid V] - c[U \mid V][K \mid L] = \sum, \alpha_i [K_i \mid L_i][U_i \mid V_i]$$

with $\alpha_i \in k$ and $[K_i \mid L_i] <_{\text{st}} [U_i \mid V_i]$ while $[K_i \mid L_i] <_{\text{st}} [K \mid L], [U \mid V]$.

Once again we apply θ to this equation and cancel $[M]^{-2}$ from the resulting equation to obtain

$$[I][J] - c[J][I] = \sum, \alpha_i (-q)^{\bullet} [I_i][J_i]$$

noting that $[I_i] <_s [J_i]$ and $[I_i] <_s [I], [J]$ for each i by once again appealing to Proposition 5.1.3. Hence property (5) is established.

Thus, $G_q(m, n)$ is a graded quantum algebra with a straightening law with respect to the poset Π_s. □

Chapter 6

The Poisson Structure Associated to Quantum Grassmannians

We describe in detail the bijection between the prime (resp. primitive) ideals of the 2×4 Quantum Grassmannian, $G_q(2,4)$, and the prime Poisson (resp. symplectic) ideals of its commutative analogue $G(2,4)$. We follow the technique given by Oh in [25] and identify the symplectic ideals of the algebra $G(2,4)$. Throughout this chapter let k denote an algebraically closed field.

6.1 Poisson Algebras

Definition 6.1.1. A *Poisson algebra* is a commutative k-algebra R equipped with a bilinear map

$$\{-,-\} : R \times R \longrightarrow R$$

such that

$$\begin{aligned} &(i)\ \ \{a,b\} = -\{b,a\}, \\ &(ii)\ \ \{a,\{b,c\}\} + \{b,\{c,a\}\} + \{c,\{a,b\}\} = 0, \\ &(iii)\ \ \{ab,c\} = a\{b,c\} + b\{a,c\}, \end{aligned}$$

for all $a,b,c \in R$.

An ideal I of R is called a *Poisson ideal* provided

$$\{I,R\} \subseteq I.$$

A Poisson ideal I of R is *symplectic* if there exists a maximal ideal M of R such that I is the largest Poisson ideal contained in M.

The *Poisson centre* of a Poisson algebra R is the subalgebra

$$Z_p(R) = \{a \in R : \{a,R\} = 0\}$$

6.2 Poisson Ideals In $k\mathbb{Z}^n$

Consider the free abelian group $\mathbb{Z}^n$ of finite rank n. The group ring $k\mathbb{Z}^n = k[t_\lambda : \lambda \in \mathbb{Z}^n]$ has multiplication given by $t_\lambda t_\mu = t_{\lambda+\mu}$ for all $\lambda, \mu \in \mathbb{Z}^n$. An antisymmetric biadditive map

$$u : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow k$$

has the properties that

$$u(\lambda + \mu, \nu) = u(\lambda, \nu) + u(\mu, \nu)$$

and

$$u(\lambda, \mu) = -u(\mu, \lambda).$$

Given such a map u, we have a Poisson bracket on $k\mathbb{Z}^n$ induced by the map u. This bracket is given by

$$\{t_\lambda, t_\mu\} = u(\lambda, \mu) t_{\lambda+\mu}.$$

Let

$$\mathbb{Z}^n_u = \{\lambda \in \mathbb{Z}^n : u(\lambda, \mu) = 0, \quad \forall \mu \in \mathbb{Z}^n\}.$$

Lemma 6.2.1.

$$Z_p(k\mathbb{Z}^n) = k\mathbb{Z}^n_u.$$

Proof. See [25], 2.1. □

Lemma 6.2.2. *There is an inclusion preserving bijection between the Poisson ideals of $k\mathbb{Z}^n$ and the ideals of $k\mathbb{Z}^n_u$. If I is a Poisson ideal of $k\mathbb{Z}^n$ then $I = (I \cap k\mathbb{Z}^n_u)k\mathbb{Z}^n$, and if J is an ideal of $k\mathbb{Z}^n_u$ then $J = (Jk\mathbb{Z}^n) \cap k\mathbb{Z}^n_u$.*

Proof. See [25], 2.2. □

Lemma 6.2.3. *(1) Every prime Poisson ideal of $k\mathbb{Z}^n$ is of the form $Jk\mathbb{Z}^n$ and conversely every ideal of the form $Jk\mathbb{Z}^n$ is a prime Poisson ideal, where J is a prime ideal of $k\mathbb{Z}^n_u$.*
(ii) Every symplectic ideal of $k\mathbb{Z}^n$ is of the form $Jk\mathbb{Z}^n$ and conversely every ideal of the form $Jk\mathbb{Z}^n$ is a symplectic ideal, where J is a maximal ideal of $k\mathbb{Z}^n_u$.

Proof. See [25], 2.3. □

Definition 6.2.4. Set $\{e_i : 1 \leq i \leq n\}$ to be the standard basis of $\mathbb{Z}^n$ and let

$$\sigma : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow k^*$$

be an antisymmetric bicharacter. Explicitly by this we mean that, for all $\lambda, \mu, \nu \in \mathbb{Z}^n$,

$$\begin{aligned} &\sigma(\lambda, \lambda) = 1, \\ &\sigma(\lambda, \mu) = \sigma(\mu, \lambda)^{-1}, \\ &\sigma(\lambda, \mu + \nu) = \sigma(\lambda, \mu)\sigma(\lambda, \nu). \end{aligned}$$

We define $R^n(\sigma)$ to be the k-algebra generated by the elements $x_1^{\pm 1}, ..., x_n^{\pm 1}$ subject to the relations

$$x_i x_j = \sigma(e_i, e_j) x_j x_i$$

for all i, j.

Lemma 6.2.5.

$$Z(R^n(\sigma)) \cong k\mathbb{Z}^n_\sigma$$

where

$$\mathbb{Z}^n_\sigma := \{\lambda : \sigma(\lambda, \mu) = 1, \quad \forall \mu \in \mathbb{Z}^n\}.$$

Proof. See [10] 1.2. □

Lemma 6.2.6. *When $\mathbb{Z}^n_\sigma = \mathbb{Z}^n_u$ there is a bijection between the prime (resp. primitive) ideals of $R^n(\sigma)$ and the prime Poisson (resp. symplectic) ideals of the group ring $k\mathbb{Z}^n$. Explicitly, the map*

$$\mathrm{Spec}(R^n(\sigma)) \longrightarrow p\mathrm{Spec}(k\mathbb{Z}^n)$$

given by

$$P \mapsto k\mathbb{Z}^n(P \cap Z^n_\sigma)$$

is a bijection and P is a primitive ideal of $R^n(\sigma)$ if and only if $k\mathbb{Z}^n(P \cap Z^n_\sigma)$ is a symplectic ideal of $k\mathbb{Z}^n$.

Proof. See [25] 2.6. □

6.3 The Poisson Structure on $G(2, 4)$

In this section, we recall the fact from [2] that the process of quantization can be regarded as the construction of a noncommutative algebra in which the noncommutativity is seen as encoding the Poisson structure of a commutative Poisson algebra. In [2], the details of this construction are given as follows :

Construction

Let R be a commutative k-algebra and let $h \in R$. Let A be an R-algebra such that h is regular in A. Suppose that

$$\bar{A} := A/hA$$

is commutative.
For each $a \in A$, we will write $\bar{a}$ for the image of a in the factor algebra $\bar{A}$. We are now ready to define our Poisson bracket on $\bar{A}$. Given $\alpha := \bar{a}, \beta := \bar{b} \in \bar{A}$, we have that the commutator

$$[a, b] = h\gamma(a, b)$$

for some unique element $\gamma(a,b)$ of A by the commutativity of the factor algebra $\bar{A}$ and the fact that h is a regular element. We define

$$\{\alpha, \beta\} := \overline{\gamma(a,b)}.$$

It is easy to verify that this definition is independent of the choice of a and b.

6.3.1 Example : $\mathcal{O}_q(M_{2,2})$

In 6.3, take $R := k[q^{\pm 1}]$ and $h := q-1$. Then we have that

$$\mathcal{O}_q(M_{2,2}) \diagup \langle q-1 \rangle \cong \mathcal{O}(M_{2,2})$$

is commutative and so we may calculate the Poisson bracket on $\mathcal{O}(M_{2,2})$ using the relations in the algebra $\mathcal{O}_q(M_{2,2})$. We obtain the bracket given by

$$\begin{array}{lll} \{x_{11}, x_{12}\} = x_{11}x_{12}, & \{x_{11}, x_{21}\} = x_{11}x_{21}, & \{x_{12}, x_{21}\} = 0, \\ \{x_{12}, x_{22}\} = x_{12}x_{22}, & \{x_{21}, x_{22}\} = x_{21}x_{22}, & \{x_{11}, x_{22}\} = 2x_{12}x_{21}. \end{array}$$

6.4 A Poisson Bracket on $G(2,4)$ using $G_q(2,4)$

Following on from the previous example, we may calculate the Poisson bracket on the commutative algebra

$$G(2,4) \cong G_q(2,4) \diagup \langle q-1 \rangle .$$

As before, we take $R = k[q^{\pm 1}]$ and $h = q-1$. We have

$$[[12],[13]] = [12][13] - [13][12] = [12][13] - q^{-1}[12][13] = (q-1)q^{-1}[12][13]$$

Hence

$$\{[12],[13]\} = [12][13]$$

We have a similar Poisson bracket for q-commuting minors. Hence

$$\{[ij],[kl]\} = [ij][kl]$$

when $|\{i,j\} \cap \{k,l\}| = 1$.

We now consider the Lie bracket for commuting minors [14] and [23]. We have

$$[[14],[23]] = [14][23] - [23][14] = 0.$$

Thus,

$$\{[14],[23]\} = 0$$

Or in the more general $G(2,n)$ case,

$$\{[ij],[kl]\} = 0$$

whenever $i < k < l < j$.

We now consider the minors [12] and [34] which q^2-commute in $G_q(2,4)$. We have

$$[[12],[34]] = [12][34] - [34][12] = (1-q^{-2})[12][34] = (q-1)(q^{-1}+q^{-2})[12][34].$$

Hence

$$\{[12],[34]\} = 2[12][34].$$

Or in the $G(2,n)$ case,

$$\{[ij],[kl]\} = 2[ij][kl]$$

for $i < j < k < l$.

Finally, we consider the bracket of the minors [13] and [24]. In this case we have

$$\begin{aligned}
[[13],[24]] &= [13][24] - [24][13] \\
&= [13][24] - ([13][24] - (q-q^{-1})[14][23]) \\
&= (q-1)(1+q^{-1})[14][23]
\end{aligned}$$

Hence

$$\{[13],[24]\} = 2[14][23]$$

or in $G(2,n)$,

$$\{[ij],[kl]\} = 2[il][jk]$$

whenever $i < k < j < l$.

6.5 The Poisson Structures on $G(2,4)$ and $\mathcal{O}(M_{2,2})$

Recall that, in commutative dehomogenisation theory, we have the homomorphism

$$\phi : G(2,4) \longrightarrow \mathcal{O}(M_{2,2})$$

given by

$$
\begin{aligned}
[34] &\mapsto 1, \\
[12] &\mapsto [12|12], \\
[13] &\mapsto x_{11}, \\
[14] &\mapsto x_{21}, \\
[23] &\mapsto x_{12}, \\
[24] &\mapsto x_{22}.
\end{aligned}
$$

Definition 6.5.1. Let A and B be Poisson algebras with associated Poisson brackets $\{-,-\}_A$ and $\{-,-\}_B$ respectively. We say that an algebra homomorphism

$$\theta : A \longrightarrow B$$

is a *Poisson homomorphism* if

$$\theta(\{a_1, a_2\}_A) = \{\theta(a_1), \theta(a_2)\}_B$$

for all $a_1, a_2 \in A$.

Proposition 6.5.2. *The dehomogenisation map ϕ given above is a Poisson homomorphism.*

Proof. Recalling the Poisson bracket on $\mathcal{O}(M_{2,2})$, we have

$$\phi(\{[13],[14]\}_G) = \phi([13][14]) = x_{11}x_{21}$$

and

$$\{\phi([13]), \phi([14])\}_M = \{x_{11}, x_{21}\}_M = x_{11}x_{21}.$$

Secondly, we have

$$\phi(\{[13],[23]\}_G) = \phi([13][23]) = x_{11}x_{12}$$

and

$$\{\phi([13]), \phi([23])\}_M = \{x_{11}, x_{12}\}_M = x_{11}x_{12}.$$

Next, we have

$$\phi(\{[13],[24]\}_G) = \phi(2[14][23]) = 2x_{21}x_{12}.$$

and

$$\{\phi([13]), \phi([24])\}_M = \{x_{11}, x_{22}\}_M = 2x_{21}x_{12}.$$

We also have

$$\phi(\{[14],[23]\}_G) = \phi(0) = 0$$

and

$$\{\phi([14]), \phi([23])\}_M = \{x_{21}, x_{12}\}_M = 0.$$

In the next case, we have

$$\phi(\{[14],[24]\}_G) = \phi([14][24]) = x_{21}x_{22}$$

and

$$\{\phi([14]), \phi([24])\}_M = \{x_{21}, x_{22}\}_M = x_{21}x_{22}.$$

Finally, we have

$$\phi(\{[23],[24]\}_G) = \phi([23][24]) = x_{12}x_{22}$$

and

$$\{\phi([23]), \phi([24])\}_M = \{x_{12}, x_{22}\}_M = x_{12}x_{22}.$$

□

Let S be the multiplicative set of $G_q(2,4)$ generated by [12],[14],[23] and [34]. Each of these four consecutive minors are normal elements in the algebra $G_q(2,4)$ and so the multiplicative subset of $G_q(2,4)$ generated by these four elements is a right Ore set in $G_q(2,4)$. Now recall from Corollary 2.1.5 that the multiplicative set consisting of the nonnegative powers of the element [13] forms a right Ore set in $G_q(2,4)$. Hence by 1.1.8, the multiplicative set S generated by the elements $[12],[13],[14],[23]$ and $[34]$ is also right Ore. Set

$$R := G_q(2,4)[S^{-1}].$$

Now let T denote the subalgebra of R generated by the elements

$$[12]^{\pm 1}, [13]^{\pm 1}, [14]^{\pm 1}, [23]^{\pm 1}, [34]^{\pm 1}.$$

Then we have

$$[24] = [13]^{-1}(q^{-1}[12][34] + q[14][23]) \in T$$

via the Quantum Plücker relation. Hence $T = R$.
Now we may express R as

$$R = R^5(\sigma)$$

where σ is the antisymmetric bicharacter of $\mathbb{Z}^5$ defined by

$$\begin{array}{lll} \sigma(e_1,e_2) = q, & \sigma(e_1,e_3) = q, & \sigma(e_1,e_4) = q, \\ \sigma(e_1,e_5) = q^2, & \sigma(e_2,e_3) = q, & \sigma(e_2,e_4) = q, \\ \sigma(e_2,e_5) = q, & \sigma(e_3,e_4) = 1, & \sigma(e_3,e_5) = q, \\ \sigma(e_4,e_5) = q. & & \end{array}$$

Explicitly, the map

$$R \longrightarrow R^5(\sigma)$$

given by

$$\begin{aligned}[12] &\mapsto x_1,\\ [13] &\mapsto x_2,\\ [14] &\mapsto x_3,\\ [23] &\mapsto x_4,\\ [34] &\mapsto x_5,\end{aligned}$$

is an isomorphism.

We are also able to define an antisymmetric biadditive map

$$u : \mathbb{Z}^5 \times \mathbb{Z}^5 \longrightarrow k$$

defined by

$$\sigma(\mu, \nu) = q^{u(\mu,\nu)}.$$

Explicitly, we take

$$\begin{array}{lll} u(e_1, e_2) = 1, & u(e_1, e_3) = 1, & u(e_1, e_4) = 1, \\ u(e_1, e_5) = 2, & u(e_2, e_3) = 1, & u(e_2, e_4) = 1, \\ u(e_2, e_5) = 1, & u(e_3, e_4) = 0, & u(e_3, e_5) = 1, \\ u(e_4, e_5) = 1. & & \end{array}$$

Since q is not a root of unity, we have that

$$\mathbb{Z}^5_\sigma = \mathbb{Z}^5_u.$$

Hence, by 6.2.6 there is a bijection between the primes of R and the prime Poisson ideals of $k\mathbb{Z}^5$.

6.6 $G(2, 4)$

Let B denote the localization of $G(2, 4)$ with respect to the multiplicative set generated by $[12], [13], [14], [23], [34]$. Let C be the subalgebra of B generated by

$$[12]^{\pm 1}, [13]^{\pm 1}, [14]^{\pm 1}, [23]^{\pm 1}, [34]^{\pm 1}.$$

Since

$$[24] = [13]^{-1}([[12][34] - [14][23]) \in C,$$

$B = C$. Hence B can be presented as the group ring $k\mathbb{Z}^5$. Explicitly, the map

$$\theta : B \longrightarrow k\mathbb{Z}^5$$

defined by

$$
\begin{aligned}
[12] &\mapsto t_{e_1},\\
[13] &\mapsto t_{e_2},\\
[14] &\mapsto t_{e_3},\\
[23] &\mapsto t_{e_4},\\
[34] &\mapsto t_{e_5},
\end{aligned}
$$

is an isomorphism.

6.7 Primes In $G_q(2,4)$

In this section we analyse the primes of $G_q(2,4)$ and determine which of them have non-trivial intersection with the set

$$\wp := \{[12],[13],[14],[23],[24],[34]\}.$$

First, we note that the relation

$$[13][24]-[24][13]=(q-q^{-1})[14][23]$$

implies that, for all primes P of $G_q(2,4)$, if we have either $[13]\in P$ or $[24]\in P$, then we must also have either $[14]\in P$ or $[23]\in P$.

Similarly, from the Quantum Plücker relations

$$[12][34]-q[13][24]+q^2[14][23]=0$$

and

$$q^4[34][12]-q^3[24][13]+q^2[23][14]=0$$

we get the relation

$$(1-q^2)[12][34]=q[13][24]-q^3[24][13].$$

Hence for all primes P of $G_q(2,4)$ with either $[13]\in P$ or $[24]\in P$, we must also have either $[12]\in P$ or $[34]\in P$.
We have the following important lemma showing us a relationship between primes and H-primes of $G_q(2,4)$.

Lemma 6.7.1. *Let P be a prime ideal of $G_q(2,4)$ and set $Y := P\cap\wp$. Then*

$$\{\langle P\cap\wp\rangle : P\in\operatorname{Spec}(G_q(2,4))\} = H-\operatorname{Spec}(G_q(2,4))$$

Proof. Take

$$Q := (P:H)=\bigcap_{h\in H} h(P)\subseteq P.$$

We claim that $Y = Q \cap \wp$.
Clearly, we have

$$Q \cap \wp \subseteq P \cap \wp = Y.$$

Now let $[ij] \in Y$. Then certainly $[ij] \in P$.
Let $h \in H$. Then

$$h([ij]) = \lambda[ij]$$

for some $\lambda \in k^*$.
Hence

$$[ij] = \lambda^{-1}h([ij]) \in h(P).$$

Since $h \in H$ was arbitrary, we have

$$[ij] \in \bigcap_{h \in H} h(P) = Q$$

so $[ij] \in Q \cap \wp$ as required.

So $\langle P \cap \wp \rangle$ must be one of the 34 H-invariant ideals identified in Chapter 2. $\square$

Lemma 6.7.2. *For each subset, $X \subseteq \wp$ which generates an H-invariant prime of $G_q(2,4)$, there is an Ore set S of the factor algebra $G_q(2,4) \diagup \langle X \rangle$ such that*

(i)

$$(G_q(2,4) \diagup \langle X \rangle)[S^{-1}] \cong R^n(\sigma)$$

for some σ and $n = 0, 1, 2, 3, 4, 5$.

(ii)

$$S \cap (P \diagup \langle X \rangle) \neq \phi$$

for every prime ideal P with $X \subseteq P \cap \wp$.

(iii)

$$\sigma(\mu, \nu) = q^{u(\mu,\nu)}$$

for all $\mu, \nu \in \mathbb{Z}^n$.

Proof. For part (i), the only problem we encounter is with the cases of $X = \{[12]\}$ or $X = \{[34]\}$ since otherwise all images of generators in the associated factor rings $G_q(2,4) \diagup \langle X \rangle$ are easily seen to be normal in this new factor ring. Each of the two remaining problematic cases is resolved by appealing to the quantum Plücker relations.

Let us assume that $X = \{[12]\}$. Then from the quantum Plücker relations

$$[12][34] - q[13][24] + q^2[14][23] = 0$$

we obtain the relation

$$\widehat{[14]}\widehat{[23]} = q^{-1}\widehat{[13]}\widehat{[24]}$$

in the factor algebra $G_q(2,4)\diagup\langle X\rangle$.
Hence, the problematic relation

$$[13][24]-[24][13]=(q-q^{-1})[14][23]$$

becomes

$$\widehat{[13]}\widehat{[24]}-\widehat{[24]}\widehat{[13]}=(q-q^{-1})q^{-1}\widehat{[13]}\widehat{[24]}$$

in $G_q(2,4)\diagup\langle X\rangle$.

Hence,

$$\widehat{[13]}\widehat{[24]}-\widehat{[24]}\widehat{[13]}=\widehat{[13]}\widehat{[24]}-q^{-2}\widehat{[13]}\widehat{[24]}$$

i.e.

$$\widehat{[13]}\widehat{[24]}=q^2\widehat{[24]}\widehat{[13]}.$$

Hence all images of generators are normal in the case $X=\{[12]\}$. It is easily seen that the same relation is produced on passing to the factor of $G_q(2,4)$ by the ideal generated by [34]. □

Explicitly, we must examine the case for each possible choice of the ideal $\langle X\rangle$. Let $G:=G(2,4)$. We will use the notation $\widehat{[-]}$ to denote the various images of generators in factor rings which will will construct.

(1) $X=\{[12]\}$
in $G\diagup\langle X\rangle$, consider the multiplicative set S generated by $\widehat{[13]},\widehat{[14]},\widehat{[23]},\widehat{[24]},\widehat{[34]}$. Each of the above five generators is normal in $G\diagup\langle X\rangle$ so S is a right Ore set in this new factor ring. Further, we have

$$q^{-1}\widehat{[13]}\widehat{[24]}=q^2\widehat{[14]}\widehat{[23]}$$

from the quantum plücker relation. Hence in the localization of $G\diagup\langle X\rangle$ with respect to S, we have

$$\widehat{[13]}=q\widehat{[14]}\widehat{[23]}\widehat{[24]}^{-1}.$$

and so we observe that we need only the elements $\widehat{[14]}^{\pm1},\widehat{[23]}^{\pm1},\widehat{[24]}^{\pm1}$ and $\widehat{[34]}^{\pm1}$ to generate $(G\diagup\langle X\rangle)[S^{-1}]$.
Now we have

$$(G\diagup\langle X\rangle)[S^{-1}]\cong R^4(\sigma)$$

with

$$\begin{array}{lll}\sigma(e_1,e_2)=1, & \sigma(e_1,e_3)=q, & \sigma(e_1,e_4)=q,\\ \sigma(e_2,e_3)=q, & \sigma(e_2,e_4)=q, & \sigma(e_3,e_4)=q.\end{array}$$

(2) $X=\{[23]\}$
In $G\diagup\langle X\rangle$, let S be the multiplicative set generated by $\widehat{[12]},\widehat{[13]},\widehat{[14]},\widehat{[24]},\widehat{[34]}$. Then S is right Ore and as in the previous case we can generate the localization

$(G\diagup\langle X\rangle)[S^{-1}]$ by $\widehat{[13]}^{\pm 1}, \widehat{[14]}^{\pm 1}, \widehat{[24]}^{\pm 1}$ and $\widehat{[34]}^{\pm 1}$ due to the relation

$$\widehat{[12]} = q\widehat{[13]}\widehat{[24]}\widehat{[34]}^{-1}$$

once again arising from the quantum Plücker relation. Now

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^4(\sigma)$$

where

$$\begin{aligned} &\sigma(e_1, e_2) = q, && \sigma(e_1, e_3) = q^2, && \sigma(e_1, e_4) = q, \\ &\sigma(e_2, e_3) = q, && \sigma(e_2, e_4) = q, && \sigma(e_3, e_4) = q. \end{aligned}$$

(3) $X = \{[34]\}$
In $G\diagup\langle X\rangle$, let S denote the multiplicative set generated by $\widehat{[12]}, \widehat{[13]}, \widehat{[14]}, \widehat{[23]}, \widehat{[24]}$. Then S is right Ore and as in Case (1) we may appeal to the relation

$$q\widehat{[13]}\widehat{[24]} = q^2\widehat{[14]}\widehat{[23]}$$

arising from the quantum Plücker relation to see that

$$\widehat{[13]} = q\widehat{[14]}\widehat{[23]}\widehat{[24]}^{-1}$$

and so we can use the elements $\widehat{[12]}^{\pm 1}, \widehat{[14]}^{\pm 1}, \widehat{[23]}^{\pm 1}$ and $\widehat{[24]}^{\pm 1}$ to generate the localization $(G\diagup\langle X\rangle)[S^{-1}]$. Now we have

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^4(\sigma)$$

where

$$\begin{aligned} &\sigma(e_1, e_2) = q, && \sigma(e_1, e_3) = q, && \sigma(e_1, e_4) = q, \\ &\sigma(e_2, e_3) = 1, && \sigma(e_2, e_4) = q, && \sigma(e_3, e_4) = q. \end{aligned}$$

(4) $X = \{[14]\}$
In the factor algebra $G\diagup\langle X\rangle$, let S be the multiplicative set generated by the elements $\widehat{[12]}, \widehat{[13]}, \widehat{[23]}, \widehat{[24]}, \widehat{[34]}$. Then S is right Ore and once again we see that

$$\widehat{[13]} = q\widehat{[12]}\widehat{[34]}\widehat{[24]}^{-1}$$

and so the localization $(G\diagup\langle X\rangle)[S^{-1}]$ can be generated by the elements $\widehat{[12]}^{\pm 1}, \widehat{[23]}^{\pm 1}, \widehat{[24]}^{\pm 1}$ and $\widehat{[34]}^{\pm 1}$. We have

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^4(\sigma)$$

where we have

$$\begin{aligned} &\sigma(e_1, e_2) = q, && \sigma(e_1, e_3) = q, && \sigma(e_1, e_4) = q^2, \\ &\sigma(e_2, e_3) = q, && \sigma(e_2, e_4) = q, && \sigma(e_3, e_4) = q. \end{aligned}$$

(5) $X = \{[12], [13], [23]\}$
In $G/\langle X\rangle$, let S be the multiplicative set generated by $\widehat{[14]}, \widehat{[24]}, \widehat{[34]}$. S is right Ore and

$$(G/\langle X\rangle)[S^{-1}] \cong R^3(\sigma)$$

where

$$\sigma(e_1, e_2) = q, \qquad \sigma(e_1, e_3) = q, \qquad \sigma(e_2, e_3) = q.$$

(6) $X = \{[12], [23], [24]\}$
In $G/\langle X\rangle$, set S to be the multiplicative set generated by the elements $\widehat{[13]}, \widehat{[14]}, \widehat{[34]}$. S is right Ore and we have

$$(G/\langle X\rangle)[S^{-1}] \cong R^3(\sigma)$$

where we have

$$\sigma(e_1, e_2) = q, \qquad \sigma(e_1, e_3) = q, \qquad \sigma(e_2, e_3) = q.$$

(7) $X = \{[23], [24], [34]\}$
In $G/\langle X\rangle$ let S be the right Ore set generated by the three elements $\widehat{[12]}, \widehat{[13]}, \widehat{[14]}$. Then

$$(G/\langle X\rangle)[S^{-1}] \cong R^3(\sigma)$$

with

$$\sigma(e_1, e_2) = q, \qquad \sigma(e_1, e_3) = q, \qquad \sigma(e_2, e_3) = q.$$

(8) $X = \{[13], [23], [34]\}$
In $G/\langle X\rangle$ let S be the right Ore set generated by the elements $\widehat{[12]}, \widehat{[14]}, \widehat{[24]}$. Then

$$(G/\langle X\rangle)[S^{-1}] \cong R^3(\sigma)$$

where

$$\sigma(e_1, e_2) = q, \qquad \sigma(e_1, e_3) = q, \qquad \sigma(e_2, e_3) = q.$$

(9) $X = \{[14], [23]\}$
In $G/\langle X\rangle$, let S be the multiplicative set generated by the elements $\widehat{[[12]}, \widehat{[13]}, \widehat{[24]}, \widehat{[34]}$. Then S is right Ore and we have

$$(G/\langle X\rangle)[S^{-1}] \cong R^4(\sigma)$$

where

$$\begin{aligned} &\sigma(e_1, e_2) = q, && \sigma(e_1, e_3) = q, && \sigma(e_1, e_4) = q^2, \\ &\sigma(e_2, e_3) = 1, && \sigma(e_2, e_4) = q, && \sigma(e_3, e_4) = q. \end{aligned}$$

(10) $X = \{[12], [34]\}$

In $G/\langle X\rangle$, set S to be the right Ore set generated by the four elements $\widehat{[13]}, \widehat{[14]}, \widehat{[23]}, \widehat{[24]}$. Then we have

$$(G/\langle X\rangle)[S^{-1}] \cong R^4(\sigma)$$

where

$$\sigma(e_1, e_2) = q, \qquad \sigma(e_1, e_3) = q, \qquad \sigma(e_1, e_4) = q^2,$$
$$\sigma(e_2, e_3) = 1, \qquad \sigma(e_2, e_4) = q, \qquad \sigma(e_3, e_4) = q.$$

(11) $X = \{[12], [14], [24]\}$
In $G/\langle X\rangle$, let S be the multiplicative set generated by $\widehat{[13]}, \widehat{[23]}, \widehat{[34]}$. Then S is right Ore and

$$(G/\langle X\rangle)[S^{-1}] \cong R^3(\sigma)$$

where

$$\sigma(e_1, e_2) = q, \qquad \sigma(e_1, e_3) = q, \qquad \sigma(e_2, e_3) = q.$$

(12) $X = \{[12], [13], [14]\}$
In the factor algebra $G/\langle X\rangle$, let S be the multiplicative set generated by $\widehat{[23]}, \widehat{[24]}, \widehat{[34]}$. S is right Ore and we have

$$(G/\langle X\rangle)[S^{-1}] \cong R^3(\sigma)$$

where

$$\sigma(e_1, e_2) = q, \qquad \sigma(e_1, e_3) = q, \qquad \sigma(e_2, e_3) = q.$$

(13) $X = \{[13], [14], [34]\}$
In the factor algebra $G/\langle X\rangle$, set S to be the right Ore set generated by the three elements $\widehat{[12]}, \widehat{[23]}, \widehat{[24]}$. Then we have

$$(G/\langle X\rangle)[S^{-1}] \cong R^3(\sigma)$$

with

$$\sigma(e_1, e_2) = q, \qquad \sigma(e_1, e_3) = q, \qquad \sigma(e_2, e_3) = q.$$

(14) $X = \{[14], [24], [34]\}$
In $G/\langle X\rangle$, let S be the right Ore set generated by $\widehat{[12]}, \widehat{[13]}, \widehat{[23]}$. Then we have

$$(G/\langle X\rangle)[S^{-1}] \cong R^3(\sigma)$$

where

$$\sigma(e_1, e_2) = q, \qquad \sigma(e_1, e_3) = q, \qquad \sigma(e_2, e_3) = q.$$

(15) $X = \{[12], [13], [23], [24]\}$
In $G/\langle X\rangle$, set S to be the multiplicative set generated by the elements $\widehat{[14]}, \widehat{[34]}$.

Then S is right Ore and we have that

$$(G/\langle X\rangle)[S^{-1}] \cong R^2(\sigma)$$

where

$$\sigma(e_1, e_2) = q.$$

(16) $X = \{[12], [13], [23], [34]\}$
In $G/\langle X\rangle$, let S be the right Ore set generated by $\widehat{[14]}, \widehat{[24]}$. We have

$$(G/\langle X\rangle)[S^{-1}] \cong R^2(\sigma)$$

where

$$\sigma(e_1, e_2) = q.$$

(17) $X = \{[12], [23], [24], [34]\}$
In the factor $G/\langle X\rangle$, set S to be the right Ore set generated by the two elements $\widehat{[13]}, \widehat{[14]}$. Then we have

$$(G/\langle X\rangle)[S^{-1}] \cong R^2(\sigma)$$

where

$$\sigma(e_1, e_2) = q.$$

(18) $X = \{[12], [13], [14], [23]\}$
In $G/\langle X\rangle$, let S denote the multiplicative set generated by $\widehat{[24]}$ and $\widehat{[34]}$. Then S is right Ore and

$$(G/\langle X\rangle)[S^{-1}] \cong R^2(\sigma)$$

where

$$\sigma(e_1, e_2) = q.$$

(19) $X = \{[13], [23], [24], [34]\}$
In $G/\langle X\rangle$, let S be the right Ore set generated by the elements $\widehat{[12]}, \widehat{[14]}$. We have

$$(G/\langle X\rangle)[S^{-1}] \cong R^2(\sigma)$$

where

$$\sigma(e_1, e_2) = q.$$

(20) $X = \{[12], [14], [23], [24]\}$
In $G/\langle X\rangle$ let S be the multiplicative set generated by the elements $\widehat{[13]}$ and $\widehat{[34]}$. Then S is right Ore and we have

$$(G/\langle X\rangle)[S^{-1}] \cong R^2(\sigma)$$

where

$$\sigma(e_1, e_2) = q.$$

(21) $X = \{[13], [14], [23], [34]\}$
In the factor $G/\langle X\rangle$, we take S to be the right Ore set generated by $\widehat{[12]}, \widehat{[24]}$.

We then have

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^2(\sigma)$$

with

$$\sigma(e_1, e_2) = q.$$

(22) $X = \{[12], [13], [14], [24]\}$
In $G\diagup\langle X\rangle$, let S be the right Ore set generated by the elements $\widehat{[23]}, \widehat{[34]}$. Then

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^2(\sigma)$$

where

$$\sigma(e_1, e_2) = q.$$

(23) $X = \{[14], [23], [24], [34]\}$
In $G\diagup\langle X\rangle$, set S to be the right Ore set generated by $\widehat{[12]}, \widehat{[13]}$. Then we have

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^2(\sigma)$$

where

$$\sigma(e_1, e_2) = q.$$

(24) $X = \{[12], [13], [14], [34]\}$
In the factor algebra $G\diagup\langle X\rangle$, this time we take S to be the right Ore set generated by $\widehat{[23]}, \widehat{[24]}$. We have

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^2(\sigma)$$

where

$$\sigma(e_1, e_2) = q.$$

(25) $X = \{[12], [14], [24], [34]\}$
In the factor $G\diagup\langle X\rangle$, let S be the right Ore set generated by $\widehat{[13]}, \widehat{[23]}$. Then

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^2(\sigma)$$

where

$$\sigma(e_1, e_2) = q.$$

(26) $X = \{[13], [14], [24], [34]\}$
In $G\diagup\langle X\rangle$, let S be the right Ore set generated by the elements $\widehat{[12]}, \widehat{[23]}$. We have

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^2(\sigma)$$

with

$$\sigma(e_1, e_2) = q.$$

(27) $X = \{[12], [13], [23], [24], [34]\}$
In the factor algebra $G\diagup\langle X\rangle$ let S be the right Ore set generated by $\widehat{[14]}$. Then we have

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^1(\sigma)$$

where

$$\sigma(e_1, e_1) = 1.$$

(28) $X = \{[12], [13], [23], [14], [24]\}$
In $G\diagup\langle X\rangle$, let S be the right Ore set generated by the element $\widehat{[34]}$. We have

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^1(\sigma)$$

where

$$\sigma(e_1, e_1) = 1.$$

(29) $X = \{[12], [13], [14], [23], [34]\}$
In $G\diagup\langle X\rangle$, take S to be the right Ore set generated by the element $\widehat{[24]}$. We have

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^1(\sigma)$$

with

$$\sigma(e_1, e_1) = 1.$$

(30) $X = \{[12], [14], [23], [24], [34]\}$
In the factor $G\diagup\langle X\rangle$, set S to be the right Ore set generated by $\widehat{[13]}$. Then

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^1(\sigma)$$

with

$$\sigma(e_1, e_1) = 1.$$

(31) $X = \{[13], [14], [23], [24], [34]\}$
In $G\diagup\langle X\rangle$, let S be the right Ore set generated by $\widehat{[12]}$. Then we have

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^1(\sigma)$$

where

$$\sigma(e_1, e_1) = 1.$$

(32) $X = \{[12], [13], [14], [24], [34]\}$
In $G\diagup\langle X\rangle$ let S be the right Ore set generated by the element $\widehat{[23]}$. We have

$$(G\diagup\langle X\rangle)[S^{-1}] \cong R^1(\sigma)$$

where

$$\sigma(e_1, e_1) = 1.$$

(33) $X = \phi$
Take S to be the multiplicative set of G generated by $[12], [13], [14], [23], [34]$. If we localize G at S, then as we have already shown, we may present $G[S^{-1}]$ as $R^5(\sigma)$.
(34) $X = \{[12], [13], [14], [23], [24], [34]\}$
This case is trivial since we have

$$G\diagup\langle X\rangle \cong k$$

For each set X which generates an H-invariant ideal of $G_q(2,4)$, let

$$\mathrm{spec}_X(G) = \{P \in \mathrm{spec}(G) : P \cap \wp = X\}$$

and similarly let

$$\mathrm{pspec}_X(G) = \{P \in \mathrm{pspec}(G) : P \cap \wp = X\}.$$

Theorem 6.7.3. *There is a bijection between the prime ideals of $G_q(2,4)$ and the prime Poisson ideals of $G(2,4)$. Precisely, we have that*

$$\mathrm{spec}(G_q(2,4)) = \bigcup_X \mathrm{spec}_X(G_q(2,4))$$

and

$$\mathrm{pspec}(G(2,4)) = \bigcup_X \mathrm{pspec}_X(G(2,4)).$$

Further, the map given in the previous Lemma and 6.2.6 is a bijection.

6.8 Symplectic Ideals in $G(2,4)$

In this section, we find all the symplectic ideals of the algebra $G(2,4)$. To find these ideals, we simply need to identify the symplectic ideals of $(G(2,4)/\langle X\rangle)[S^{-1}]$ for each H-invariant prime $\langle X\rangle$ of the deformed algebra $G_q(2,4)$. We recall that

$$(G(2,4)/\langle X\rangle)[S^{-1}] \cong k\mathbb{Z}^n$$

where the Poisson bracket is induced by an antisymmetric biadditive map u. All symplectic ideals of $k\mathbb{Z}^n$ are in bijective correspondence with maximal ideals of the Poisson centre, $Z_P(k\mathbb{Z}^n) = k\mathbb{Z}^n_u$, and so we are able to identify all the symplectic ideals of $G(2,4)$.

(1) $X = \langle[12]\rangle$
$\mathbb{Z}^4_u$ is generated by $e_1 - e_2$ and hence

$$\mathrm{symp}_{\{[12]\}}(\mathrm{G}) = \{\langle [12], [14] - \alpha[23] : \alpha \in \mathrm{k}^*\rangle\}.$$

(2) $X = \{[23]\}$
$\mathbb{Z}^4_u$ is trivial and so

$$\mathrm{symp}_{\{[23]\}}(\mathrm{G}) = \{\langle [23]\rangle\}.$$

(3) $X = \{[34]\}$
$\mathbb{Z}^4_u$ is generated by $e_2 - e_3$ and hence

$$\mathrm{symp}_{\{[34]\}}(\mathrm{G}) = \{\langle [34], [14] - \alpha[23] : \alpha \in \mathrm{k}^*\rangle\}.$$

(4) $X = \{[14]\}$
$\mathbb{Z}_u^4$ is once again trivial and so

$$\text{symp}_{\{[14]\}}(\text{G}) = \{\langle[14]\rangle\}.$$

(5) $X = \{[12], [13], [23]\}$
$\mathbb{Z}_u^3$ is trivial so

$$\text{symp}_{\{[12],[13],[23]\}}(\text{G}) = \{\langle[12], [13], [23]\rangle\}.$$

(6) $X = \{[12], [23], [24]\}$
$\mathbb{Z}_u^3$ is trivial so

$$\text{symp}_{\{[12],[23],[24]\}}(\text{G}) = \{\langle[12], [23], [24]\rangle\}.$$

(7) $X = \{[23], [24], [34]\}$
$\mathbb{Z}_u^3$ is trivial so

$$\text{symp}_{\{[23],[24],[34]\}}(\text{G}) = \{\langle[23], [24], [34]\rangle\}.$$

(8) $X = \{[13], [23], [34]\}$
$\mathbb{Z}_u^3$ is trivial so

$$\text{symp}_{\{[13],[23],[34]\}}(\text{G}) = \{\langle[13], [23], [34]\rangle\}.$$

(9) $X = \{[14], [23]\}$
$\mathbb{Z}_u^3$ is generated by $e_2 - e_3$ so

$$\text{symp}_{\{[14],[23]\}}(\text{G}) = \{\langle[14], [23], [13] - \alpha[24] : \alpha \in \text{k}^*\rangle\}.$$

(10) $X = \{[12], [34]\}$
$\mathbb{Z}_u^2$ is generated by $e_2 - e_3$. Hence

$$\text{symp}_{\{[12],[34]\}}(\text{G}) = \{\langle[12], [34], [14] - \alpha[23] : \alpha \in \text{k}^*\rangle\}.$$

(11) $X = \{[12], [14], [24]\}$
$\mathbb{Z}_u^3$ is trivial so

$$\text{symp}_{\{[12],[14],[24]\}}(\text{G}) = \{\langle[12], [14], [24]\rangle\}.$$

(12) $X = \{[12], [13], [14]\}$
We have that $\mathbb{Z}_u^3$ is once again trivial so

$$\text{symp}_{\{[12],[13],[14]\}}(\text{G}) = \{\langle[12], [13], [14]\rangle\}.$$

(13) $X = \{[13], [14], [34]\}$
$\mathbb{Z}_u^2$ is trivial so

$$\text{symp}_{\{[13],[14],[34]\}}(\text{G}) = \{\langle[13], [14], [34]\rangle\}.$$

(14) $X = \{[14], [24], [34]\}$
We have that $\mathbb{Z}_u^2 = \mathbb{Z}$ so

$$\text{symp}_{\{[14],[24],[34]\}}(\text{G}) = \{\langle[14], [13], [24], [34]\rangle\}.$$

(15) $X = \{[12], [13], [23], [24]\}$
$\mathbb{Z}_u^4$ is trivial so

$$\text{symp}_{\{[12],[13],[23],[24]\}}(\text{G}) = \{\langle[12], [13], [23], [24]\rangle\}.$$

(16) $X = \{[12], [13], [23], [34]\}$
$\mathbb{Z}_u^3$ is trivial so we have

$$\text{symp}_{\{[12],[13],[13],[34]\}}(\text{G}) = \{\langle[12], [13], [23], [34]\rangle\}$$

(17) $X = \{[12], [23], [24], [34]\}$
$\mathbb{Z}_u^3$ is trivial so

$$\text{symp}_{\{[12],[23],[24],[34]\}}(\text{G}) = \{\langle[12], [23], [24], [34]\rangle\}.$$

(18) $X = \{[12], [13], [14], [23]\}$
We have that $\mathbb{Z}_u^2$ is trivial and hence

$$\text{symp}_{\{[12],[13],[14],[23]\}}(\text{G}) = \{\langle[12], [13], [14], [34]\rangle\}.$$

(19) $X = \{[13], [23], [24], [34]\}$
$\mathbb{Z}_u^2$ is once again trivial so we have

$$\text{symp}_{\{[13],[23],[24],[34]\}}(\text{G}) = \{\langle[13], [23], [24], [34]\rangle\}.$$

(20) $X = \{[12], [14], [23], [24]\}$
$\mathbb{Z}_u^2$ is trivial so

$$\text{symp}_{\{[12],[14],[23],[24]\}}(\text{G}) = \{\langle[12], [14], [23], [24]\rangle\}.$$

(21) $X = \{[13], [14], [23], [34]\}$
$\mathbb{Z}_u^2$ is trivial and hence

$$\text{symp}_{\{[13],[14],[23],[34]\}}(\text{G}) = \{\langle[13], [14], [23], [34]\rangle\}.$$

(22) $X = \{[12], [13], [14], [24]\}$
$\mathbb{Z}_u^3$ is trivial so

$$\text{symp}_{\{[12],[13],[14],[24]\}}(\text{G}) = \{\langle[12], [13], [14], [24]\rangle\}.$$

(23) $X = \{[14], [23], [24], [34]\}$
We have that $\mathbb{Z}_u^2$ is trivial. Hence

$$\text{symp}_{\{[14],[23],[24],[34]\}}(\text{G}) = \{\langle[14], [23], [24], [34]\rangle\}.$$

(24) $X = \{[12], [13], [14], [34]\}$
$\mathbb{Z}_u^2$ is trivial and so

$$\text{symp}_{\{[12],[13],[14],[34]\}}(\text{G}) = \{\langle[12], [13], [14], [34]\rangle\}.$$

(25) $X = \{[12],[14],[24],[34]\}$
$\mathbb{Z}_u^2$ is trivial so

$$\mathrm{symp}_{\{[12],[14],[24],[34]\}}(\mathrm{G}) = \{\langle[12],[14],[24],[34]\rangle\}.$$

(26) $X = \{[13].[14],[24],[34]\}$
Here $\mathbb{Z}_u^2$ is trivial and so

$$\mathrm{symp}_{\{[13],[14],[24],[34]\}}(\mathrm{G}) = \{\langle[13],[14],[24],[34]\rangle\}.$$

(27) $X = \{12],[13],[23],[24],[34]\}$
$\mathbb{Z}_u^2 = \mathbb{Z}$ so

$$\mathrm{symp}_{\{[12],[13],[23],[24],[34]\}}(\mathrm{G}) = \{\langle[12],[13],[14]-\alpha,[23],[24],[34] : \alpha \in \mathrm{k}^*\rangle\}.$$

(28) $X = \{[12],[13],[14],[23],[24]\}$
$\mathbb{Z}_u^2 = \mathbb{Z}$ so

$$\mathrm{symp}_{\{[12],[13],[14],[23],[24]\}}(\mathrm{G}) = \{\langle[12],[13],[14],[23],[24],[34]-\alpha : \alpha \in \mathrm{k}^*\rangle\}.$$

(29) $X = \{[12],[13],[14],[23],[34]\}$
$\mathbb{Z}_u^2 = \mathbb{Z}$ so

$$\mathrm{symp}_{\{[12],[13],[14],[23],[34]\}}(\mathrm{G}) = \{\langle[12],[13],[14],[23],[24]-\alpha,[34] : \alpha \in \mathrm{k}^*\rangle\}.$$

(30) $X = \{[12],[14],[23],[24],[34]\}$
$\mathbb{Z}_u^2 = \mathbb{Z}$ and so

$$\mathrm{symp}_{\{[12],[14],[23],[24],[34]\}}(\mathrm{G}) = \{\langle[12],[13]-\alpha,[14],[23],[24],[34] : \alpha \in \mathrm{k}^*\rangle\}.$$

(31) $X = \{[13],[14],[23],[24],[34]\}$
We have $\mathbb{Z}_u^2 = \mathbb{Z}$ and hence

$$\mathrm{symp}_{\{[13],[14],[23],[24],[34]\}}(\mathrm{G}) = \{\langle[12]-\alpha,[13],[14],[23],[24],[34] : \alpha \in \mathrm{k}^*\rangle\}.$$

(32) $X = \{[12],[13],[14],[24],[34]\}$
$\mathbb{Z}_u^2 = \mathbb{Z}$ and so

$$\mathrm{symp}_{\{[12],[13],[14],[24],[34]\}}(\mathrm{G}) = \{\langle[12],[13],[14],[23]-\alpha,[24],[34] : \alpha \in \mathrm{k}^*\rangle\}.$$

(33) $X = \phi$
$\mathbb{Z}_u^5$ is generated by $e_3 - e_4$ and hence

$$\mathrm{symp}_{\phi}(\mathrm{G}) = \{\langle[14]-\alpha[23] : \alpha \in \mathrm{k}^*\rangle\}$$

(34) $X = \{[12],[13],[14],[23],[24],[34]\}$
$\mathbb{Z}_u^5$ is trivial and so

$$\mathrm{symp}_{\{[12],[13],[14],[23],[24],[34]\}}(\mathrm{G}) = \{\langle[12],[13],[14],[23],[24],[34]\rangle\}.$$

Theorem 6.8.1. *The symplectic ideals of $G(2,4)$ of finite codimension consist of the ideal*

$$\langle [12], [13], [14], [23], [24], [34] \rangle$$

together with the ideals

$$\begin{aligned}
&\langle [12] - \alpha, [13], [14], [23], [24], [34] : \alpha \in k^* \rangle \,, \\
&\langle [12], [13] - \alpha, [14], [23], [24], [34] : \alpha \in k^* \rangle \,, \\
&\langle [12], [13], [14] - \alpha, [23], [24], [34] : \alpha \in k^* \rangle \,, \\
&\langle [12], [13], [14], [23] - \alpha, [24], [34] : \alpha \in k^* \rangle \,, \\
&\langle [12], [13], [14], [23], [24] - \alpha, [34] : \alpha \in k^* \rangle \,, \\
&\langle [12], [13], [14], [23], [24], [34] - \alpha : \alpha \in k^* \rangle \,.
\end{aligned}$$

Chapter 7

The Prime Spectrum of the 2×2 Reflection Equation Algebra

7.1 The k-algebra $A_q(2)$

In this section we define the algebra which we will study in detail.

Definition 7.1.1. Let k be an algebraically closed field and let $0 \neq q \in k$ such that q is not a root of unity. Majid's process of forming what is called the covariantised product produces *Reflexion Equation Algebras* from quantum matrices. For details of this construction, see [24] p.368 Ex.19. For 2×2 quantum matrices, the corresponding REA can be defined as the k-algebra $A_q(2)$ generated by four elements a, b, c, d subject to the six relations:

$$ad = da, \;\; bd = q^{-2}db, \;\; cd = q^2dc$$

$$bc - cb = (q^{-2} - 1)d(a - d)$$
$$ab - ba = (q^{-2} - 1)bd$$
$$ac - ca = (1 - q^{-2})dc$$

The algebra $A_q(2)$ has many possible expressions as an Ore extension. We present and prove one of them here.

Proposition 7.1.2. *The algebra* $A_q(2)$ *can be expressed as*

$$A_q(2) = k[a, d][c; \sigma_1][b; \sigma_2, \delta_2].$$

Sketch. Clearly, a and d commute so we can begin our construction with an ordinary polynomial ring in these two indeterminates. Now, we aim to add b to our extension. As a preliminary calculation, we have

$$ca = ac - (1 - q^{-2})dc = (a - (1 - q^{-2})d)c$$

and so we want our automorphsim

$$\sigma_1 : k[a,d] \longrightarrow k[a,d]$$

to be such that

$$\sigma_1(a) = a - (1 - q^{-2})d, \quad \sigma_1(d) = q^2 d.$$

For the next stage of our extension, we see that

$$ba = ab - (q^{-2} - 1)bd = ab - (q^{-2} - 1)q^{-2}db = (a - (q^{-4} - q^{-2})d)b$$

and

$$bc = cb + (q^{-2} - 1)d(a - d).$$

Hence we want

$$\sigma_2 : k[a,d][c;\sigma_1] \longrightarrow k[a,d][c;\sigma_1]$$

to be such that

$$\sigma_2(a) = a - (q^{-4} - q^{-2})d, \quad \sigma_2(d) = q^{-2}d, \quad \sigma_2(c) = c.$$

For this final stage of our extension, a σ_2-derivation

$$\delta_2 : k[a,d][c;\sigma_1] \longrightarrow k[a,d][c;\sigma_1]$$

is also needed. We take

$$\delta_2(a) = \delta_2(d) = 0$$

while

$$\delta_2(c) = (q^{-2} - 1)d(a - d).$$

□

Corollary 7.1.3. *$A_q(2)$ is a noetherian domain.*

Proof. Noting that the base ring is the field k and invoking induction on Theorem 1.2.3 gives the result. □

7.2 Primes Of $A_q(2)$ Containing d

We first notice that the element $d \in A_q(2)$ is normal, meaning that $dA_q(2) = A_q(2)d$. This is observed from the defining relations on the generators.

Our first aim is to classify those primes of $A_q(2)$ which contain d. These primes arise as the primes of the factor algebra $A_q(2) / \langle d \rangle$. Examining the relations closely allows one to see that

$$A_q(2) / \langle d \rangle \cong k[a,b,c]$$

Hence the primes of $A_q(2)$ which contain d are identifiable as the primes of the commutative polynomial algebra $k[a, b, c]$. For example, as a result of Hilbert's Nullstellensatz, we have

Proposition 7.2.1. *As k is algebraically closed, the maximal ideals of $A_q(2)$ containing d are the ideals*

$$\langle a - \alpha, b - \beta, c - \gamma, d \rangle$$

where $\alpha, \beta, \gamma \in k$.

Note now that since d is normal, we are able to apply Lemma 1.1.7 and form the localization of $A_q(2)$ with respect to the multiplicative set consisting of the non-negative powers of d. The task of describing the prime ideals of $A_q(2)$ which do not contain d proves to be considerably more difficult. Firstly note that these primes correspond to the primes of the localization $A_q(2)[d^{-1}]$. This correspondence is given as follows

Proposition 7.2.2. *Let R be a ring and let $X \subseteq R$ be a right Ore set of regular elements. Set $S = RX^{-1}$. The maps*

$$\phi : \mathrm{Spec}(R \backslash X) \longrightarrow \mathrm{Spec}(S)$$

given by

$$P \mapsto PS$$

and

$$\eta : \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R \backslash X)$$

given by

$$Q \mapsto Q \cap R$$

are bijections between the set of primes of S and the set of primes of R disjoint from X.

Proof. See [11], chapter 10. □

Hence, when attempting to gain insight into these primes, we would think that it should make sense to begin analysing the latter localization.

7.3 H-Action on $A_q(2)$

Taking the standard 2×2 quantum matrix algebra as our motivation, we identify an action of a k-torus on $A_q(2)$ by automorphisms. Another look at the defining relations in $A_q(2)$ leads us to the conclusion that any action by a k-torus H must have the same H-eigenvalue for the generator a as for the generator d as any such

action must be a morphism of algebras.
Hence, we assume there should be a 3-torus action on $A_q(2)$ of the form

$$(\alpha, \beta, \gamma).a = \alpha a, \;\; (\alpha, \beta, \gamma).b = \beta b,$$

$$(\alpha, \beta, \gamma).c = \gamma c, \;\; (\alpha, \beta, \gamma).d = \alpha d.$$

However, we in fact have

Proposition 7.3.1. *There is an $H = (k^*)^2$ action on $A_q(2)$ given by*

$$(\alpha, \beta).a = \alpha a, \;\; (\alpha, \beta).b = \beta b,$$

$$(\alpha, \beta).c = \beta^{-1}\alpha^2 c, \;\; (\alpha, \beta).d = \alpha d.$$

Proof. The key stage where the relationship between γ and α and β is revealed is with the relation

$$bc - cb = (q^{-2} - 1)d(a - d).$$

In order for the action of H to define an algebra morphism, we need

$$\beta\gamma(bc - cb) = (q^{-2} - 1)\alpha^2(d(a - d)).$$

Hence since $A_q(2)$ is a domain, we have

$$\beta\gamma = \alpha^2.$$

Hence

$$\gamma = \beta^{-1}\alpha^2.$$

It is a routine check that, with this problem out of the way, an action is defined. □

Since d is an H eigenvector under this action,, we have an induced H-action on the localization $A_q(2)[d^{-1}]$.

Proposition 7.3.2. *The action of H on $A_q(2)$ gives the induced action on $A_q(2)[d^{-1}]$ given by the following :*

$$(\alpha, \beta).\bar{a} = \bar{a}, \;\; (\alpha, \beta).\bar{b} = \beta\alpha^{-1}\bar{b},$$

$$(\alpha, \beta).\bar{c} = \beta^{-1}\alpha\bar{c}, \;\; (\alpha, \beta).d = \alpha d,$$

where $\bar{a} = ad^{-1}, \;\; \bar{b} = bd^{-1}, \;\; \bar{c} = cd^{-1}$.

We would like to establish finiteness on either H-spec$A_q(2)$ or H-spec$A_q(2)[d^{-1}]$ applying methods of Goodearl detailed in [2], Chapter 2.5. This method works beautifully in the $\mathcal{O}_q(M_2(k))$ case. However, one of the conditions required in the Goodearl situation is that no H-eigenvalue is a root of unity. This condition clearly fails for the localization $A_q(2)[d^{-1}]$. By easy calculation, the setup also fails for $A_q(2)$.

One might also hope that the localization $A_q(2)[d^{-1}]$ expressed as an Ore extension would be a q-skew extension, as is the case for $\mathcal{O}_q(M_{2,2})$. If this were the case, then Goodearl has results which, upon splitting the primes in the Ore extension into three different cases, provides a detailed description of all the primes in the Ore extension. This would provide us with a description of those primes P of $A_q(2)$ such that $d \notin P$. Sadly, this is not the case so another approach is required.

7.4 Primes Of $A_q(2)$ Not Containing b or d

Proposition 7.4.1. *Set*

$$t := qa + q^{-1}d, \quad \Delta := q^{-2}ad - bc.$$

Then t and Δ belong to $Z(A_q(2))$, the centre of $A_q(2)$.

Proof. We have

$$at = qa^2 + q^{-1}ad = qa^2 + q^{-1}da = ta,$$

$$bt = qba + q^{-1}bd = q(ab - (q^{-2} - 1)bd) + q^{-1}q^{-2}db$$

$$= qab - (q^{-3} - q^{-1})db + q^{-3}db = qab + q^{-1}db = tb,$$

$$ct = qca + q^{-1}cd = q(ac - (1 - q^{-2})dc) + q^{-1}q^2dc = qac - (q - q^{-1})dc + qdc = qac + q^{-1}dc = tc,$$

$$dt = qda + q^{-1}d^2 = qad + q^{-1}d^2 = td.$$

Hence $t \in Z(A_q(2))$. Similar calculations give $\Delta \in Z(A_q(2))$. □

Now we aim to show that the set

$$\{b^m d^n : m, n \geq 0\}$$

is a right Ore set in $A_q(2)$.
Note that if $S := k^* b^m$ and $T := k^* d^n$, then $ST = TS$.

Notice first that $A_q(2)$ is graded in the same sense that $\mathcal{O}_q(M_{2,2})$ is. Hence

$$A_q(2) = \bigoplus_{i \geq 0} A_i$$

where we identify each component A_i to be the subspace of $A_q(2)$ spanned by all monomials $a^j b^k c^l d^m$ where $j + k + l + m = i$.
We have the additional property that

$$A_n = (A_1)^n$$

for all $n \geq 0$.

Next, note that

$$A_1 = C + X$$

where $C = ck$ and $X = ak + bk + dk$.
Upon examining the relations once again, it follows that

$$Cb^2 \subseteq bCb + bX^2.$$

We also have $Xb = bX$ Hence

$$Xb^2 \subseteq bCb + b^2X \subseteq bCb + bX^2.$$

Thus

$$A_1b^2 \subseteq bCb + bX^2 \subseteq bCX + bX^2 \subseteq b(C + X)X = bA_1X.$$

Now we have

$$(A_1X)b^2 \subseteq bA_1XX \subseteq bA_1X^2$$

and also, since $Xb = bX$ we have

$$(A_1X^t)b^2 \subseteq bA_1X^{t+1}$$

for all $t \geq 0$.

Proposition 7.4.2. *For all $n \in \mathbb{N}$,*

$$A_1X^nb^{2n} \subseteq b^nA_1X^{2n}.$$

Proof. The $n = 1$ case is true by the above discussion. Now assume that, for $k \geq 1$, we have that

$$A_1X^kb^{2k} \subseteq b^kA_1X^{2k}.$$

Then

$$\begin{aligned} A_1X^{k+1}b^{2k+2} &= A_1X^kXb^{2k+2} \\ &= A_1X^kb^{2k}Xb^2 \\ &\subseteq b^kA_1X^{2k}Xb^2 \\ &\subseteq b^k(A_1b^2)X^{2k+1} \\ &\subseteq b^k(bA_1X)X^{2k+1} \\ &= b^{k+1}A_1X^{2(k+1)}. \end{aligned}$$

Hence the result. □

Lemma 7.4.3. *For all $n \in \mathbb{N}$, we have*

$$A_nb^{2^n} \subseteq b\prod_{i=0}^{n-1}(A_1X^{2^i}).$$

Proof. The $n = 1$ case says that $A_1b^2 \subseteq bA_1X$ which was noted earlier. Now assume that $k \geq 1$ and that

$$A_k b^{2^k} \subseteq b \prod_{i=0}^{k-1} (A_1 X^{2^i}).$$

Then

$$\begin{aligned}
A_{k+1} b^{2^{k+1}} &= A_1 (A_k b^{2^k}) b^{2^k} \\
&\subseteq A_1 b \prod_{i=0}^{k-1} (A_1 X^{2^i}) b^{2^k} \\
&= A_1 b \prod_{i=0}^{k-2} (A_1 X^{2^i})(A_1 X^{2^{k-1}}) b^{2^k} \\
&\subseteq A_1 b \prod_{i=0}^{k-2} (A_1 X^{2^i}) b^{2^{k-1}} (A_1 X^{2^k}) \\
&= A_1 b \prod_{i=0}^{k-3} (A_1 X^{2^i})(A_1 X^{2^{k-2}}) b^{2^{k-1}} (A_1 X^{2^k}) \\
&\subseteq A_1 b \prod_{i=0}^{k-3} (A_1 X^{2^i}) b^{2^{k-2}} (A_1 X^{2^{k-1}})(A_1 X^{2^k}) \\
&\subseteq ... \subseteq A_1 b (A_1 X) b^2 (A_1 X^4) ... (A_1 X^{2^{k-1}})(A_1 X^{2^k}) \\
&\subseteq A_1 b^2 (A_1 X^2)(A_1 X^4) ... (A_1 X^{2^{k-1}})(A_1 X^{2^k}) \\
&\subseteq b (A_1 X)(A_1 X^2)(A_1 X^4) ... (A_1 X^{2^{k-1}})(A_1 X^{2^k}) \\
&= b \prod_{i=0}^{k} (A_1 X^{2^i}).
\end{aligned}$$

□

Armed with this, we can show that

Corollary 7.4.4. *The set*

$$\{b^n : n \geq 0\}$$

satisfies the right Ore condition in $A_q(2)$.

Lemma 7.4.5. *The set*

$$X := \{b^m d^n : m, n \geq 0\}$$

is a right Ore set in $A_q(2)$.

Proof. This follows from 1.1.8 on taking $S = k^*\{b^m : m \geq 0\}$ and $T = k^*\{d^n : n \geq 0\}$ and by noting that we do have $ST = TS$ in this case from the relation $bd = q^{-2} db$. □

It is similarly checked that $\{x_{11}^m x_{21}^n : m, n \geq 0\}$ is a right Ore set in $\mathcal{O}_q(M_{2,2})$, using the local nilpotence of the derivation involved in the expression of $\mathcal{O}_q(M_{2,2})$ as an iterated Ore extension, and the fact that this Ore expression is of q-skew type.

Now set

$$u := qad^{-1} + q^{-1}, \quad v := d^{-1}$$

working in the localization $A_q(2)[b^{-1}, d^{-1}]$.

Note that $uv^{-1} = t \in Z(A_q(2))$.

Proposition 7.4.6. *We have the commuation relations*

$$uv = vu, \;\; bv = q^2 vb, \;\; cv = q^{-2} vc,$$

$$ub = q^{-2} bu, \;\; uc = q^2 cu.$$

Proof. We have

$$uv = qad^{-2} + q^{-1}d^{-1} = qd^{-1}ad^{-1} + qd^{-1} = vu,$$

$$bv = bd^{-1} = q^2 d^{-1} b = q^2 vb,$$

$$cv = cd^{-1} = q^{-2} d^{-1} c = q^{-2} vc,$$

$$ub = qad^{-1}b + q^{-1}b = qq^{-2}abd^{-1} + q^{-1}b = q^{-1}(ba + (q^{-2} - 1)bd)d^{-1} + q^{-1}b$$

$$= q^{-1}bad^{-1} + q^{-1}(q^{-2}-1)b + q^{-1}b = q^{-1}bad^{-1} + q^{-3}b = q^{-2}b(qad^{-1} + q^{-1}) = q^{-2}bu,$$

$$uc = qad^{-1}c + q^{-1}c = q^3 acd^{-1} + q^{-1}c = q^3(ca + (1 - q^{-2})dc)d^{-1} + q^{-1}c$$

$$= q^3 cad^{-1} + (q^3 - q)dcd^{-1} + q^{-1}c = q^3 cad^{-1} + (q - q^{-1})c + q^{-1}c = q^3 cad^{-1} + qc$$

$$= q^2 c(qad^{-1} + q^{-1}) = q^2 cu.$$

□

We now aim to analyse the localization $A_q(2)[b^{-1}, d^{-1}]$ in order to find those primes of $A_q(2)$ which do not contain b or d. We establish a connection between this localization and a localization of the better understood algebra $\mathcal{O}_{q^2}(M_{2,2})$.

For the next Lemma, we need to slightly adjust the definition of $\mathcal{O}_q(M_{2,2})$ which we have given by defining

$$\bar{x_{12}} := -q^4 x_{12}$$

and then setting

$$\mathcal{O}_q(M_{2,2}) = k[x_{11}, \bar{x_{12}}, x_{21}, x_{22}]$$

subject to the relations given earlier.

Lemma 7.4.7. *The map*

$$\theta : \mathcal{O}_{q^2}(M_{2,2})[x_{11}^{-1}, x_{21}^{-1}] \longrightarrow A_q(2)[b^{-1}, d^{-1}]$$

defined by

$$\theta(x_{11}) = b,\ \ \theta(\bar{x}_{12}) = -q^4 u = -q^4(qad^{-1} + q^{-1}),$$
$$\theta(x_{21}) = v = d^{-1},\ \ \theta(x_{22}) = \Delta b^{-1} - q^2 uvb^{-1},$$

is an isomorphism of algebras.

Proof. We begin by checking that θ is an algebra morphism. We have

$$\theta(x_{11})\theta(\bar{x}_{12}) = -q^4 bu = -q^4 q^2 ub = q^2\theta(\bar{x}_{12})\theta(x_{11}),$$

$$\theta(x_{11})\theta(x_{21}) = bv = q^2 vb = q^2\theta(x_{21})\theta(x_{11}),$$
$$\theta(\bar{x}_{12})\theta(x_{21}) = -q^4 uv = -q^4 vu = \theta(x_{21})\theta(\bar{x}_{12}),$$
$$\theta(\bar{x}_{12})\theta(x_{22}) = -q^4 u\Delta b^{-1} + q^4 q^2 u^2 vb^{-1} = -q^4\Delta ub^{-1} + q^4 q^2 uvub^{-1}$$
$$= -q^4 q^2 \Delta b^{-1} u + q^4 q^4 uvb^{-1} u = q^2\theta(x_{22})\theta(\bar{x}_{12}),$$
$$\theta(x_{21})\theta(x_{22}) = v\Delta b^{-1} - q^2 vuvb^{-1} = \Delta vb^{-1} - q^2 uv^2 b^{-1} = q^2\Delta b^{-1} v - q^4 uvb^{-1} v = q^2\theta(x_{22})\theta(x_{21}),$$
$$\theta(x_{11})\theta(x_{22}) - \theta(x_{22})\theta(x_{11}) = b\Delta b^{-1} - q^2 buvb^{-1} - \Delta + q^2 uv = \Delta - q^4 ubvb^{-1} - \Delta + q^2 uv$$
$$= q^2 uv - q^6 uv = -(q^6 - q^2)uv = (q^2 - q^{-2})\theta(\bar{x}_{12})\theta(x_{21}).$$

Secondly, we note that θ is an epimorphism since we have

$$a = -q^{-5}\theta(\bar{x}_{12})\theta(x_{21}^{-1}) - q^{-2}\theta(x_{21}^{-1}),$$

$$b = \theta(x_{11}),\quad d = \theta(x_{21}^{-1}).$$

The consideration of c requires a bit more effort. Note that

$$\Delta = q^{-2}ad - bc.$$

Hence in $A_q(2)[b^{-1}, d^{-1}]$,

$$c = b^{-1}(q^{-2}ad - \Delta).$$

Now a little exploration of the identity

$$\theta(x_{22}) = \Delta b^{-1} - q^2 uvb^{-1}$$

together with the facts that

$$v = \theta(x_{21}),\quad u = \theta(-q^{-4}\bar{x}_{12})$$

yields

$$\Delta = \theta(x_{22})\theta(x_{11}) - q^{-2}\theta(\bar{x}_{12})\theta(x_{21}).$$

Thus

$$c = \theta(x_{11}^{-1})(-q^{-7}\theta(\bar{x}_{12})\theta(x_{21}^{-2}) - q^{-4}\theta(x_{21}^{-2}) - \theta(x_{22})\theta(x_{11}) + q^{-2}\theta(\bar{x}_{12})\theta(x_{21})) \in \mathrm{im}(\theta).$$

Hence θ is an epimorphism as claimed.
For the final stage of the proof, we use Gelfand-Kirillov dimension to prove that

θ is a monomorphism.
Assume that $\ker(\theta) \neq 0$. Then

$$\begin{aligned}\text{GKdim}(A_q(2)[b^{-1}, d^{-1}]) &= \text{GKdim}(\mathcal{O}_{q^2}(M_{2,2})[x_{11}^{-1}, x_{21}^{-1}] \diagup \ker(\theta)) \\ &\leq \text{GKdim}(\mathcal{O}_{q^2}(M_{2,2})[x_{11}^{-1}, x_{21}^{-1}]) - 1 \\ &= \text{GKdim}(\mathcal{O}_{q^2}(M_{2,2}) - 1 = 3\end{aligned}$$

which is a contradiction since $\text{GKdim}(A_q(2)[b^{-1}, d^{-1}]) = \text{GKdim}(A_q(2)) = 4$.
Hence $\ker(\theta) = 0$ and the result follows. □

Hence we can use results about the prime spectrum of quantum 2x2 matrices to pull back information about the primes in $A_q(2)$. More precisely, by analysing the primes of $\mathcal{O}_{q^2}(M_{2,2})$ which do not contain x_{11} or x_{21}, we can identify those primes of A which do not contain b or d.

It is a well known result that there are only 14 H-primes in $\mathcal{O}_{q^2}(M_{2,2})$. We are only interested in those H-primes which do not contain x_{11} or x_{21} and there are only four of these. Hence the H-primes of interest are :

$$\langle 0 \rangle, \ \langle x_{12} \rangle, \ \langle x_{12}, x_{22} \rangle, \ \langle D_{q^2} \rangle.$$

We find all the primes of $\mathcal{O}_{q^2}(M_{2,2})$ not containing x_{11} or x_{21} by evaluating the corresponding H-strata to each of these H-primes.

Proposition 7.4.8. *Set $R = \mathcal{O}_{q^2}(M_{2,2})$. Then the primes of R which do not contain x_{11} and do not contain x_{21} are*

$$\text{Spec}_{\langle 0 \rangle}(R) = \{\langle 0 \rangle, \langle x_{12} - \alpha x_{21}, \ D_{q^2} - \beta \rangle, \ \langle f \rangle \cap R\},$$

$$\text{Spec}_{\langle x_{12} \rangle}(R) = \{\langle x_{12} \rangle, \ \langle x_{11}x_{22} - \alpha \rangle\},$$

$$\text{Spec}_{\langle x_{12}, x_{22} \rangle}(R) = \{\langle x_{12}, x_{22} \rangle\},$$

$$\text{Spec}_{\langle D_{q^2} \rangle}(R) = \{\langle D_{q^2} \rangle, \ \langle D_{q^2}, x_{12} - \alpha x_{21} \rangle\},$$

where $0 \neq \alpha, \beta \in k$ and $f \in k[(x_{12}x_{21})^{\pm 1}, \ (D_{q^2})^{\pm 1}]$ is irreducible.

Proof. See [13], Chapter 4. □

Hence, the primes of $A_q(2)$ which do not contain b and which do not contain d can be obtained by extending the primes listed above to the localzation $\mathcal{O}_{q^2}(M_{2,2})[x_{11}^{-1}, x_{21}^{-1}]$, applying θ and finally contracting the primes obtained in the localization $A_q(2)[b^{-1}, d^{-1}]$ by this process back to primes in the k-algebra $A_q(2)$.

7.5 Primes Of $A_q(2)$ Containing b but not d

So far, we have dealt with those primes P of $A_q(2)$ such that $d \in P$ and those primes Q of $A_q(2)$ such that $d \notin Q$, $b \notin Q$. Hence we have dealt with all possible occurrences of primes of $A_q(2)$ except the primes of $A_q(2)$ which contain b but not d.

Proposition 7.5.1. *Let P be a prime ideal of $A_q(2)$ such that $b \in P$ but $d \notin P$. Then a is congruent to d modulo P.*

Proof. We have

$$(q^{-2} - 1)d(a - d) = bc - cb \in P.$$

Hence $d(a - d) \in P$.
Thus

$$A_q(2)d(a - d) \subseteq P$$

i.e.

$$dA_q(2)(a - d) \subseteq P$$

since d is normal.
But $d \notin P$ and P is a prime ideal. Hence we must have

$$a - d \in P.$$

Hence the result. □

Now let $\{P_\lambda\}$ be the set of all those primes P_λ of $A_q(2)$ such that $b \in P_\lambda$ but $d \notin P_\lambda$.
Set

$$I := \bigcap P_\lambda, \quad S := A_q(2)/I.$$

Then the primes of $A_q(2)$ containing b but not d correspond to primes of S that do not contain $\bar{d}$, where $\bar{d}$ denotes the image of d in S.
Now note that in evaluating the primes in S that we need only consider the relations which involve d and c. Here we mean their images in the factor ring but we use the same notation for simplicity. Since the ideal $A_q(2)bA_q(2) \subseteq I$ and by the proposition, d is congruent to a in S, we are left with the relation

$$ac - ca = (1 - q^{-2})ac$$

i.e.

$$ca = q^{-2}ac.$$

Hence we can identify S with the quantum plane $\mathcal{O}_{q^{-2}}(k^2)$ generated by a and c.

Proposition 7.5.2.

$$\mathrm{Spec}(\mathcal{O}_{q^{-2}}(k^2)) = \{\langle 0\rangle,\ \langle a\rangle,\ \langle c\rangle,\ \langle a, c\rangle,\ \langle a - \alpha, c\rangle,\ \langle a, c - \gamma\rangle\}$$

where $\alpha, \gamma \in k$.

Proof. See [2], Part II.1 for details. □

Proposition 7.5.3. *The primes of $A_q(2)$ containing b but not d are*

$$\langle b \rangle, \;\; \langle b, c \rangle,$$

$$\langle a - \alpha, \; b, c \rangle$$

for $\alpha \in k$.

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