Well-posedness of the one-dimensional derivative nonlinear Schrödinger equation

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Abstract

This thesis is concerned with the well-posedness of the one-dimensional derivative nonlinear Schrödinger equation (DNLS). In particular, we study the initial-value problem associated to DNLS with low-regularity initial data in two settings: (i) on the torus (namely with the periodic boundary condition) and (ii) on the real line.

Our first main goal is to study the global-in-time behaviour of solutions to DNLS in the periodic setting, where global well-posedness is known to hold under a small mass assumption. In Chapter 2, we relax the smallness assumption on the mass and establish global well-posedness of DNLS for smooth initial data. In Chapter 3, we then extend this result for rougher initial data. In particular, we employ the I-method introduced by Colliander, Keel, Staffilani, Takaoka, and Tao and show the global well-posedness of the periodic DNLS at the end-point regularity. In the implementation of the I-method, we apply normal form reductions to construct higher order modified energy functionals.

In Chapter 4, we turn our attention to the uniqueness of solutions to DNLS on the real line. By using an infinite iteration of normal form reductions introduced by Guo, Kwon, and Oh in the context of one-dimensional cubic NLS on the torus, we construct solutions to DNLS without using any auxiliary function space. As a result, we prove the unconditional uniqueness of solutions to DNLS on the real line in an almost end-point regularity.

Lay summary

Nonlinear Schrödinger equations arise naturally as models describing wave phenomena in various branches of physics. They belong to the class of nonlinear dispersive partial differential equations, which are broadly characterized by the property that solutions tend to spread out spatially (disperse) as time evolves.

Among the important problems in mathematical analysis, but also of interest in other fields (such as applied mathematics and theoretical physics) are the *local well-posedness* of these equations. Roughly speaking, by local well-posedness we mean that for given initial data, there *exists* a *unique* solution solving the equation which satisfies the initial condition, and moreover we have *stability* under perturbations of initial data (that is, a small change in the initial data incurs a small change in the solution). Once these desired properties are known to hold, we can begin to deepen our understanding of the behavior of solutions to such equations. For example, one might very much be interested to rule out the possibility that some solutions develop pathological behavior in finite time (and thus establish *global-in-time well-posedness*), or as it is the case when using numerical simulations, one might want to guarantee the uniqueness of solutions among the largest possible class of solutions (that is the *unconditional well-posedness*). Furthermore, for Schrödinger-type equations, such properties of solutions might depend non-intuitively on the underlying domain of the physical variables.

In this work, we study the one-dimensional derivative nonlinear Schrödinger equation (DNLS) which is used as a model equation in plasma physics. In particular, we focus on showing: (i) the global-in-time well-posedness of DNLS on the circle with relaxed conditions both on the size (Chapter 2) and on the regularity of initial data (Chapter 3), and (ii) the unconditional well-posedness of DNLS on the real line (Chapter 4).

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Contents

Notation

$\mathbb{R}$: the set of real numbers

$\mathbb{Z}$: the set of integer numbers

$\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$

$\mathbb{T}_\lambda := \mathbb{R}/2\pi\lambda\mathbb{Z}$ $(\lambda > 0)$

$\mathbb{Z}_\lambda := \frac{1}{\lambda}\mathbb{Z}$ $(\lambda > 0)$

$\#A$: the number of elements of the finite set A

$d\#$: the counting measure on $\mathbb{Z}$

$[x]$: the largest integer smaller than x

$\langle x\rangle := \sqrt{1+|x|^2}$

$\mathrm{Re}(z)/\mathrm{Im}(z)$: the real/imaginary part of $z \in \mathbb{C}$

$\displaystyle ⨍_{\mathbb{T}_\lambda} [\,\cdot\,]dx := \frac{1}{2\pi\lambda}\int_{\mathbb{T}_\lambda} [\,\cdot\,]dx$ $(\lambda > 0)$

$\mathcal{F}$ or $\mathcal{F}_x$: the Fourier transform operator in the spatial variable

$\mathcal{F}_{t,x}$: the Fourier transform operator in both the temporal and spatial variables

$\mathcal{S}(\mathbb{R})$: the class of Schwartz functions on $\mathbb{R}$

$\mathcal{S}(\mathbb{T}_\lambda)$: the class of $2\pi\lambda$-periodic C^∞-functions on $\mathbb{T}_\lambda$

J_x^s: $\mathcal{F}(J_x^s f)(\xi) = \langle\xi\rangle^s\mathcal{F}(f)(\xi)$

D_x^s: $\mathcal{F}(D_x^s f)(\xi) = |\xi|^s\mathcal{F}(f)(\xi)$

$A = o(B)$ (litle-o): $\lim_{x\to\infty}\frac{A(x)}{B(x)} = 0$

$A = O(B)$ (big-o): there exist $C > 0$ and x_0 such that $|A(x)| \le CB(x)$ for all $x \ge x_0$

η: compactly supported C^∞-function

Chapter 1

Introduction

This thesis studies some of the well-posedness properties of the initial-value problem for the one-dimensional derivative nonlinear Schrödinger equation (DNLS):

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x(|u|^2 u)\ , \quad (t,x) \in \mathbb{R}\times\mathbb{M}\,, \\ u|_{t=0} = u_0\,. \end{cases} \tag{1.1}$$

Throughout this work, the initial data u_0 is assumed to belong to the Sobolev space $H^s(\mathbb{M})$. Here, we are concerned with the following two settings:

(i) $\mathbb{M} = \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, i.e. DNLS on the torus (namely (1.1) with the periodic boundary condition), and

(ii) $\mathbb{M} = \mathbb{R}$, i.e. DNLS on the real line.

The index s denotes "the number of derivatives" required from the initial data. We aim to work with the smallest possible value for s, or in other words, with low-regularity initial data. This desideratum does not only allow one to work within a larger class of initial data, but oftentimes the analysis in low-regularity spaces also uncovers qualitative properties of solutions to problems such as (1.1) that are not present for smooth solutions.

Generally speaking, a well-posedness theory establishes that for given initial data u_0, there exists a unique solution $u = u(t,x)$ to the initial-value problem and that the solution depends continuously on the initial data. This thesis is focused on two of the properties that go beyond the bare-minimum requirements of well-posedness. More specifically, we provide answers to the following questions:

1. Can the solutions be extended for all times?

2. Is the uniqueness of solutions guaranteed among the largest possible class of solutions?

We recall that DNLS arises as a model equation in plasma physics [42, 33, 34, 50] when describing the propagation of Alfvén waves in magnetized plasma with constant magnetic field. We refer to the monograph [43] for a more recent derivation of this equation.

One of the important features of DNLS is that it conserves the following quantities (referred to as the *mass*, *momentum*, and *energy* of a solution u to DNLS):

$$M(u) := \int_{\mathbb{M}} |u|^2 dx \,, \tag{1.2}$$

$$P(u) := \int_{\mathbb{M}} \operatorname{Im}(u\partial_x \overline{u}) + \frac{1}{2}|u|^4 dx \,, \tag{1.3}$$

$$E(u) := \int_{\mathbb{M}} |\partial_x u|^2 + \frac{3}{2}|u|^2 \operatorname{Im}(u\partial_x \overline{u}) + \frac{1}{2}|u|^6 \, dx \,. \tag{1.4}$$

More precisely, if u satisfies (1.1) and has enough regularity, then $E(u(t)) = E(u_0)$ for all existence times t of u (and similarly for the other two quantities).

In the Euclidean setting, i.e. $\mathbb{M} = \mathbb{R}$, the family of solutions to DNLS is invariant under the following scaling transformation:

$$u(t,x) \mapsto \frac{1}{\lambda^{\frac{1}{2}}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) =: u^\lambda(t,x). \tag{1.5}$$

We have that $\|u_0^\lambda\|_{L^2_x} = \|u_0\|_{L^2_x}$, where $u_0^\lambda(x) = u^\lambda(0,x)$. Thus, $s_c = 0$ is the scaling critical Sobolev index. In the periodic setting, i.e. $\mathbb{M} = \mathbb{T}$, the above transformation changes the underlying domain and thus the scaling transformation is no longer a symmetry of the equation on $\mathbb{T}$. Nevertheless, in both settings, reasonable well-posedness theory is to be expected for $s > s_c$, and possibly for $s = s_c$.

1.1 DNLS in the periodic setting

In [21], Herr showed that the initial-value problem (1.1) is locally well-posed in $H^s(\mathbb{T})$, for any $s \geq \frac{1}{2}$. Moreover, also in [21], the author pointed out that the H^1-solutions extend globally in time, provided initial data with mass less than $\frac{2}{3}$.

In Chapter 2, we relax the smallness assumption on the mass and establish global well-posedness of the periodic DNLS for smooth initial data. More precisely, we show the following:

Theorem 1.1. *Let $\lambda > 0$. Then, the initial-value problem* (1.1) *on $\mathbb{T}_\lambda$ is globally well-posed in $H^1(\mathbb{T}_\lambda)$, provided initial data with mass less than 4π.*

Note: Section 2.3 presents a proof by contradiction which is based on joint work with Tadahiro Oh published in

[36] R. Mosincat and T. Oh, *A remark on global well-posedness of the derivative nonlinear Schrödinger equation on the circle*, C.R. Acad. Sci. Paris, Ser. I 353 (2015), pp. 837–841,

Alternatively, in Section 2.4 we give a direct proof based on an inequality established in Section 3 of the following article:

[35] R. Mosincat, *Global well-posedness of the derivative nonlinear Schrödinger equation with periodic boundary condition in* $H^{\frac{1}{2}}$, J. Differential Equations 263 (2017), 4658-4722.

Theorem 1.1 improves the known mass threshold in [21] for global well-posedness in $H^1(\mathbb{T})$. Moreover, we note that the mass threshold 4π is independent of the period $L = 2\pi\lambda$. It is worthwhile to mention that on compact intervals, as well as on the half line $\mathbb{R}_+ = [0, \infty)$, under the Dirichlet boundary condition, (1.1) possesses finite time blowup solutions above the mass threshold 2π (provided $E(u) < 0$ and some additional conditions). See [47, 54].

Chapter 3 is devoted to extending Theorem 1.1 to rougher initial data. More precisely, we establish:

Theorem 1.2. *The derivative nonlinear Schrödinger equation with periodic boundary condition* (1.1) *is globally well-posed in* $H^{\frac{1}{2}}$ *for initial data* u_0 *with* $M[u_0] < 4\pi$.

Note: Theorem 1.2 is the main result of the following article:

[35] R. Mosincat, *Global well-posedness of the derivative nonlinear Schrödinger equation with periodic boundary condition in* $H^{\frac{1}{2}}$, J. Differential Equations 263 (2017), 4658-4722.

The main tool for proving Theorem 1.2 is the "I-method" introduced by Colliander, Keel, Staffilani, Takaoka, and Tao [9]. In order to reach the end-point regularity $s = \frac{1}{2}$ we use a refinement of this method by employing a third generation modified energy, which we briefly describe in what follows.

In the first instance, the I-method aims to prove that $\mathcal{E}^1(u(t)) := E(Iu(t))$ is "*almost* conserved", i.e. that it changes *slowly* in time, for solutions[1] u of DNLS. Here, the operator I is the identity operator for functions supported on small frequencies and "smooths out" high frequency components. This method provides a robust scheme which can be refined, for example by iteratively taking $\mathcal{E}^j(u) := \mathcal{E}^{j-1}(u) +$ "correction terms", where the *small* correction terms are chosen such that $\mathcal{E}^j$ changes yet even slower than $\mathcal{E}^{j-1}$ in time. For the proof of Theorem 1.2, we use:

$$\mathcal{E}^3(u) := \mathcal{E}^1(u) + \Lambda_4(\sigma_4; u) + \Lambda_6(\sigma_6; u) + M(u)\Lambda_4(\widetilde{\sigma_4}; u).$$

We have to resort to this third generation modified energy since the second generation modified energy $\mathcal{E}^2(u) := \mathcal{E}^1(u) + \Lambda_4(\sigma_4; u)$ is not "nice enough" for reaching $s = \frac{1}{2}$ in the I-method scheme. The correction multipliers have the form $\sigma_k = \frac{M_k}{\alpha_k}$, where M_k is chosen such that we eliminate a badly behaved term and α_k is determined by the

[1]First, one shows the almost conservation property for smooth solutions, and then by a standard approximation argument infers the property for H^s-solutions.

structure of the equation. If M_k vanishes on the zero set of α_k, then we can still define the correction multiplier σ_k (by setting it zero whenever $\alpha_k = 0$). This is indeed the case for σ_4 and $\widetilde{\sigma_4}$. However, for the sixth order correction multiplier σ_6, $\alpha_6 = 0$ does not imply $M_6 = 0$. In this case, we use the *resonant decomposition* idea appeared first in [12] for the cubic NLS on $\mathbb{R}^2$. Roughly, the idea is to split $M_6 = M_6^{(1)} + M_6^{(2)}$ corresponding to $|\alpha_6| \leq R$ (*resonant contribution*) and $|\alpha_6| > R$ (*nonresonant contribution*). Due to the restriction $|\alpha_6| \leq R$, the multiplier $M_6^{(1)}$ has a better pointwise estimate than M_6. The multiplier $M_6^{(2)}$ is used to define the correction multiplier σ_6, i.e. $\sigma_6 = \frac{M_6^{(2)}}{\alpha_6}$. Finally, we optimize the choice of the threshold R. This *resonant decomposition* step is equivalent to *applying normal form reductions to the evolution equation satisfied by* $\mathcal{E}^2(u(t))$. The core part of the argument proving Theorem 1.2 is estimating (time averages of) the nonlinear terms of $\frac{d}{dt}\mathcal{E}^j(u(t))$ for which we use Fourier restriction norms. Compared to the global well-posedness of DNLS in $H^{\frac{1}{2}}(\mathbb{R})$, result obtained by Miao, Wu, and Xu in [32] also using the third generation modified energy, in the periodic setting, the difficulty in proving nonlinear estimates stems from fewer linear estimates available: we *only* have the L^4-Strichartz estimate due to Bourgain [5] and its bilinear refinement due to De Silva, Pavlović, Staffilani, and Tzirakis [13] for the interaction of two frequency-separated linear evolutions.

1.2 DNLS in the Euclidean setting

Let us review some of the well-posedness results for DNLS in the Euclidean setting. Kaup and Newell [24] showed that DNLS is completely integrable, in the sense that it is the compatibility condition for a certain pair of linear differential equations. In particular, it possesses an infinite family of conservation laws, as well as a two-parameter family of solitons. Throughout this work, we do not employ the complete integrability structure of DNLS. For a certain class of Schwartz initial data, by using the inverse scattering method, Lee [30, 31] obtained local and global solvability.

In low-regularity spaces, Takaoka [44] used the Fourier restriction norm spaces introduced by Bourgain [5] and proved

$$\|v^2 \partial_x \overline{v}\|_{X^{s,b-1}(\mathbb{R}\times\mathbb{R})} \lesssim \|v\|^3_{X^{s,b}(\mathbb{R}\times\mathbb{R})}, \tag{1.6}$$

for $\frac{1}{2} \leq s < 1$ and $\frac{1}{2} < b \leq \frac{5}{8}$, in a fashion similar to the estimate for the KdV equation [25]. On the other hand, Takaoka in [44] noted that for estimates of the form

$$\||v|^2 \partial_x v\|_{X^{s,b-1}(\mathbb{R}\times\mathbb{R})} \lesssim \|v\|^3_{X^{s,b}(\mathbb{R}\times\mathbb{R})}, \tag{1.7}$$

"the Fourier restriction norm method seems inapplicable." However, the transformation[2] $v = \mathcal{G}_1(u)$ removes the nonlinearity $|u|^2\partial_x u$ from (1.1), i.e. v solves

$$i\partial_t v + \partial_x^2 v = -iv^2\partial_x \overline{v} - \frac{1}{2}|v|^4 v. \tag{1.8}$$

Therefore, Takaoka established the local well-posedness of (1.1) down to $H^{\frac{1}{2}}(\mathbb{R})$. and also showed that the above estimate (1.6) does not hold if $s < \frac{1}{2}$, for any $b \in \mathbb{R}$. Moreover, for $0 \leq s < \frac{1}{2}$, the solution map $u_0 \in H^s(\mathbb{R}) \mapsto u(t) \in H^s(\mathbb{R})$ fails to be C^3, for any $t \neq 0$. Another mild ill-posedness result for DNLS in $H^s(\mathbb{R})$ $(0 \leq s < \frac{1}{2})$ was given by Biagioni and Linares in [4]. They used the solitary waves of DNLS [24, 50] and showed that the local uniform continuity of the same solution map does not hold. Hence, the fixed point argument for the gauge equivalent equation (1.8) is no longer the tool to construct $H^s(\mathbb{R})$-solutions for DNLS in the range $0 \leq s < \frac{1}{2}$.

In Chapter 4, we establish the unconditional (local) well-posedness of DNLS in $H^s(\mathbb{R})$, for any $s > \frac{1}{2}$. Unconditional well-posedness is a notion of well-posedness which does not depend on how solutions are constructed. If well-posedness is obtained by employing an auxiliary function space, then the uniqueness of solutions holds *conditionally.* If, instead, uniqueness holds in the entire class of continuous-in-time H^s-valued functions, we say that the initial-value problem is *unconditionally* (locally) well-posed in H^s.

Theorem 1.3. *Let $s > \frac{1}{2}$. Then, DNLS is unconditionally (locally) well-posed in $H^s(\mathbb{R})$. More precisely, for any $u_0 \in H^s(\mathbb{R})$ and interval I containing $t = 0$, if $u_1, u_2 \in C(I; H^s(\mathbb{R}))$ are solutions to* (1.1) *with $u_1|_{t=0} = u_2|_{t=0} = u_0$, then $u_1(t) = u_2(t)$ for all $t \in I$.*

Note: Theorem 1.3 is based on joint work with Haewon Yoon:

[37] R. Mosincat and H. Yoon, *Unconditional uniqueness for the derivative nonlinear Schrödinger equation on the real line*, preprint, 2018.

Theorem 1.3 removes the auxiliary function spaces used by Takaoka [44] when proving the local well-posedness of DNLS in $H^s(\mathbb{R})$, $s > \frac{1}{2}$. Previously, Yin Yin Su Win [52] showed the unconditional well-posedness of DNLS in the energy space $H^1(\mathbb{R})$. In Subsection 4.6 we point out why this strategy does not work below the energy space.

The proof of Theorem 1.3 is based on the recent normal form approach to unconditional well-posedness of Kwon, Oh, and Yoon [28] and on ideas due to Kishimoto [27, 26] to use certain trilinear forms and the $H^{s-1}(\mathbb{R})$-norm to show convergence to zero (in the sense of distributions) of a remainder term (see subsections 4.2 and 4.3, respectively). In [26], Kishimoto proved the unconditional well-posedness of the *periodic* DNLS in $H^s(\mathbb{T})$, for $s > \frac{1}{2}$. In the periodic case, the normal form approach exploits in a non-trivial manner the discrete structure of the spatial frequency space. In the

[2] This reduction of DNLS to (1.8) was also employed by Lee [31], to which he attached a certain spectral problem.

Euclidean case, however, number theoretic tools such as the divisor counting argument are no longer available.

In [28], the authors developed an infinite iteration scheme of normal form reductions in an abstract form for nonlinear dispersive PDEs on the real line. We recall that in the context of dispersive PDEs, the normal form method has been introduced by Babin, Ilyin, and Titi [2] for the unconditional well-posedness of KdV on the torus. For the cubic NLS on $\mathbb{T}$, however, Guo, Kwon, and Oh [17] needed to perform normal form reductions infinitely many times (whereas in [2], two iterations sufficed). In particular, they introduced the notion of "ordered trees" to handle the resulting multilinear terms. We also make use of this bookkeeping tool as well as related notions, see Appendix A.

Chapter 2

Mass threshold improvement for global well-posedness in $H^1(\mathbb{T})$

When proving that local H^s-solutions u can be extended globally in time, one seeks to have control on the growth of $\|u(t)\|_{H^s}$ in time. If there is a functional well-defined for H^s-functions, invariant under the flow of the equation, and which has a good coercivity property (i.e. controls the blow-up norm), then global well-posedness follows routinely by iterating the local well-posedness result. For global well-posedness in $H^1(\mathbb{T})$ under the mass condition $M(u_0) < 2\pi$, one needs only the energy conservation law. However, to obtain Theorem 1.1 with the improved mass condition $M(u_0) < 4\pi$, we use a combination of the energy and momentum functionals.

2.1 Gagliardo-Nirenberg inequalities on $\mathbb{T}_\lambda$

In this subsection, we prove inequalities of the form

$$\|f\|_{L^q(\mathbb{T})} \leq \big(c(s,p,q)+\varepsilon\big)\|f\|^{1-\theta}_{L^p(\mathbb{T})}\|D^s_x f\|^{\theta}_{L^2(\mathbb{T})} + K_\epsilon \|f\|^{1-\theta}_{L^p(\mathbb{T})}\|f\|^{\theta}_{L^2(\mathbb{T})},\ \varepsilon>0,$$

where the positive constant K_ϵ blows up as $\varepsilon \searrow 0$ (here, $D^s_x := (-\partial_x^2)^{\frac{s}{2}}$). On $\mathbb{R}$, Gagliardo-Nirenberg inequalities (i.e. $\varepsilon=0$, $K_\varepsilon=0$) are known to hold with sharp, explicit constants $c(s,p,q)$ (see for example the article of Bellazzini, Frank, and Visciglia [3]), but on $\mathbb{T}$ constant functions provide counterexamples.

We recall that on the real line, we have the sharp Gagliardo-Nirenberg inequalities

$$\|f\|_{L^6(\mathbb{R})} \leq \left(\tfrac{2}{\pi}\right)^{\frac{1}{3}} \|\partial_x f\|^{\frac{1}{3}}_{L^2(\mathbb{R})}\|f\|^{\frac{2}{3}}_{L^2(\mathbb{R})}, \tag{2.1}$$

$$\|f\|_{L^6(\mathbb{R})} \leq C_{\mathrm{GN}}\|\partial_x f\|^{\frac{1}{9}}_{L^2(\mathbb{R})}\|f\|^{\frac{8}{9}}_{L^4(\mathbb{R})}, \tag{2.2}$$

where $C_{\mathrm{GN}} := 3^{\frac{1}{6}}(2\pi)^{-\frac{1}{9}}$. For (2.1) we refer to [51], whereas for (2.2), see [1].

On bounded domains, inequalities of the above form cannot hold, simply for the fact that constant functions provide counterexamples. However, the situation is similar to the Poincaré inequality, and in fact, using elementary arguments, [1] it was shown in [21, Appendix C] the following inequality:

$$\big\|\big(|f|^2-\mu(f)\big)f\big\|_{L^2(\mathbb{T})} \le \|\partial_x f\|_{L^2(\mathbb{T})}\|f\|^2_{L^2(\mathbb{T})}\,, \tag{2.3}$$

for any 2π-periodic function f, where

$$\mu(f) := \fint_{\mathbb{T}} |f(x)|^2 dx = \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{T})}\,. \tag{2.4}$$

Although (2.3) can be used to study the coercivity of E (see Lemma 2.6 below), we use here the following result (see e.g. Lebowitz, Rose and Speer [29, Lemma 4.1]) since it yields the same mass threshold $M(u_0) < 2\pi$ as in the Euclidean setting.

We first establish the following version of the Gagliardo-Nirenberg inequality on $\mathbb{T}_\lambda$ which incorporates the sharp constant from (2.2). The proof is based on an argument from [29].

Lemma 2.1. *Let $\lambda, \varepsilon > 0$. Then, for any $f \in H^1(\mathbb{T}_\lambda)$, we have*

$$\|f\|_{L^6(\mathbb{T}_\lambda)} \le C_{\mathrm{GN}}\Big(1+\frac{\varepsilon}{5\pi\lambda}\Big)^{\frac{2}{9}}\Big(\|\partial_x f\|^2_{L^2(\mathbb{T}_\lambda)} + \frac{\sqrt{2}}{\varepsilon\sqrt{\pi\lambda}}\|f\|^2_{L^4(\mathbb{T}_\lambda)}\Big)^{\frac{1}{18}}\|f\|^{\frac{8}{9}}_{L^4(\mathbb{T}_\lambda)}\,, \tag{2.5}$$

$$\|f\|_{L^6(\mathbb{T}_\lambda)} \le C_{GN}\Big(1+\frac{\varepsilon}{5\pi\lambda}\Big)^{\frac{2}{9}}\Big(\|\partial_x f\|^2_{L^2(\mathbb{T}_\lambda)} + \frac{1}{\pi\lambda\varepsilon}\|f\|^2_{L^2(\mathbb{T}_\lambda)}\Big)^{\frac{1}{18}}\|f\|^{\frac{8}{9}}_{L^4(\mathbb{T}_\lambda)}\,, \tag{2.6}$$

$$\|f\|^6_{L^6(\mathbb{T}_\lambda)} \le \Big(\frac{4}{\pi^2}+\varepsilon\Big)\|\partial_x f\|^2_{L^2(\mathbb{T}_\lambda)}\|f\|^4_{L^2(\mathbb{T}_\lambda)} + K_\varepsilon\|f\|^6_{L^2(\mathbb{T}_\lambda)}\,, \tag{2.7}$$

for some constant $K_\varepsilon > 0$ (independent of λ).

Proof. Let $f \in H^1(\mathbb{T}_\lambda) \subset C(\mathbb{T}_\lambda)$. By periodicity, we assume that

$$|f(0)| = |f(L)| \le L^{-\frac{1}{4}}\|f\|_{L^4(\mathbb{T}_\lambda)} \tag{2.8}$$

without loss of generality, where $L := 2\pi\lambda$. Let F be an extension of f on $[0, L]$ to $\mathbb{R}$ such that (i) $\operatorname{supp} F \subset [-\delta, L+\delta]$ and (ii) F linearly interpolates 0 and $f(0)$ on $[-\delta, 0]$ and $f(L)$ and 0 on $[L, L+\delta]$. Then, by a direct calculation, we have

$$\|f\|^6_{L^6(\mathbb{T}_\lambda)} \le \|F\|^6_{L^6(\mathbb{R})}, \tag{2.9}$$

$$\|F\|^4_{L^4(\mathbb{R})} \le \|f\|^4_{L^4(\mathbb{T}_\lambda)} + \frac{2\delta}{5}|f(0)|^4 \le \Big(1+\frac{2\delta}{5L}\Big)\|f\|^4_{L^4(\mathbb{T}_\lambda)}, \tag{2.10}$$

$$\|\partial_x F\|^2_{L^2(\mathbb{R})} \le \|\partial_x f\|^2_{L^2(\mathbb{T}_\lambda)} + 2\frac{|f(0)|^2}{\delta} \le \|\partial_x f\|^2_{L^2(\mathbb{T}_\lambda)} + \frac{2}{\delta L^{\frac{1}{2}}}\|f\|^2_{L^4(\mathbb{T}_\lambda)}. \tag{2.11}$$

Then, the estimate (2.5) follows from (2.2) with (2.9), (2.10), and (2.11).

[1]The inequality (2.3) is also true on $\mathbb{T}_\lambda$ and this can be easily checked by scaling considerations. The same result can be obtained by using the pointwise Poincaré inequality followed by an application of the Hölder inequality.

The inequalities (2.6) and (2.7) follow via analogous arguments.

□

2.2 Gauge trasformations on $\mathbb{T}_\lambda$

The adaptation of the gauge transformation is due to Herr [21] where he proved the local well-posedness of (1.1) in $H^s(\mathbb{T})$ for $s \geq \frac{1}{2}$. For $\beta \in \mathbb{R}$, we consider

$$\mathcal{G}_\beta : L^2_x(\mathbb{T}_\lambda) \to L^2_x(\mathbb{T}_\lambda) \quad , \quad \mathcal{G}_\beta(f)(x) := e^{-i\beta\mathcal{J}(f)(x)} f(x) \quad , \tag{2.12}$$

where $\mathcal{J}(f)$ is the mean-zero antiderivative of $|f|^2 - \mu(f)$, i.e.

$$\mathcal{J}(f)(x) := \frac{1}{2\pi\lambda} \int_0^{2\pi\lambda} \int_\theta^x |f(y)|^2 - \mu(f)\, dy\, d\theta\,, \tag{2.13}$$

where

$$\mu(f) := \fint_{\mathbb{T}_\lambda} |f(x)|^2 dx = \frac{1}{2\pi\lambda} \|f\|^2_{L^2(\mathbb{T}_\lambda)}\,. \tag{2.14}$$

Remark 2.2. Consider $g = \mathcal{G}_\beta(f)$. Then $|g| = |f|$ and thus $\|g\|_{L^p} = \|f\|_{L^p}$ for any p; in particular $\mu(\mathcal{G}_\beta f) = \mu(f)$. We also note that $\mathcal{G}_\beta$ is inverted by $\mathcal{G}_{-\beta}$.

Moreover, we have

$$\partial_x f = e^{i\beta\mathcal{J}(g)}\Big(\partial_x g + i\beta(|g|^2 - \mu(g))g\Big).$$

By using (2.3), it follows that

$$\begin{aligned} \|\partial_x f\|^2_{L^2} &= \|\partial_x g\|^2_{L^2} + \beta^2 \|(|g|^2 - \mu(g))g\|^2_{L^2} - 2\beta \int (|g|^2 - \mu(g))\, \mathrm{Im}(g\partial_x \overline{g})\, dx \\ &\leq \Big(1 + \beta^2 \|g\|^4_{L^2} + 2|\beta| \|g\|^2_{L^2}\Big) \|\partial_x g\|^2_{L^2}. \end{aligned}$$

By interpolating with the trivial estimate $\|f\|_{L^2} = \|g\|_{L^2}$, we get that for any $0 \leq s \leq 1$,

$$\|D^s_x f\|_{L^2(\mathbb{T}_\lambda)} \lesssim_{M(f)} \|D^s_x g\|_{L^2(\mathbb{T}_\lambda)}$$

Clearly, if we apply the above lines of argument for $\mathcal{G}_{-\beta}$ instead of $\mathcal{G}_\beta$ we in fact get that

$$\|D^s_x f\|_{L^2(\mathbb{T}_\lambda)} \sim \|D^s_x g\|_{L^2(\mathbb{T}_\lambda)} \tag{2.15}$$

with implicit constants depending on s, β, and $M(f) = M(g)$.

By setting $w(t,x) = \mathcal{G}_\beta(u(t))(x)$, the derivative nonlinear Schrödinger equation

(1.1) becomes

$$
\begin{aligned}
i\partial_t w + \partial_x^2 w - 2i\beta\mu[w]\,\partial_x w = & \, 2i(1-\beta)|w|^2\partial_x w + i(1-2\beta)w^2\partial_x\overline{w} + \beta\mu(w)|w|^2 w \\
& + \Big(\frac{\beta}{2} - \beta^2\Big)|w|^4 w - \psi(w)w,
\end{aligned}
\tag{2.16}
$$

where

$$
\psi(w) := \frac{\beta}{2\pi\lambda}\int_{\mathbb{T}_\lambda}\Big(2\,\mathrm{Im}(w\overline{w}_x) + (\frac{3}{2} - 2\beta)|w|^4\Big)\,dx + \beta^2\mu(w)^2. \tag{2.17}
$$

Correspondingly, the momentum and energy functionals are

$$
P(\mathcal{G}_{-\beta}(w)) = \int_{\mathbb{T}_\lambda}\Big(\mathrm{Im}(w\overline{w}_x) + (\frac{1}{2} - \beta)|w|^4\Big)\,dx + \beta\mu[w]M[w] =: P_\beta(w), \tag{2.18}
$$

$$
\begin{aligned}
E(\mathcal{G}_{-\beta}(w)) = & \int_{\mathbb{T}_\lambda}\Big(|w_x|^2 + (\frac{3}{2} - 2\beta)|w|^2\,\mathrm{Im}(w\overline{w}_x) + (\beta^2 - \frac{3}{2}\beta + \frac{1}{2})|w|^6\Big)\,dx \\
& + \frac{\beta}{2}\mu(w)\|w\|_{L_x^4}^4 + 2\beta\mu(w)P_\beta(w) - \beta^2\mu(w)^2 M(w) =: E_\beta(w).
\end{aligned}
\tag{2.19}
$$

We point out that in the periodic setting, the terms coupled with $\mu(w)$ and $\psi(w)$ are new terms when comparing (2.16) to the corresponding equation in the Euclidean setting.

We can eliminate the auxiliary linear term on the left hand side of (2.16) by the translation transformation

$$
w(t,x) \mapsto v(t, x + 2\beta\mu(w(t))t). \tag{2.20}
$$

Correspondingly, we introduce the gauge transformation of spacetime functions

$$
\mathcal{G}^\beta : C_t^0 L_x^2(J\times\mathbb{T}_\lambda) \to C_t^0 L_x^2(J\times\mathbb{T}_\lambda)\ ,\ \mathcal{G}^\beta(u)(t,x) := \mathcal{G}_\beta(u(t))(x - 2\beta\mu(u(t))t). \tag{2.21}
$$

For the local well-posedness theory, it is necessary to use the gauge parameter $\beta = 1$ so that the "bad" nonlinear term $|w|^2\partial_x w$ in (2.16) is eliminated. Hence, in the sequel, we consider the equation on $\mathbb{T}_\lambda$ corresponding to this gauge choice, namely

$$
i\partial_t v + \partial_x^2 v = -iv^2\partial_x\overline{v} - \frac{1}{2}|v|^4 v + \mu(v)|v|^2 v - \psi(v)v, \tag{2.22}
$$

where we recall that $\mu(v) = \frac{1}{2\pi\lambda}\|v\|_{L^2(\mathbb{T}_\lambda)}^2$ and

$$
\psi(v) := \frac{1}{2\pi\lambda}\int_{\mathbb{T}_\lambda}\Big(2\,\mathrm{Im}(v\partial_x\overline{v}) - \frac{1}{2}|v|^4\Big)\,dx + \mu(v)^2. \tag{2.23}
$$

The following lemma provides the continuity properties of the gauge transformation. We note that in order to have the Lipschitz continuity of $\mathcal{G}^\beta$ (rather than of $\mathcal{G}_\beta$) one needs to fix the L^2-norm of the functions at all times t.

Lemma 2.3. *[21, Lemma 2.3] Let $s, r, \mu_0 \geq 0$, $T > 0$. There exists $c = c(r, s, \lambda) > 0$ such that:*

1. *If $f, g \in B_r := \{f \in H^s(\mathbb{T}_\lambda) : \|f\|_{H^s(\mathbb{T}_\lambda)} \leq r\}$, then*

$$\|\mathcal{G}_\beta(f) - \mathcal{G}_\beta(g)\|_{H^s(\mathbb{T}_\lambda)} \leq c\|f - g\|_{H^s(\mathbb{T}_\lambda)}. \tag{2.24}$$

2. *If $u, v \in B^{r,\mu_0}$, where*

$$B^{r,\mu_0} := \{u \in C([-T, T]; H^s(\mathbb{T}_\lambda)) : \|u\|_{L_t^\infty H_x^s} \leq r, \mu(u(t)) = \mu_0 \text{ for all } t \in [-T, T]\},$$

then

$$\|\mathcal{G}^\beta(u)(t) - \mathcal{G}^\beta(v)(t)\|_{H^s(\mathbb{T}_\lambda)} \leq c\|u(t) - v(t)\|_{H^s(\mathbb{T}_\lambda)} \tag{2.25}$$

for all $t \in [-T, T]$.

2.3 Proof by contradiction for Theorem 1.1

In this subsection, we prove by contradiction that $\|u(t)\|_{H^1(\mathbb{T}_\lambda)}$ stays bounded on any finite time interval. Let $\lambda, \delta > 0$ and note that the conclusion of Theorem 1.1 follows once we prove the global well-posedness of DNLS in $H^1(\mathbb{T}_\lambda)$ provided initial data with mass less than $4\pi\big(1 + \frac{\delta}{5\pi\lambda}\big)^{-2}$. The proof below follows ideas of Wu [55].

By time reversibility, we restrict our attention to positive times. First, we recall that the local well-posedness result due to Herr [21] yields a simple blowup alternative: either (i) the solution u to (1.1) exists globally or (ii) there exists a finite time T_* such that $\lim_{t \uparrow T_*} \|u(t)\|_{\dot{H}^1} = \infty$.

Fix $\delta, \lambda > 0$. Suppose that there exists a solution u to (1.1) such that

$$M(u) < 4\pi\big(1 + \frac{\delta}{5\pi\lambda}\big)^{-2}$$

and

$$\lim_{t \uparrow T_*} \|u(t)\|_{\dot{H}^1(\mathbb{T}_\lambda)} = \infty\,,$$

for some finite time $T_* > 0$. Let $v = \mathcal{G}^{\frac{3}{4}}(u)$ and consider the functional

$$\begin{aligned} \mathcal{E}(v) &:= \int_{\mathbb{T}_\lambda} \Big(|\partial_x v|^2 - \frac{1}{16}|v|^6 + \frac{3}{8}\mu(v)|v|^4\Big)\, dx \\ &= E_{\frac{3}{4}}(v) - \frac{3}{2}\mu(v)P_{\frac{3}{4}}(v) + \frac{9}{16}\mu(v)^2 M(v)\,, \end{aligned} \tag{2.26}$$

where for the second expression we used (2.19). We have $E_{\frac{3}{4}}(v) = E(u)$, $P_{\frac{3}{4}}(v) = P(u)$, $M(v) = M(u)$ and therefore $\mathcal{E}(v)$ is conserved. Since the gauge transformation is continuous on $C([-T, T] : H^1(\mathbb{T}_\lambda))$, our assumption and (2.15) imply that there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$\lim_{n\to\infty} \|v(t_n)\|_{\dot{H}^1(\mathbb{T}_\lambda)} = \infty$$

while $M(v) = M(u) < 4\pi\big(1 + \frac{\delta}{5\pi\lambda}\big)^{-2}$. Then, it follows from the conservation of $\mathcal{E}(v)$ that

$$\lim_{n\to\infty} \|v(t_n)\|_{L^6(\mathbb{T}_\lambda)} = \infty\,. \tag{2.27}$$

As in [55], we define $\{f_n\}_{n\in\mathbb{N}}$ by

$$f_n := \frac{\|v(t_n)\|^4_{L^4(\mathbb{T}_\lambda)}}{\|v(t_n)\|^3_{L^6(\mathbb{T}_\lambda)}}\,.$$

Then, we have the following lemma.

Lemma 2.4. *Let $\delta, \lambda > 0$. There exists $\varepsilon_n = \varepsilon_n(\lambda, \delta) \to 0$ as $n \to \infty$ such that*

$$2C_{\mathrm{GN}}^{-\frac{9}{2}}\left(1 + \frac{\delta}{5\pi\lambda}\right)^{-1} + \varepsilon_n \le f_n \le M(v)^{\frac{1}{2}}\,. \tag{2.28}$$

In particular, $\|v(t_n)\|_{L^4(\mathbb{T}_\lambda)} \to \infty$ as $n \to \infty$.

Proof. The upper bound in (2.28) follows from Hölder's inequality.

In what follows, for notational simplicity, we suppress the domain of integration $\mathbb{T}_\lambda$ with the understanding that all the norms are taken over $\mathbb{T}_\lambda$.

From the upper bound in (2.28) together with (2.27) it follows that

$$\gamma_n := \left(\frac{2}{\delta L^{\frac{1}{2}}} - \frac{3}{8}\mu(v(t_n))\|v(t_n)\|^2_{L^4}\right)\frac{\|v(t_n)\|^2_{L^4}}{\|v(t_n)\|^6_{L^6}} \longrightarrow 0, \tag{2.29}$$

as $n \to \infty$. By Lemma 2.1, we have

$$\begin{aligned} f_n &\ge C_{\mathrm{GN}}^{-\frac{9}{2}}\left(1 + \frac{2\delta}{5L}\right)^{-1}\left(\|\partial_x v(t_n)\|^2_{L^2} + \frac{2}{\delta L^{\frac{1}{2}}}\|v(t_n)\|^2_{L^4}\right)^{-\frac{1}{4}}\|v(t_n)\|^{\frac{3}{2}}_{L^6} \\ &= 2C_{\mathrm{GN}}^{-\frac{9}{2}}\left(1 + \frac{2\delta}{5L}\right)^{-1}\left(1 + 16\frac{\mathcal{E}(v)}{\|v(t_n)\|^6_{L^6}} + 16\gamma_n\right)^{-\frac{1}{4}}, \end{aligned} \tag{2.30}$$

where in the last step we simply used the definitions of $\mathcal{E}(v)$ (see (2.26)) and of γ_n (see (2.29)). Then, the lower bound in (2.28) follows from (2.27), (2.29), and (2.30) with the conservation of $\mathcal{E}(v)$.

The claim that $\|v(t_n)\|_{L^4(\mathbb{T}_\lambda)} \to \infty$ as $n \to \infty$ follows from (2.27) and (2.28).

□

In the following, we use the conservation of the functional

$$\begin{aligned}\mathcal{P}(v) &:= \operatorname{Im}\int_{\mathbb{T}_\lambda} v\overline{v}_x dx - \frac{1}{4}\int_{\mathbb{T}_\lambda} |v|^4 dx \\ &= P_{\frac{3}{4}}(v) - \frac{3}{4}\mu(v)M(v)\,.\end{aligned} \tag{2.31}$$

We consider modulated functions $\phi_n(x,t) = e^{i\alpha_n x}v(x,t)$ for some non-zero $\alpha_n \in \mathbb{Z}_\lambda$ (to be chosen later). On the one hand, we have

$$\mathcal{P}(v) + \frac{1}{4}\int_{\mathbb{T}_\lambda} |v|^4 dx = \operatorname{Im}\int_{\mathbb{T}_\lambda} v\overline{v}_x dx = -\frac{1}{2\alpha_n}\mathcal{E}(\phi_n) + \frac{\alpha_n}{2}M(v) + \frac{1}{2\alpha_n}\mathcal{E}(v). \tag{2.32}$$

On the other hand, by Lemma 2.1 with (2.29), we have

$$\mathcal{E}\big((\phi_n(t_n)\big) \geq -(\eta_n + \gamma_n)\|v(t_n)\|_{L^6}^6 \tag{2.33}$$

where η_n is defined by

$$\eta_n := \frac{1}{16} - \left(1 + \frac{2\delta}{5L}\right)^{-4} C_{\mathrm{GN}}^{-18} f_n^{-4}. \tag{2.34}$$

Case 1: $\eta_n + \gamma_n \leq 0$ for infinitely many n.

In this case, we simply set $\alpha_n = \frac{1}{\lambda}$. Then, for those values of n with $\eta_n + \gamma_n \leq 0$, it follows from (2.32) and (2.33) with (2.29) that

$$\begin{aligned}\frac{1}{4}\|v(t_n)\|_{L^4}^4 &\leq \frac{\lambda}{2}(\eta_n + \gamma_n)\|v(t_n)\|_{L^6}^6 - \mathcal{P}(v) + \frac{1}{2\lambda}M(v) + \frac{\lambda}{2}\mathcal{E}(v) \\ &\leq -\mathcal{P}(v) + \frac{1}{2\lambda}M(v) + \frac{\lambda}{2}\mathcal{E}(v).\end{aligned}$$

Then, from the conservation of M, $\mathcal{P}$, and $\mathcal{E}$, we conclude that $\|v(t_n)\|_{L^4} = O(1)$. This is a contradiction to Lemma 2.4.

Case 2: $\eta_n + \gamma_n > 0$ for all sufficiently large n.

In this case, we choose

$$\alpha_n := \frac{1}{\lambda}\Big[\frac{L}{2\pi}\big(M(v)^{-1}(\eta_n + \gamma_n)\big)^{\frac{1}{2}}\|v(t_n)\|_{L^6}^3\Big] + \frac{1}{\lambda} \in \mathbb{Z}_\lambda\,,$$

where γ_n and η_n are as in (2.29) and (2.34). Here, $[x]$ denotes the integer part of x. Then, from (2.32) and (2.33), we have

$$\frac{1}{4}\|v(t_n)\|_{L^4}^4 \leq \Big(M(v)(\eta_n + \gamma_n)\Big)^{\frac{1}{2}}\|v(t_n)\|_{L^6}^3 - \mathcal{P}(v) + \frac{1}{2\lambda}M(v) + \frac{1}{2\alpha_n}\mathcal{E}(v).$$

Then, by Lemma 2.4, (2.27), (2.29), and (2.34) along with the conservation of M, $\mathcal{P}$, and $\mathcal{E}$, we obtain

$$f_n^6 \leq M(v) f_n^4 - 16\left(1 + \frac{2\delta}{5L}\right)^{-4} C_{\mathrm{GN}}^{-18} M(v) + o(1) \tag{2.35}$$

as $n \to \infty$. Arguing as in [55], we see that (2.35) is impossible if

$$M(u) = M(v) < 4\pi\left(1 + \frac{\delta}{5\pi\lambda}\right)^{-2}.$$

This completes the proof by contradiction for Theorem 1.1.

2.4 Direct proof for Theorem 1.1

We begin this section by revisiting the energy functional corresponding to the gauge equivalent DNLS equation (2.22) on $\mathbb{T}_\lambda$, namely

$$E_1(v) := \int_{\mathbb{T}_\lambda} \left(|\partial_x v|^2 - \frac{1}{2}|v|^2 \operatorname{Im}(v\partial_x \overline{v}) + \frac{1}{2}\mu(v)|v|^4\right) dx + 2\mu(v)P_1(v) - \mu(v)^2 M(v). \tag{2.36}$$

Compared to the real line case, due to the particularity of the gauge transformation in the periodic setting, the terms coupled with $\mu(v)$ are new. The last two terms are conserved, by discarding them we still have a conservation law. However, the term $\frac{1}{2}\mu(v)\|v\|_{L^4(\mathbb{T}_\lambda)}^4$ is not conserved by the flow of (2.22).

Remark 2.5. If v is a smooth solution of (2.22), $\|v\|_{L^4(\mathbb{T}_\lambda)}$ is not necessarily conserved. Indeed, using integration by parts, we find

$$\begin{aligned} \partial_t \|v\|_{L^4(\mathbb{T}_\lambda)}^4 &= 4\operatorname{Re}\int_{\mathbb{T}_\lambda} |v|^2\overline{v}\Big(i\partial_x^2 v - i\big(-iv^2\partial_x\overline{v} - \frac{1}{2}|v|^4 v + \mu(v)|v|^2 v - \psi(v)v\big)\Big)\, dx \\ &= 4\operatorname{Im}\int_{\mathbb{T}_\lambda} \partial_x(v\overline{v}^2)\partial_x v\, dx - 4\operatorname{Im}\psi(v)\|v\|_{L^4(\mathbb{T}_\lambda)}^4 + \text{h.o.t.} \\ &= 4\operatorname{Im}\int_{\mathbb{T}_\lambda} \overline{v}^2(\partial_x v)^2\, dx + \text{h.o.t.}, \end{aligned}$$

where we used the fact that $\psi(v)$ is $\mathbb{R}$-valued; see (2.23). In general, the higher order terms (h.o.t.) cannot cancel the fourth order term $4\operatorname{Im}\int_{\mathbb{T}_\lambda} \overline{v}^2(\partial_x v)^2\, dx$.

Nevertheless, by Sobolev embedding and interpolation of H^s-norms, we have

$$\|v\|_{L^4(\mathbb{T}_\lambda)} \lesssim \|v\|_{H^{\frac{1}{4}}(\mathbb{T}_\lambda)} \leq \|v\|_{L^2(\mathbb{T}_\lambda)}^{\frac{3}{4}} \|v\|_{H^1(\mathbb{T}_\lambda)}^{\frac{1}{4}}$$

and therefore, for any $\varepsilon > 0$,

$$\frac{1}{2}\mu(v)\|v\|_{L^4(\mathbb{T}_\lambda)}^4 \lesssim \|v\|_{L^2(\mathbb{T}_\lambda)}^5 \|v\|_{H^1(\mathbb{T}_\lambda)} \lesssim \varepsilon\|\partial_x v\|_{L^2(\mathbb{T}_\lambda)}^2 + \varepsilon\|v\|_{L^2(\mathbb{T}_\lambda)}^2 + \frac{1}{\varepsilon}\|v\|_{L^2(\mathbb{T}_\lambda)}^{10}. \tag{2.37}$$

Therefore, we consider the essential part of the energy functional in (2.36), namely

$$\mathcal{E}(v) := \int_{\mathbb{T}_\lambda} \left(|\partial_x v|^2 - \frac{1}{2}|v|^2 \operatorname{Im}(v\overline{v}_x) \right) dx. \tag{2.38}$$

This is the same expression as the energy corresponding to (1.8) on the real line (see [9]). In view of (2.37) and the conservation of mass, when controlling the $\dot{H}^1$-norm of a solution v to (2.22), the above $\mathcal{E}(v)$ is just as good as the conservation law $E_1(v)$.

Applying the same strategy to the mixed term $|v|^2 \operatorname{Im}(v\overline{v}_x)$ and by using the Gagliardo-Nirenberg inequality (2.7), we get

$$\mathcal{E}(v) + 1 \gtrsim_{\delta, M(v)} \left(4\pi^2(1-\varepsilon-\delta)\delta - M(v)^2\right) \|\partial_x v\|^2_{L^2(\mathbb{T}_\lambda)}$$

for any $\varepsilon, \delta > 0$, where the constant 1 in the left-hand side above hides a polynomial in $M(v)$. Since $\sup_{\varepsilon,\delta>0}(1-\varepsilon-\delta)\delta = \frac{1}{4}$, this would yield the mass threshold condition $M(v) < \pi$.

However, as was noticed by Hayashi and Ozawa [19, 20] in the Euclidean case, the choice $\beta = \frac{3}{4}$ for the gauge transformation yields a neat expression for the corresponding energy functional and a better mass threshold condition, namely $M(v) < 2\pi$. By using the adaptation of a Gagliardo-Nirenberg inequality (see Section 2.1), we show that this threshold also carries over to the periodic setting.

In the proofs below all the norms are taken over $\mathbb{T}_\lambda$.

Lemma 2.6. *Let $\lambda \geq 1$. For any $f \in H^1(\mathbb{T}_\lambda)$ with $M[f] = \|f\|^2_{L^2(\mathbb{T}_\lambda)} < 2\pi$, we have:*

$$\|\partial_x f\|^2_{L^2(\mathbb{T}_\lambda)} \lesssim \mathcal{E}(f) + 1. \tag{2.39}$$

The implicit constant depends only on $M[f]$and blows up as $M[f] \nearrow 2\pi$.

Proof. Consider $g = \mathcal{G}_{-\frac{1}{4}}(f)$ and we note that

$$\begin{aligned}\mathcal{E}(f) &= \|\partial_x g\|^2_{L^2} - \frac{1}{16}\|g\|^6_{L^6} + \frac{1}{16}\left(\mu(g)^2 - 2\mu(g)\right)\|g\|^2_{L^2} - \frac{1}{2}\mu(g)\int \operatorname{Im}(g\partial_x\overline{g})dx + \frac{1}{8}\mu(g)\|g\|^4_{L^4} \\ &\geq \|\partial_x g\|^2_{L^2} - \frac{1}{16}\|g\|^6_{L^6} - \frac{1}{8}\mu(g)\|g\|^2_{L^2} - \frac{1}{2}\mu(g)\int \operatorname{Im}(g\partial_x\overline{g})dx\end{aligned}$$

and note that for any $\varepsilon > 0$ and any $\lambda \geq 1$,

$$\left|\frac{1}{2}\mu(g)\int \operatorname{Im}(g\partial_x\overline{g})dx\right| \leq \|g\|^3_{L^2}\|\partial_x g\|_{L^2} \leq \varepsilon\|\partial_x g\|^2_{L^2} + C_\varepsilon\|g\|^6_{L^2},$$

for some $C_\varepsilon \sim \varepsilon^{-1}$. We choose $\varepsilon > 0$ such that $\|f\|^4_{L^2}(\frac{1}{4\pi^2} + \frac{\varepsilon}{16}) < 1 - \varepsilon$, and by (2.7) [2]

[2] By using (2.3) at this point, the coercivity of $\mathcal{E}[\,\cdot\,] + 1$ would be obtained under $M[u_0] < 2\sqrt{2}$.

we then get

$$\begin{aligned}\mathcal{E}(f) \geq (1-\varepsilon)\|\partial_x g\|_{L^2}^2 - \frac{1}{16}(\frac{4}{\pi^2}+\varepsilon)\|\partial_x g\|_{L^2}^2\|g\|_{L^2}^4 - \frac{1}{16}K_\varepsilon\|g\|_{L^2}^6 \\ - \frac{1}{8}\mu(g)\|g\|_{L^2}^2 - C_\varepsilon\|g\|_{L^2}^6\end{aligned}$$

and thus

$$\mathcal{E}(f) + M(f)^3 \gtrsim \left((1-\varepsilon) - (\frac{1}{4\pi^2}+\frac{\varepsilon}{16})\|f\|_{L^2}^4\right)\|\partial_x g\|_{L^2}^2. \tag{2.40}$$

By combining (2.15) and (2.40), we deduce (2.39) and the proof is complete.

□

Inspired by the paper of Guo and Wu [18], we can improve the mass threshold below which we can control the $\dot{H}^1$-norm of f by using both the energy $\mathcal{E}(f)$ and the momentum

$$\mathcal{P}(f) := \int_{\mathbb{T}_\lambda} \operatorname{Im}(f\partial_x\overline{f})dx - \frac{1}{2}\|f\|_{L^4(\mathbb{T}_\lambda)}^4 \tag{2.41}$$

associated to (2.22), where we dropped the conserved term from (2.18). The key observation is to notice that by modulating f, the change in kinetic energy incurred resembles the main part of the momentum $\mathcal{P}(f)$.

Lemma 2.7. *Let $\lambda \geq 1$. For any $f \in H^1(\mathbb{T}_\lambda)$ with $M(f) = \|f\|_{L^2(\mathbb{T}_\lambda)}^2 < 4\pi$, we have:*

$$\|\partial_x f\|_{L^2(\mathbb{T}_\lambda)}^2 \lesssim |\mathcal{E}(f)| + \mathcal{P}(f)^2 + 1. \tag{2.42}$$

The implicit constant depends only on $M(f)$ and blows up as $M(f) \nearrow 4\pi$.

Proof. As in the proof of Lemma 2.6 above, let us consider $g = \mathcal{G}_{-\frac{1}{4}}(f)$ for which, according to (2.15), we have

$$\|\partial_x f\|_{L^2(\mathbb{T}_\lambda)} \sim \|\partial_x g\|_{L^2(\mathbb{T}_\lambda)}.$$

The main part is showing that

$$\|\partial_x g\|_{L^2(\mathbb{T}_\lambda)}^2 \lesssim |E_{\frac{3}{4}}(g)| + P_{\frac{3}{4}}(g)^2 + 1. \tag{2.43}$$

Indeed, this suffices to get (2.42) as we have

$$\begin{aligned}|P_{\frac{3}{4}}(g)| =& |P_1(f)| \leq |\mathcal{P}(f)| + \mu(f)M(f), \\ |E_{\frac{3}{4}}(g)| =& |E_1(f)| \lesssim |\mathcal{E}(f)| + \frac{1}{2}\mu(f)\|f\|_{L^4}^4 + \mu(f)|\mathcal{P}(f)| + \mu(f)^2 M(f),\end{aligned}$$

and we can use (2.37).

From (2.18)-(2.19), we recall that

$$P_{\frac{3}{4}}(g) = \int_{\mathbb{T}_\lambda} \mathrm{Im}(g\partial_x\overline{g})dx - \frac{1}{4}\|g\|_{L^4}^4 + \frac{3}{4}\mu(g)M(g),$$
$$E_{\frac{3}{4}}(g) = \|\partial_x g\|_{L^2}^2 - \frac{1}{16}\|g\|_{L^6}^6 + \frac{3}{8}\mu(g)\|g\|_{L^4}^4 + \frac{3}{2}\mu(g)P_{\frac{3}{4}}(g) - \frac{9}{16}\mu(g)^2M(g).$$

In order to get (2.43), we consider the modulated function $g_\alpha(x) := e^{i\alpha x}g(x)$ with $\alpha \in \mathbb{Z}_\lambda$ and $\alpha > 0$ to be chosen later. We have

$$\|\partial_x g_\alpha\|_{L^2}^2 = \|\partial_x g\|_{L^2}^2 + \alpha^2\|g\|_{L^2}^2 - 2\alpha\int_{\mathbb{T}_\lambda} \mathrm{Im}(g\partial_x\overline{g})dx \tag{2.44}$$

and therefore

$$E_{\frac{3}{4}}(g_\alpha) - E_{\frac{3}{4}}(g) = \alpha^2M(g) - 2\alpha\int_{\mathbb{T}_\lambda} \mathrm{Im}(g\partial_x\overline{g})dx + \frac{3}{8}\mu(g)\Big(P_{\frac{3}{4}}(g_\alpha) - P_{\frac{3}{4}}(g)\Big). \tag{2.45}$$

Since

$$\mathrm{Im}(g_\alpha\partial_x\overline{g_\alpha}) = \mathrm{Im}(g\partial_x\overline{g}) - \alpha|g|^2,$$

we also have

$$P_{\frac{3}{4}}(g_\alpha) - P_{\frac{3}{4}}(g) = -\alpha M(g). \tag{2.46}$$

Therefore

$$\frac{1}{2\alpha}E_{\frac{3}{4}}(g_\alpha) - \frac{1}{2\alpha}E_{\frac{3}{4}}(g) = \frac{\alpha}{2}M(g) - \int_{\mathbb{T}_\lambda} \mathrm{Im}(g\partial_x\overline{g})dx - \frac{3}{4}\mu(g)M(g)$$

and thus we find that

$$\frac{1}{2\alpha}E_{\frac{3}{4}}(g_\alpha) - \frac{1}{2\alpha}E_{\frac{3}{4}}(g) - \frac{\alpha}{2}M(g) + \frac{1}{4}\|g\|_{L^4}^4 = -P_{\frac{3}{4}}(g). \tag{2.47}$$

We now use the Gagliardo-Nirenberg inequality (2.6) to give a lower bound to the first term in (2.47); we drop the positive term $\frac{3}{8}\mu[g_\alpha]\|g_\alpha\|_{L^4}^4$. Also, we use (2.46), and taking into account that the Lebesgue norms of g_α and g coincide, we have

$$\begin{aligned} E_{\frac{3}{4}}(g_\alpha) \geq &\|g\|_{L^6}^6\left(C_{GN}^{-18}\left(1+\frac{\delta}{5\pi\lambda}\right)^{-4}\frac{\|g\|_{L^6}^{12}}{\|g\|_{L^4}^{16}} - \frac{1}{16}\right) - \frac{1}{\pi\lambda\delta}M(g) \\ &+ \frac{3}{2}\mu(g)\left(P_{\frac{3}{4}}(g) - \alpha M(g)\right) - \frac{9}{16}\mu(g)^2M(g) \end{aligned}$$

By (2.47), we then get

$$\begin{aligned} |P_{\frac{3}{4}}(g)| + \frac{3}{4}\mu(g)M(g) \geq &\frac{1}{4}\|g\|_{L^4}^4 - \frac{\alpha}{2}M(g) - \frac{1}{2\alpha}\varphi\Big(\frac{\|g\|_{L^6}^6}{\|g\|_{L^4}^8}\Big)\|g\|_{L^4}^8 \\ &- \frac{1}{2\alpha}\left(|E_{\frac{3}{4}}(g)| + \frac{3}{2}\mu(g)|P_{\frac{3}{4}}(g)| + \frac{9}{16}\mu(g)^2M(g) + \frac{1}{\pi\lambda\delta}M(g)\right), \end{aligned}$$

where

$$\varphi(x) := \left(\frac{1}{16} - C_{\mathrm{GN}}^{-18}\left(1+\frac{\delta}{5\pi\lambda}\right)^{-4} x^2\right) x$$

and for which we have

$$\max_{x>0}\varphi(x) \geq \max_{x>0}\left(\frac{1}{16} - C_{\mathrm{GN}}^{-18}x^2\right) x = \frac{1}{64\pi}.$$

We now balance the terms $\frac{\alpha}{2}M(g)$ and $\frac{1}{128\pi\alpha}\|g\|_{L^4}^8$ by choosing

$$\alpha_* := \frac{\|g\|_{L^4}^4}{8\sqrt{\pi}\|g\|_{L^2}}.$$

However, in order to correctly define g_α as a periodic function on $\mathbb{T}_\lambda$, we take

$$\alpha := \frac{1}{\lambda}\Big([\lambda\alpha_*] + 1\Big) \tag{2.48}$$

(here, by $[x]$ we denote the integer part of x). Then

$$\begin{aligned} -\frac{\alpha}{2}M(g) - \frac{1}{128\pi\alpha}\|g\|_{L^4}^8 &\geq -\frac{\alpha_*}{2}M(g) - \frac{1}{128\pi\alpha_*}\|g\|_{L^4}^8 - \frac{1}{2\lambda}M(g) \\ &= -\alpha_* M(g) - \frac{1}{2\lambda}M(g) \end{aligned}$$

and taking into account that $\lambda \geq 1$, we deduce

$$\begin{aligned} |P_{\frac{3}{4}}(g)| + M(g)^2 + M(g) \geq& \frac{1}{4}\|g\|_{L^4}^4 - \alpha_* M(g) \\ &- \frac{1}{2\alpha_*}\Big(|E_{\frac{3}{4}}(g)| + M(g)|P_{\frac{3}{4}}(g)| + M(g)^3 + \delta^{-1}M(g)\Big). \end{aligned} \tag{2.49}$$

We consider the following positive reals

$$\begin{aligned} a &:= \frac{1}{4}\left(1 - \frac{1}{2\sqrt{\pi}}\|g\|_{L^2}\right), \\ b &:= 4\sqrt{\pi}\|g\|_{L^2}\Big(|E_{\frac{3}{4}}(g)| + M(g)|P_{\frac{3}{4}}(g)| + M(g)^3 + \delta^{-1}M(g)\Big), \\ c &:= |P_{\frac{3}{4}}(g)| + M(g)^2 + M(g). \end{aligned}$$

Thus, the inequality (2.49) provides the following

$$c \geq a\|g\|_{L^4}^4 - \frac{b}{\|g\|_{L^4}^4}.$$

It follows that

$$\|g\|_{L^4}^4 \leq \frac{c + \sqrt{c^2 + 4ab}}{2a} \lesssim c + b^{\frac{1}{2}}$$

and so we obtain

$$\|g\|_{L^4}^8 \lesssim c^2 + b \lesssim |E_{\frac{3}{4}}(g)| + P_{\frac{3}{4}}(g)^2 + 1. \tag{2.50}$$

Therefore, by using again (2.6),

$$\begin{aligned}
&\|\partial_x g\|_{L^2}^2 + \frac{2}{\delta}\mu(g) \\
&\quad = E_{\frac{3}{4}}(g) + \frac{1}{16}\|g\|_{L^6}^6 - \frac{3}{8}\mu(g)\|g\|_{L^4}^4 - \frac{3}{2}\mu(g)P_{\frac{3}{4}}(g) + \frac{9}{16}\mu(g)^2 M(g) + \frac{2}{\delta}\mu(g) \\
&\quad \lesssim |E_{\frac{3}{4}}(g)| + |P_{\frac{3}{4}}(g)| + 1 + \left(\|\partial_x g\|_{L^2}^2 + \frac{2}{\delta}\mu(g)\right)^{\frac{1}{3}} \|g\|_{L^4}^{\frac{16}{3}}
\end{aligned}$$

where the implicit constant can be taken to depend only on $M(g)$. Then either

$$\|\partial_x g\|_{L^2}^2 + \frac{2}{\delta}\mu(g) \lesssim |E_{\frac{3}{4}}(g)| + |P_{\frac{3}{4}}(g)| + 1$$

or

$$\left(\|\partial_x g\|_{L^2}^2 + \frac{2}{\delta}\mu(g)\right)^{\frac{2}{3}} \lesssim \|g\|_{L^4}^{\frac{16}{3}}$$

and we use (2.50). In both cases, (2.43) holds and the proof is completed.

□

Chapter 3

Global well-posedness below the energy space

Let us outline here the content of this chapter. In Section 3.1, we introduce function spaces and review linear estimates (including a revised bilinear L^4-Strichartz estimate). After applying the gauge transformation augmented with a translation operator, Theorem 1.2 is reduced to Proposition 3.9 concerning the global solutions of the periodic gauge equivalent equation (2.22). Then, we build up the I-method apparatus. We recall that in Section 2.4 we showed the coercivity property of the energy functional in the periodic setting. In particular, by incorporating the momentum functional, we have $\dot{H}^1$-norm control under the improved mass threshold $M[u_0] < 4\pi$.

In Section 3.2, we provide a modified local well-posedness result based on existing local multi-linear estimates and an interpolation lemma for the I-operator. In this instantiation of the I-method scheme, we construct a third generation modified energy functional in Section 3.3 after revisiting the first and second generation energies, as well as discussing the frequency regions that previously did not allow reaching the regularity $s = \frac{1}{2}$. In the same section, we also revisit the crafting of the resonant set from the real line setting and we provide pointwise bounds on multipliers which are used in the following two sections.

In Section 3.4 we analyze the growth of the third generation modified energy and conclude with its almost conservation property, whereas in Section 3.5 we show that it stays close to the first generation modified energy. The almost conservation of the modified momentum follows similarly to the Euclidean case and is also established in Section 3.5.

In Section 3.6, we modify the usual I-method argument to include the almost conserved momentum and we finish the proof of Proposition 3.9.

3.1 Preliminaries

In this subsection we review the basic properties in the Fourier restriction norm method that are by now well-known – see e.g. [49, Subsection 2.6].

The convention we use for the Fourier transform of a $2\pi\lambda$-periodic function is

$$\widehat{f}(k) = \int_0^{2\pi\lambda} e^{-ikx} f(x)dx \quad , \quad k \in \mathbb{Z}_\lambda$$

which is inverted by

$$\breve{g}(x) = \frac{1}{2\pi\lambda} \sum_{k\in\mathbb{Z}_\lambda} e^{ikx} g(k) \quad , \quad x \in [0, 2\pi\lambda].$$

The convolution products on $\mathbb{T}_\lambda$ and $\mathbb{Z}_\lambda$ are given by

$$f * g(x) = \int_0^{2\pi\lambda} f(x-y)g(y)\,dy,$$
$$a \star b(k) = \frac{1}{2\pi\lambda} \sum_{h\in\mathbb{Z}_\lambda} a(k-h)b(h),$$

respectively. We have $\widehat{fg}(k) = \widehat{f} \star \widehat{g}(k)$, and by endowing $\mathbb{Z}_\lambda$ with the scaled counting measure $(dk)_\lambda := \frac{1}{2\pi\lambda} d\#$, the inner products on $L^2(\mathbb{T}_\lambda)$ and $L^2(\mathbb{Z}_\lambda)$ are

$$\langle f, g\rangle_{L^2(\mathbb{T}_\lambda)} = \int_0^{2\pi\lambda} f(x)\overline{g(x)}\,dx,$$
$$\langle a, b\rangle_{L^2(\mathbb{Z}_\lambda)} = \frac{1}{2\pi\lambda} \sum_{k\in\mathbb{Z}_\lambda} a(k)\overline{b(k)} = \int_{\mathbb{Z}_\lambda} a(k)\overline{b(k)}(dk)_\lambda,$$

respectively. Then, the Parseval and Plancherel identities are written as

$$\langle f, \breve{a}\rangle_{L^2(\mathbb{T}_\lambda)} = \langle \widehat{f}, a\rangle_{L^2(\mathbb{Z}_\lambda)},$$
$$\|f\|_{L^2(\mathbb{T}_\lambda)} = \|\widehat{f}\|_{L^2(\mathbb{Z}_\lambda)}.$$

The Sobolev space $H^s(\mathbb{T}_\lambda)$, respectively the Fourier Lebesgue space $\mathcal{F}L^{s,r}(\mathbb{T}_\lambda)$ are the completion of the $2\pi\lambda$-periodic C^∞ functions with respect to the norms

$$\|f\|_{H^s(\mathbb{T}_\lambda)} := \|\langle k\rangle^s \widehat{f}(k)\|_{L^2(\mathbb{Z}_\lambda)}, \tag{3.1}$$
$$\|f\|_{\mathcal{F}L^{s,r}(\mathbb{T}_\lambda)} := \|\langle k\rangle^s \widehat{f}(k)\|_{L^r(\mathbb{Z}_\lambda)}, \tag{3.2}$$

where $\langle k\rangle := (1+|k|^2)^{\frac{1}{2}}$, $k \in \mathbb{Z}_\lambda$, for any $s \geq 0$, $r \geq 1$. We also use the homogeneous Sobolev norm:

$$\|f\|_{\dot{H}^s(\mathbb{T}_\lambda)} := \||k|^s \widehat{f}(k)\|_{L^2(\mathbb{Z}_\lambda)}.$$

Remark 3.1. Notice that for any $k \neq 0$, uniformly in $\lambda \geq 1$, we have

$$|k| \leq \langle k \rangle \lesssim \lambda |k| \,. \tag{3.3}$$

Also, we have

$$\#\{k \in \mathbb{Z}_\lambda : \langle k \rangle \sim 1\} = O(\lambda)\,.$$

By $\mathcal{S}_\lambda$ we denote the class of functions $u^\lambda : \mathbb{R} \times \mathbb{T}_\lambda \to \mathbb{C}$ which are Schwartz in t, $2\pi\lambda$-periodic and C^∞ in x. With a slight abuse of notation, the time-space Fourier transform and its inverse are

$$\widehat{u}(\tau, k) = \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} e^{-i(\tau t + kx)} u(t,x)\, dx\, dt \quad , \quad \tau \in \mathbb{R}, k \in \mathbb{Z}_\lambda,$$
$$\check{v}(t,x) = \int_{\mathbb{Z}_\lambda} \int_{\mathbb{R}} e^{i(\tau t + kx)} v(\tau, k)\, d\tau\, (dk)_\lambda \quad , \quad t \in \mathbb{R}, x \in \mathbb{T}_\lambda.$$

Nonlinear interactions take on the Fourier side the form

$$\widehat{uv}(\tau,k) = \widehat{u} \star \widehat{v}(\tau,k) = \frac{1}{2\pi\lambda} \sum_{k_1 \in \mathbb{Z}_\lambda} \int_{\mathbb{R}} \widehat{u}(\tau_1, k_1) \widehat{v}(\tau - \tau_1, k - k_1)\, d\tau_1$$
$$= \int_{k_1 + k_2 = k} \int_{\tau_1 + \tau_2 = \tau} \widehat{u}(\tau_1, k_1) \widehat{v}(\tau_2, k_2)\, d\tau_1\, (dk_1)_\lambda.$$

The unitary group on $L^2(\mathbb{T}_\lambda)$ determined by the linear Schrödinger equation on $\mathbb{T}_\lambda$ is given by

$$(U_\lambda(t) f)(x) = \frac{1}{2\pi\lambda} \sum_{k \in \mathbb{Z}_\lambda} e^{ikx + itk^2} \widehat{f}(k). \tag{3.4}$$

For $s, b \in \mathbb{R}$ (spatial and temporal regularity indices), we define the $X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)$ space as the completion of $\mathcal{S}_\lambda$ under the norm

$$\|u\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)} := \|\langle k \rangle^s \langle \tau + k^2 \rangle^b \widehat{u}(t,k)\|_{L^2_\tau L^2_k(\mathbb{R} \times \mathbb{Z}_\lambda)}. \tag{3.5}$$

It is well known that the (continuous) embedding $X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda) \subset C_t H^s_x(\mathbb{R} \times \mathbb{T}_\lambda)$ holds if and only if $b > \frac{1}{2}$.

From the work of Herr [21], we know that the trilinear estimate needed for the local well-posedness theory (see Lemma 3.13 below) holds only with $b = \frac{1}{2}$ since the local smoothing and maximal function estimates for the linear Schrödinger propagator are no longer available – one has to rely merely on the L^4-Strichartz estimate of Bourgain [5] and on Sobolev inequalities. We introduce the spaces $Y^{s,b}$ and Z^s via the norms

$$\|u\|_{Y^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)} := \|\langle k \rangle^s \langle \tau + k^2 \rangle^b \widehat{u}(t,k)\|_{L^2_k L^1_\tau(\mathbb{Z}_\lambda \times \mathbb{R})}, \tag{3.6}$$

$$\|u\|_{Z^s(\mathbb{R} \times \mathbb{T}_\lambda)} := \|u\|_{X^{s,\frac{1}{2}}(\mathbb{R} \times \mathbb{T}_\lambda)} + \|u\|_{Y^{s,0}(\mathbb{R} \times \mathbb{T}_\lambda)}, \tag{3.7}$$

and the companion space $\widetilde{Z}^s$ by

$$\|u\|_{\widetilde{Z}^s(\mathbb{R}\times\mathbb{T}_\lambda)} := \|u\|_{X^{s,-\frac{1}{2}}(\mathbb{R}\times\mathbb{T}_\lambda)} + \|u\|_{Y^{s,-1}(\mathbb{R}\times\mathbb{T}_\lambda)}. \tag{3.8}$$

We have $Y^{s,0}(\mathbb{R}\times\mathbb{T}_\lambda) \subset C_tH_x^s(\mathbb{R}\times\mathbb{T}_\lambda)$ and therefore $Z^s = X^{s,\frac{1}{2}} \cap Y^{s,0} \subset C_tH_x^s$.

For a given time interval J, the time-localized versions of the Fourier restriction norms are defined via

$$\|u\|_{X^{s,b}(J\times\mathbb{T}_\lambda)} := \inf\{\|v\|_{X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)} : v_{|J} = u\}, \tag{3.9}$$

and similarly for $Y^{s,b}(J\times\mathbb{T}_\lambda)$, $Z^s(J\times\mathbb{T}_\lambda)$, and $\widetilde{Z}^s(J\times\mathbb{T}_\lambda)$.

By the Riemann-Lebesgue lemma and Hölder inequality, we have

$$\begin{aligned}\|u\|_{L^\infty_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} &\lesssim \big\| \|\widehat{u}\|_{L^1_\tau(\mathbb{R})}\big\|_{L^1_k(\mathbb{Z}_\lambda)}\\ &\le \left(\frac{1}{2\pi\lambda}\sum_{k\in\mathbb{Z}_\lambda}\langle k\rangle^{-1-}\right)^{\frac{1}{2}}\left(\frac{1}{2\pi\lambda}\sum_{k\in\mathbb{Z}_\lambda}\langle k\rangle^{1+}\|\widehat{u}(\tau,k)\|^2_{L^1_\tau(\mathbb{R})}\right)^{\frac{1}{2}}\\ &\lesssim \left\|\langle k\rangle^{\frac{1}{2}+}\|\widehat{u}\|_{L^1_\tau(\mathbb{R})}\right\|_{L^2_k(\mathbb{Z}_\lambda)}\end{aligned}$$

and thus

$$\|u\|_{L^\infty_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \|u\|_{Y^{\frac{1}{2}+,0}(\mathbb{R}\times\mathbb{T}_\lambda)}. \tag{3.10}$$

Similarly, by Minkowski's integral inequality, Riemann-Lebesgue lemma and Plancherel's identity, one obtains

$$\|u\|_{L^\infty_t H^s_x(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \|u\|_{Y^{s,0}(\mathbb{R}\times\mathbb{T}_\lambda)}, \tag{3.11}$$

for any $s\in\mathbb{R}$.

Additionally, we have the following linear estimates. Here and throughout this thesis, $\eta(t)$ denotes a smooth compactly supported cut-off in time.

Lemma 3.2. *[21, Lemma 3.6] Let $s\in\mathbb{R}$. There exists $c>0$ such that*

$$\|\eta(t)U_\lambda(t)f\|_{Z^s(\mathbb{R}\times\mathbb{T}_\lambda)} \le c\|f\|_{H^s(\mathbb{T}_\lambda)} \tag{3.12}$$

$$\left\|\eta(t)\int_0^t U_\lambda(t-\tau)F(\tau,\cdot)d\tau\right\|_{Z^s(\mathbb{R}\times\mathbb{T}_\lambda)} \le c\|F\|_{\widetilde{Z}^s(\mathbb{R}\times\mathbb{T}_\lambda)} \tag{3.13}$$

for all $f\in H^s$ and all $F\in\mathcal{S}_\lambda$.

Lemma 3.3. *Let $2\le p,q<\infty$, $b\ge\frac{1}{2}-\frac{1}{p}$, $s\ge\frac{1}{2}-\frac{1}{q}$, $\lambda\ge1$. For $u\in\mathcal{S}_\lambda$, we have*

1. Sobolev estimates:

$$\|u\|_{L_t^p H_x^s(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)}, \tag{3.14}$$

$$\|u\|_{L_t^\infty H_x^s(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \|u\|_{X^{s,\frac{1}{2}+}(\mathbb{R}\times\mathbb{T}_\lambda)}, \tag{3.15}$$

$$\|u\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)}, \tag{3.16}$$

$$\|u\|_{L_t^\infty L_x^\infty(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \|u\|_{X^{\frac{1}{2}+,\frac{1}{2}+}(\mathbb{R}\times\mathbb{T}_\lambda)}; \tag{3.17}$$

2. Strichartz estimates:

$$\|u\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \|u\|_{X^{0,\frac{3}{8}}(\mathbb{R}\times\mathbb{T}_\lambda)}, \tag{3.18}$$

$$\|u\|_{L_{t,x}^6(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \lambda^{0+}\|u\|_{X^{0+,\frac{1}{2}+}(\mathbb{R}\times\mathbb{T}_\lambda)} \tag{3.19}$$

with implicit constants independent of $\lambda \geq 1$.

One can prove the first part by using the interaction representation

$$\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)} = \|U_\lambda(-t)u(t,x)\|_{H_x^s H_t^b(\mathbb{T}_\lambda\times\mathbb{R})}, \tag{3.20}$$

the classical Sobolev inequalities, Minkowski's integral inequality and the fact that the operators $U_\lambda(t)$ are unitary on $H_x^s(\mathbb{T}_\lambda)$. For the second part, we recall that the $L^4(\mathbb{T})$- and $L^6(\mathbb{T})$-Strichartz estimates on finite length intervals are due to Bourgain [5]; however, the global-in-time versions also hold – see e.g. [22, Proposition 2.2.4], [49, Proposition 2.13]. The corresponding estimates on the scaled torus (i.e. (3.18) and (3.19)) can be justified by going over the Stichartz estimates due to Bourgain [5] and revisiting the counting arguments, but now accounting for Fourier modes in $\mathbb{Z}_\lambda$ rather than $\mathbb{Z}$ (e.g. there are $O(\lambda M)$ elements k in $\mathbb{Z}_\lambda$ satisfying $|k| \lesssim M$, there is a normalizing factor in the measure placed on $\mathbb{Z}_\lambda$, etc.). It turns out that the $L^4(\mathbb{T}_\lambda)$-Strichartz estimate has an implicit constant independent of λ, while the $L^6(\mathbb{T}_\lambda)$-Strichartz estimate has a logarithmic loss in λ (in addition to the loss in derivative).

By interpolating the Strichartz estimate (3.19) with the Sobolev inequality (3.16) (for $p = q = 6$), we also have

$$\|u\|_{L_{t,x}^6(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \lambda^{0+}\|u\|_{X^{0+,\frac{1}{2}-}(\mathbb{R}\times\mathbb{T}_\lambda)}. \tag{3.21}$$

We note that the estimates (3.14)-(3.21) also hold for Fourier restriction norms on a time interval J rather than on the entire real line.

We record the following scaling properties of the space-time norms introduced above when using (1.5) and a parameter $\lambda \geq 1$:

$$\|u^\lambda\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{T}_\lambda)} = \lambda^{\frac{2}{p}+\frac{1}{q}-\frac{1}{2}}\|u\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{T})},$$
$$\|u^\lambda\|_{L_t^p \dot{H}_x^s(\mathbb{R}\times\mathbb{T}_\lambda)} = \lambda^{-s+\frac{2}{p}}\|u\|_{L_t^p \dot{H}_x^s(\mathbb{R}\times\mathbb{T})},$$

and

$$\lambda^{-s+\frac{2}{p}}\|u\|_{L^p_t H^s_x(\mathbb{R}\times\mathbb{T})} \lesssim \|u^\lambda\|_{L^p_t H^s_x(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \lambda^{\frac{2}{p}}\|u\|_{L^p_t H^s_x(\mathbb{R}\times\mathbb{T})}.$$

For $s, b \geq 0$, we have

$$\lambda^{-1}\|u^\lambda\|_{X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T})} \lesssim \lambda^{-1+s+2b}\|u^\lambda\|_{X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)},$$

while for $s \geq 0$, $b < 0$, we record

$$\lambda^{-1+2b}\|u^\lambda\|_{X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T})} \lesssim \lambda^{-1+s}\|u^\lambda\|_{X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)}.$$

We also use the following lemma when dealing with sharp time-cutoff functions:

Lemma 3.4. *Let $s \in \mathbb{R}$ and $0 < b' < b < \frac{1}{2}$. Suppose that $\phi \in H^b_t(\mathbb{R})$. Then,*

$$\|\phi u\|_{X^{s,b'}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \|\phi\|_{H^b_t(\mathbb{R})}\|u\|_{X^{s,\frac{1}{2}}(\mathbb{R}\times\mathbb{T}_\lambda)}.$$

Proof. By (3.20),

$$\|\phi u\|_{X^{s,b'}(\mathbb{R}\times\mathbb{T}_\lambda)} = \|\phi(t)U_\lambda(-t)u(t,x)\|_{H^s_x H^{b'}_t(\mathbb{T}_\lambda\times\mathbb{R})}$$

and let $J_t := \langle \partial_t \rangle$. Then, via the fractional Leibniz rule, we have

$$\|\phi(t)U_\lambda(-t)u(t)\|_{H^{b'}_t} \lesssim \|J^{b'}_t\phi\|_{L^p_t}\,\|U_\lambda(-t)u(t)\|_{L^q_t} + \|\phi\|_{L^q_t}\left\|J^{b'}_t\big(U_\lambda(-t)u(t)\big)\right\|_{L^p_t}, \quad (3.22)$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. We take $p > 2$ so that we have the continuous Sobolev embedding $H^{b-b'}_t(\mathbb{R}) \subset L^p_t(\mathbb{R})$. By also using the Sobolev embedding $H^b_t(\mathbb{R}) \subset L^q_t(\mathbb{R})$, the conclusion follows from (3.22) and the triangle inequality for the $H^s_x(\mathbb{R})$-norm.

□

3.1.1 A bilinear L^4-Strichartz estimate

The following result is a key ingredient in the analysis of the almost conservation estimates as it is a refinement of the L^4-Strichartz estimate that provides a decaying factor in λ. Such an estimate is similar to the bilinear L^4-estimate in the non-periodic setting [9, Lemma 7.1], and we point out that for $\lambda \to \infty$, we recover the same decay rate. For Schrödinger evolutions on the one-dimensional torus, this estimate (but without pointing out the alternative (ii)) was first proved in [13]. See also [38].

Lemma 3.5. *Let $\lambda \geq 1$, $N_1, N_2 \in 2^{\mathbb{Z}}$ and suppose ϕ_1, ϕ_2 are smooth functions on $\mathbb{T}_\lambda$ with* $\operatorname{supp}(\widehat{\phi_j}) \subset \{k \in \mathbb{Z}_\lambda : |k| \sim N_j\}$, $j = 1, 2$. *Assume that either*

(i) *$N_1 \gg N_2$, or*

(ii) *$N_1 \sim N_2$ and $k_1 k_2 < 0$ for all $k_1 \in \operatorname{supp}(\widehat{\phi_1})$, $k_2 \in \operatorname{supp}(\widehat{\phi_2})$.*

Then

$$\big\| \big(\eta(t) U_\lambda(t)\phi_1\big)\big(\eta(t)U_\lambda(t)\phi_2\big)\big\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim C(\lambda, N_1)\|\phi_1\|_{L^2_x(\mathbb{T}_\lambda)}\|\phi_2\|_{L^2_x(\mathbb{T}_\lambda)} \tag{3.23}$$

where

$$C(\lambda, N_1) = \begin{cases} 1 & , \text{ if } N_1 \lesssim 1 \\ (\frac{1}{\lambda} + \frac{1}{N_1})^{\frac{1}{2}} & , \text{ if } N_1 \gg 1 \end{cases}. \tag{3.24}$$

Moreover, suppose that u_1, u_2 *are Fourier supported in* $\{|k_1| \sim N_1\}$ *and* $\{|k_2| \sim N_2\}$*, respectively, for all times* t*. Then, under the same assumption on the two frequency supports, we have*

$$\|\eta(t)u_1 \cdot \eta(t)u_2\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim_\varepsilon C(\lambda, N_1)^{1-2\varepsilon}\|u_1\|_{X^{0,\frac{1}{2}-\varepsilon}(\mathbb{R}\times\mathbb{T}_\lambda)}\|u_2\|_{X^{0,\frac{1}{2}-\varepsilon}(\mathbb{R}\times\mathbb{T}_\lambda)}, \tag{3.25}$$

for any $\varepsilon > 0$ *sufficiently small.*

Remark 3.6. In [53, Proposition 2.1], there seems to be a mistake in the case $N_1 \sim N_2$: the two Fourier supports should be localized on opposite sides of the origin on the real line in order for (3.23) to be true. The estimate with this additional assumption was used in proving Cases (2) and (3) of [53, Lemma 7.5]. Although with similar ideas as in the proof of [13, Proposition 3.7], we present its proof here so that this observation becomes clear.

Proof. By Plancherel's identity, the left hand side of (3.23) becomes

$$\left\| \int_{\tau_1+\tau_2=\tau}\int_{k_1+k_2=k} \widehat{\eta}(\tau_1 + k_1^2)\widehat{\eta}(\tau_2 + k_2^2)\widehat{\phi_1}(k_1)\widehat{\phi_2}(k_2)(dk_1)_\lambda d\tau_1 \right\|_{L^2_\tau L^2_k}.$$

We denote $\psi := \widehat{\eta} * \widehat{\eta}$, and without loss of generality, we can assume that ψ is $\mathbb{R}$-valued and non-negative. [1] Then

$$\int_{\mathbb{R}} \widehat{\eta}(\tau_1 + k_1^2)\widehat{\eta}(\tau - \tau_1 + k_2^2)d\tau_1 = \psi(\tau + k_1^2 + k_2^2) \geq 0$$

and we denote

$$M := \left(\sup_{k,\tau} \int_{k_1+k_2=k} \psi(\tau + k_1^2 + k_2^2)(dk_1)_\lambda \right)^{\frac{1}{2}}.$$

[1] In general, we can write $\psi = \psi_+ - \psi_- + i\psi^+ - i\psi^-$ with the four components satisfying the non-negativity assumption, from where we can carry on analogous arguments for each of these terms.

By Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned}
&\left\|\int_{k_1+k_2=k}\psi(\tau+k_1^2+k_2^2)\widehat{\phi_1}(k_1)\widehat{\phi_2}(k_2)(dk_1)_\lambda\right\|_{L^2_\tau L^2_k}\\
&\quad\le\left\|\left(\int_{k_1+k_2=k}\psi(\tau+k_1^2+k_2^2)(dk_1)_\lambda\right)^{\frac12}\right.\\
&\qquad\left.\times\left(\int_{k_1+k_2=k}\psi(\tau+k_1^2+k_2^2)|\widehat{\phi_1}(k_1)|^2|\widehat{\phi_2}(k_2)|^2(dk_1)_\lambda\right)^{\frac12}\right\|_{L^2_\tau L^2_k}\\
&\quad\le M\left(\int_{\mathbb{Z}_\lambda}\int_{\mathbb{Z}_\lambda}\int_{\mathbb{R}}\psi(\tau+k_1^2+k_2^2)|\widehat{\phi_1}(k_1)|^2|\widehat{\phi_2}(k_2)|^2\,d\tau\,(dk_1)_\lambda\,(dk)_\lambda\right)^{\frac12}\\
&\quad\le M\|\psi\|^{\frac12}_{L^1(\mathbb{R})}\|\phi_1\|_{L^2(\mathbb{T}_\lambda)}\|\phi_2\|_{L^2(\mathbb{T}_\lambda)}\,.
\end{aligned}$$

Thus, in order to obtain (3.23), it remains to show that $M\lesssim C(\lambda,N_1)$.

Since ψ is a Schwartz function, it is rapidly decaying, and so we can split $\mathbb{R}$ into disjoint intervals I_j $(j\in\mathbb{Z})$ [2] such that for all j we have $|I_j|\sim 1$ and $\|\psi_{|I_j}\|_{L^\infty}\lesssim 2^{-|j|}$. Given $k\in\mathbb{Z}_\lambda$, $\tau\in\mathbb{R}$, and $j\in\mathbb{Z}$, we consider the set

$$S_{k,\tau,j}=\{k_1\in\mathbb{Z}_\lambda : k_1\in\operatorname{supp}(\widehat{\phi_1})\,,\ k-k_1\in\operatorname{supp}(\widehat{\phi_2})\,,\ \tau+k_1^2+(k-k_1)^2\in I_j\}$$

and we estimate

$$M\lesssim\left(\sup_{k,\tau}\sum_{j\in\mathbb{Z}}\left(\frac{1}{\lambda}\#S_{k,\tau,j}\right)2^{-|j|}\right)^{\frac12},$$

where $\#S_{k,\tau,j}$ denotes the cardinality of $S_{k,\tau,j}$.

If $N_1\lesssim 1$, then clearly

$$\#S_{k,\tau,j}\le\#\left\{k_1\in\frac{1}{\lambda}\mathbb{Z} : |k_1|\lesssim 1\right\}\lesssim\lambda$$

and thus $M\lesssim 1$.

Now let us assume $N_1\gg 1$. To estimate the cardinality of a nonempty set $S_{k,\tau,j}$, we denote

$$f_{k,\tau}(k_1):=\tau+k_1^2+(k-k_1)^2.$$

Notice that

$$|f'_{k,\tau}(k_1)|=2|k_1-(k-k_1)|\sim N_1, \tag{3.26}$$

and that this property holds not only when $N_1\gg N_2$ but also when k_1 and $k-k_1$ have opposite signs, and this is ensured by assumption (ii). From (3.26) and the mean value theorem, we get that

$$\#S_{k,\tau,j}\lesssim 1+\frac{\lambda}{N_1},$$

[2] If ψ were compactly supported, it would be enough to consider only one such interval, namely a finite-length interval which includes the support of ψ.

uniformly in j (if $\lambda \lesssim N_1$ there might be only one element in $S_{k,\tau,j}$).

For the last part, by the transference principle for $X^{s,b}$ spaces (see for example [49, Lemma 2.9]), the estimate (3.23) implies

$$\|\eta(t)u_1 \cdot \eta(t)u_2\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim C(\lambda, N_1)\|u_1\|_{X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{T}_\lambda)}\|u_2\|_{X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{T}_\lambda)}. \tag{3.27}$$

On the other hand, by Hölder inequality and the L^4-Strichartz estimate, we have

$$\|\eta(t)u_1 \cdot \eta(t)u_2\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \|u_1\|_{X^{0,\frac{3}{8}}(\mathbb{R}\times\mathbb{T}_\lambda)}\|u_2\|_{X^{0,\frac{3}{8}}(\mathbb{R}\times\mathbb{T}_\lambda)}. \tag{3.28}$$

By interpolating (3.27) and (3.28), we obtain (3.25) for $\varepsilon > 0$ sufficiently small.

□

Remark 3.7. We point out that the implicit constant in (3.25) depends on ε. Hence, we cannot disregard the logarithmic loss in the constant $C(\lambda, N_1)$. This loss is essentially the reason for which we need to introduce the second correction term in (3.79) in the third iteration of the I-method (see also Remark 3.28).

Remark 3.8. Notice that, under assumption (i) of the above lemma, the estimate (3.25) holds if we replace one of the functions on the left hand side with its conjugate.

We use the above bilinear estimate essentially in the regime $1 \leq \lambda \lesssim N_1$, and thus, in our estimates, $C(\lambda, N_1) \sim \lambda^{-\frac{1}{2}}$.

We recall that the local well-posedness theory for (2.22) via a fixed point argument in the space Z^1 was developed in [21, 22] (see the estimates in Lemma 3.13 below). Therefore, in order to get Theorem 1.2, we aim to prove that the H^s-solutions of (2.22) exist globally in time in the following sense:

Proposition 3.9. *Let $\frac{1}{2} \leq s < 1$ and $v_0 \in H^s(\mathbb{T})$ with $M(v_0) < 4\pi$. Then for any $\varepsilon > 0$, there exists $c = c(\|v_0\|_{H^s(\mathbb{T})}, M(v_0), \varepsilon) < \infty$ such that for all $T > 0$, the solution v of (2.22) with $v(0) = v_0$ satisfies*

$$\sup_{0\leq t\leq T} \|v(t)\|_{H^s_x(\mathbb{T})} \leq c(1+T)^{2-2s+\varepsilon}.$$

Since the equation (2.22) has the time reversibility symmetry $v(t,x) \mapsto \overline{v(-t,-x)}$ and the L^2_x-norm is conserved along the evolution, the above result implies that the H^s_x-norm of any solution v of (2.22) does not blow up in finite time.

3.1.2 The I-operator

Following the papers by Colliander,Keel, Staffilani, Takaoka, and Tao [9, 10], for $0 \leq s < 1$ and $N \gg 1$ a fixed dyadic number, we define the Fourier multiplication operator[3]

$$I : H^s(\mathbb{T}_\lambda) \to H^1(\mathbb{T}_\lambda) \ , \ \widehat{If}(k) = m(k)\widehat{f}(k) \ , \ k \in \mathbb{Z}_\lambda \tag{3.29}$$

where $m : \mathbb{R} \to (0, 1]$ is an even, smooth, non-increasing function on $[0, \infty)$, chosen such that

$$m(\xi) = \begin{cases} 1 & , \text{ if } |\xi| \leq N \\ \left(\frac{N}{|\xi|}\right)^{1-s} & , \text{ if } |\xi| \geq 2N \end{cases}$$

and a smooth interpolant for $N \leq |\xi| \leq 2N$. Furthermore, for any $s \geq \frac{1}{2}$, the Fourier multiplier $m(\,\cdot\,)$ can be chosen such that it satisfies the monotonicity property

$$\xi \mapsto m(\xi)\xi^{\frac{1}{2}} \text{ is non-decreasing on } [0, \infty). \tag{3.30}$$

One easily checks that, for any $0 \leq \theta < 1$ and any $\theta \leq s < 1$, we have

$$m(k)\langle k\rangle^{1-\theta} \gtrsim \begin{cases} N^{1-\theta} & , \text{ if } |k| \gg N \\ 1 & , \text{ if } |k| \lesssim N \end{cases} \tag{3.31}$$

with implicit constants independent of λ.

We note that I behaves like the identity operator on frequencies smaller than N and integrates of order $1 - s$ on frequencies much bigger than N. Indeed,

$$\sum_{k \lesssim N} \langle k\rangle^{2s} |\widehat{u}(k)|^2 \lesssim \sum_{k \lesssim N} \langle k\rangle^2 m(k)^2 |\widehat{u}(k)|^2 \lesssim N^{2(1-s)} \sum_{k \lesssim N} \left(\frac{\langle k\rangle}{N}\right)^{2-2s} \langle k\rangle^{2s} m(k)^2 |\widehat{u}(k)|^2$$

and

$$\sum_{k \gg N} \langle k\rangle^2 \frac{1}{\langle k\rangle^{2-2s}} |\widehat{u}(k)|^2 \lesssim \sum_{k \gg N} \langle k\rangle^2 m(k)^2 |\widehat{u}(k)|^2 \lesssim \sum_{k \gg N} \langle k\rangle^2 \left(\frac{N}{\langle k\rangle}\right)^{2-2s} |\widehat{u}(k)|^2 \ .$$

Therefore, we have

$$\|u\|_{H^s(\mathbb{T}_\lambda)} \lesssim \|Iu\|_{H^1(\mathbb{T}_\lambda)} \lesssim N^{1-s} \|u\|_{H^s(\mathbb{T}_\lambda)}, \tag{3.32}$$

as well as

$$\|u\|_{\dot{H}^s(\mathbb{T}_\lambda)} \lesssim \|Iu\|_{\dot{H}^1(\mathbb{T}_\lambda)} \lesssim N^{1-s} \|u\|_{\dot{H}^s(\mathbb{T}_\lambda)}. \tag{3.33}$$

[3] The operator I depends on the regularity index s and the parameters N and λ, but we choose to omit them as indices of I whenever possible. However, in Lemma 3.15 it becomes necessary to point them out explicitly.

3.1.3 Multilinear forms

As in [9, 10, 13, 32], we use the shorthand notations $k_{12\ldots n} := k_1 + k_2 + \ldots + k_n$, $k_{1-2} := k_1 - k_2$, etc., as well as $m_j := m(k_j), m_{jh} := m(k_{jh})$, etc. Also, we set

$$\Gamma_n(\mathbb{T}_\lambda) := \{\mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}_\lambda)^n : k_{12\ldots n} = 0\},$$
$$\Gamma_n(\mathbb{R}) := \{(\tau_1, \ldots, \tau_n) \in \mathbb{R}^n : \tau_1 + \ldots + \tau_n = 0\}$$

and we endow them with the measure induced from the scaled counting measure $\frac{1}{(2\pi\lambda)^{n-1}} d\#$ and, respectively, from the Lebesgue measure $d\tau_1 \ldots d\tau_{n-1}$, by pushing forward under the map $(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, -x_1 - \ldots - x_{n-1})$.

For n even integer, we define the n-linear form of $f_1, \ldots, f_n : \mathbb{T}_\lambda \to \mathbb{C}$ associated to the multiplier $M_n : \mathbb{R}^n \to \mathbb{C}$ by

$$\Lambda_n(M_n; f_1, \ldots, f_n) := \int_{\Gamma_n(\mathbb{T}_\lambda)} M_n(k_1, k_2, ..., k_n) \prod_{j=1}^{n} \widehat{f}_j(k_j)$$

and the shorthand $\Lambda_n(M_n; f) := \Lambda_n(M_n; f, \overline{f}, \ldots, f, \overline{f})$. For example, we have

$$\int_{\mathbb{T}_\lambda} |v_x^2| dx = -\Lambda_2(k_1 k_2; v),$$
$$\operatorname{Im} \int_{\mathbb{T}_\lambda} |v|^2 v \bar{v}_x dx = -\frac{1}{4} \Lambda_4(k_{13-24}; v).$$

Remark 3.10. We note that

$$\overline{\Lambda(M_n; f)} = \int_{\Gamma_n(\mathbb{T}_\lambda)} \overline{M_n}(k_1, k_2, \ldots, k_n) \widehat{\overline{f}}(-k_1) \widehat{f}(-k_2) \cdots \widehat{\overline{f}}(-k_{n-1}) \widehat{f}(-k_n)$$

and thus, if the multiplier M_n is such that

$$\overline{M_n}(-k_2, -k_1, \ldots, -k_n, -k_{n-1}) = \sigma M_n(k_1, k_2, \ldots, k_{n-1}, k_n),$$

then we have that $\Lambda_n(M_n; f)$ is $\mathbb{R}$-valued ($i\mathbb{R}$-valued), provided that $\sigma = +1$ ($\sigma = -1$).

If n, ℓ are even integers and $1 \leq j \leq n$, the elongation at index j with ℓ positions of the multiplier M_n is defined by

$$\mathbb{X}_j^\ell(M_n)(k_1, k_2, \ldots, k_{n+\ell}) := M_n(k_1, \ldots, k_{j-1}, k_j + k_{j+1} + \ldots + k_{j+\ell}, k_{j+\ell+1}, \ldots, k_{n+\ell}).$$

By comparing the differentiation rule below with the similar rule in the Euclidean setting (see [10, Proposition 3.5]), we note that the additional term (i.e. the one coupled with $\mu(v)$) is due to the particularity of the gauge transformation (2.12)-(2.13).

Proposition 3.11. *Let v be a smooth solution of* (2.22)*, $n \geq 1$, and $M_n : \mathbb{R}^n \to \mathbb{C}$.*

Then,

$$\partial_t \Lambda_n(M_n; v(t)) = i\Lambda_n\left(M_n \sum_{j=1}^{n}(-1)^j k_j^2; v(t)\right)$$
$$- i\Lambda_{n+2}\left(\sum_{j=1}^{n}\mathbb{X}_j^2(M_n)k_{j+1}; v(t)\right) - i\mu(v)\Lambda_{n+2}\left(\sum_{j=1}^{n}\mathbb{X}_j^2(M_n); v(t)\right)$$
$$+ \frac{i}{2}\Lambda_{n+4}\left(\sum_{j=1}^{n}(-1)^{j-1}\mathbb{X}_j^4(M_n); v(t)\right). \tag{3.34}$$

Proof. This follows by direct computation and the definitions introduced above. Indeed, on the Fourier side, the equation (2.22) is written as

$$\partial_t \widehat{v}(k) = -ik^2\widehat{v}(k) - i\int_{k_{123}=k} k_2\widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3) - i\mu(v)\int_{k_{123}=k}\widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3)$$
$$+ \frac{i}{2}\int_{k_{12345}=k}\widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3)\widehat{\overline{v}}(k_4)\widehat{v}(k_5) + i\psi(v)\widehat{v}(k).$$

Also, by first taking the complex conjugate in (2.22), we have as well

$$\partial_t \widehat{\overline{v}}(k) = +ik^2\widehat{\overline{v}}(k) - i\int_{k_{123}=k} k_2\widehat{\overline{v}}(k_1)\widehat{v}(k_2)\widehat{\overline{v}}(k_3) - i\mu(v)\int_{k_{123}=k}\widehat{\overline{v}}(k_1)\widehat{v}(k_2)\widehat{\overline{v}}(k_3)$$
$$- \frac{i}{2}\int_{k_{12345}=k}\widehat{\overline{v}}(k_1)\widehat{v}(k_2)\widehat{\overline{v}}(k_3)\widehat{v}(k_4)\widehat{\overline{v}}(k_5) - i\psi(v)\widehat{\overline{v}}(k).$$

Since ψ is $\mathbb{R}$-valued, the terms corresponding to the $\psi(v)v$ term in (2.22) cancel each other.

□

We introduce the following notation for the factor corresponding to the term $\partial_x^2 v$ in the equation (2.22):

$$\alpha_n(\mathbf{k}) := -i(k_1^2 - k_2^2 + \ldots + k_{n-1}^2 - k_n^2). \tag{3.35}$$

Note that $\alpha_2 = 0$ on $\Gamma_2(\mathbb{T}_\lambda)$. A key property for the analysis of the second and third generation modified energies is the factorization of α_4 on $\Gamma_4(\mathbb{T}_\lambda)$:

$$\alpha_4(\mathbf{k}) = -i\left((k_1 - k_2)k_{12} + (k_3 - k_4)k_{34}\right) = -2ik_{12}k_{14}. \tag{3.36}$$

Furthermore, we introduce the modulations:

$$\omega_j := \tau_j + k_j^2 \quad , \quad \text{for } j \text{ odd},$$
$$\omega_j := \tau_j - k_j^2 \quad , \quad \text{for } j \text{ even},$$

for all $(\tau_1, \dots, \tau_n) \in \Gamma_n(\mathbb{R})$, and we note that

$$\begin{aligned}\omega_1 + \omega_2 + \omega_3 + \omega_4 &= \tau_{1234} + k_1^2 - k_2^2 + k_3^2 - k_4^2 \\ &= 2k_{12}k_{14}\end{aligned}$$

which implies

$$\max_{1\le j\le 4} |\omega_j| \gtrsim |k_{12}k_{14}|. \tag{3.37}$$

3.2 Local well-posedness for the I-system

Given v (sufficiently smooth) solution of (2.22), since Iv does not solve the gauge equivalent equation (2.22), $P_1(Iv)$ and $E_1(Iv)$ are not conservation laws. Instead, v satisfies the following I-system

$$\begin{cases} i\partial_t(Iv) + \partial_x^2(Iv) = -iI(v^2\partial_x\overline{v}) - \frac{1}{2}I(|v|^4v) + \mu(v)I(|v|^2v) - \psi(v)(Iv) \ , \ x \in \mathbb{T} \\ (Iv)_{|t=0} = Iv_0. \end{cases} \tag{3.38}$$

We modify the local well-posedness proof for (2.22) to obtain the following result for (3.38).

Proposition 3.12. *Let $B > 0$. There exist $\delta \sim B^{-\theta}$ (for some $\theta > 0$) and $D > 0$ (both independent of N and λ) such that if $v_0 \in H^s(\mathbb{T}_\lambda)$ is such that $\|Iv_0\|_{H^1(\mathbb{T}_\lambda)} \le B$, then*

$$\|Iv\|_{Z^1([0,\delta]\times\mathbb{T}_\lambda)} \le D. \tag{3.39}$$

In order to prove this result, we use the estimates of the local well-posedness theory for (2.22) due to Herr [21] and an interpolation lemma of Colliander, Keel, Staffilani, Takaoka, and Tao [11, Lemma 12.1] for translation invariant multi-linear operators.

Lemma 3.13. *[21, Section 4] Let $\delta \in (0,1)$ and $\lambda \ge 1$. There exist $c, \varepsilon > 0$ such that*

$$\|u_1(\partial_x\overline{u_2})u_3\|_{\widetilde{Z}^{\frac{1}{2}}(\mathbb{R}\times\mathbb{T}_\lambda)} \le c\delta^\varepsilon \prod_{j=1}^{3} \|u_j\|_{X^{\frac{1}{2},\frac{1}{2}}(\mathbb{R}\times\mathbb{T}_\lambda)}, \tag{3.40}$$

$$\|u_1\overline{u_2}u_3\overline{u_4}u_5)\|_{\widetilde{Z}^{\frac{1}{2}}(\mathbb{R}\times\mathbb{T}_\lambda)} \le c\delta^\varepsilon \prod_{j=1}^{5} \|u_j\|_{X^{\frac{1}{2},\frac{1}{2}}(\mathbb{R}\times\mathbb{T}_\lambda)}, \tag{3.41}$$

$$\|u_1\overline{u_2}u_3\|_{\widetilde{Z}^{\frac{1}{2}}(\mathbb{R}\times\mathbb{T}_\lambda)} \le c\delta^\varepsilon \prod_{j=1}^{3} \|u_j\|_{X^{\frac{1}{2},\frac{1}{2}}(\mathbb{R}\times\mathbb{T}_\lambda)}, \tag{3.42}$$

for all $u_j \in \mathcal{S}_\lambda$ with $\mathrm{supp}(u_j) \in \{(t,x) \in \mathbb{R}\times\mathbb{T}_\lambda : |t| \le \delta\}$, $1 \le j \le 5$.

Remark 3.14. One can check that the pointwise weights bounds provided by [21, Lemma 4.1, Lemma 4.3] hold uniformly in $\lambda \ge 1$ (although, in view of Remark 3.1, further sub-cases have to be addressed). Then, the multi-linear estimates above use only the L^4-Strichartz and Sobolev inequalities of Lemma 3.3 above, which are all scaling invariant.

In order to state the interpolation lemma, let I_N^s denote the I-operator introduced in (3.29). Also, following [11], we let S_x to denote the shift operator $S_x u(t,y) = u(t, y-x)$. A Banach space X of functions $u : J \times \mathbb{T}_\lambda \to \mathbb{C}$ (where $J \subset \mathbb{R}$ is some time interval) is translation invariant if $\|S_x u\|_X = \|u\|_X$ for all $u \in X$ and all x. We use the spaces $X = X^{1,\frac{1}{2}}(J \times \mathbb{T}_\lambda)$ and $Z = \widetilde{Z}^1(J \times \mathbb{T}_\lambda)$ which clearly satisfy this requirement. An n-linear operator $T : X \times \ldots \times X \to Z$ is translation invariant if $S_x T(u_1, \ldots, u_n) = T(S_x u_1, \ldots, S_x u_n)$ for all $u_j \in X$.

Lemma 3.15. *Let $s_0 > 0$, $n \geq 1$ and let $T : X \times \ldots \times X \to Z$ be a translation invariant n-linear operator. Suppose*

$$\|I_1^s T(u_1, \ldots u_n)\|_Z \leq C \prod_{j=1}^{n} \|I_1^s u_j\|_X \tag{3.43}$$

for all $s_0 \leq s \leq 1$ and all $u_j \in X$, for some $C > 0$. Then, we also have

$$\|I_N^s T(u_1, \ldots u_n)\|_Z \leq DC \prod_{j=1}^{n} \|I_N^s u_j\|_X \tag{3.44}$$

for all $s_0 \leq s \leq 1$ and all $u_j \in X$, for some $D > 0$ independent of N and λ.

To convince the reader that the proof of [11, Lemma 12.1] yields the constant D independent of the parameter λ (as well as N), we provide the following remark that uses the "periodization" procedure also encountered in the Poisson summation formula.

Remark 3.16. We know that the Littlewood-Paley projection operators $P_{\lesssim N} f := \phi_N * f$ are uniformly bounded in N, where $\phi_N := N\phi(N\cdot)$ and $\widehat{\phi}$ is a symmetric function on $\mathbb{Z}_\lambda$ equal to one on $\{|k| \leq 1\}$ and vanishes outside $\{|k| < 2\}$. However, the bound $\|\phi\|_{L^1(\mathbb{T}_\lambda)}$ depends on λ. We modify slightly this usual definition in order to ensure uniform boundedness in the scaling parameter λ as well. Thus, let ψ be a Schwartz function on $\mathbb{R}$ such that $\widehat{\psi}$ is a symmetric bump function compactly supported in $\{\xi \in \mathbb{R} : |\xi| \leq 2\}$ and identically one for $|\xi| \leq 1$. Define $\psi_N := N\psi(N\cdot)$ and for any $x \in \mathbb{T}_\lambda$ we set

$$\varphi_N(x) := \sum_{k \in \mathbb{Z}} \psi_N(x + 2\pi\lambda k).$$

Note that $\widehat{\varphi_N}(k) = \widehat{\psi_N}(k)$ for any $k \in \mathbb{Z}_\lambda$, and thus the operator $P_{\lesssim N} f = \varphi_N * f$ acts as the identity operator when $\operatorname{supp}(\widehat{f}) \subset \{k \in \mathbb{Z}_\lambda : |k| \lesssim N\}$ (this is compatible with the region where the operators I_N^s also behave like the identity operator). Also,

$$\|\varphi_N\|_{L^1(\mathbb{T}_\lambda)} = \|\psi_N\|_{L^1(\mathbb{R})} = \|\psi\|_{L^1(\mathbb{R})}$$

and therefore

$$\|P_{\lesssim N}\|_{X\to X}, \|P_{\lesssim N}\|_{Z\to Z} \lesssim 1,$$

uniformly in N and λ. Finally, by arguing as in [11] that $I_1^s I_N^{2-s}$ and $N^{s-1} I_N^s I_1^{2-s}$ are bounded (uniformly in N and λ), by splitting $u_j = P_{\lesssim N} u_j + P_{\gg N} u_j$ for each j, and by estimating each contribution separately, we obtain (3.44).

We apply the above interpolation lemma to the trilinear and quintilinear terms corresponding to the right hand side of (2.22), namely

$$\begin{aligned} \mathcal{N}(v) &= -iv^2 \partial_x \overline{v} - \tfrac{1}{2}|v|^4 v + \mu(v)|v|^2 v - \psi(v)v & (3.45)\\ &=: \mathcal{N}_1(v) + \mathcal{N}_2(v) + \mu(v)\mathcal{N}_3(v) - \psi(v)v. & (3.46) \end{aligned}$$

Note that the estimates of Lemma 3.13 give (3.43) for $s_0 = \frac{1}{2}$. Since $I_N^1 = \mathrm{Id}$ for any N, we obtain the estimate (3.43) for $s = 1$ via the Leibniz rule and Lemma 3.13. For example,

$$\begin{aligned} \|u_1(\partial_x \overline{u_2})u_3\|_{\widetilde{Z}^1} &\lesssim \|\langle\partial_x\rangle^{\frac{1}{2}} u_1(\partial_x \overline{u_2})u_3\|_{\widetilde{Z}^{\frac{1}{2}}} + \|u_1(\partial_x \langle\partial_x\rangle^{\frac{1}{2}} \overline{u_2})u_3\|_{\widetilde{Z}^{\frac{1}{2}}} + \|u_1(\partial_x \overline{u_2})(\langle\partial_x\rangle^{\frac{1}{2}} u_3)\|_{\widetilde{Z}^{\frac{1}{2}}} \\ &\lesssim \delta^\varepsilon \prod_{j=1}^3 \|\langle\partial_x\rangle^{\frac{1}{2}} u_j\|_{X^{\frac{1}{2},\frac{1}{2}}} \sim \delta^\varepsilon \prod_{j=1}^3 \|u_j\|_{X^{1,\frac{1}{2}}}. \end{aligned}$$

One argues analogously for the other multi-linear estimates of Lemma 3.13. Hence, by applying Lemma 3.15, we obtain

$$\begin{aligned} \|I\Big(u_1(\partial_x \overline{u_2})u_3\Big)\|_{\widetilde{Z}^1} &\lesssim \delta^\varepsilon \textstyle\prod_{j=1}^3 \|Iu_j\|_{X^{1,\frac{1}{2}}}, & (3.47)\\ \|I\Big(u_1 \overline{u_2} u_3 \overline{u_4} u_5\Big)\|_{\widetilde{Z}^1} &\lesssim \delta^\varepsilon \textstyle\prod_{j=1}^5 \|Iu_j\|_{X^{1,\frac{1}{2}}}, & (3.48)\\ \|I\Big(u_1 \overline{u_2} u_3\Big)\|_{\widetilde{Z}^1} &\lesssim \delta^\varepsilon \textstyle\prod_{j=1}^3 \|Iu_j\|_{X^{1,\frac{1}{2}}}. & (3.49) \end{aligned}$$

We also need the following Lipschitz continuity properties for the coupling coefficients $\mu(v)$ and $\psi(v)$. We easily have

$$|\mu(f) - \mu(g)| \le \frac{1}{2\pi\lambda} \left(\|f\|_{L^2(\mathbb{T}_\lambda)} + \|g\|_{L^2(\mathbb{T}_\lambda)}\right) \|f-g\|_{L^2(\mathbb{T}_\lambda)}, \tag{3.50}$$

as well as

$$\begin{aligned} &|\psi(f) - \psi(g)| \\ &\quad \lesssim \frac{1}{2\pi\lambda} \left(\|f\|^3_{H^{\frac{1}{2}}(\mathbb{T}_\lambda)} + \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{T}_\lambda)} + \|g\|^3_{H^{\frac{1}{2}}(\mathbb{T}_\lambda)} + \|g\|_{\dot{H}^{\frac{1}{2}}(\mathbb{T}_\lambda)} \right) \|f-g\|_{H^{\frac{1}{2}}(\mathbb{T}_\lambda)}. \end{aligned} \tag{3.51}$$

Indeed, by following [21, Lemma 2.5] for (3.51), we use (2.17) (the definition of ψ), Hölder's inequality, Parseval's identity, and the L^6-Sobolev inequality:

$$\begin{aligned}
|\psi(f)-\psi(g)| &\le 2\Big|\fint_{\mathbb{T}_\lambda} f\partial_x\overline{f}-g\partial_x\overline{g}\Big|+\frac{1}{2}\Big|\fint_{\mathbb{T}_\lambda}|f|^4-|g|^4\Big|+\big|\mu(f)^2-\mu(g)^2\big|\\
&\lesssim \frac{1}{2\pi\lambda}\big|\big(D_x^{\frac12}(f-g),D_x^{\frac12}f\big)_{L^2}+\big(D_x^{\frac12}g,D_x^{\frac12}(f-g)\big)_{L^2}\big|\\
&\quad+\frac{1}{2\pi\lambda}\big(\|f\|^3_{L^6(\mathbb{T}_\lambda)}+\|g\|^3_{L^6(\mathbb{T}_\lambda)}\big)\|f-g\|_{L^2(\mathbb{T}_\lambda)}\\
&\quad+\frac{1}{(2\pi\lambda)^2}\big(\|f\|^2_{L^2(\mathbb{T}_\lambda)}+\|g\|^2_{L^2(\mathbb{T}_\lambda)}\big)\Big|\|f\|^2_{L^2(\mathbb{T}_\lambda)}-\|g\|^2_{L^2(\mathbb{T}_\lambda)}\Big|\\
&\lesssim \frac{1}{2\pi\lambda}\Big(\|f\|_{\dot H^{\frac12}(\mathbb{T}_\lambda)}+\|g\|_{\dot H^{\frac12}(\mathbb{T}_\lambda)}\Big)\|f-g\|_{\dot H^{\frac12}(\mathbb{T}_\lambda)}\\
&\quad+\frac{1}{2\pi\lambda}\big(\|f\|^3_{H^{\frac12}(\mathbb{T}_\lambda)}+\|g\|^3_{H^{\frac12}(\mathbb{T}_\lambda)}\big)\|f-g\|_{L^2(\mathbb{T}_\lambda)}\\
&\quad+\frac{1}{(2\pi\lambda)^2}\big(\|f\|^3_{L^2(\mathbb{T}_\lambda)}+\|g\|^3_{L^2(\mathbb{T}_\lambda)}\big)\|f-g\|_{L^2(\mathbb{T}_\lambda)}\\
&\lesssim \frac{1}{2\pi\lambda}\Big(\|f\|_{\dot H^{\frac12}(\mathbb{T}_\lambda)}+\|g\|_{\dot H^{\frac12}(\mathbb{T}_\lambda)}+\|f\|^3_{H^{\frac12}(\mathbb{T}_\lambda)}+\|g\|^3_{H^{\frac12}(\mathbb{T}_\lambda)}\Big)\|f-g\|_{H^{\frac12}(\mathbb{T}_\lambda)}
\end{aligned}$$

We can now proceed with the proof of Proposition 3.12 by using the fixed point argument in a closed ball of the space $W=\{v:\eta_\delta(t)Iv(t,x)\in Z^1(\mathbb{R}\times\mathbb{T}_\lambda)\}$ with norm

$$\|v\|_W:=\|\eta_\delta Iv\|_{Z^1(\mathbb{R}\times\mathbb{T}_\lambda)},$$

with $\delta\in(0,1)$ and $D>0$ to be chosen later, and $\eta_\delta(t):=\eta(\frac{t}{\delta})$. By denoting $\mathcal{N}(v)$ the right-hand side of (2.22) and by the Duhamel formula, solutions of (3.38) are those v that satisfy

$$Iv(t)=U_\lambda(t)Iv_0-i\int_0^t U_\lambda(t-t')I\mathcal{N}(v(t'))dt' \tag{3.52}$$

in the $C([0,T];H^1(\mathbb{T}_\lambda))$ topology, for some $T>0$. Consider the mapping $v\mapsto\Gamma(v)$ given by

$$\Gamma(v):=\eta(t)U_\lambda(t)v_0-i\eta(t)\int_0^t U_\lambda(t-t')\mathcal{N}(\eta_\delta(t')v(t'))dt'.$$

By (3.12)-(3.13) and (3.47)-(3.49), we have

$$\begin{aligned}
\|\Gamma(v)\|_W &\le \|\eta(t)U_\lambda(t)Iv_0\|_{Z^1}+\Big\|\eta(t)\int_0^t U_\lambda(t-\tau)I\mathcal{N}(\eta_\delta(t')v(t'))dt'\Big\|_{Z^1}\\
&\le c_1\left(\|Iv_0\|_{H^1(\mathbb{T}_\lambda)}+\|I\mathcal{N}(\eta_\delta v)\|_{\widetilde{Z}^1}\right)\\
&\le c_1B+c_2\delta^\varepsilon\left(\|\eta_\delta Iv\|^3_{X^{1,\frac12}}+\|\eta_\delta Iv\|^5_{X^{1,\frac12}}+\|\eta_\delta Iv\|_{\widetilde{Z}^1}\right).
\end{aligned} \tag{3.53}$$

Also,

$$\begin{aligned}\|\Gamma(v_1)-\Gamma(v_2)\|_W &= \left\|\eta(t)\int_0^t U_\lambda(t-\tau)\left(I\mathcal{N}(\eta_\delta(t')v_1(t'))-I\mathcal{N}(\eta_\delta(t')v_2(t'))\right)dt'\right\|_{Z^1}\\ &\lesssim \|I\left(\mathcal{N}_1(\eta_\delta v_1)-\mathcal{N}_1(\eta_\delta v_2)\right)\|_{\widetilde{Z}^1}+\|I\left(\mathcal{N}_2(\eta_\delta v_1)-\mathcal{N}_2(\eta_\delta v_2)\right)\|_{\widetilde{Z}^1}\\ &\quad+\|I\left(\mathcal{N}_3(\eta_\delta v_1)-\mathcal{N}_3(\eta_\delta v_2)\right)\|_{\widetilde{Z}^1}+\|I\left(\eta_\delta v_1-\eta_\delta v_2\right)\|_{\widetilde{Z}^1}.\end{aligned}$$

We write

$$\mathcal{N}_1(u_1)-\mathcal{N}_1(u_2)=u_1(\partial_x\overline{u_1})(u_1-u_2)+u_1\partial_x(\overline{u_1-u_2})u_2+(u_1-u_2)(\partial_x\overline{u_2})u_2$$

and by using (3.47), we obtain

$$\|I\left(\mathcal{N}_1(\eta_\delta v_1)-\mathcal{N}_1(\eta_\delta v_2)\right)\|_{\widetilde{Z}^1}\lesssim\delta^\varepsilon(\|\eta_\delta Iv_1\|_{Z^1}^2+\|\eta_\delta Iv_2\|_{Z^1}^2)\|\eta_\delta I(v_1-v_2)\|_{Z^1}.$$

By using (3.48), (3.49), (3.50), (3.51), together with

$$\|v\|_{C_tH_x^{\frac{1}{2}}(\mathbb{R}\times\mathbb{T}_\lambda)}\lesssim\|Iv\|_{C_tH_x^1(\mathbb{R}\times\mathbb{T}_\lambda)}\lesssim\|Iv\|_{Z^1(\mathbb{R}\times\mathbb{T}_\lambda)}$$

(which follows from (3.32)), we obtain

$$\begin{aligned}\|I\left(\mathcal{N}_2(\eta_\delta v_1)-\mathcal{N}_2(\eta_\delta v_2)\right)\|_{\widetilde{Z}^1}&\lesssim\delta^\varepsilon(\|\eta_\delta Iv_1\|_{Z^1}^4+\|\eta_\delta Iv_2\|_{Z^1}^4)\|\eta_\delta I(v_1-v_2)\|_{Z^1},\\ \|I\left(\mathcal{N}_3(\eta_\delta v_1)-\mathcal{N}_3(\eta_\delta v_2)\right)\|_{\widetilde{Z}^1}&\lesssim\delta^\varepsilon(\|\eta_\delta Iv_1\|_{Z^1}^2+\|\eta_\delta Iv_2\|_{Z^1}^2)\|\eta_\delta I(v_1-v_2)\|_{Z^1}.\end{aligned}$$

It follows that

$$\|\Gamma(v_1)-\Gamma(v_2)\|_W\lesssim\delta^\varepsilon\left(\|v_1\|_W^2+\|v_2\|_W^2+\|v_1\|_W^4+\|v_2\|_W^4\right)\|v_1-v_2\|_W. \tag{3.54}$$

By taking $D=2c_1B+1$ and δ such that $\delta^\varepsilon D^5\sim 1$, from (3.53) and (3.54), we get

$$\|\Gamma(v)\|_W\le D\quad\text{and}\quad\|\Gamma(v_1)-\Gamma(v_2)\|_W\le\frac{1}{2}\|v_1-v_2\|_W,$$

for all $v,v_1,v_2\in\{w\in W:\|w\|_W\le D\}$. Hence, by Banach's fixed point theorem, there exists a unique v with $\|v\|_W\le D$ such that $v=\Gamma(v)$ in W. Thus,

$$\|Iv\|_{Z^1([0,\delta]\times\mathbb{T}_\lambda)}\le\|\eta_\delta Iv\|_{Z^1(\mathbb{R}\times\mathbb{T}_\lambda)}\le D$$

and it follows that (3.52) holds on $[0,\delta]$. The proof of Proposition 3.12 is completed.

3.3 Modified energy functionals via the I-operator and correction terms

In view of the discussion in Section 2.4, we consider the essential part of the energy functional associated to (2.22), namely

$$\mathcal{E}(v) := \int_{\mathbb{T}_\lambda} \Big(|\partial_x v|^2 - \frac{1}{2}|v|^2 \operatorname{Im}(v\overline{v}_x) \Big)\, dx. \tag{3.55}$$

The *first modified energy* is defined to be the $\mathbb{R}$-valued functional

$$\mathcal{E}^1(v) := \mathcal{E}(Iv) = -\Lambda_2(k_1k_2m_1m_2; v) + \frac{1}{4}\Lambda_4(k_{13}m_1m_2m_3m_4; v) \tag{3.56}$$

and for v sufficiently smooth solution of (2.22), one can compute its time increment from the fundamental theorem of calculus

$$\mathcal{E}^1(v(t_0+\delta)) - \mathcal{E}^1(v(t_0)) = \int_{t_0}^{t_0+\delta} \frac{d}{dt}\mathcal{E}^1(v(t))\, dt. \tag{3.57}$$

Using (3.34), we have

$$\frac{d}{dt}\mathcal{E}^1(v(t)) = \Lambda_4(M_4^1; v) + \Lambda_6(M_6^1; v) + \Lambda_8(M_8^1; v) - i\mu(v)\Big(\Lambda_4(K_4^1; v) + \Lambda_6(K_6^1; v)\Big), \tag{3.58}$$

with the multipliers M_4^1, M_6^1, M_8^1 given by [9, Proposition 4.1], e.g.

$$M_4^1(\mathbf{k}) := -\frac{i}{2}m_1m_2m_3m_4k_{12}k_{13}k_{14} - \frac{i}{2}(m_1^2k_1^2k_3 + m_2^2k_2^2k_4 + m_3^2k_3^2k_1 + m_4^2k_4^2k_2). \tag{3.59}$$

Here, it is not particularly important to have the precise expression of the multipliers M_6^1, M_8^1. The multipliers K_4^1, K_6^1 are new to the periodic setting (due to a different expression of the gauge transformation) and are given by

$$K_4^1(\mathbf{k}) := \frac{1}{2}\sum_{j=1}^{4}(-1)^j m_j^2 k_j^2\ , \tag{3.60}$$

$$K_6^1(\mathbf{k}) := \frac{2}{3}\sum_{\substack{\{a,c,e\}=\{1,3,5\}\\ \{b,d,f\}=\{2,4,6\}}} m_am_bm_cm_{def} - m_dm_em_fm_{abc}\ . \tag{3.61}$$

Note that by Remark 3.10, $\Lambda_4(K_4^1; v)$ and $\Lambda_6(K_6^1, v)$ are purely imaginary, and that $\Lambda_4(M_4^1; v)$, $\Lambda_6(M_6^1; v)$ and $\Lambda_8(M_8^1; v)$ are $\mathbb{R}$-valued.

The rule of thumb when one tries to prove estimates on the various terms of (3.57) is that "different pieces of Λ_n appearing in the right hand side of $\partial_t\mathcal{E}^1(v)$ are easier for n larger" [10, p. 72]. This motivates the following procedure when one tries to refine the I-method.

A *second instantiation of the I-method* modifies further the expression of the energy functional by taking

$$\mathcal{E}^2(v) := \mathcal{E}^1(v) + \Lambda_4(\sigma_4; v) \tag{3.62}$$

where the "correction" multiplier σ_4 is chosen so that in the expression of $\frac{d}{dt}\mathcal{E}^2(v)$, no fourth order term $\Lambda_4(\,\cdot\,; v)$ appears. For the sake of keeping the equations compact, we choose to drop the reference to v from $\Lambda_n(M_n; v)$, and the frequency arguments $\mathbf{k} = (k_1, \ldots, k_n)$ when the formulae get too long.

By the differentiation rule (3.34), we have

$$\begin{aligned}\frac{d}{dt}\Lambda_4(\sigma_4) =& \Lambda_4(\sigma_4\alpha_4) - i\Lambda_6\left(\sum_{j=1}^{4}\mathbb{X}_j^2(\sigma_4)k_{j+1}\right) + \frac{i}{2}\Lambda_8\left(\sum_{j=1}^{4}(-1)^{j-1}\mathbb{X}_j^4(\sigma_4)\right)\\ &- i\mu(v)\Lambda_6\left(\sum_{j=1}^{4}\mathbb{X}_j^2(\sigma_4)\right).\end{aligned} \tag{3.63}$$

Note that if $\alpha_4(\mathbf{k}) = 0$, then either $k_{12} = 0$ or $k_{14} = 0$, and both imply that $\widetilde{M}_4(\mathbf{k}) = 0$. We define the first correction $\sigma_4 = \sigma_4(\mathbf{k})$ for $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$ by setting

$$\sigma_4 := -\frac{M_4^1}{\alpha_4} = -\frac{1}{4}\left(m_1m_2m_3m_4k_{13} + \frac{m_1^2k_1^2k_3 + m_2^2k_2^2k_4 + m_3^2k_3^2k_1 + m_4^2k_4^2k_2}{k_{12}k_{14}}\right) \tag{3.64}$$

when $\alpha_4 \neq 0$, and $\sigma_4 = 0$ when $\alpha_4 = 0$. Thus, in the second iteration of the I-method there are no resonances for the correction term as we have $|M_4^1(\mathbf{k})| \lesssim |\alpha_4(\mathbf{k})|$ for all $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$.

Therefore, by (3.56), (3.62) and (3.64), the second generation modified energy is given by

$$\mathcal{E}^2(v) = -\Lambda_2(k_1k_2m_1m_2) + \frac{1}{2}\Lambda_4(M_4), \tag{3.65}$$

where we set

$$M_4 := -\frac{m_1^2k_1^2k_3 + m_2^2k_2^2k_4 + m_3^2k_3^2k_1 + m_4^2k_4^2k_2}{2k_{12}k_{14}} \tag{3.66}$$

when the denominator does not vanish. Note that since $k_{12}k_{14} = 0$ in $\Gamma_4(\mathbb{T}_\lambda)$ implies $m_1^2k_1^2k_3 + m_2^2k_2^2k_4 + m_3^2k_3^2k_1 + m_4^2k_4^2k_2 = 0$, we can set in this cases $M_4 := 0$.

Hence from (3.58) and (3.63), we get

$$\frac{d}{dt}\mathcal{E}^2(v(t)) = \Lambda_6(M_6^2) + \Lambda_8(M_8^2) - i\mu(v)\Big(\Lambda_4(K_4^1) + \Lambda_6(K_6^1) + \Lambda_6(K_6^2)\Big), \tag{3.67}$$

where M_6^2 and M_8^2 are the multipliers given by

$$\begin{aligned} M_6^2 :=& \frac{i}{6}\sum_{j=1}^{6}(-1)^j m_j^2 k_j^2 \qquad (3.68)\\ &- \frac{i}{72}\sum_{\substack{\{a,c,e\}=\{1,3,5\}\\ \{b,d,f\}=\{2,4,6\}}}\Big(M_4(k_{abc},k_d,k_e,k_f)k_b + M_4(k_a,k_{bcd},k_e,k_f)k_c\\ &\qquad + M_4(k_a,k_b,k_{cde},k_f)k_d + M_4(k_a,k_b,k_c,k_{def})k_e\Big),\\ M_8^2 :=& C_8\sum_{\substack{\{a,c,e,g\}=\{1,3,5,7\}\\ \{b,d,f,h\}=\{2,4,6,8\}}}\Big(M_4(k_{abcde},k_f,k_g,k_h) - M_4(k_a,k_{bcdef},k_g,k_h) \qquad (3.69)\\ &\qquad + M_4(k_a,k_b,k_{cdefg},k_h) - M_4(k_a,k_b,k_c,k_{defgh})\Big) \end{aligned}$$

(as in [10, Proposition 3.7] or [32, p. 2173]). Also,

$$K_6^2 := \sum_{j=1}^{4}\mathbb{X}_j^2(\sigma_4). \qquad (3.70)$$

We note that when proving the estimates on M_6 (see Lemma 3.20), cancelations between the large terms coming from the first term in (3.68) and the large terms coming from the sum of M_4's are exploited, and thus the coefficients of the two pieces of M_6 are critical, whereas the constant $C_8 = -\frac{i}{2(5!)^2}$ is irrelevant in our analysis.

Remark 3.17 (small frequencies remark). Notice that if $|k_j| \ll N$ for all j, we have $m(k_j) = 1$ and thus

$$M_4(\mathbf{k}) = -\frac{k_1k_3k_{13} + k_2k_4(-k_{13})}{2k_{12}k_{14}} = \frac{k_{13}}{2}, \text{ for all } \mathbf{k}\in\Gamma_4(\mathbb{T}_\lambda). \qquad (3.71)$$

One can similarly check that if $|k_j| \ll N$ for all j, all the multipliers M_n^g, K_n^g ($n = 4, 6, 8$, $g = 1, 2$) vanish.

On $\Gamma_n(\mathbb{T}_\lambda)$, the largest two frequencies must have comparable sizes and thus, without loss of generality, we may assume that

$$\mathbf{k}\in\Upsilon_n(\mathbb{T}_\lambda) := \{(k_1,\dots,k_n)\in\mathbb{Z}_\lambda^n : |k_1^*|\sim|k_2^*|\gtrsim N\}, \qquad (3.72)$$

where N is the frequency size threshold of the I-operator as defined in Subsection 3.1.2, and $(k_1^*,\dots,k_n^*)$ denotes a rearrangement of $(k_1,\dots,k_n)$ such that

$$|k_1^*|\geq|k_2^*|\geq\ldots\geq|k_n^*|.$$

We'll also adopt the notation $N_j = |k_j^*|$.

Due to Remark 3.17, when proving the necessary estimates, it is enough to consider $\mathbf{k}\in\Upsilon_n(\mathbb{T}_\lambda)$, i.e. only the region $N_1\sim N_2\gtrsim N$.

Remark 3.18 (**Symmetry Remark**). We point out that the multipliers M_n^g's that appear throughout this chapter, and consequently the associated multilinear forms $\Lambda_n(M_n^g; v_1, v_2, \ldots, v_n)$ are invariant under permutations of the even or of the odd k_j (or v_j) indices. Also, the same is true (up to sign) if one swaps the set of all odd k_j's (or v_j's) with the set of all even k_j's (respectively v_j's).

Hence, in addition to (3.72), without loss of generality we may assume that

$$|k_1| \geq |k_3| \geq \ldots \geq |k_{n-1}| \quad , \quad |k_2| \geq |k_4| \geq \ldots \geq |k_n|$$

and

$$|k_1| \geq |k_2|.$$

If all these are in place, we have $k_1^* = k_1$, but either $k_2^* = k_2$ or $k_2^* = k_3$.

3.3.1 Pointwise bounds on the multipliers

We provide here the multiplier estimates that are relevant in our analysis, namely for the almost conservation estimates of the modified energy functional in Section 3.4 and in the estimates of the correction terms in Section 3.5. We recall that we work under the symmetry assumptions on the multipliers M_n^g, K_n^g mentioned in Remark 3.18. Also, since we rely on (3.30), the assumption $s \geq \frac{1}{2}$ is needed for all of the results below.

Although the multiplier M_4 is not involved directly in (3.67), the refined bounds (ii) and (iii) below are crucial for M_6^2 and M_8^2.

Lemma 3.19. *[10, Lemma 4.1, 4.2] For M_4 defined by* (3.66) *and $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$, we have:*

(i) $|M_4(\mathbf{k})| \lesssim m(N_1)^2 N_1$;

(ii) *if* $|k_1| \sim |k_3| \gtrsim N \gg N_3$, *then* $|M_4(\mathbf{k})| \lesssim m(N_1)^2 N_3$;

(iii) *if* $|k_1| \sim |k_2| \gtrsim N \gg N_3$, *then* $M_4(\mathbf{k}) = \frac{m(k_1)^2 k_2^2}{2k_1} + O(N_3)$.

By using the estimate (i) above, one can immediately obtain a crude bound for the symbol M_6^2 (see (i) below). We recall that in [10], the refined estimate (ii) below, as well as using Bourgain's trick to provide additional denominators, make possible to prove the global well-posedness of DNLS on the real line for $s > \frac{1}{2}$, but not at the end-point $s = \frac{1}{2}$. It is worth mentioning that for (ii), in the case $N_3 \ll N$ and the largest two frequencies have same parity, it was exploited the cancellation "between the large terms coming from β_6 and the large terms of the sum of the M_4." Hence the almost conservation estimate of $\mathcal{E}^2$ owes to the specific nonlinear structure $-iv^2\partial_x\overline{v} - \frac{1}{2}|v|^4 v$ of the gauged DNLS equation (2.22) in the Euclidean case.

Lemma 3.20. *[10, Lemma 6.2] For M_6^2 defined by* (3.68) *and $\mathbf{k} \in \Gamma_6(\mathbb{T}_\lambda)$, we have:*

(i) $|M_6^2(\mathbf{k})| \lesssim m(N_1)^2 N_1^2$;

(ii) *if* $N_3 \ll N$, *then* $|M_6^2(\mathbf{k})| \lesssim N_1 N_3$.

Lemma 3.21. *For* σ_4 *defined by* (3.64) *and* $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$, *we have:*

$$|\sigma_4(\mathbf{k})| \lesssim m(N_1)^2 N_1.$$

Proof. For σ_4, one easily notes that $\sigma_4^1 := -\frac{1}{4} m_1 m_2 m_3 m_4 k_{13}$ is bounded by $m(N_1)^2 N_1$ and for $\sigma_4^2 := \sigma_4 - \sigma_4^1$, we have Lemma 3.19 which gives $|\sigma_4^2| \sim |M_4| \lesssim m(N_1)^2 N_1$.

□

Another immediate consequence of Lemma 3.19 is the following:

Lemma 3.22. *For* M_8^2 *defined by* (3.69) *and* $\mathbf{k} \in \Gamma_8$, *we have:*

(i) $|M_8^2(\mathbf{k})| \lesssim m(N_1)^2 N_1$;

(ii) *if* $N_3 \ll N$, *then* $|M_8^2(\mathbf{k})| \lesssim N_3$.

Lemma 3.23. *For* K_4^1 *defined by* (3.60) *and* $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$, *we have*

(i) $|K_4^1(\mathbf{k})| \lesssim m(N_1)^2 N_1^2$;

(ii) *if* $|k_1| \sim |k_2| \gtrsim N \gg N_3$, *then* $|K_4^1(\mathbf{k})| \lesssim m(N_1)^2 N_1 N_3$.

Proof. The first statement is immediate as $\xi \mapsto m(\xi)^2 \xi^2$ is increasing. For the second statement, $|m'(\xi)| \sim m(\xi)|\xi|^{-1}$ when $|\xi| \gg N$, and by the mean value theorem

$$|m(k_1)^2 k_1^2 - m(k_2)^2 k_2^2| \sim m^2(\theta)|\theta||k_1 - (-k_2)|$$

for some θ between k_1 and $-k_2$; hence $|\theta| \sim N_1$ and $m(\theta)^2 \sim m(N_1)^2$. Since we also have $|k_{12}| = |k_{34}| \lesssim N_3$, we get $|m(k_1)^2 k_1^2 - m(k_2)^2 k_2^2| \lesssim m(N_1)^2 N_1 N_3$. Then, the crude bound

$$|m(k_3)^2 k_3^2 - m(k_4)^2 k_4^2| \le m(k_3)^2 k_3^2 + m(k_4)^2 k_4^2 \lesssim m(N_3)^2 N_3^2$$

together with $m(N_3)^2 N_3 \le m(N_1)^2 N_1$, concludes the proof.

□

Lemma 3.24. *For* K_6^1, K_6^2 *defined by* (3.61), (3.70) *respectively, and* $\mathbf{k} \in \Gamma_6(\mathbb{T}_\lambda)$, *we have*

(i) $|K_6^1(\mathbf{k})| \lesssim 1$;

(ii) $|K_6^2(\mathbf{k})| \lesssim m(N_1)^2 N_1$.

Proof. The first statement is immediate from $0 < m(\cdot) \le 1$, while the second follows from Lemma 3.21.

□

3.3.2 Necessity of the third iteration of the I-method

To make the matters clear why we need to implement a third generation I-method, we prove here the decay estimate for $\int \Lambda_6(M_6^2)dt$. This part serves two purposes: first, to see how one applies the bilinear estimate in order to recover the result of [53, Lemma 7.5], and second to uncover the worst case scenarios and hence motivate the non-resonant subregions of Subsection 3.3.4.

Proposition 3.25. *For $s > \frac{1}{2}$ and M_6^2 defined by* (3.68)*, we have the estimate*

$$\left| \int_0^\delta \Lambda_6(M_6^2; v(t))\, dt \right| \lesssim N^{-1+} \lambda^{-1+} \|Iv\|^6_{Z^1([0,\delta]\times\mathbb{T}_\lambda)}. \tag{3.73}$$

Proof. We write $v = \sum_{k\in\mathbb{Z}_\lambda} v_j$, with $\operatorname{supp}(\widehat{v_j}) \subset \{(\tau,k) \in \mathbb{R}\times\mathbb{Z}_\lambda : |k| \sim N_j\}$ for each N_j dyadic number. Thus, it is enough to estimate

$$\int_{\mathbb{R}} \Lambda_6(M_6^2; v_1, v_2, \ldots, v_6)\, dt \tag{3.74}$$

where without loss of generality we can assume, in addition to the frequency localization, that each $\widehat{v_j}$ is real valued and non-negative. This step, as well as why it is enough to consider the time integral on $\mathbb{R}$ rather than on $[0,\delta]$ can be justified by standard arguments as in Section 3.4.

Case 1. $N_1 \sim N_2 \gtrsim N \gg N_3$. By Lemma 3.20 (ii) we have $|M_6^2| \lesssim N_1 N_3$. Notice that for $s > \frac{1}{2}$ and $\varepsilon > 0$ small enough, one obtains

$$m(N_1)^2 N_1 = N^{2-2s} N_1^{2s-1} = N^{1-\varepsilon}\left(\frac{N_1}{N}\right)^{2s-1-\varepsilon} N_1^{\varepsilon} \gtrsim N^{1-\varepsilon} N_1^{\varepsilon}. \tag{3.75}$$

In this case, $m(N_j) = 1$ for $j \geq 3$ and therefore by (3.25) and (3.10), we get

$$\begin{aligned}
(3.74) &\lesssim \int_* \int_{**} \frac{1}{m(N_1)^2 N_1 \prod_{j=4}^6 \langle k_j\rangle} \prod_{j=1}^6 \widehat{J_x I v_j} \\
&\lesssim \frac{N^{-1+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} (J_x I v_1)(J_x I v_3)(J_x I v_2)(J_x I v_4)(I v_5)(I v_6) dx dt \\
&\lesssim \frac{N^{-1+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_3)\|_{L^2_{t,x}} \|(J_x I v_2)(J_x I v_4)\|_{L^2_{t,x}} \|I v_5\|_{L^\infty_{t,x}} \|I v_6\|_{L^\infty_{t,x}} \\
&\lesssim \frac{N^{-1+}\lambda^{-1+}}{N_1^{0+}} \prod_{j=1}^4 \|I v_j\|_{X^{1,\frac{1}{2}}} \prod_{j=5,6} \|I v_j\|_{Y^{\frac{1}{2}+,0}} \\
&\lesssim \frac{N^{-1+}\lambda^{-1+}}{N_1^{0+}} \prod_{j=1}^6 \|I v_j\|_{Z^1}
\end{aligned} \tag{3.76}$$

where $\int_*$ and $\int_{**}$ stand for integration on $\Gamma_6(\mathbb{R})$ and on $\Gamma_6(\mathbb{T}_\lambda)$, respectively (see Section 3.1.3). The operator J_x denotes the Bessel potential operator, i.e.

$$\widehat{J_x f}(k) = \langle k \rangle \widehat{f}(k)\,.$$

Remark 3.26. For $s = \frac{1}{2}$, we only have $m(N_1)^2 N_1 \gtrsim N$ as we cannot afford to borrow an N_1^{ε} factor as in (3.75) above. Notice that since there are no other tools to obtain additional decaying factors, to make up for the logarithmic loss in λ, as well as to ensure summability, one would need to obtain a better estimate, for example

$$|M_6^2| \lesssim N_1^{1-\theta} N_3^{1+\theta}, \tag{3.77}$$

which gives the following factor in (3.76):

$$N^{-1}\lambda^{-1}\frac{N_3^{\theta}\lambda^{0+}}{N_1^{\theta}} \lesssim \frac{N^{-1}\lambda^{-1-}}{N_3^{0+}}$$

(recall that since $s \geq \frac{1}{2}$, we have $1 \leq \lambda \leq N$). We note that the decaying factor $N^{-1}\lambda^{-1-}$ would allow us to obtain the global well-posedness result at $s = \frac{1}{2}$ (see Section 3.6). Although the bound (3.77) is not conceivable on the entire $\Gamma_6(\mathbb{T}_\lambda)$, such an estimate can be established on a carefully chosen subset (see Section 3.3.4).

Case 2. $N_3 \gtrsim N \gg N_4$. By Lemma 3.20 (i) we have $|M_6^2| \lesssim m(N_1)^2 N_1^2$, and for $s \geq 0$, $m(N_3)N_3 \gtrsim N^{-1+} N_3^{0+}$. We then have

$$\begin{aligned}(3.74) &\lesssim \int_* \int_{**} \frac{1}{m(N_3)N_3 \prod_{j=4}^{6}\langle k_j\rangle} \prod_{j=1}^{6} \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-1+}}{N_3^{0+}} \int_* \int_{**} \widehat{J_x I v_1}\,\widehat{J_x I v_2}\,\widehat{J_x I v_3}\,\widehat{I v_4}\,\widehat{I v_5}\,\widehat{I v_6}.\end{aligned}$$

At this point we have to discuss the frequency separation of the first three factors.

Subcase 2.1 If $N_3 \sim N_1$, then since $N_3 \gg N_4$, two out of the three frequencies k_1, k_2, k_3 must have opposite signs, say k_1 and k_2. Thus $J_x I v_1$ and $J_x I v2$ are separated in frequency, and so are $J_x I v_3$ and $J_x I v_4$. We have

$$\begin{aligned}(3.74) &\lesssim \frac{N^{-1+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} (J_x I v_1)(J_x I v_2)(J_x I v_3)(J_x I v_4)(I v_5)(I v_6)\,dxdt \\ &\lesssim \frac{N^{-1+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_2)\|_{L^2_{t,x}} \|(J_x I v_3)(J_x I v_4)\|_{L^2_{t,x}} \|I v_5\|_{L^\infty_{t,x}} \|I v_6\|_{L^\infty_{t,x}} \\ &\lesssim \frac{N^{-1+}\lambda^{-1+}}{N_1^{0+}} \prod_{j=1}^{6} \|I v_j\|_{Z^1}.\end{aligned}$$

Subcase 2.2 If $N_3 \ll N_1$, then as in Case 1, we can clearly apply the bilinear estimate (3.25) to the $L^2_{t,x}$-norms of both $(J_xIv_1)(J_xIv_3)$ and $(J_xIv_2)(J_xIv_4)$ and obtain

$$(3.74) \lesssim \frac{N^{-1+}\lambda^{-1+}}{N_3^{0+}} \|Iv_1\|_{X^{1,\frac{1}{2}}} \|Iv_2\|_{X^{1,\frac{1}{2}}} \prod_{j=3}^{6} \|Iv_j\|_{Z^1}.$$

Notice that in this sub case the factor $1/N_3^{0+}$ does not allow direct summation over the dyadic numbers $N_1 \sim N_2$. However, exploiting the L^2-based norm of the space $X^{1,\frac{1}{2}}$ of the first two factors, one can recover the claim (see Section 3.4) without any setback.

Remark 3.27. Notice that although in Case 2 we have three large frequencies ($N_3 \gtrsim N \gg N_4$), the bound on the weight M_6^2 is worse than in Case 1, and overall we obtain the same (insufficient) decaying factor of $N^{-1+}\lambda^{-1+}$. Therefore we also need to correct for this case.

Case 3. $N_4 \gtrsim N$. By Lemma 3.20 (i) we have $|M_6^2| \lesssim m(N_1)^2 N_1^2$, and for $s \geq 0$, $m(N_j)N_j \gtrsim N^{-1+}N_j^{0+}$, $j = 3, 4$. It follows that

$$\begin{aligned}(3.74) &\lesssim \int_* \int_{**} \frac{1}{m(N_3)N_3 m(N_4)N_4 \prod_{j=5,6}\langle k_j\rangle} \prod_{j=1}^{6} \widehat{J_xIv_j} \\ &\lesssim \frac{N^{-2+}}{N_3^{0+}} \int_* \int_{**} \widehat{J_xIv_1}\widehat{J_xIv_2}\widehat{J_xIv_3}\widehat{J_xIv_4}\widehat{Iv_5}\widehat{Iv_6}.\end{aligned}$$

Although when $\lambda \sim N$, the decaying factor obtained above is just as good as that in the previous cases, we can gain here another decaying factor $\lambda^{-\frac{1}{2}+}$ by separating the analysis into subcases $N_3 \sim N_1$ and $N_3 \ll N_1$, as we did in Case 2. We obtain

$$(3.74) \lesssim \frac{N^{-2+}\lambda^{-\frac{1}{2}+}}{N_3^{0+}} \|Iv_1\|_{X^{1,\frac{1}{2}}} \|Iv_2\|_{X^{1,\frac{1}{2}}} \prod_{j=3}^{6} \|Iv_j\|_{Z^1}$$

and since we choose the parameters so that $1 \leq \lambda \leq N$, we have in this case a better decaying factor.

□

The other sixth order term in (3.67) is $\mu(v)\Lambda_4(K_4^1; v)$. The coefficient $\mu(v) = \frac{1}{2\pi\lambda}\|v\|^2_{L^2(\mathbb{T}_\lambda)}$ already provides a decaying factor of λ^{-1}. In the remark below, we investigate the worst case scenario corresponding to this term.

Remark 3.28. The pointwise bound $|K_4^1(\mathbf{k})| \lesssim m(N_1)^2 N_1^2$ is optimal in the case $N_3 \ll N$ and the largest two frequencies have the same parity (as we have, for example, $|K_4^1(N_1, 0, -N_1, 0)| = m(N_1)^2 N_1^2$). In this case, the best estimate that can be obtained is

$$
\begin{aligned}
\int_{\mathbb{R}} \Lambda_4(K_4^1; v_1, \dots, v_4)\, dt &\lesssim \int_* \int_{**} \frac{1}{\langle k_3\rangle\langle k_4\rangle} \prod_{j=1}^{4} \widehat{J_x I v_j} \\
&\lesssim \frac{1}{N_3^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} (J_x I v_1)(J_x^{0+} I v_3)(J_x I v_2)(I v_4) dx dt \\
&\lesssim \frac{\lambda^{-1+}}{N_3^{0+}} \|(J_x I v_1)(J_x^{0+} I v_3)\|_{L^2_{t,x}} \|(J_x I v_2)(I v_4)\|_{L^2_{t,x}} \\
&\lesssim \frac{\lambda^{-1+}}{N_3^{0+}} \prod_{j=1}^{4} \|I v_j\|_{X^{1,\frac{1}{2}-}}. \qquad (3.78)
\end{aligned}
$$

Hence, we have the estimate[4]

$$
\mu(v) \left| \int_0^\delta \Lambda_4(K_4^1; v(t)) dt \right| \lesssim \lambda^{-2+} \|Iv\|^6_{Z^1([0,\delta]\times\mathbb{T}_\lambda)}.
$$

This decay rate is insufficient to reach the regularity index $s = \frac{1}{2}$. Since the bound of K_4^1 is optimal and the available tools cannot yield a better estimate, we have to provide a second correction term that removes (at least) this case.

3.3.3 The third generation modified energy

We refine further the choice of modified energy for the I-method as a refinement of $\mathcal{E}^2$ of the form

$$
\mathcal{E}^3(v) := \mathcal{E}^2(v) + \Lambda_6(\sigma_6; v) + i\mu(v)\Lambda_4(\widetilde{\sigma_4}; v). \qquad (3.79)
$$

In the same manner as above, we are lead to define the "correction" term σ_6 by imposing $M_6 + \sigma_6\alpha_6 = 0$. In contrast to the situation of α_4 discussed above, the set on which α_6 vanishes is not small, in particular $\alpha_6 = 0$ does not imply $M_6 = 0$. The idea around this is to define a region Ω in the hyperplane $\Gamma_6(\mathbb{T}_\lambda)$ referred to as *the non-resonant set of* σ_6 where α_6 clearly does not vanish, but also with the property that on $\Omega^c := \Gamma_6(\mathbb{T}_\lambda) \setminus \Omega$ we have satisfactory pointwise estimates on M_6^2. We can then take

$$
\sigma_6 := -\frac{M_6^2}{\alpha_6} \cdot \mathbb{1}_\Omega\,, \qquad (3.80)
$$

where $\mathbb{1}_\Omega$ denotes the characteristic function of the set Ω which is defined in Subsection 3.3.4.

For the second correction term in (3.79), the situation is simpler (since $\alpha_4 = 0$ implies $K_4^1 = 0$) and we can define

$$
\widetilde{\sigma_4} := \frac{K_4^1}{\alpha_4} \qquad (3.81)
$$

[4] In the region where we have the refined estimate $|K_4^1(\mathbf{k})| \lesssim m(N_1)^2 N_1 N_3$, one obtains the pre-factor $N^{-1+}\lambda^{-2+}$ in (3.78).

when $\alpha_4 \neq 0$, and $\widetilde{\sigma_4} := 0$ when $\alpha_4 = 0$.

Using (3.34), we find that

$$\begin{aligned}\frac{d}{dt}\mathcal{E}^3(v(t)) =& \Lambda_6(M_6^2 \cdot \mathbb{1}_{\Omega^c}) + \Lambda_8(M_8^2) + \Lambda_8(M_8^3) + \Lambda_{10}(M_{10}^3) \\ & - i\mu(v)\Big(\Lambda_4(K_4^1) + \Lambda_6(K_6^1) + \Lambda_6(K_6^2) + \Lambda_6(\widetilde{K_6^3}) + \Lambda_8(K_8^3) + \Lambda_8(\widetilde{K_8^3})\Big) \\ & + \mu(v)^2\Lambda_6(\widetilde{K_6^4})\end{aligned} \tag{3.82}$$

where the additional terms (i.e. the ones corresponding to the two correction terms σ_6 and $\widetilde{\sigma_4}$) are given by

$$M_8^3 := -i\sum_{j=1}^{6}\mathbb{X}_j^2(\sigma_6)k_{j+1}, \tag{3.83}$$

$$K_8^3 := \sum_{j=1}^{6}\mathbb{X}_j^2(\sigma_6), \tag{3.84}$$

$$M_{10}^3 := \frac{i}{2}\sum_{j=1}^{6}(-1)^{j+1}\mathbb{X}_j^4(\sigma_6), \tag{3.85}$$

$$\widetilde{K_6^3} := i\sum_{j=1}^{4}\mathbb{X}_j^2(\widetilde{\sigma_4})k_{j+1}, \tag{3.86}$$

$$\widetilde{K_6^4} := \sum_{j=1}^{4}\mathbb{X}_j^2(\widetilde{\sigma_4}), \tag{3.87}$$

$$\widetilde{K_8^3} := \frac{i}{2}\sum_{j=1}^{4}(-1)^{j}\mathbb{X}_j^4(\widetilde{\sigma_4}). \tag{3.88}$$

3.3.4 A non-resonant set for α_6

We now turn to describing the set Ω, as it was introduced in [32]. With the simplifying assumptions of Remark 3.18 in place, let us analyze the expression

$$i\alpha_6 = k_1^2 - k_2^2 + k_3^2 - k_4^2 + k_5^2 - k_6^2.$$

If precisely two frequencies have sizes above the threshold N, we distinguish the following two cases.

Case 1. If the largest two frequencies have the same parity, then clearly $|\alpha_6| \gtrsim N_1^2$. The corresponding non-resonant region is defined to be

$$\Omega_1 := \{\mathbf{k} \in \Upsilon_6(\mathbb{T}_\lambda) : |k_1| \sim |k_3| \gtrsim N \gg N_3\}. \tag{3.89}$$

This definition is just slightly different from the analogous one in [32, Section 3] and it does not affect the estimates.

Case 2. If the largest two frequencies have opposite parity, say k_1 and k_2, then on $\Gamma_6(\mathbb{T}_\lambda)$ it must be that $k_1 = -k_2 + O(N_3)$ and

$$i\alpha_6 = k_{12}(k_1 - k_2) + O(N_3^2).$$

While $k_1 - k_2 = O(N_1)$, it is possible to have $k_{12} = 0$ and $\alpha_6 = 0$. Even if the latter does not happen, a too weak lower bound on α_6 renders an insufficiently good upper bound on M_8^3 (one of the multipliers that involve $\sigma_6 = -\frac{M_6^2}{\alpha_6}$, see (3.83)). As in [32], we consider the following subregion

$$\Omega_2 := \left\{ \mathbf{k} \in \Upsilon_6(\mathbb{T}_\lambda) : |k_1| \sim |k_2| \gtrsim N \gg N_3 \text{ and } |k_{12}| \gtrsim \left(\frac{N_3}{N_1}\right)^{\frac{1}{2}} N_3 \right\}. \tag{3.90}$$

Remark 3.29. Notice that in the case $|k_1| \sim |k_2| \gtrsim N \gg N_3$, we have $|k_{12}| = |k_{3456}| \lesssim N_3$. On the other hand, by looking to ensure $\alpha_6 \neq 0$, the natural bound to impose is $|k_{12}| \gtrsim \frac{N_3^2}{N_1}$. However, while the latter gives a better bound on the remaining part $M_6^2 \mathbf{1}_{\Omega^c}$ of $\frac{d}{dt}\mathcal{E}^3$, it does not allow for a satisfactory bound on the correction multiplier σ_6 (which appears, for example, in M_8^3). At the other extreme, correcting only in the region $|k_{12}| \sim N_3$ does not produce a small enough bound on $M_6^2 \mathbf{1}_{\Omega^c}$. We would like to point out that (here, as well as in the Euclidean setting [32]), the choice of $\frac{1}{2}$ in the exponent is not essential, as any lower bound of the form

$$|k_{12}| \gtrsim \left(\frac{N_3}{N_1}\right)^{\theta} N_3$$

(for some $0 < \theta < 1$) produces the extra $N^{-\theta}$ decay factor needed to reach $s = \frac{1}{2}$.

Case 3. Finally, since the decay factors in the estimate of $\Lambda_6(M_6^2)$-term were also critical in the case $N_3 \gtrsim N \gg N_4$ (see Case 2 in the proof of Proposition 3.25), we need to correct for it in this region as well. When three frequency sizes are much larger than the remaining frequency sizes, α_6 does not vanish as we have $|\alpha_6| \gtrsim N_3^2$. Therefore, we define

$$\Omega_3 := \{\mathbf{k} \in \Upsilon_6(\mathbb{T}_\lambda) : N_3 \gg N_4\} \tag{3.91}$$

We point out that the correction is deliberately intended for the larger region $N_3 \gg N_4$ (i.e. Ω_3) rather than $N_3 \gtrsim N \gg N_4$, since on Ω_3 we have

$$|k_1^* + k_2^*| = |k_3^* + k_4^* + k_5^* + k_6^*| \sim N_3 \gtrsim \left(\frac{N_3}{N_1}\right)^{\frac{1}{2}} N_3. \tag{3.92}$$

Correcting for M_6^2 in these three subregions of $\Upsilon_6(\mathbb{T}_\lambda)$ is enough for our goal, hence we consider $\Omega := \Omega_1 \cup \Omega_2 \cup \Omega_3$ to be the non-resonant set of α_6, and in what follows we denote $\Omega^c := \Upsilon_6(\mathbb{T}_\lambda) \setminus \Omega$.

3.3.5 Pointwise bounds on the multipliers (continued)

In this section we first recall the pointwise estimates obtained by Miao, Wu, and Xu [32], and then we establish the bounds needed to handle the second correction term in (3.79).

Lemma 3.30. *[32, Corollary 4.1] For M_6^2 defined by* (3.68) *and* $\mathbf{k} \in \Gamma_6$*, we have:*

(i) *if $N_3 \ll N$, then $|M_6^2(\mathbf{k})| \lesssim N_1 |k_1^* + k_2^*| + N_3^2$;*

(ii) *if $N_3 \ll N$ and $\mathbf{k} \in \Omega^c$, then $|M_6^2(\mathbf{k})| \lesssim N_1^{\frac{1}{2}} N_3^{\frac{3}{2}}$.*

Lemma 3.31. *[32, Lemma 4.9] For σ_6 defined by* (3.80) *and* $\mathbf{k} \in \Gamma_6(\mathbb{T}_\lambda)$*, we have:*

(i) *$|\sigma_6(\mathbf{k})| \lesssim 1$;*

(ii) *if $\mathbf{k} \in \Omega_1 \cap \{N_3 \ll N\}$, then $|\sigma_6(\mathbf{k})| \lesssim \frac{N_3}{N_1}$.*

Lemma 3.32. *[32, Proposition 4.3] For M_8^3 defined by* (3.83) *and* $\mathbf{k} \in \Gamma_8$*, we have:*

(i) *$|M_8^3(\mathbf{k})| \lesssim N_1$;*

(ii) *if $N_3 \ll N$, then $|M_8^3(\mathbf{k})| \lesssim N_1^{\frac{1}{2}} N_3^{\frac{1}{2}}$.*

Also, as direct consequences of the above Lemma 3.31, we have the same bounds for K_8^3 and M_{10}^3 (see (3.84) and (3.85)) as for σ_6. Finally, we provide the pointwise estimates corresponding to the second correction term in (3.79).

Lemma 3.33. *For $\widetilde{\sigma_4}$ defined by* (3.81) *and* $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$*, we have:*

(i) *$|\widetilde{\sigma_4}(\mathbf{k})| \lesssim m(N_1)^2 N_1$;*

(ii) *if $N_3 \ll N$, then $|\widetilde{\sigma_4}(\mathbf{k})| \lesssim m(N_1)^2$.*

Proof. Let β_4 denote the numerator in (3.81). We have the crude estimate $|\beta_4| \lesssim m(N_1)^2 N_1^2$, and note that either $\alpha_4 = 0$ (in which case $\widetilde{\sigma_4} = 0$) or $|\alpha_4| \geq 2N_1$. Depending on the parity of the largest two frequencies, we distinguish two cases.

If $k_1^* = k_1$ and $k_2^* = k_3$, then $|\alpha_4| \sim |k_{12}||k_{14}| \sim N_1^2$ and $|\beta_4| \sim m(N_1)^2 N_1^2$.

If $k_1^* = k_1$ and $k_2^* = k_2$, then $|\alpha_4| \sim N_1 |k_{34}|$, k_1 and k_2 have opposite signs and by the mean value theorem, we have

$$\begin{aligned} |\beta_4| \leq & |m(k_1)^2 k_1^2 - m(-k_2)^2 (-k_2)^2| + |k_{34}| \cdot |k_3 - k_4| \\ \leq & |k_{12}| \cdot |(m(\xi)^2 \xi^2)'| + |k_{34}| \cdot |k_3 - k_4|, \end{aligned}$$

where $|\xi| \sim N_1$ and thus

$$\left| \frac{d}{d\xi} (m(\xi)^2 \xi^2) \right| \sim |m(\xi)^2 \xi| \sim m(N_1)^2 N_1.$$

Since

$$|k_3 - k_4| \lesssim N_3 \ll N \lesssim m(N_1)^2 N_1,$$

we get

$$|\beta_4| \lesssim m(N_1)^2 N_1 |k_{34}|$$

and the conclusion follows.

□

Consequently, by simply referring to their definitions in (3.87) and (3.88), we also have the same bounds for $\widetilde{K_6^4}$ and $\widetilde{K_8^3}$, respectively, as for $\widetilde{\sigma_4}$, In the same manner, we have the following lemma.

Lemma 3.34. *For $\widetilde{K_6^3}$ defined by* (3.86) *and $\mathbf{k} \in \Gamma_6(\mathbb{T}_\lambda)$, we have:*

(i) $|\widetilde{K_6^3}(\mathbf{k})| \lesssim m(N_1)^2 N_1^2$*;*

(ii) *if $N_3 \ll N$, then $|\widetilde{K_6^3}(\mathbf{k})| \lesssim m(N_1)^2 N_1$.*

3.4 Almost conservation estimates for the third generation modified energy

The scope of this section is to show that for a smooth solution v of (2.22), the possible increase of $\mathcal{E}^3(v(\cdot))$ can be made arbitrary small by appropriately choosing the parameters N and λ, i.e. that we have an estimate of the form

$$\left|\mathcal{E}^3(v(\delta)) - \mathcal{E}^3(v(0))\right| \lesssim N^{-\gamma}\lambda^{-\kappa} \tag{3.93}$$

for some $\gamma, \kappa > 0$. [5] On the right hand side we use (powers of) the Z^1-norm of Iv who, we recall, lives on the scaled spatial domain $\mathbb{T}_\lambda$ and whose energy on frequencies $\gtrsim N$ is damped by the operator I.

We decompose the solution v using the Littlewood-Paley projectors in spatial frequencies:

$$v = \sum_{j=0}^{\infty} P_{2^j} v \quad , \quad \widehat{P_{2^j} v}(\tau, k) = \mathbf{1}_{I_j}(n)\widehat{v}(\tau, k),$$

where $I_0 := \{k \in \mathbb{Z}_\lambda : |k| < 1\}$ and $I_j := \{k \in \mathbb{Z}_\lambda : 2^{j-1} \leq |k| < 2^j\}$ for $j \geq 1$. By the fundamental theorem of calculus, the proof of (3.93) reduces to estimating expressions of the form

$$\int_0^\delta \Lambda_n(M_n; v(t))\, dt$$

[5] The powers γ, κ are responsible for the level of regularity at which the global existence via the I-method is obtained. Subsequent iterations of the I-method aim at finding a functional that can provide good enough decay rates in order to reach $s = \frac{1}{2}$. Eventually, one approximates an $H^s(\mathbb{T})$-solution v of (2.22) by a sequence of smooth solutions $(v_n)_n$ and then (3.93) is derived for v.

corresponding to the multipliers M_n that appear in (3.82). It is enough[6] to obtain estimates for

$$\int_{\mathbb{R}} \mathbf{1}_{[0,\delta]}(t)\Lambda_n(M_n; v_1(t), \dots, v_n(t))\, dt, \tag{3.94}$$

where each v_j has Fourier support in the band $\{(\tau, k) : |k| \sim N_j\}$, with $N_j \sim |I_j|$. If $N_j \ll N$ for all j, the multiplier M_n vanishes, hence we assume $N_1 \sim N_2 \gtrsim N$ (see Remark 3.17). Due to Remark 3.18, we can also assume that $N_1 \geq N_2 \geq \dots \geq N_n$.

Regarding the sharp time-cutoff, we note that in each case, we are able to place at least a few factors in the $X^{1,\frac{1}{2}-}$-norm (rather than in the Z^1-norm) and since we know that

$$\|\mathbf{1}_{[0,\delta]}(t)\|_{H_t^{\frac{1}{2}-}} \lesssim \delta^{0+},$$

by Lemma 3.4, we have

$$\|\mathbf{1}_{[0,\delta]} Iv\|_{X^{1,\frac{1}{2}-}} \lesssim \delta^{0+}\|Iv\|_{X^{1,\frac{1}{2}}}. \tag{3.95}$$

Therefore, in proving the results of this section, we are concerned with estimates of the form

$$\int_{\mathbb{R}} \Lambda_n(M_n; v_1(t), \dots, v_n(t))\, dt \lesssim N^{-\gamma}\lambda^{-\kappa} \prod_{j=1}^{n} \|Iv_j\|_{Z^1(\mathbb{R}\times\mathbb{T}_\lambda)}, \tag{3.96}$$

where $v_j = P_{N_j} v_j$ for all j.

Before starting to prove estimates of the form (3.96) for each term that appears in (3.82), we make some further reductions common to all of them.

Remark 3.35. Since the norms on the right hand side of (3.96) depend on $|\widehat{v_j}|$, for the sake of simplified writing, we assume that all $\widehat{v_j}$'s are real valued and non-negative.

Remark 3.36. To ensure summability over all dyadics $N_1 \geq N_2 \geq \dots \geq N_n$, we can most of the times obtain a factor of $1/N_1^{0+}$ on the right hand side above since then

$$\frac{1}{N_1^{0+}} \prod_{j=1}^{n} \|IP_{N_j} v_j\|_{Z^1(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \left(\prod_{j=1}^{n} \frac{1}{N_j^{0+}} \right) \|Iv_j\|^n_{Z^1(\mathbb{R}\times\mathbb{T}_\lambda)} ,$$

and the summation (first over N_n, lastly over N_1) is straightforward. However, having $L^2_{\tau,k}$-based norms on the largest two frequency factors Iv_1 and Iv_2 allows one to relax the summability factor to $1/N_3^{0+}$ in the region $N_1 \sim N_2$. This essentially follows from an application of Cauchy-Schwarz inequality. Indeed, suppose that we have

$$|\mathcal{L}(P_{N_1}v_1, P_{N_2}v_2)| \leq A\|IP_{N_1}v_1\|_{X^{1,\frac{1}{2}}}\|IP_{N_2}v_2\|_{X^{1,\frac{1}{2}}}$$

for the bilinear functional $\mathcal{L}$ defined by fixing $v_3, \dots, v_n$ in the left hand side of (3.96).

[6] Indeed, one can take the functions v_j such that the time restrictions $v_{j\,|[0,\delta]} = P_j v$ and $\|v_j\|_{Z^1(\mathbb{R}\times\mathbb{T}_\lambda)} \leq \|P_j v\|_{Z^1([0,\delta]\times\mathbb{T}_\lambda)} + \varepsilon$ for odd j's, and similarly with $P_j\overline{v}$ for even j's. Eventually one takes $\varepsilon \to 0$ to obtain the estimate.

Let $N_1 = 2^{j_1}$ and $N_2 = 2^{j_2}$. Summing over the pair of dyadic numbers (N_1, N_2) in the region $N_1 \sim N_2$ amounts to summing over the pair of integers (j_1, j_2) with $|j_1 - j_2| \leq 4$. [7] Therefore, by taking $\widehat{w_j}(\tau, k) = m(k)\langle k\rangle\langle \tau + k^2\rangle^{\frac{1}{2}}\widehat{v_j}(\tau, k)$, we obtain

$$\begin{aligned}\sum_{N_1\sim N_2} |\mathcal{L}(P_{N_1}v_1, P_{N_2}v_2)| &\leq \sum_{(j_1,j_2):|j_1-j_2|\leq 4} |\mathcal{L}(P_{2^{j_1}}v_1, P_{2^{j_2}}v_2)| \\ &\leq A\Big(\sum_{j_1\in\mathbb{Z}} \|P_{2^{j_1}}w_1\|^2_{L^2_{t,x}}\Big)^{\frac{1}{2}}\Big(\sum_{j_2\in\mathbb{Z}} \|P_{2^{j_2}}w_2\|^2_{L^2_{t,x}}\Big)^{\frac{1}{2}} \\ &\lesssim A\|w_1\|_{L^2_{t,x}}\|w_2\|_{L^2_{t,x}}.\end{aligned}$$

Remark 3.37. When applying the bilinear estimate (3.25), we argue as follows. For two $v_1, v_2 : [0,\delta]\times\mathbb{T}_\lambda \to \mathbb{C}$, taking into account (3.9), we consider extensions $v_{1,n}, v_{2,n} : \mathbb{R}\times\mathbb{T}_\lambda \to \mathbb{C}$ (i.e. $v_{j,n}|_{[0,\delta]} = v_j$, $j = 1, 2$) such that $\|v_{j,n}\|_{Z^s(\mathbb{R}\times\mathbb{T}_\lambda)} \to \|v_j\|_{Z^s([0,\delta]\times\mathbb{T}_\lambda)}$ as $n \to \infty$, $j = 1, 2$. Also, we consider a time cut-off η such that $\eta(t) = 1$ for all $t \in [-K, K]$ for some $K \gg \max\{\delta, 1\}$. Then,

$$\begin{aligned}\|v_1v_2\|_{L^2_{t,x}([0,\delta]\times\mathbb{T}_\lambda)} &\leq \|\eta(t)v_{1,n}\cdot\eta(t)v_{2,n}\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \\ &\lesssim \|v_{1,n}\|_{X^{0,\frac{1}{2}-\varepsilon}(\mathbb{R}\times\mathbb{T}_\lambda)}\|v_{2,n}\|_{X^{0,\frac{1}{2}-\varepsilon}(\mathbb{R}\times\mathbb{T}_\lambda)}\end{aligned}$$

and after taking $n \to \infty$, we get

$$\|v_1v_2\|_{L^2_{t,x}([0,\delta]\times\mathbb{T}_\lambda)} \lesssim \|v_1\|_{X^{0,\frac{1}{2}-\varepsilon}([0,\delta]\times\mathbb{T}_\lambda)}\|v_2\|_{X^{0,\frac{1}{2}-\varepsilon}([0,\delta]\times\mathbb{T}_\lambda)}. \tag{3.97}$$

With these reduction remarks at hand, we can proceed to the proof of almost conservation estimates. We denote by J_x the Bessel potential operator in the space variable, i.e. $\widehat{J_xf}(k) = \langle k\rangle\widehat{f}(k)$. For simplicity, $\int_*$ and $\int_{**}$ stand for integration on $\Gamma_6(\mathbb{R})$ and on $\Gamma_6(\mathbb{T}_\lambda)$, respectively (see Section 3.1.3).

Lemma 3.38. *Let $s \geq \frac{1}{2}$ and $\delta > 0$. For M_6^2 defined by (3.68), and Ω^c as in Subsection 3.3.4, we have the estimate*

$$\left|\int_0^\delta \Lambda_6(M_6^2\mathbf{1}_{\Omega^c}; v(t))\,dt\right| \lesssim N^{-\frac{3}{2}+}\lambda^{-1+}\delta^{0+}\|Iv\|^6_{Z^1([0,\delta]\times\mathbb{T}_\lambda)}. \tag{3.98}$$

Proof. We distinguish several subregions of Ω^c, but first note that for all $\mathbf{k} \in \Upsilon_6(\mathbb{T}_\lambda)\backslash\Omega_3$ we have $N_3 \sim N_4$.

Case 1: $N_1 \sim N_2 \gtrsim N \gg N_3$. Note that $m(N_j) = 1$ for $j \geq 3$, and $m(N_1)^2N_1 \gtrsim N$.

[7] For $n \leq 10$, on $\Gamma_n(\mathbb{T}_\lambda)$ we have $\frac{1}{9}N_2 \leq N_1 \leq 9N_2$, so there is a universal upper bound on $|j_1 - j_2|$.

By Lemma 3.30, we have $|M_6^2 \mathbb{1}_{\Omega^c}| \lesssim N_1^{\frac{1}{2}} N_3^{\frac{3}{2}}$ and by using (3.30), we get

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, \dots, v_6)\, dt &\lesssim \int_* \int_{**} \frac{1}{m(N_1)^2 N_1^{\frac{3}{2}} N_3^{\frac{1}{2}} \prod_{j=5,6} \langle k_j \rangle} \prod_{j=1}^{6} \widehat{J_x I v_j} \\
&\lesssim \frac{N^{-\frac{3}{2}+}}{N_1^{0+}} \int_* \int_{**} \frac{1}{\langle k_5 \rangle \langle k_6 \rangle} \prod_{j=1}^{6} \widehat{J_x I v_j} \\
&\lesssim \frac{N^{-\frac{3}{2}+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} (J_x I v_1)(J_x I v_3)(J_x I v_2)(J_x I v_4)(I v_5)(I v_6) dx dt \\
&\lesssim \frac{N^{-\frac{3}{2}+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_3)\|_{L^2_{t,x}} \|(J_x I v_2)(J_x I v_4)\|_{L^2_{t,x}} \prod_{j=5,6} \|I v_j\|_{L^\infty_{t,x}}.
\end{aligned}$$

By (3.25) and (3.10), we thus get

$$\int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, \dots, v_6)\, dt \lesssim \frac{N^{-\frac{3}{2}+} \lambda^{-1+}}{N_1^{0+}} \prod_{j=1}^{4} \|I v_j\|_{X^{1,\frac{1}{2}}} \prod_{j=5,6} \|I v_j\|_{Y^{\frac{1}{2}+,0}}.$$

The case $N_3 \gtrsim N \gg N_4$ is vacuous on Ω^c and thus the next case we have to consider is the one in which precisely four of the frequencies have sizes larger than the threshold N.

Case 2: $N_4 \gtrsim N \gg N_5$. We also have $N_1 \sim N_2$, $N_3 \sim N_4$ and for $j = 3, 4$,

$$m(N_j) N_j = N^{1-} \left(\frac{N_j}{N} \right)^{s-} N_j^{0+} \gtrsim N^{1-} N_j^{0+}. \tag{3.99}$$

By using the crude estimate $|M_6^2| \lesssim m(N_1)^2 N_1^2$ of Lemma 3.20, we estimate

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, \dots, v_6)\, dt &\lesssim \int_* \int_* \frac{1}{m(N_3)^2 N_3^2 \langle k_5 \rangle \langle k_6 \rangle} \prod_{j=1}^{6} \widehat{J_x I v_j} \\
&\lesssim \frac{N^{-2+}}{N_3^{0+}} \int_* \frac{1}{\langle k_5 \rangle \langle k_6 \rangle} \prod_{j=1}^{6} \widehat{J_x I v_j}.
\end{aligned} \tag{3.100}$$

We now discuss two subcases.

Subcase 2.1. If $N_3 \sim N_1$, since $N_5 \ll N_4$, two out of the four frequencies k_1, k_2, k_3, k_4 must have opposite signs, say k_1 and k_2. Therefore v_1 and v_2 are separated in frequency and we use the bilinear estimate (3.25), and together with the L^4-Strichartz estimate (3.18), we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, \dots, v_6)\, dt &\lesssim \frac{N^{-2+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_2)\|_{L^2_{t,x}} \prod_{j=3,4} \|J_x I v_j\|_{L^4_{t,x}} \prod_{j=5,6} \|I v_j\|_{L^\infty_{t,x}} \\
&\lesssim \frac{N^{-2+} \lambda^{-\frac{1}{2}+}}{N_1^{0+}} \prod_{j=1}^{4} \|J_x I v_j\|_{X^{0,\frac{1}{2}}} \prod_{j=5,6} \|I v_j\|_{Y^{\frac{1}{2}+,0}}.
\end{aligned}$$

Subcase 2.2. If $N_3 \ll N_1$, then we apply the bilinear estimate (3.25) twice and get

$$\begin{aligned}
&\int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, v_2, \dots, v_6)\, dt \\
&\quad \lesssim \frac{N^{-2+}}{N_3^{0+}} \|(J_x I v_1)(J_x I v_3)\|_{L^2_{t,x}} \|(J_x I v_2)(J_x I v_4)\|_{L^2_{t,x}} \prod_{j=5,6} \|I v_j\|_{L^\infty_{t,x}} \\
&\quad \lesssim \frac{N^{-2+}\lambda^{-1+}}{N_3^{0+}} \prod_{j=1}^{4} \|J_x I v_j\|_{X^{0,\frac{1}{2}}} \prod_{j=5,6} \|I v_j\|_{Y^{\frac{1}{2}+,0}}.
\end{aligned}$$

Case 3: $N_5 \gtrsim N$. We use (3.99) for $j = 3,4,5$, $m(k_6)\langle k_6\rangle^{\frac{1}{2}} \gtrsim 1$, and $N_5 \geq N_6$ to deduce

$$\begin{aligned}
&\int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, \dots, v_6)\, dt \\
&\quad \lesssim \int_* \int_{**} \frac{1}{\prod_{j=3}^{6} m(k_j)\langle k_j\rangle} \prod_{j=1}^{6} \widehat{J_x I v_j} \\
&\quad \lesssim \frac{N^{-3+}}{N_3^{0+}} \int_* \int_{**} \widehat{J_x I v_1}\widehat{J_x I v_2} \prod_{j=3}^{5} \left(\frac{\lambda^{0+}}{\langle k_j\rangle^{0+}} \widehat{J_x I v_j} \right) \left(\frac{1}{\langle k_6\rangle^{\frac{1}{2}+}} \widehat{J_x I v_6} \right) \\
&\quad \lesssim \frac{N^{-3+}\lambda^{0+}}{N_3^{0+}} \prod_{j=1,2} \|J_x I v_j\|_{L^4_{t,x}} \prod_{j=3}^{5} \|J_x^{1-} I v_j\|_{L^6_{t,x}} \|J_x^{\frac{1}{2}-} I v_6\|_{L^\infty_{t,x}}.
\end{aligned}$$

The factors λ^{0+} above appear due to the application of (3.3). By using the Strichartz estimates (3.18) and (3.21), as well as the embedding (3.10), we have

$$\begin{aligned}
&\int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, \dots, v_6)\, dt \\
&\quad \lesssim \frac{N^{-3+}\lambda^{0+}}{N_3^{0+}} \prod_{j=1,2} \|J_x I v_j\|_{X^{0,\frac{3}{8}}} \prod_{j=3}^{5} \|J_x^{1-} I v_j\|_{X^{0+,\frac{1}{2}}} \|J_x^{\frac{1}{2}-} I v_6\|_{Y^{\frac{1}{2}+,0}} \\
&\quad \lesssim \frac{N^{-3+}\lambda^{0+}}{N_3^{0+}} \prod_{j=1}^{5} \|I v_j\|_{X^{1,\frac{1}{2}}} \|I v_6\|_{Y^{1,0}}.
\end{aligned}$$

Since in Section 3.6 we choose λ, N such that $1 \leq \lambda \leq N$ (for $s \geq \frac{1}{2}$), in the second and third cases we have faster decaying factors than in Case 1.

□

Lemma 3.39. *Let $s \geq \frac{1}{2}$ and $\delta > 0$. For M_8^3 defined by* (3.83)*, we have the estimate*

$$\left| \int_0^\delta \Lambda_8(M_8^3; v(t))\, dt \right| \lesssim N^{-\frac{3}{2}+}\lambda^{-1+}\delta^{0+} \|I v\|^8_{Z^1([0,\delta]\times\mathbb{T}_\lambda)}. \tag{3.101}$$

The same estimate holds if M_8^3 is replaced by M_8^2.

Proof. By Lemma 3.32, we have $|M_8^3(\mathbf{k})| \lesssim N_1$ for all $\mathbf{k} \in \Gamma_8(\mathbb{T}_\lambda)$, and if $N_3 \ll N$, then $|M_8^3(\mathbf{k})| \lesssim N_1^{\frac{1}{2}} N_3^{\frac{1}{2}}$.

We distinguish three cases and in all of them we use that $m(N_1)^2 N_1 \gtrsim N$, and when $N_3 \gtrsim N$, $m(N_3)N_3 \gtrsim N^{1-} N_3^{0+}$ as in (3.99).

Case 1: $N_1 \sim N_2 \gtrsim N \gg N_3$. We have

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_8(M_8^3; v_1, \dots, v_8)\, dt &\lesssim \int_* \int_{**} \frac{1}{m(N_1)^2 N_1^{\frac{3}{2}} N_3^{\frac{1}{2}} \prod_{j=4}^{8} \langle k_j \rangle} \prod_{j=1}^{8} \widehat{J_x I v_j} \\
&\lesssim \frac{N^{-\frac{3}{2}+}}{N_1^{0+}} \int_* \int_{**} (\widehat{J_x I v_1} \widehat{J_x I v_3})(\widehat{J_x I v_2} \widehat{J_x I v_4}) \prod_{j=5}^{8} \widehat{I v_j} \\
&\lesssim \frac{N^{-\frac{3}{2}+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_3)\|_{L^2_{t,x}} \|(J_x I v_2)(J_x I v_4)\|_{L^2_{t,x}} \prod_{j=5}^{8} \|I v_j\|_{L^\infty_{t,x}} \\
&\lesssim \frac{N^{-\frac{3}{2}+} \lambda^{-1+}}{N_1^{0+}} \prod_{j=1}^{4} \|I v_j\|_{X^{1,\frac{1}{2}}} \prod_{j=5}^{8} \|I v_j\|_{Y^{\frac{1}{2}+,0}}.
\end{aligned}$$

Case 2: $N_3 \gtrsim N \gg N_4$. Here, we get

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_8(M_8^3; v_1, \dots, v_8)\, dt &\lesssim \int_* \int_{**} \frac{1}{m(N_1)^2 N_1 m(N_3) N_3 \prod_{j=4}^{8} \langle k_j \rangle} \prod_{j=1}^{8} \widehat{J_x I v_j} \\
&\lesssim \frac{N^{-2+}}{N_3^{0+}} \int_* \int_{**} (\widehat{J_x I v_1} \widehat{J_x I v_4})(\widehat{J_x I v_2})(\widehat{J_x I v_3}) \prod_{j=5}^{8} \widehat{I v_j} \\
&\lesssim \frac{N^{-2+}}{N_3^{0+}} \|(J_x I v_1)(J_x I v_4)\|_{L^2_{t,x}} \|J_x I v_2\|_{L^4_{t,x}} \|J_x I v_3\|_{L^4_{t,x}} \prod_{j=5}^{8} \|I v_j\|_{L^\infty_{t,x}} \\
&\lesssim \frac{N^{-2+} \lambda^{-\frac{1}{2}+}}{N_3^{0+}} \prod_{j=1}^{4} \|I v_j\|_{X^{1,\frac{1}{2}}} \prod_{j=5}^{8} \|I v_j\|_{Y^{\frac{1}{2}+,0}}.
\end{aligned}$$

Case 3: $N_4 \gtrsim N$. In this case, we additionally have that $m(N_4)N_4 \gtrsim N$. For $5 \le j \le 8$, since $m(k_j)\langle k_j \rangle^{\frac{1}{2}} \gtrsim 1$, by taking into account (3.3), we have

$$N_3^{0+} m(k_j) \langle k_j \rangle \gtrsim \lambda^{0-} \langle k_j \rangle^{\frac{1}{2}+}. \tag{3.102}$$

Thus, we obtain

$$\begin{aligned}\int_{\mathbb{R}} \Lambda_8(M_8^3; v_1, \dots, v_8)\, dt &\lesssim \frac{N^{-3+}}{N_3^{0+}} \int_* \int_{**} \prod_{j=1}^{4} \widehat{J_x I v_j} \prod_{j=5}^{8} \frac{\lambda^{0+}}{\langle k_j \rangle^{\frac{1}{2}+}} \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-3+}\lambda^{0+}}{N_3^{0+}} \prod_{j=1}^{4} \|J_x I v_j\|_{L^4_{t,x}} \prod_{j=5}^{8} \|J_x^{\frac{1}{2}-} I v_j\|_{L^\infty_{t,x}} \\ &\lesssim \frac{N^{-3+}\lambda^{0+}}{N_3^{0+}} \prod_{j=1}^{4} \|J_x I v_j\|_{X^{0,\frac{3}{8}}} \prod_{j=5}^{8} \|J_x^{\frac{1}{2}-} I v_j\|_{Y^{\frac{1}{2}+,0}} \\ &\lesssim \frac{N^{-3+}\lambda^{0+}}{N_3^{0+}} \prod_{j=1}^{4} \|I v_j\|_{X^{1,\frac{1}{2}}} \prod_{j=5}^{8} \|I v_j\|_{Y^{1,0}}.\end{aligned}$$

We recall that for the multiplier M_8^2 we have better bounds than for M_8^3 (see Lemma 3.20 and Lemma 3.30), hence it is enough to consider only the latter.

□

Lemma 3.40. *Let $s \geq \frac{1}{2}$ and $\delta > 0$. For M_{10}^3 defined by* (3.85)*, we have the estimate*

$$\left| \int_0^\delta \Lambda_{10}(M_{10}^3; v(t))\, dt \right| \lesssim N^{-2+}\lambda^{-1+}\delta^{0+} \|Iv\|^{10}_{Z^1([0,\delta]\times\mathbb{T}_\lambda)}. \tag{3.103}$$

Proof. By (3.85) and Lemma 3.31, we have $|M_{10}^3(\mathbf{k})| \lesssim 1$ and thus we gain the factor N^{-2+} from $m(N_j)N_j \gtrsim N^{1-}N_j^{0+}$, $j = 1, 2$. For additional decaying factors, it is enough to discuss two cases.

Case 1: $N_2 \gtrsim N \gg N_3$. We have

$$\begin{aligned}\int_{\mathbb{R}} \Lambda_{10}(M_{10}^3; v_1, \dots, v_{10})\, dt &\lesssim \frac{N^{-2+}}{N_1^{0+}} \int_* \int_{**} (\widehat{J_x I v_1} \widehat{J_x I v_3})(\widehat{J_x I v_2} \widehat{J_x I v_4}) \prod_{j=5}^{10} \widehat{I v_j} \\ &\lesssim \frac{N^{-2+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_3)\|_{L^2_{t,x}} \|(J_x I v_2)(J_x I v_4)\|_{L^2_{t,x}} \prod_{j=5}^{10} \|I v_j\|_{L^\infty_{t,x}} \\ &\lesssim \frac{N^{-2+}\lambda^{-1+}}{N_1^{0+}} \prod_{j=1}^{4} \|I v_j\|_{X^{1,\frac{1}{2}}} \prod_{j=5}^{10} \|I v_j\|_{Y^{\frac{1}{2}+,0}}\end{aligned}$$

Case 2: $N_3 \gtrsim N$. In this case, we additionally have $m(N_3)N_3 \gtrsim N$. Also, we use $m(k_4)\langle k_4 \rangle \gtrsim 1$, and $m(k_j)\langle k_j \rangle^{\frac{1}{2}} \gtrsim 1$ for $5 \leq j \leq 10$. By using $1/N_1^\varepsilon \leq \prod_{j=5}^{10} 1/N_j^{\varepsilon/6}$, we get

$$\begin{aligned}\int_{\mathbb{R}} \Lambda_{10}(M_{10}^3; v_1, \dots, v_{10})\, dt &\lesssim \frac{N^{-3+}}{N_1^{0+}} \prod_{j=1}^{4} \|J_x I v_j\|_{L^4_{t,x}} \prod_{j=5}^{10} \|J_x^{\frac{1}{2}-} I v_j\|_{L^\infty_{t,x}} \\ &\lesssim \frac{N^{-3+}}{N_1^{0+}} \prod_{j=1}^{4} \|I v_j\|_{X^{1,\frac{3}{8}}} \prod_{j=5}^{10} \|I v_j\|_{Y^{1,0}}.\end{aligned}$$

Note that in Case 2 (by discussing various subregions), we could provide at least an additional $\lambda^{-\frac{1}{2}+}$ factor, but since $N^{-1+} \lesssim \lambda^{-1+}$ and the decaying factor in Case 1 is optimal, we limit ourselves to the above estimate.

$\square$

For the remaining terms that appear in (3.82) (i.e. the ones due to the gauge transformation in the periodic setting), we have a decaying factor λ^{-1} thanks to the coupling coefficient $\mu(v)$. Indeed, by (3.11) and by using $1 \leq m(k)\langle k\rangle$, we have

$$\mu(v) = \frac{1}{2\pi\lambda}\|v\|^2_{L^\infty_t L^2_x} \lesssim \lambda^{-1}\|J_x I v\|^2_{Y^{0,0}} \leq \lambda^{-1}\|Iv\|^2_{Z^1}.$$

Lemma 3.41. *Let $s > 0$ and $\delta > 0$. For K_4^1 as defined by* (3.60)*, we have the estimate*

$$\left|\int_0^\delta \Lambda_4(K_4^1; v(t))\, dt\right| \lesssim N^{-1+}\lambda^{-1+}\delta^{0+}\|Iv\|^4_{X^{1,\frac{1}{2}}([0,\delta]\times\mathbb{T}_\lambda)}. \tag{3.104}$$

Proof. By Lemma 3.23 we have $|K_4^1(\mathbf{k})| \lesssim m(N_1)^2 N_1^2$ for all $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$, and if $N_3 \ll N$ then $|K_4^1(\mathbf{k})| \lesssim m(N_1)^2 N_1 N_3$. We need to discuss three cases.

Case 1: $N_1 \sim N_2 \gtrsim N \gg N_3$. Due to the refined estimate, we have

$$\begin{aligned}\int_{\mathbb{R}} \Lambda_4(K_4^1; v_1, \ldots, v_4)\, dt &\lesssim \int_* \int_{**} \frac{1}{N_1\langle k_4\rangle} \prod_{j=1}^4 \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-1+}}{N_1^{0+}} \int_{\mathbb{R}}\int_{\mathbb{T}_\lambda} (J_x I v_1)(J_x I v_3)(J_x I v_2)(J_x I v_4) dx dt \\ &\lesssim \frac{N^{-1+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_3)\|_{L^2_{t,x}} \|(J_x I v_2)(J_x I v_4)\|_{L^2_{t,x}} \\ &\lesssim \frac{N^{-1+}\lambda^{-1+}}{N_1^{0+}} \prod_{j=1}^4 \|Iv_j\|_{X^{1,\frac{1}{2}-}}.\end{aligned}$$

Case 2: $N_3 \gtrsim N \gg N_4$. By using (3.99), we obtain

$$\begin{aligned}\int_{\mathbb{R}} \Lambda_4(K_4^1; v_1, \ldots, v_4)\, dt &\lesssim \int_* \int_{**} \frac{1}{m(N_3)N_3 m(k_4)\langle k_4\rangle} \prod_{j=1}^4 \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-1+}}{N_3^{0+}} \int_{\mathbb{R}}\int_{\mathbb{T}_\lambda} (J_x I v_1)(J_x I v_2)(J_x I v_3)(v_4) dx dt.\end{aligned}$$

Subcase 2.1. If $N_3 \sim N_1$, then since $N_3 \gg N_4$, two out of the three frequencies k_1, k_2, k_3 must have opposite signs, say k_1 and k_2. Thus $J_x I v_1$ and $J_x I v_2$ are separated in frequency, and so are J_xIv_3 and J_xIv_4. By also using $m(k_4)\langle k_4\rangle \gtrsim 1$, we have

$$\begin{aligned}\int_{\mathbb{R}} \Lambda_4(K_4^1; v_1, \dots, v_4)\, dt &\lesssim \frac{N^{-1+}}{N_1^{0+}} \|(J_xIv_1)(J_xIv_2)\|_{L^2_{t,x}} \|(J_xIv_3)(J_xIv_4)\|_{L^2_{t,x}} \\ &\lesssim \frac{N^{-1+}\lambda^{-1+}}{N_3^{0+}} \prod_{j=1}^4 \|Iv_j\|_{X^{1,\frac{1}{2}-}}. \end{aligned} \tag{3.105}$$

Subcase 2.2. If $N_3 \ll N_1$, then as in Case 1, we can clearly apply the bilinear estimate (3.25) to the $L^2_{t,x}$-norms of both $(J_xIv_1)(J_xIv_3)$ and $(J_xIv_2)(J_xIv_4)$ and obtain the same bound as in (3.105).

Case 3: $N_4 \gtrsim N$. We have $m(N_j)N_j \gtrsim N^{-1+}N_j^{0+}$ for $j = 3, 4$ and thus

$$\begin{aligned}\int_{\mathbb{R}} \Lambda_4(K_4^1; v_1, \dots, v_4)\, dt &\lesssim \int_* \int_{**} \frac{1}{m(N_3)N_3 m(N_4)N_4} \prod_{j=1}^4 \widehat{J_xIv_j} \\ &\lesssim \frac{N^{-2+}}{N_3^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} (J_xIv_1)(J_xIv_2)(J_xIv_3)(J_xIv_4) dxdt \\ &\lesssim \frac{N^{-2+}}{N_3^{0+}} \prod_{j=1}^4 \|J_xIv_j\|_{L^4_{t,x}} \\ &\lesssim \frac{N^{-2+}}{N_3^{0+}} \prod_{j=1}^4 \|Iv_j\|_{X^{1,\frac{3}{8}}}. \end{aligned}$$

□

Lemma 3.42. *Let $s \geq \frac{3}{8}$ and $\delta > 0$. For K_6^1 defined by* (3.61)*, we have the estimate*

$$\left| \int_0^\delta \Lambda_6(K_6^1; v(t))\, dt \right| \lesssim N^{-2+}\delta^{0+} \|Iv\|^6_{X^{1,\frac{1}{2}}([0,\delta]\times\mathbb{T}_\lambda)}. \tag{3.106}$$

Proof. By Lemma 3.24, we have $|K_6^1(\mathbf{k})| \lesssim 1$ for all $\mathbf{k} \in \Gamma_6(\mathbb{T}_\lambda)$. By using (3.31), (3.16) and (3.18), we estimate

$$\begin{aligned}\int_{\mathbb{R}} \Lambda_6(K_6^1; v_1, \dots, v_6)\, dt &\lesssim \int_* \int_{**} \frac{1}{(m(N_1)N_1)^2} \prod_{j=1,2} \widehat{J_xIv_j} \prod_{j=3}^6 \widehat{v_j} \\ &\lesssim \frac{N^{-2+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} \prod_{j=1,2} J_xIv_j \prod_{j=3}^6 J_x^{\frac{5}{8}} Iv_j\, dxdt \\ &\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1,2} \|J_xIv_j\|_{L^4_{t,x}} \prod_{j=3}^6 \|J_x^{\frac{5}{8}} Iv_j\|_{L^8_{t,x}} \\ &\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1}^6 \|Iv_j\|_{X^{1,\frac{3}{8}}}. \end{aligned}$$

□

Lemma 3.43. *Let $s \geq \frac{1}{2}$ and $\delta > 0$. For K_6^2 defined by* (3.70)*, we have the estimate*

$$\left| \int_0^\delta \Lambda_6(K_6^2; v(t))\, dt \right| \lesssim N^{-1+} \lambda^{-1+} \delta^{0+} \|Iv\|^6_{Z^1([0,\delta]\times\mathbb{T}_\lambda)}. \tag{3.107}$$

Proof. By Lemma 3.24, we have $|K_6^2| \lesssim m(N_1)^2 N_1$.

Case 1: $N_2 \gtrsim N \gg N_3$. By using $1 \lesssim m(k_j)\langle k_j \rangle$ for $j = 3, 4$, and $\frac{1}{N_1^{0+}} \lesssim m(k_j)\langle k_j\rangle^{\frac{1}{2}-}$ for $j = 5, 6$, we estimate

$$\begin{aligned}
&\int_{\mathbb{R}} \Lambda_6(K_6^2; v_1, \ldots, v_6) dt \\
&\lesssim \int_* \int_{**} \frac{1}{N_1} \prod_{j=1,2} \widehat{J_x I v_j} \prod_{j=3}^6 \widehat{v_j} \\
&\lesssim \frac{N^{-1+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} \prod_{j=1}^4 J_x I v_j \prod_{j=5,6} J_x^{\frac{1}{2}-} I v_j \, dx dt \\
&\lesssim \frac{N^{-1+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_3)\|_{L^2_{t,x}} \|(J_x I v_2)(J_x I v_4)\|_{L^2_{t,x}} \prod_{j=5,6} \|J_x^{\frac{1}{2}-} I v_j\|_{L^\infty_{t,x}} \\
&\lesssim \frac{N^{-1+}\lambda^{-1+}}{N_1^{0+}} \prod_{j=1}^4 \|Iv_j\|_{X^{1,\frac{3}{8}}} \prod_{j=5,6} \|Iv_j\|_{Y^{1,0}}.
\end{aligned}$$

Case 2: $N_3 \gtrsim N$. We make use of $m(N_3)N_3 \gtrsim N$ and thus we get

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_6(K_6^2; v_1, \ldots, v_6) dt &\lesssim \frac{N^{-2+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} \prod_{j=1}^3 J_x I v_j \prod_{j=4}^6 v_j \, dx dt \\
&\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1}^3 \|J_x I v_j\|_{L^4_{t,x}} \prod_{j=4}^6 \|v_j\|_{L^{12}_{t,x}} \\
&\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1}^3 \|Iv_j\|_{X^{1,\frac{3}{8}}} \prod_{j=4}^6 \|v_j\|_{X^{\frac{5}{12},\frac{5}{12}}}.
\end{aligned}$$

□

For the next lemma, we make the following remark. The proof follows identically in Case 1, but we only have $|\widetilde{K_6^3}| \lesssim m(N_1)^2 N_1^2$ in Case 2. By splitting the discussion into the subcases $N_3 \sim N_1$ and $N_3 \ll N_1$ as in Case 2 of the proof of Lemma 3.38, we can provide at least an additional $\lambda^{-\frac{1}{2}+}$ factor. Hence, we have:

Lemma 3.44. *Let $s \geq \frac{1}{2}$ and $\delta > 0$. For $\widetilde{K_6^3}$ defined by* (3.70)*, we have the estimate*

$$\left| \int_0^\delta \Lambda_6(\widetilde{K_6^3}; v(t))\, dt \right| \lesssim N^{-1+} \lambda^{-\frac{1}{2}+} \delta^{0+} \|Iv\|^6_{Z^1([0,\delta]\times\mathbb{T}_\lambda)}. \tag{3.108}$$

The estimates for $\int_0^\delta \Lambda_6(\widetilde{K_6^4})dt$ and $\int_0^\delta \Lambda_8(\widetilde{K_8^3})dt$ follow identically to that of Lemma 3.43

above, since we have the same upper bound (see Lemma 3.33 and the subsequent comment).

Lemma 3.45. *Let $s \geq \frac{5}{12}$ and $\delta > 0$. For K_8^3 defined by* (3.84)*, we have the estimate*

$$\left| \int_0^\delta \Lambda_8(K_8^3; v(t))\, dt \right| \lesssim N^{-2+} \delta^{0+} \|Iv\|^8_{X^{1,\frac{1}{2}}([0,\delta]\times\mathbb{T}_\lambda)}. \tag{3.109}$$

Proof. By Lemma 3.31, we have $|K_8^3(\mathbf{k})| \lesssim 1$ for all $\mathbf{k} \in \Gamma_8(\mathbb{T}_\lambda)$. Hence, similarly to the proof of Lemma 3.42, we get

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_8(K_8^3; v_1, \ldots, v_8)\, dt &\lesssim \frac{N^{-2+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} \prod_{j=1,2} J_x I v_j \prod_{j=3}^{8} J_x^{\frac{7}{12}} I v_j \, dxdt \\
&\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1,2} \|J_x I v_j\|_{L^4_{t,x}} \prod_{j=3}^{8} \|J_x^{\frac{7}{12}} I v_j\|_{L^{12}_{t,x}} \\
&\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1}^{8} \|Iv_j\|_{X^{1,\frac{1}{2}}}.
\end{aligned}$$

□

We put all the results of this section together in the following:

Proposition 3.46. *Let $s \geq \frac{1}{2}$ and $\delta > 0$. Suppose v is a solution to* (2.22) *on $[0, \delta]$. For $\mathcal{E}^3$ defined by* (3.79)*, we have*

$$\left| \mathcal{E}^3(v(\delta)) - \mathcal{E}^3(v(0)) \right| \leq N^{-\frac{3}{2}+} \lambda^{-1+} \delta^{0+} P(\|Iv\|_{Z^1([0,\delta]\times\mathbb{T}_\lambda)}), \tag{3.110}$$

for some polynomial P with non-negative coefficients.

3.5 Control of the almost conserved energy and of the almost conserved momentum

In this section we show that $\mathcal{E}(Iv(t))$ stays close to $\mathcal{E}^3(v(t))$ (which is very slowly varying in time) and that $\mathcal{P}(Iv(t))$ stays close to $\mathcal{P}(v(t)) = \mathcal{P}(v_0)$, at any time t. For the sake of efficiency, we adopt in the proofs below the reduction remarks from the previous section.

Lemma 3.47. *Let $s \geq \frac{1}{2}$. For σ_4 defined by* (3.64)*, we have*

$$|\Lambda_4(\sigma_4; f)| \lesssim N^{-1+} \|If\|^4_{H^1(\mathbb{T}_\lambda)}. \tag{3.111}$$

Proof. By Lemma 3.21, we have $|\sigma_4(\mathbf{k})| \lesssim m(N_1)^2 N_1$ for all $k \in \Gamma_4(\mathbb{T}_\lambda)$. Then, by

Hölder and Sobolev inequalities, and using $\frac{1}{N_1^{0+}} \lesssim m(k_j)\langle k_j\rangle^{\frac{1}{2}-}$ for $j = 3, 4$, we have

$$\begin{aligned}
\Lambda_4(\sigma_4; f_1, \dots, f_4) &\lesssim \int_{**} \frac{m(N_1)^2 N_1}{\prod_{j=1}^4 m(k_j)\langle k_j\rangle} \prod_{j=1}^4 \widehat{J_x I f_j} \\
&\lesssim \frac{1}{N_1} \int_{\mathbb{T}_\lambda} (J_x I f_1)(J_x I f_2)(J_x^{\frac{1}{2}-} I f_3)(J_x^{\frac{1}{2}-} I f_4) dx \\
&\lesssim \frac{N^{-1+}}{N_1^{0+}} \|J_x I f_1\|_{L^2_x} \|J_x I f_2\|_{L^2_x} \|J_x^{\frac{1}{2}-} I f_3\|_{L^\infty_x} \|J_x^{\frac{1}{2}-} I f_4\|_{L^\infty_x} \\
&\lesssim \frac{N^{-1+}}{N_1^{0+}} \prod_{j=1}^4 \|I f_j\|_{H^1_x}.
\end{aligned}$$

□

The estimate for $\Lambda_4(\widetilde{\sigma_4}; f)$ follows similarly since, by Lemma 3.33 (i), we have the same pointwise bound, that is $|\widetilde{\sigma_4}(\mathbf{k})| \lesssim m(N_1)^2 N_1$.

Lemma 3.48. *Let $s \geq \frac{1}{2}$. For σ_6 defined by* (3.80)*, we have*

$$|\Lambda_6(\sigma_6; f)| \lesssim N^{-2+} \|If\|^6_{H^1(\mathbb{T}_\lambda)}. \tag{3.112}$$

Proof. By Lemma 3.31, we have $|\sigma_6(\mathbf{k})| \lesssim 1$ for all $k \in \Gamma_6(\mathbb{T}_\lambda)$. Similarly to the proof of Lemma 3.47, we have

$$\begin{aligned}
\Lambda_6(\sigma_6; f_1, \dots, f_4) &\lesssim \frac{N^{-2+}}{N_1^{0+}} \int_{\mathbb{T}_\lambda} (J_x I f_1)(J_x I f_2) \prod_{j=3}^6 (J_x^{\frac{1}{2}-} I f_j) dx \\
&\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1}^6 \|I f_j\|_{H^1_x}.
\end{aligned}$$

□

Hence, we proved that all the correction terms are small, and thus we obtain:

Proposition 3.49. *Let $s \geq \frac{1}{2}$. For $\mathcal{E}$ and $\mathcal{E}^3$ defined by* (3.55) *and* (3.79)*, we have*

$$\left|\mathcal{E}(If) - \mathcal{E}^3(f)\right| \lesssim N^{-1+} \left(\|If\|^4_{H^1_x(\mathbb{T}_\lambda)} + \|If\|^6_{H^1_x(\mathbb{T}_\lambda)} \right), \tag{3.113}$$

for all $f \in H^s_x(\mathbb{T}_\lambda)$.

Next, we turn to the analysis of $\mathcal{P}[I(\cdot)]$ for which, as in [18], we prove:

Proposition 3.50. *Let $s \geq \frac{1}{2}$. For $\mathcal{P}$ defined by* (2.41)*, we have*

$$|\mathcal{P}[If] - \mathcal{P}[f]| \lesssim N^{-1} \left(\|If\|^2_{H^1(\mathbb{T}_\lambda)} + \|If\|^4_{H^1(\mathbb{T}_\lambda)} \right), \tag{3.114}$$

for all $f \in H^s_x(\mathbb{T}_\lambda)$.

Proof. We have

$$|\mathcal{P}[If]-\mathcal{P}[f]| \le \left|\operatorname{Im}\int_{\mathbb{T}_\lambda}\left(If\partial_x(\overline{If})-f\partial_x\overline{f}\right)dx\right| + \frac{1}{2}\left|\int_{\mathbb{T}_\lambda}\left(|If|^4-|f|^4\right)dx\right| \quad (3.115)$$

and we can estimate the two terms separately.

First, using integration by parts, we write

$$\begin{aligned}\operatorname{Im}\int_{\mathbb{T}_\lambda}\left(If\partial_x(\overline{If})-f\partial_x\overline{f}\right)dx &= \operatorname{Im}\int_{\mathbb{T}_\lambda} If\partial_x\left(\overline{If}-\overline{f}\right)dx + \operatorname{Im}\int_{\mathbb{T}_\lambda}\partial_x\overline{f}\,(If-f)\,dx\\ &= \operatorname{Im}\int_{\mathbb{T}_\lambda} If\partial_x\left(\overline{If}-\overline{f}\right)dx + \operatorname{Im}\int_{\mathbb{T}_\lambda} f\partial_x\left(\overline{If-f}\right)dx\\ &= \operatorname{Im}\int_{\mathbb{T}_\lambda}(If+f)\partial_x\left(\overline{If-f}\right)dx.\end{aligned}$$

Notice that $I-\mathrm{Id}=P_{\mathrm{hi}}(I-\mathrm{Id})$, where Id is the identity operator and we take $P_{\mathrm{hi}}:=P_{\gtrsim N}$. Thus, by commuting Fourier multiplier operators, using the self-adjointness of Littlewood-Paley operators and duality properties of Sobolev norms, we have

$$\begin{aligned}\left|\operatorname{Im}\int_{\mathbb{T}_\lambda}(If+f)\partial_x\left(\overline{If-f}\right)dx\right| &\le \left|\langle P_{\mathrm{hi}}(If+f),(I-\mathrm{Id})\partial_x\overline{f}\rangle_{L^2(\mathbb{T}_\lambda)}\right|\\ &\le \|P_{\mathrm{hi}}(If+f)\|_{H^{\frac{1}{2}}}\|P_{\mathrm{hi}}(I-\mathrm{Id})\partial_x f\|_{H^{-\frac{1}{2}}}\\ &\le \left(\|P_{\mathrm{hi}}If\|_{H^{\frac{1}{2}}}+\|P_{\mathrm{hi}}f\|_{H^{\frac{1}{2}}}\right)^2.\end{aligned}$$

Since $1\lesssim N^{-\frac{1}{2}}\langle k\rangle^{\frac{1}{2}}$, $1\lesssim N^{-\frac{1}{2}}m(k)\langle k\rangle^{\frac{1}{2}}$ for all $|k|\gtrsim N$, we have

$$\|P_{\mathrm{hi}}If\|_{H^{\frac{1}{2}}}\lesssim N^{-\frac{1}{2}}\|If\|_{H^1}, \quad (3.116)$$

$$\|P_{\mathrm{hi}}f\|_{H^{\frac{1}{2}}}\lesssim N^{-\frac{1}{2}}\|If\|_{H^1}. \quad (3.117)$$

Thus the first term in the right hand side of (3.115) is bounded by $N^{-1}\|If\|_{H^1}^2$.

For the second term in the right hand side of (3.115), we write

$$|If|^4-|f|^4=|If|^2If(\overline{If}-\overline{f})+|If|^2(If-f)\overline{f}+If(\overline{If}-\overline{f})|f|^2+(If-f)\overline{f}|f|^2$$

and we treat, for example, the second term (modulo complex conjugation, it has all three possible factors involved); the others can be argued for analogously. By Hölder's inequality,

$$\begin{aligned}\left|\int_{\mathbb{T}_\lambda}|If|^2\overline{f}P_{\mathrm{hi}}(I-\mathrm{Id})f\,dx\right| &\le \left|\langle (I-\mathrm{Id})f,P_{\mathrm{hi}}(|If|^2f)\rangle_{L^2}\right|\\ &\lesssim \|(I-\mathrm{Id})f\|_{L^6}\|P_{\mathrm{hi}}(|If|^2f)\|_{L^{\frac{6}{5}}}.\end{aligned}$$

Then, by Sobolev embedding,

$$\|(I - \mathrm{Id})f\|_{L^6} \lesssim \|P_{\mathrm{hi}}(I - \mathrm{Id})f\|_{H^{\frac{1}{3}}} \leq \|P_{\mathrm{hi}} If\|_{H^{\frac{1}{2}}} + \|P_{\mathrm{hi}} f\|_{H^{\frac{1}{2}}} \tag{3.118}$$

and we can use the estimates (3.116)-(3.117) to gain a factor of $N^{-\frac{1}{2}}$. Another decaying factor is obtained via a Bernstein estimate, and then by Leibniz and Hölder inequalities, we get

$$\begin{aligned}\|P_{\mathrm{hi}}(|If|^2 f)\|_{L^{\frac{6}{5}}} &\lesssim N^{-\frac{1}{2}} \|J_x^{\frac{1}{2}} P_{\mathrm{hi}}(|If|^2 f)\|_{L^{\frac{6}{5}}} \\ &\lesssim N^{-\frac{1}{2}} \left(\|J_x^{\frac{1}{2}} If\|_{L^2} \|If\|_{L^6} \|f\|_{L^6} + \|J_x^{\frac{1}{2}} f\|_{L^2} \|If\|_{L^6}^2 \right) \\ &\lesssim N^{-\frac{1}{2}} \|If\|_{H^1}^3,\end{aligned} \tag{3.119}$$

where in the last step we used the Sobolev embedding as in (3.118) and $\|If\|_{H^{\frac{1}{2}}} \lesssim \|f\|_{H^{\frac{1}{2}}} \lesssim \|If\|_{H^1}$. Notice that if we do not drop the frequency restriction when passing to (3.119), at least one factor (in both terms) has to be supported on frequencies $\gtrsim N$, hence by arguing as for (3.116), we could get another factor of $N^{-\frac{1}{2}}$. Therefore, we obtain that the second term of (3.115) is bounded by $N^{-\frac{3}{2}} \|If\|_{H^1}^4$.

□

3.6 Proof of Proposition 3.9 via the I-method

In order to prove that blowup of the $H^{\frac{1}{2}}$-norm of a solution v of (2.22) does not occur in finite time, we adapt the I-method of [9, 10] (therein also referred to as "the almost conserved energy method") to also incorporate the almost conservation of $\mathcal{P}(Iv)$.

For initial data $v_0 \in \mathcal{H}^s(\mathbb{T}) := \{f \in H^s(\mathbb{T}) : M[f] < 4\pi\}$, $s < 1$, its energy $E(v_0)$ might not even be defined. However, the functionals $\mathcal{E}(Iv(t))$ and $\mathcal{P}(Iv(t))$ are well-defined and via Lemma 2.7,

$$\|Iv(t)\|_{H^1}^2 \lesssim |\mathcal{E}(Iv(t))| + \mathcal{P}(Iv(t))^2 + 1, \tag{3.120}$$

where the smoothing operator I is defined by (3.29) in Section 3.1.2 and v is a (local) solution of (2.22) with $v(0) = v_0$. This control allows us to iterate the local well-posedness theory for any initial data in $\mathcal{H}^s(\mathbb{T})$ and prove that the solution v exists for arbitrarily large times.

Since (3.33) allows for $\|Iv_0\|_{\dot{H}^1} \sim N^{1-s}$, which would give a time of existence $\delta \downarrow 0$ as $N \uparrow \infty$, we use the scaling transformation (1.5) and we note that

$$\|Iv_0^\lambda\|_{\dot{H}^1(\mathbb{T}_\lambda)} \lesssim N^{1-s} \lambda^{-s} \|v_0\|_{\dot{H}^s(\mathbb{T})}. \tag{3.121}$$

We choose the scaling parameter

$$\lambda = N^{\frac{1-s}{s}} \tag{3.122}$$

to ensure that $\delta \gtrsim 1$ uniformly in N and λ. We then have $1 \ll \lambda \le N$ in the regularity range $\frac{1}{2} \le s < 1$, (in particular, $\lambda = N$ for $s = \frac{1}{2}$). We also record that $\|v_0^\lambda\|_{H^s(\mathbb{T}_\lambda)}$, $P(Iv_0^\lambda)$, $E(Iv_0^\lambda)$ are bounded by constants depending only on $\|v_0\|_{H^s(\mathbb{T})}$.

A slightly modified iteration argument concludes the proof of Proposition 3.9. Indeed, consider $B > 0$ such that

$$B^2 \sim \|v_0\|_{H^s(\mathbb{T})}^2 + |\mathcal{E}(Iv_0)| + \mathcal{P}(Iv_0)^2 + 1$$

and suppose that at step j, we have

$$\|Iv^\lambda(j\delta)\|_{H^1(\mathbb{T}_\lambda)} \le B.$$

Then, by Proposition 3.12,

$$\|Iv^\lambda\|_{Z^1([j\delta, j\delta+\delta]\times\mathbb{T}_\lambda)} \le D$$

and according to Proposition 3.46,

$$|\mathcal{E}^3(v^\lambda(j\delta+\delta))| \le |\mathcal{E}^3(v^\lambda(j\delta))| + \delta^{0+} N^{-\gamma} \lambda^{-\kappa} C_1(D)$$

with $\gamma = \frac{3}{2}-$, $\kappa = 1-$. Assuming that we run this iteration J times so that we cover the scaled time interval $[0, \lambda^2 T]$, i.e. assuming that we choose J such that

$$J \gtrsim \lambda^2 T, \tag{3.123}$$

we have

$$|\mathcal{E}^3(v^\lambda(J\delta))| \le |\mathcal{E}^3(v^\lambda(0))| + J\delta^{0+} N^{-\gamma} \lambda^{-\kappa} C_1(D).$$

Notice that $|\mathcal{E}^3(v^\lambda(t))|$ stays bounded (e.g. $|\mathcal{E}^3(v^\lambda(t))| \le 2|\mathcal{E}^3(v^\lambda(0))|$) over the entire $[0, \lambda^2 T]$ if we further impose that N is chosen such that

$$J \lesssim N^\gamma \lambda^\kappa. \tag{3.124}$$

At each iteration step, due to Proposition 3.49 and Proposition 3.50, we have in particular that

$$|\mathcal{E}(Iv^\lambda((j+1)\delta))| \le 2|\mathcal{E}^3(v^\lambda(0))| + N^{-1+} C_2(D), \tag{3.125}$$

$$|\mathcal{P}(Iv^\lambda((j+1)\delta))| \le |\mathcal{P}(v^\lambda(0))| + N^{-1} C_3(D), \tag{3.126}$$

where we used a version of (3.11) restricted to the time interval $[j\delta, (j+1)\delta]$. We get

$$\|Iv^\lambda((j+1)\delta)\|_{H^1(\mathbb{T}_\lambda)} \lesssim D.$$

We choose N large enough so that in (3.125) and (3.126) the second terms are dominated by the first terms. By Lemma 2.7, we then deduce

$$\|Iv^\lambda((j+1)\delta)\|_{H^1(\mathbb{T}_\lambda)} \le B$$

and thus we get to perform the iteration again.

Note that (3.123), (3.124) and $s \ge \frac{1}{2}$ yield

$$T \lesssim N^{\gamma-(\kappa-2)+\frac{1}{s}(\kappa-2)} \lesssim N^{\gamma+\kappa-2}.$$

In our case, $\gamma+\kappa-2 = \frac{1}{2}-$; hence, given any large T, we can choose a frequency threshold $N = N(T) \gg 1$ for the I-operator.

Notice that for all $t \in [0, \lambda^2 T] \subset [0, J\delta]$, we have $\mathcal{E}(Iv^\lambda(t)) \lesssim \mathcal{E}(Iv_0^\lambda) \lesssim 1$ and $\mathcal{P}(Iv^\lambda(t)) \lesssim \mathcal{P}(Iv_0^\lambda) \lesssim 1$, thus $\|Iv^\lambda(t)\|_{H^1(\mathbb{T}_\lambda)} \lesssim 1$. Also, we recall that we still need to undo the scaling:

$$\|v(t)\|_{H^s(\mathbb{T})} \lesssim \lambda^s \|v^\lambda(\lambda^2 t)\|_{H^s(\mathbb{T}_\lambda)} \lesssim \lambda^s \|Iv^\lambda(t)\|_{H^1(\mathbb{T}_\lambda)} \lesssim N^{1-s},$$

for all $t \in [0, T]$, where we used (3.32) and (3.122). The above numerology allows us to take $N \sim T^{2+}$ and thus

$$\sup_{t\in[0,T]} \|v(t)\|_{H^s(\mathbb{T})} \lesssim T^{2-2s+}$$

for any $\frac{1}{2} \le s < 1$.

3.7 Comments and remarks

In view of the local well-posedness result in the scale of Fourier-Lebesgue spaces by Grünrock and Herr [16], it would be interesting to investigate via the I-method the global dynamics of the DNLS flow in $\mathcal{F}L^{\frac{1}{2},r}(\mathbb{T})$, for the appropriate range in r, to complement the almost sure global well-posedness result of Nahmod, Oh, Rey-Bellet, and Staffilani [39]. This is also to be studied in the Euclidean case, where the local well-posedness was established by Grünrock in [15]. In the same direction of thought, we mention that Takaoka [46] proved the existence of local $H^s(\mathbb{T})$-solutions with small (unquantified) mass in the range $\frac{12}{25} < s < \frac{1}{2}$ by establishing a priori estimates for the gauge equivalent equation (4.2).

Above the mass threshold 4π, the question of whether all solutions to DNLS extend globally in time is not settled for low-regularity initial data. By relying on the inverse scattering method, Jenkins, Liu, Perry, and Sulem [23] proved that all solutions started with initial data in the weighted Sobolev space $H^{2,2}(\mathbb{R})$ exist for all times. For $H^1(\mathbb{R})$-initial data, by using variational analysis of soliton solutions, Fukaya, Hayashi, and Inui [14] gave a sufficient condition for the global well-posedness of DNLS covering the result of Wu [55]. Also, the work of Takaoka [45] on the energy exchange behavior for a variant of (4.2) might provide further insight on the DNLS dynamics.

Chapter 4

Unconditional uniqueness of solutions in the Euclidean setting

In this chapter, we use a method to prove well-posedness of nonlinear dispersive equations which avoids a heavy machinery from harmonic analysis. We implement an infinite iteration of normal form reductions (namely, integration by parts in time) and reformulate the equation in terms of an infinite series of multilinear terms. This allows us to prove Theorem 1.3, i.e. the unconditional well-posedness of solutions to (1.1) in an almost end-point space.

4.1 The normal form equation

In this section, we *formally* derive a normal form equation for a gauged DNLS equation on $\mathbb{R}$. First, we use a gauge transformation to remove the nonlinear term $2i|u|^2\partial_x u$ from (1.1) at the expense of introducing a (pure power) quintic nonlinear term – see (4.2) below. Then, we apply an infinite iteration of normal form reductions to transform the gauged DNLS into a new equation involving infinite series of nonlinearities of arbitrarily high degrees. To this end, we employ the normal form method developed in [28].

We use the following gauge transformation

$$u(t,x) \mapsto w(t,x) := \exp\left(-i\int_{-\infty}^{x} |u(t,y)|^2 dy\right) u(t,x). \tag{4.1}$$

Notice that this an autonomous transformation, i.e. it does not depend explicitly on the time variable. Thus, DNLS is transformed into the *gauged DNLS*:

$$i\partial_t w + \partial_x^2 w = -iw^2\partial_x \overline{w} - \frac{1}{2}|w|^4 w. \tag{4.2}$$

This nonlinear transformation (4.1) goes back to the works of Hayashi [19] and Hayashi and Ozawa [20]. See also [31]. It is well known by now (see [44]) that the cubic nonlinearity with the derivative falling on the complex-conjugate factor can be handled using the Fourier restriction norm method, whereas the cubic term $|u|^2\partial_x u$ fails to have a useful estimate. It turns out that this is also the case when employing the normal form approach, namely we have to remove the bad nonlinearity before renormalizing the equation. Nevertheless, we can transfer a well-posedness result on the gauged DNLS equation back to the original DNLS equation with the following:

Lemma 4.1 ([10])**.** *Let $s \geq 0$. The mapping $u \mapsto w$ defined by* (4.1) *is bi-Lipschitz on $H^s(\mathbb{R})$.*

Next, we denote $S(t) := e^{it\partial_x^2}$ and we use the change of variable $v(t) = S(-t)w(t)$ (the interaction representation variable). Then, equation (4.2) becomes

$$\partial_t v = \mathcal{Q}(v) + \mathcal{T}(v), \tag{4.3}$$

where we denoted the quintic and the cubic nonlinear terms respectively by:

$$\mathcal{Q}(v) := -\frac{1}{2}\big|S(t)v(t)\big|^4 S(t)v(t), \tag{4.4}$$

$$\mathcal{T}(v) := -i\big(S(t)v(t)\big)^2\partial_x\overline{S(t)v(t)}. \tag{4.5}$$

In what follows we will exploit the oscillatory nature of the Fourier transform of $\mathcal{T}$.

With a slight abuse of notation[1], let us introduce the trilinear operator $\mathcal{T}$ defined by

$$\mathcal{F}\Big[\mathcal{T}(v_1, v_2, v_3)\Big](t,\xi) = \int_{\xi=\xi_1-\xi_2+\xi_3} e^{i\Phi(\bar{\xi})t}\xi_2\widehat{v_1}(\xi_1)\overline{\widehat{v_2}(\xi_2)}\widehat{v_3}(\xi_3)d\xi_1 d\xi_2\,, \tag{4.6}$$

where the phase is given by

$$\Phi(\bar{\xi}) := \xi^2 - \xi_1^2 + \xi_2^2 - \xi_3^2. \tag{4.7}$$

Notice that on the convolution hyperplane $\xi = \xi_1 - \xi_2 + \xi_3$, we have

$$\Phi(\bar{\xi}) = 2(\xi - \xi_1)(\xi - \xi_3) = 2(\xi_2 - \xi_1)(\xi_2 - \xi_3). \tag{4.8}$$

Since it is determined by the linear part of the equation, the function $\Phi(\bar{\xi})$ is the same as the *modulation function* for the cubic NLS equation in [28], but the trilinear operator is different due to the presence of the derivative in the cubic nonlinearity.

Since for $s > \frac{1}{2}$, $H^s(\mathbb{R})$ is a Banach algebra, the quintic term can be estimated easily:

$$\|\mathcal{Q}(v)\|_{H^s(\mathbb{R})} \lesssim \|v\|^5_{H^s(\mathbb{R})}. \tag{4.9}$$

Due to the derivative loss in the cubic term, $\mathcal{T}$ does not have a similar estimate in $H^s(\mathbb{R})$, even though $s > \frac{1}{2}$. Therefore we proceed to renormalize this nonlinearity by

[1] Note that when all the entries of the trilinear operator are the same, we write $\mathcal{T}(v)$ instead of $\mathcal{T}(v, v, v)$.

means of normal form reductions (NFR).

4.1.1 The first step of NFR

The idea is to exploit the oscillatory factor of the convolution integral in (4.6), and so we apply integration by parts on a domain of integration where $|\Phi(\bar{\xi})| > N$, for some threshold N to be chosen later. We first decompose

$$\mathcal{T}(v) = \mathcal{T}_1(v) + \mathcal{T}_2(v), \tag{4.10}$$

where $\mathcal{T}_2^{(1)}(v)$ is defined as $\mathcal{T}(v)$ (see (4.6) above), but the integration is further restricted to the domain

$$C_0 := C_0(\xi) := \Big\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \big|\Phi(\xi, \xi_1, \xi_2, \xi - \xi_1 + \xi_2)\big| > N\Big\} \tag{4.11}$$

and $\mathcal{T}_1(v) := \mathcal{T}(v) - \mathcal{T}_2(v)$. Thanks to the modulation restriction, the term $\mathcal{T}_1(v)$ enjoys a sufficiently good $H^s(\mathbb{R})$-estimate – see Lemma 4.2 below. For the remainder term $\mathcal{T}_2(v)$, we apply differentiation by parts[2] in order to renormalize it. To ease the writing, we drop the complex conjugate, the Fourier transform notation, and the complex constants of modulus one in front of the nonlinearities. We have:

$$\begin{aligned}\mathcal{T}_2(v)(t,\xi) &= \partial_t\Bigg[\int_{\substack{\xi=\xi_1-\xi_2+\xi_3\\ |\Phi(\bar{\xi})|>N}} \frac{e^{i\Phi(\bar{\xi})t}}{\Phi(\bar{\xi})}\xi_2 v(t,\xi_1)v(t,\xi_2)v(t,\xi_3)d\xi_1 d\xi_2\Bigg]\\ &\qquad - \int_{\substack{\xi=\xi_1-\xi_2+\xi_3\\ |\Phi(\bar{\xi})|>N}} \frac{e^{i\Phi(\bar{\xi})t}}{\Phi(\bar{\xi})}\xi_2 \partial_t\big(v(t,\xi_1)v(t,\xi_2)v(t,\xi_3)\big)d\xi_1 d\xi_2\\ &=: \partial_t\Big[\mathcal{T}_0^{(2)}(v)(t,\xi)\Big] + \mathcal{T}^{(2)}(v)(t,\xi).\end{aligned}$$

Let us start employing the ordered tree notation from Appendix A. At this stage, we can express everything in terms of T_1, the sole ternary tree of first generation. With $\mu_1 := \Phi(\bar{\xi})$, the nonlinearities $\mathcal{T}_0^{(2)}(v)$, $\mathcal{T}^{(2)}(v)$ can be written as follows:

$$\mathcal{T}_0^{(2)}(v)(t,\xi) = \int_{\boldsymbol{\xi}\in\Xi_\xi(T_1)} \mathbf{1}_{C_0}\frac{e^{i\mu_1 t}}{\mu_1}\xi_2^{(1)} \prod_{a\in T_1^\infty} v(t,\xi_a)\,, \tag{4.12}$$

$$\mathcal{T}^{(2)}(v)(t,\xi) = \int_{\boldsymbol{\xi}\in\Xi_\xi(T_1)} \mathbf{1}_{C_0}\frac{e^{i\mu_1 t}}{\mu_1}\xi_2^{(1)}\partial_t\Bigg(\prod_{a\in T_1^\infty} v(t,\xi_a)\Bigg)\,. \tag{4.13}$$

[2]Here, "differentiation by parts" means usual integration by parts (with respect to the time variable) in the Duhamel formulation of (4.3), without writing explicitly the time integration. In other words,

$$\mathcal{T}_2(v)(t,\xi) = \partial_t\Big[\mathcal{T}_0^{(2)}(v)(t',\xi)\Big] + \mathcal{T}^{(2)}(v)(t,\xi)$$

stands for

$$\int_0^t \mathcal{T}_2(v)(t',\xi)dt' = \Big[\mathcal{T}_0^{(2)}(v)(t',\xi)\Big]_{t'=0}^{t'=t} + \int_0^t \mathcal{T}^{(2)}(v)(t',\xi)dt'.$$

By using the product rule and supposing v is a smooth solution of (4.3), we get

$$\mathcal{T}^{(2)}(v) = \mathcal{T}^{(2)}_{\mathcal{Q}}(v) + \mathcal{T}^{(2)}_{\mathcal{T}}(v)\,. \tag{4.14}$$

On the right side above, $\mathcal{T}^{(2)}_{\mathcal{Q}}(v)$ is the sum of three septic terms, corresponding to replacing $\partial_t v(t,\xi_b)$ by $\mathcal{Q}(v)(t,\xi_b)$, $b \in T_1^\infty$. Similarly, $\mathcal{T}^{(2)}_{\mathcal{T}}(v)$ is the sum of three quintic terms, corresponding to replacing $\partial_t v(t,\xi_b)$ by $\mathcal{T}(v)(t,\xi_b)$, $b \in T_1^\infty$. More precisely, we have

$$\mathcal{T}^{(2)}_{\mathcal{Q}}(v)(\xi) := \sum_{b\in T_1^\infty} \int_{\boldsymbol{\xi}\in\Xi_\xi(T_1)} \mathbf{1}_{C_0^c} \frac{e^{it\mu_1}}{\mu_1} \xi_2^{(1)} \mathcal{Q}(v)(\xi_b) \prod_{a\in\mathcal{T}_1^\infty\setminus\{b\}} v(\xi_a) \tag{4.15}$$

$$\mathcal{T}^{(2)}_{\mathcal{T}}(v)(\xi) := \sum_{b\in T_1^\infty} \int_{\boldsymbol{\xi}\in\Xi_\xi(T_1)} \mathbf{1}_{C_0^c} \frac{e^{it\mu_1}}{\mu_1} \xi_2^{(1)} \mathcal{T}(v)(\xi_b) \prod_{a\in\mathcal{T}_1^\infty\setminus\{b\}} v(\xi_a) \tag{4.16}$$

Thus, if v is a smooth solution of (4.3), then it is also a solution of

$$\partial_t v = \mathcal{Q}(v) + \partial_t \mathcal{T}^{(2)}_0(v) + \mathcal{T}^{(1)}_{\mathcal{T},1}(v) + \mathcal{T}^{(2)}_{\mathcal{Q}}(v) + \mathcal{T}^{(2)}_{\mathcal{T}}(v), \tag{4.17}$$

where we set $\mathcal{T}^{(1)}_{\mathcal{T},1}(v) := \mathcal{T}_1(v)$ for the sake of consistency with subsequent NFR steps. It turns out that we can establish sufficiently good estimates for all of the nonlinear terms of (4.17), except for those in $\mathcal{T}^{(2)}_{\mathcal{T}}(v)$. Therefore, we proceed to renormalize them.

4.1.2 The second step of NFR

For the sake of clarity, let us write $\mathcal{T}^{(2)}_{\mathcal{T}}(v)$ defined in (4.16) first without appealing to the terminology of Appendix A, and then in the compact writing facilitated by the ordered trees notation:

$$\begin{aligned}
\mathcal{T}^{(2)}_{\mathcal{T}}(v)(\xi) &= \int_{\substack{\xi=\xi_1-\xi_2+\xi_3\\ \xi_1=\xi_{11}-\xi_{12}+\xi_{13}}} \mathbf{1}_{C_0^c} \frac{e^{i\Phi(\bar\xi)t}\xi_2}{\Phi(\bar\xi)} \left(e^{i\Phi(\overline{\xi_1})t}\xi_{12}\right) v(\xi_{11})v(\xi_{12})v(\xi_{13})v(\xi_2)v(\xi_3) \\
&\quad + \int_{\substack{\xi=\xi_1-\xi_2+\xi_3\\ \xi_2=\xi_{21}-\xi_{22}+\xi_{23}}} \mathbf{1}_{C_0^c} \frac{e^{i\Phi(\bar\xi)t}\xi_2}{\Phi(\bar\xi)} \left(e^{i\Phi(\overline{\xi_2})t}\xi_{22}\right) v(\xi_1)v(\xi_{21})v(\xi_{22})v(\xi_{23})v(\xi_3) \\
&\quad + \int_{\substack{\xi=\xi_1-\xi_2+\xi_3\\ \xi_3=\xi_{31}-\xi_{32}+\xi_{33}}} \mathbf{1}_{C_0^c} \frac{e^{i\Phi(\bar\xi)t}\xi_2}{\Phi(\bar\xi)} \left(e^{i\Phi(\overline{\xi_3})t}\xi_{32}\right) v(\xi_1)v(\xi_2)v(\xi_{31})v(\xi_{32})v(\xi_{33}) \\
&= \sum_{T\in\mathfrak{T}(2)} \int_{\boldsymbol{\xi}\in\Xi_\xi(T)} \mathbf{1}_{C_0^c} \frac{e^{i\mu_1 t}\xi_2^{(1)}}{\mu_1} \left(e^{i\mu_2 t}\xi_2^{(2)}\right) \prod_{a\in T^\infty} v(\xi_a),
\end{aligned} \tag{4.18}$$

where $\Phi(\overline{\xi_j}) = \Phi(\xi_j,\xi_{j1},\xi_{j2},\xi_{j3})$ for $1 \le j \le 3$. Notice that, in (4.18), the phase is $\mu_1+\mu_2$, where μ_1 is the same as in the first step of NFR, i.e. $\mu_1 = \Phi(\bar\xi)$, and

$$\mu_2 := \Phi(\overline{\xi^{(2)}}) = 2(\xi_2^{(2)} - \xi_1^{(2)})(\xi_2^{(2)} - \xi_3^{(2)})\,,$$

for $\boldsymbol{\xi} \in \Xi_\xi(T)$. We now decompose

$$\mathcal{T}_{\mathcal{T}}^{(2)}(v) = \mathcal{T}_{\mathcal{T},1}^{(2)}(v) + \mathcal{T}_{\mathcal{T},2}^{(2)}(v)\,,$$

i.e. each term of the sum in (4.18) is split into two parts corresponding to further restricting the domain of integration to

$$C_1 = C_1(\xi; T) := \big\{\boldsymbol{\xi} \in \Xi_\xi(T) : |\mu_1 + \mu_2| \leq \beta_1|\mu_1|\big\}$$

and its complement, respectively, where $\beta_1 \geq 2$ is to be chosen later. By Lemma 4.10 below, we have $H^s(\mathbb{R})$-estimates for the terms in $\mathcal{T}_{\mathcal{T},1}^{(2)}(v)$. For the remainder $\mathcal{T}_{\mathcal{T},2}^{(2)}(v)$, we apply differentiation by parts for all of its three terms. Thus by working with the ordered trees notation, we have[3]

$$\begin{aligned}
\mathcal{T}_{\mathcal{T},2}^{(2)}(v)(t,\xi) &= \partial_t\Bigg[\sum_{T\in\mathfrak{T}(2)}\int_{\boldsymbol{\xi}\in\Xi_\xi(T)} \mathbf{1}_{C_0\cap C_1^c}\frac{e^{i(\mu_1+\mu_2)t}}{\mu_1(\mu_1+\mu_2)}\xi_2^{(1)}\xi_2^{(2)}\prod_{a\in T^\infty} v(\xi_a)\Bigg] \\
&\qquad - \sum_{T\in\mathfrak{T}(2)}\int_{\boldsymbol{\xi}\in\Xi_\xi(T)} \mathbf{1}_{C_0\cap C_1^c}\frac{e^{i(\mu_1+\mu_2)t}}{\mu_1(\mu_1+\mu_2)}\xi_2^{(1)}\xi_2^{(2)}\partial_t\Big(\prod_{a\in T^\infty} v(\xi_a)\Big) \\
&=: \partial_t\Big[\mathcal{T}_0^{(3)}(v)(t,\xi)\Big] + \mathcal{T}^{(3)}(v)(t,\xi).
\end{aligned}$$

By using the product rule and the assumption that v is a smooth solution of (4.3), we get

$$\mathcal{T}^{(3)}(v) = \mathcal{T}_{\mathcal{Q}}^{(3)}(v) + \mathcal{T}_{\mathcal{T}}^{(3)}(v)\,,$$

and the equation for v becomes

$$\partial_t v = \mathcal{Q}(v) + \sum_{j=2}^{3}\mathcal{T}_0^{(j)}(v) + \sum_{j=1}^{2}\mathcal{T}_{\mathcal{T},1}^{(j)}(v) + \sum_{j=2}^{3}\mathcal{T}_{\mathcal{Q}}^{(j)}(v) + \mathcal{T}_{\mathcal{T}}^{(3)}(v).$$

The last term $\mathcal{T}_{\mathcal{T}}^{(3)}(v)$ is passed to the next step in the iterative procedure. As we believe the iterative procedure became clear, let us present the general step of normal form reductions.

4.1.3 The Jth step of NFR

We now write down the terms that appear in the Jth step of normal form reductions. We decompose $\mathcal{T}_{\mathcal{T}}^{(J)}(v) = \mathcal{T}_{\mathcal{T},1}^{(J)}(v) + \mathcal{T}_{\mathcal{T},2}^{(J)}(v)$, corresponding to further restricting the

[3]Given an ordered tree T_2 with T_1 denoting its first generation tree, for $A_1 \subseteq \Xi(T_1)$, $A_2 \subset \Xi(T_2)$, we define by a slight abuse of notation, $A_1 \cap A_2 := \{\boldsymbol{\xi} \in A_2 : \boldsymbol{\xi}|_{T_1} \in A_1\}$. Inductively, this definition is generalized to higher generation ordered trees as follows: if T_{J+1} is an ordered tree with chronicle $\{T_j\}_{j=1}^{J+1}$ and $A_j \subseteq \Xi(T_j)$, $j = 1, 2, \ldots, J+1$, then $A_1 \cap A_2 \cap \ldots \cap A_{J+1} := \{\xi \in \Xi(T_{J+1}) : \xi|_{T_J} \in A_1 \cap A_2 \cap \ldots \cap A_J\}$.

domain of integration of $\mathcal{T}^{(J)}_{\mathcal{T}}(v)$ to

$$C_{J-1} = C_{J-1}(\xi; T) := \Big\{ \boldsymbol{\xi} \in \Xi_\xi(T) : \big|\widetilde{\mu}_{J-1} + \mu_J\big| \leq \beta_{J-1}|\widetilde{\mu}_{J-1}|\Big\}$$

and its complement, respectively, where $\beta_{J-1} \geq 2$ is to be chosen later (See 4.9). After differentiation by parts and by using the equation (4.3), we are led to

$$\mathcal{T}^{(J)}_{\mathcal{T},2}(v)(t,\xi) = \partial_t\Big[\mathcal{T}^{(J+1)}_0(v)(t,\xi)\Big] + \mathcal{T}^{(J+1)}_{\mathcal{Q}}(v)(t,\xi) + \mathcal{T}^{(J+1)}_{\mathcal{T}}(v)(t,\xi)\,, \tag{4.19}$$

where the terms on the right-hand side are given by the following formulae:

$$\mathcal{T}^{(J+1)}_0(v)(\xi) = \sum_{T\in\mathfrak{T}(J)} \int_{\boldsymbol{\xi}\in\Xi_\xi(T)} \mathbf{1}_{F_J}\Big(\prod_{j=1}^{J} \frac{e^{i\mu_j t}\xi_2^{(j)}}{\widetilde{\mu}_j}\Big)\Big(\prod_{a\in T^\infty} v(\xi_a)\Big) \tag{4.20}$$

$$\mathcal{T}^{(J+1)}_{\mathcal{Q}}(v)(\xi) = \sum_{T\in\mathfrak{T}(J)} \sum_{b\in T^\infty} \int_{\boldsymbol{\xi}\in\Xi_\xi(T)} \mathbf{1}_{F_J}\Big(\prod_{j=1}^{J} \frac{e^{i\mu_j t}\xi_2^{(j)}}{\widetilde{\mu}_j}\Big)\Big(\mathcal{Q}(v)(\xi_b)\prod_{\substack{a\in T^\infty\\ a\neq b}} v(\xi_a)\Big) \tag{4.21}$$

$$\mathcal{T}^{(J+1)}_{\mathcal{T}}(v)(\xi) = \sum_{T\in\mathfrak{T}(J+1)} \int_{\boldsymbol{\xi}\in\Xi_\xi(T)} \mathbf{1}_{F_J}\Big(\prod_{j=1}^{J} \frac{e^{i\mu_j t}\xi_2^{(j)}}{\widetilde{\mu}_j}\Big)\big(e^{i\mu_{J+1}t}\xi_2^{(J+1)}\big)\Big(\prod_{a\in T^\infty} v(\xi_a)\Big) \tag{4.22}$$

where we have sets $F_1 := C_0$ and $F_J := C_0 \cap C_1^c \cap \ldots \cap C_{J-1}^c$ for $J \geq 2$.

The equation (4.3) becomes

$$\partial_t v = \mathcal{Q}(v) + \sum_{j=2}^{J+1} \partial_t \mathcal{T}^{(j)}_0(v) + \sum_{j=2}^{J+1} \mathcal{T}^{(j)}_{\mathcal{Q}}(v) + \sum_{j=1}^{J} \mathcal{T}^{(j)}_{\mathcal{T},1}(v) + \mathcal{T}^{(J+1)}_{\mathcal{T}}(v). \tag{4.23}$$

We record the formula for the term $\mathcal{T}^{(J+1)}_{\mathcal{T},1}(v)$ appeared in the next step of NFR:

$$\mathcal{T}^{(J+1)}_{\mathcal{T},1}(v)(\xi) = \sum_{T\in\mathfrak{T}(J+1)} \int_{\boldsymbol{\xi}\in\Xi_\xi(T)} \mathbf{1}_{F_J\cap C_J}\Big(\prod_{j=1}^{J} \frac{e^{i\mu_j t}\xi_2^{(j)}}{\widetilde{\mu}_j}\Big)\big(e^{i\mu_{J+1}t}\xi_2^{(J+1)}\big)\Big(\prod_{a\in T^\infty} v(\xi_a)\Big), \tag{4.24}$$

where F_J is defined above, and

$$C_J = C_J(\xi; T) := \Big\{ \boldsymbol{\xi} \in \Xi_\xi(T) : \big|\widetilde{\mu}_J + \mu_{J+1}\big| \leq \beta_J|\widetilde{\mu}_J|\Big\} \tag{4.25}$$

with $\beta_J \geq 2$ to be determined later.

4.1.4 The limit equation

By iterating the normal form reduction step indefinitely, we formally derive the following limit equation:

$$\partial_t v = \mathcal{Q}(v) + \partial_t\bigg(\sum_{j=2}^{\infty}\mathcal{T}_0^{(j)}(v)\bigg) + \sum_{j=2}^{\infty}\mathcal{T}_{\mathcal{Q}}^{(j)}(v) + \sum_{j=1}^{\infty}\mathcal{T}_{\mathcal{T},1}^{(j)}(v), \tag{4.26}$$

where $\mathcal{T}_{\mathcal{Q}}^{(j)}$ and $\mathcal{T}_{\mathcal{T},1}^{(j)}$ are $(2j+1)$-multilinear term, and $\mathcal{T}_0^{(j)}$ is $(2j-1)$-multilinear term. These multilinear terms $\mathcal{T}_{\mathcal{Q}}^{(j)}$, $\mathcal{T}_{\mathcal{T},1}^{(j)}$, and $\mathcal{T}_0^{(j)}$ appear as a result of $(j-1)$-many iterations of normal form reductions.

4.2 The strong estimates

We consider the trilinear operator $\mathcal{T}_\Phi$ defined by

$$\mathcal{F}\Big[\mathcal{T}_\Phi(v_1,v_2,v_3)\Big](t,\xi) = \int_{\xi=\xi_1-\xi_2+\xi_3} \frac{|\xi_2|}{\langle\Phi(\bar{\xi})\rangle^{\frac{1}{2}}}\widehat{v_1}(\xi_1)\overline{\widehat{v_2}(\xi_2)}\widehat{v_3}(\xi_3)d\xi_1 d\xi_2\,, \tag{4.27}$$

where $\Phi(\bar{\xi})$ is given by (4.7). We can prove the $H^s(\mathbb{R})$-estimates for all higher order terms that appear in (4.26) once we establish the following lemma:

Lemma 4.2 (Basic trilinear estimate in the $H^s(\mathbb{R})$-norm)**.** *Let $s > \frac{1}{2}$. Then there exists a finite constant $C = C(s) > 0$ such that*

$$\|\mathcal{T}_\Phi(v_1,v_2,v_3)\|_{H^s_x(\mathbb{R})} \le C\prod_{j=1}^{3}\|v_j\|_{H^s_x(\mathbb{R})}\,.$$

Proof. By duality, the desired estimate follows once we prove that

$$\int_{\xi=\xi_1-\xi_2+\xi_3} m(\bar{\xi})\widehat{v_1}(\xi_1)\widehat{v_2}(\xi_2)\widehat{v_3}(\xi_3)\widehat{v_4}(\xi)d\xi_1 d\xi_2 d\xi \le C\prod_{j=1}^{4}\|v_j\|_{L^2_x(\mathbb{R})}, \tag{4.28}$$

for any $v_1,\dots,v_4 \in L^2(\mathbb{R})$ with $\widehat{v_j} \ge 0$ $(1 \le j \le 4)$, where the multiplier is given by

$$m(\bar{\xi}) := \frac{|\xi_2|}{\langle\Phi(\bar{\xi})\rangle^{\frac{1}{2}}}\cdot\frac{\langle\xi\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s}. \tag{4.29}$$

Case 1: $\min(|\xi_2-\xi_1|,|\xi_2-\xi_3|) \le 1$.

Without loss of generality, let us assume that $|\xi_2-\xi_1| \le 1$. Since $\langle\xi_1\rangle \sim \langle\xi_2\rangle$ and $\langle\xi_3\rangle \sim \langle\xi\rangle$, we have $m(\bar{\xi}) \lesssim 1$. Denote $\zeta := \xi_2-\xi_1 = \xi_3-\xi$ and thus by using Hölder's inequality, we get that

$$
\begin{aligned}
\text{LHS of (4.28)} &\le \int_{|\zeta|\le 1}\int_{\xi_1}\widehat{v_1}(\xi_1)\widehat{v_2}(\xi_1+\zeta)d\xi_1\int_{\xi_3}\widehat{v_3}(\xi_3)\widehat{v_4}(\xi_3-\zeta)d\xi_3\,d\zeta\\
&\le \Big\|\int_{\xi_1}\widehat{v_1}(\xi_1)\widehat{v_2}(\xi_1+\zeta)d\xi_1\Big\|_{L^\infty_\zeta}\Big\|\int_{\xi_3}\widehat{v_3}(\xi_3)\widehat{v_4}(\xi_3-\zeta)d\xi_3\Big\|_{L^\infty_\zeta}\\
&\lesssim \prod_{j=1}^{4}\|v_j\|_{L^2}.
\end{aligned}
$$

For all of the remaining cases we assume that $|\xi_2-\xi_1|>1$ and $|\xi_2-\xi_3|>1$. Also, we note that the largest two frequencies necessarily have comparable sizes and that the multiplier m is symmetric in ξ_1,ξ_3.

We are using the following known fact:

$$
\int_{\mathbb{R}}\frac{1}{\langle\eta-\xi\rangle^a\langle\xi\rangle^b}\,d\xi\lesssim 1\,, \tag{4.30}
$$

for any $a,b\ge 0$ such that $a+b>1$, with implicit constant independent of $\eta\in\mathbb{R}$. Indeed, this follows immediately from Young's convolution inequality:

$$
\big\|\big(\langle\cdot\rangle^{-a}*\langle\cdot\rangle^{-b}\big)(\eta)\big\|_{L^\infty_\eta(\mathbb{R})}\le\|\langle\xi\rangle^{-a}\|_{L^p_\xi(\mathbb{R})}\|\langle\xi\rangle^{-b}\|_{L^q_\xi(\mathbb{R})}\,,
$$

with $p=\frac{a+b}{a}$ and $q=\frac{a+b}{b}$ (if a or b is zero, then (4.30) is trivially true).

By Cauchy-Schwarz inequality, for (4.28), it is enough to show that

$$
\mathcal{M}_j:=\sup_{\xi_j\in\mathbb{R}}\int_{\xi=\xi_1-\xi_2+\xi_3}m(\bar{\xi})^2d\xi_k d\xi_l\le C^2 \tag{4.31}
$$

for some mutually distinct $1\le j,k,l\le 4$ (with the convention that $\xi_4=\xi$). Indeed, by the Cauchy-Schwarz inequality with respect to $d\xi_k d\xi_l$ (with the index r such that $\{j,k,l,r\}=\{1,2,3,4\}$),

$$
\begin{aligned}
\text{LHS of (4.28)} &\le \int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}m(\bar{\xi})^2d\xi_k d\xi_l\right)^{\frac12}\left(\int_{\mathbb{R}^2}\widehat{v_j}(\xi_j)^2\widehat{v_k}(\xi_k)^2\widehat{v_l}(\xi_l)^2\widehat{v_r}(\xi_r)^2d\xi_k d\xi_l\right)^{\frac12}d\xi_j\\
&\le \mathcal{M}_j^{\frac12}\int_{\mathbb{R}}\widehat{v_j}(\xi_j)\left(\int_{\mathbb{R}^2}\widehat{v_k}(\xi_k)^2\widehat{v_l}(\xi_l)^2\widehat{v_r}(\xi_r)^2d\xi_k d\xi_l\right)^{\frac12}d\xi_j\\
&\le \mathcal{M}_j^{\frac12}\|\widehat{v_j}\|_{L^2(\mathbb{R})}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^2}\widehat{v_k}(\xi_k)^2\widehat{v_l}(\xi_l)^2\widehat{v_r}(\xi_r)^2d\xi_k d\xi_l d\xi_j\right)^{\frac12}
\end{aligned}
$$

where in the last step we used the Cauchy-Schwarz inequality with respect to $d\xi_j$ and then (4.28) follows from (4.31) by possibly changing the order of integration on the right-hand side above (and taking into account the linear dependence $\xi_4=\xi_1-\xi_2+\xi_3$).

Next, we discuss several cases based on the frequency size of the derivative factor $\partial_x\overline{v_2}$.

Case 2: $|\xi_2| \sim |\xi_1| \gg |\xi_3|, |\xi|$ (symmetric to $|\xi_2| \sim |\xi_3| \gg |\xi_1|, |\xi|$). Then

$$\langle \Phi(\bar{\xi}) \rangle \sim |\Phi(\xi)| \sim |\xi_2 - \xi_1| \cdot |\xi_2| \sim \langle \xi_2 - \xi_1 \rangle \langle \xi_2 \rangle$$

and therefore

$$m(\bar{\xi}) \sim \frac{|\xi_2|^{\frac{1}{2}-s} \langle \xi \rangle^s}{\langle \xi_2 - \xi_1 \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^s \langle \xi_3 \rangle^s} \lesssim \frac{1}{\langle \xi_2 - \xi_1 \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^{s-\frac{1}{2}}} \cdot \frac{1}{\langle \xi_3 \rangle^s} .$$

By using (4.30), it follows that

$$\mathcal{M}_2 \lesssim \sup_{\xi_2 \in \mathbb{R}} \int_{\mathbb{R}} \frac{1}{\langle \xi_2 - \xi_1 \rangle \langle \xi_1 \rangle^{2s-1}} d\xi_1 \cdot \int_{\mathbb{R}} \frac{1}{\langle \xi_3 \rangle^{2s}} d\xi_3 \lesssim 1 .$$

Case 3: $|\xi_2| \sim |\xi| \gg |\xi_1|, |\xi_3|$. Then $|\Phi(\bar{\xi})| \sim \xi_2^2$ and therefore

$$m(\bar{\xi}) \sim \frac{1}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s}$$

which is square integrable on $(\mathbb{R}^2, d\xi_1 d\xi_3)$.

Case 4: $|\xi_1| \sim |\xi_3| \gg |\xi_2|, |\xi|$. Then $|\Phi(\bar{\xi})| \sim \xi_1^2$ and therefore

$$m(\bar{\xi}) \sim \frac{|\xi_2| \langle \xi \rangle^s}{\langle \xi_1 \rangle^{1+2s} \langle \xi_2 \rangle^s} \lesssim \frac{1}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s}$$

which is square integrable on $(\mathbb{R}^2, d\xi_1 d\xi_3)$.

Case 5: $|\xi_1| \sim |\xi| \gg |\xi_2|, |\xi_3|$ (symmetric to $|\xi_3| \sim |\xi| \gg |\xi_1|, |\xi_2|$). Then

$$\langle \Phi(\bar{\xi}) \rangle \sim \langle \xi_1 \rangle \langle \xi_2 - \xi_3 \rangle$$

and therefore

$$m(\bar{\xi}) \sim \frac{|\xi_2|}{\langle \xi_1 \rangle^{\frac{1}{2}} \langle \xi_2 - \xi_3 \rangle^{\frac{1}{2}}} \cdot \frac{1}{\langle \xi_2 \rangle^s \langle \xi_3 \rangle^s} \lesssim \frac{1}{\langle \xi_2 - \xi_3 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{s-\frac{1}{2}} \langle \xi_3 \rangle^s} .$$

By using (4.30), it follows that

$$\mathcal{M}_1 \lesssim \sup_{\xi_1 \in \mathbb{R}} \int_{\mathbb{R}} \frac{1}{\langle \xi_3 \rangle^{2s}} \left(\int_{\mathbb{R}} \frac{1}{\langle \xi_2 - \xi_3 \rangle \langle \xi_2 \rangle^{2s-1}} d\xi_2 \right) d\xi_3 \lesssim 1 .$$

Case 6: $|\xi_1| \sim |\xi_2| \sim |\xi_3| \gg |\xi|$. Then $|\Phi(\bar{\xi})| \sim \xi_1^2$ and therefore

$$m(\bar{\xi}) \sim \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s} \lesssim \frac{1}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s}$$

which is square integrable on $(\mathbb{R}^2, d\xi_1 d\xi_3)$.

Case 7: $|\xi_1| \sim |\xi_2| \sim |\xi| \gg |\xi_3|$ (symmetric to $|\xi_3| \sim |\xi_2| \sim |\xi| \gg |\xi_1|$). Then

$|\Phi(\bar{\xi})| \sim \xi_2^2$ and therefore

$$m(\bar{\xi}) \sim \frac{1}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s}$$

which is square integrable on $(\mathbb{R}^2, d\xi_1 d\xi_3)$.
Case 8: $|\xi_1| \sim |\xi_3| \sim |\xi| \gg |\xi_2|$ Then $|\Phi(\bar{\xi})| \sim \xi_1^2$ and therefore

$$m(\bar{\xi}) \sim \frac{|\xi_2|}{\langle \xi_1 \rangle} \cdot \frac{1}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s} \lesssim \frac{1}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s}$$

which is square integrable on $(\mathbb{R}^2, d\xi_1 d\xi_2)$.
Case 9: $|\xi_1| \sim |\xi_2| \sim |\xi_3| \sim |\xi|$. With $\langle \Phi(\bar{\xi}) \rangle \sim \langle \xi_2 - \xi_1 \rangle \langle \xi_2 - \xi_3 \rangle$, we get

$$m(\bar{\xi}) \sim \frac{1}{\langle \xi_2 - \xi_1 \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^s} \cdot \frac{1}{\langle \xi_2 - \xi_3 \rangle^{\frac{1}{2}} \langle \xi_3 \rangle^s}$$

and thus by applying (4.30) twice, $\mathcal{M}_2 \lesssim 1$.

□

Remark 4.3. By comparing the estimate of Lemma 4.2 with the similar estimate for the cubic NLS on $\mathbb{R}$ (see [28, Lemma 2.3]), we note that whenever $m(\bar{\xi}) \lesssim 1$ (e.g. when $\min(|\xi_2 - \xi_1|, |\xi_2 - \xi_3|) \leq 1$ or when $|\xi_1| \sim |\xi_2| \sim |\xi_3|$), our operator $\mathcal{T}_\Phi$ acts as the operator $\mathcal{N}^0_{\leq M}$ from [28] (with displacement parameter $\alpha = 0$ and localization size $M \sim 1$), and thus we can appeal to the arguments used therein. For the sake of completeness we have also included the argument for Case 1 in the proof of Lemma 4.2 above.

Remark 4.4. Notice that in the above proof, the case when $|\xi_2| \sim |\xi| \gg |\xi_1|, |\xi_3|$ in Case 3 informs us why the derivative falling on the conjugate factor in the cubic nonlinearity $v^2 \partial_x \overline{v}$ can be handled: in the worst case scenario of the low×high×low → high frequency interaction, we can use the $\frac{1}{2}$-power of the modulation to cancel the factor ξ_2 in the numerator. This motivates the need to use the gauge transformation (4.1) to eliminate the nonlinearity $2|v|^2 \partial_x v$ from the right-hand side of (1.1).

Remark 4.5. At the end-point regularity $s = \frac{1}{2}$, with minor changes in the proof, we can also obtain an estimate as in Lemma 4.2, but for $\mathcal{T}_\Phi$ defined by

$$\mathcal{F}\Big[\mathcal{T}_\Phi(v_1, v_2, v_3)\Big](t, \xi) = \int_{\xi = \xi_1 - \xi_2 + \xi_3} \frac{|\xi_2|}{\langle \Phi(\bar{\xi}) \rangle^{\frac{1}{2}+\varepsilon}} \widehat{v_1}(\xi_1) \overline{\widehat{v_2}(\xi_2)} \widehat{v_3}(\xi_3) d\xi_1 d\xi_2 \,, \tag{4.32}$$

where $\varepsilon > 0$ can be taken arbitrarily small. However, in this case $C = C(\varepsilon) \nearrow \infty$ as $\varepsilon \searrow 0$. This remark also applies to Corollaries 4.6 and 4.8, Lemmata 4.11, and 4.12, but not to Lemma 4.10.

In the proofs of the following lemmata, we freely use the Fourier lattice property of $H^s(\mathbb{R})$, i.e.

$$\big\| \mathcal{F}^{-1}\big(|\mathcal{F}(v)|\big) \big\|_{H^s(\mathbb{R})} = \|v\|_{H^s(\mathbb{R})} \,,$$

and thus we drop the modulus notation on factors such as $v(\xi)$ (which henceforth we assume to be non-negative).

Corollary 4.6. *Let $s > \frac{1}{2}$. Then for $\mathcal{T}^{(1)}_{\mathcal{T},1} = \mathcal{T}_1(v)$ given by* (4.10)*, we have*

$$\big\|\mathcal{T}^{(1)}_{\mathcal{T},1}(v)\big\|_{H^s_x(\mathbb{R})} \lesssim N^{\frac{1}{2}} \|v\|^3_{H^s_x(\mathbb{R})} .$$

Proof. We have

$$\begin{aligned}
\Big|\mathcal{F}\big[\mathcal{T}^{(1)}_{\mathcal{T},1}(v)\big](\xi)\Big| &\leq \int_{\substack{\xi=\xi_1-\xi_2+\xi_3\\ |\Phi(\bar{\xi})|\leq N}} N^{\frac{1}{2}} N^{-\frac{1}{2}} |\xi_2| \widehat{v}(t,\xi_1)\widehat{v}(t,\xi_2)\widehat{v}(t,\xi_3) d\xi_1 d\xi_2 \\
&\lesssim N^{\frac{1}{2}} \int_{\xi=\xi_1-\xi_2+\xi_3} \frac{|\xi_2|}{\langle \Phi(\bar{\xi})\rangle^{\frac{1}{2}}} \widehat{v}(t,\xi_1)\widehat{v}(t,\xi_2)\widehat{v}(t,\xi_3) d\xi_1 d\xi_2 \\
&= N^{\frac{1}{2}} \Big|\mathcal{F}\big[\mathcal{T}_\Phi(v)\big](\xi)\Big|
\end{aligned}$$

and therefore the estimate follows from Lemma 4.2.

□

For estimating the remaining nonlinear terms of (4.26), it is convenient to introduce the mapping $\mathfrak{S}(T;\,\cdot\,)$ associated to an ordered tree T, say of generation J, which essentially applies the operator $\mathcal{T}_\Phi$ iteratively taking into account the structure of T. We define these mappings by the following bottom-up algorithmic procedure.

Definition 4.7. Let $J \geq 1$ and $T \in \mathfrak{T}(J)$. We define the $(2J+1)$-linear map $\mathfrak{S}(T;\,\cdot\,)$ on space-time functions $v_j \in C(I; H^s(\mathbb{R}))$ $(1 \leq j \leq 2J+1 = |T^\infty|)$ by the following rules.

(i) Replace the jth terminal node of T by v_j, for all $j \in \{1,\dots,2J+1\}$.

(ii) For $j = J, J-1, \dots, 1$, replace the jth root node $r^{(j)}$ by the trilinear operator $\mathcal{T}_\Phi$ whose arguments are given by the functions associated with its three children.

For such mappings, we have the following corollary which is a consequence of Lemma 4.2.

Corollary 4.8. *Let $s > \frac{1}{2}$, $J \geq 1$ and $T \in \mathfrak{T}(J)$. Then*

$$\big\|\mathfrak{S}(T; v_1,\dots,v_{2J+1})\big\|_{H^s_x(\mathbb{R})} \leq C^J \prod_{j=1}^{2J+1} \|v_j\|_{H^s_x(\mathbb{R})} ,$$

where C is the constant given by Lemma 4.2.

Proof. It follows immediately by successively applying Lemma 4.2. Namely, we start with the root node $r^{(1)}$ of T and we move top-down on T. Since T is a tree of generation J, it has J many root nodes and thus we pick up the constant C^J.

□

Next, for simplicity we set $\beta_0 := 1$ and for any $J \geq 1$ we put

$$b_J := \prod_{j=0}^{J-1} \beta_j \,. \tag{4.33}$$

Remark 4.9. For each $s > \frac{1}{2}$, we choose the constants β_j's such that we ensure

$$\sup_{J\geq 1} \frac{c_{J+1}\beta_J(10C)^{J+1}(2J+6)}{b_1^\theta \cdots b_{J-1}^\theta} \lesssim 1\,,$$

where $c_{J+1} = 1\cdot 3\cdot 5\cdot\dots\cdot(2J+1)$ (see (A.1)) and $\theta = \theta(s) := \min\{2s-1, \frac{1}{2}\}$. For instance, we may take

$$\beta_j = (2j+3)^{\frac{2}{\theta}}, \quad j \geq 1.$$

Then, one can observe that the factorial decay of denominator $5^{2J-2}\cdot 7^{2J-4}\cdot\dots\cdot(2J-1)^4\cdot(2J+1)^2$ is enough to compensate the factorial growth term c_{J+1} and the exponential growth term $(10C)^J$.

We are now ready to prove the estimates for all nonlinear terms of (4.26), which we treat in decreasing order of difficulty.

Lemma 4.10. *Let $s > \frac{1}{2}$ and $J \geq 1$. Then, for $\mathcal{T}^{(J+1)}_{\mathcal{T},1}$ given by* (4.24) *we have*

$$\|\mathcal{T}^{(J+1)}_{\mathcal{T},1}(v)\|_{H^s_x(\mathbb{R})} \lesssim N^{-\frac{1}{2}(J-1)}\|v\|^{2J+3}_{H^s_x(\mathbb{R})}\,, \tag{4.34}$$

$$\|\mathcal{T}^{(J+1)}_{\mathcal{T},1}(v) - \mathcal{T}^{(J+1)}_{\mathcal{T},1}(w)\|_{H^s_x(\mathbb{R})} \lesssim N^{-\frac{1}{2}(J-1)}\Big(\|v\|^{2J+2}_{H^s_x(\mathbb{R})} + \|w\|^{2J+2}_{H^s_x(\mathbb{R})}\Big)\|v-w\|_{H^s_x(\mathbb{R})}\,. \tag{4.35}$$

Proof. With $\mathcal{T}^{(J+1)}_{\mathcal{T},1}(T;v)$ simply denoting the summand in (4.24), we have

$$\mathcal{T}^{(J+1)}_{\mathcal{T},1}(v) = \sum_{T\in\mathfrak{T}(J+1)} \mathcal{T}^{(J+1)}_{\mathcal{T},1}(T;v).$$

and thus

$$\|\mathcal{T}^{(J+1)}_{\mathcal{T},1}(v)\|_{H^s} \leq c_{J+1} \sup_{T\in\mathfrak{T}(J+1)} \|\mathcal{T}^{(J+1)}_{\mathcal{T},1}(T;v)\|_{H^s}\,. \tag{4.36}$$

Now fix $T \in \mathfrak{T}(J+1)$. We recall that the frequency support of $\mathcal{T}^{(J+1)}_{\mathcal{T},1}(T;v)$ is

$$C_0 \cap C_1^c \cap \dots \cap C_{J-1}^c \cap C_J\,.$$

Hence, we have

$$|\mu_1| > N\,, \quad |\widetilde{\mu}_j| > \beta_{j-1}|\widetilde{\mu}_{j-1}| \text{ for } j = 2,\dots,J\,, \quad \text{and} \quad |\widetilde{\mu}_{J+1}| \leq \beta_J|\widetilde{\mu}_J|\,.$$

In particular, $|\widetilde{\mu}_j| > b_jN$ for $j = 1,\dots,J$. Note that $\beta_{j-1} \geq 2$ for $j = 2,\dots,J$. Then, from $|\mu_j| \leq |\widetilde{\mu}_j| + |\widetilde{\mu}_{j-1}| < \frac{3}{2}|\widetilde{\mu}_j|$ and $|\widetilde{\mu}_j| \leq |\mu_j| + |\widetilde{\mu}_{j-1}| < |\mu_j| + \frac{1}{2}|\widetilde{\mu}_j|$, we deduce $|\widetilde{\mu}_j| \sim |\mu_j|$, for $j = 2, \ldots, J$. Also, since $|\mu_{J+1}| \leq |\widetilde{\mu}_{J+1}| + |\widetilde{\mu}_J| \leq (\beta_J + 1)|\widetilde{\mu}_J|$, we get $|\mu_{J+1}| \leq 2\beta_J|\widetilde{\mu}_J|$. Thus we have

$$\begin{aligned}
\mathcal{T}^{(J+1)}_{\mathcal{T},1}(T; v) &\leq \int_{\boldsymbol{\xi} \in \Xi_\xi(T)} \mathbf{1}_{C_J \cap F_J} \bigg(\prod_{j=1}^{J} \frac{|\xi_2^{(j)}|}{|\widetilde{\mu}_j|} \bigg) |\xi_2^{(J+1)}| \bigg(\prod_{a \in T^\infty} v(\xi_a) \bigg) \\
&\lesssim \int_{\boldsymbol{\xi} \in \Xi_\xi(T)} \bigg(\prod_{j=1}^{J-1} \frac{|\xi_2^{(j)}|}{(b_j N)^{\frac{1}{2}} \langle \mu_j \rangle^{\frac{1}{2}}} \bigg) \frac{|\xi_2^{(J)}|}{(2\beta_J)^{-\frac{1}{2}} \langle \mu_J \rangle^{\frac{1}{2}} \langle \mu_{J+1} \rangle^{\frac{1}{2}}} \\
&\qquad \times |\xi_2^{(J+1)}| \bigg(\prod_{a \in T^\infty} v(\xi_a) \bigg) \\
&\lesssim \beta_J^{\frac{1}{2}} \prod_{j=1}^{J-1} b_j^{-\frac{1}{2}} N^{-\frac{1}{2}(J-1)} \cdot \mathfrak{S}(T; v)
\end{aligned}$$

Therefore, by Corollary 4.8 and (4.36), we get

$$\|\mathcal{T}^{(J+1)}_{\mathcal{T},1}(v)\|_{H^s(\mathbb{R})} \lesssim \frac{c_{J+1} \beta_J^{\frac{1}{2}} C^{J+1}}{b_1^{\frac{1}{2}} \cdots b_{J-1}^{\frac{1}{2}}} N^{-\frac{1}{2}(J-1)} \|v\|^{2J+3}_{H^s(\mathbb{R})} .$$

For the difference estimate (4.35), a similar argument applies. Namely, one writes the difference using a telescopic sum and employs the multilinear version of the operator $\mathfrak{S}(T, \cdot)$ with precisely one entry being $v - w$ and the others being either v or w. Compared to (4.34), we note that for (4.35) we pick up an extra factor of $2J + 4$ since we have the bound

$$\big|a^{2J+3} - b^{2J+3}\big| \leq \bigg(\sum_{j=1}^{2J+3} |a|^{2J+3-j} |b|^{j-1} \bigg) |a - b| \leq (2J + 4) \Big(|a|^{2J+2} + |b|^{2J+2} \Big) |a - b| .$$

Hence,

$$\|\mathcal{T}^{(J+1)}_{\mathcal{T},1}(v) - \mathcal{T}^{(J+1)}_{\mathcal{T},1}(w)\|_{H^s(\mathbb{R})} \lesssim \frac{c_{J+1} \beta_J^{\frac{1}{2}} C^{J+1} (2J + 4)}{b_1^{\frac{1}{2}} \cdots b_{J-1}^{\frac{1}{2}}} \Big(\|v\|^{2J+2}_{H^s(\mathbb{R})} + \|w\|^{2J+2}_{H^s(\mathbb{R})} \Big) \|v - w\|^{2J}_{H^s(\mathbb{R})} .$$

By taking into account Remark 4.9 we deduce (4.34) and (4.35).

□

Next, we consider the nonlinear terms coming as boundary terms when applying integration by parts with respect to the temporal variable in Section 4.1.

Lemma 4.11. *Let $s > \frac{1}{2}$ and $J \geq 1$. Then, for $\mathcal{T}_0^{(J+1)}$ given by* (4.20) *we have*

$$\|\mathcal{T}_0^{(J+1)}(v)\|_{H^s(\mathbb{R})} \lesssim N^{-\frac{1}{2}J} \|v\|^{2J+1}_{H^s(\mathbb{R})} , \tag{4.37}$$

$$\|\mathcal{T}_0^{(J+1)}(v) - \mathcal{T}_0^{(J+1)}(w)\|_{H^s(\mathbb{R})} \lesssim N^{-\frac{1}{2}J} \Big(\|v\|^{2J}_{H^s(\mathbb{R})} + \|w\|^{2J}_{H^s(\mathbb{R})} \Big) \|v - w\|_{H^s(\mathbb{R})} . \tag{4.38}$$

Proof. With $\mathcal{T}_0^{(J+1)}(T;v)$ simply denoting the summand in (4.20), we have

$$\mathcal{T}_0^{(J+1)}(v) = \sum_{T\in\mathfrak{T}(J)} \mathcal{T}_0^{(J+1)}(T;v). \tag{4.39}$$

and thus

$$\|\mathcal{T}_0^{(J+1)}(v)\|_{H^s} \le c_J \sup_{T\in\mathfrak{T}(J)} \|\mathcal{T}_0^{(J+1)}(T;v)\|_{H^s}\,. \tag{4.40}$$

Now fix $T \in \mathfrak{T}(J)$. We recall that the frequency support of $\mathcal{T}_0^{(J+1)}(T;v)$ is $F_J = C_0\cap C_1^c\cap\cdots\cap C_{J-1}^c$. Hence, we have $|\mu_1| > N$, $|\widetilde{\mu}_j| > \beta_{j-1}|\widetilde{\mu}_{j-1}|$ for $j = 2,\ldots,J$. As in the proof of Lemma 4.10, we have $|\mu_j| \sim |\widetilde{\mu}_j| > b_{j-1}N$ for $j = 2,\ldots,J$. Thus we have

$$\begin{aligned}\mathcal{T}_0^{(J+1)}(T;v) &\le \int_{\boldsymbol{\xi}\in\Xi_\xi(T)} \mathbf{1}_{F_J}\bigg(\prod_{j=1}^{J}\frac{|\xi_2^{(j)}|}{|\widetilde{\mu}_j|}\bigg)\bigg(\prod_{a\in T^\infty} v(\xi_a)\bigg)\\ &\lesssim \int_{\boldsymbol{\xi}\in\Xi_\xi(T)} \bigg(\prod_{j=1}^{J}\frac{|\xi_2^{(j)}|}{(b_jN)^{\frac12}\langle\mu_j\rangle^{\frac12}}\bigg)\bigg(\prod_{a\in T^\infty} v(\xi_a)\bigg)\\ &\lesssim \bigg(\prod_{j=1}^{J-1} b_j^{-\frac12}\bigg)N^{-\frac12 J}\cdot\mathfrak{S}(T;v)\end{aligned}$$

Therefore, by Corollary 4.8 and (4.40), we get

$$\|\mathcal{T}_0^{(J+1)}(v)\|_{H^s(\mathbb{R})} \lesssim \frac{c_J C^J}{b_1^{\frac12}\cdots b_{J-1}^{\frac12}} N^{-\frac12 J}\|v\|_{H^s(\mathbb{R})}^{2J+1}\,.$$

For the difference estimate (4.38), an observation analogous to that in the proof of Lemma 4.10 applies and thus we obtain

$$\|\mathcal{T}_0^{(J+1)}(v)-\mathcal{T}_0^{(J+1)}(w)\|_{H^s(\mathbb{R})} \lesssim \frac{c_J C^J(2J+2)}{b_1^{\frac12}\cdots b_{J-1}^{\frac12}} N^{-\frac12 J}\Big(\|v\|_{H^s(\mathbb{R})}^{2J}+\|w\|_{H^s(\mathbb{R})}^{2J}\Big)\|v-w\|_{H^s(\mathbb{R})}\,.$$

□

In the proofs of the following lemma, we skip the argument for the difference estimate altogether as the same ideas apply as for the difference estimate of Lemma 4.11.

Lemma 4.12. *Let $s > \frac12$ and $J \ge 1$. Then, for $\mathcal{T}_{\mathcal{Q}}^{(J+1)}$ given by* (4.21) *we have*

$$\|\mathcal{T}_{\mathcal{Q}}^{(J+1)}(v)\|_{H_x^s(\mathbb{R})} \lesssim N^{-\frac12 J}\|v\|_{H_x^s(\mathbb{R})}^{2J+5}, \tag{4.41}$$

$$\|\mathcal{T}_{\mathcal{Q}}^{(J+1)}(v)-\mathcal{T}_{\mathcal{Q}}^{(J+1)}(w)\|_{H_x^s(\mathbb{R})} \lesssim N^{-\frac12 J}\Big(\|v\|_{H_x^s(\mathbb{R})}^{2J+4}+\|w\|_{H_x^s(\mathbb{R})}^{2J+4}\Big)\|v-w\|_{H_x^s(\mathbb{R})} \tag{4.42}$$

Proof. The proof is similar to the proof of Lemma 4.11. We have

$$\big\|\mathcal{T}_{\mathcal{Q}}^{(J+1)}(v)\big\|_{H^s} \le c_J(2J+1) \sup_{T\in\mathfrak{T}(J)} \sup_{b\in T^\infty} \big\|\mathcal{T}_{\mathcal{Q}}^{(J+1)}(T,b;v)\big\|_{H^s}\,, \tag{4.43}$$

where $\mathcal{T}_{\mathcal{Q}}^{(J+1)}(T,b;v)$ denotes the (inner-most) summand in (4.21). Fix $T\in\mathfrak{T}(J)$ and $b\in T^\infty$. Then we have

$$\begin{aligned}
\mathcal{T}_{\mathcal{Q}}^{(J+1)}(T,b;v) &\le \int_{\boldsymbol{\xi}\in\Xi_\xi(T)} \mathbf{1}_{F_J}\bigg(\prod_{j=1}^{J}\frac{|\xi_2^{(j)}|}{|\widetilde{\mu}_j|}\bigg)\bigg(\mathcal{Q}(v)(\xi_b)\prod_{\substack{a\in T^\infty\\ a\neq b}} v(\xi_a)\bigg)\\
&\lesssim \int_{\boldsymbol{\xi}\in\Xi_\xi(T)} \bigg(\prod_{j=1}^{J}\frac{|\xi_2^{(j)}|}{(b_jN)^{\frac12}\langle\mu_j\rangle^{\frac12}}\bigg)\bigg(\mathcal{Q}(v)(\xi_b)\prod_{\substack{a\in T^\infty\\ a\neq b}} v(\xi_a)\bigg)\\
&\lesssim \bigg(\prod_{j=1}^{J-1} b_j^{-\frac12}\bigg) N^{-\frac12 J}\cdot \mathfrak{S}(T;\mathbf{v}_b)\,,
\end{aligned}$$

where if b is the jth terminal node of T, we put

$$\mathbf{v}_b := (v,\dots,v,\underbrace{\mathcal{Q}(v)}_{j\text{th spot}},v,\dots,v)\,.$$

Therefore, by Corollary 4.8, (4.9), and (4.43),

$$\big\|\mathcal{T}_{\mathcal{Q}}^{(J+1)}(v)\big\|_{H^s_x(\mathbb{R})} \lesssim \frac{c_J C^J(2J+1)}{b_1^{\frac12}b_2^{\frac12}\cdots b_{J-1}^{\frac12}} N^{-\frac12 J}\|v\|_{H^s(\mathbb{R})}^{2J+5}$$

For the difference estimate (4.42), an observation analogous to that in the proof of Lemma 4.10 (see also the proof of Lemma 4.11) applies and we take into account Remark 4.9.

□

4.3 The estimates in a weak norm

Here, we prove the estimates necessary to rigorously justify the normal form equation (4.26) for rough $H^s(\mathbb{R})$-solutions of (4.3), which is done explicitly in Section 4.4. For this purpose, we have to be able to estimate $\partial_t v$, for $v\in C(I;H^s(\mathbb{R}))$ solution to (4.3).

It is clear that due to the derivative in the cubic nonlinearity, the estimate

$$\big\|v^2\partial_x\overline{v}\big\|_{H^s_x(\mathbb{R})} \lesssim \|v\|_{H^s_x(\mathbb{R})}^3$$

fails. However, if we weaken the norm in the left-hand side above, then we might be able to obtain an estimate satisfactory to our aims in Section 4.4. Hence, with the following lemma, we identify a family of Sobolev norms weaker than the $H^s(\mathbb{R})$-norm which can serve as a weak topology used to justify the normal form equation (4.26).

Lemma 4.13. *Let $s > \frac{1}{2}$ and $\sigma \le s - 1$. Then, we have the trilinear estimate*

$$\big\|v_1\big(\partial_x\overline{v_2}\big)v_3\big\|_{H^\sigma_x(\mathbb{R})} \lesssim \prod_{j=1}^{3} \|v_j\|_{H^s_x(\mathbb{R})} .$$

Proof. By duality, the desired estimate follows once we show:

$$\int_{\xi=\xi_1-\xi_2+\xi_3} m_4(\bar{\xi})v_1(\xi_1)v_2(\xi_2)v_3(\xi_3)v_4(\xi)d\xi_1 d\xi_2 d\xi \lesssim \prod_{k=1}^{4} \|u_k\|_{L^2_\xi(\mathbb{R})} , \tag{4.44}$$

for any $v_1, \dots, v_4 \in L^2(\mathbb{R})$ with $\widehat{v_j} \ge 0$ $(1 \le j \le 4)$, with the multiplier

$$m_4(\bar{\xi}) = \frac{\langle\xi\rangle^\sigma|\xi_2|}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s} . \tag{4.45}$$

We study the boundedness of this multiplier, distinguishing which two of the four frequencies are the largest. On the convolution hyperplane, it must be that the largest two frequencies are comparable. Also, by the symmetry of m_4 with respect to ξ_1, ξ_3, we may assume without loss of generality that $|\xi_1| \ge |\xi_3|$.

Case 1: $|\xi| \sim |\xi_2| \gtrsim |\xi_1|, |\xi_3|$.

In this case, since $\sigma + 1 - s \le 0$, we have

$$m_4(\bar{\xi}) \lesssim \langle\xi\rangle^{\sigma+1-s}\frac{1}{\langle\xi_1\rangle^s\langle\xi_3\rangle^s} \le \frac{1}{\langle\xi_1\rangle^s\langle\xi_3\rangle^s} .$$

Case 2: $|\xi| \sim |\xi_1| \gtrsim |\xi_2|, |\xi_3|$.

Since $\sigma + 1 - s \le 0$, we have

$$m_4(\bar{\xi}) \lesssim \langle\xi\rangle^{\sigma+1-s}\frac{\langle\xi_2\rangle^{1-s}}{\langle\xi\rangle\langle\xi_3\rangle^s} \lesssim \frac{1}{\langle\xi_2\rangle^s\langle\xi_3\rangle^s} .$$

Case 3: $|\xi_2| \sim |\xi_1| \gtrsim |\xi|, |\xi_3|$.

Since $s + \sigma \le 2s - 1$, we have$\langle\xi\rangle^{s+\sigma} \le \langle\xi\rangle^{2s-1} \lesssim \langle\xi_2\rangle^{2s-1}$ for $s \ge \frac{1}{2}$, and

$$m_4(\bar{\xi}) \lesssim \frac{\langle\xi\rangle^\sigma}{\langle\xi_2\rangle^{2s-1}\langle\xi_3\rangle^s} \lesssim \frac{1}{\langle\xi\rangle^s\langle\xi_3\rangle^s} .$$

Case 4: $|\xi_1| \sim |\xi_3| \gtrsim |\xi|, |\xi_2|$.

Since $s + \sigma \le 2s - 1$, we have $\langle\xi_2\rangle\langle\xi\rangle^{s+\sigma} \le \langle\xi_2\rangle\langle\xi\rangle^{2s-1} \lesssim \langle\xi_1\rangle^{2s}$ for $s \ge \frac{1}{2}$, and

$$m_4(\bar{\xi}) \lesssim \frac{\langle\xi_2\rangle^{1-s}\langle\xi\rangle^\sigma}{\langle\xi_1\rangle^{2s}} \lesssim \frac{1}{\langle\xi\rangle^s\langle\xi_2\rangle^s} .$$

In each of the four cases, there exist $k_1,\, k_2 \in \{1,2,3,4\}$, $k_1 \neq k_2$ such that

$$m_4(\bar{\xi}) \lesssim \frac{1}{\langle \xi_{k_1} \rangle^{\frac{1}{2}+} \langle \xi_{k_2} \rangle^{\frac{1}{2}+}}$$

(with the convention that $\xi_4 = \xi$) and let j denote the third index . Then, by Cauchy-Schwarz inequality, the Sobolev embedding $H^s \hookrightarrow L^\infty$, and the fact that $H^s(\mathbb{R})$ is a Fourier lattice, we have

$$\text{LHS of (4.44)} \lesssim \prod_{k \in \{k_1,k_2\}} \big\| \langle \partial_x \rangle^{-s} \mathcal{F}^{-1}[|u_k|] \big\|_{L^\infty_x} \|u_j\|_{L^2_\xi} \|u_4\|_{L^2_\xi} \lesssim \prod_{k=1}^{4} \|u_k\|_{L^2_\xi}$$

and the proof is completed.

□

As a consequence of Lemma 4.13 and (4.9), we have the following:

Corollary 4.14. *Let $s > \frac{1}{2}$ and $v \in C(I; H^s(\mathbb{R}))$ be a solution to (4.3). Then, uniformly in $t \in I$, we have*

$$\|\partial_t v\|_{H^{s-1}_x(\mathbb{R})} \lesssim \|v\|^3_{H^s_x(\mathbb{R})} + \|v\|^5_{H^s_x(\mathbb{R})} \,. \tag{4.46}$$

Next, for $M \geq 1$, we consider the trilinear operator $\mathcal{T}^{\mathrm{w}}_{|\Phi|>M}$ defined by

$$\mathcal{F}\Big[\mathcal{T}^{\mathrm{w}}_{|\Phi|>M}(v_1, v_2, v_3)\Big](t,\xi) = \int_{\substack{\xi=\xi_1-\xi_2+\xi_3 \\ |\Phi(\bar{\xi})|>M}} \frac{|\xi_2|}{\langle \Phi(\bar{\xi}) \rangle} \widehat{v_1}(\xi_1) \overline{\widehat{v_2}(\xi_2)} \widehat{v_3}(\xi_3) d\xi_1 d\xi_2 \,, \tag{4.47}$$

where $\Phi(\bar{\xi})$ is given by (4.7).

Lemma 4.15 (The estimate of $\mathcal{T}^{\mathrm{w}}_{|\Phi|>M}$ in the $H^{s-1}(\mathbb{R})$-norm)**.** *Let $s > \frac{1}{2}$ and $\theta = \theta(s) := \min\{2s-1, \frac{1}{2}\}$. Then, there exists a finite constant $C = C(s) > 0$ such that*

$$\|\mathcal{T}^{\mathrm{w}}_{|\Phi|>M}(v_1, v_2, v_3)\|_{H^{s-1}(\mathbb{R})} \leq CM^{-\theta} \|v_j\|_{H^{s-1}(\mathbb{R})} \|v_k\|_{H^s(\mathbb{R})} \|v_l\|_{H^s(\mathbb{R})},$$

for any j, k, ℓ such that $\{j,k,l\} = \{1,2,3\}$ and for any $M \geq 1$.

Proof. We denote $\gamma := 2s - 1 > 0$. Similarly to the proof of Lemma 4.2, it suffices to prove that

$$\int_{\substack{\xi=\xi_1-\xi_2+\xi_3 \\ |\Phi(\bar{\xi})|>M}} m_j(\bar{\xi}) \widehat{v_1}(\xi_1) \widehat{v_2}(\xi_2) \widehat{v_3}(\xi_3) \widehat{v_4}(\xi) \, d\xi_1 d\xi_2 d\xi \;\leq\; CM^{-\frac{\gamma}{2}} \prod_{j=1}^{4} \|v_j\|_{L^2_x} \,, \tag{4.48}$$

for any $v_1, \ldots, v_4 \in L^2(\mathbb{R})$ with $\widehat{v_j} \geq 0$ $(1 \leq j \leq 4)$. Also, by Cauchy-Schwarz inequality, it suffices to check that

$$\mathcal{M}^j_k := \sup_{\xi_k \in \mathbb{R}} \left(\int_{\substack{\xi=\xi_1-\xi_2+\xi_3 \\ |\Phi(\bar{\xi})|>M}} m_j(\bar{\xi})^2 d\xi_{\ell_1} d\xi_{\ell_2} \right)^{\frac{1}{2}} \leq CM^{-\frac{\gamma}{2}} \,, \tag{4.49}$$

for some $1 \le k \le 4$, where the multiplier is given by

$$m_j(\bar{\xi}) := \frac{|\xi_2|}{\langle \Phi(\bar{\xi}) \rangle} \cdot \frac{\langle \xi_j \rangle^{1-s}}{\langle \xi \rangle^{1-s} \langle \xi_k \rangle^s \langle \xi_\ell \rangle^s} = \frac{\langle \xi_j \rangle}{\langle \xi \rangle \langle \Phi(\bar{\xi}) \rangle^{\frac{1}{2}}} m(\bar{\xi}) \tag{4.50}$$

with $\{j, k, \ell\} = \{1, 2, 3\}$ and $m(\bar{\xi})$ given by (4.29).

Let us first prove the lemma for $j = 1$.

Case 1: $\min(|\xi_2 - \xi_1|, |\xi_2 - \xi_3|) \le 1$. Since m_1 is not symmetric in ξ_1, ξ_3 we treat the following two subcases.

Subcase 1.1: $|\xi_2 - \xi_1| \le 1$. Then $\langle \xi_1 \rangle \sim \langle \xi_2 \rangle$ and also $\langle \xi_3 \rangle \sim \langle \xi \rangle$. We have

$$m_1(\bar{\xi}) \sim \frac{|\xi_2|}{\langle \Phi(\bar{\xi}) \rangle \langle \xi_3 \rangle \langle \xi_1 \rangle^{2s-1}} \lesssim \frac{|\xi_2|^{2-2s}}{\langle \Phi(\bar{\xi}) \rangle} .$$

Assume for now that $|\xi_2| \gg \langle \xi_3 \rangle$. Then $\langle \Phi(\bar{\xi}) \rangle \sim \langle \xi_2(\xi_2 - \xi_1) \rangle$ and thus

$$m_1(\bar{\xi}) \lesssim \frac{|\xi_2(\xi_2 - \xi_1)|^{2-2s}}{\langle \Phi(\bar{\xi}) \rangle^\gamma \langle \xi_2(\xi_2 - \xi_1) \rangle^{2-2s} |\xi_2 - \xi_1|^{2-2s}} \lesssim \frac{M^{-\gamma}}{|\xi_2 - \xi_1|^{2-2s}} .$$

Similarly to Case 1 in the proof of Lemma 4.2, we denote $\zeta := \xi_2 - \xi_1 = \xi_3 - \xi$ and by using Hölder's inequality, we get that

$$\begin{aligned}
\text{LHS of (4.48)} &\lesssim \int_{|\zeta| \le 1} \frac{M^{-\gamma}}{|\zeta|^{1-\gamma}} \int_{\xi_1} \widehat{v_1}(\xi_1) \widehat{v_2}(\xi_1 + \zeta) d\xi_1 \int_{\xi_3} \widehat{v_3}(\xi_3) \widehat{v_4}(\xi_3 - \zeta) d\xi_3 \, d\zeta \\
&\le \left(\int_{|\zeta| \le 1} \frac{M^{-\gamma} d\zeta}{|\zeta|^{1-\gamma}} \right) \\
&\qquad \times \left\| \int_{\xi_1} \widehat{v_1}(\xi_1) \widehat{v_2}(\xi_1 + \zeta) d\xi_1 \right\|_{L^\infty_\zeta} \left\| \int_{\xi_3} \widehat{v_3}(\xi_3) \widehat{v_4}(\xi_3 - \zeta) d\xi_3 \right\|_{L^\infty_\zeta} \\
&\lesssim M^{-\gamma} \prod_{j=1}^4 \|v_j\|_{L^2} .
\end{aligned}$$

If $|\xi_2| \lesssim \langle \xi_3 \rangle$, then $m_1(\bar{\xi}) \lesssim M^{-1}$ and in the argument above we use $\int_{|\zeta| \le 1} d\zeta \lesssim 1$.

Subcase 1.2: $|\xi_2 - \xi_3| \le 1$. Then $\langle \xi_2 \rangle \sim \langle \xi_3 \rangle$ and also $\langle \xi_1 \rangle \sim \langle \xi \rangle$. We have

$$m_1(\bar{\xi}) \sim \frac{|\xi_2|}{\langle \Phi(\bar{\xi}) \rangle \langle \xi_2 \rangle^{2s}} \lesssim M^{-1}$$

and we argue as in Subcase 1.1 above.

In all the cases below, we assume that $|\xi_2 - \xi_1| > 1$ and $|\xi_2 - \xi_3| > 1$.

Case 2: $|\xi_2| \sim |\xi_1| \gg |\xi_3|, |\xi|$. Then

$$\langle \Phi(\bar{\xi}) \rangle \sim |\Phi(\bar{\xi})| \sim |\xi_2| \cdot |\xi_2 - \xi_1| \sim \langle \xi_1 \rangle \langle \xi - \xi_3 \rangle$$

and
$$m_1(\bar{\xi}) \sim \frac{|\xi_1|^{2-2s}}{\langle \Phi(\bar{\xi})\rangle \langle \xi\rangle^{1-s}\langle \xi_3\rangle^s} \sim \frac{1}{\langle \Phi(\bar{\xi})\rangle^{\gamma}} \cdot \frac{1}{\langle \xi-\xi_3\rangle^{2-2s}\langle \xi\rangle^{1-s}\langle \xi_3\rangle^s}\,.$$

Note that by directly using (4.30) to study the square integrability of m_1, we have

$$\mathcal{M}_1^1 \lesssim M^{-\gamma}\left(\int_{\mathbb{R}} \frac{1}{\langle \xi_3\rangle^{2s}} \int_{\mathbb{R}} \frac{1}{\langle \xi_3-\xi\rangle^{4-4s}\langle \xi\rangle^{2-2s}} d\xi\, d\xi_3\right)^{\frac{1}{2}} \lesssim M^{-\gamma}\,,$$

provided $s<\frac{5}{6}$. However, we can cover the entire range of $s\in(\frac{1}{2},1)$ if we discuss two separate subcases.

Subcase 2.1: If $\langle \xi_3\rangle \gtrsim \langle \xi\rangle$, then by using (4.30) we have

$$\mathcal{M}_1^1 \lesssim M^{-\gamma}\left(\int_{\mathbb{R}} \frac{1}{\langle \xi_3\rangle^{1+\varepsilon}} \int_{\mathbb{R}} \frac{1}{\langle \xi\rangle^{2s-1-\varepsilon}} \cdot \frac{1}{\langle \xi_3-\xi\rangle^{4-4s}\langle \xi\rangle^{2-2s}} d\xi\, d\xi_3\right)^{\frac{1}{2}} \lesssim M^{-\gamma}\,,$$

provided that we choose $\varepsilon>0$ such that $4-4s-\varepsilon>0$.

Subcase 2.2: If $\langle \xi_3\rangle \ll \langle \xi\rangle$, then we have $\langle \xi-\xi_3\rangle \sim \langle \xi\rangle$ and

$$m_1(\bar{\xi}) \lesssim M^{-\frac{\gamma}{2}} \frac{1}{\langle \xi_1-\xi\rangle^{s-\frac{1}{2}}\langle \xi\rangle^{\frac{5}{2}-2s}\langle \xi_3\rangle^s}$$

which is square integrable provided on $(\mathbb{R}^2, d\xi_3 d\xi)$.

Case 3: $|\xi_2|\sim|\xi_3|\gg|\xi_1|,|\xi|$. Then $\langle \Phi(\bar{\xi})\rangle \sim \langle \xi_2\rangle\langle \xi_2-\xi_3\rangle$ and

$$m_1(\bar{\xi}) \sim \frac{1}{\langle \Phi(\bar{\xi})\rangle^{\gamma}} \cdot \frac{\langle \xi_1\rangle^{1-s}}{\langle \xi_2\rangle\langle \xi-\xi_1\rangle^{2-2s}\langle \xi\rangle^{1-s}}\,.$$

Subcase 3.1: If $\langle \xi_1\rangle \lesssim \langle \xi\rangle$, then

$$m_1(\bar{\xi}) \lesssim \frac{M^{-\gamma}}{\langle \xi_2\rangle^{\frac{1}{2}+\varepsilon}\langle \xi_3\rangle^{\frac{1}{2}-\varepsilon}\langle \xi_2-\xi_3\rangle^{2-2s}}$$

which is square integrable (via (4.30)) provided that we choose $0<\varepsilon<2-2s$.

Subcase 3.2: If $\langle \xi_1\rangle \gg \langle \xi\rangle$, then $\langle \Phi(\bar{\xi})\rangle \sim \langle \xi_2\rangle\langle \xi_1\rangle$ and

$$m_1(\bar{\xi}) \lesssim \frac{M^{-\gamma}}{\langle \xi_2\rangle\langle \xi_1\rangle^{1-s}\langle \xi\rangle^{1-s}} \lesssim \frac{M^{-\gamma}}{\langle \xi_2\rangle^{\frac{1}{2}+\varepsilon}\langle \xi\rangle^{\frac{1}{2}-\varepsilon}\langle \xi\rangle^{2-2s}}$$

which is square integrable on $(\mathbb{R}^2, d\xi_2 d\xi)$ once we choose $0<\varepsilon<2-2s$.

Case 4: $|\xi_2|\sim|\xi|\gg|\xi_1|,|\xi_3|$. Then $\langle \Phi(\bar{\xi})\rangle \sim \langle \xi_2\rangle^2$ and thus

$$m_1(\bar{\xi}) \sim \frac{1}{\langle \Phi(\bar{\xi})\rangle^{\frac{1}{2}}} \cdot \frac{\langle \xi_1\rangle^{1-s}}{\langle \xi_2\rangle\langle \xi_3\rangle^s} \lesssim \frac{M^{-\frac{1}{2}}}{\langle \xi_2\rangle^s\langle \xi_3\rangle^s}\,,$$

which is square integrable on $(\mathbb{R}^2, d\xi_2 d\xi_3)$ since $s>\frac{1}{2}$.

Case 5: $|\xi_1| \sim |\xi_3| \gg |\xi_2|, |\xi|$. Then $\langle \Phi(\bar{\xi})\rangle \sim \langle \xi_1\rangle^2$ and

$$m_1(\bar{\xi}) \sim \frac{|\xi_2|}{\langle \xi_1\rangle^2} \cdot \frac{\langle \xi_1\rangle^{1-s}}{\langle \xi\rangle^{1-s}\langle \xi_2\rangle^s\langle \xi_3\rangle^s} \lesssim \frac{|\xi_2|}{\langle \xi_1\rangle^{1+s}} \cdot \frac{1}{\langle \xi_2\rangle^s\langle \xi_3\rangle^s} \lesssim \frac{1}{\langle \Phi(\bar{\xi})\rangle^{\frac{s}{2}}} \cdot \frac{1}{\langle \xi_2\rangle^s\langle \xi_3\rangle^s} \lesssim \frac{M^{-\frac{s}{2}}}{\langle \xi_2\rangle^s\langle \xi_3\rangle^s}$$

which is square integrable on $(\mathbb{R}^2, d\xi_2 d\xi_3)$ since $s > \frac{1}{2}$.

Case 6: $|\xi_1| \sim |\xi| \gg |\xi_2|, |\xi_3|$. Then $\langle \Phi(\bar{\xi})\rangle \sim \langle \xi_1\rangle\langle \xi - \xi_1\rangle$ and

$$\begin{aligned} m_1(\bar{\xi}) &\sim \frac{|\xi_2|}{\langle \xi_1\rangle\langle \xi - \xi_1\rangle} \cdot \frac{1}{\langle \xi_2\rangle^s\langle \xi_3\rangle^s} \lesssim \frac{\langle \xi_2\rangle^{1-s}}{\langle \xi_1\rangle\langle \xi - \xi_1\rangle\langle \xi_3\rangle^s} \lesssim \frac{1}{\langle \xi_1\rangle^s\langle \xi - \xi_1\rangle\langle \xi_3\rangle^s} \\ &\sim \frac{1}{\big(\langle \xi_1\rangle\langle \xi - \xi_1\rangle\big)^{2s-1}} \cdot \frac{1}{\langle \xi_1\rangle^{1-s}\langle \xi - \xi_1\rangle^{2-2s}\langle \xi_3\rangle^s} \lesssim M^{-\gamma} \frac{1}{\langle \xi_1\rangle^{1-s}\langle \xi - \xi_1\rangle^{2-2s}} \cdot \frac{1}{\langle \xi_3\rangle^s}. \end{aligned}$$

By using (4.30), it follows that

$$\mathcal{M}_4^1 \lesssim M^{-\gamma} \sup_{\xi\in\mathbb{R}} \left(\int_{\mathbb{R}} \frac{d\xi_1}{\langle \xi_1\rangle^{2-2s}\langle \xi - \xi_1\rangle^{4-4s}} \int_{\mathbb{R}} \frac{d\xi_3}{\langle \xi_3\rangle^{2s}} \right)^{\frac{1}{2}} \lesssim M^{-\gamma},$$

provided that $s < \frac{5}{6}$. However, with

$$m_1(\bar{\xi}) \lesssim \frac{1}{\big(\langle \xi_1\rangle\langle \xi - \xi_1\rangle\big)^{s-\frac{1}{2}}} \cdot \frac{1}{\langle \xi_1\rangle^{\frac{1}{2}}\langle \xi - \xi_1\rangle^{\frac{3}{2}-s}\langle \xi_3\rangle^s} \lesssim M^{-\frac{\gamma}{2}} \frac{1}{\langle \xi_1\rangle^{\frac{1}{2}}\langle \xi - \xi_1\rangle^{\frac{3}{2}-s}} \cdot \frac{1}{\langle \xi_3\rangle^s},$$

and (4.30), we get

$$\mathcal{M}_4^1 \lesssim M^{-\frac{\gamma}{2}},$$

for any $s \in (\frac{1}{2}, 1)$

Case 7: $|\xi_3| \sim |\xi| \gg |\xi_1|, |\xi_2|$. Then $\langle \Phi(\bar{\xi})\rangle \sim \langle \xi\rangle\langle \xi - \xi_3\rangle$ and thus

$$\begin{aligned} m_1(\bar{\xi}) &\sim \frac{|\xi_2|\langle \xi_1\rangle^{1-s}}{\langle \xi\rangle^2\langle \xi - \xi_3\rangle\langle \xi_2\rangle^s} \lesssim \frac{\langle \xi_2\rangle^{1-s}\langle \xi_1\rangle^{1-s}}{\langle \xi\rangle^2\langle \xi - \xi_3\rangle} \lesssim \frac{1}{\langle \xi\rangle^{2s}\langle \xi - \xi_3\rangle} \\ &= \frac{1}{\big(\langle \xi\rangle\langle \xi - \xi_3\rangle\big)^{2s-1}} \cdot \frac{1}{\langle \xi\rangle\langle \xi - \xi_3\rangle^{2-2s}} \lesssim M^{-\gamma} \frac{1}{\langle \xi\rangle^{\frac{1}{2}+\varepsilon}} \cdot \frac{1}{\langle \xi_3\rangle^{\frac{1}{2}-\varepsilon}\langle \xi - \xi_3\rangle^{2-2s}} \end{aligned}$$

Therefore, via (4.30), we obtain

$$\mathcal{M}_1^1 \lesssim M^{-\gamma} \sup_{\xi_1\in\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{\langle \xi\rangle^{1+2\varepsilon}} \int_{\mathbb{R}} \frac{1}{\langle \xi_3\rangle^{1-2\varepsilon}\langle \xi - \xi_3\rangle^{4-4s}} d\xi_3\, d\xi \right)^{\frac{1}{2}} \lesssim M^{-\gamma}.$$

provided that we choose $0 < \varepsilon < 4 - 4s$.

Case 8: $|\xi_1| \sim |\xi_2| \sim |\xi_3| \gg |\xi|$. Then $\langle \Phi(\bar{\xi})\rangle \sim \langle \xi_1\rangle^2$ and

$$m_1(\bar{\xi}) \sim \frac{1}{\langle \xi_1\rangle} \cdot \frac{\langle \xi_1\rangle^{1-s}}{\langle \xi\rangle^{1-s}\langle \xi_1\rangle^{2s}} \sim \frac{1}{\langle \xi_1\rangle^{3s}\langle \xi\rangle^{1-s}} = \frac{1}{\langle \xi_1\rangle^{2s-\frac{1}{2}}} \cdot \frac{1}{\langle \xi_1 - \xi\rangle^{\frac{1}{2}}\langle \xi\rangle^{1-s}} \cdot \frac{1}{\langle \xi_1\rangle^s}$$

By (4.30), it follows that

$$\mathcal{M}_2^1 \lesssim M^{\frac{1}{4}-s} \sup_{\xi_2\in\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{\langle \xi_1\rangle^{2s}} \int_{\mathbb{R}} \frac{1}{\langle \xi_1-\xi\rangle\langle\xi\rangle^{2-2s}} d\xi\, d\xi_1 \right)^{\frac{1}{2}} \lesssim M^{\frac{1}{4}-s},$$

provided that $s \in (\frac{1}{2}, 1)$.
Case 9: $|\xi_1| \sim |\xi_2| \sim |\xi| \gg |\xi_3|$. Then $\langle \Phi(\bar\xi)\rangle \sim \langle \xi_1\rangle^2$ and

$$m_1(\bar\xi) \sim \frac{1}{\langle \Phi(\bar\xi)\rangle^{\frac{1}{2}}} \cdot \frac{1}{\langle\xi_1\rangle^s\langle\xi_3\rangle^s} \lesssim M^{-\frac{1}{2}} \frac{1}{\langle\xi_1\rangle^s\langle\xi_3\rangle^s}$$

which is square integrable on $(\mathbb{R}^2, d\xi_1 d\xi_3)$ since $s > \frac{1}{2}$.
Case 10: $|\xi_1| \sim |\xi_3| \sim |\xi| \gg |\xi_2|$. Then $\langle \Phi(\bar\xi)\rangle \sim \langle \xi_1\rangle^2$ and

$$m_1(\bar\xi) \sim \frac{|\xi_2|}{\langle\xi_1\rangle^{2+s}\langle\xi_2\rangle^s} \lesssim \frac{\langle\xi_2\rangle^{1-s}}{\langle\xi_1\rangle^{2+s}} \lesssim \frac{1}{\langle\xi_1\rangle^{1+2s}} \sim \frac{1}{\langle \Phi(\bar\xi)\rangle^{\frac{1}{2}}} \cdot \frac{1}{\langle\xi_1\rangle^s\langle\xi_3\rangle^s} \lesssim M^{-\frac{1}{2}} \frac{1}{\langle\xi_1\rangle^s\langle\xi_3\rangle^s}$$

which is square integrable on $(\mathbb{R}^2, d\xi_1 d\xi_3)$ since $s > \frac{1}{2}$.
Case 11: $|\xi_2| \sim |\xi_3| \sim |\xi| \gg |\xi_1|$. Then $\langle \Phi(\bar\xi)\rangle \sim \langle \xi_2\rangle^2$ and

$$m_1(\bar\xi) \sim \frac{|\xi_1|^{1-s}}{\langle\xi_2\rangle^{2+s}} \lesssim \frac{1}{\langle\xi_2\rangle^{1+2s}} \sim \frac{1}{\langle \Phi(\bar\xi)\rangle^{\frac{1}{2}}} \cdot \frac{1}{\langle\xi_2\rangle^s\langle\xi_3\rangle^s} \lesssim M^{-\frac{1}{2}} \frac{1}{\langle\xi_2\rangle^s\langle\xi_3\rangle^s}$$

which is square integrable on $(\mathbb{R}^2, d\xi_2 d\xi_3)$ since $s > \frac{1}{2}$.
Case 12: $|\xi_1| \sim |\xi_2| \sim |\xi_3| \sim |\xi|$. We have

$$\begin{aligned} m_1(\bar\xi) &\sim \frac{|\xi_1|}{\langle \Phi(\bar\xi)\rangle} \cdot \frac{1}{\langle\xi_1\rangle^{2s}} \lesssim \frac{1}{\langle \Phi(\bar\xi)\rangle} \cdot \frac{1}{\langle\xi_1\rangle^{2s-1}} \\ &\sim \frac{1}{\langle \Phi(\bar\xi)\rangle^{\gamma}} \cdot \frac{1}{\langle\xi_2-\xi_1\rangle^{2-2s}\langle\xi_1\rangle^{s-\frac{1}{2}}} \cdot \frac{1}{\langle\xi_2-\xi_3\rangle^{2-2s}\langle\xi_3\rangle^{s-\frac{1}{2}}} \end{aligned}$$

and by using (4.30), we deduce

$$\mathcal{M}_2^1 \lesssim M^{-\gamma} \sup_{\xi_2\in\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{\langle\xi_2-\xi_1\rangle^{4-4s}\langle\xi_1\rangle^{2s-1}} d\xi_1 \int_{\mathbb{R}} \frac{1}{\langle\xi_2-\xi_3\rangle^{4-4s}\langle\xi_3\rangle^{2s-1}} d\xi_3 \right)^{\frac{1}{2}} \lesssim M^{-\gamma}$$

provided that $\frac{1}{2} \le s < 1$.

Thus, this finishes the proof for $j = 1$. Notice that the case $j = 3$ is symmetric to the case $j = 1$. It remains to discuss the case $j = 2$. In this case, by the symmetry of m_2 with respect to ξ_1, ξ_3, we may assume without loss of generality that $|\xi_1| \ge |\xi_3|$. If $\langle\xi_2\rangle \lesssim \langle\xi_1\rangle$, then it is easy to check that $m_2(\bar\xi) \lesssim m_1(\bar\xi)$ and thus (4.48) for $j = 2$ follows from (4.48) for $j = 1$.

Now, let us assume that $j = 2$ and that $\langle\xi_2\rangle \gg \langle\xi_1\rangle$. In fact, in this case, we have

$\langle \xi \rangle \sim \langle \xi_2 \rangle \gg \langle \xi_1 \rangle \geq \langle \xi_3 \rangle$ which implies $\langle \Phi(\bar{\xi}) \rangle \sim \langle \xi_2 \rangle^2$ and

$$m_2(\bar{\xi}) \sim \frac{1}{\langle \Phi(\bar{\xi}) \rangle^{\frac{1}{2}}} \cdot \frac{1}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s} \lesssim \frac{M^{-\frac{1}{2}}}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s}$$

which is square integrable on $(\mathbb{R}^2, d\xi_2 d\xi_3)$ since $s < 1$. This concludes the proof of Lemma 4.15 for all three possible values of j.

□

Lemma 4.16 (The estimate of $\mathcal{T}^{\mathrm{w}}_{|\Phi|>M}$ in the $H^s(\mathbb{R})$-norm)**.** *Let $s > \frac{1}{2}$. Then, there exists a finite constant $C = C(s) > 0$ such that*

$$\|\mathcal{T}^{\mathrm{w}}_{|\Phi|>M}(v_1, v_2, v_3)\|_{H^s(\mathbb{R})} \leq CM^{-\frac{1}{2}} \prod_{j=1}^{3} \|v_j\|_{H^s(\mathbb{R})},$$

for any $M \geq 1$.

Proof. It is an immediate consequence of Lemma 4.2 taking into account that the multiplier of the operator $\mathcal{T}^{\mathrm{w}}_{|\Phi|>M}$ has an additional $\frac{1}{2}$-power of $\langle \Phi(\bar{\xi}) \rangle$ in the denominator as compared to the multiplier of $\mathcal{T}_\Phi$ and that in the domain of integration we have $|\Phi(\bar{\xi})| > M$.

This finishes the proof for $j = 1$. Notice that the case $j = 3$ is symmetric to the case $j = 1$. It remains to discuss the case $j = 2$. In this case, by the symmetry of m_2 with respect to ξ_1, ξ_3, we may assume without loss of generality that $|\xi_1| \geq |\xi_3|$. If $\langle \xi_2 \rangle \lesssim \langle \xi_1 \rangle$, then it is easy to check that $m_2(\bar{\xi}) \lesssim m_1(\bar{\xi})$ and thus (4.48) for $j = 2$ follows from (4.48) for $j = 1$.

Now, let us assume that $j = 2$ and that $\langle \xi_2 \rangle \gg \langle \xi_1 \rangle$. In fact, in this case, we have $\langle \xi \rangle \sim \langle \xi_2 \rangle \gg \langle \xi_1 \rangle \geq \langle \xi_3 \rangle$ which implies $\langle \Phi(\bar{\xi}) \rangle \sim \langle \xi_2 \rangle^2$ and

$$m_2(\bar{\xi}) \sim \frac{1}{\langle \Phi(\bar{\xi}) \rangle^{\frac{1}{2}}} \cdot \frac{1}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s} \lesssim \frac{M^{-\frac{1}{2}}}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s}$$

which is square integrable on $(\mathbb{R}^2, d\xi_1 d\xi_3)$ for $s > \frac{1}{2}$. This concludes the proof of Lemma 4.15 for all three possible values of j.

□

Lemma 4.17 (The estimate of $\mathcal{T}^{\mathrm{w}}_{|\Phi|>M}$ in the $H^s(\mathbb{R})$-norm)**.** *Let $s > \frac{1}{2}$. Then, there exists a finite constant $C = C(s) > 0$ such that*

$$\|\mathcal{T}^{\mathrm{w}}_{|\Phi|>M}(v_1, v_2, v_3)\|_{H^s(\mathbb{R})} \leq CM^{-\frac{1}{2}} \prod_{j=1}^{3} \|v_j\|_{H^s(\mathbb{R})},$$

for any $M \geq 1$.

Proof. It is an immediate consequence of Lemma 4.2 taking into account that the multiplier of the operator $\mathcal{T}^{\mathrm{w}}_{|\Phi|>M}$ has an additional $\frac{1}{2}$-power of $\langle\Phi(\bar{\xi})\rangle$ in the denominator as compared to the multiplier of $\mathcal{T}_{\Phi}$ and that in the domain of integration we have $|\Phi(\bar{\xi})| > M$.

□

Definition 4.18. Let $J \geq 1$ and $T \in \mathfrak{T}(J)$. We define the $(2J+1)$-linear map $\mathfrak{S}^{\mathrm{w}}(T;\,\cdot\,)$ on space-time functions $v_j \in C(I; H^s(\mathbb{R}))$ $(1 \leq j \leq 2J+1 = |T^{\infty}|)$ by the following rules.

(i) Replace the jth terminal node of T by v_j, for all $j \in \{1, \ldots, 2J+1\}$.

(ii) For $j = J, J-1, \ldots, 1$, replace the jth root node $r^{(j)}$ by the trilinear operator $\mathcal{T}^{\mathrm{w}}_{|\Phi|>b_jN/2}$ whose arguments are given by the functions associated with its three children.

We have the following immediate consequence of Lemmata 4.15 and 4.17.

Corollary 4.19. *Let $s > \frac{1}{2}$, $\theta = \theta(s) = \min\{2s-1, \frac{1}{2}\}$, $J \geq 1$, and $T \in \mathfrak{T}(J)$. Then, for any $1 \leq j \leq 2J+1$ we have*

$$\left\|\mathfrak{S}^{\mathrm{w}}(T; v_1, \ldots, v_{2J+1})\right\|_{H^s_x(\mathbb{R})} \leq \frac{(2^{\theta}C)^J}{b_1^{\theta} b_2^{\theta} \cdots b_{J-1}^{\theta}} N^{-\theta J} \|v_j\|_{H^{s-1}_x(\mathbb{R})} \prod_{\substack{k=1\\k\neq j}}^{2J+1} \|v_k\|_{H^s_x(\mathbb{R})}\,,$$

where C is the maximum between the two constants given by Lemmata 4.15 and 4.17.

Proof. We apply iteratively Lemma 4.15 or Lemma 4.17. Let a_j denote the jth terminal node of T. Since T is a tree of generation J, it has J many root nodes $r^{(1)}, r^{(2)}, \ldots, r^{(J)}$, where $r^{(j)} \in \pi_j(T)$, $1 \leq j \leq J$. Let $1 \leq k \leq J$ such that the root node $r^{(k)} \in \pi_k(T)$ is the parent of the jth terminal node a_j. We recall (see Remark A.6) that there exists the shortest path $P(r^{(1)}, r^{(k)}) = r^{(k_1)}, r^{(k_2)}, \ldots, r^{(k_\ell)}$ of root nodes from $r^{(1)} =: r^{(k_1)}$ to $r^{(k)} =: r^{(k_\ell)}$, $1 = k_1 < k_2 < \ldots < k_\ell = k$.

We prove the desired estimate by moving top-down on T with a chronicle $\{T_j\}_{j=1}^J$. Starting with $j = 1$, if a_j is a child of $r^{(1)}$, then we just apply Lemma 4.15. Otherwise, T_1 has one child (and only one) that belongs to $P(r^{(1)}, r^{(k)})$ which is $r^{(k_2)} \in \pi_{k_2}(T)$, $1 < k_2 \leq k$. So we use Lemma 4.15, placing the subtree with root node $r^{(k_2)}$ in the $H^{s-1}(\mathbb{R})$-norm and the other two subtrees (possibly, it can be just one node) in the $H^s(\mathbb{R})$-norm. In a similar manner, we continue to move down the path $r^{(k_2)}, \ldots, r^{(k_{\ell-1})}, r^{(k)}$ and each time we apply Lemma 4.15 analogously. For any subtree of T whose root node does not belong to $\{r^{(1)}, r^{(k_2)}, \ldots, r^{(k_{\ell-1})}, r^{(k)}\}$, we use Lemma 4.17 in chronological order. Notice that (modulo the constant C), the coefficient provided by the latter lemma is smaller than the one provided by the former. In the worst cases scenario (i.e. the tree is "linear" so that $k = J$, and $P(r^{(1)}, r^{(k)}) = r^{(1)}, r^{(2)}, \ldots, r^{(J)}$), we only apply

Lemma 4.15 to pick up the coefficient

$$C^J\left(\prod_{j=1}^{J-1}\left(\frac{b_jN}{2}\right)^{-\theta}\right) = (2^\theta C)^J\left(\prod_{j=1}^{J-1} b_j^{-\theta}\right)N^{-\theta J}\,,$$

with b_j given by (4.33).

□

4.3.1 Convergence to zero of the remainder term

Here, we argue that for fixed $N > 1$, the remainder term $\mathcal{T}_{\mathcal{T}}^{(J+1)}(v)$ of (4.23) converges to zero in the $H^{s-1}(\mathbb{R})$-norm as $J \to \infty$.

Lemma 4.20. *Let $s > \frac{1}{2}$ and $\theta = \theta(s) = \min\{2s-1, \frac{1}{2}\}$. Then, for $\mathcal{T}_{\mathcal{T}}^{(J+1)}(v)$ given by (4.22), we have*

$$\|\mathcal{T}_{\mathcal{T}}^{(J+1)}(v)\|_{H^{s-1}(\mathbb{R})} \lesssim N^{-\theta J}\|v\|_{H^s(\mathbb{R})}^{2J+3}\,. \tag{4.51}$$

Proof. The formula (4.22) for $\mathcal{T}_{\mathcal{T}}^{(J+1)}(v)$ was obtained by replacing $\partial_t v$ with $\mathcal{T}(v)$ in $\mathcal{T}^{(J+1)}(v)$. On the other hand, the same formula (4.22) can also obtained by replacing one v in $\mathcal{T}_0^{(J)}$ with $\mathcal{T}(v)$. More precisely, we can write

$$\begin{aligned}\mathcal{T}_{\mathcal{T}}^{(J+1)}(v) &= \sum_{T\in\mathfrak{T}(J)}\sum_{j=1}^{2J+1}\int_{\boldsymbol{\xi}\in\Xi_\xi(T)}\mathbf{1}_{F_J}\left(\prod_{j=1}^{J}\frac{e^{i\mu_J t}\xi_2^{(j)}}{\widetilde{\mu}_j}\right)\left(\mathcal{T}(v)(\xi_{a_k})\prod_{\substack{a\in T^\infty\\ a\neq a_k}} v(\xi_a)\right)\\ &=: \sum_{T\in\mathfrak{T}(J)}\sum_{k=1}^{2J+1}\mathcal{T}_0^{(J+1)}(T, a_k; \mathbf{v}_k)\,,\end{aligned} \tag{4.52}$$

where a_j denotes the kth terminal node of T, and for simplicity, we put

$$\mathbf{v}_k = (v, \dots, v, \underbrace{\mathcal{T}(v)}_{k\text{th spot}}, v, \dots, v)\,.$$

We then have

$$\|\mathcal{T}_{\mathcal{T}}^{(J+1)}(v)\|_{H_x^{s-1}(\mathbb{R})} \le c_J(2J+1)\sup_{T\in\mathfrak{T}(J)}\sup_{1\le k\le 2J+1}\|\mathcal{T}_0^{(J+1)}(T, a_k; \mathbf{v}_k)\|_{H^{s-1}}\,. \tag{4.53}$$

Proceeding as in the proof of Lemma 4.11, we have $\frac{1}{2}|\widetilde{\mu}_j| < |\mu_j| < 2|\widetilde{\mu}_j|$ for $j = 1, \dots, J$ (due to the of integration being restricted to F_J) and $|\mu_j| > \frac{1}{2}|\widetilde{\mu}_j| > \frac{b_{j-1}N}{2}$. Therefore, we have

$$\begin{aligned}\mathcal{T}_0^{(J+1)}(T, a_j; \mathbf{v}_j) &\leq \int_{\boldsymbol{\xi}\in\Xi_\xi(T)} \mathbf{1}_{F_J}\bigg(\prod_{j=1}^{J}\frac{|\xi_2^{(j)}|}{|\widetilde{\mu}_j|}\bigg)\bigg(\mathcal{T}(v)(\xi_{a_k})\prod_{\substack{a\in T^\infty\\ a\neq a_k}} v(\xi_a)\bigg)\\ &\leq 2^J \int_{\boldsymbol{\xi}\in\Xi_\xi(T)} \mathbf{1}_{F_J}\bigg(\prod_{j=1}^{J}\frac{|\xi_2^{(j)}|}{|\mu_j|}\bigg)\bigg(\mathcal{T}(v)(\xi_{a_k})\prod_{\substack{a\in T^\infty\\ a\neq a_k}} v(\xi_a)\bigg)\\ &\leq (2\sqrt{2})^J \mathfrak{S}^{\mathrm{w}}(T; \mathbf{v}_k)\end{aligned}$$

With Corollary 4.19 and $\theta = \min\{2s-1, \frac{1}{2}\}$, we get

$$\|\mathcal{T}_0^{(J+1)}(T, a_k; \mathbf{v}_k)\|_{H_x^{s-1}(\mathbb{R})} \leq (4C)^J \bigg(\prod_{j=1}^{J} b_j\bigg)^{-\theta} N^{-\theta J}\|\mathcal{T}(v)\|_{H_x^{s-1}(\mathbb{R})}\|v\|_{H_x^s(\mathbb{R})}^{2J} \tag{4.54}$$

for each $T \in \mathfrak{T}(J)$ and $1 \leq j \leq 2J+1$. Then, by (4.53) and Lemma 4.13 we get

$$\|\mathcal{T}_{\mathcal{T}}^{(J+1)}(v)\|_{H_x^{s-1}(\mathbb{R})} \leq \frac{c_J(2J+1)(4C)^J}{b_1^\theta \cdots b_{J-1}^\theta} N^{-\theta J}\|v\|_{H_x^s(\mathbb{R})}^{2J+3}$$

The desired estimate (4.51) follows by taking into account Remark 4.9.

□

4.4 Justification of the normal form reductions for rough solutions

In each step of the infinite iteration in Section 4.1 we performed normal form reductions (NFR) which relied on two formal operations which obviously hold if v is assumed to be a smooth solution to (4.3). Namely, (i) we applied the product rule when distributing the time derivative over products of several factors of v (see e.g. (4.57) below), and (ii) we switched the time derivative with integrals in spatial frequencies (see e.g. (4.58) below). In this section, we justify these operations for a rough solution v to (4.3).

Let $s > \frac{1}{2}$, $\theta = \theta(s) = \min\{2s-1, \frac{1}{2}\}$, and let I be an interval containing $t = 0$. Suppose that $v \in C(I; H^s(\mathbb{R}))$ is a solution to (4.3), namely it satisfies (in the sense of distributions) the Duhamel formula

$$v(t) = v_0 + \int_0^t \mathcal{Q}(v)(t')dt' + \int_0^t \mathcal{T}(v)(t')dt', \tag{4.55}$$

with $\mathcal{Q}$, $\mathcal{T}$ as in (4.4), (4.5), respectively. By Lemma 4.14, we have $v \in C_t^1\big(I; H_x^{s-1}(\mathbb{R})\big)$. With p, $q \in (1, \infty)$ such that $\frac{2}{p} + \frac{1}{q} = 1$ and $\frac{1}{2} - \frac{1}{q} \leq s - 1$, by Hölder inequality and

Sobolev embedding, we also have that

$$\begin{aligned}\|v_1(\partial_x\overline{v_2})v_3\|_{L^1_x(\mathbb{R})} &\le \|v_1\|_{L^p_x(\mathbb{R})}\|\partial_x\overline{v_2}\|_{L^q_x(\mathbb{R})}\|v_3\|_{L^p_x(\mathbb{R})}\\ &\lesssim \|v_1\|_{H^s_x(\mathbb{R})}\|v_2\|_{H^s_x(\mathbb{R})}\|v_3\|_{H^s_x(\mathbb{R})}\,.\end{aligned}$$

Note that the condition $\frac{1}{2}-\frac{1}{p}\le s$ is automatically satisfied. Therefore, we have

$$\|\mathcal{T}(v)\|_{H^{s-1}_x(\mathbb{R})}+\|\mathcal{T}(v)\|_{L^1_x(\mathbb{R})}\lesssim \|v\|^3_{H^s_x(\mathbb{R})}\,.$$

Note that all of the above estimates hold uniformly in $t\in I$. For the quintic term in (4.55), we immediatelly have

$$\|\mathcal{Q}(v)\|_{H^s_x(\mathbb{R})}+\|\mathcal{Q}(v)\|_{L^1_x(\mathbb{R})}\lesssim \|v\|^5_{H^s_x(\mathbb{R})}.$$

Moreover, by the Riemann-Lebesgue lemma, it follows that

$$\widehat{\mathcal{Q}(v)}\,,\,\widehat{\mathcal{T}(v)}\in C_t(I;C_\xi(\mathbb{R}))$$

with

$$\begin{aligned}\|\widehat{\mathcal{T}(v)}\|_{L^\infty_\xi(\mathbb{R})}&\lesssim \|v\|^3_{H^s_x(\mathbb{R})}\,,\\ \|\widehat{\mathcal{Q}(v)}\|_{L^\infty_\xi(\mathbb{R})}&\lesssim \|v\|^5_{H^s_x(\mathbb{R})}\,.\end{aligned}$$

By taking the Fourier transform of (4.55), by Fubini's theorem, we get

$$\widehat{v}(t,\xi)=\widehat{v_0}(\xi)+\int_0^t\widehat{\mathcal{Q}(v)}(t',\xi)dt'+\int_0^t\widehat{\mathcal{T}(v)}(t',\xi)dt'.$$

and by taking time derivative for fixed $\xi\in\mathbb{R}$, we have

$$\partial_t\widehat{v}(t,\xi)=\widehat{\mathcal{Q}(v)}(t,\xi)+\widehat{\mathcal{T}(v)}(t,\xi),$$

for each $(t,\xi)\in I\times\mathbb{R}$. It follows that

$$\widehat{v}\in C^1_t\big(I;C_\xi(\mathbb{R})\big). \tag{4.56}$$

4.4.1 Justification of the first step of NFR

Here, we carefully justify that v is also a solution to (4.17), namely that the Duhamel formula

$$\begin{aligned}v(t)=v_0&+\int_0^t\mathcal{Q}(v)(t')dt'+\int_0^t\mathcal{T}^{(1)}_{\mathcal{T},1}(v)(t')dt'+\mathcal{T}^{(2)}_0(v)(t)-\mathcal{T}^{(2)}_0(v)(0)\\ &+\int_0^t\mathcal{T}^{(2)}_{\mathcal{Q}}(v)(t')dt'+\int_0^t\mathcal{T}^{(2)}_{\mathcal{T}}(v)(t')dt'\end{aligned}$$

is satisfied in the sense of distributions. Due to (4.56), it is immediate that the application of the product rule

$$\partial_t\Big(\widehat{v}(t,\xi_1)\widehat{\partial_x v}(t,\xi_2)\widehat{v}(t,\xi_3)\Big) = \big(\partial_t\widehat{v}(t,\xi_1)\big)\widehat{\partial_x v}(t,\xi_2)\widehat{v}(t,\xi_3) + \widehat{v}(t,\xi_1)\big(\partial_t\widehat{\partial_x v}(t,\xi_2)\big)\widehat{v}(t,\xi_3) + \widehat{v}(t,\xi_1)\widehat{\partial_x v}(t,\xi_2)\big(\partial_t\widehat{v}(t,\xi_3)\big) \tag{4.57}$$

is justified for all $t \in I$ and all $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$.

Next, we would like to justify the following:

$$\partial_t\left[\int_{\mathbb{R}^2} f(t,\xi,\xi_1,\xi_2)d\xi_1 d\xi_2\right] = \int_{\mathbb{R}^2} \partial_t f(t,\xi,\xi_1,\xi_2)d\xi_1 d\xi_2\,, \tag{4.58}$$

where the function $f : I \times \mathbb{R}^3 \to \mathbb{C}$ is given by

$$f(t,\xi,\xi_1,\xi_2) = \mathbf{1}_{C_0}\frac{e^{i\Phi(\bar{\xi})t}}{i\Phi(\bar{\xi})}\widehat{v}(t,\xi_1)\overline{\widehat{\partial_x v}(t,\xi_2)}\widehat{v}(t,\xi-\xi_1+\xi_2)\,,$$

i.e. the integrand for $\mathcal{T}_0^{(2)}(v)$ – see (4.12). We have that

$$\begin{aligned}
\partial_t f(t,\xi,\xi_1,\xi_2) &= \mathbf{1}_{C_0}e^{i\Phi(\bar{\xi})t}\widehat{v}(t,\xi_1)\overline{\widehat{\partial_x v}(t,\xi_2)}\widehat{v}(t,\xi-\xi_1+\xi_2) \\
&\quad + \mathbf{1}_{C_0}\frac{e^{i\Phi(\bar{\xi})t}}{i\Phi(\bar{\xi})}\big(\partial_t\widehat{v}(t,\xi_1)\big)\overline{\widehat{\partial_x v}(t,\xi_2)}\widehat{v}(t,\xi-\xi_1+\xi_2) \\
&\quad + \mathbf{1}_{C_0}\frac{e^{i\Phi(\bar{\xi})t}}{i\Phi(\bar{\xi})}\widehat{v}(t,\xi_1)\big(\overline{\partial_t\widehat{\partial_x v}(t,\xi_2)}\big)\widehat{v}(t,\xi-\xi_1+\xi_2) \\
&\quad + \mathbf{1}_{C_0}\frac{e^{i\Phi(\bar{\xi})t}}{i\Phi(\bar{\xi})}\widehat{v}(t,\xi_1)\overline{\widehat{\partial_x v}(t,\xi_2)}\big(\partial_t\widehat{v}(t,\xi-\xi_1+\xi_2)\big) \\
&=: g_1(t,\xi,\xi_1,\xi_2) + g_2(t,\xi,\xi_1,\xi_2) + g_3(t,\xi,\xi_1,\xi_2) + g_4(t,\xi,\xi_1,\xi_2)
\end{aligned}$$

By omitting any complex constants of modulus one, we can write

$$\begin{aligned}
\int_{\mathbb{R}^2} f(t,\xi,\xi_1,\xi_2)d\xi_1 d\xi_2 &= \mathcal{F}\big[\mathcal{T}_0^{(2)}(v)\big](t,\xi) \\
\int_{\mathbb{R}^2} g_1(t,\xi,\xi_1,\xi_2)d\xi_1 d\xi_2 &= \mathcal{F}\big[\mathcal{T}_2(v)\big](t,\xi) \\
\int_{\mathbb{R}^2} g_2(t,\xi,\xi_1,\xi_2)d\xi_1 d\xi_2 &= \mathcal{F}\big[\mathcal{T}_0^{(2)}(\partial_t v, v, v)\big](t,\xi) \\
\int_{\mathbb{R}^2} g_3(t,\xi,\xi_1,\xi_2)d\xi_1 d\xi_2 &= \mathcal{F}\big[\mathcal{T}_0^{(2)}(v, \partial_t v, v)\big](t,\xi) \\
\int_{\mathbb{R}^2} g_4(t,\xi,\xi_1,\xi_2)d\xi_1 d\xi_2 &= \mathcal{F}\big[\mathcal{T}_0^{(2)}(v, v, \partial_t v)\big](t,\xi)\,,
\end{aligned}$$

where $\mathcal{T}_0^{(2)}(v)$, $\mathcal{T}_2(v)$ are given by (4.12), respectively. Furthermore, we set

$$F := \mathcal{T}_0^{(2)}(v)\,,$$
$$G_1 := \mathcal{T}_2(v)\,,\ G_2 := \mathcal{T}_0^{(2)}(\partial_t v, v, v)\,,\ G_3 := \mathcal{T}_0^{(2)}(v, \partial_t v, v)\,,\ G_4 := \mathcal{T}_0^{(2)}(v, v, \partial_t v)\,,$$

$g := g_1 + g_2 + g_3 + g_4$, and $G := G_1 + G_2 + G_3 + G_4$. Thus (4.58) follows once we show that $\partial_t F = G$ holds in the sense of distributions.

By Lemma 4.15, we deduce[4] that $F \in C(I; H^{s-1}(\mathbb{R}))$ with

$$\|F(t)\|_{H_x^{s-1}} \lesssim N^{-\theta}\|v\|_{H_x^s}^3\,. \tag{4.59}$$

Similarly, we have that $G \in C(I; H^{s-1}(\mathbb{R}))$ since by Lemma 4.13 and Lemma 4.15, we have

$$\begin{aligned}\|G(t)\|_{H_x^{s-1}} &\le \big\|\mathcal{T}_2(v)\big\|_{H_x^{s-1}} + \big\|\mathcal{T}^{\mathrm{w}}_{|\Phi|>N}(\partial_t v, v, v)\big\|_{H_x^{s-1}} + \big\|\mathcal{T}^{\mathrm{w}}_{|\Phi|>N}(v, \partial_t v, v)\big\|_{H_x^{s-1}}\\ &\qquad + \big\|\mathcal{T}^{\mathrm{w}}_{|\Phi|>N}(v, v, \partial_t v)\big\|_{H_x^{s-1}}\\ &\lesssim \|v\|_{H_x^s}^3 + \|\partial_t v\|_{H_x^{s-1}}\|v\|_{H_x^s}^2\\ &\lesssim \|v\|_{H_x^s}^3 + \|v\|_{H_x^s}^5 + \|v\|_{H_x^s}^7\,,\end{aligned} \tag{4.60}$$

where in the last step we applied Lemma 4.14.

Now fix $t \in I$ and let $\varphi \in \mathcal{S}(\mathbb{R})$. By the Plancherel formula, we have

$$\int_{\mathbb{R}} F(t,x)\varphi(x)dx = \int_{\mathbb{R}^3} f(t,\xi,\xi_1,\xi_2)\widehat{\varphi}(\xi)d\xi_1 d\xi_2 d\xi\,,$$
$$\int_{\mathbb{R}} G(t,x)\varphi(x)dx = \int_{\mathbb{R}^3} g(t,\xi,\xi_1,\xi_2)\widehat{\varphi}(\xi)d\xi_1 d\xi_2 d\xi\,.$$

By appealing to the Fourier lattice property of the Sobolev spaces H^{s-1}, H^{1-s}, to the Riemann-Lebesgue lemma and by using (4.60), we have

$$|g(t,\xi,\xi_1,\xi_2)\widehat{\varphi}(\xi)| \lesssim \|G\|_{H_x^{s-1}}\big\|\mathcal{F}^{-1}\big[|\widehat{\varphi}|^{\frac{1}{2}}\big]\big\|_{H_x^{1-s}}|\widehat{\varphi}(\xi)|^{\frac{1}{2}} \lesssim_{\|v\|_{C(I;H^s(\mathbb{R}))}} |\widehat{\varphi}(\xi)|^{\frac{1}{2}}\,.$$

and thus the dominated convergence theorem implies:

$$\partial_t \int_{\mathbb{R}} F(t,x)\varphi(x)dx = \int_{\mathbb{R}} G(t,x)\varphi(x)dx\,.$$

4.4.2 Justification of the Jth step of NFR

In justifying the first step of NFR, the main ingredients[5] are the estimates (4.59) and (4.60). For a generic step J, we briefly show how to derive the corresponding estimates.

[4]For the continuity in time of F, one uses the multilinear version of the estimate provided by Lemma 4.15.

[5]Whenever we apply the product rule, we appeal to (4.56).

To this end, fix $T \in \mathfrak{T}(J)$ and note that for (4.19), we have used the following:

$$\partial_t \bigg[\int_{\boldsymbol{\xi} \in \Xi_\xi(T)} f(t, \xi, \boldsymbol{\xi}) \bigg] = \int_{\boldsymbol{\xi} \in \Xi_\xi(T)} \partial_t f(t, \xi, \boldsymbol{\xi}) \,, \tag{4.61}$$

where the function $f : I \times \Xi(T) \to \mathbb{C}$ is given by

$$f(t, \xi, \boldsymbol{\xi}) = \mathbf{1}_{F_J} \bigg(\prod_{j=1}^{J} \frac{e^{i\mu_j t \xi_2^{(j)}}}{\widetilde{\mu}_j} \bigg) \bigg(\prod_{a \in T^\infty} v(\xi_a) \bigg) \,,$$

i.e. the integrand for $\mathcal{T}_0^{(J+1)}(v)$ – see (4.20). Note that

$$\begin{aligned} \int_{\boldsymbol{\xi} \in \Xi_\xi(T)} f(t, \xi, \boldsymbol{\xi}) &= \mathcal{F}\big[\mathcal{T}_0^{(J+1)}(T; v)\big](t, \xi) =: \mathcal{F}\big[F\big](t, \xi) \,, \\ \int_{\boldsymbol{\xi} \in \Xi_\xi(T)} \partial_t f(t, \xi, \boldsymbol{\xi}) &= \mathcal{F}\bigg[\mathcal{T}_{\mathcal{T},2}^{(J)}(T; v) + \sum_{k=1}^{2J+1} \mathcal{T}_0^{(J+1)}(T, a_k; \widetilde{\mathbf{v}}_k)\bigg](t, \xi) =: \mathcal{F}\big[G\big](t, \xi) \,, \end{aligned}$$

where $\mathcal{T}_0^{(J+1)}(T, a_k; \widetilde{\mathbf{v}}_k)$ in the summation above is defined by replacing $\mathbf{v}_k$ in (4.52) by

$$\widetilde{\mathbf{v}}_k = (v, \ldots, v, \underbrace{\partial_t v}_{k\text{th spot}}, v, \ldots, v),$$

and a_k is the kth terminal node of $T \in \mathfrak{T}(J)$.

Similarly to (4.54) in the proof of Lemma with Corollaries 4.14, we have

$$\begin{aligned} \big\|\mathcal{T}_0^{(J+1)}(T; v)\|_{H_x^{s-1}(\mathbb{R})} &\lesssim N^{-\theta J} \|v\|_{H_x^s(\mathbb{R})}^{2J+1} \,, \\ \big\|\mathcal{T}_0^{(J+1)}(T, a_k; \widetilde{\mathbf{v}}_k)\|_{H_x^{s-1}(\mathbb{R})} &\lesssim N^{-\theta J} \|v\|_{H_x^s(\mathbb{R})}^{2J+3} \big(1 + \|v\|_{H_x^s(\mathbb{R})}^2\big) \,, \quad k = 1, \ldots, 2J+1. \end{aligned}$$

Also, similarly to the proof of Lemma 4.10, with Corollary 4.19 and Lemma 4.13, we get

$$\big\|\mathcal{T}_{\mathcal{T},2}^{(J)}(T; v)\|_{H_x^{s-1}(\mathbb{R})} \lesssim N^{-\theta(J-1)} \|v\|_{H_x^s(\mathbb{R})}^{2J+1} \,.$$

It follows that F, $G \in C(I; H^{s-1}(\mathbb{R}))$ with

$$\|F\|_{H_x^{s-1}(\mathbb{R})} \lesssim \|v\|_{H_x^s(\mathbb{R})}^{2J+1} \,, \tag{4.62}$$

$$\|G\|_{H_x^{s-1}(\mathbb{R})} \lesssim \|v\|_{H_x^s(\mathbb{R})}^{2J+1} + \|v\|_{H_x^s(\mathbb{R})}^{2J+3} + \|v\|_{H_x^s(\mathbb{R})}^{2J+5} \,. \tag{4.63}$$

Similarly to the previous subsection, by appealing to the dominated convergence theorem and (4.62), (4.63) one justifies (4.61).

Together with Lemma 4.4.2, we conclude that the Duhamel formula of the equation (4.26) is satisfied in the sense of distributions, provided that $v \in C(I; H_x^s(\mathbb{R}))$ is a solution to (4.3).

4.5 Proof of Theorem 1.3

First, we summarilly go over the fixed point argument for (4.26) with prescribed initial data $v(0) = v_0 \in H^s(\mathbb{R})$, $s > \frac{1}{2}$. Integrating the limit equation (4.26) in time, we obtain the folllowing Duhamel formulation:

$$v(t) = v_0 + \int_0^t \mathcal{Q}(v)(t')dt' + \sum_{j=2}^{\infty} \Big(\mathcal{T}_0^{(j)}(v)(t) - \mathcal{T}_0^{(j)}(v)(0) \Big) + \sum_{j=2}^{\infty} \int_0^t \mathcal{T}_{\mathcal{Q}}^{(j)}(v)(t')dt' + \sum_{j=1}^{\infty} \int_0^t \mathcal{T}_{\mathcal{T},1}^{(j)}(v)(t')dt' . \tag{4.64}$$

Let us denote the right-hand side of (4.64) by $\Gamma(v)$, and for simplicity we write $C_T H^s$ instead of $C([-T,T]; H^s(\mathbb{R}))$.

Having the estimates of Section 4.2, one can show that Γ is a contraction on the ball $\mathcal{B}_T := \{v \in C_T H^s : \|v\|_{C_T H^s} \leq 2\|v_0\|_{H^s}\}$, provided that $T > 0$ and $N > 1$ are appropriately chosen. Indeed, we set $R := 2\|v_0\|_{H^s}$, and thus by Lemmata 4.2, 4.11, 4.12, and 4.10, we get

$$\begin{aligned}
\|\Gamma(v)\|_{C_T H^s} &\leq \frac{1}{2}R + TR^5 + c\sum_{j=2}^{\infty} N^{-\frac{1}{2}(j-1)} R^{2(j-1)+1} + cT\sum_{j=2}^{\infty} N^{-\frac{1}{2}(j-1)} R^{2(j-1)+5} \\
&\quad + cT\sum_{j=1}^{\infty} N^{-\frac{1}{2}(j-2)} R^{2(j-1)+3} \\
&\leq \frac{1}{2}R + TR^5 + c\frac{N^{-\frac{1}{2}}R^3}{1 - N^{-\frac{1}{2}}R^2} + cT\frac{N^{-\frac{1}{2}}R^7}{1 - N^{-\frac{1}{2}}R^2} \\
&\quad + cTN^{\frac{1}{2}}R^3 + cTR^5 + cT\frac{N^{-\frac{1}{2}}R^7}{1 - N^{-\frac{1}{2}}R^2} \\
&\leq \frac{1}{2}R + (1+c)TR^5 + 2c(1 + 2TR^4)N^{-\frac{1}{2}}R^3 + cTN^{\frac{1}{2}}R^3.
\end{aligned}$$

for some $c = c(s) > 0$, when $N \geq 4R^4$ so that $(1 - N^{-\frac{1}{2}}R^2)^{-1} \leq 2$. First, we choose $T_1 = T_1(R) > 0$ such that $(1+c)T_1R^4 \leq \frac{1}{6}$, then we choose $N = N(R) \geq 1 + 4R^4$ such that $2c(1 + 2T_1R^2)N^{-\frac{1}{2}}R^2 \leq \frac{1}{6}$, and finally we choose $T = \min\big\{T_1, \frac{1}{6}(cN^{\frac{1}{2}}R^2)^{-1}\big\}$.

By possibly choosing smaller T and bigger N and by using the difference estimates of Lemmata 4.11, 4.12, 4.2, and 4.10, the contraction property of Γ follows analogously. Therefore, by the contraction mapping principle, for given $v_0 \in H^s(\mathbb{R})$, there exists a unique $v \in C_T H^s$ satisfying (4.64). Moreover, $\|v\|_{C_T H^s} \lesssim \|v_0\|_{H^s}$.

Now let us consider two solutions $u_1, u_2 \in C_T H^s$ of DNLS. By Lemma 4.1, $w_1, w_2 \in C_T H^s$ and

$$\|u_1 - u_2\|_{C_T H^s} \lesssim \|w_1 - w_2\|_{C_T H^s} = \|v_1 - v_2\|_{C_T H^s} ,$$

where $v_j(t) := S(-t)w_j(t)$, $t \in [-T,T]$, are solutions to (4.3). Then, by the arguments of Section 4.4, v_1, v_2 are solutions of the normal form equation (4.26) derived in Section 4.1. Similarly to the above lines of reasoning, we deduce

$$\|v_1 - v_2\|_{C_T H^s} = \|\Gamma(v_1) - \Gamma(v_2)\|_{C_T H^s} \lesssim \|v_1(0) - v_2(0)\|_{H^s} = \|u_1(0) - u_2(0)\|_{H^s}$$

and thus any two solutions $u_1, u_2 \in C_T H^s$ started from the same initial data must coincide on the time interval $[-T, T]$. By appealing to the time translation symmetry of DNLS, we conclude that any initial data $u_0 \in H^s(\mathbb{R})$ determines a unique solution to DNLS which is continuous in time with values in $H^s(\mathbb{R})$.

4.6 Comments and remarks

For DNLS on the real line, Yin Yin Su Win [52] established its unconditional well-posedness in the energy space, i.e., for $s = 1$. Indeed, by modifying the $X^{s,b}$-multilinear estimates in [44], the author of [52] showed the uniqueness of solutions to DNLS in $X_T^{\frac{1}{2},\frac{1}{2}}$ (here, $X_T^{s,b}$ simply denotes a local in time version of $X^{s,b}$ – see (3.9)). Now, uniqueness of solutions in $X_T^{\frac{1}{2},\frac{1}{2}}$ implies unconditional uniqueness of solutions to DNLS in $H^1(\mathbb{R})$. Indeed, this follows from arguing by interpolation (of $X^{s,b}$-spaces): first, if $u \in C([-T, T]; H^1(\mathbb{R}))$, then clearly $u \in X_T^{1,0} = L^2([-T, T]; H^1(\mathbb{R}))$; second, by the algebra property of $C([-T, T]; H^1(\mathbb{R}))$ we have $\partial_x(|u|^2 u) \in C([-T, T]; L^2(\mathbb{R})) \subset L^2([-T, T]; L^2(\mathbb{R}))$ and thus $u = (i\partial_t + \partial_x^2)^{-1}\big(i\partial_x(|u|^2 u)\big) \in X_T^{0,1}$; third, by interpolation, any solution $u \in C([-T, T]; H^1(\mathbb{R}))$ to (1.1) is contained in $X_T^{\frac{1}{2},\frac{1}{2}}$ and thus it must be unique. This strategy does not work for $s < 1$ because the key trilinear estimate is known to fail in $X^{s,b}$ with $s < \frac{1}{2}$, for any $b \in \mathbb{R}$ (see [44, Proposition 3.3]).

Lastly, regarding the global well-posedness of DNLS on the real line, we have the following corollary to Theorem 1.3:

Corollary 4.21. *Let $s > \frac{1}{2}$, $u_0 \in H^s(\mathbb{R})$ with $M(u_0) < 2\pi$. Then, DNLS is unconditionally globally well-posed in $H^s(\mathbb{R})$.*

This is an immediate consequence of Theorem 1.3 together with the main result of Colliander, Keel, Staffilani, Takaoka, and Tao [10], i.e. the (conditional) global well-posedness of DNLS in $H^s(\mathbb{R})$, $s > \frac{1}{2}$, provided that $M(u_0) < 2\pi$.

Appendix A

Ordered ternary trees and associated multilinear operators

We include here the notation and terminology used in [28, Section 3.1] regarding the cubic NLS equation on the real line.

Definition A.1. Given a partially ordered set T with partial order $\leq$, we say that $b \in T$ with $b \leq a$ and $b \neq a$ is a child of $a \in T$, if $b \leq c \leq a$ implies either $c = a$ or $c = b$. If the latter condition holds, we also say that a is the parent of b.

As in [7, 40], the trees refer to a particular subclass of ternary trees.

Definition A.2. A ternary tree T is a finite partially ordered set satisfying the following properties:

Let $a_1, a_2, a_3, a_4 \in T$. If $a_4 \leq a_2 \leq a_1$ and $a_4 \leq a_3 \leq a_1$, then we have $a_2 \leq a_3$ or $a_3 \leq a_2$.

A node $a \in T$ is called terminal, if it has no child. A non-terminal node $a \in T$ is a node with exactly three children denoted by a_1, a_2 and a_3.[1]

There exists a maximal element $r \in T$ (called the root node) such that $a \leq r$ for all $a \in T$. We assume that the root node is non-terminal.

T consists of the disjoint union of T^0 and T^∞, where T^0 and T^∞ denote the collection of parental (non-terminal) nodes and terminal nodes, respectively.

Note that the number $|T|$ of nodes in a tree T is $3j + 1$ for some $j \in \mathbb{N}$, where $|T^0| = j$ and $|T^\infty| = 2j + 1$. Next, we recall the notion of ordered trees introduced in [17]. Roughly speaking, an ordered tree "remembers how it grew".

Definition A.3. We say that a sequence $\{T_j\}_{j=1}^{J}$ is a chronicle of J generations, if

[1] Note that the order of children plays an important role in our discussion. We refer to a_j as the jth child of a non-terminal node $a \in T$. In terms of the planar graphical representation of a tree, we set the jth node from the left as the jth child a_j of $a \in T$.

T_j has j parental nodes for each $j = 1, \dots, J$,

T_{j+1} is obtained by changing one of the terminal nodes in T_j, denoted by $p^{(j)}$, into a non-terminal node (with three children), $j = 1, \dots, J-1$.

Given a chronicle $\{T_j\}_{j=1}^J$ of J generations, we refer to T_J as an *ordered tree of the Jth generation*. We use $\mathfrak{T}(J)$ to denote the collection of the ordered trees of the Jth generation.

Note that the cardinality of $\mathfrak{T}(J)$ is given by

$$|\mathfrak{T}(J)| = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2J-1) =: c_J \tag{A.1}$$

Remark A.4. Given two ordered trees T_J and $\widetilde{T}_J$ of the Jth generation, it may happen that $T_J = \widetilde{T}_J$ as trees (namely as graphs) while $T_J \neq \widetilde{T}_J$ as ordered trees according to Definition A.3. Henceforth, when we refer to an ordered tree T_J of the Jth generation, it is understood that there is an underlying chronicle $\{T_j\}_{j=1}^J$.

Definition A.5. (i) Given an ordered tree $T_J \in \mathfrak{T}(J)$ with a chronicle $\{T_j\}_{j=1}^J$, we define a "projection" π_j, $j = 1, \dots, J$, from T_J to subtrees in T_J of one generation by setting

$\pi_1(T_J) = T_1$,

$\pi_j(T_J)$ to be the tree formed by the three terminal nodes in $T_j \backslash T_{j-1}$ and its parent, $j = 2, \dots, J$. Intuitively speaking, $\pi_j(T_J)$ is the tree added in transforming T_{j-1} into T_j.

We use $r^{(j)}$ to denote the root node of $\pi_j(T_J)$ and refer to it as the *jth root node*. By definition, we have

$$r^{(j)} = p^{(j-1)}. \tag{A.2}$$

Note that $p^{(j-1)}$ is not necessarily a node in $\pi_{j-1}(T_J)$.
(ii) Given $j \in \{1, \dots, J-1\}$, $p^{(j)}$ appears as a terminal node of $\pi_k(T)$ for exactly one $k \in \{1, 2 \dots, j-1\}$. In particular, $p^{(j)}$ is the ℓth child of the kth root note $r^{(k)}$ for some $\ell \in \{1, 2, 3\}$. We define *the order of $p^{(j)}$*, denoted by $\#p^{(j)}$, to be this number $\ell \in \{1, 2, 3\}$.
(iii) We define the *essential terminal nodes* $\pi_j^\infty(T_J)$ of the jth generation by setting

$$\pi_j^\infty(T_J) := \pi_j(T_J)^\infty \cap T_J^\infty = (T_j \setminus T_{j-1}) \cap T_J^\infty.$$

By definition, $\pi_j^\infty(T_J)$ may be empty. Note that $\{\pi_j^\infty(T_J)\}_{j=1}^J$ forms a partition of T_J^∞.

We record the following simple observation.

Remark A.6. Let $T \in \mathfrak{T}(J)$ be an ordered tree. Then, for each fixed $j = 2, \dots, J$, there exists a path[2] $a_1, a_2, \dots, a_K$, starting at the root node $r = r^{(1)}$ and ending at the jth root node $r^{(j)}$ such that $a_k \neq r^{(\ell)}$ for any $k = 1, \dots, K$ and $\ell \geq j+1$. Namely, we can move from $r^{(1)}$ to $r^{(j)}$ without hitting a root node of a higher generation.

More concretely, given $r^{(j)}$, we know that it appears as a terminal node of $\pi_{j_1}(T)$ for exactly one $j_1 \in \{1, 2 \dots, j-1\}$. Similarly, $r^{(j_1)}$ appears as a terminal node of $\pi_{j_2}(T)$ for exactly one $j_2 \in \{1, 2 \dots, j_1 - 1\}$. We can iterate this process, which must terminate in a finite number of steps with $j_k = 1$. This generates the shortest path $r^{(j_k)}, r^{(j_{k-1})}, \dots, r^{(j_1)}, r^{(j)}$ from $r^{(1)}$ to $r^{(j)}$ and we denote it by $P(r^{(1)}, r^{(j)})$. Similarly, given $a \in T \setminus \{r^{(1)}\}$, one can easily construct the shortest path from $r^{(1)}$ to a since a is a terminal node of $\pi_k(T)$ for some k. We denote this shortest path by $P(r^{(1)}, a)$.

Given an ordered tree, we need to consider all possible frequency assignments to nodes that are "consistent".

Definition A.7. Given an ordered tree $T \in \mathfrak{T}(J)$, we define an *index function* $\boldsymbol{\xi} : T \to \mathbb{R}$ such that

$$\xi_a = \xi_{a_1} - \xi_{a_2} + \xi_{a_3} \tag{A.3}$$

for $a \in T^0$, where a_1, a_2, and a_3 denote the children of a. Here, we identified $\boldsymbol{\xi} : T \to \mathbb{R}$ with $\{\xi_a\}_{a \in T} \in \mathbb{R}^T$. We use $\Xi(T) \subset \mathbb{R}^T$ to denote the collection of such index functions $\boldsymbol{\xi}$. Also, the collection of index functions $\boldsymbol{\xi} \in \Xi(T)$ with fixed frequency $\xi \in \mathbb{R}$ at the root node of T is denoted by $\Xi_\xi(T)$

Remark A.8. If we associate functions $v_a = v_a(\xi_a)$ to each node $a \in T$, then the relation (A.3) implies that $v_a = v_{a_1} * \overline{v_{a_2}} * v_{a_3}$.

Given an ordered tree $T_J \in \mathfrak{T}(J)$ with a chronicle $\{T_j\}_{j=1}^J$ and associated index functions $\boldsymbol{\xi} \in \Xi(T_J)$, we use superscripts to keep track of "generations" of frequencies.

Consider T_1 of the first generation. We define the first generation of frequencies by

$$\big(\xi^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}\big) := (\xi_r, \xi_{r_1}, \xi_{r_2}, \xi_{r_3}),$$

where r_j denotes the three children of the root node r.

In general, the ordered tree T_j of the jth generation is obtained from T_{j-1} by changing one of its terminal nodes $a \in T_{j-1}^\infty$ into a non-terminal node. Then, we define the jth generation of frequencies by

$$\big(\xi^{(j)}, \xi_1^{(j)}, \xi_2^{(j)}, \xi_3^{(j)}\big) := (\xi_a, \xi_{a_1}, \xi_{a_2}, \xi_{a_3}),$$

where a_j denotes the three children of the node $a \in T_{j-1}^\infty$. Note that the parent node a is nothing but the jth root node $r^{(j)}$ defined in Definition A.5.

[2] A path is a sequence of nodes $a_1, a_2, \dots, a_K$ such that a_k and a_{k+1} are adjacent.

Our main analytical tool is the localized modulation estimate of Lemma 4.2. Hence, it is important to keep track of the modulation for frequencies in each generation. We use μ_j to denote the corresponding modulation function introduced at the jth generation. Namely, we set

$$\begin{aligned}\mu_j = \mu_j\big(\xi^{(j)}, \xi_1^{(j)}, \xi_2^{(j)}, \xi_3^{(j)}\big) &:= \big(\xi^{(j)}\big)^2 - \big(\xi_1^{(j)}\big)^2 + \big(\xi_2^{(j)}\big)^2 - \big(\xi_3^{(j)}\big)^2 \\ &= 2\big(\xi_2^{(j)} - \xi_1^{(j)}\big)\big(\xi_2^{(j)} - \xi_3^{(j)}\big) = 2\big(\xi^{(j)} - \xi_1^{(j)}\big)\big(\xi^{(j)} - \xi_3^{(j)}\big),\end{aligned}$$

where the last two equalities hold in view of (A.3). We also use the following short-hand notation:

$$\widetilde{\mu}_j := \sum_{k=1}^{j} \mu_k.$$

Given $\xi \in \mathbb{R}$ and $T \in \mathfrak{T}(J)$, we use a short-hand notation for iterated integrals of the form

$$\int_{\boldsymbol{\xi} \in \Xi_\xi(T)} [\,\cdot\,] := \underbrace{\int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2}}_{J \text{ times}} [\,\cdot\,] \, d\xi_1^{(J)} d\xi_2^{(J)} \ldots d\xi_1^{(1)} d\xi_2^{(1)}.$$

Appendix B

Mild ill-posedness below $H^{\frac{1}{2}}(\mathbb{T})$

In the periodic setting, the Lipschitz continuity of the solution map of DNLS on bounded subsets of $H^s(\mathbb{T})$ is further restricted to subsets with prescribed L^2-norm due to the use of a translation operator when reversing the gauge transformation of DNLS (see Lemma 2.3). In fact, the local uniform continuity of the solution map of the periodic DNLS fails without fixing the mass on bounded subsets of $H^s(\mathbb{T})$, at any regularity level (see [22, Theorem 3.1.1.(ii)]). However, for the gauge equivalent equation (4.2), one does not face the local uniform continuity bottleneck due to the translation operator and it was for this equation that the contraction mapping argument was applied in [21].

We provide here the following mild ill-posedness result. The mild sense refers to the fact that the result shows that the contraction mapping argument cannot be applied for the gauge equivalent equation (2.22). The proof uses ideas similar to those in [6, 8], to construct smooth solutions that show the failure of uniform continuity of the solution map of (2.22) on bounded subsets of $H^s(\mathbb{T})$, for $0 \leq s < \frac{1}{2}$.

Proposition B.1. *Let* $0 \leq s < \frac{1}{2}$ *and* $T > 0$. *For any* $0 < \delta \ll \varepsilon < 1$, *there exist smooth initial data* $v_0, \widetilde{v}_0$ *such that*

$$\|v_0\|_{H^s(\mathbb{T})},\ \|\widetilde{v}_0\|_{H^s(\mathbb{T})} \lesssim \varepsilon, \tag{B.1}$$

$$\|v_0 - \widetilde{v}_0\|_{H^s(\mathbb{T})} \lesssim \delta, \tag{B.2}$$

and for which the corresponding solutions $v, \widetilde{v}$ *to* (2.22) *have the property*

$$\|v - \widetilde{v}\|_{L^\infty_t([-T,T];H^s_x(\mathbb{T}))} \gtrsim \varepsilon. \tag{B.3}$$

In particular, if it exists, the solution map $v_0 \in H^s(\mathbb{T}) \mapsto v \in C([-T,T]; H^s(\mathbb{T}))$ *of* (2.22) *can not be uniformly continuous on bounded subsets of* $H^s(\mathbb{T})$.

Proof. Let $a \in \mathbb{C}$ and $N \in \mathbb{Z}$, $N \gg 1$ (to be chosen later) and consider functions supported on a single frequency of the form

$$v_{N,a}(t,x) = ae^{i(Nx+\theta(N)t)},$$

for some $\mathbb{R}$-valued $\theta(\cdot)$. We have

$$\mu(v_{N,a}) = |a|^2 \ , \ \psi(v_{N,a}) = -2|a|^2 N + \frac{1}{2}|a|^4$$

and thus we compute the corresponding nonlinearity of (2.22):

$$\mathcal{N}(v_{N,a}) = |a|^2 aNe^{i(Nx+\theta(N)t)}.$$

Then, by taking $\theta(N) = -N^2 - |a|^2 N$, the function

$$v_{N,a}(t,x) = ae^{i(Nx-N^2t-|a|^2Nt)} \tag{B.4}$$

is a solution of (2.22) with

$$\|v_{N,a}(t,x)\|_{L^2_x(\mathbb{T})} \sim |a| \ , \ \|v_{N,a}(t,x)\|_{\dot{H}^s_x(\mathbb{T})} \sim |a|N^s$$

and since $s \geq 0$, we also have

$$\|v_{N,a}(t,x)\|_{H^s_x(\mathbb{T})} \sim |a|N^s.$$

Now let $a = bN^{-s}$ and $\widetilde{a} = \widetilde{b}N^{-s}$ with $b, \widetilde{b} \in \mathbb{C}$ such that $|b| \sim |\widetilde{b}| \sim \varepsilon$ and $|b - \widetilde{b}| \lesssim \delta$. We find

$$\|v_{N,a}(0,x) - v_{N,\tilde{a}}(0,x)\|_{H^s_x(\mathbb{T})} = |b - \tilde{b}|N^{-s}\|e^{iNx}\|_{H^s_x(\mathbb{T})} \lesssim \delta.$$

On the other hand, by setting $\varphi(N,b) := |bN^{-s}|^2 N$ to simplify the writing, we obtain

$$\begin{aligned}
\|v_{N,a}(t,x) - v_{N,\tilde{a}}(t,x)\|_{H^s_x(\mathbb{T})} &= \left| be^{-i\varphi(N,b)t} - \tilde{b}e^{-i\varphi(N,\tilde{b})t} \right| N^{-s}\|e^{iNx}\|_{H^s_x(\mathbb{T})} \\
&\gtrsim |b| \left| e^{-i\varphi(N,b)t} - e^{-i\varphi(N,\tilde{b})t} \right| - |b - \tilde{b}| \\
&\gtrsim \varepsilon \left| e^{i(\varphi(N,b)-\varphi(N,\tilde{b}))t} - 1 \right| - \delta.
\end{aligned}$$

Note that

$$\varphi(N,b) - \varphi(N,\tilde{b}) = N^{1-2s}(|b|^2 - |\tilde{b}|^2) \tag{B.5}$$

and that at $t = t_N$, where

$$t_N := \frac{\pi}{\varphi(N,b) - \varphi(N,\tilde{b})}, \tag{B.6}$$

the two solutions have opposite phases, and thus

$$\|v_{N,a}(t_N,x) - v_{N,\tilde{a}}(t_N,x)\|_{H^s_x(\mathbb{T})} \gtrsim \varepsilon - \delta \sim \varepsilon.$$

Indeed, since the power of N in (B.5) is positive, we can choose an integer $N = N(\varepsilon, T)$ (independent of δ) such that $|t_N| \leq T/2$, or equivalently

$$|\varphi(N,b) - \varphi(N,\tilde{b})| \gtrsim T^{-1}.$$

$\square$

We note that the same is true for any equation obtained from DNLS through a gauge transformation (2.12) with any other parameter β.

Remark B.2. One can easily adapt the argument in the proof of Proposition B.1 to any other gauge equivalent equation, including DNLS itself. Indeed, it is enough to take

$$\theta_\beta(N) = \theta(N) - (\beta^2 - \frac{3}{2}\beta + \frac{1}{2})|a|^4.$$

Correspondingly, we take

$$\varphi_\beta(N,b) = \varphi(N,b) - (\beta^2 - \frac{3}{2}\beta + \frac{1}{2})|b|^4 N^{-4s}$$

and note that for $N \gg 1$, the difference in phase is essentially as above, i.e.

$$\varphi_\beta(N,b) - \varphi_\beta(N,\tilde{b}) \sim \varphi(N,b) - \varphi(N,\tilde{b}).$$

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