CURVE-FITTING

ACCORDING TO THE METHOD OF A. EKKE

AND R. SCHMIDT.

by

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When fitting a curve to a given set of data, it is usual, having chosen the curve-type which will best suit the data, to take as abscissae for the plotted points of the curve the observations themselves, and then 'smooth' the frequencies or weights of these observations to the correct ordinates, as if considering these frequencies to be subject to 'error' but the points at which the observations were made to be exact.

In the present method, however, we intend to make use of the inverse process. That is, we work as if considering the frequencies to be exact, but the observations to be in error. Thus we make no alteration in the frequencies as they are presented to us, but combine two processes for 'smoothing' the observations as follows,

(i) We take, as our abscissae, not the given observations, but a certain set of points calculated from the frequencies and the chosen curve-type, which have been called 'Ekke's Best Values', after some work done by A. Ekke in connection with them in a Kiel dissertation, 1934.

(ii) We now further smooth these abscissae to as close an approximation to the given observations as is required, by combining the given observations in a set of polynomials with coefficients to be determined, (a process suggested by R. Schmidt in an article appearing in the Annals of Mathematical Statistics, Vol.v, P.30).
Consider a set of observations arranged in the $n$ class-intervals, $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$, not necessarily equally spaced.

Let each of these intervals be equally weighted, i.e., let each contain $1/n$ of the total observations. Erect rectangles on these intervals as bases, each being of area $1/n$. Then the resulting histogram is a rough representation of the distribution of the data.

Now consider a curve drawn through the tops of the rectangles. We want this curve to represent as closely as possible the distribution of the data, but we do not want to alter the equal weighting of the class intervals.

Our problem, therefore, is to adjust the positions of the $x_0, x_1, x_2, \ldots, x_n$, so that, being still equally weighted, the difference of the areas under the curve and under the histogram may be a minimum.

To avoid difficulties arising out of positive and negative differences of area, it will be convenient
to take the square of the difference.
The square of the difference between the areas under curve and histogram to the left of an ordinate erected at any point, \( x \), is given by,

\[
\left[ \int_{-\infty}^{x} \phi(t) dt - S(x, x_0, x_1, \ldots, x_n) \right]^2
\]

where, \( y = \phi(x) \), is the equation of the curve, and \( S(x, x_0, x_1, \ldots, x_n) \), represents the step integral from \( x_0 \) to \( x \).

We wish to take this squared difference over the whole range of the curve and histogram, hence we require to make,

\[
\Delta = \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{x} \phi(t) dt - S(x, x_0, x_1, \ldots, x_n) \right]^2 dx
\]

a minimum, by adjusting the positions of the \( x_r \).

Now, \( \Delta = \int_{-\infty}^{+\infty} \left\{ F(x) - \left( \frac{r}{n} + \frac{x - x_r}{n} \right) \right\}^2 dx \) \( (r = 0, 1, \ldots, n) \)

where \( F(x) \) is the cumulative integral,

\[
F(x) = \int_{-\infty}^{x} \phi(t) dt
\]

and \( x \) is situated in the class-interval, \( (x_r, x_{r+1}) \).

Differentiating this at the point \( x = x_r \),

we have,

\[
\frac{\partial \Delta}{\partial x_r} = \left[ F(x) - \left( \frac{r}{n} + 0 \right) \right]^2
\]
Now put $\Delta/\Delta x$ equal to zero, to obtain minimum conditions, and we have,

$$F(x_r) = \frac{r}{n}.$$ 

So that the ordinates at the points $x_r$ must divide the area under the curve into equal portions, each of area $1/n$.

i.e. the points $x_r$ are to be the 'n-iles' of the distribution.

2.1

We now extend this to the case of unequally weighted data. Let the weights in successive intervals be, $p_1, p_2, \ldots, p_n$, and suppose the rectangles now to be of areas $p_1, p_2, \ldots, p_n$, respectively.

Our integral now becomes,

$$\Delta = \int \left\{ F(x) - \left[ p_1 + p_2 + \ldots + p_r + \frac{x - x_r}{x_r - x} \right] \right\} dx.$$ 

So that,

$$\frac{\partial \Delta}{\partial x_r} = \left[ F(x_r) - (p_1 + p_2 + \ldots + p_r + 0) \right],$$ 

and, putting this equal to zero, we have,

$$F(x_r) = p_1 + p_2 + \ldots + p_r.$$ 

Now let $\psi$ be a function such that,

$$\psi(Y) = x$$

where,

$$Y = F(x).$$

i.e. $\psi$ is the inverse function of $F$, for,

$$\psi\{F(x)\} = x.$$
Then, when $\Delta$ is a minimum, our 'best' abscissae are given by,

$$x_r = \psi[p_1 + p_2 + \ldots + p_r].$$

These $x_r$ are, however, situated at the ends of their respective class-intervals, so that, to centralise our abscissae, let us take the set of points $\xi_1, \xi_2, \ldots, \xi_n$, such that the ordinate at $\xi_r$ divides the area above the class-interval $(x_r, x_{r+1})$ into two equal parts, so that,

$$\xi_r = \psi[p_1 + p_2 + \ldots + p_{m-1} + \frac{1}{2} p_r].$$

Then the set of abscissae $\xi_1, \xi_2, \ldots, \xi_n$ are Ekke's Best Values for the representation of the data.
Suppose that we are given a set of $n$ observations at points $x = \{x_1, x_2, ..., x_n\}$, having weights $p_1, p_2, ..., p_n$, where $\sum p_i = 1$. We wish to obtain a set of ideal abscissae $\hat{x} = \{\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\}$, corresponding to ordinates $p_1, p_2, ..., p_n$, of points on a curve of a given type, so that this curve may represent as closely as possible the distribution of the data.

First of all, following Ekke's suggestion, we will obtain from the given weights, as in 2.1, Ekke's best values $\xi_1, \xi_2, ..., \xi_n$, for the curve-type which we chose as most suitable to the given data.

Now, following Schmidt's suggestion, let us put

$$\hat{x}_i = a_1 + a_2 \xi_i + a_3 \xi_i^2 + ... + a_k \xi_i^k,$$

a set of $n$ polynomials of degree $k$ in the $n$ Ekke values, having arbitrary coefficients $a_1, a_2, ..., a_k$, the same for each polynomial.

The vector $\hat{x}$ is thus represented by the sum of a set of vectors.

The well known advantages of orthogonality and normality in such methods of polynomial representation lead us to normalise and orthogonalise the vectors $1, \xi_1, \xi_2, ..., \xi_n$, obtaining a set of vectors $z_0, z_1, ..., z_n$. 
We can then write,
\[ \hat{x} = a_1 z_1 + a_2 z_2 + \ldots + a_k z_k \]  
(1)
where the coefficients, \( a_1, a_2, \ldots, a_k \), are still arbitrary and to be determined.

Thus we require to transform the column vectors,
\[ u_j = \{ z_1^j, z_2^j, \ldots, z_k^j \} \quad (j=0,1,\ldots,k) \]  
(2)
into column vectors, \( z_j \), in such a way that,
\[ Z'PZ = I \]
where,
\[ Z = \begin{bmatrix} z_{00} & z_{01} & \cdots & z_{0k} \\ z_{10} & z_{11} & \cdots & z_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ z_{k0} & z_{k1} & \cdots & z_{kk} \end{bmatrix} = \begin{bmatrix} D_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & D_k \end{bmatrix} \]
and,
\[ P = \begin{bmatrix} \vdots & \cdots & \cdots & \vdots \\ D_2 & & & \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & D_k \end{bmatrix} \]
i.e. in such a way that,
\[ z_j' P z_i = \begin{cases} 0 & \text{when } i \neq j \\ 1 & i = j \end{cases} \]  
(3)
these being the conditions of weighted normality and orthogonality.

It is known that the first of these conditions (3)
can be satisfied by a linear transformation of the form,

\[ z_j = b_j u_0 + b_j u_1 + \ldots + b_j u_k \quad (j=0,1, \ldots, k) \quad (4) \]

where the \( b_j \)'s have to be determined.

Inverting this system, we obtain the system of equations,

\[ u_i = c_i z_0 + c_i z_1 + \ldots + c_i z_j + z_j \quad (i=0,1, \ldots, k) \quad (5) \]

where the \( c_i \)'s will be functions of the \( b_j \)'s in (4).

Now premultiply all terms in each of the first \((j+1)\) of the equations (5) by \( z_j' \), and we obtain, having regard to the first of the conditions (3),

\[ z_j' P u_j = 0 \quad (i = 0,1,2, \ldots, j-1) \quad (6) \]

and

\[ z_j' P u_j = \ldots \]

Now premultiply all terms of the \((j+1)\)th of the equations (4) successively by \( u_j' \), \( u_j' \), \ldots, \( u_j' \), and we obtain, on account of the results (6),

\[ 0 = b_j u_j' u_0 + b_j u_j' u_1 + \ldots + b_j u_j' u_k \quad (i = 0,1,2, \ldots, j-1) \quad (7) \]

But from the nature of the vectors (2), we see that,

\[ u_j' u_0 = p_0 \xi_1 + p_0 \xi_2 + \ldots + p_0 \xi_n = S_0 = 1 \]
\[ u_j' u_1 = p_1 \xi_1 + p_1 \xi_2 + \ldots + p_1 \xi_n = S_1 \]
\[ u_j' u_k = p_k \xi_1 + p_k \xi_2 + \ldots + p_k \xi_n = S_k \]

and in general,

\[ u_j' u_m = p_m \xi_1 + p_m \xi_2 + \ldots + p_m \xi_n = S_m \quad \ldots \quad (8) \]
Hence equations (7) become, on substitution from (8),
\[
\begin{align*}
\left\{ \begin{array}{l}
\begin{align*}
&b_0^i S_0 + b_1^i S_1 + \ldots + b_{j-1}^i S_{j-1} + S_i^i = 0 \\
&(i = 0, 1, 2, \ldots, j-1)
\end{align*}
\end{array} \right. \\
\begin{align*}
&b_0^i S_0 + b_1^i S_1 + \ldots + b_{j-1}^i S_{j-1} + S_i^i = z_i^i P z_i^i
\end{align*}
\end{align*}
\]
Now, having given the weights, \( p_i \), and having formed the set of Ekke values, \( \xi_i \), as in 2.1, the \( S_0, S_1, \ldots, S_{j-1} \) can easily be obtained, particularly if a calculating machine is available, from the relation (8).
Substitute these values in the first \( j \) equations of (9), and then solve these \( j \) equations for \( b_0^i, b_1^i, \ldots, b_{j-1}^i \).
Substituting these values of the \( b \)'s in equations (4) we obtain a set of vectors, \( z_i^j \), which satisfy the first of the conditions (3), i.e. they are orthogonal.
We now have to normalise them, i.e. we require,
\[
\sum z_i^j P z_i^j = 1
\]
the second condition of (3).
A normalising factor, \( \mu_i^j = \sqrt{z_i^j P z_i^j} \), can be obtained from the last equation of (9).
Then, if we multiply each of the vectors, \( z_i^j \), by \( 1/\mu_i^j \), the resulting set of vectors will be weighted normal and orthogonal.

We have, then, finally, collecting all the above results:—
\[ z_r = \frac{1}{\mu^2} \left\{ b_{10} u_0 + b_{11} u_1 + \ldots + b_{1n} u_n + u_r \right\} \ldots \ldots \ldots \quad (10) \]

where, \( u_r = \{ \xi_1^r, \xi_2^r, \ldots, \xi_n^r \} \)

\[
\mu^2 = \left\{ b_{10} S_0 + b_{11} S_1 + \ldots + b_{1n} S_n + S_r \right\}
\]

\[
S_r = p_1 \xi_1^r + p_2 \xi_2^r + \ldots + p_n \xi_n^r
\]

and,

\[
\begin{bmatrix}
S_0 & S_1 & S_2 & \ldots & S_n \\
S_1 & S_2 & S_3 & \ldots & S_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
S_{n-1} & S_n & S_1 & \ldots & S_n \\
\end{bmatrix} \begin{bmatrix}
b_{10} \\
b_{11} \\
\vdots \\
b_{1n} \\
\end{bmatrix} = \begin{bmatrix}
S_0 \\
S_1 \\
\vdots \\
S_n \\
\end{bmatrix}
\]

the result of solving the first \( j \) of the equations (9).
4.0

THE TCHBYSHEF COEFFICIENTS.

Having now obtained the vectors, $z_i'$, as shown in section (3), we still have to find the most suitable values for the coefficients $a_1, a_2, \ldots, a_{K+1}$ in the Schmidt polynomials.

We consider again, therefore, the smoothing of the points $x_i$.

Suppose that we have given an observation at $x_i$ of weight $p_i$, giving rise to a point $A_i$ in a plotted graph of the distribution.

The point $D$, on the chosen curve, having the ordinate $p_i$, has for abscissa the ideal point $x_i$. Thus for the closest fit of the curve to the data, we require that the sum of the areas like $ABCD$ should be a minimum.

i.e. we require

$$\sum_{i=1}^{n} \{p_i(x_i - \hat{x}_i)\}^2$$

to be a minimum, taking the square of each area to avoid difficulties arising from the presence of positive and negative areas.

Now

$$\sum_{i=1}^{n} \{D_i x_i - D_i \hat{x}_i\}^2 \leq \sum_{i=1}^{n} \{D_i x_i - D_i (a_1 z_i + a_2 z_i + \ldots + a_{K+1} z_i)\}^2$$
substituting Schmidt polynomials for the $\hat{x}_i$.

Putting this into matrix notation, we have,

$$\sum_{i} [p_i, x_i - p_i, \hat{x}_i]^2 = [x - Za]^T [P] [x - Za] [P]$$

where $a = \{a_1, a_2, a_3, \ldots, a_n\}$

and $x$, $Z$, $P$, have already been defined in 3.0

But, $[x - Za]^T [P] [x - Za]$  

$$= x^T P x - x^T P Z a - a^T Z^T P x + a^T Z^T P Z a$$  

$$= x^T P x - a^T Z^T P x - x^T P Z a + a^T a$$

since $Z^T P Z = I$, from conditions 3.0 (3).

$$= x^T P x - x^T P Z Z^T P x + [a - Z^T P x]^T [a - Z^T P x]$$

going the quadratic form.

Now the $x_i$ and $p_i$ are given, and the $z_{ji}$ have already been chosen, hence the required minimum is obtained when

$$[a - Z^T P x]^T [a - Z^T P x] = 0$$

i.e. when, $a_i = z^T_i P x = \sum z_{ij} p_i x_j$.  ..................... (1)

These coefficients so defined have been called by Schmidt 'Tchebychef Coefficients'.

We may now obtain a set of 'best' abscissae for the given ordinates from the Schmidt polynomials,

$$\hat{x}_i = a_1 z_{i0} + a_2 z_{i1} + \ldots + a_n z_{ik}$$

finding the $z_{ij}$ again 3.0 and the $a_i$'s as in 4.0,(1).

Hence our inverse method of curve-fitting is complete.
Suppose that we change our scale and origin in such a way that the new Ekke values are given by,

$$\zeta_i = a \xi_i + \delta$$

Then, from 3.0 (2), we see that our new $u$'s are

$$\tilde{\xi}_i = \delta u_i + (\frac{j}{2}) \xi_i a u_i + (\frac{j}{2}) \xi_i a u_i + ... + a \xi_i u_i \ldots \ldots$$ \hspace{1cm} (1)

Now it is known that if two sets of vectors, $u_i$ and $\tilde{\xi}_i$, are linearly connected as in (1), then the weighted normal orthogonal systems, $z_i$ and $\tilde{\xi}_i$, formed from them in the manner of 3.0 are identical. Hence our vectors, $z_i$, are invariant under such a scale and origin transformation. Again, under this transformation, the new values of the $x_i$ are given by

$$\tilde{x}_i = a x_i + \delta,$$

or, in matrix notation,

$$\tilde{x} = a x + \delta z_{10}.$$ 

Hence the Tchebycheff coefficients become

$$\tilde{a}_{i0} = z_{10}^i P(a x + \delta z_{10})$$

$$= a z_{10}^i P x + \delta z_{10}^i P z_{10}.$$ 

But, from the conditions of 3.0 (3),

$$z_{10}^i P z_{10} = 0 \text{ unless } j = 0,$$

in which case it is unity.
Hence our new Tchebychef coefficients are

\[ a_n = a_n + \delta \]
\[ a_{n+1} = a_{n+1} \]
\[ a_{n+2} = a_{n+2} \]

... ...

\[ a_{n+m} = a_{n+m} \]

And hence our new \( x^*_k \) are given by

\[ x^*_k = a \cdot \left( (a_1 + \delta/a) z_0 + a_2 z_1 + a_3 z_2 + ... + a_{m+1} z_{m+1} \right) \]

Thus our fitting is independent of the original origin and scale of \( x \) chosen for the data.
In section 4.0 we obtained the following result,
\[
\sum (p,x_i - \hat{p},x_i)^2 = (x'P x - x'P Z Z'P x + [a - Z'P x] [a - Z'P x]) P
\]
which, under the prescribed minimum conditions, was found to be equal to,
\[
(x'P x - x'P Z Z'P x) P \quad \text{.................................................. (1)}
\]
\[
= (x'P x - a'a) P \quad \text{ (for } a = Z'P x) \]
\[
= \left[ (p_1 x_1^2 + p_2 x_2^2 + \ldots + p_k x_k^2) - (a_1^2 + a_2^2 + \ldots + a_k^2) \right] P \quad \text{.................................................. (2)}
\]
Now if \( k = n-1 \), then the system of vectors, \( z_i \), is complete, as well as normal and orthogonal,
\[
\therefore P Z Z' = Z'P Z = I.
\]
Hence from (1),
\[
\sum (p,x_i - \hat{p},x_i)^2 = (x'P x - x'I P x) P = 0
\]
i.e. every area such as ABCD in 4.0 must be zero, which means that the fit of the curve is exact. Also, since all terms in (2) are positive, the nearer \((k+1)\) approaches to \(n\) the smaller does this sum become, and, the closer the fit.

Thus the closer the ratio
\[
\frac{a_1^2 + a_2^2 + \ldots + a_k^2}{p_1 x_1^2 + p_2 x_2^2 + \ldots + p_k x_k^2}
\]
is to unity the closer is our fit.
Or, removing the term in \(a_1\), which alone is dependent on the origin chosen, we have as a convenient
measure of the goodness of our fit,

\[ M_k = \sqrt{\frac{\sum a_i^2 + a_2^2 + \ldots + a_{k+1}^2}{\sum \alpha d} - a_i^2} \]

where \( M_k \) lies between zero and unity.

**PRACTICAL APPLICATION OF THE METHOD OF FITTING.**

It will be obvious from 6.0 that the more terms we retain in our Schmidt polynomials the more accurate will be our fit.

Practical considerations, however, usually limit us to four terms, as anything further involves a large amount of extra labour, including the solution of a matrix equation of fourth or higher order, for a small gain in accuracy.

It will be found in the following examples that four terms, and even three, are usually sufficient to produce a satisfactory fit.

A point of importance to notice is that by this method we are able to fit a large number of varying types of distribution using in each case the simple binomial as the chosen curve type from which to obtain our \( \alpha_k \) values. This means that we may dispense with the cumbersome Pearson curves and even with the curves of Type A and B, except in cases of extreme skewness.

We proceed to give examples of (i) a Binomial, (ii) a Type A, and (iii) a Type II curve all fitted to a good degree of accuracy with either 3 or 4 terms of a Schmidt polynomial, and all commencing by the
assumption of the Binomial Type.

Example (i). — Binomial Distribution of Data.

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<th>np</th>
<th>nF(ξ)</th>
<th>F(ξ)</th>
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<td>0.3800</td>
<td>1.18 1.39 1.64 1.94</td>
<td>1.23 42</td>
<td>1.603</td>
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<tr>
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<td>0.4378</td>
<td>1.54 2.37 3.65 5.62</td>
<td>1.60 1.20</td>
<td>1.922</td>
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<tr>
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<td>0.4739</td>
<td>1.94 3.76 7.30 14.16</td>
<td>2.02 2.31</td>
<td>2.405</td>
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<tr>
<td>3.0 343.5</td>
<td>0.4927</td>
<td>2.44 5.95 14.53 35.45</td>
<td>2.53 4.04</td>
<td>3.004</td>
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</tbody>
</table>

We first list in column (1) the observations \( x \), and in column (2) their respective frequencies \( np \).

Now, as in 2.1, we form \( nF(\xi) \), where,

\[
nF(\xi) = n\{p_1 + p_2 + \ldots + p_n + \frac{1}{2}p_1\}
\]

Dividing by \( n \) and subtracting \( .5 \) we obtain \( F(\xi) = .5 \)

in column (4), the proportion of the curve to be
between an ordinate at $\xi$ and that at $x = 0$.

(ii) Columns (5) to (10) contain Ekke's best values together with their squares, cubes, ..., sixth powers.

In this example we assume a binomial distribution for the data, so that the Ekke values in column (5) can be obtained from the results of column (4) in a table of binomial integrals, for,

$$F(\xi) - .5 = \int_{0}^{\xi} \phi(t) \, dt,$$

where the central ordinate is taken as $\xi = 0$, and $\phi(t) \neq \xi$, is the binomial curve.

If an untabulated curve type be chosen to represent the data, then the figures of column (5) will have to be calculated from those of column (4) by the somewhat lengthy process of inverse integration.

In this example we intend to take three terms only of the Schmidt polynomials so that the columns (9) and (10) are in this case not needed.

(iii) We now obtain the quantities $nS_1, nS_2, \ldots, nS_n$ by multiplying each figure in column (5), (6), ..., (10) by the corresponding figure in XX col. (2) and adding the resultant products, for we saw in 3.0 (8) that

$$nS_r = n \cdot p \cdot u(r), \quad \text{where} \quad p = [p_1, p_2, \ldots, p_n].$$

Dividing these results by $\sqrt{n}$ we have the values of $S_1, S_2, \ldots, S_n$.

This can be done in one process on a calculating machine, and so needs no tabulation.
The results in this case are:

\[ S_0 = 1 \]  
(this, of course, is always the case since \( u_0 = \{1, 1, \ldots, 1\} \),
and \( p_0 + p_1 + \ldots + p_n = 1 \))

\[ S_1 = -0.008 \]
\[ S_2 = +0.935 \]
\[ S_3 = -0.067 \]
\[ S_4 = +2.512 \]

(iv) We now apply the results of 3.0(9) to find the coefficients, \( b \).

\[
\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} \begin{bmatrix} b_{10} \\ b_{20} \\ b_{30} \\ b_{40} \\ b_{50} \end{bmatrix} = - \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}
\]

In this example we will not require the third relation.

From the first two of these equations we have,

\[ b_{10} = 0.008 \]

\[
\begin{bmatrix} 1 & -0.008 \\ -0.008 & 0.935 \end{bmatrix} \begin{bmatrix} b_{20} \\ b_{30} \end{bmatrix} = \begin{bmatrix} -0.935 \\ 0.067 \end{bmatrix}
\]

Whence,

\[ b_{20} = -0.935 \]
\[ b_{30} = 0.064 \]

(v) Our next step is to use the last of equations 3.0(9) to obtain the \( \mu \)'s, or normalising factors.
\[\mu^2 = z^1 P z = b_1 S_1 + \ldots + b_\delta S_\delta + \sum_{\delta+1}^\delta b_\delta S_\delta + S_1.\]

Thus, \[\mu^2 = S_0 = 1.0\]
\[\mu^1 = b_1 S_1 + S_2 = 0.935\]
\[\mu^2 = b_2 S_2 + \sum_{j=1}^{\delta} b_j S_j + S_1 = 1.633\]
\[\mu^3 = b_2 S_2 + \sum_{j=1}^{\delta} b_j S_j + S_1 = \text{(not needed in this case.)}\]

These calculations, again, are done in one operation on a machine and so need no tabulation.

From these we have,
\[
\begin{align*}
\mu_0 &= 1.000 \\
\mu_1 &= 0.967 \\
\mu_2 &= 1.278
\end{align*}
\]

(vi) Now to obtain the vectors, \(z_{ij}\), we use the relations of 3.0 (10).

\[
\begin{align*}
z_0 &= u_0/\mu_0 \\
z_1 &= (b_{10} u_0 + u_1)/\mu_1 \\
z_2 &= (b_{20} u_0 + \delta_0 u_1 + u_2)/\mu_2 \\
z_3 &= (b_{30} u_0 + \delta_0 u_1 + \delta_0 u_2)/\mu_3
\end{align*}
\]

To form col. (11), we add \(b_{10}\) to each figure in col. (5) and multiply the result by \(1/\mu_1\).

To form col. (12), we add \(b_{20}\) together with \(b_{21}\) times the figures of col. (5) to the figures of col. (6), and divide the result by \(\mu_2\).

Similarly for col. (13) which we do not need in this example.

(vii) Now, making use of the result of 4.0 (1) we are able to obtain the Tchebycheff Coefficients.
For, \( n_a \) is simply the sum of the products of the figures in cols. (1) and (2).

\[ n_{a_2} = \text{sum of products of cols. (1), (2), and (11)}. \]

\[ n_{a_3} = \ldots \ldots \ldots \ldots \text{ (1), (2), and (12)}. \]

\[ n_{a_4} = \ldots \ldots \ldots \ldots \text{ (1), (2), and (13)}. \]

Whence, dividing by \( n \), we obtain \( a_1, a_2, a_3, a_4 \).

All these totals can be obtained in one operation on a machine, and so need no tabulation.

The results are in this case:

\[ a_1 = .183 \]
\[ a_2 = 1.062 \]
\[ a_3 = .043 \]
\[ a_4 = \text{(does not appear in this case.)} \]

(viii) We now have to add the products,

\[ a_1 z_1 + a_2 z_1 + a_3 z_1 + a_4 z_3. \]

giving the column vector, \( \mathbf{a} \), which appears in column (14).

Each figure in this last column can be obtained in one step on a machine by adding together \( a_1 \), \( a_2 \) times figures of col. (11), \( a_3 \) times the figures of col. (12), and \( a_4 \) times the figures of col. (13).

This completes the fitting of the curve.

(ix) We need now to find \( M_2, M_3, M_4 \), the measures of the exactness of our fit.
We have, 
\[ a_1^2 = 0.033 \]
\[ a_2^2 = 1.127 \]
\[ a_3^2 = 0.002 \]
\[ a_4^2 = \ldots \]

Thus,
\[ M = \sqrt{\frac{a_2^2}{\sum px^2 - a_1^2}} = \sqrt{\frac{1.127}{1.130}} = 0.9986 \]
\[ M = \sqrt{\frac{a_2^2 + a_3^2}{\sum px^2 - a_1^2}} = \sqrt{\frac{1.129}{1.130}} = 0.9995 \]
\[ M = \sqrt{\frac{a_2^2 + a_3^2 + a_4^2}{\sum px^2 - a_1^2}} \]

The closeness of \( M \) to unity shows the needlessness of proceeding to further terms of the Schmidt Polynomials.

The sum, \( \sum px^2 \), can be obtained directly on the machine from cols. (1) and (2), and is, in this example, \( 1.163 \).
### Example (ii). Type A Distribution of Data.

| x  | np | nF(\(\xi\))F(\(\xi\))-5 | \(u_a\) | \(u_b\) | \(u_c\) | \(z_a\) | \(z_b\) | \(z_c\) | \(x\) |
|----|----|---------------------------|------|------|------|------|------|------|------|------|
| -4 | 1  | 0.5 - 0.4996 | -3.38 | 11.42 | -38.61 | 130.50 | -3.53 | 3.63 | -3.84 |
| -3 | 44 | 23.0 - 0.4809 | -2.07 | 4.28 | -8.87 | 18.36 | -2.16 | 2.36 | -2.83 |
| -2 | 244 | 167.0 - 0.3614 | -1.09 | 1.19 | -1.30 | 1.42 | -1.14 | 0.32 | -2.18 |
| -1 | 292 | 435.0 - 0.1389 | -0.36 | 0.13 | -0.05 | 0.02 | -0.33 | -0.59 | -1.18 |
| 0  | 226 | 694.0 + 0.0760 | 0.19 | 0.04 | +0.01 | 0.00 | +0.19 | -0.72 | +0.07 |
| 1  | 134 | 874.0 + 0.2254 | 0.60 | 0.36 | 0.22 | 0.13 | 0.61 | -0.51 | 1.23 |
| 2  | 82  | 982.0 + 0.3151 | 0.90 | 0.31 | 0.73 | 0.66 | 0.93 | -0.19 | 2.34 |
| 3  | 57  | 1051.5 + 0.3727 | 1.14 | 1.30 | 1.48 | 1.69 | 1.18 | +0.13 | 3.23 |
| 4  | 37  | 1093.5 + 0.4113 | 1.35 | 1.83 | 2.46 | 3.32 | 1.39 | 0.56 | 4.13 |
| 5  | 25  | 1129.5 + 0.4375 | 1.53 | 2.34 | 3.58 | 5.43 | 1.58 | 0.96 | 4.95 |
| 6  | 19  | 1151.5 + 0.4557 | 1.70 | 2.89 | 4.91 | 8.35 | 1.76 | 1.37 | 5.76 |
| 7  | 14  | 1168.0 + 0.4694 | 1.97 | 3.50 | 6.54 | 12.23 | 1.93 | 1.84 | 6.60 |
| 8  | 9   | 1179.5 + 0.4790 | 2.04 | 4.16 | 8.49 | 17.32 | 2.11 | 2.54 | 7.48 |
| 9  | 6   | 1187.0 + 0.4852 | 2.18 | 4.75 | 10.36 | 22.56 | 2.26 | 2.79 | 8.26 |
| 10 | 4   | 1192.0 + 0.4894 | 2.30 | 5.29 | 12.17 | 27.99 | 2.38 | 3.21 | 8.93 |
| 11 | 3   | 1195.5 + 0.4923 | 2.42 | 5.86 | 14.17 | 34.29 | 2.51 | 3.65 | 9.64 |
| 12 | 3   | 1198.5 + 0.4943 | 2.56 | 6.55 | 16.78 | 42.96 | 2.65 | 4.26 | 10.53 |
| 13 | 2   | 1201.0 + 0.4963 | 2.73 | 7.45 | 20.35 | 55.56 | 2.83 | 4.87 | 11.52 |
| 14 | 1   | 1202.5 + 0.4981 | 2.90 | 8.41 | 24.39 | 70.73 | 3.01 | 5.61 | 12.63 |
| 15 | 1   | 1203.5 + 0.4989 | 3.06 | 9.36 | 28.65 | 87.67 | 3.17 | 6.42 | 13.75 |

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Example (ii), (calculation)

(iii) \[ S_0 = 1.00 \]
\[ S_1 = 0.01 \]
\[ S_2 = 0.92 \]
\[ S_3 = 0.13 \]
\[ S_4 = 2.45 \]

(iv) \[
\begin{bmatrix}
S_0 & S_1 \\
S_1 & S_2
\end{bmatrix}
\begin{bmatrix}
\mu_0 \\
\mu_1
\end{bmatrix}
= -
\begin{bmatrix}
S_4 \\
S_3
\end{bmatrix} \]
\[ b_{10} = -0.01 \]
\[ b_{20} = -0.93 \]
\[ b_{31} = -0.13 \]

(v) \[
\begin{align*}
\mu_0^2 &= S_0 \\
\mu_1^2 &= S_1 b_{10} + S_2 \\
\mu_2^2 &= S_2 b_{20} + S_3 b_{31} + S_4
\end{align*}
\]
\[ \mu_0 = 1.00 \]
\[ \mu_1 = 0.96 \]
\[ \mu_2 = 1.26 \]

(vi) The vectors \( z_0, z_4 \), appear in the table.

(vii) \[ a_1 = 0.27 \]
\[ a_2 = 2.41 \]
\[ a_3 = 0.91 \]

(viii) The fitting values, \( \hat{x}_i \), appear in the table.

(ix) \[ a_1^2 = 0.073 \]
\[ s_2^2 = 5.808 \]
\[ a_3^2 = 0.828 \]
\[ \Sigma px^2 = 6.854 \]
Example (ii).

Actual Observations, thus ——

Fitted Curve, thus ——x—
Thus, \[ M_2 = \sqrt{\frac{a_2}{2 \, px^2 - a_1^2}} = \sqrt{\frac{5.808}{6.781}} = .9255 \]

\[ M_3 = \sqrt{\frac{a_2^2 + a_3^2}{2 \, px^2 - a_1^2}} = \sqrt{\frac{6.636}{6.781}} = .9893 \]

A satisfactory fit being again obtained with three terms of the Schmidt Polynomials, even the skewness of the original distribution would have necessitated in the ordinary way a Type A fitting, whereas we have again assumed the simple binomial only.
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<td>7.22</td>
<td>9.99</td>
<td>4.84</td>
<td>3.29</td>
<td>9.33</td>
<td>15.18</td>
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<td>6.39</td>
<td>6.89</td>
<td>4.78</td>
<td>4.68</td>
<td>4.49</td>
<td>4.85</td>
<td>4.77</td>
<td>4.80</td>
<td>4.28</td>
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<td>6.06</td>
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<td>8.13</td>
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<td>1.84</td>
<td>1.65</td>
<td>3.67</td>
<td>0.10</td>
<td>0.14</td>
<td>0.20</td>
<td>0.64</td>
<td>1.46</td>
</tr>
</tbody>
</table>

Example (Mill): Type III Distillation of Data.
Actual Observations, thus ——

Curve Fitted from the Values $\hat{x}_{(a)}$, thus ——

Values $\hat{x}_{(b)}$, thus x
Example (iii).

(iii) \( S_0 = 1.00 \)
\( S_1 = .02 \) \( S_4 = 1.92 \)
\( S_2 = .79 \) \( S_5 = 1.90 \)
\( S_3 = .30 \) \( S_6 = 6.87 \)

(iv) \( S_{b1} = -S_1 \)
\( \begin{bmatrix} S_0 & S_1 \\ S_1 & S_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = - \begin{bmatrix} S_3 \\ S_4 \end{bmatrix} \)
\( \begin{bmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{bmatrix} \begin{bmatrix} b_{10} \\ b_{20} \\ b_{31} \end{bmatrix} = - \begin{bmatrix} S_5 \\ S_6 \end{bmatrix} \)

adj. \( S = \)
\[ \begin{bmatrix} S_3 - S_2 & S_2 - S_1 & S_1 \\ S_2 - S_3 & S_3 & S_2 \\ S_1 - S_4 & S_0 - S_2 & S_2 - S_1 \end{bmatrix} \]

\[ = \begin{bmatrix} 1.43 & .20 & -.61 \\ .20 & 1.30 & -.28 \\ -.61 & -.28 & .79 \end{bmatrix} \]

\( |S| = 1.43S_0 + .20S_1 - .61S_2 \)
\( = .95 \)
\( = 1/1.05 \)
Thus,
\[
\begin{bmatrix}
 b_{32} \\
 b_{31} \\
 b_{30}
\end{bmatrix}
= 1.05
\begin{bmatrix}
 1.43 & .20 & -.61 \\
 .20 & 1.30 & -.28 \\
 -.61 & -.28 & .79
\end{bmatrix}
\begin{bmatrix}
 -.30 \\
 -1.92 \\
 -1.90
\end{bmatrix}
= \begin{bmatrix}
 .36 \\
 -2.13 \\
 -1.82
\end{bmatrix}
\]
i.e. \( b_{30} = .36 \)
\( b_{31} = -2.13 \)
\( b_{32} = -1.82 \)

(v) \( \mu_0^2 = S_0 = 1.00 \)
\( \mu_1^2 = b_0 S_1 + S_1 = .79 \)
\( \mu_2^2 = b_2 S_2 + b_{21} S_{21} + S_{21} = 1.20 \)
\( \mu_3^2 = b_{30} S_3 + b_{31} S_{31} + b_{32} S_{32} + S_5 = 1.33 \)

Hence, \( \mu_0 = 1.00 \)
\( \mu_1 = .79 \)
\( \mu_2 = 1.10 \)
\( \mu_3 = 1.15 \)

(vi) The vectors, \( z_0, z_1, z_2, z_3 \), appear in the table.

(vii) \( a_1 = 2.31 \)
\( a_2 = 1.10 \)
\( a_3 = .34 \)
\( a_4 = .12 \)

(viii) We have in this example two sets of fitting-values, \( \hat{x}_{(ii)} \), \( \hat{x}_{(uv)} \).

The \( \hat{x}_{(ii)} \) are values obtained from three terms of the Schmidt polynomials, and the \( \hat{x}_{(uv)} \) from four.
(ix) \[ a_1^2 = 5.336 \]
\[ a_2^4 = 1.210 \]
\[ a_3^2 = .116 \]
\[ a_4^2 = .014 \]
\[ \Sigma px^2 = 6.324 \]

\[ M_2 = \sqrt{\frac{a_2^2}{\Sigma px^2 - a_1^2}} = \sqrt{\frac{1.210}{1.488}} = .9017 \]

\[ M_3 = \sqrt{\frac{a_2^2 + a_3^2}{\Sigma px^2 - a_1^2}} = \sqrt{\frac{1.326}{1.488}} = .9439 \]

\[ M_4 = \sqrt{\frac{a_2^2 + a_3^2 + a_4^2}{\Sigma px^2 - a_1^2}} = \sqrt{\frac{1.340}{1.488}} = .9488 \]

It will be noticed that the fit is not so satisfactory in this example, owing to our having assumed a simple binomial for such a skew distribution.

The values of the \( a_i z_i \) were as follows:

\[ -.09, +.08, -.17, -.18, -.06, +.13, +.43, +.91 \]

so that the \( x_{(4)} \) were obtained from the \( x_{(1)} \) thus:

<table>
<thead>
<tr>
<th>( x_{(3)} )</th>
<th>( a_i z_i )</th>
<th>( x_{(4)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.09</td>
<td>-.09</td>
<td>1.00</td>
</tr>
<tr>
<td>1.89</td>
<td>+.08</td>
<td>1.97</td>
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<td>3.25</td>
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</tr>
</tbody>
</table>
The method of curve-fitting detailed in this paper is, we consider, not likely to become of very great practical importance, owing mainly to the following defects.

(i) The inverse method is less direct and theoretically more cumbersome than the usual method of approach, and does not appear to have any very great advantages over, or to supersede in way, the usual method of fitting a curve by means of the integral, \( F(x) \), instead of its inverse, \( \psi(Y) \), which we have used.

(ii) The method lacks satisfactory tests of significance. Our \( M_k \)-Test, elaborated in section 6.0, does give some idea of the goodness of our fit, but, without a complete knowledge of the sampling distribution of \( M_k \), we are in no way able to say from its value alone how many terms of the Schmidt Polynomials will be required to produce any given accuracy, or whether the fit obtained from any given number of terms is satisfactory or not.

We do not, therefore, consider the method to be a profitable subject for further research.
AN ITERATIVE METHOD
FOR APPROXIMATING TO THE ROOTS
OF A POLYNOMIAL.
INTRODUCTORY

In this paper we are concerned with obtaining an iterative process for approximating to the roots of a polynomial.

Consider a polynomial, \( P \), of degree \( n \), which has a polynomial factor, \( P' \), of degree \( m \); and, let \( P \) be a polynomial of degree \( m \) which is an approximation to \( P' \). We will show that, if \( P \) be divided by \( P' \), the division being cut short when the remainder is of degree \( m \), then under certain conditions, this 'penultimate remainder' \( P \), is a closer approximation to the factor \( P' \), than was \( P \).

Thus if \( P \) be now taken as a new divisor and the process repeated, the penultimate remainder \( P \), obtained is a still closer approximation to the factor \( P' \), than was \( P \).

Hence by continued iterations of this process we may arrive finally at a polynomial \( P_r \), of degree \( m \), which is as close an approximation as we please to the required factor \( P' \), of the polynomial \( P \).

The theorem will be shown to be perfectly general, but it will be observed that its practical use is in most cases limited to the cases in which \( m = 1 \) or 2.

The case \( m = 1 \) will provide a useful alternative to the already existing methods of Horner and others for approximating to the real roots of a polynomial.

The case \( m = 2 \) is of importance in that it provides a means of obtaining the complex roots of a polynomial, for we may take out by this process a quadratic factor.
made up of a complex root and its conjugate, and then factorise this quadratic.

If, therefore, we require the factors of a polynomial we will obtain, graphically or otherwise, an approximation to a factor and use this approximation as the first divisor in the iterative process. When this factor is obtained to a sufficient degree of accuracy, the last quotient in the division may then be taken as the new polynomial, of degree \((n - m)\), from which to extract the next factor by another application of the same method.

The process is admirably adapted for use where a calculating machine is available.
2.0 GENERAL FORM OF THE PENULTIMATE REMAINDER.

Since we are going to be concerned with the penultimate remainder throughout, we shall first discuss the form which this remainder takes in the general case.

2.1 Linear Divisor.

Let us divide a polynomial,

\[ x^n - ax^{n-1} + ax^{n-2} - \ldots + (-1)^n a, \]

by \((x - b_1)\).

Successive remainders will be of degrees \((n-1)\), \((n-2)\), \ldots, and the penultimate remainder of the \((n-n+1)^{th}\) or first degree.

Let \(R_k\) be the coefficient of \(x^k\) in the \(k^{th}\) remainder.

Then we have immediately, by direct division,

\[
\begin{array}{c|cc}
R_1 &=& b_1 - a_1, \\
1 & a & b_1 \\
\hline
\end{array}
\]

\[
\begin{array}{c|ccc}
R_2 &=& b_2 - ab_1 + a, \\
1 & a & b_1 & a \\
\hline
\end{array}
\]

\[
\begin{array}{c|ccc}
R_3 &=& b_3 - ab_2 + ab_1 - a_3, \\
1 & a & b_1 & a_3 \\
\hline
\end{array}
\]

From the form of these results, let us assume that,

\[
R_k = \\
\begin{array}{c|cc}
1 & a & a_2 & \ldots & a_k \\
1 & b_1 & \ldots & \ldots \\
\hline
\end{array} 
\]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1) \]
Expanding the determinant in (1) by its last column, we obtain,

\[
\begin{vmatrix}
1 & a_1 & a_2 & \ldots & a_{n-1} \\
1 & b_1 & . & & . \\
. & . & 1 & b_1 & . \\
. & . & . & . & . \\
. & . & . & . & b_1 \\
\end{vmatrix}
\]

Thus we have the recurrence relation,

\[
R_k = b_1 R_{k+1} + (-1)^k a_k.
\]

From which, noting that,

\[
\begin{align*}
R_x &= b_1 R_x + a_x \\
R_a &= b_1 R_a - a_1
\end{align*}
\]

we deduce that (1) is the general form of \( R_x \).

Thus our penultimate remainder, the \((n-1)\) in the series, is given by,

\[
R_x + (-1)^a a = \begin{vmatrix}
1 & a_1 & a_2 & \ldots & a_{n-1} \\
1 & b_1 & . & & . \\
. & . & 1 & b_1 & . \\
. & . & . & . & . \\
. & . & . & . & b_1 \\
\end{vmatrix}
\]

2.2 Quadratic Divisor.

Let us now divide \( x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots + (-1)^{n-1} a_n \),

by \( (x^2 - b_1 x + b_2) \).

Let \( R_x, R_x' \) be the coefficients of \( x^{n-K}, x^{n-K-1} \), respectively, in the \( k \)th remainder in this division.
we see that (2) is the general form of $R_k$ in this case. Also, from the form of a division sum, we know that

$$R'_{K_{k+1}} = (-1)^{K_{k+1}} a_{K_{k+1}} - b R_{K_{k+1}}$$

Thus our penultimate term is given by,

$$R_n x^2 + R'_{n+1} x + (-1)^n a_n$$
General Case.

The above two cases will in general be all that we will require, but if divisors of higher degree are used, we might proceed by an exactly similar proof to the above.

Thus if 

\[ x^n - a_1 x^{n-1} + \ldots + (-1)^n a_n \]

be divided by 

\[ x^m - b_1 x^{m-1} + \ldots + (-1)^m b_m \]

then it might be shown that if

\[
\begin{array}{cccccccc}
1 & a_1 & a_2 & a_3 & \ldots & a_m & a_{m+1} & a_{m+2} & \ldots & a_{k-1} & a_k \\
1 & b_1 & b_2 & b_3 & \ldots & b_m & b_{m+1} & \ldots & \ldots \\
1 & b_1 & b_2 & \ldots & b_{m-1} & b_m & b_{m+1} & \ldots & \ldots \\
1 & b_1 & \ldots & b_{m-2} & b_{m-1} & b_m & \ldots & \ldots \\
1 & b_1 & \ldots & b_{m-3} & b_{m-2} & b_{m-1} & b_m & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & b_1 & \ldots & & & & \ldots & \ldots \\
\end{array}
\]

then our penultimate remainder is given by,

\[
R x^{m-n} + \left[ (-1)^{n+m+1} a_{n+m+1} - b R x^{m-1} + b R x^{m-2} - \ldots - (-1)^{m+1} b R x^{m-n} \right] x^{m-1} + \left[ (-1)^{n+m+2} a_{n+m+2} + b R x^{m-1} - b R x^{m-2} + \ldots - (-1)^n b R x^{m-n-1} \right] x^{m-2} + \ldots + \left[ (-1)^n a - (-1)^m b R \right] x + (-1)^n a_n
\]
THE THEOREM FOR THE CASE OF A LINEAR DIVISOR.

Suppose that \( \alpha \) is a real root of the polynomial \( P \), so that \((x - \alpha)\) is a factor.

And suppose our approximation to this factor is \((x - \alpha + \varepsilon)\), where \( \varepsilon \) is small.

If the penultimate remainder in this first division is \((x - \alpha + \varepsilon)\), and in the next division \((x - \alpha + \varepsilon)\), and so on, then we are concerned with the ultimate convergence of the series \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \).

We will show that, if \( \varepsilon \) is sufficiently small for second and higher powers to be neglected, then this series tends to become a Geometrical Progression whose convergence depends on the relative magnitudes of \( \alpha \) and the other roots of the polynomial.

We begin by examining two particular cases.

**Polynomial of the Third Degree.**

Let \( \alpha_1, \alpha_2, \alpha_3 \) be the roots of the polynomial \( x^3 - a_1 x^2 + a_2 x - a_3 \).

And let us divide the polynomial by \((x - \alpha_i + \varepsilon)\), where \( \varepsilon \) is small.

From 2.1 we see that the penultimate remainder reduced, (i.e. divided by the coefficient of \( x \)), is

\[
\begin{array}{c|ccc}
\alpha_3 \\
\hline
1 & a_1 & a_2 \\
\hline
1 & a_1 + \varepsilon_1 & \varepsilon_2 \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

i.e.

\[
x = \frac{a_3}{a_1 a_2 a_3} \frac{\varepsilon_2^2 + \varepsilon_1 (a_1 - a_2 a_3) + a_2 a_3}{a_1 a_2 a_3}
\]

Replacing \( a_1, a_2, a_3 \) by their corresponding symmetric functions...
of the roots of the polynomial.

\[ x - \left\{ a_1 + \frac{-a_1 (a_1 - a_2 - a_3) \varepsilon_1}{a_2 a_3 + \varepsilon_1 (a_1 - a_2 - a_3)} \right\} \]

neglecting terms in \( \varepsilon_1^2 \).

Thus we have,

\[ \varepsilon_2 = \frac{a_1 (a_1 - a_2 - a_3)}{a_2 a_3} \varepsilon_1, \]

neglecting also \( \varepsilon_1 (a_1 - a_2 - a_3) \) in the denominator, which is small compared with \( a_2 a_3 \).

We now divide the polynomial by \( (x - a + \varepsilon_2) \), and so on, continuing the process, and at each stage we find

\[ \varepsilon_n = \frac{a_1 (a_1 - a_2 - a_3)}{a_2 a_3} \varepsilon_{n-1}. \]

Thus, provided \( \varepsilon_n \) is small enough, our series approximates to a G.P. with common ratio \( \frac{a_1 (a_1 - a_2 - a_3)}{a_2 a_3} \).

Note that if one root (say \( a_3 \)) be much larger than the other two then our ratio, which may be written,

\[ \frac{a_1 (a_1 - a_2 - a_3)}{a_2 a_3} \]

\[ = \frac{a_1 (a_1 - a_2 - 1)}{a_2 a_3}, \]

approaches \( \frac{a_1}{a_3} \).

Thus we will usually obtain a convergent series for the numerically smallest root of the polynomial.

It will be seen, however, that cases will arise in which our ratio is greater than \( \frac{a_1}{a_3} \), and therefore the series is divergent, even when \( a_3 \) is the numerically smallest root. Such cases may be dealt with by first increasing the roots of the polynomial, e.g. by the root-squaring method, so that \( a_3 \) becomes large enough to outweigh the others and so make the ratio approach \( \frac{a_1}{a_3} \), which is, of course, less than \( \frac{a_1}{a_3} \).

Note that our ratio may be written in the determinantal form:

\[ \frac{-a_1}{a_2 a_3} \begin{vmatrix} 1 & a_2 + a_3 \\ a_2 & a_3 \end{vmatrix}. \]
Example (i).

Find a root of the polynomial \( x^3 - 5x^2 - 2x + 24 \).

It can easily be verified that the factors of this polynomial are \((x + 2)(x - 4)(x - 3)\).

So that, if we are approximating to the root, 3, we would expect the \( \varepsilon_n \), after sufficient iterations, to tend to become a G.P. whose common ratio is \( \frac{3}{8} \).

\[
\begin{array}{c|ccc|c}
1 & 4 & 2 & \varepsilon_n & \varepsilon_{n-1} \\
\hline
1 & 3 & 1 & -2.89 & + 24 \\
4 & 3 & 1 & -7.95643 & + 24 \\
8 & 3 & 1 & -7.9833019 & + 24 & 393 \\
16 & 3 & 1 & -7.9936657 & + 24 & 382 \\
32 & 3 & 1 & -7.9976247 & + 24 & 378 \\
64 & 3 & 1 & -7.99910821 & + 24 & 376 \\
128 & 3 & 1 & -7.99956543 & + 24 & 375 \\
256 & 3 & 1 & -7.99995609 & + 24 & 375 \\
512 & 3 & 1 & -7.9999999999 & + 24 & 375 \\
\end{array}
\]

Suppose that \((x - 1.9)\) has been obtained as an approximation to a root. We divide by this and note the reduced penultimate remainder.

The work may be arranged as follows, suppressing all unnecessary figures:

\[
\begin{array}{c|ccc|c}
1 & 4 & 2 & \varepsilon_n & \varepsilon_{n-1} \\
\hline
1 & 3.04182 & -1.95818 & -7.95543 & + 24 \\
4 & 3.0164282 & -1.9357138 & -7.9833019 & + 24 & 393 \\
8 & 3.0062749 & -1.997251 & -7.9936657 & + 24 & 382 \\
16 & 3.0023697 & -1.9976303 & -7.9976247 & + 24 & 378 \\
32 & 3.0014931 & -1.9991090 & -7.99910821 & + 24 & 376 \\
64 & 3.0006934 & -1.9995654 & -7.99956543 & + 24 & 375 \\
128 & 3.0003910 & -1.99995609 & -7.99995609 & + 24 & 375 \\
256 & 3.00004931 & -1.9999999999 & -7.9999999999 & + 24 & 375 \\
512 & 3.0000001 & -1.9999999999 & -7.9999999999 & + 24 & 375 \\
\end{array}
\]

Thus our last penultimate remainder is \( x - 3.000001849 \)

And the last quotient is \( x^2 - 7.99995609x - 7.99995609 \)

The ratios on the right show the tendency towards a G.P. of common ratio \( 3/3 \).

Example (ii).

If in the above example we had tried to approximate to the root \((x + 2)\), although this is the numerically smallest root, we should have obtained a divergent series.
For in this case the ratio of the G.P. is

\[
\frac{-2}{3 \times 4} = \frac{3+4}{1-2} \quad \text{i.e.} \quad -\frac{3}{2}.
\]

Suppose, for instance, that we obtain \((x + 2.1)\) as our first approximation and perform the division as before:

\[
\begin{array}{c|ccc}
1 & 5 & 2 & +24 \\
\hline
& 1 & 5 & -2 & +24 \\
+ & 2.1 & 6 & 7.1 & +14.91 \\
& -7.1 & -12.91 & +24 & x + 1.859 \\
1 & -6.859 & -10.751 & +24 & x + 2.232
\end{array}
\]

Deviations from the true root are respectively, +100, -141, and +0.232, being terms of an approximate G.P. of common ratio -1.5.

In this case, therefore, we take the polynomial whose roots are the squares of those of the given polynomial and apply the process to it, starting with the suspected root, \([x - (-2.1)^2]\), i.e. \((x - 4.41)\).

The polynomial whose roots are the squares of those of \(x^3 - a_1 x^2 + a_2 x - a_3\) is

\[x^3 - (a_1^2 - 2a_2) x^2 + (a_2^2 - 2a_1 a_3) x - a_3^2\]

Thus our required polynomial is,

\[x^3 - 29x^2 + 244x - 576\]

The factors of this are, \((x - 4)(x - 9)(x - 16)\).

Thus the series for \(\varepsilon\) should tend towards a convergent G.P. whose common ratio is

\[-\frac{4(4 - q - 16)}{144}, \text{i.e.} \frac{7}{15} \text{ or } 0.467\]

Performing the iterative divisions as before we have the following results:
Thus our required approximation to the factor is, 

\[ x - \sqrt[4]{0.000704} \]

i.e. \( x - 2.000176 \).

3.2 Polynomial of the Fourth Degree.

Let \( a_1, a_2, a_3, a_4 \) be roots of the polynomial, 

\[ x^4 - a_1 x^3 + a_2 x^2 - a_3 x + a_4 \]

and let us divide the polynomial by \( (x - a_i + \epsilon_i) \), where \( \epsilon_i \) is small.

The penultimate remainder reduced is, by 2.1, 

\[
\begin{array}{cccc}
1 & a_1 & a_2 & a_3 \\
1 & a_1 + \epsilon_1 & & \\
& 1 & a_1 + \epsilon_1 & \\
& & 1 & a_1 + \epsilon_1
\end{array}
\]

Replacing coefficients by their respective symmetric functions of the roots, and neglecting terms in \( \epsilon_1^2, \epsilon_1^3 \), this becomes,
\[ x + \frac{a_1 \, a_2 \, a_3 \, a_4}{\varepsilon_1 (a_1^2 - a_1 a_2 - a_1 a_3 - a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4) - a_1 a_2 a_3 a_4} \]

which, neglecting the term in \( \varepsilon_1 \) in the denominator, may be written,

\[ x = \left\{ a_1 + \frac{a_1 (a_1^2 - a_1 a_2 - a_1 a_3 - a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4)}{a_1 a_2 a_3 a_4} \right\} \]

Thus, \( \varepsilon_2 = \rho \varepsilon_1 \), where

\[ \rho = \frac{a_1 (a_1^2 - a_1 a_2 - a_1 a_3 - a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4)}{a_1 a_2 a_3 a_4} \]

In a similar way we could now obtain the same ratio between \( \varepsilon_3 \) and \( \varepsilon_2 \), and then between \( \varepsilon_4 \) and \( \varepsilon_3 \), and so on.

Hence, ultimately, we find that the series for \( \varepsilon_n \) tends towards a G.P. whose common ratio is given by,

\[ \frac{\varepsilon_n}{\varepsilon_{n-1}} = \rho \]

No that \( \rho \) may be written,

\[ \frac{a_1}{a_2 a_3 a_4} \begin{array}{c|cccc} 1 & a_2 + a_3 + a_4 & a_2 a_3 + a_2 a_4 + a_3 a_4 \\ \hline a_1 a_2 a_3 a_4 & 1 & a_1 \\ \hline & 1 & a_1 \\ \end{array} \]

We note also that, as before, if \( a_3, a_4 \) be large compared with \( a_1, a_2 \), then our ratio, which may be written,

\[ \frac{a_1}{a_2} \left\{ \frac{a_1 (a_1 - a_2)}{a_2 a_4} - \frac{a_1 - a_2}{a_3} - \frac{a_1 - a_2}{a_4} + 1 \right\} \]
approaches the value \( a_n/a_{n+1} \).

So that usually we will obtain a convergent series when approximating to the smallest root of a polynomial.

In cases when we do not, then we must first increase the roots as before, and then apply the iterative process.

Example(i).

Find the factor of \( x^5 - 2x^3 - 25x^2 + 26x + 120 \), to which \((x + 1.5)\) is a first approximation.

<table>
<thead>
<tr>
<th></th>
<th>-2</th>
<th>-25</th>
<th>+26</th>
<th>+120</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 + 1.5</td>
<td>-3.5</td>
<td>-19.75</td>
<td>+55.625</td>
<td>+120</td>
</tr>
<tr>
<td>1 + 2.16</td>
<td>-4.16</td>
<td>-16.014</td>
<td>+60.520</td>
<td>+120</td>
</tr>
<tr>
<td>1 + 1.980525</td>
<td>-3.980525</td>
<td>-17.116471</td>
<td>+59.899999</td>
<td>+120</td>
</tr>
<tr>
<td>1 + 2.003352</td>
<td>-4.003352</td>
<td>-16.979877</td>
<td>+60.016671</td>
<td>+120</td>
</tr>
<tr>
<td>1 + 1.999444</td>
<td>-3.999444</td>
<td>-17.003336</td>
<td>+59.997218</td>
<td>+120</td>
</tr>
</tbody>
</table>

Thus the required approximation to a factor is,

\[ x + 2.000093 \]

and the remaining quotient is,

\[ x^3 - 3.999444x^2 - 17.003336x + 59.997218 \].
It may be verified that the factors of the given polynomial are, \((x - 3)(x - 5)(x + 2i)(x + 4)\).

Thus we would expect the series for \(E\) to be a G.P. whose common ratio is,

\[
\begin{array}{ccc}
-2 & 3-4+5 & -20+15-12 \\
3(-4) & 5 & 1 \\
1 & -2 & -2 \\
\end{array}
\]

i.e. \(-1/6\) or \(-\frac{i}{6}\).

The values of \(E_n/E_{n-1}\) at each step are shown in the right hand column of the example.

**General Case.**

The division if a polynomial of degree \(n\) by an approximate linear factor.

Let the roots of the polynomial, \(P\), be

\[a_1, a_2, a_3, \ldots, a_n\]

Noting the determinantal form of the results of 3.1 and 3.2, let us assume that in the general case the series of \(E\)’s tends to become a G.P. with common ratio \(p_n\), where,

\[
g_n = \left(-1\right)^n \frac{a_1}{T_r} \begin{array}{cccc}
1 & T_1 & T_2 & \cdots & T_{n-3} & T_{n-2} \\
1 & a_1 & \cdots & \cdots & \cdots \\
. & 1 & a_1 & \cdots & \cdots \\
. & . & . & \cdots & \cdots \\
. & . & . & \cdots & \cdots \\
. & . & . & \cdots & 1 & a_1 \\
\end{array} \tag{I}
\]

\(T_r = \text{the sum } r \text{ at a time of the roots } a_2, a_3, \ldots, a_n\) without \(a_1\),

and \((x - a_1)\) is the factor to which our linear divisor is an approximation.
Expand the determinant on the right of (I) by its last column, and we get,

\[
(-1)^n \rho_n = \frac{a_i}{T_{n-1}} \left\{ \begin{array}{cccc}
1 & T_1 & T_2 & \cdots & T_{n-w} \\
1 & a_1 & & & \\
& & 1 & a_1 & \\
& & & \ddots & \ddots \\
& & & & 1 & a_1 \\
\end{array} \right\} T_{n-1} + (-1)^{n-w} T_{n-2}
\]

\[
\frac{a_i}{a_n T'} \left\{ \begin{array}{cccc}
1 & T' a_n & T' + aT' & \cdots & T' + aT' \\
1 & a_1 & & & \\
& & 1 & a_1 & \\
& & & \ddots & \ddots \\
& & & & 1 & a_1 \\
\end{array} \right\} T + (-1)^{n-w} \{ T' + aT' \}
\]

where \( T_r = \sum r \) at a time of the roots, \( a_2 \neq a_3 \), \ldots, \( a_{n-1} \), without \( a_i \) of the polynomial, \( P' \), of degree \((n-1)\) which is, \( P/(x-a_n) \).

\[
eq \frac{a_i}{a_n} \left\{ (-1)^{n-1} T' \rho_{n-1} - a_n \{ a_{n-3} - T' a_{n-5} + T' a_{n-7} - \cdots + (-1)^n \rho_{n-3} \} \right. \\
\left. \quad + (-1)^{n-w} \{ T' + aT' \} \right\}
\]

where \( \rho_{n-1} \) = the ratio for \( P' \) which corresponds to \( \rho_n \) for \( P \).

\[
eq \frac{a_i}{a_n} \left[ (-1)^{n-1} \rho_{n-1} + (-1)^{n-2} \rho_{n-3} \right] - \frac{a_i}{T_{n-2}} \left[ \begin{array}{c}
a_{n-3} - T' a_{n-5} + T' a_{n-7} - \cdots \\
\end{array} \right. \\
\left. \quad + (-1)^{n-w} T' a_{n-3} + (-1)^{n-3} T' a_{n-3} \right]
\]

Thus we have the recurrence relation,

\[
(-1)^n \rho_n = (-1)^{n-1} \frac{a_i}{a_n} \left[ \rho_{n-1} \right] - \frac{a_i}{T_{n-2}} \left[ \rho_{n-3} - T' a_{n-5} + T' a_{n-7} - \cdots + (-1)^{n-1} T' a_{n-3} \right].
\]
From this relation, noting from 3.1 and 3.2 that

\[ \rho_\pi = \frac{a_1}{T_3} \begin{vmatrix} 1 & T_1 & T_2 \\ 1 & a_1 & c \\ . & . & . \end{vmatrix} = -\frac{a_1}{T_3} (\rho_{\pi} - 1) - \frac{a_1}{T_3} (\chi_1 - T_1') \]

where \( -\rho_{\pi} = \frac{a_1}{T_3} \begin{vmatrix} 1 & T_1' \\ 1 & a_1 \\ . & . & . \end{vmatrix} \),

we deduce that (I) is the general form of the required ratio, \( \rho_n \).

It will be seen that, if \( a_1, a_2, \ldots, a_n \), are large compared with \( a_1, a_2, \) then, dividing \( a_1, a_2, \ldots, a_n \), into the determinant in (I) the ratio approaches \( a_1/a_2 \), so that, as seen in the particular cases, a convergent series will usually be obtained for the smallest root of a polynomial. If not, then, as before, the root-squaring process may first be applied.

It may be noted also that if the roots of the polynomial \( x^n - a_n x^{n-1} + a_2 x^n - \ldots + (-1)^n a_n \)

be not known, (as is usually the case), then

\[ \rho_2 = (-1)^n \frac{a_1}{a_n} \begin{vmatrix} 1 & a_1, a_2 - a_1(a_2 - a_1), a_3 - a_2(a_3 - a_2), \ldots & \end{vmatrix} \]

adding to the first row, 2 row \( x(n - 2) \)

\[ + 3 \text{rd row} x(n - 3)(a_1 - a) \]
and so on, the above expression becomes,

\[
(-1)^n a_i^2 \begin{vmatrix}
(n-1) & (n-2)a_i & (n-3)a_i & \cdots & a_i \\
1 & a_1 & \cdots & \\
\vdots & \vdots & \ddots & \\
1 & \cdots & \cdots & a_i
\end{vmatrix}
\]

Thus if we have an approximation to one root, \(a_1\), of the polynomial, we may, from the above expression, obtain an approximation to the common ratio of the G.P. that we may expect to get.

**Complex Roots.**

Complex roots in the polynomial, other than the root to which we are approximating, do not in any way affect the above work.

For, if we are approximating to \(a_1, a_2, a_3\) are a pair of conjugate complex roots, then we notice from the result 3.3(I), that \(a_1\) and \(a_3\) appear in the ratio only as \((a_1 a_3)\) or as \((a_1^2 a_3^2)\), so imaginary quantities do not occur.

**Example.**

Find a closer approximation to a root of

\[x^4 - 7x^3 + 15x^2 + 41x - 50\]

given \((x-2)\) as a first approximation.

It may be verified that the roots in this case are,

\[(x-1)(x+2)(x-4-3i)(x-4+3i)\]

Hence we have, \(a_1 = 1\), \(a_2 = -2\)

\[a_3 + a_4 = 3\quad\text{and}\quad a_3 a_4 = 4^2 + 3^2 = 25\]
And thus,

\[ \gamma = \frac{1}{\sqrt{-25}} \begin{vmatrix} 1, & -2+3, & (-2\times3)+25 \\ 1, & 1, & . \\ ., & 1, & 1 \end{vmatrix} \]

\[ = -\frac{2}{25} = -0.08. \]

The working of the example is as follows:

\[
\begin{array}{c|cccc|c}
1 & -7 & +15 & +41 & -50 & \varepsilon_n/\varepsilon_m \\
1-2 & -5 & +5 & +51 & -50 \\
1-0.999872 & -6.019602+9.098454+49.920022 & -50 \\
1-1.001602 & -5.993983+9.999393+50.006393 & -50 \\
1-0.999872 & -6.000128+9.000660+49.999488 & -50 \\
1-1.000102 & -6.000128+9.000660+49.999488 & -50 \\
\end{array}
\]

-0.0799
-0.0797
THEOREM IN THE CASE OF A QUADRATIC DIVISOR

In this case we suppose that \( a, a \) are roots of the polynomial \( P \), so that \( \{x^2-(a+a)x + a, a\} \) is a factor.

Suppose that our approximate factor for the division is,

\[
x^2 - (a+a+\varepsilon)x + (a\varepsilon + \varepsilon')
\]

where \( \varepsilon, \varepsilon' \) are small.

and that the penultimate remainder reduced is,

\[
x^2 - (a+a+\varepsilon)x + (a\varepsilon + \varepsilon')
\]

and so on.

We are concerned with the ultimate convergence of the two series, \( \varepsilon, \varepsilon, \varepsilon \ldots \ldots \), and \( \varepsilon, \varepsilon, \varepsilon' \ldots \ldots \), and will show that when \( a, a \) are real, both series tend to become G.P.'s having their common ratios equal and depending as before on the values of the roots of the polynomial.

We begin as before by investigating a particular case:

**Polynomial of the Fourth Degree.**

Let \( a, a, a, a \) be roots of the polynomial,

\[
x^4 - a_1x^3 + a_2x^2 - a_3x + a_4
\]

And let us take for our first divisor,

\[
x^2 - (a+a+\varepsilon)x + (a\varepsilon + \varepsilon')
\]

From the result of 2.2, the penultimate, reduced, will be,

\[
x^2 - \frac{a_5 + b_1}{1 + a_1} x + \frac{a_4}{1 + a_1}
\]
where, \( b_1 = a_1 + a_2 + \varepsilon \)
\[ b = a_1 + a_2 + \varepsilon' \]
Replacing \( a_1, a_2, a_3, \ldots \) by the corresponding symmetric functions of the roots of the polynomial, and neglecting terms of the second degree in \( \varepsilon, \varepsilon' \), this becomes,

\[ x^2 - \frac{aa(a+a) + a\varepsilon - (a+a)\varepsilon'}{aa + \varepsilon(a+a-a-a) - \varepsilon'} x + \frac{a_1 a_2 a_3 a_4}{a_1 a_2 a_3 a_4} \]

Neglecting the terms in \( \varepsilon, \varepsilon' \) in the denominator, we may write this in the form,

\[ x^2 - \left( a + a + \frac{(\varepsilon - \varepsilon(a+a))}{a_1 a_2 a_3 a_4} \right) x + \frac{a_1 a_2 a_3 a_4}{a_1 a_2 a_3 a_4} \]

which, in the next stage of the iteration, we take as our new divisor,

\[ x^2 - (a + a + \varepsilon') x + (a + a + \varepsilon') \]

obtaining a new penultimate remainder,

\[ x^2 - (a + a + \varepsilon') x + (a + a + \varepsilon') \]

and so on.

We see that at each stage of the process,

\[ \varepsilon = \frac{\varepsilon_1 (a+a-a-a) - \varepsilon (a+a)(a+a-a-a) - aa}{a_1 a_2 a_3 a_4} \]

\[ \varepsilon' = \frac{\varepsilon'_1 a_1 a_2 a_3 (a+a-a-a)}{a_1 a_2 a_3 a_4} \]

Rearranging (1), we have,

\[ \varepsilon'_1 = \frac{\varepsilon_1 a_1 a_2 a_3 (a+a-a-a) - a_1 a_2 a_3 a_4}{a_1 a_2 a_3 a_4} \]
Substituting this value for $\varepsilon'_{p-1}$ in (2) we obtain

$$
\varepsilon_p = \frac{a_1 a_p \varepsilon'_{p-1} - a_p a_1 \varepsilon'_{p-1}}{a_1 a_p (a+a-\alpha-\alpha)}
$$

Substituting this value for $\varepsilon'_{p-1}$ and the corresponding result for $\varepsilon'_{p}$ in (1), we obtain

$$
\varepsilon'_{p} = \frac{a_1 a_p \varepsilon'_{p-1} - a_p a_1 \varepsilon'_{p-1}}{a_1 a_p (a+a-\alpha-\alpha)}
$$

i.e. $\varepsilon'_{p} = \frac{a_1 a_p \varepsilon'_{p-1} - a_p a_1 \varepsilon'_{p-1}}{a_1 a_p (a+a-\alpha-\alpha)}$

The same recurrence relation between the $\varepsilon'_{p}$ as exists between the $\varepsilon_p \varepsilon'_{p-1} \varepsilon'_{p-1}$ in (3).
4.2

General Case.

Division of a polynomial of the \( n \)th degree by an approximate quadratic factor.

Let the roots of the polynomial, \( P \), be \( a_1, a_2, \ldots \). Guided by the form of the results of 4.1, let us assume that we may obtain in the general case a recurrence relation between the \( C \)'s as follows:

\[
C_p - (\rho_n + \rho'_n) C_{p-1} + \rho_n \rho'_n C_{p-1} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)
\]

where, \( \rho_n = (-1)^{n-1} \frac{a_1}{T_{n-2}} \)

\[
\begin{vmatrix}
1 & T_1 & T_2 & \ldots & T_{n-2} \\
1 & a_1 & & & \\
& 1 & a_2 & & \\
& & \ddots & \ddots & \\
& & & 1 & a_n \\
& & & & \ddots
\end{vmatrix}
\]

and, \( \rho'_n = (-1)^{n-1} \frac{a_2}{T_{n-2}} \)

\[
\begin{vmatrix}
1 & T_1 & T_2 & \ldots & T_{n-2} \\
1 & a_2 & & & \\
& 1 & a_3 & & \\
& & \ddots & \ddots & \\
& & & 1 & a_n \\
& & & & \ddots
\end{vmatrix}
\]

where \( T_r \) = the sum \( p \) at a time of the roots, \( a_3, a_4, \ldots, a_n \), without \( a_1, a_2 \).

and \( \{x^2 - (a_1 + a_2)x + a_1 a_2\} \) is the factor to which our divisor is an approximation.

Consider \( \rho_n \).

Expand the determinant on the right hand side by its last column, and write \( T_r = T_r' + \rho_n T_{r-1}' \).
where \( T_r' \) is the sum \( r \) at a time of the roots \( a_1, a_2, \ldots, a_n \), of the polynomial \( P/(x - a_n) \).

We obtain,

\[
(-1)^{n-1} \rho_n = \frac{a_1}{a_n T_n'} \left\{ \frac{1}{a_1} \begin{vmatrix} 1 & a_1 & \cdots & 1 \end{vmatrix} \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \end{vmatrix} + \frac{a_1}{a_1} \begin{vmatrix} 1 & a_1 & \cdots & 1 \end{vmatrix} \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \end{vmatrix} \right. 
\]

Thus we have the relation,

\[
(-1)^{n-1} \rho_n = (-1)^{n-1} \rho_{n-1}^{a_1} \left\{ a_1 \begin{vmatrix} 1 & a_1 & \cdots & 1 \end{vmatrix} \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \end{vmatrix} \right. 
\]

Now we have already proved, 4.1, that,

\[
\rho_n = -\frac{a_1}{a_n} \frac{1}{a_1} \begin{vmatrix} 1 & a_1 + a_r' \end{vmatrix} \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \end{vmatrix} 
\]

Thus from the relation (3),

\[
\rho_n = -\frac{a_1}{a_n} \left\{ \frac{1}{a_1} \begin{vmatrix} 1 & a_1 + a_r' \end{vmatrix} \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \end{vmatrix} - \frac{a_1}{a_1} \begin{vmatrix} 1 & a_1 + a_r' \end{vmatrix} \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \end{vmatrix} \right. 
\]

Thus from the relation (3),

\[
\rho_n = -\frac{a_1}{a_n} \left\{ \frac{1}{a_1} \begin{vmatrix} 1 & a_1 + a_r' \end{vmatrix} \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \end{vmatrix} - \frac{a_1}{a_1} \begin{vmatrix} 1 & a_1 + a_r' \end{vmatrix} \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \end{vmatrix} \right. 
\]
Thus from the recurrence relation (3) we may establish the general result for $\rho_n$.

Similarly by putting $\epsilon'_0$ for $\epsilon'$ we may establish the result for $\rho'_n$, and hence we have in general the recurrence relation (1), between the $s'$'s of the series.

Again from (4.1) (4) we see that an exactly similar result applies to the $s'$'s, namely,

$$\epsilon'_p - (\rho + \rho'_n) \frac{\epsilon'_{p-1}}{\epsilon'_{p-n}} + p \rho'_n \epsilon'_p = 0 \quad \cdots \cdots \cdots (4)$$

where $\rho_n, \rho'_n$ have the same meanings as before.

Now divide (1) by $\epsilon'_{p-n}$ and (4) by $\epsilon'_{p-n}$, and we get,

$$\frac{\epsilon_p}{\epsilon'_{p-n}} \frac{\epsilon_{p-1}}{\epsilon'_{p-n}} - (\rho + \rho'_n) \frac{\epsilon_{p-1}}{\epsilon'_{p-n}} + p \rho'_n \epsilon_p = 0$$

$$\frac{\epsilon'_p}{\epsilon'_{p-n}} \frac{\epsilon'_{p-1}}{\epsilon'_{p-n}} - (\rho + \rho'_n) \frac{\epsilon'_{p-1}}{\epsilon'_{p-n}} + p \rho'_n \epsilon'_p = 0$$

Now let $p$ become large, and we have ultimately, factorising the above equations,

$$\lim \frac{\epsilon_p}{\epsilon_{p-1}} = \lim \frac{\epsilon'_p}{\epsilon'_{p-1}} = \rho_n \text{ or } \rho'_n, \quad \text{ whichever is the larger.}$$

Thus both series tend to become G.P.'s having for common ratio either $\rho_n$ or $\rho'_n$, as defined in the relations (2), whichever is the larger.

Divide $a_1, a_2, \ldots, a_n$ into the determinant in either of the ratios (2).

We then note that, if $a_1, a_2, \ldots, a_n$ are large
compared with \( a_1, a_2, a_3 \), the ratios \( \rho_1, \rho_2 \) approach \( a_1/a_3, a_2/a_3 \) respectively.

Thus we will usually obtain convergent series for the coefficients of a factor made up of the two numerically smallest roots of the polynomial.

If \( a_1, a_2, \ldots, a_n \) be not sufficiently large compared with \( a_1, a_2, a_3 \), then, provided these latter are the numerically smallest roots, we may produce convergent series by first powering the roots as shown before.

We note also that,

\[
\rho_n = (-1)^n \frac{a_i^2 a_r}{a_n}, \quad 1, a_1-(a+a), a_2-(a+a)\{a_1-(a+a)}], \ldots
\]

Adding to row (1), row(2) \( \times (n-3) \)

\[+ \text{row}(3) \times (n-4)(a_1-a) + a_r\]

\[+ \text{row}(4) \times (n-5)\{a_2-a_1(a_2-a)\} + a_r(a_2-a)\]

and so on, the above result becomes,

\[
(-1)^n \frac{a_i^2 a_r}{a_n}, \quad 1, (n-2)a, (n-3)a, \ldots \quad a_{n-3}\]

Thus if we do not know the roots of the polynomial, we may approximate to the common ratio of the G.P. we expect to get, by substituting the values of \( a_1, a_2 \) from our approximate factor in the above formula.
Example (1).

Find a quadratic factor of \( x^2 - 2x^2 - 11x + 12 \), given \( (x^2 + x - 4) \) as a first approximation.

\[
\begin{array}{c|ccc}
1 & 2 & -11 & +12 \\
-4 & -5 & -7 & +12 \\
\end{array}
\]

Actually the factors of this polynomial are,

\( (x - 1)(x + 3)(x - 4) \)

So that the factor to which we are approximating is \( (x^2 + 2x - 3) \).

Applying the relations (2) to this example we find

\[
\rho_3 = \frac{a_1}{a_3} = 1/4 \\
\rho'_3 = \frac{a_2}{a_3} = -3/4
\]

Thus the series for both \( \varepsilon_p \) and \( \varepsilon_p' \) should be convergent G.P.'s, both having the common ratio \(-.75\).

The series for \( \varepsilon_p \) and \( \varepsilon_p' \) from the actual working, and the ratio of term to term in each are as follows,

<table>
<thead>
<tr>
<th>( \varepsilon_p )</th>
<th>( \varepsilon_p / \varepsilon_{p-1} )</th>
<th>( \varepsilon_p' )</th>
<th>( \varepsilon_p' / \varepsilon_{p-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-203967</td>
<td>-194033</td>
<td>-331</td>
<td>-710</td>
</tr>
<tr>
<td>+.153530</td>
<td>-.777</td>
<td>+.161197</td>
<td>-.331</td>
</tr>
<tr>
<td>-.115029</td>
<td>-.725</td>
<td>-.114400</td>
<td>-.710</td>
</tr>
<tr>
<td>+.083665</td>
<td>-.771</td>
<td>+.088222</td>
<td>-.776</td>
</tr>
<tr>
<td>-.065096</td>
<td>-.734</td>
<td>-.065057</td>
<td>-.732</td>
</tr>
<tr>
<td>+.049620</td>
<td>-.762</td>
<td>+.049630</td>
<td>-.763</td>
</tr>
<tr>
<td>-.036761</td>
<td>-.741</td>
<td>-.036759</td>
<td>-.741</td>
</tr>
<tr>
<td>+.027626</td>
<td>-.757</td>
<td>+.027636</td>
<td>-.757</td>
</tr>
<tr>
<td>-.020725</td>
<td>-.745</td>
<td>+.020725</td>
<td>-.745</td>
</tr>
<tr>
<td>+.015625</td>
<td>-.754</td>
<td>+.015625</td>
<td>-.754</td>
</tr>
</tbody>
</table>
Example (ii).

Find a quadratic factor of \( x^4 + 11x^3 + 21x^2 - 59x - 70 \), given \( (x^2 + 7x + 4) \) as a first approximation.

\[
\begin{array}{c|ccc}
1 & +11 & +21 & -59 & -70 \\
1+7 & 4 & +17 & +28 & -11 & -75 & -70 \\
1+5.999956 +5.000282 & & & & & & & \\
\end{array}
\]

The factors of this polynomial are actually 
\[
(x+1)(x-2)(x+5)(x+7),
\]
so that the factor to which we are approximating is, 
\[
(x+1)(x+5) = x^2 + 6x + 5.
\]

Applying relations (2), we have,

\[
\rho \psi = -\frac{-1}{-14} \begin{vmatrix} 1 & 2-7 \\ 1 & -1 \end{vmatrix} = -2/7 = -0.286,
\]

\[
\rho' \psi = -\frac{-5}{-14} \begin{vmatrix} 1 & 2-7 \\ 1 & -5 \end{vmatrix} = 0.
\]

The series for \( \epsilon_{p} \), \( \epsilon'_{p} \) are as follows,

\[
\begin{array}{c|c|c|c|c|c|c}
\epsilon_{p} & \epsilon_{p} / \epsilon_{p-1} & \epsilon'_{p} & \epsilon'_{p} / \epsilon'_{p-1} \\
-0.000025 & -0.011994 & -0.000028 & -0.283 & -0.000028 & -0.283 \\
+0.000028 & +0.003457 & +0.000285 & +0.283 & +0.000285 & +0.283 \\
-0.000028 & -0.287 & -0.000028 & -0.283 & -0.000028 & -0.283 \\
+0.000028 & +0.000028 & +0.000028 & +0.283 & +0.000028 & +0.283 \\
\end{array}
\]
Example (iii).

Find a quadratic factor of \( x^5 + 5x^4 - 45x^3 + 55x^2 + 44x - 60 \) given \( (x^2 - 3x + 1) \) as a first approximation.

<table>
<thead>
<tr>
<th>l</th>
<th>5</th>
<th>-45</th>
<th>+55</th>
<th>+44</th>
<th>-60</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 3</td>
<td>+1</td>
<td>8</td>
<td>-46</td>
<td>-24</td>
<td>+47</td>
</tr>
<tr>
<td>1 - 3.5</td>
<td>+3.2</td>
<td>8.5</td>
<td>-48.2</td>
<td>-29.8</td>
<td>+27.8</td>
</tr>
<tr>
<td>1 - 3.106554+2.207011</td>
<td>+1.639344</td>
<td>7.811475-46.639344</td>
<td>-21.961767</td>
<td>-24.977577+42.194305</td>
<td>+69.380391</td>
</tr>
<tr>
<td>1 - 3.020495+2.040823</td>
<td>+2.040823</td>
<td>3.020495-47.040823</td>
<td>-24.225865</td>
<td>-22.814956+38.631589</td>
<td>+68.912467</td>
</tr>
<tr>
<td>1 - 2.990709+1.981443</td>
<td>+1.981443</td>
<td>7.990709-46.981443</td>
<td>-23.997832</td>
<td>-23.083566+39.166828</td>
<td>+68.036229</td>
</tr>
<tr>
<td>1 - 3.004375+2.008745</td>
<td>+2.008745</td>
<td>3.004375-47.008745</td>
<td>-24.048144</td>
<td>-22.960601+38.921252</td>
<td>+68.982256</td>
</tr>
<tr>
<td>1 - 2.997970+1.995941</td>
<td>+1.995941</td>
<td>2.997970-47.995941</td>
<td>-24.08642</td>
<td>-22.889401+39.738886</td>
<td>+68.810049+90.181992-60</td>
</tr>
</tbody>
</table>
The true factors in this case are,

\[(x+1)(x-1)(x-2)(x-3)(x+10)\]

So that the factor to which we are approximating is,

\[(x-1)(x-2) = x^2 - 3x + 2.\]

Applying the relations (2), we have,

\[
\begin{align*}
\rho_5 &= \frac{1}{30} \begin{vmatrix} 1 & -3 & -30+10 & -3 \\ 1 & 1 & . & . \\ . & 1 & 1 & . \\ . & . & . & . \\
\end{vmatrix} = -14/30 = -0.467 \\
\rho_5' &= \frac{2}{30} \begin{vmatrix} 1 & -3 & -30+10 & -3 \\ 1 & 2 & . & . \\ . & 1 & 2 & . \\
\end{vmatrix} = -3/15 = -0.200
\end{align*}
\]

The series for \(\epsilon_p\) and \(\epsilon'_p\) are as follows,

<table>
<thead>
<tr>
<th>(\epsilon_p)</th>
<th>(\epsilon_p/\epsilon_{p-1})</th>
<th>(\epsilon'_p)</th>
<th>(\epsilon'<em>p/\epsilon'</em>{p-1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.188525</td>
<td>-0.188525</td>
<td>-0.360656</td>
<td>-0.360656</td>
</tr>
<tr>
<td>+0.106554</td>
<td>-0.556</td>
<td>+0.207011</td>
<td>-0.574</td>
</tr>
<tr>
<td>-0.042151</td>
<td>-0.0400</td>
<td>-0.083596</td>
<td>-0.404</td>
</tr>
<tr>
<td>+0.020495</td>
<td>-0.486</td>
<td>+0.040823</td>
<td>-0.433</td>
</tr>
<tr>
<td>-0.009291</td>
<td>-0.485</td>
<td>-0.018552</td>
<td>-0.484</td>
</tr>
<tr>
<td>+0.004375</td>
<td>-0.471</td>
<td>+0.008745</td>
<td>-0.471</td>
</tr>
<tr>
<td>-0.002030</td>
<td>-0.464</td>
<td>-0.004059</td>
<td>-0.464</td>
</tr>
</tbody>
</table>
Two Theorems.

(a) The ratio \( \epsilon'_b / \epsilon_p \) tends to become equal to one or other of the roots of the quadratic factor to which we are approximating.

For, divide equation 4.1(1) by equation 4.1(2),

\[
\epsilon'_b = \frac{\epsilon'_p \left( \alpha + a - a - a \right) - \epsilon'_p \left( (a+a) \left( a - a - a \right) - a, a \right)}{\epsilon'_p \left( a, a - a, a \right) - \epsilon_p \left( a, a - a, a \right)}
\]

Now let \( E = \lim. \epsilon'_b / \epsilon_p = \lim. \epsilon'_p \theta / \epsilon_p \) as \( p \) becomes large, and we get

\[
\frac{1}{E} = \frac{E \left( \alpha + a - a - a \right) - (a+a) \left( a + a - a - a \right) + a, a}{E \left( a, a - a, a \right) - \alpha, a \left( a + a - a - a \right)}
\]

i.e. \( E^2 \left( \alpha + a - a - a \right) - E \left( a+a) \left( a + a - a - a \right) + a, a \left( a + a - a - a \right) = 0 \)

or

\( E^2 - (a, a) E + a, a = 0 \).

Thus

\( E = a_1 \) or \( a_2 \).

This proof could obviously be extended to the general case.

We note in the examples of section 4.2 the values of \( E \) at the various stages of the work.

(i) \( E = 1.000 \) for the last 3 terms of the two series, corresponding to the root, 1, of the quadratic factor.

(ii) \( E = 5.036 \) for the last 3 terms, corresponding to the root, 5, of the factor.

(iii) \( E = 1.943 \) corresponding to the root, 2, of the quadratic factor.
(b) The differences from their true values of the roots of successive approximate quadratic factors form G.P.'s which have for common ratios the two alternatives for the ratios of the G.P.'s formed by the coefficients.

That is, if \( a_1, a_2 \) are roots of a polynomial and we represent an approximate quadratic factor in the following two ways,

\[
x^2 - (a_1 + a_2 + \epsilon) x + (a_1 a_2 + \epsilon')
\]

and

\[
(x - a_1)(x - a_2)
\]

i.e.

\[
x^2 - (a_1 + a_2 + \epsilon + i\epsilon') x + (a_1 a_2 + a_1 \epsilon' + a_2 \epsilon')
\]

if we neglect \( ii' \).

Then the \( i_p \) and \( i_p' \) form G.P.'s whose common ratios are respectively \( p \) and \( p' \) as given in 4.2(2).

Taking the case of a polynomial of the fourth degree put

\[
\epsilon = i + i'
\]

\[
\epsilon' = a_1 i' + a_2 i
\]

in 4.1, and our penultimate remainder becomes

\[
x^2 - \left( a_1 + a_2 + \frac{ia_1(a_3 a_4 - a) + i'a_2(a_3 a_4 - a)}{a_1 a_2} \right) x
\]

\[
+ \left( \frac{a_1 a_2}{a_3 a_4} \right) \frac{i(a_3 a_4 - a) + i'(a_3 a_4 - a)}{a_1 a_2}
\]

i.e.

\[
x - a_1 - \frac{ia_1(a_3 a_4 - a)}{a_3 a_4} \left[ x - a_2 - \frac{i'a_2(a_3 a_4 - a)}{a_3 a_4} \right],
\]

adding a term in \( ii' \) which is negligible.

Thus we see that, i

\[
i_p = i_{p-1} \frac{a_1(a_3 a_4 - a)}{a_3 a_4} = \frac{a_1}{a_3 a_4} \frac{1}{i_{p-1}} \quad a_1 \left| a_3 + a_4 \right|
\]

\[
i_p' = i_{p-1} \frac{a_2(a_3 a_4 - a)}{a_3 a_4} = \frac{a_2}{a_3 a_4} \frac{1}{i_{p-1}} \quad a_2 \left| a_3 + a_4 \right|
\]

the same ratios as were obtained as alternatives for the \( \epsilon_p, \epsilon_p' \).
It is obvious that the processes of articles 3 and 4 could be now extended to the general case of approximating to a factor, \( P' \), of the \( m \)th degree of a polynomial, \( P \), of degree \( n \).

We do not give the proof in detail as the cases in which \( m = 1 \) and \( m = 2 \) are quite sufficient for our needs.

We may note, however, that we should obtain a recurrence relation for the \( c' \)s,

\[
\xi - \sum \xi c' + \sum \xi c - \ldots + (-1)^m \sum \xi c = 0 \quad \text{.........................} \quad (1)
\]

where \( \Sigma_c = \text{sum } r \text{ at a time of } p_n, p'_n, \ldots, p^{(m-1)}_n \).

\[
\rho^{(m)}_n = (-1)^{n-m+1} \sum \frac{a_{s+1}}{T_{n-m}}
\]

and \( T = \text{sum } r \text{ at a time of the roots of } P \text{ without those of } P' \).

Dividing (1) by \( \xi_{p-m} \) and putting \( \xi_{p}/\xi_{p-m} = \xi_{p+1}/\xi_{p+1} = \ldots = \xi_{p-n+1}/\xi_{p-n} \), in the limit when \( p \) is large, we obtain an equation of the \( m \)th degree in \( \xi_{p}/\xi_{p-1} \), whose solutions are \( p_n, p'_n, p^{(m-1)}_n \).

The value of \( \xi_{p}/\xi_{p-1} \) will be the greatest of these.

As before we will find that the ratios of the series for \( \xi_{p}', \xi_{p}'' \ldots, \xi^{(m-1)}_{p} \), connected with the \( 2^{rd}, 3^{rd}, \ldots, m^{th} \) coefficients in the approximate factor, are all the same and equal to the greatest of the above alternatives.
Example (i).
Find a cubic factor of

\[ x^6 + 25.3x^5 + 19x^4 - 1083.8x^3 + 1180x^2 + 1060x - 1200, \]
given that \((x^3 - 4x^2 - 2x + 1)\) is a first approximation.

<table>
<thead>
<tr>
<th>1</th>
<th>23.8</th>
<th>+19</th>
<th>-1063.8</th>
<th>+1180</th>
<th>+1060</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>27.8</td>
<td>+21</td>
<td>-1084.8</td>
<td>+27.8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-111.2</td>
<td></td>
<td>-55.6</td>
<td>+1152.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+132.2</td>
<td></td>
<td>-1029.2</td>
<td>+132.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-528.8</td>
<td></td>
<td>-264.4</td>
<td>+927.8</td>
<td></td>
</tr>
<tr>
<td>-2.3</td>
<td>26.6</td>
<td>+20.9</td>
<td>-1086.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-74.5</td>
<td></td>
<td>-50.5</td>
<td>+63.8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+95.4</td>
<td></td>
<td>-1035.7</td>
<td>+1116.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-267.1</td>
<td></td>
<td>-181.3</td>
<td>+229.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-768.6</td>
<td></td>
<td>+1297.5</td>
<td>+681.0</td>
<td></td>
</tr>
<tr>
<td>-1.7</td>
<td>25.5</td>
<td>+20.9</td>
<td>-1085.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>+43.4</td>
<td></td>
<td>-28.1</td>
<td>+40.8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+64.3</td>
<td></td>
<td>-1057.3</td>
<td>+1159.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-109.3</td>
<td></td>
<td>-70.7</td>
<td>+102.9</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-948.0</td>
<td></td>
<td>+1209.9</td>
<td>+957.1</td>
<td></td>
</tr>
</tbody>
</table>

\[
\text{The series for } \varepsilon_p, \varepsilon'_p, \varepsilon''_p \text{ are as follows, together with their respective ratios:—}
\]

<table>
<thead>
<tr>
<th>(\varepsilon_p)</th>
<th>(\varepsilon'_p)</th>
<th>(\varepsilon''_p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.076266</td>
<td>0.009599</td>
<td>0.006583</td>
</tr>
<tr>
<td>0.002790</td>
<td>0.116</td>
<td>0.003800</td>
</tr>
<tr>
<td>0.000949</td>
<td>0.108</td>
<td>0.000860</td>
</tr>
</tbody>
</table>

It being easily verified that the factor to which we are approximating is, \(x^3 - 1.2x^2 - x + 1.2\).
The factors of the polynomial are actually:

\[(x-1)(x+1)(x-1.2)(x-5)(x+10)(x+20)\]

So that, applying formula (2), we have:

\[
\begin{align*}
\rho' &= \frac{-1}{1000} \left| \begin{array}{ccc}
1 & -25 & +50 \\
1 & -1 \\
\cdot & 1 & -1
\end{array} \right| = -\frac{26}{1000} = -0.026, \\
\rho'' &= \frac{+1}{1000} \left| \begin{array}{ccc}
1 & -25 & +50 \\
1 & +1 \\
\cdot & 1 & +1
\end{array} \right| = +\frac{76}{1000} = +0.076, \\
\rho''' &= \frac{1.2}{1000} \left| \begin{array}{ccc}
1 & -25 & +50 \\
1 & 1.2 \\
\cdot & 1 & 1.2
\end{array} \right| = +\frac{97.728}{1000} = +0.098
\end{align*}
\]
6. COMPLEX ROOTS IN THE CASE OF A QUADRATIC DIVISOR.

It will be seen in this section that the above work applies only to divisors which are entirely real. We will find that if we wish to approximate to a complex factor, the same method indeed will be used, but the series obtained will be of a different character.

Consider the following example;—

Find a quadratic factor of \( x^3 - 9x^2 + 35x - 65 \), given \((x^2 - 3x + 7)\) as a first approximation.

\[
\begin{array}{c|ccc}
1-3 & 9 & +33 & -65 \\
-3 & 1-4.333 & 1+0.333 & +2.667 \\
-4 & 1-4.750 & 1+3.928 & +4.512 \\
-5 & 1-4.433 & 1+1.294 & +5.076 \\
-6 & 1-3.924 & 1+4.406 & +5.337 \\
-7 & 1-3.663 & 1+2.805 & +5.816 \\
-8 & 1-3.734 & 1+1.179 & +6.007 \\
-9 & 1-3.992 & 1+2.462 & +6.398 \\
-10 & 1-4.101 & 1+2.512 & +6.398 \\
-11 & 1-4.015 & 1+2.991 & +6.648 \\
-12 & 1-3.966 & 1+3.099 & +6.935 \\
-13 & 1-3.955 & 1+3.446 & +7.265 \\
-14 & 1-3.935 & 1+3.920 & +7.615 \\
-15 & 1-3.905 & 1+4.319 & +7.995 \\
-16 & 1-3.865 & 1+4.721 & +8.395 \\
-17 & 1-3.820 & 1+5.128 & +8.815 \\
-18 & 1-3.763 & 1+5.532 & +9.245 \\
-19 & 1-3.695 & 1+5.946 & +9.685 \\
-20 & 1-3.618 & 1+6.367 & +10.125 \\
-21 & 1-3.531 & 1+6.794 & +10.576 \\
-22 & 1-3.435 & 1+7.225 & +11.037 \\
-23 & 1-3.330 & 1+7.661 & +11.498 \\
-24 & 1-3.216 & 1+8.102 & +11.960 \\
-25 & 1-3.094 & 1+8.547 & +12.422 \\
-26 & 1-2.963 & 1+8.996 & +12.885 \\
-28 & 1-2.673 & 1+9.897 & +13.813 \\
\end{array}
\]

It may be verified that the factor to which we are approximating is \((x^2 - 4x + 13)\).
Graph to accompany example 6.1.

Series for $\varepsilon_p$, thus,  

Series for $\varepsilon'_p$, thus,  

The series for the $\varepsilon_p$, $\varepsilon'_p$ and the corresponding ratios of term to term are as follows,

<table>
<thead>
<tr>
<th>$\varepsilon_p$</th>
<th>$\varepsilon_p'/\varepsilon_{p-1}$</th>
<th>$\varepsilon'_p$</th>
<th>$\varepsilon'<em>p'/\varepsilon'</em>{p-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-.033513</td>
<td></td>
<td>-.034777</td>
<td></td>
</tr>
<tr>
<td>+.034777</td>
<td>+1.038</td>
<td>-.036554</td>
<td>+1.037</td>
</tr>
<tr>
<td>-.010453</td>
<td>+.300</td>
<td>-.039796</td>
<td>+.302</td>
</tr>
<tr>
<td>+.009589</td>
<td>-.919</td>
<td>-.027032</td>
<td>+.922</td>
</tr>
<tr>
<td>+.013113</td>
<td>+1.568</td>
<td>+.024979</td>
<td>-.922</td>
</tr>
<tr>
<td>+.005509</td>
<td>+.420</td>
<td>+.034133</td>
<td>+1.368</td>
</tr>
<tr>
<td>-.002452</td>
<td>-.441</td>
<td>+.014339</td>
<td>+.419</td>
</tr>
<tr>
<td>-.004811</td>
<td>+1.978</td>
<td>-.006520</td>
<td>+.441</td>
</tr>
<tr>
<td>-.002582</td>
<td>+.537</td>
<td>-.013497</td>
<td>+1.977</td>
</tr>
<tr>
<td>+.000434</td>
<td>-.168</td>
<td>-.006710</td>
<td>+.537</td>
</tr>
<tr>
<td>+.001689</td>
<td>+.392</td>
<td>+.001123</td>
<td>-.168</td>
</tr>
<tr>
<td>+.001126</td>
<td>+.667</td>
<td>+.002928</td>
<td>+.667</td>
</tr>
</tbody>
</table>

It will be seen that, in this example, our series for $\varepsilon_p$ and $\varepsilon'_p$ are not G.P.'s, but are both 'damped' oscillating series as shown in the accompanying diagram, converging slowly to the required coefficients. Both series are apparently of the same form, (as is shown by the two sets of ratios $\varepsilon_p/\varepsilon_{p-1}$ and $\varepsilon'_p/\varepsilon'_{p-1}$), but differ in amplitude and epoch.

It appears, then, that we are not justified in assuming G.P.'s for our series when the factor to which we are approximating contains complex roots, for the factors of the polynomial in this case are,

$$(x^2 - 4x + 13)(x - 5)$$

i.e. $(x - 2 + 3i)(x - 2 - 3i)(x - 5)$.

We note, however, that the relations 4.2 (1), (4) are approximately satisfied, for we have in this case,

$$\varepsilon'_p - \frac{a_1 + a_2}{a_3} \varepsilon'_n + \frac{a_1 a_2}{a_3^2} \varepsilon'_m =$$
\[
\frac{4}{5} \cdot 0.004393 + \frac{13}{25} \cdot 0.001128 = 0.002928
\]
\[
= 0.002928 - 0.003512 + 0.000587 = 0.000003
\]

and
\[
\epsilon_{1} = \frac{a_{i} + a_{j}}{\alpha_{b}} \epsilon_{11} + \frac{a_{i} a_{j}}{\alpha_{b}} \epsilon_{10}
\]
\[
\frac{4}{5} \cdot 0.001126 = 0.001639 + \frac{13}{25} \cdot 0.000434 = 0.001126 - 0.001351 + 0.00026 = 0.000001
\]

In fact, in the general case of a polynomial of degree \( n \), the recurrence relation

\[
\epsilon_{p} - (\rho_{n} + \rho'_{n}) \epsilon_{p-1} + \rho_{n} \rho'_{n} \epsilon_{p-2} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)
\]

holds whether the roots \( \alpha_{i}, \alpha_{j} \) be real or complex, for, up to this point, \( \alpha_{i}, \alpha_{j} \) always occur as either their sum or their product, so that imaginary quantities do not appear. Also no limits have been taken, and no assumptions made, except that \( \epsilon_{p}, \epsilon'_{p} \) be small.

At this point, however, we took the limit of the ratio \( \epsilon_{p} / \epsilon_{p-1} \) and found it to be equal to \( \rho_{n} \) or \( \rho'_{n} \). But both \( \rho_{n} \) and \( \rho'_{n} \) are \textit{complex} if \( \alpha_{i}, \alpha_{j} \) be a complex pair. Thus, although all the terms in the series above are real, it would appear that the ratio of one term to the next is complex. The explanation of this obvious fallacy is that we are not justified in taking the limit of \( \epsilon_{p} / \epsilon_{p-1} \) when \( \alpha_{i}, \alpha_{j} \) are complex.
For no such limit exists in this case, as is clearly demonstrated by the oscillatory nature of the terms in the above example.

Let us, then, consider the case when the roots of equation (1) are complex.

Let them be, \( r(\cos \mu \pm i \sin \mu) \)

i.e. let \( \rho_n = r(\cos \mu - i \sin \mu) \)

and \( \rho_n' = r(\cos \mu + i \sin \mu) \)

Equation (1) then becomes,

\[
\epsilon_p - 2r \cos \mu \epsilon_{p-1} + r^2 \epsilon_{p-2} = 0 \quad \ldots \quad (2)
\]

This equation is satisfied if we put,

\[
\epsilon_p = Ar^p \cos (p \mu + \xi).
\]

For then we have,

\[
\frac{1}{r \epsilon_{p-2}} = \frac{\cos(p-2 \mu + \xi)}{\cos(p-3 \mu + \xi)} \quad (a)
\]

\[
\frac{1}{r^2 \epsilon_{p-3}} = \frac{\cos(p-1 \mu + \xi)}{\cos(p-3 \mu + \xi)} \quad (b)
\]

\[
\frac{1}{r^3 \epsilon_{p-4}} = \frac{\cos(p \mu + \xi)}{\cos(p-3 \mu + \xi)} \quad (c)
\]

Adding (a) and (c),

\[
\frac{1}{r \epsilon_{p-2}} \left[ \frac{1}{\epsilon_{p-2}} + \frac{1}{r^2 \epsilon_p} \right] = \frac{2 \cos(p-1 \mu + \xi) \cos \mu}{\cos(p-3 \mu + \xi)}
\]

which, from (b),

\[
= 2 \cdot \frac{1}{r^2} \frac{\epsilon_{p-1}}{\epsilon_{p-2}} \cdot \cos \mu.
\]

Therefore,

\[
\epsilon_p - 2r \cos \mu \epsilon_{p-1} + r^2 \epsilon_{p-2} = 0.
\]
We note also that,

\[ \rho_n \rho_n' = r^2 \]

\[ \frac{\rho_n - \rho_n'}{\rho_n + \rho_n'} = -i \tan \mu. \]

Thus we may represent the terms of our series by,

\[ e_p = Ar \cos(\mu + \xi) \]

where,

\[ r = \sqrt{\frac{\rho_n - \rho_n'}{\rho_n + \rho_n'}} \]

and

\[ \tan \mu = i \frac{\rho_n - \rho_n'}{\rho_n + \rho_n'} \]

\( A, \xi \) are terms connected with the amplitude and epoch respectively of the oscillation, and are to be determined from two terms of the series itself in any particular case.

6.3 To find the values of \( A, \xi \) in any particular case.

From 6.2 (3) we have,

\[ e_0 = A \cos \xi. \]

\[ e_1 = A r \cos(\mu + \xi) \]

\[ = A r \{ \cos \mu \cos \xi - \sin \mu \sin \xi \} \]

\[ = A \sqrt{\frac{\rho_n - \rho_n'}{\rho_n + \rho_n'}} \left\{ \frac{e_0 (\rho_n + \rho_n')}{A} \cdot \frac{1}{2} \left( 1 - \frac{e_0^2}{A^2} \cdot \frac{\rho_n - \rho_n'}{2\sqrt{\rho_n \rho_n'}} \right) \right\} \]

\[ = \frac{1}{2} \left\{ e_0 (\rho_n + \rho_n') + \sqrt{A^2 - e_0^2} (\rho_n - \rho_n') \right\} \]

i.e. \( \sqrt{A^2 - e_0^2} = \frac{2e_1 - e_0 (\rho_n + \rho_n')}{(\rho_n - \rho_n')} \)

Therefore, \( A^2 = e_0^2 \left( \frac{2e_1 - e_0 (\rho_n + \rho_n')}{\rho_n - \rho_n'} \right)^2 \)

and \( \cos \xi = \frac{e_0}{A} \).

where \( e_0, e_1 \) are any two consecutive terms in the series.
For instance, in the example of 6.1, we have,

\[
\rho_3 = \frac{2 - 31}{5}; \quad \rho'_3 = \frac{2 + 31}{5}.
\]

Thus

\[
r = \sqrt{\rho_3 \rho'_3} = \sqrt{15}/5 = .7211
\]

\[
\tan \mu = i \frac{\rho_3 - \rho'_3}{\rho_3 + \rho'_3} = \frac{1(-61)}{4} = 1.5
\]

\[
= \tan 56^\circ 19'.
\]

Now take as our two consecutive terms in the series for \( \epsilon'_0 \),

\[
\epsilon'_0 = -6710 \quad \text{i.e. the 4th and 3rd terms from the end},
\]

\[
\epsilon'_1 = +1128
\]

Then

\[
A^2 = 45024100 - \left( \frac{2256 + \frac{1}{5}6710}{-61/5} \right)^\nu
\]

\[
= 45024100 + \left[ \frac{5640 + 13420}{3} \right]^\nu
\]

\[
= 85383944.
\]

\[
\therefore \quad A = 9240.6
\]

\[
\therefore \quad \cos \xi = \frac{-6710}{-9240.6} = .7261 = \cos 43^\circ 27'.
\]

We want \( \xi \) to have its principal value, therefore we take the sign of \( A \) to be the same as that of \( \epsilon'_0 \).

Thus,

\[
\mu = 56^\circ 19',
\]

\[
\mu + \xi = 99^\circ 46' = 180^\circ - 23^\circ 55'
\]

\[
'2\mu + \xi = 156^\circ 5' = 180^\circ - 23^\circ 55'
\]

\[
3\mu + \xi = 212^\circ 24' = 180^\circ + 32^\circ 24'
\]

\[
\therefore \mu + \xi = -12^\circ 52',
\]

\[
-2\mu + \xi = -69^\circ 11'.
\]
Also, \( \log r = 1.8580 \)
\( \log A = 3.9657. \)

Now, \( \log \epsilon' = \log A + p \log r + \log \cos(p\mu + \xi). \)
Thus, \( \log \epsilon' = 3.9657 + 1.8580 + 1.2290 \)
\[
= \log 3.0527
= \log 1129.
\]
\( \log \epsilon'_2 = 3.9657 + 1.7160 + 1.9610 \)
\[
= 3.6427
= \log 4392.
\]
\( \log \epsilon'_3 = 3.9657 + 1.5740 + 1.9265 \)
\[
= 3.4562
= \log 2925.
\]
\( \log \epsilon'_4 = 3.9657 + 0.1420 + 1.9890 \)
\[
= 4.0967
= \log 12490.
\]
\( \log \epsilon'_5 = 3.9657 + 0.2840 + 1.5507 \)
\[
= 3.8004
= \log 6316.
\]
Thus, \( \epsilon'_1 = +1129 \)
\( \epsilon'_2 = +4392 \)
\( \epsilon'_3 = +2925 \)
\( \epsilon'_4 = -12490 \)
\( \epsilon'_5 = -6316. \)

Continuing in this way, we could reproduce the whole series for \( \epsilon'_p. \)

The accuracy of the results may be checked by reference to the example itself.

Similar calculations could be performed for the series for \( \epsilon_p \) in the same example.
### Example

Find the quadratic factor of

\[ x^4 + x^3 - 9x^2 - 29x - 60, \]

to which \((x^2 + 3x + 5.4)\) is an approximation.

<table>
<thead>
<tr>
<th>1</th>
<th>+1</th>
<th>+9</th>
<th>-29</th>
<th>-60</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 + 3 + 5.4</td>
<td>-2</td>
<td>-14.4</td>
<td>-6</td>
<td>-18.2</td>
</tr>
<tr>
<td>1 + 2.11666 + 5.16254</td>
<td>-1.11666</td>
<td>-14.16254</td>
<td>-2.36359</td>
<td>-11.79256</td>
</tr>
<tr>
<td>1 + 1.96926 + 5.08520</td>
<td>-1.96926</td>
<td>-14.08520</td>
<td>-1.90372</td>
<td>-12.17643</td>
</tr>
<tr>
<td>1 + 2.01614 + 5.00148</td>
<td>-2.01614</td>
<td>-14.00148</td>
<td>-2.04868</td>
<td>-11.95280</td>
</tr>
<tr>
<td>1 + 2.001021 + 5.019744</td>
<td>-1.001021</td>
<td>-14.019744</td>
<td>-2.003064</td>
<td>-12.016680</td>
</tr>
<tr>
<td>1 + 2.000901 + 5.002981</td>
<td>-1.000901</td>
<td>-14.002981</td>
<td>-2.002704</td>
<td>-12.000277</td>
</tr>
</tbody>
</table>
and the last reduced remainder is,

\[ x^2 + 2.000196x + 5.000295. \]

The factors of the polynomial are actually,

\[ (x + 1 + 2i)(x + 1 - 2i)(x + 3)(x - 4) \]

so that the factor to which we are approximating is

\[ (x + 1 + 2i)(x + 1 - 2i) = x^2 + 2x + 5. \]

The series for \( \xi_p \) and \( \xi'_p \) are as follows,

\[
\begin{array}{ll}
\xi_p & \xi'_p \\
.11666 & .16254 \\
- .03074 & + .03520 \\
- .02315 & - .07247 \\
+ .01614 & + .00148 \\
+ .001021 & + .019784 \\
- .004846 & = \xi_a \\
+ .001331 & = \xi_b \\
+ .000901 & = \xi_c \\
- .000670 & = \xi_d \\
- .000287 & = \xi_e \\
+ .000196 & = \xi_f \\
\end{array}
\]

These series conform to the equation,

\[ \xi_p = \text{Ar}^p \cos(p\mu + \xi) \]

as may be shown by taking, for example,

\[
\begin{align*}
\xi_0 &= +1331 \\
\xi_1 &= +901 \\
\end{align*}
\]

the 5th and 4th terms from the end.

We have

\[ \rho_4 = -\frac{1 - 2i}{-12} \begin{vmatrix} 1 & 1 \\ 1, & 1-2i \end{vmatrix} = \frac{1 - 3i}{-6} \]

and

\[ \rho'_4 = -\frac{1 + 2i}{-12} \begin{vmatrix} 1 & 1 \\ 1, & 1+2i \end{vmatrix} = \frac{1 + 3i}{-6} \]

Thus,

\[ r = \sqrt{\rho_4 \rho'_4} = \frac{\sqrt{10}}{-6} = -0.5270 \]

\[ \tan \mu = \frac{i \rho_4 - \rho'_4}{\rho_4 + \rho'_4} = i \frac{6i}{2} = 3.0 = \tan 71.34^\circ. \]
\[ A^2 = (1331)^2 - \left( \frac{1802 + \frac{1}{2} \times 1331}{61/6} \right) \]
\[ = 6814580. \]

\[ A = + 2610.47 \quad \text{(taking the +ve. sign since } \epsilon_1 \text{ is +ve.)} \]

\[ \cos \xi = \frac{1331}{2610.47} = 0.5099 = \cos 59^\circ 20'. \]

Thus, \( \mu = 71^\circ 34' \)

\( \xi + \mu = 130^\circ 54' = 180^\circ - 49^\circ 6' \)
\( \xi + 2\mu = 202^\circ 28' = 180^\circ + 22^\circ 28' \)
\( \xi + 3\mu = 274^\circ 2' = 360^\circ - 35^\circ 58' \)
\( \xi + 4\mu = 346^\circ 36' = 360^\circ - 14^\circ 24' \)
\( \xi - \mu = -12^\circ 14'. \)

Also, \( \log A = 5.4167 \)

\[ \log (-r) = 1.7218. \]

Hence we have

\[ \log \epsilon_1 = 3.4167 + 1.7281 + 1.8161 = 2.9546 \]
\[ = \log 900.7. \]
\[ \log \epsilon_2 = 3.4167 + 1.4436 + 1.9657 = 2.8260 \]
\[ = \log 669.9. \]
\[ \log \epsilon_3 = 3.4167 + 1.1654 + 2.6479 = 1.4300 \]
\[ = \log 26.92. \]
\[ \log \epsilon_4 = 3.4167 + 0.8872 + 1.9861 = 2.2900 \]
\[ = \log 195.0. \]
\[ \log \epsilon_5 = 3.4167 + 0.2732 + 1.9900 = 3.5849 \]
\[ = \log 4341. \]
From which, taking account of the signs of the constituent terms, we obtain,

\[
\begin{align*}
\epsilon_1 &= + 901 \\
\epsilon_2 &= - 670 \\
\epsilon_3 &= - 27 \\
\epsilon_4 &= + 195 \\
\epsilon_5 &= - 4341
\end{align*}
\]

and so on, the terms all agreeing to six places of decimals with the corresponding terms obtained in the example.
Thus we have,

\[
\Delta^{n-1} b_p
\]

Now this is an approximate value of \( a \), since our assumption that \( \Delta e_p / e_p = p = \Delta e_{p+1} / e_p \), was only approximately true.

Hence we are led to consider the series of approximations to \( a \) given by,

\[
\Delta^{n-1} b_p
\]

Dr. Aitken has shown in his articles that the series of first differences of these \( b_p \) also tends to become a G.P. whose ratio is the square of the ratio of the original series.

Thus the convergence of the derived series is more rapid than that of the original series, and we find also that it commences at a more advanced stage of approximation.

It can also be shown that from this first derived series we may form in the same manner a second derived series of still more rapid convergence, and so on, but in practice it will be found unnecessary to go beyond the first derived series.

It will be seen also that, if we put \( (c + b_p) \) instead of \( b_p \), where \( c \) is any constant, and denote by an operator \( P \), the process of forming the above determinant and dividing it by \( \Delta b_p \), then,

\[
P(c + b_p) = c + P(b_p)
\]
So that before applying the $\delta^2$-process we may first subtract from all terms of the original series any convenient constant, or, in particular, remove all figures common to the terms already obtained.

We now apply the $\delta^2$-process to improve the approximation in the various examples already worked in sections 3, 4, and 5.

3.1 Example(i), P. 44

<table>
<thead>
<tr>
<th>$b_p$</th>
<th>$\Delta b_p$</th>
<th>$\Delta^2 b_p$</th>
<th>$b_p - 3.0$</th>
<th>$b'_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>164282</td>
<td>101533</td>
<td>62481</td>
<td>-0.000711</td>
<td>2.999929</td>
</tr>
<tr>
<td>062749</td>
<td>39052</td>
<td>24265</td>
<td>-0.000101</td>
<td>2.999990</td>
</tr>
<tr>
<td>023697</td>
<td>14787</td>
<td>9222</td>
<td>-0.000013</td>
<td>2.999999</td>
</tr>
<tr>
<td>003345</td>
<td>5565</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The value of $\rho$ expected is $(.375)^2 = .141$.

3.1 Example(ii), P. 44

<table>
<thead>
<tr>
<th>$b_p$</th>
<th>$\Delta b_p$</th>
<th>$\Delta^2 b_p$</th>
<th>$b_p - 4$</th>
<th>$b'_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>148870</td>
<td>60745</td>
<td>24474</td>
<td>-0.001900</td>
<td>3.998100</td>
</tr>
<tr>
<td>088125</td>
<td>36271</td>
<td>14820</td>
<td>-0.000646</td>
<td>3.999354</td>
</tr>
<tr>
<td>051354</td>
<td>21451</td>
<td>8836</td>
<td>-0.000222</td>
<td>3.999778</td>
</tr>
<tr>
<td>030403</td>
<td>12615</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>017788</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The value $\rho$ expected is $(.583)^2 = .340$.

The remaining calculation, being performed directly on a machine, needs no tabulation.
**4.2 Example (i), P. 6.**

<table>
<thead>
<tr>
<th>$b_p$</th>
<th>$\Delta b_p$</th>
<th>$\Delta^2 b_p$</th>
<th>$b'_p$</th>
<th>$\Delta b'_p$</th>
<th>$\Delta^2 b'_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-194033</td>
<td>-355230</td>
<td>-630827</td>
<td>+.006003</td>
<td>3.006003</td>
<td></td>
</tr>
<tr>
<td>+161197</td>
<td>+275597</td>
<td>+478824</td>
<td>+.002572</td>
<td>3.002572</td>
<td></td>
</tr>
<tr>
<td>-114400</td>
<td>-203227</td>
<td>-257111</td>
<td>+.001254</td>
<td>3.001254</td>
<td></td>
</tr>
<tr>
<td>+088827</td>
<td>+153334</td>
<td>+268571</td>
<td>+.000656</td>
<td>3.000656</td>
<td></td>
</tr>
<tr>
<td>-065057</td>
<td>-114687</td>
<td>-149630</td>
<td>+0.002572</td>
<td>3.002572</td>
<td></td>
</tr>
</tbody>
</table>

The square of the ratio of the original series in this case is, $(.75)^2 = .5625$.

The above series is derived from the $\epsilon_p$. A similar series could, of course, be derived from the $\epsilon_p$.

**4.2 Example (ii), P. 6.**

<table>
<thead>
<tr>
<th>$b_p$</th>
<th>$\Delta b_p$</th>
<th>$\Delta^2 b_p$</th>
<th>$b'_p$</th>
<th>$\Delta b'_p$</th>
<th>$\Delta^2 b'_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+20882</td>
<td>+23217</td>
<td>+26833</td>
<td>+.00344</td>
<td>6.000344</td>
<td></td>
</tr>
<tr>
<td>-02325</td>
<td>+03016</td>
<td>+03905</td>
<td>+.00004</td>
<td>6.000004</td>
<td></td>
</tr>
<tr>
<td>+00691</td>
<td>+00839</td>
<td>-00193</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**5.0 Example (i), P. 6.**

<table>
<thead>
<tr>
<th>$b_p$-1.2</th>
<th>$\Delta b_p$</th>
<th>$\Delta^2 b_p$</th>
<th>$b'_p$-1.2</th>
<th>$\Delta b'_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+76286</td>
<td>67476</td>
<td>+8635</td>
<td>-.000032</td>
<td>1.199918</td>
</tr>
<tr>
<td>+08790</td>
<td>07541</td>
<td>59635</td>
<td></td>
<td></td>
</tr>
<tr>
<td>+00949</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Approximation to Complex Roots.

If $a$ is complex we cannot put $\frac{\varepsilon_p^*/\varepsilon_p}{\Delta \varepsilon_{p+1}/\Delta \varepsilon_p} = \rho = \Delta \varepsilon_{p+1}/\Delta \varepsilon_p$, as in 7.1, because we now have a cosine factor entering as explained in section 6 above.

In this case we have, as in 6.2,

$$\varepsilon_{p+2} - 2r \cos \mu \varepsilon_{p+1} + r^2 \varepsilon_p = 0$$

whence, \(\cos \mu = \frac{\varepsilon_{p+2} + r^2 \varepsilon_p}{2r \varepsilon_{p+1}}\)

$$= \frac{r^2(b_p - a) + (b_{p+2} - a)}{2r(b_{p+1} - a)}$$

Now, putting \((p + 1)\) for \(p\) throughout, we have,

\[
\cos \mu = \frac{r^2(b_{p+1} - a) + (b_{p+3} - a)}{2r(b_{p+2} - a)}
\]

\[
\therefore \frac{r^2(b_p - a) + (b_{p+2} - a)}{b_{p+1} - a} = \frac{r^2(b_{p+1} - a) + (b_{p+3} - a)}{b_{p+2} - a}
\]

i.e. \(r^2[(b_{p+1} - a)^2 - (b_p - a)(b_{p+2} - a)] = (b_{p+2} - a)^2 - (b_{p+1} - a)(b_{p+3} - a)\).

i.e. \(r^2\left\{a^2b_p - \begin{vmatrix} b_p & b_{p+1} \\ b_{p+1} & b_{p+2} \end{vmatrix}\right\} = a^2b_{p+1} - \begin{vmatrix} b_{p+1} & b_{p+2} \\ b_{p+1} & b_{p+2} \end{vmatrix}\)

Thus, \(a = \frac{r^2 \left\{b_p b_{p+1} - \begin{vmatrix} b_{p+1} & b_{p+2} \\ b_{p+1} & b_{p+2} \end{vmatrix}\right\}}{r^2 \Delta^2 b_p - \Delta^2 b_{p+1}}\)

Now this value of \(a\) is again an approximation, since it depends on the assumption that,

\(\varepsilon_p = A \cdot r^p \cos(p \mu + \xi)\),

which was itself an approximation.
We are thus led to consider the series of approximations to a given by,

\[ b'_p = \frac{r^2 \begin{vmatrix} b_p & b_{p+1} \\ b_{p+1} & b_{p+2} \end{vmatrix}}{r^2 \Delta^2 b_p - \Delta^2 b_{p+1}} \]

And this again, as indicated in Dr. Aitken's articles, is more rapidly convergent and starts at a more advanced stage of approximation than the original series for \( b_p \).

If we denote the above formula by,

\[ b'_p = P(b_p) \]

and add a constant, \( c \), to \( b'_p \), then we have as before,

\[ P(c + b_p) = c + P(b_p) \]

so that as before we may subtract from the series of \( b'_p \)'s any convenient constant before forming the derived series of \( b'_p \)'s.

We now apply this variation of the \( c^2 \)-process to the examples worked above in section 6.

**6.1 Example (i), P.72**

We take six terms of the series for \( \xi_p \), and from them derive three terms of a new series, which will be found to be closer approximations to the true root than were the original terms.

A similar procedure could, of course, be applied to the series for \( \xi'_p \).

We recall that, \( r^2 = 15/25 = .52 \)
<table>
<thead>
<tr>
<th>$b_p$</th>
<th>$\Delta b_p$</th>
<th>$\Delta^2 b_p$</th>
<th>$r^2 \Delta^2 b_p$</th>
<th>$\frac{b_p b_{pm}}{b_{pm} b_{pm-1}}$</th>
<th>$b_p^{\prime} - 4$</th>
<th>$b_p^{\prime}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+.101357</td>
<td>+14966</td>
<td>-56206</td>
<td>-51167</td>
<td>-00594896</td>
<td>+001282</td>
<td>4.001282</td>
</tr>
<tr>
<td>+.086891</td>
<td>+71172</td>
<td>+21940</td>
<td>-36559</td>
<td>-00315907</td>
<td>-000740</td>
<td>3.999260</td>
</tr>
<tr>
<td>+.015719</td>
<td>+49232</td>
<td>+47968</td>
<td>-00660</td>
<td>-00166978</td>
<td>+000131</td>
<td>4.000131</td>
</tr>
<tr>
<td>-.033513</td>
<td>+1264</td>
<td>+25603</td>
<td>-24339</td>
<td>-0085963</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-.034777</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-.010438</td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The remainder of the work, being performed directly on a calculating machine, needs no tabulation.

6.4 Example (i), P. 79.

We take this time the series for $\varepsilon_p$, and derive a second series from six of its terms.

We recall that, \[ r^2 = \frac{40}{144} = .2778. \]