# UNIVERSITY OF EDINBURGH THESIS

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ON RADially symmetric gravitational fields in continuous matter.

J. Ghosh.
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X. Summary.
I. Introduction.

§1. The gravitational equations in the most general form as given by Einstein in 1917 are

$$K_{pq} - \frac{1}{2} g_{pq} K + \beta g_{pq} = -8 \pi T_{pq},$$

where

- $K_{pq}$ is the contracted Riemann-Christoffel tensor,
- $K$ is the Riemann curvature,
- $\beta$ is a small universal constant,
- $T_{pq}$ is the material-energy-tensor.

The simplest and most important case arises when the field is radially symmetric and in this case we may take the metric of space-time as

$$ds^2 = -e^{\lambda} dt^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + e^\nu dt^2,$$

where $(r, \theta, \phi, t)$ are the coordinates, the velocity of light is unity and $\lambda$ and $\nu$ are functions of $r$ only. We have then, for $p = q$, $g_{pq} = K_{pq} = T_{pq} = 0$, and for the metric (1.2), the equations (1.1) becomes

$$-8 \pi \left( T_{tt}^{\prime} + \frac{\beta}{8 \pi} \right) = e^{-\lambda} \left( \frac{\nu'}{n} - \frac{\nu}{n^2} \right),$$

$$-8 \pi \left( T_{rr}^{\prime} + \frac{\beta}{8 \pi} \right) = e^{-\lambda} \left\{ \frac{1}{2} \nu'' - \frac{1}{n} \nu' + \frac{1}{n} \nu' + \frac{1}{2} (\nu - \nu')/\nu \right\}.$$
where the dashes denote differentiation with respect to $n$.

§2. The above equations have been solved in a few particular cases, viz., the field of an isolated particle, the field of empty space, the field occupied by homogeneous incoherent matter filling all space and the field occupied by what Schwarzschild defines as a perfect fluid. A case of heterogeneous fluid has also been discussed by the present writer.

§3. The object of the present paper is to investigate certain types of exact radially symmetric solutions of the above equations for fields occupied by continuous matter possessing internal stresses. When the field is symmetrical about the origin the two transverse components of stress are identically equal. This is implied in the equation (1.5). The radial and the transverse components may be given as functions of $n$, but, in general, a relation between them is sufficient for the determination of the field when $T_4$ is given. We shall assume that $T_2$ and $T_1$ are connected by a linear relation, for the following reasons:

---

1 Schwarzschild, Berlin Sitz., 1916, p.189.
2 De Sitter, M.N.R.A.S. (1916-17)
3 Einstein, Berlin Sitz., 1917, p.142.
(i) It is the simplest functional relation between the two components of stress which lead to exact solutions of the gravitational equations.

(ii) This relation includes several actual distributions of stress, e.g., (a) when an isotropic homogeneous sphere is subjected to uniform normal pressure, the radial and transverse stresses are equal, (b) when a homogeneous sphere is subjected to its own gravitation, the three stresses are equal, (c) when a homogeneous sphere is subjected to a constant central force together with a suitable normal pressure on the surface, the radial and transverse stresses are proportional, (d) the three normal stresses are equal in a perfect fluid.

(iii) The solutions obtained on this assumption are of a more general nature than those hitherto obtained and contain as particular cases the solutions of Einstein, Schwzschild, and De Sitter.

(iv) If we start with a functional relation between $T_r^2$ and $T_r'$, it is probable that the linear relation is the only one which leads to exact solutions.

§4. We shall first obtain certain types of exact solutions of the equations (1.3). . . . (1.6) with the only assumption that $T_r^2$ and $T_r'$ are linearly related and then proceed to apply some of the results to obtain the fields of certain definite distributions of matter.

The gravitational potentials in the Newtonian theory depend only on the mass-density while those in Einstein's theory depend on the density as well as on the internal stresses. It is therefore not possible to compare the results with
one another when the fields are occupied by matter possessing internal stresses.

§5. According to the latest theory of Einstein, the gravitational equations assume a new form, viz.,

\[ K_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \kappa = -8\pi T_{\alpha\beta}, \]

which is equivalent to the older form in empty space, but is different for spaces possessing an energy-tensor. A note on these equations and their solutions corresponding to the radially symmetric field forms the last section of the paper.

\[ a^2 \left( v' = \frac{v}{x} - \frac{v'}{x^2} \right) + \left( a^2 + 1 \right) v' = 0 \]

From (1.6), we get

\[ \pi T_{\alpha\beta} = \pi \left( \frac{\partial f}{\partial a} + \frac{\partial f}{\partial \beta} \right) \]

§7. The equation (6.2) is satisfied, if the following two are satisfied:

\[ \pi' + \pi v \left( \frac{1}{x} + \frac{a}{x^2} \right) = \pi \]

or, putting

\[ p' = \frac{d}{dx} \left( \frac{1}{x} p \right) + \frac{1}{x} \pi = 0 \]

From (7.1), we have

1 Mathematische Annalen, December, 1926.
II. The Solutions of the First Type.

§6. In accordance with the remarks of §3, we assume that \( \tau_i \) and \( \tau_k \) are connected by the relation

\[
\tau_k = m \tau_i + n , \tag{6.1}
\]

Eliminating \( \tau_k \) and \( \tau_i \) from (1.3), (1.4) and (6.1), we get

\[
\begin{align*}
\frac{1}{2} v'' - \frac{1}{4} \lambda^2 v' + \frac{1}{4} v'' + \frac{1}{2} (v', \lambda) + \frac{1}{2} (V, \lambda) & + \frac{1}{2} (v', \lambda) + \frac{1}{2} (V, \lambda) \\tag{6.2}
\end{align*}
\]

or,

\[
\begin{align*}
\frac{1}{2} v'' - \frac{1}{4} \lambda^2 v' + \frac{1}{4} v'' + \frac{1}{2} (v', \lambda) + \frac{1}{2} (V, \lambda) & + \frac{1}{2} (v', \lambda) + \frac{1}{2} (V, \lambda) \\tag{6.3}
\end{align*}
\]

where

\[
N = (m-1) \beta - \pi \gamma \eta \\tag{6.4}
\]

§7. The equation (6.2) is satisfied, if the following two are satisfied:

\[
\begin{align*}
- \frac{1}{2} \frac{\lambda}{\eta} + \frac{m (\frac{\lambda}{\eta})}{\eta^2} - \lambda \epsilon \lambda & = 0 \\tag{7.1}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{2} v'' + \frac{1}{4} \lambda^2 v' + v' \left( \frac{1}{2} - \frac{m \lambda}{\eta} - \frac{1}{2} \lambda \right) & = 0 , \\tag{7.2}
\end{align*}
\]

From (7.1), we have

\[
\begin{align*}
e^{-\lambda} \lambda^1 - \frac{q m}{\eta} v \left( 1 - e^{-\lambda} \right) + 2 N \lambda & = 0
\end{align*}
\]

or, putting

\[
1 - e^{-\lambda} = \frac{q}{\eta} \quad \tag{7.3}
\]

\[
\begin{align*}
p' - \frac{q m}{\eta} \lambda + 2 N \lambda & = 0 \,
\end{align*}
\]
whence \( \lambda = \frac{N}{m-1} \lambda^2 + \beta \lambda^m \)

where \( \beta \) is a constant of integration.

Thus

\[
e^{\lambda} = \frac{1}{1 - \frac{N}{m-1} \lambda^2 - \beta \lambda^m}
\]

From (7.2), we have

\[
e^{\frac{1}{2}v} \left( \frac{1}{q} \frac{d}{d x} + \frac{1}{q} \frac{d}{d x} \right) + \frac{1}{2} v' e^{\frac{1}{2}v} \left( \frac{1-\lambda_1^m}{\lambda} - \frac{1}{2} \lambda' \right) = 0
\]

or, putting \( e^{\frac{1}{2}v} = u \),

\[
u'' + \nu' \left( \frac{1-\lambda_1^m}{\lambda} - \frac{1}{2} \lambda' \right) = 0
\]

or,

\[
u'' + \nu \left( \frac{1-\lambda_1^m}{\lambda} \right) - \frac{1}{2} \lambda' \nu = 0
\]

where \( u' = u \).

Integrating, we have

\[
u' = u = \beta \lambda^m \lambda_{n-1}
\]

where \( \beta \) is a constant of integration.

Hence

\[
e^{\lambda} = u^2 = \left[ \frac{E}{C + \int \frac{B \lambda^m \lambda_{n-1} \lambda_{n-1} \lambda_{n-1}}{(1 - \frac{N}{m-1} \lambda^2 - \beta \lambda^m)^{1/2}}} \right]^2
\]

where \( C \) is a constant.

§8. Case (a). Let \( N = 0 \), i.e., \( \sigma = \beta \).\( \sigma = (n-1) \beta = 0 \)

This condition will be satisfied if

(1) \( n = 0 \), \( \beta = 0 \), \( m \) arbitrary,

or, (ii) \( n = 0 \), \( m = 1 \)

or, (iii) \( \beta = \beta \).

In either of these cases, we have

\[
e^{\lambda} = \frac{1}{1 - \beta \lambda^m}
\]

or,

\[
e^{\frac{1}{2}v} = \left[ C - \frac{\beta}{\lambda^m} \left( 1 - \beta \lambda^m \right) \right]^2
\]

or,

\[
e^{\lambda} = \left( C - \beta \left( 1 - \beta \lambda^m \right) \right)^2
\]
Case (i). We have $T^z = n T^z$, and the matter is supposed to be placed in a galilean world.

Since $e^\Lambda = 1 - A \frac{z^m}{n}$, and

$$v' = \frac{2 m B' A \frac{z^m}{n}}{(1 - A \frac{z^m}{n})^\frac{1}{2}} \left\{ C - B' (1 - A \frac{z^m}{n}) \right\}^\frac{1}{2},$$

we get, from (1.3),

$$-8\pi T^z = \frac{e^\Lambda v'}{n} - \frac{1 - e^{-\Lambda}}{n^2},$$

or,

$$8\pi T^z = \frac{e^{-\Lambda} v'}{n} + \frac{1 - e^{-\Lambda}}{n^2} = (2 m + 1) A \frac{z^m}{n^2 -},$$

The Riemann-curvature $K$ is given by

$$K = 8\pi T = 8\pi T^b$$

$$= 8\pi \left\{ (2 m + 1) T^1 + T^4 \right\}$$

$$= 2 (2 m + 1) A \frac{z^m - 1}{(1 - A \frac{z^m}{n})^\frac{1}{2}} \left\{ C - B' (1 - A \frac{z^m}{n}) \right\}^\frac{1}{2},$$

Case (ii). We have $T^z = n T^z$, so that all the normal stresses are equal. In this case

$$e^\Lambda = \frac{1}{1 - A \frac{z^m}{n}}$$

$$e^\nu = \left\{ C - B' (1 - A \frac{z^m}{n}) \right\}^\frac{1}{2}$$

and

$$T^z = \frac{1}{8\pi} \left\{ (A - \beta) - \frac{B' A (1 - A \frac{z^m}{n})}{C - B' (1 - A \frac{z^m}{n})} \right\}$$

$$T^4 = \frac{3 A - \beta}{8\pi},$$
In the special case, when the natural curvature is zero,

\[ \beta = 0, \]

we get

\[ \mathcal{T}_n^2 = \mathcal{T}_n^3 = \mathcal{T}_o^3 \]

\[ = \frac{A}{8\pi} \left[ 1 - \frac{\beta'(1 - \alpha^2)}{C - \beta'(1 - \alpha^2)^{1/2}} \right], \tag{8.12} \]

\[ \mathcal{T}_o^4 = \frac{3A}{8\pi}, \tag{8.13} \]

and

\[ \kappa = 2A \left[ 3 - \frac{\beta'(1 - \alpha^2)^{1/2}}{C - \beta'(1 - \alpha^2)^{1/2}} \right]. \tag{8.14} \]

Case (i). In this case, we have

\[ \mathcal{T}_n^2 = n \mathcal{T}_n^1 + \mathcal{R} \]

\[ = n \mathcal{T}_n^1 + \frac{n-1}{8\pi} \beta. \]

Thus, from (1.3) and (1.4), we get

\[ 8\pi \mathcal{T}_n^1 = \mathcal{A} \mathcal{R}^{2m-1} \left[ 1 - \frac{2\mathcal{R} \beta'(1 - \mathcal{A}^2)^{1/2}}{C - \mathcal{B} (1 - \mathcal{A}^2)^{1/2}} \right] - \beta, \tag{8.15} \]

\[ 8\pi \mathcal{T}_n^2 = \mathcal{A} \mathcal{R}^{2m-2} \left[ 1 - \frac{2 \mathcal{R} \beta'(1 - \mathcal{A}^2)^{1/2}}{C - \mathcal{B} (1 - \mathcal{A}^2)^{1/2}} \right] - \beta, \tag{8.16} \]

and

\[ \kappa = 2(2m+1) \mathcal{A} \mathcal{R}^{2m-2} \left[ 1 - \frac{\mathcal{B} \beta'(1 - \mathcal{A}^2)^{1/2}}{C - \mathcal{B} (1 - \mathcal{A}^2)^{1/2}} \right] - 3\beta. \tag{8.17} \]

§9. Case (b). Consider next the case \( \mathcal{R} = 0 \), so that

\[ \mathcal{T}_n^3 = \mathcal{T}_n^2 = \mathcal{R}, \text{ a constant.} \]

We have

\[ \mathcal{E} = \frac{1}{1 - \mathcal{A} + \mathcal{A}^2} = \frac{1}{\mathcal{A} + \mathcal{A}^2}, \tag{9.1} \]

where

\[ \mathcal{A}' = 1 - \mathcal{A}. \tag{9.2} \]

\[ \mathcal{E}^{1/2} = \mathcal{C} + \int \frac{\mathcal{B} \mathcal{D} \mathcal{N}}{2 \sqrt{\mathcal{A}' + \mathcal{A}^2 \mathcal{L}}}, \]

or,

\[ \mathcal{E}^{1/2} = \mathcal{C} + \mathcal{B}' \log \left( \frac{\mathcal{N}}{\mathcal{A}' + \mathcal{A}^2 \mathcal{L}} \right), \]

\[ \mathcal{E} = \left\{ \mathcal{C} + \mathcal{B}' \log \left( \frac{\mathcal{N}}{\mathcal{A}' + \mathcal{A}^2 \mathcal{L}} \right) \right\}^2. \tag{9.3} \]
Substituting in (1.3) and (1.6), we get

\[ 8 \pi T_1' = \varepsilon \pi n + \frac{A}{\lambda^2} - \frac{2B}{\lambda^2} \frac{\sqrt{A^2 + \pi n^2}}{\varepsilon} \left\{ c + B' \log \left( \frac{\pi n}{A^2 + \pi n^2} \right) \right\} \quad (9.4) \]

\[ 8 \pi T_4'' = \frac{A}{\lambda^2} + 2(\beta + 12 \pi n) \quad (9.5) \]

The curvature \( \kappa \) is given by

\[ \kappa = \frac{2A}{\pi^2} - \frac{2B}{\pi^2} \frac{\sqrt{A^2 + \pi n^2}}{\varepsilon^2} \left\{ c + B' \log \left( \frac{\pi n}{A^2 + \pi n^2} \right) \right\} + 48 \pi n + \beta \quad (9.6) \]

In the particular case when \( \beta = 0 \), we have \( T_2 = T_3 = 0 \), so that the medium possesses only a radial stress. In this case, we get

\[ e^\lambda = \frac{l}{A^2 - \pi n^2} \quad (9.7) \]

\[ e^\mu = c + \frac{B}{\sqrt{A^2 + \pi n^2}} \log \frac{n}{\sqrt{A^2 + \pi n^2} - \beta} \quad (9.8) \]

From (1.3), we get

\[ 8 \pi T_1' = \frac{A}{\lambda^2} - \frac{2B}{\lambda^2} \frac{(A + \beta n)}{\varepsilon} \frac{1}{\sqrt{A^2 - \beta n^2}} \quad (9.9) \]

If further \( \beta = 0 \), we have

\[ e^\lambda = \text{const.} = \frac{l}{1 - A} \quad (9.10) \]

\[ e^\mu = (c + B' \log r)^2 \quad (9.11) \]

\[ 8 \pi T_1' = \frac{A}{\lambda^2} - \frac{2B}{\lambda^2} \frac{B'}{\varepsilon} \frac{1}{(c - B \log r)} \quad (9.12) \]

and

\[ 8 \pi T_4'' = \frac{A}{\lambda^2} \quad (9.13) \]

The solutions (9.7), ..., (9.12) correspond to fields in which only the radial and the temporal components of the energy-tensor exist.
III. The Solutions of the Second Type.

§10. A second class of solutions may be obtained in the following manner: the equations (6.2) is satisfied if the following two equations are satisfied:

\[ \frac{1}{2} y'' + \frac{1}{4} y' + y' \left\{ \frac{1}{x} - \frac{m}{2} \right\} = 0, \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (10.1) \]

\[ -\lambda' \left\{ \frac{1}{4} y' + \frac{1}{2} y \right\} + \frac{2m(\alpha^2 - 1)}{\alpha^2} - N e^\lambda = 0, \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (10.2) \]

Let \( x = e^{\frac{1}{4} y} \). We have \( x' = \frac{1}{4} e^{\frac{1}{4} y} y' \),

\[ x'' = e^{\frac{1}{4} y} \left( \frac{1}{2} y'' + \frac{1}{4} y' \right) \]

Hence the equation (10.1) reduces to

\[ x'' + x' \left\{ \frac{1}{2} \alpha^2 \right\} = 0, \]

whence, on integration,

\[ x = C + B e^{2m} \]

so that

\[ e^y = (C + B e^{2m})^2. \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (10.3) \]

Substituting \( y' = \frac{4m B e^{2m-1}}{C + B e^{2m}} \)

in (10.2), we get

\[ -e^{\lambda} \lambda' \left\{ \frac{4m B e^{2m-1}}{C + B e^{2m}} + \frac{1}{e^{\lambda}} \right\} + \frac{4m(1 - e^\lambda)}{\alpha^2} - N = 0. \]

Let \( 1 - e^\lambda = y' \), so that \( e^\lambda = \frac{1}{y'} \) and the above equation becomes

\[ y' \left\{ \frac{4m B e^{2m-1}}{C + B e^{2m}} + \frac{1}{e^{\lambda}} \right\} - \frac{4m y'}{\alpha^2} + N = 0, \]

which is of the form

\[ x' - P x + Q = 0, \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (10.4) \]

where

\[ P = 2m \left\{ \frac{1}{C} - \frac{2m B e^{2m-1}}{C + (m+1)B e^{2m}} \right\}, \]

and

\[ Q = 2m \left\{ \frac{1}{C} + \frac{m B e^{2m-1}}{C + (m+1)B e^{2m}} \right\}. \]
The solution of (10.4) is

$$y = e^{\int P dx} \left[-\int Q e^{\int P dx} + H\right],$$

where $H$ is an arbitrary constant. Now, we have

$$-\int P dx = \frac{1}{\eta^{2m}} \left\{ C + (2m+1)B \right\}^{\frac{2m}{2m+1}}.$$

Thus,

$$y = \frac{\eta^{2m}}{\left\{ C + (2m+1)B \right\}^{\frac{2m}{2m+1}}} \left[ H - \int \frac{2N (C + B \eta^{m})}{\left\{ C + (2m+1)B \right\}^{\frac{2m}{2m+1}}} \right],$$

whence

$$\eta^{-\lambda} = 1 - \frac{\eta^{2m}}{\left\{ C + (2m+1)B \right\}^{\frac{2m}{2m+1}}} \left[ H - \int \frac{2N (C + B \eta^{m})}{\left\{ C + (2m+1)B \right\}^{\frac{2m}{2m+1}}} \right], \quad \ldots (10.5)$$

The solution will be exact for those values of $m$, for which the integral within the square brackets is integrable.

In the more important case, viz., the case in which $\tau_2^1$ and $\mathcal{R}$ are proportional, we have $\eta = 0$ and therefore, when the natural curvature is negligible, $N = 0$. In this case, the solution (10.5) is always exact.

§11. We proceed to obtain the solutions in a few special cases:

Case (a). Let $m = 0$, so that $\tau_2^1 = \tau_3^1 = 0$, a constant.

We have then

$$e^\nu = (C + B)^{1/2}, \quad \text{a constant}.$$  \hspace{1cm} (10.1)

and

$$\eta^{-\lambda} = 1 - N \eta^2.$$  \hspace{1cm} (10.2)
where $\mu$ is a constant and $N = -\beta - 2 \pi n$.

From (1.3) and (1.6), we get

\[ 8 \pi T_{f} = \frac{8 \pi n + H}{2} \hspace{2cm} \text{(11.3)} \]
\[ 8 \pi T_{v} = 4 (6 \pi n + \beta) + \frac{H}{2} \hspace{2cm} \text{(11.4)} \]

In the particular case when we assume $\beta = 0$, the solutions (11.1).....(11.4) become

\[ e^{v} = t, \quad e^{-\lambda} = C - 8 \pi n \]
\[ 8 \pi T_{f} = 8 \pi n + \frac{H}{2} \]
\[ T_{2} = T_{3} = n \]
\[ 8 \pi T_{v} = 4 \pi n + \frac{H}{2} \]

and the curvature $K$ is given by

\[ K = 8 \pi T - 4 \pi n + \frac{2H}{2} \]

§12. Let $n = \frac{1}{2}$.

We have

\[ 2 T_{2} = T_{1} + 2n \]

\[ e^{v} = (C + Br)^{2} \hspace{2cm} \text{(12.1)} \]

\[ e^{-\lambda} = 1 - \frac{r}{(C + 2Br)^{2}} \left[ H - \int \frac{2N(C + Br) dr}{(C + 2Br)^{2}} \right] \]

\[ = 1 - \frac{Hr}{(C + 2Br)^{2}} + \frac{2N(C + Br)}{3B} \hspace{2cm} \text{(12.2)} \]

From (1.3) and (1.6), we get

\[ 8 \pi T_{f} + \beta = \frac{H}{r(C + 2Br)^{2}} - \frac{2N(C + Br)}{3B} \]

\[ - \left\{ \frac{1}{r} - \frac{H}{(C + 2Br)^{2}} \right\} + \frac{2N(C + Br)}{3B} \frac{(C + Br)^{2} + 3B}{C + Br} \hspace{2cm} \text{(12.3)} \]

\[ 8 \pi T_{v} + \beta = \frac{H}{(C + 2Br)^{2}} - \frac{HBn}{(C + 2Br)^{2}} - \frac{4N}{3B} (C + Br) \]

\[ = \frac{H}{(C + 2Br)^{2}} \left( 1 - \frac{Br}{C + 2Br} \right) - \frac{4N}{3B} (C + Br) \hspace{2cm} \text{(12.4)} \]
In the particular case in which \( n = 0 \) and \( \beta = 0 \), we have \( N = 0 \). We have \( \tau_1'' = \tau_2' \) and the matter is supposed to be placed in a galilean space. The corresponding solutions are

\[ e^\gamma = (C + B \alpha)^2 \]

\[ e^\lambda = 1 - \frac{H \alpha}{(C + B \alpha)^2} \]

\[ \delta \tau_1'' = \frac{H}{\alpha(C + B \alpha)^2} - \frac{B}{C + B \alpha}\left\{ \frac{1}{\alpha} - \frac{H}{(C + B \alpha)^2} \right\} \]

\[ \tau_2'' = \tau_3'' = \frac{1}{2} \tau_1'' \]

\[ \delta \tau_4'' = \frac{H}{(C + B \alpha)^2} \left( 1 - \frac{B}{C + B \alpha} \right) \]

Art. 13. Case (c). Let \( n = 1 \), \( \alpha = 0 \), so that \( \tau_2'' = \tau_1'' \)

\[ N = 0 \]. The corresponding solutions are

\[ e^\gamma = (C + B \alpha^2)^2 \]

\[ e^\lambda = 1 - \frac{H \alpha^2}{(C + B \alpha^2)^2} \]

From (1.3) and (1.6) we find

\[ \delta \tau_1'' = \frac{1}{(B \alpha^2 + C)(B \alpha^2 + C)^3} \left[ \frac{H(B \alpha^2 + C)}{C + B \alpha} (5B \alpha^2 + C) \right] \]

\[ \tau_2'' = \tau_3'' = \tau_1'' \]

\[ \delta \tau_4'' = \frac{H(sB \alpha^2 + 3C)}{(3B \alpha^2 + C)^{5/3}} \]

The solutions (13.1) . . . . (13.4) are applicable to the medium in which all the three stresses are equal. The form of the solutions are the same whether we assume the natural curvature of space to be zero or not. The difference will be only in the values of the constants.

Art. 14. Case (d). Let \( m = -1 \). We have \( \tau_2'' + \tau_1'' = n, N = -8\alpha \tau_2'' \beta \).

In this case \( e^\gamma = (\frac{1}{C + \beta \alpha^2})^2 \)

(14.1)
\[ \dot{e}^\lambda = 1 - \frac{\lambda^2}{(C^{\lambda^2-B})^2} \left[ H - 2N \int \frac{(C^{\lambda^2+B})(C^{\lambda^2-B})}{\zeta} \, d\eta \right] \]

\[ = 1 - \frac{\lambda^2}{(C^{\lambda^2-B})^2} \left[ H - 2N \left( \frac{1}{6} C^{\lambda^2-B^2} \log \eta \right) \right] \quad \ldots (14.2) \]

In the special case when the radial and the transverse stresses are equal and opposite and the matter is supposed to be placed in a galilean space, we have \( n = 0 \) and \( \beta = 0 \), so that \( \tau_2 = \tau_3 = -\tau_1' \).

We get
\[ e^\gamma = \left( C^{\lambda^2+B} \right)^2 \quad \ldots (14.1) \]

The solution is
\[ \dot{e}^\lambda = 1 - \frac{H\lambda^2}{(C^{\lambda^2-B})^2} \quad \ldots (14.3) \]

Also
\[ 8\pi \tau_1' = \frac{1}{(C^{\lambda^2-B})^2} \left[ H + \frac{B}{\lambda^2(C^{\lambda^2+B})} \left\{ (C^{\lambda^2+B})^2 - H\lambda^2 \right\} \right] \]

and
\[ 8\pi \tau_4' = \frac{\dot{e}^\lambda \lambda^1}{\alpha} + \frac{1 - e^\lambda}{\alpha} \]

Substituting this value of \( \lambda' \) in (5.2), we get
\[ \frac{\dot{\alpha}^2}{\alpha^2} + \frac{\dot{\alpha}^2}{\alpha} + \left( \frac{\lambda^2}{\alpha} - \frac{\beta^2}{\alpha} \right) + \frac{(\lambda_0 - \lambda)}{\alpha} = 0 \quad \ldots (5.4) \]

Multiplying the left-hand side of the equation (15.2) by \( e^{\lambda^2 + \beta^2} \) and putting \( e^{\lambda^2 + \beta^2} = \alpha^2 \), we have
\[ \frac{\dot{\alpha}^2}{\alpha^2} + \frac{\dot{\alpha}^2}{\alpha} + \left( \frac{\lambda^2}{\alpha} - \frac{\beta^2}{\alpha} \right) + \frac{(\lambda_0 - \lambda)}{\alpha} = 0 \]

...
IV. Solutions of the Third Type.

§15. Another type of solutions may be obtained on the assumption that the component $T_{4}^{4}$ of the material-energy-tensor is constant.

From the equation (1.6), we get

$$\frac{e^{-\lambda} r^1}{r} + \frac{1 - e^{-\lambda}}{r^2} = \text{const.}$$

The solution is

$$1 - e^{-\lambda} = A r^2 + \frac{A'}{r}$$

To obtain solutions regular at the origin, we neglect $A'$, and we have

$$e^{-\lambda} = \frac{1}{1 - A r^2}$$

Substituting this value of $\lambda$ in (6.2), we get

$$\frac{1}{2} v'' + \frac{q}{r} v'^2 + v' \left[ \frac{1}{2} - \frac{m}{n} - \frac{1}{q} \frac{A'}{1 - A r^2} \right] + \frac{m A}{1 - A r^2} - \frac{1}{2} \frac{2 A}{1 - A r^2} - \frac{N}{1 - A r^2} = 0$$

where

$$N = (m-1) \beta - 8 \pi n$$

or,

$$\frac{1}{2} v'' + \frac{q}{r} v'^2 + \frac{1}{r} v' \left[ 1 - \frac{m}{n} - \frac{A'}{1 - A r^2} \right] + \frac{(m-1) A - N}{1 - A r^2} = 0 \quad \cdots \cdots \cdots (15.2)$$

Multiplying the left hand side of the equation (15.2) by $e^{\frac{1}{2} v}$ and putting $e^{\frac{1}{2} v} = \xi$, we have

$$\xi'' + \xi' \left\{ \frac{(m-1)(1-A r^2)-A r^3}{2 (1-A r^2)} \right\} + \frac{\xi}{1-A r^2} \left\{ (m-1) A - N \right\} = 0$$

or,

$$(\xi-A r^3) \xi'' + \left\{ \frac{(m-1)-2(m-1) A r^2}{2 (1-A r^2)} \right\} \xi' + \frac{\xi}{2 (m-1) A - N} \xi = 0 \quad \cdots \cdots \cdots (15.3)$$
Assume a solution to be

\[ z = c_0 + c_1 z + c_2 z^2 + \cdots + c_k z^k + \cdots \]  \hspace{1cm} (15.4)

Substituting in (15.3), we have

\[ (\lambda - \alpha z^3) \left\{ \cdots + c_{k-1} (k-1)(k-2) z^{k-3} + \cdots + c_{k+1} (k+1) z^{k+1} \right\} \]

\[ - \left\{ \frac{(m-1)-z (m-1) \alpha z^2}{\lambda - z} \right\} \left\{ \cdots + c_{k-1} (k-1) z^{k-2} + \cdots + c_{k+1} (k+1) z^{k+1} \right\} \]

\[ + \left\{ \frac{(m-1) A - N}{\lambda - z} \right\} \left\{ \cdots + c_{k-1} z^{k-1} + \cdots \right\} = 0 \]

Equating the coefficients of \( z^k \) to zero, we have

\[ c_{k+1} \left\{ K(k+1) - (m-1)(k+1) \right\} \]

\[ - c_{k-1} \left\{ (k-1)(k-2) A - (m-1)(k-1) A + \Lambda \right\} = 0 \]

or,

\[ c_{k+1} (k+1)(k+2-2m) = c_{k-1} \left\{ (k-1)A - (m-1)A + N \right\} \]

where

\[ N' = \frac{(m-1)-z \Lambda}{\Lambda} \hspace{1cm} (15.5) \]

From (15.5), we obtain the series of relations:

\[ k = 1, \quad c_2 A (2-2m) = c_0 A (-N') \]

\[ k = 2, \quad c_3 A (3-2m) = c_1 A \left\{ (2-2m) - N' \right\} \]

\[ k = 3, \quad c_4 A (4-2m) = c_2 A \left\{ 2(3-2m) - N' \right\} \]

\[ k = 4, \quad c_5 A (5-2m) = c_3 A \left\{ 3(4-2m) - N' \right\} \]

Hence we obtain

\[ z = c_0 S_0 + c_1 S_1 + \cdots \]  \hspace{1cm} (15.7)

where

\[ S_0 = 1 - \frac{A N'}{2(2-2m)} z + \frac{A N'}{2(2-2m)} \frac{A (2(3-2m) - N')}{4 (4-2m)} z^2 - \cdots \]  \hspace{1cm} (15.8)

\[ S_1 = z + A \left\{ \frac{(2-2m) - N'}{2 (3-2m)} \right\} z^2 + A \left\{ (2(3-2m) - N') \right\} \frac{A (3(4-2m) - N')}{8 (5-2m)} z^3 - \cdots \]  \hspace{1cm} (15.9)

As we are concerned with exact solutions only, the series must terminate. Hence we must have

\[ (K-1)(k-2m) - N' = 0 \]
for some integral value of $k$; i.e., we must have

$$A(k-1)(k-2\mu) - \beta(m-1)\beta - \beta = 0.$$ \hspace{1cm} (15.10)

For any integral value of $k$, this equation amounts to a linear relation between $m$ and $n$, which must be satisfied in order that either of the series (15.8) or (15.9) may terminate, so that the gravitational equations may have an exact solution of the form (15.4) on the assumption that $\tau = \text{const}$, and $\tau^2 = m \tau + n$.

§16. As will appear from §3, the most case arises when $\tau^2$ and $\tau'$ are proportional. If we also assume that the natural curvature of space is negligible compared with $\mu$ that due to the presence of matter, we may take $\nu = 0$ and $\beta = 0$, so that $N = 0$.

The equation (15.10) then becomes

$$(k-1)(k-2\mu) - (m-1)\beta = 0.$$ \hspace{1cm} (16.1)

The values of $k$ for integral values of are as follows:

$$k = 1, 2, 3, 4, 5, \ldots$$

$$m = 1, 2, \frac{13}{5}, \frac{7}{3}, \ldots$$

where $\rho$ = proper density of matter.

[References]

1. Berlin Sitz., 1914.
4. Theory of Relativity, Ch. XIV.
V. The Problem of the Material Sphere.

§17. There seems to exist a certain amount of difference of opinion regarding the interpretation of the components of the material energy tensor as well as their mutual relations for different kinds of actual matter. The final settlement of the question evidently depends on the ultimate structure of matter and therefore any assumption made regarding them must necessarily be of a somewhat tentative nature. We shall follow the procedure adopted by Einstein, Schwarzschild, Bauer, Silverstein, and Eddington, and assume that the material energy tensor is given by

\[ T^\rho_\nu = \rho \frac{dx_\rho}{ds} \frac{dx_\nu}{ds}, \quad (\rho, \nu = 1, 2, 3, 4) \]  

for any material medium, and in the special case of a fluid medium

\[ T^\rho_\nu = -\delta_\rho^\nu \rho + q \frac{dx_\rho}{ds} \frac{dx_\nu}{ds} \rho, \quad (\rho, \nu = 1, 2, 3, 4) \]  

where \( \rho = \) proper density of matter,

and

\[ \delta_\rho^\nu = \begin{cases} 0, & \text{for } \rho \neq \nu \\ 1, & \text{for } \rho = \nu \end{cases} \]  

1 Berlin Sitz., 1914.
3 Vienna Sitz., 1918, p. 2141.
4 Theory of Relativity, Ch. XIV.
5 Mathematical Theory of Relativity, §§ 53.54.
For a stationary radially symmetric field, the components are, in the first case,

$$\mathbf{T}^0 = - \dot{\Phi} - \frac{1}{\rho} \mathbf{r} \cdot \mathbf{r}$$

where $\dot{\Phi} = \Phi$ identically.

In the case of the fluid, we have

$$\mathbf{p} = - \rho + \nabla \mathbf{r}$$

where $\rho$ is the pressure at any point.

In the first case, we have

$$\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$$

and in the second case

$$\mathbf{T} = \rho - 4 \rho$$

In the case of a medium in which the compressibility is ideal, we may take the two invariants representing the density and the pressure to be equal and thus define such a medium by the equation

$$\rho = \rho$$

and call it an ideal gas.

To sum up, we make the following assumptions:

1. For a material medium, $\mathbf{T}_1 = \dot{\Phi}$, $\mathbf{T}_2 = \mathbf{T}_3 = \frac{1}{\rho} \mathbf{r} \cdot \mathbf{r}$, $\mathbf{T}_4 = \text{mass-density}$.

2. For a fluid medium, $\mathbf{T}_1 = \mathbf{T}_2 = \mathbf{T}_3 = - \rho$, $\mathbf{T}_4 = \text{pressure}$.
(3) For an ideal gas, \( T_1' = T_2' = T_3' = -\beta = -\rho \), and \( T_4' = 0 \)

§ 18. The paths of light in any space-time are the geodesic null lines. In the space-time \((1,2)\), they are given by the equations

\[
\frac{ds}{ds} = 0, \quad \frac{d^2x}{ds^2} + \{b\varphi, \alpha\} \frac{dx_y}{ds} \frac{dx_\varphi}{ds} = 0 \quad (b\psi = 1,2,3,4)
\]

where \((x_1, x_2, x_3, x_4) = (r, \theta, \varphi, t)\)

It is found that the geodesics in the plane \(\theta = \frac{\pi}{4}\) are given by

\[
-\hat{\alpha}^2 \left( \frac{dr}{d\psi} \right)^2 - r^2 + e^\eta \left( \frac{dt}{d\psi} \right)^2 = 0, \quad \ldots \quad (18.1)
\]

\[
\frac{d^2\varphi}{ds^2} + \frac{\theta^2}{r} \frac{dr}{ds} \frac{d\varphi}{ds} = 0, \quad \ldots \quad (18.2)
\]

\[
\frac{d^2t}{ds^2} + \nu \frac{dt}{ds} \frac{d\psi}{ds} = 0, \quad \ldots \quad (18.3)
\]

From (18.2) and (18.3), we have

\[
h^2 \frac{d\psi}{ds} = k, \quad \frac{dt}{ds} = c_1 e^{-\nu}, \quad \ldots \quad (18.4)
\]

where \(h, k, c_1\) are constants. Since \(ds = 0\), \(k, c_1\) are infinite and but \(h, c_1\) is finite and equal to \(c\), say. Then

\[
h^2 \frac{d\psi}{ds} = ce^\nu
\]

Substituting this in (18.1), we obtain the differential equations of the light-paths in the plane \(\theta = \frac{\pi}{4}\) as

\[
e^{\lambda} \left( \frac{dr}{d\psi} \right)^2 + r^2 - e^\nu \left( \frac{\lambda}{c_1 e^\nu} \right)^2 = 0
\]

or,

\[
e^{\lambda} \left( \frac{dr}{d\psi} \right)^2 + u^2 - \frac{e^\nu}{c_1} = 0, \quad \ldots \quad (18.5)
\]

where

\[
u = \frac{1}{2}
\]

§ 19. We proceed to find the gravitational field of a material sphere of radius \(a\) whose boundary is free from tractions. Since the natural curvature is negligible compared with the curvature produced by matter, we shall assume \(\beta = 0\). We also assume that the radial and transverse stresses are proportional, so that
Solution of the first type:

An appropriate solution is given by (8.1), (8.2), (8.5).

We have
\[ e^\lambda = \frac{1}{1 - A \alpha^{2m}} \]
\[ e^\nu = \left\{ C - (1 - A \alpha^{2m}) \right\} \frac{2}{3} \]

and
\[ 8\pi r_0^4 = (2m+1)A \alpha^{2m-2} \]

The density at the centre vanishes or becomes infinite, except when \( m = 1 \), a case to be considered later (Sec. VII.)

Let the density at \( r = a_0 \) be \( \rho_0 \). Then
\[ A = \frac{8\pi r_0}{(2m+1)A_0^{2m-2}} \]

Since \( T_1' = 0 \) when \( r = a \), we have
\[ \frac{8\pi r_0}{(2m+1)A_0^{2m-2}} \left\{ (1 - A a_0^{2m}) \frac{2}{3} - (1 - A a_0^{2m}) \right\} = 0 \]

Thus we find
\[ \frac{\gamma'}{T_1'} = \frac{\rho_0}{(2m-1)(1 - A a_0^{2m})^{2m-2}} \]
and the law of density is
\[ \rho = \rho_0 \left( \frac{a}{a_0} \right)^{2m-2} \]

From (18.1), we see that the stresses will be infinite either if
\[ A \alpha^{2m} = 1 \]
or if
\[ (1 - A \alpha^{2m}) \frac{2}{3} = (2m+1) \]

Hence the natural limit to the radius of the sphere is given by the lesser of the two values of \( a \) given by the above equations, i.e., the lesser of the values

\[ \left\{ \frac{(2m+1)a_0^{2m-1}}{8\pi \rho_0} \right\}^{\frac{1}{2m}} \]

and
\[ \left[ \frac{(1-(2m+1)^{2})}{(2m+1)a_0^{2m-1}} \right]^{\frac{1}{2m}} \]
provided either of them is real.

If \( m \) is such that \((2m + 1)^{2} > 1\), the second limit is imaginary and the first is imaginary if \( 2m + 1 < 0 \). If \( 2m + 1 = 0 \), there is no natural limit.

The differential equation of the light-paths in the above medium in the plane \( \theta = \frac{\pi}{2} \) is found to be, from (18.5),

\[
\frac{1}{1 - \beta \cos^{2} \theta} \left( \frac{d^{2} \mathbf{r}}{dv^{2}} \right)^{2} + \frac{\beta^{2}}{c^{2}} \frac{1}{\left( C - (1 - \beta \cos^{2} \theta) \frac{1}{2} \right)^{2}} = 0, \quad \ldots \ldots (19.3)
\]

§20. Solution of the second type:

Another solution is given by the formulae (10.3), (10.5), provided that \( N = 0 \). We get

\[
\mathbf{r} = (\mathbf{A} + \mathbf{B} \cos^{2} \theta - 1) \left( \frac{1}{H} \right) \quad \ldots \ldots (20.1)
\]

\[
-\frac{1}{\mathbf{a}} = \left( \frac{C + (2m + 1) B \cos^{2} \theta}{C + B \cos^{2} \theta} \right)^{2m + 1} \quad \ldots \ldots (20.2)
\]

From (1.3), (1.6), we find

\[
8 \pi T_{1}' = \frac{H r^{2m-2}}{\left( C + (2m + 1) B \cos^{2} \theta \right)^{2m+1}} + \frac{4m B r^{2m-1}}{C + B \cos^{2} \theta} \left\{ 1 - \frac{H r^{2m}}{\left( C + (2m + 1) B \cos^{2} \theta \right)^{2m+1}} \right\} \ldots \ldots (20.3)
\]

\[
8 \pi T_{4}' = \frac{H r^{2m-2}}{\left( C + (2m + 1) B \cos^{2} \theta \right)^{2m+1}} \left\{ \frac{2m B r^{2m}}{C + (2m + 1) B \cos^{2} \theta} \right\} \ldots \ldots (20.4)
\]

There are three arbitrary constants, viz., \( B, C \) and \( H \).

From (20.4), we see that when \( T_{b} = 0 \), \( B = H = 0 \). Since the space-time must then be galilean, we may take \( C = 1 \). The constants \( B \) and \( H \) depend on the following conditions:

1. \( T_{4}' = \rho = 0 \), when \( r = a \).
2. \( T_{1}' = -\sqrt{r} = 0 \), when \( r = a \).
§21. A particular case is of interest, viz., when
\[ m = -1 \]. In this case \( T_2 = T_3 = -T_1' \), i.e., the radial and the transverse stresses are equal and opposite.

The solutions of the first type reduce to the following:

\[ e^\lambda = \frac{r^2}{r^2 - A} \]

\[ e^\nu = \left\{ C - \left(1 - \frac{A}{r^2}\right)^{1/2} \right\}^2 \]

\[ p = p_0 \left( \frac{a^2}{r^2} \right)^q \]

and

\[ T_1' = \frac{p_0 a^4}{r^2 (r^2 - A)^{1/2}} \left\{ \frac{(r^2 - A)^{1/2} - (1 - \frac{A}{r^2})^{1/2}}{1 + (1 - \frac{A}{r^2})^{1/2}} \right\} \]

It will be seen that there is no natural limit to the radius of the sphere.

The second type of solutions gives

\[ e^\nu = \frac{(r^2 + B)^2}{r^2} \]

\[ e^\lambda = 1 - \frac{B r^2}{(r^2 - B)^2} \]

\[ 8\pi p = \frac{H}{(a^2 - B)^2} \left\{ 3 - \frac{4a^2}{a^2 - B} \right\} \]

\[ 8\pi T_1' = \frac{1}{(r^2 - B)^2} \left[ \frac{H + \frac{B}{r^2} (a^2 - B)}{a^2 - B} \left\{ (a^2 - B)^2 - H a^2 \right\} \right] \]

If \( p_0 \) be the density at the centre, we have

\[ 8\pi p_0 = \frac{H}{B^2} \]

If the radius of the sphere be \( a \), then \( T_1' = 0 \) when \( r = a \)

\[ H a^2 (a^2 + B) + B \left\{ (a^2 - B)^2 - H a^2 \right\} = 0 \]

or,

\[ 8\pi p_0 a^2 (a^2 + B) + B (a^2 - B)^2 - B H a^2 = 0 \]

or,

\[ B^2 + 2a^2 B (4\pi p_0 a^2 - 1) + a^4 = 0 \]

Hence a natural limit is imposed upon the radius of the free boundary of the surface by the condition

\[ (4\pi p_0 a^2 - 1)^2 > a^4 \]
VI. The Gravitational Field of a Sphere of Perfect Fluid.

§22. In a perfect fluid, we take $\mathcal{T}_1' = \mathcal{T}_2^2 = \mathcal{T}_3^3 = -\rho$, the pressure of the fluid and $\mathcal{T}_{4}^4 = \tau - \rho$, where $\rho$ is the density of the fluid.

In this case, the relation $\mathcal{T}_2^2 = m \mathcal{T}_1' + n$ is reduced to $\mathcal{T}_2^2 = \mathcal{T}_1'$, so that $m = 1$ and $n = 0$. We may also neglect the natural curvature of space, so that $\beta = 0$. The appropriate solutions are given by (8.12), (8.13), and (8.7) and (8.9). We see that $\mathcal{T}_4^4 = \tau - \rho = \text{const.}$ This condition has been interpreted by Schwarzschild as expressing the incompressibility of the fluid and so, in his opinion, the above solution refers to the field of a perfect ideal liquid. Eddington does not approve of this assumption (Mathematical Theory of Relativity, §72.)

§23. Another type of solutions can be obtained from the second class of solutions considered in Sec. III.

We have

$$\mathcal{E}_1 = \frac{(3\alpha n^2 + 8)}{(3\beta n^2 + 8)^{\frac{3}{2}} - cn^2}, \quad \ldots \quad (23.1)$$

$$\mathcal{E}_2 = \frac{(\beta n^2 + 8)^{\frac{1}{2}}}{c}, \quad \ldots \quad (23.2)$$

where $c, \beta, n$ are arbitrary constants. Also

$$\mathcal{G} \mathcal{E}_1 \mathcal{T}_4^4 = \frac{c(5\beta n^2 + 8)}{(3\beta n^2 + 8)^{\frac{5}{2}}},$$

$$\rho = -\mathcal{T}_1' = -\mathcal{T}_2^2 = -\mathcal{T}_3^3$$

$$= \frac{1}{s \pi (\rho n^2 + 8)(3\beta n^2 + 8)^{\frac{3}{2}}} \left[ \frac{4\beta [(3\beta n^2 + 8)^{\frac{3}{2}} - cn^2] - c(\beta n^2 + 8)}{s \pi (\rho n^2 + 8)(3\beta n^2 + 8)^{\frac{3}{2}}} \right], \quad (23.3)$$
and
\[ \phi = T_u^n + \beta \]
\[ = \frac{c \left( 5y^2 + z \delta \right)}{8\pi \left( 5y^2 + 3z \right)} + \beta \]  \hspace{1cm} (23.4)

The three arbitrary constants \( c, \beta, \delta \) are determined by the following conditions:

1. the pressure \( \rho = 0 \) at \( r = a \), the radius of the sphere,
2. the density at the centre \( \phi_0 \), a given quantity,
3. the field must reduce to the galilean when there is no matter, i.e., when \( \rho = 0 \) everywhere.

\[ \text{§24. The problem of a spherical mass of heterogeneous fluid can also be treated in the following manner:} \]

Let \( \tau_u^n = \psi(r) \), where \( \psi \) is a function of \( r \) only.

From (1.6), we get, on integration,
\[ (1 - e^\lambda) r = \beta + 8\pi \int \psi(r) r^2 dr, \]
whence
\[ e^\lambda = \left[ 1 - \frac{\beta}{8\pi} \int \psi(r) r^2 dr \right]^{-1} \]
\[ = \phi(r), \text{ say}. \]  

The condition \( \tau_1' = \tau_2 = \tau_3 \) gives
\[ v'' + v^2 - \left( \frac{2}{\alpha} + \lambda \right) v' - \frac{2}{\alpha} \lambda' + \frac{\psi(\lambda)}{\alpha^2} = 0 \]

Multiplying by \( e^{\lambda v} \) and putting \( e^{\lambda v} = z \), we get
\[ 4z'' - 2 \left( \frac{2}{\alpha} + \lambda \right) z' - \left\{ \frac{2}{\alpha} \lambda' - \frac{\psi(\lambda)}{\alpha^2} \right\} z = 0, \]
or,
\[ z'' + v_z z' + v_z z = 0 \]  \hspace{1cm} (24.1)

where
\[ v_z = -\frac{1}{\alpha} \left( \frac{2}{\alpha} + \lambda \right) \]
\[ v_z = -\frac{1}{\alpha} \left\{ \frac{2}{\alpha} \lambda' - \frac{\psi(\lambda)}{\alpha^2} \right\} \]  \hspace{1cm} (24.3)

When \( \psi(n) \) is given, we obtain \( \lambda \) from (24.1), \( \alpha \) from (24.2) and \( z \) from (24.3). Having obtained \( \lambda \) and \( \nu \), we get \( \beta \) from (1.3) and then \( \rho \) from the equation \( \phi = \beta + \psi(n) \). Hence the field is completely determined. The
constants of integration can be determined from the conditions at the centre and at the boundary and from the fact that the field reduces to a galilean one when then there is no matter.

We shall discuss one example.

Let \( \psi(r) = \frac{B}{8\pi r^2} \) \hspace{1cm} (24.4)

where \( B \) is a constant, and let the arbitrary constant \( A \) in (24.1) be equal to zero. Then

\[ \phi(r) = e^\lambda = \frac{1}{1 - B} \quad \text{a const.} \] \hspace{1cm} (24.5)

From (24.3), we have

\[ \psi_1 = \frac{l}{r}, \quad \psi_2 = \frac{c}{r^2} \]

where

\[ c = \frac{B}{B - 1} \]

and the equation (24.2) becomes

\[ \lambda'' - \frac{l}{2} \lambda' + \frac{c}{r^2} \lambda = 0 \] \hspace{1cm} (24.6)

The solution of (24.6) is evidently

\[ \lambda = A_1 r^{k_1} + A_2 r^{k_2} \]

where \( k_1 \), \( k_2 \) are the roots of the quadratic

\[ k^2 - 2k + c = 0 \]

i.e.,

\[ k_1, k_2 = \frac{1}{2} \left( 1 \pm \sqrt{1 - c} \right) \]

Hence

\[ e^\psi = \lambda_1 = \left( A_1 r^{k_1} + A_2 r^{k_2} \right)^2 \] \hspace{1cm} (24.8)

Now, \( B \) is assumed to be a positive quantity. If \( B > 1 \), \( k_1 \) and \( k_2 \) are imaginary. If \( B = 1 \), \( e^\lambda \) becomes infinite. Hence \( B \) must be a positive quantity less than unity.

If we put \( A_2 = 0 \), we get \( e^\psi = A_1 r^{2k_1} \)

Since \( B \) is positive and less than 1, \( \frac{l}{\sqrt{1 - B}} \) is positive and greater than 1. Hence from (24.7), \( k_1 \) is an essentially negative quantity. Let \( k_1 = -k \), where \( k \) is positive. Also
take $A_{1} = 1$, which implies merely a change in the unit of time.

Thus,

$$e^{\gamma} = \frac{J}{n^{k}}$$

The pressure at the maximum at any point is given by

$$p = \frac{1-B}{8\pi} \cdot \frac{-2k(1-B)-B}{(1-B)^{2}}$$
or,

$$p = \frac{C}{8\pi n^{2}}$$

where

$$C = 2k(1-B) + B$$

Hence

$$p = T^{4} + p = \frac{B}{8\pi n^{2}} + \frac{C}{8\pi n^{2}} = \frac{D}{8\pi n^{2}}$$

where

$$D = B + C = 2k(1-B) + 2B$$

Thus we see that the solutions (24.5), (24.8) correspond to the field of a perfect fluid whose density at any point is proportional to the inverse square of the distance from the centre.

Since the density becomes infinite at the centre we must regard the centre to be outside the field. It is also clear from (24.3) that the solution has a singularity at the origin. We may take $\rho_{0}$ to be the density at $n = a$. Then $D = 8\pi a^{2} \rho_{0}$ and $B$ is given by (24.11).

The curvature of space-time $K$ is given by

$$K = 8\pi (3n^{4} + T_{0}^{4}) = -\frac{E}{2n}$$

where

$$E = -B + 3C = 6k(1-B) + 2B$$

We see that when the density varies as $(\text{distance})^{-2}$, the pressure as well as the Riemann curvature obey the same law.

The Newtonian attraction at any point of a spherical mass of fluid with the law of density $\rho = \frac{D}{8\pi n^{2}}$, is given by

$$\varphi = -\frac{1}{n^{2}} \int_{0}^{n} 4\pi n^{2} \rho \, dn = -\frac{D}{2n}$$
If \( p \propto \rho \) and we assume isothermal constant \( k_T = \frac{D}{4} \), we get

\[ p = \frac{f}{\sqrt{\rho}} = \frac{D^2}{2 \pi n^2} \]

whence

\[ dp = -\frac{D^2}{16 \pi n^3} dr = -\rho K dr, \]

which is the hydrostatic condition of equilibrium. We notice that the distribution of pressure and density as well as their proportionality are in this case similar in form both in the Newtonian and in the Einsteinian gravitational fields.

From equation (18.5), we find that the path of light in the above medium is given by

\[ a_1 \left( \frac{da}{dx} \right)^2 + u^2 - a_2 u^{-\kappa} = 0, \]

where

\[ a_1 = \frac{1}{1-B}, \]

\[ a_2 \]

is an arbitrary constant,

and

\[ \kappa = \frac{\sqrt{1-B}}{1-\sqrt{1-B}}. \]

From (25.2) we get

Simplifying this in (25.3) we have

\[ y^{\frac{1}{2}} v^\frac{1}{2} = \frac{1}{\frac{x}{v} + \frac{1}{v}} \]

or, if \( \frac{y}{v} = \frac{1}{v} \),

\[ x^2 = \frac{1}{1-B} \]

so that

\[ x = \beta \sqrt{x^2} = \text{Hence} \quad v = (\beta + \beta x^2)^2. \]

From (1.3), (25.4) and (25.5), we get

\[ \beta = \beta - \gamma \beta = \frac{1}{x^2} \left( \frac{x^2}{2} - \beta x^2 \right) = \frac{\beta}{x^2} \frac{1}{\beta + \beta x^2} \]

Thus the field of an ideal gas, defined by the relation

\[ \rho = \rho \]

is given by

\[ \rho = -\frac{\beta}{x^2} \frac{1}{\beta + \beta x^2} \rho^{\frac{1}{2}} x^2 + (\beta + \beta x^2)^2 \rho \frac{1}{x^2} \cdots (38.7) \]
VII. The Field of an Ideal Gas.

§25. As in §17, we define an ideal perfect gas by the equation \( \rho = \rho \), so that \( \tau = \tau' \) and \( \tau' = \tau'' = -\rho \), and \( \tau = -3\rho \).

We shall first discuss the problem on the assumption that the natural curvature is zero, i.e., \( \beta = 0 \).

The equations (1.6), (7.1), (7.2) become

\[
\frac{1}{2} \lambda' + \frac{\varepsilon^2 \lambda}{h^2} = 0, 
\]

\[
-\frac{1}{2} \lambda' + 2\frac{\varepsilon^2 \lambda}{h^2} = 0, 
\]

and

\[
\frac{1}{2} v'' - \frac{1}{6} \kappa' v' + \frac{1}{6} v'^2 - \frac{1}{2\kappa} v' = 0, 
\]

From (25.1) and (25.2) we get

\[ e^\lambda = 1 \]

Substituting this in (25.3) we have

\[ v'' + \frac{1}{2} v'^2 - \frac{1}{2} v' = 0 \]

or, if \( e^{\frac{1}{2}v} = z \),

\[ z'' - \frac{1}{h} z' = 0 \]

so that \( z = A + B h^2 \). Hence \( e^\lambda = (A + B h^2)^2 \).

From (1.3), (25.4) and (25.5), we get

\[ \rho = \rho = -\tau' = \frac{1}{8\pi} e^\lambda \left( \frac{y'}{h} - \frac{\varepsilon^2 \lambda}{h^2} \right) = \frac{B}{2\pi} \frac{1}{A + B h^2} \]

Thus the field of an ideal gas, defined by the relation \( \rho = \rho \), is given by

\[ ds^2 = -dt^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + (A + B h^2)^2 dt^2, \]

\[ \cdots \]
the pressure and density at any point being given by
\[ \rho = \rho = \frac{B}{2\pi (A + B\alpha^2)} \]


Suppose that the gas is enclosed in a spherical envelope of radius \( a \) and the pressure on the envelope is \( \rho \). We have \( \rho = \frac{B}{2\pi (A + B\alpha^2)} \), whence \( B = \frac{2\pi \rho A}{1 - 2\pi \rho \alpha^2} \). Also, if \( \rho = 0 \) we have \( B = 0 \) and therefore from (25.6), \( \rho = \rho = 0 \) everywhere, so that the space is empty and \( e^v \) must then be unity.

This gives \( A = 1 \). Hence we have

\[ e^v = \left\{ 1 + \frac{2\pi \rho \alpha^2}{1 - 2\pi \rho \alpha^2} \right\} \]

and

\[ \rho = \rho = \frac{\rho}{(1 - 2\pi \rho \alpha^2) + 2\pi \rho \alpha^2} \]

The pressure at the origin = \( \frac{\rho}{1 - 2\pi \rho \alpha^2} \), if this is to be positive and \( \rho \) is positive, we must have \( \rho < \frac{1}{2\pi \alpha^2} \) or \( \alpha^2 < \frac{1}{2\pi \rho} \).

If \( 2\pi \rho \alpha^2 = 1 \), \( e^v \) is infinite. If \( \rho > \frac{1}{2\pi \alpha^2} \), let

\[ 2\pi \alpha^2 \rho - 1 = \frac{2\pi \rho c^2}{2\pi \rho (\alpha^2 - c^2)} , \] so that

\[ \rho = \rho = \frac{\rho}{2\pi \rho (\alpha^2 - c^2)} \]

which shew that, in this case, the solution has a singularity at \( r = \infty \) (which is less than ). Thus we see that, when \( \rho < \frac{1}{2\pi \alpha^2} \) the solution is regular and the pressure is positive everywhere.

Case (b).

If the space occupied by the gas extends from \( r = a \) to \( r = \infty \), and if \( \rho \) be the pressure at \( r = a \), we have, as before

\[ \rho = \rho = \frac{\rho}{1 + 2\pi \rho (\alpha^2 \rho)} . \]
Since \( r > a \), the pressure is positive everywhere. It vanishes at \( r = a \), where \( e^\nu \) becomes infinite. \( e^\nu \) is also infinite when \( p = \frac{1}{2\pi a^2} \). Thus, in this case, unless \( p = \frac{1}{2\pi a^2} \), the solution is regular everywhere except at infinity.

Case (c).

Suppose now that the whole space is enclosed by the gas, the pressure at the origin being given by \( p = p_0 \), a known constant.

Putting \( \alpha = 0 \) in the solution of case (b), we have

\[
\begin{align*}
\rho &= \frac{p_0}{1 + 2\pi p_0 a^2} \\
e^\nu &= \left(1 + \frac{\pi}{2} p_0 a^2\right)^2,
\end{align*}
\]

If \( p_0 \) be assumed to be positive, the pressure and density are positive everywhere and vanish at infinity.

The solution is regular everywhere except at infinity.

\( \S27 \). The spatial part of the metric (25.7) represents a euclidean 3-space, while the Riemann curvature of the space-time (16) is given by

\[
K = 8\pi T = -2\pi \frac{1}{\rho} \frac{1}{1 - 2\pi p_0 a^2 + 2\pi p \lambda^2}
\]

in cases (a) and (b), and

\[
K = -2\pi \frac{p_0}{1 + 2\pi p_0 a^2},
\]

in case (c).

The curvature is of opposite sign to the pressure and vanishes at infinity.

Proceeding as in \( \S18 \), we find the equation of the geodesics in the plane \( \theta = \frac{\pi}{2} \) to be

\[
d^2 \frac{u}{d\xi^2} + u = \frac{k^2}{\lambda^2} \cdot \frac{2Bu^3}{(B + u^2)^3},
\]

(27.1)
where the constants \( n \) and \( \lambda \) are \( \frac{d^2 \phi}{ds^2} = \lambda \), \( \frac{d\phi}{ds} = \frac{1}{(1+3\rho)^{\frac{1}{2}}} \)

For null lines \( ds = 0 \), so that \( \lambda \) and \( \kappa \) are infinite, but \( \frac{\lambda}{\kappa} \) is finite and equal to a constant.

Hence the paths of light are given by the differential equation

\[
\frac{d^2 \phi}{d\sigma^2} + \frac{d\phi}{d\sigma} = -\frac{\lambda}{(\rho^2 + \lambda^2)^{\frac{1}{2}}} \]

where \( \lambda \) is an arbitrary parameter and \( \rho^2 \) is put for the positive quantity \( \beta \).

\[ \text{§28. We now proceed to obtain the solutions corresponding to the general form of Einstein's equations, viz., those containing} \ \beta \ \text{, the cosmological constant. As the gas may be of any density at the origin and it may also be very thin at large distances from the origin, the curvature due to the presence of matter may not, in this case, be so large as to justify the neglect of } \beta \ \text{altogether.} \]

In place of the equations (25.1), (25.2), (25.3), we now have

\[ e^\lambda \left( \frac{\lambda}{\rho^2} + \frac{\lambda - 1}{\rho^2} \right) = \beta \]

\[ -\lambda' - \frac{1}{3}(1-\lambda') = 0 \]

\[ \frac{1}{2}v'' - \frac{1}{4} \lambda v' + \frac{1}{4} \lambda v'' - \frac{1}{2} v' = 0 \]

The solution of (28.1) is

\[ e^\lambda = (1 + C \rho^2)^{-1} \]

which satisfies (28.2), if \( C = -\frac{1}{2} \beta \)

Hence

\[ e^\lambda = (1 - \frac{1}{2} \beta \rho^2)^{-1} \]

Substituting from (28.4) in (28.3) and putting \( \frac{1}{2}v' = z \),

we have

\[ z'' - \frac{1}{n(1 - \frac{1}{3} \beta \rho^2)} z' = 0 \]
which gives

\[ z = A - B \left( 1 - \frac{1}{2} \beta n^2 \right)^{\frac{1}{2}} \]

where \( A \) and \( B \) are constants of integration. Hence

\[ e^{\nu} = \left\{ A - B \left( 1 - \frac{1}{2} \beta n^2 \right)^{\frac{1}{2}} \right\}^2 \]

From (1.3), we get

\[ p = \frac{1}{12 \pi} \left( A - B \left( 1 - \frac{1}{2} \beta n^2 \right)^{\frac{1}{2}} \right) \]

Thus, the field of a mass of ideal gas placed in the De Sitter world is given by the metric

\[ ds^2 = -\left( 1 - \frac{1}{2} \beta n^2 \right)^{-1} \, dt^2 - r^2 \, d\Omega^2 + \left\{ A - B \left( 1 - \frac{1}{2} \beta n^2 \right)^{\frac{1}{2}} \right\}^2 \, d\Omega^2 \]

and the pressure and density at any point by

\[ p = \frac{\rho A}{12 \pi} / \left\{ A - B \left( 1 - \frac{1}{2} \beta n^2 \right)^{\frac{1}{2}} \right\} \]

The solution (28.7) is of the same form as the solution of Schwarzschild for a medium which he defines as an incompressible liquid on the assumption that the cosmological constant is zero. But, as we have already mentioned (§22), Schwarzschild's condition of incompressibility is open to objection and so his form of solution really refers to the field, not of a liquid placed in a galilean world, but of an ideal gas placed in the naturally curved world. This seems to be a more correct interpretation of the form of solution obtained by Schwarzschild.

It should be observed, however, that, though the form of the solution is identical in the two cases, the properties of the field (e.g., the distribution of pressure) are entirely different. We shall therefore proceed to discuss the solution briefly with reference to the medium under our consideration.

§29. Case (a).

Suppose that the external boundary of the fluid is a sphere.
of radius $\alpha$ under a given pressure $p$. We have from (28.6),
\[ p = \frac{\rho}{12\pi} \cdot \frac{1}{A-B \left(1 - \frac{1}{2} \beta \alpha^2 \right)^\ell} \]
and from (28.6), $\beta = 0$ everywhere, we get an empty space. Hence $e^y$ must be unity. Thus from (28.5), we have $A-B = 1$.

We therefore get
\[
A = \frac{p \left(1 - \frac{1}{2} \beta \alpha^2 \right)^\ell}{p \left(1 - \frac{1}{2} \beta \alpha^2 \right)^\ell - 1 \left(1 - k^2 \right) + \kappa}
\]
\[
B = \frac{P_k}{p \left(1 - \frac{1}{2} \beta \alpha^2 \right)^\ell - 1 \left(1 - k^2 \right) + \kappa},
\]
where $\kappa = \frac{p}{12\pi}$.

From (28.6), we have
\[
\beta = \frac{\kappa p \left(1 - \frac{1}{2} \beta \alpha^2 \right)^\ell}{p \left(1 - \frac{1}{2} \beta \alpha^2 \right)^\ell - (P_k)(1 - \frac{1}{2} \beta \alpha^2)^\ell}
\]
The pressure will be finite everywhere unless
\[
a^2 > \frac{1}{\beta^2} \left(1 - \frac{1}{2} \beta \alpha^2 \right)^\ell
\]
or,
\[
a^2 > \frac{1}{2 \pi p} - \frac{\rho}{48\pi^2 p^2}.
\]
This gives an upper limit to the size of the sphere of the fluid under a given pressure. If we neglect $\beta$, this reduces to
\[
a^2 = \frac{1}{2 \pi p}
\]
(Cf. §26, Case (a))

Case (b).

Suppose that the fluid fills all space and that the pressure at the origin is given to be $\beta_0$. We then have
\[
a-B = 1, \quad \beta_0 = \frac{k \alpha}{\alpha-B}
\]
whence
\[
A = \frac{\beta_0}{k}, \quad B = \frac{\beta_0 - \kappa}{k}
\]
and
\[
\beta = \frac{\kappa \beta_0}{\beta_0 - (\beta_0 - k)(1 - \frac{1}{2} \beta \alpha^2)^\ell}
\]
The pressure is positive and finite everywhere.

Since $1 - \frac{1}{2} \beta \alpha^2$ is never negative, the limiting value of $\beta$ is equal to $\kappa = \frac{p}{12\pi}$, and depends on the natural curvature of space and not on the value at the origin.
§30. In case (a), if $p=0$, we find from (29.2), that $p=0$ everywhere, so that the space is empty. In this case, $A=0, B=-1$ and since $e'=\{A - B (1-\frac{1}{3} p n^2)^{1/2}\}^2$, we get $e'=1-\frac{1}{3} p n^2$, which is the solution for De Sitter world.

In case (b), if $p=0$, we find $p=0$ everywhere from (29.5) and the space is empty. From (29.4), we get $A=0, B=-1$. Thus the solution reduces to De Sitter's space-time.

The spatial part of the metric is identical with that of De Sitter world and the momentary 3-space $(\text{at}=0)$ is therefore a non-euclidean hyperbolic space. It will be observed that the distribution of pressure and density at any point is bound up with the constant natural curvature of space and the presence of the fluid changes the constant curvature to a variable one.

If $K$ be the Riemann curvature, we have

$$K = 4p = 8\pi T,$$

or

$$K = 4p \left\{ 1 - \frac{1}{A-B (1-\frac{1}{3} p n^2)} \right\}.$$  (30.1)

In case (b), i.e., when the fluid fills all space, we have

$$K = 4p - \frac{24\pi p_0 \beta}{12\pi p_0 - (12\pi p_0 \beta) (1-\frac{1}{3} p n^2)^2}.$$  (30.2)

If $\beta$ is neglected we have

$$K = 4p - \frac{24\pi p_0}{1+2\pi p_0 n^2},$$

whence it follows that the curvature changes sign at $r = \sqrt{\frac{6\pi p_0 \beta}{2\pi p_0}}$ and if $p_0 < \frac{\beta}{4\pi}$, the curvature is of the same sign throughout.

Proceeding as in §17, the paths of light in the plane of the above medium are found to be given by

$$\frac{3u^2}{3u^2-\beta} \left( \frac{du}{d\phi} \right)^2 + n^2 = \frac{3m^2 u^2}{\{m/\sqrt{\beta - B (3u^2 - \beta)^2}\}^2},$$  (30.2)

where $u = \frac{1}{h}$ and $m^3$ is a variable parameter.
If the squares and higher powers of $\beta$ are neglected, we get, after a little reduction

$$
\left( \frac{du}{d\varphi} \right)^2 = \frac{3u^2u^2(3u^2 + \beta) - (A-B)u^2\{3(A-B)u^2 + B\beta\}}{(A-B)\{u^2 + A\beta\}}
$$

which is of the form

$$
\frac{du}{d\varphi} = \pm \sqrt{u^2(pu^2 + q)}
$$

(30.3)

VII. Consider the solutions (7.3) and (7.4), which

$$
\psi = \int \frac{\theta \exp \left( -\frac{1}{2} \theta^2 \right)}{\left( 1 - \theta^2 \right)^{1/2}} d\theta
$$

where $A, A, C$ are arbitrary constants of integration.

First, put $\gamma = 0$ and $\beta = 0$. We get

$$
\gamma = \gamma_0 \frac{1}{\beta - \beta_0^2} \gamma_0^2 + \gamma_0^2
$$

Now let $\gamma = 1 - \beta_0^2$. Also let $\varphi = \gamma_0$. Then

$$
\gamma = \gamma_0 \frac{1}{\beta - \beta_0^2} \gamma_0^2 + \gamma_0^2
$$

which is Einstein's cylindrical world.

Secondly, let $\gamma = 1 - \beta_0^2$. So that $\gamma = 0$

We get

$$
\gamma = \gamma_0 \frac{1}{\beta - \beta_0^2} \gamma_0^2 + \gamma_0^2
$$

$$
\gamma = \left[ \gamma_0 \frac{1}{\beta - \beta_0^2} \gamma_0^2 + \gamma_0^2 \right]^{1/2}
$$

(31.1)

Now let $\gamma = 1 - \beta_0^2$ and perform the integration. We get

$$
\gamma = \gamma_0 \frac{1}{\beta - \beta_0^2} \gamma_0^2 + \gamma_0^2
$$

$$
\gamma = \gamma_0 \left[ \gamma_0 \frac{1}{\beta - \beta_0^2} \gamma_0^2 + \gamma_0^2 \right]^{1/2}
$$

This is Schwarzschild's solution, with

$$
A = \frac{2\pi \beta}{\gamma_0^2}, \quad \beta = \frac{1}{\gamma_0^2}, \quad C = \frac{1}{\gamma_0^2}
$$

where $\gamma_0$ is the radius of the boundary of the liquid sphere.
VIII. Deduction of Einstein, Schwarzschild and De Sitter's Solutions.

§31. Consider the solutions (7.3) and (7.4), viz.,

\[
\begin{align*}
\varepsilon^\lambda &= \frac{1}{\sqrt{1 - N_r n^2 - A_r n^m}}, \\
\varepsilon^\nu &= \left[ C + \int \frac{B r^{m-1} dr}{(1 - N_r n^2 - A_r n^m)^{\frac{1}{2}}} \right]^2,
\end{align*}
\]

where \( A, B, C \) are arbitrary constants of integration.

**First**, put \( n = 0 \) and \( B = 0 \). We get

\[
\varepsilon^\lambda = \frac{1}{1 - \rho n^2 - A n^m}, \quad \varepsilon^\nu = C^2
\]

Now let \( m = 1 \), \( A = \frac{1}{2} \beta \). Also let \( C' = 1 \). Then \( \varepsilon^\lambda = (1 - \frac{1}{2} \rho n^2)^{-1} \), \( \varepsilon^\nu = 1 \)

Hence

\[
ds^2 = - \left(1 - \frac{1}{2} \rho n^2\right)^{-1} ds^2 - n^2 d\theta^2 - n^2 \sin^2 \theta d\phi^2 + dt^2
\]

which is Einstein's Cylindrical World.

**Secondly**, let \( n = \beta = 0 \), so that \( N = 0 \)

We get

\[
\begin{align*}
\varepsilon^\lambda &= \frac{1}{1 - \rho n^2}, \\
\varepsilon^\nu &= \left[ C + \int \frac{B r^{m-1} dr}{(1 - \rho n^2)^{\frac{1}{2}}} \right]^2
\end{align*}
\]

Now let \( m = 1 \) and perform the integration. We get

\[
\begin{align*}
\varepsilon^\lambda &= \frac{1}{1 - \rho n^2}, \\
\varepsilon^\nu &= \left[ C - B' (1 - \rho n^2)^{\frac{1}{2}} \right]^2
\end{align*}
\]

This is Schwarzschild's solution, with

\[
A = \frac{8\pi n^4}{3}, \quad \text{where } T_n^4 \text{ is const.}
\]

\[
C = \frac{3}{2} (1 - \rho \alpha^2)^{\frac{1}{2}}, \quad B' = \frac{1}{2},
\]

where \( \alpha \) is the radius of the boundary of the liquid sphere.
Thirdly, in the solutions (31.1), let \( C = 0, \ m = 1 \). We get

\[
\begin{align*}
e^\lambda &= \frac{1}{1 - \alpha_0^2} \\
e^\nu &= \frac{\beta_1^2}{\beta_2^2} (1 - \alpha_0^2)
\end{align*}
\]

Let \( A = \frac{1}{3} \beta = B \), so that \( e^\lambda = \frac{1}{1 - \frac{1}{3} \beta \alpha^2} = e^{-\nu} \)

Hence

\[
dS^2 = - \left(1 - \frac{1}{3} \beta \alpha^2\right) dt^2 - \alpha^2 d\phi^2 - \alpha^2 \sin^2 \phi \, d\phi^2 + \left(1 - \frac{1}{3} \beta \alpha^2\right) dt^2
\]

Einstein in a recent issue of the Mathematische Annalen (Dec., 1925) is

\[
K_{\mu \nu} = \frac{1}{4} g_{\mu \nu} \nabla^2 - \nabla g_{\mu \nu} \nabla \beta + \frac{1}{4} g_{\mu \nu} T_{\beta \beta} - \nabla_\mu \nabla_\nu \beta \quad (32.1)
\]

Performing transformation with \( g_{\mu \nu} \), we find

\[
T_{\mu \nu} = 0 \quad (32.1)
\]

In the equation (32.1), \( T_{\mu \nu} \) is the electromagnetic energy tensor whose scalar vanishes.

In free space, i.e., in space where the components \( T_{\mu \nu} \) are zero, the equation (32.1) reduces to the older form, but in fields where \( T_{\mu \nu} \) are not zero, the equations are obviously different.

It is found that in radially symmetric fields, the remaining equations (32.1) admit of solutions similar to those of equation (1.1), though their application to material media is dealt with the difficulty that the tensor \( T_{\mu \nu} \) represents only the electromagnetic energy tensor.

§33. For a radially symmetric field, we take the space-time to be given by (1.2), so that

\[
\begin{align*}
K &= e^{-\frac{1}{2} \phi} \left( \phi' - \frac{1}{2} \phi^2 + \frac{1}{2} (\phi')^2 / \alpha + \frac{(\phi')^2}{\alpha} \right) \\
K_0 &= \frac{1}{4} \phi - \frac{1}{2} \phi' + \frac{1}{4} \phi^2 - \frac{1}{2} \phi''
\end{align*}
\]
IX. Note on Einstein's New Gravitational Equations.

§32. The new form of the equations as proposed by Einstein in a recent issue of the Mathematische Annalen (Dec., 1926) is

\[ K_{pq} - \gamma \gamma_{pq} \kappa = -8\pi T_{pq} \tag{32.1} \]

Performing transvection with \( \gamma_{pq} \), we find

\[ T = 0 \tag{32.2} \]

In the equation (32.1), \( T_{pq} \) is the electromagnetic energy tensor whose scalar vanishes.

In free space, i.e., in space where the components \( T_{pq} \) are zero, the equation (32.1) reduces to the older form, but in fields where \( T_{pq} \) are not zero, the equations are obviously different.

It is found that in radially symmetric fields, the \textit{new} equations (32.1) admit of solutions similar to those of equation (1.1), though their application to material media is best with the difficulty that the tensor \( T_{pq} \) represents only the electromagnetic energy tensor.

§33. For a radially symmetric field, we take the space-time to be given by (1.2), so that

\[ K = -z^2 \left\{ v'' - \frac{1}{2} \lambda' v' + \frac{1}{2} v'^2 + 2 (v' - \lambda')/\kappa + 2(1-e^2)/\lambda^2 \right\} \]

\[ K_{n} = \frac{1}{2} v'' - \frac{1}{2} \lambda' v' + \frac{1}{2} v'^2 - \frac{\lambda'}{\lambda} \]
\[ k_{32} = e^{-\frac{\ell'}{2}} \{ -\frac{1}{2} \lambda (v' - \lambda') \} - 1 = k_{33} / \delta \omega^2 \theta \]

\[ k_{44} = e^{\lambda} \{ -\frac{1}{2} v'' + \frac{1}{4} \lambda' v' - \frac{1}{6} v'^2 - v'/\ell \} \]

Hence

\[ k_{1} - \frac{1}{4} g_{11} K = \frac{1}{4} \left( v'' - \frac{1}{2} \lambda' v' + \frac{1}{2} v'^2 \right) - \frac{1}{2} v'/\lambda - \frac{1}{4} \lambda'/\ell - \frac{1}{2} \frac{1-e^\lambda}{\ell^2} \]

\[ k_{22} - \frac{1}{4} g_{33} K = -\frac{1}{4} e^{-\lambda} \left( \frac{1}{2} v'' - \frac{1}{4} \lambda' v' + \frac{1}{4} v'^2 - \frac{1-e^\lambda}{\ell^2} \right) \]

\[ k_{33} - \frac{1}{4} g_{33} K = s \omega^2 \theta \left( k_{22} - \frac{1}{4} g_{44} K \right) \]

\[ k_{44} - \frac{1}{4} g_{44} K = \frac{1}{4} e^{\lambda} \left( -v'' + \frac{1}{2} \lambda' v' - \frac{1}{2} v'^2 - 2 v' v \lambda' + \frac{1-e^\lambda}{\ell} \right) \]

Since \[ k_{1} - \frac{1}{4} g_{11} K = -8 \pi T_{1} = 8 \pi e^{\lambda} T_{1}, \quad \lambda \neq 0, \]
the equations become

\[ 8 \pi T_{1} = \frac{1}{4} e^{-\lambda} \left\{ \frac{1}{2} v'' - \frac{1}{4} \lambda' v' + \frac{1}{4} v'^2 - \frac{\lambda' v}{\ell} - \frac{1-e^\lambda}{\ell^2} \right\} \]

\[ 8 \pi T_{2} = -\frac{1}{2} e^{-\lambda} \left\{ \frac{1}{2} v'' - \frac{1}{4} \lambda' v' + \frac{1}{4} v'^2 - \frac{1-e^\lambda}{\ell^2} \right\} \]

\[ 8 \pi T_{3} = T_{2} \]

\[ 8 \pi T_{4} = \frac{1}{4} e^{-\lambda} \left\{ v'' - \frac{1}{4} \lambda' v' + \frac{1}{4} v'^2 + \frac{2(\lambda' v)}{\ell} - \frac{2(1-e^\lambda)}{\ell^2} \right\} \]

It is easily verified that \[ T = \sum_{p} T_{p} = 0 \]

§34. We proceed to obtain the solutions of (35.1) ...(35.4) on the assumption that \[ T_{1} = MT_{1} + N, \quad \text{where} \quad M \quad \text{and} \quad N \quad \text{are constants.} \] We get, from (33.1) and (33.2)

\[ -\frac{1}{2} \left\{ \frac{1}{2} v'' - \frac{1}{4} \lambda' v' + \frac{1}{4} v'^2 - \frac{1-e^\lambda}{\ell^2} \right\} \]

\[ = M \left\{ \frac{1}{2} v'' - \frac{1}{4} \lambda' v' + \frac{1}{4} v'^2 - \frac{1-e^\lambda}{\ell^2} - \frac{\lambda' v}{\ell} \right\} + 16 \pi N e^\lambda \]

or,

\[ \frac{1}{2} v'' - \frac{1}{4} \lambda' v' + \frac{1}{4} v'^2 - \frac{1-e^\lambda}{\ell^2} - m \cdot \frac{\lambda' v}{\ell} + n e^\lambda = 0, \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (34.1) \]

where

\[ m = \frac{M}{M+1}, \quad n = \frac{16 \pi N}{M+1} \]

The First Type of Solutions:

The equations (34.1) will be satisfied, if
Integrating the first of the equations (34.3), we get

\[ e^\lambda = \frac{1}{1 - \frac{m}{2m - 1} \lambda^2 - \beta \lambda^4} \]

where \( \beta \) is an arbitrary constant.

From the second, we have, if \( e^{\frac{1}{2}v} = u \)

\[ u'' - u' \left( \frac{1}{2} \lambda^4 + \frac{2m}{m} \right) = 0, \]

whence

\[ u = \int B e^{\frac{1}{2}v} \, \omega \, dv + C, \]

where \( B \) and \( C \) are constants of integration.

Thus

\[ e^v = u^2 = \left[ \int \frac{B \omega^2 \, dv}{(1 - \frac{m}{2m - 1} \lambda^2 - \beta \lambda^4)^\frac{1}{2}} + C \right]^2 \]

Particular cases:

(1) Suppose that the arbitrary constant \( \beta = 0 \). We have

\[ e^\lambda = \frac{1}{1 - \frac{m}{2m - 1} \lambda^2} \]

where

\[ I_1 = \int_0^\pi \omega^2 \, d\theta, \quad \omega^2 \theta = \frac{2m}{2m - 1} \theta. \]

(2) Suppose that \( T_2^2 = M T_1' \), so that \( N = 0 \). Then

\[ n = 0 \]

and we have

\[ e^\lambda = \frac{1}{1 - A \omega^2} \]

where

\[ I_2 = \int (i \omega \theta)^{2m - 2m + 1} \, d\theta, \quad \omega \theta = A \omega \theta \]

and \( A' = A^{\frac{m(2m - 1)}{2}} \).

When \( M = 1 \), the solutions are exactly the same as (8.7), (8.9).
§35. The solutions of the second type.

The equations (34.1) will be satisfied, if the following are satisfied:

\[
\begin{align*}
\frac{1}{2} y'' + \frac{1}{2} y'^2 - \frac{m}{\lambda} y' &= 0 \\
- \lambda \left\{ \frac{1}{4} y' + \frac{m}{\lambda} \right\} - \frac{1 - e^{-\lambda}}{e^{2\lambda}} + n e^{\lambda} &= 0
\end{align*}
\]

Integrating the first of these equations, we get

\[
e^\nu = \left( C + B \lambda^{2m+1} \right)^2
\]

Substituting this value of \( \nu \) in the second of the equations (35.1), we have

\[
e^{-\lambda} \left\{ \frac{(2m+1) B \lambda^{2m+1}}{2(C + B \lambda^{2m+1})} + \frac{m}{\lambda} \right\} - \frac{1 - e^{-\lambda}}{\lambda} - n = 0
\]

Let \( 1 - \frac{e^{-\lambda}}{\lambda} = \xi \), so that the equation becomes

\[
\xi' - \xi = \frac{1}{m} \left\{ \frac{1}{\lambda} - \frac{(2m+1) B \lambda^{2m+1}}{2m C + (4m+1) B \lambda^{2m+1}} \right\} = \frac{2m(C + BH^{2m+1})}{2m C + (4m+1) B \lambda^{2m+1}} = 0,
\]

whence

\[
\xi = \frac{C}{\lambda} \left[ \lambda + \int \frac{2m(C + BH^{2m+1})}{2m C + (4m+1) B \lambda^{2m+1}} \right]^{\frac{1}{2}}
\]

where \( H \) is an arbitrary constant and \( S = 4m+1 \). Thus

\[
e^{-\lambda} = 1 - \frac{C}{\lambda} \left\{ \frac{2m C + (4m+1) B \lambda^{2m+1}}{2m C + (4m+1) B \lambda^{2m+1}} \right\}^{\frac{1}{2}} \left[ H + \int \frac{2m(C + BH^{2m+1})}{2m C + (4m+1) B \lambda^{2m+1}} \right]^{\frac{1}{2}} d\lambda \]

Particular cases:

(1) Let \( T = M T \), \( N = 0 \) and therefore \( n = 0 \). We have

\[
e^\nu = \left( C + B \lambda^{2m+1} \right)^2
\]

We find

\[
T' = \frac{m-1}{16m} \left\{ \frac{2m-1}{m} - \frac{(2m+1) B \lambda^{2m+1}}{C + B \lambda^{2m+1}} \right\},
\]
where \( P = \left\{ 2^m C + (\frac{4m+1}{B^2}) B^2 \right\} \frac{1}{4m+1} \)

\( T_0' = -\left( 2^{m+1} \right) T_1' \)

(2) Let \( T_2^l = T_1' + \mathcal{N} \), so that \( \mathcal{N} = 1, \; m = \frac{1}{2} \)

We have

\[
e^\nu = \left( c + B \lambda^2 \right)^2
\]

\[
e^{-\lambda} = 1 - \frac{\mu^2}{(c + 3B\lambda^2)^2} \left[ \mathcal{H} + \int_0^{2\lambda} \frac{(c + B\lambda^2) d\lambda}{\mu^2 (c + 3B\lambda^2)^2} \right]
\]

\[
= 1 - \frac{k^2}{(c + 3B\lambda^2)^2} \left[ \mathcal{H} + \left( \frac{1}{3} \mathcal{I} - (c + 3B\lambda^2)^2 \right) \right],
\]

where \( \mathcal{I} = \int_0^{2\lambda} \frac{d\lambda}{(c + 3B\lambda^2)^2} \), \( \tan^2 \theta = \frac{3B}{c} \lambda^2 \)

When \( \mathcal{N} = 0 \), the solution is the same as (13.1), (13.2).

§36. Solution of the third type.

This type of solution corresponds to the case when \( T_0' \) is assumed to be constant \( \frac{k}{2\pi} \), say.

Since \( T_1' + T_2^1 + T_3^3 + T_4^4 = 0 \) and \( T_2^1 = T_3^3 \), we have

\[
T_2^1 = -\frac{1}{2} T_1' - \frac{A}{4K}.
\]

Substituting \( -\frac{1}{2} \) for \( M \) and \( -\frac{A}{4K} \) for \( N' \), we see that

\[
m = 1, \; n = -24
\]

from (34.2). Hence from (34.4) and (34.5), we get

\[
e^\nu = \frac{1}{1 - \frac{4}{3} \lambda^2 \frac{A}{\lambda^2 - 1}} \Rightarrow \frac{1}{1 - \frac{2}{3} \lambda^2}
\]

omitting \( A \), so that the solution may be regular at the origin, and

\[
e^\nu = \frac{1}{n^2} \left\{ C + B (1 - \frac{2}{3} \lambda^2) \right\}^2,
\]

omitting \( A' \), so that the solution may be regular at the origin, and

\[
e^\nu = \frac{1}{n^2} \left\{ C + B (1 - \frac{2}{3} \lambda^2) \right\}^2
\]
X. Summary and Conclusion.

§37. Assuming that the distribution of matter is radially symmetric, and that the internal stresses are such that the radial and transverse components are linearly related, three types of solutions of Einstein's gravitational equations within matter have been obtained, viz.,

(i) Type I, (7.3) and (7.4)
(ii) Type II, (10.3) and (10.5)
(iii) Type III, (15.1) and (15.7),

together with several particular cases.

Following Einstein, Eddington and Silberstein regarding the interpretation of the material-energy-tensor, some of the above solutions have been applied to the following cases:

(1) The material sphere in which \( \rho \theta = \mu \tau \). The space-time is given by (8.1) and (8.2) and the law of density by (19.2). The natural limit to radius of the sphere is obtained in §19., and the differential equation to the light-path is given by (19.3). An alternative solution is given by (20.1) and (20.2) with a particular case (§21).

(ii) The sphere of perfect fluid. One type of solution agrees with that of Scharzschild, though the interpretation is different. A second type of solution is given by (23.1) and (23.2).

A particular case in which the density at any point varies inversely as the square of the distance from the origin is
discussed and compared with the corresponding Newtonian field.

(iii) Defining an ideal gas by the relation \( p = \rho \), two solutions are obtained, corresponding to the cases when the natural curvature of space is assumed to be zero or not.

In the former case, the space-time is given by (25.7) and the limiting radius of the sphere is \( \frac{1}{\sqrt{2\pi P}} \), where \( P \) is the pressure on the boundary.

In the second case, the space-time is given by (28.7) and the natural limit to the radius is \( \left\{ \frac{1}{2\pi P} - \frac{\beta}{4\pi^2 p^2} \right\}^{\frac{1}{2}} \) which reduces to the former limit when \( \beta = 0 \).

The differential equations to the paths of light are given by (27.2) and (30.3) respectively.

(iv) The solutions of Einstein, Schwarzschild and De Sitter are shewn to be particular cases of the solutions of the first type (§31).

The new equations of Einstein, viz., equations (5.1) are shewn to possess three types of solution somewhat similar to those of the older equation. The first type is given by (34.4), (34.5), the second type by (35.2), (35.3) and the third type by (36.1), (36.2).
II.

On Some Problems in Elasticity.
Contents:

1. Vibrations of an elliptic Plate.
3. Longitudinal Vibrations of a Hollow Cylinder.
4. Torsional Vibrations of a Circular Tube.
5. Vibrations of a Thin Rotating Rod and of a Rotating Circular Ring.
VIBRATIONS OF AN ELLIPTIC PLATE.

(Published in the Proceedings of the Edinburgh Mathematical Society)

§1. INTRODUCTORY.

Since the publication of the memoir of Mathieu* on the transverse vibrations of an elliptic membrane, the subject has been treated by many authors from different points of view. But the corresponding problem of the plate has received but little attention. Mathieu discussed the problem as early as 1869, but the method adopted is different from that followed in the present paper, the main object of which is to apply Whittaker's solutions† of Mathieu's Equation to the problem of the elliptic plate. These solutions are really better suited for numerical calculations than the evaluations of infinite determinants.

* Liouville, t. XIII, 1868.
\section{THE EQUATION OF MOTION AND ITS SOLUTION.}

Let the thickness of the plate be \( z \), the volume-density \( \rho \) and Poisson's ratio \( \sigma \). If \( \psi \) be the transverse displacement, the equation of motion is

\begin{equation}
\frac{\partial^2 \psi}{\partial t^2} + \frac{E \kappa^2}{3 \rho (1 - \sigma^2)} \nabla^2 \psi = 0. \tag{1}
\end{equation}

where

\[ \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + 2 \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}. \]

If \( \frac{2\pi}{P} \) be the period of oscillation, and \( \omega \) varies as \( \cos(\beta t + \epsilon) \), the equation (1) reduces to

\[ \nabla^2 \psi - \kappa^2 \omega^2 = 0 \tag{2} \]

where

\[ \kappa^2 = \frac{\beta^2}{c^2}, \quad c^2 = \frac{E \kappa^2}{3 \rho (1 - \sigma^2)}. \tag{3} \]

From (2), we have \( (\nabla^2 + \kappa^2)\omega = 0 \), and if \( \omega_1 \) and \( \omega_2 \) be respectively the solutions of

\begin{align*}
(\nabla^2 + \kappa^2) \omega_1 &= 0 \tag{4'} \\
(\nabla^2 - \kappa^2) \omega_2 &= 0 \tag{5'}
\end{align*}

then \( \omega = A \omega_1 + B \omega_2 \) is a solution of \( (\sigma^2 - \kappa^2)\omega = 0 \). Hence a solution of (1) is

\[ \omega = (A \omega_1 + B \omega_2) \cos(\beta t + \epsilon). \tag{6} \]

Putting \( x + iy = \frac{h}{2} \cosh(\xi + i\gamma) \), so that \( \xi = \omega_0 t \), give confocal ellipses with focal distance \( 2h \), the equation (4) becomes

* Rayleigh, Theory of Sound, Vol.I, Chap. X.
\[
\frac{d^2 \omega_r}{d \xi^2} + \frac{d^2 \omega_r}{d \eta^2} + k^2 k_r \left( \cosh^2 \frac{\xi}{2} - \cosh^2 \frac{\eta}{2} \right) = 0 \quad \ldots \ldots \quad (7)
\]

Let \( \omega_r = F(\xi)G(\eta) \), where \( F(\xi) \) is a function of \( \xi \) only and \( G(\eta) \) a function of \( \eta \) only. We get

\[
\begin{align*}
\frac{d^2 F}{d \xi^2} + (16 \eta \cosh 2\xi - N) F &= 0 \\
\frac{d^2 G}{d \eta^2} - (16 \eta \cosh \eta - N) G &= 0
\end{align*}
\]

where

\[
a = N - \frac{1}{2} k^2 k_r^2, \quad \eta = \frac{1}{3} k^2 k_r^2
\]

If we put \( \xi = \text{Re}(\xi \frac{\eta}{2}) \), \( \eta = \text{Re}(\xi \frac{\eta}{2}) \) respectively in the equations (8), they both reduce to the equation

\[
\frac{d^3 u}{d \xi^3} + (a + 16\eta \cosh 2\xi) u = 0 \quad \ldots \ldots \quad (10)
\]

The solution of this equation given by Whittaker is

\[
u = \Lambda(\xi) = e^{\mu^2 \xi} u(\xi) \quad \ldots \ldots \quad (11)
\]

where \( u(\xi) \) is a periodic function of \( \xi \), and \( \mu \) is given by the power series in \( \eta \)

\[
\mu = 4\eta \sin \xi \sigma - 12\eta^2 \sin^2 \xi \sigma - 12\eta^3 \sin^3 \xi \sigma + \ldots \quad ; \quad \ldots \ldots \quad (12)
\]

in this equation, \( \sigma \) is a parameter connected with the parameters \( a \) and \( \eta \) of the differential equation by the relation

\[
a = 1 + 8\eta \cosh \sigma + (-16\eta + 8\cosh 4\sigma) \eta^2 - 8\eta^2 \cosh 2\sigma + \ldots \quad ; \quad \ldots \ldots \quad (13)
\]

and \( u(\xi) \) is given by the Fourier Series
where the coefficients are given by the power series
in \( q \) : 
\[
\begin{align*}
\lambda_3 &= q + q^2 c_0 z \sigma + \left(-\frac{11 q}{3} + 5 c_0 4 \sigma \right) q^3 + \left(-\frac{7 q}{4} c_0 2 \sigma + 7 c_0 6 \sigma \right) q^4 + \ldots \\
\lambda_5 &= 3 q + q^3 \beta \mu 2 \sigma + 3 q^3 \mu 4 \sigma + \left(-\frac{27 q}{9} \beta \mu 2 \sigma + q \mu 6 \sigma \right) q^4 + \ldots \\
\lambda_7 &= \frac{1}{3} q^2 + \frac{2}{9} q^2 \beta \mu 2 \sigma + \frac{4 q}{27} q^4 \beta \mu 4 \sigma + \ldots \\
\end{align*}
\]
\[ \text{etc.} \]

A second solution is obtained by putting \(-\sigma\) for \( \sigma \).

Hence the complete solution is
\[
\nu = A \Lambda (x, \sigma, \varphi) + B \Lambda (x, -\sigma, \varphi)
\]

The solution is purely periodic when \( \mu = \sigma \). Let \( \sigma = s \)
be any one of the values of \( \sigma \) which makes \( \mu = \sigma \). Then the
solution (16) reduces to a single term and the general
solution must be taken as
\[
\nu = A \Lambda (x, s, \varphi) + B \Lambda_1 (x, s, \varphi)
\]
where \( \Lambda_1 (x, s, \varphi) \) is the elliptic cylinder function of the
second kind.

In our problem, \( C_r (\eta) \) must be a purely periodic func-

tion; hence we have
\[ G(\eta) = \Lambda \left( \eta - \frac{T}{k}, s, \nu \right) , \]
since the \textit{elliptic cylinder function} of the second kind is not periodic. Also
\[ F(\xi) = A \Lambda \left[ -i\xi - \frac{T}{k}, s, \nu \right] + B \Lambda_i \left[ -i\xi - \frac{T}{k}, s, \nu \right] \]
\[ = A F_i(\xi) + B F_2(\xi) \]
where, for brevity, we have put
\[ \Lambda \left[ -i\xi - \frac{T}{k}, s, \nu \right] = F_i(\xi) \]
\[ \Lambda_i \left[ -i\xi - \frac{T}{k}, s, \nu \right] = F_2(\xi) . \]

Hence
\[ \omega_1 = G(\eta) \left\{ A F_i(\xi) + B F_2(\xi) \right\} . \]  

To obtain the solution of (5), we take \( \omega_1 = P(\xi) Q(\eta) \),
where \( P(\xi) \) is a function of \( \xi \) only and \( Q(\eta) \) is a function of \( \eta \) only, and setting \( a = N + \frac{1}{2} \lambda^2 k^2, \nu = \frac{1}{3} \lambda^2 k^2 \), instead of (9), we get
\[ P(\xi) = C \Lambda \left( i\xi, s, \nu \right) + D \Lambda_i \left( i\xi, s, \nu \right) , \]
\[ Q(\eta) = \Lambda \left( \eta, s, \nu \right) , \]
since \( \Lambda(\eta) \) must be periodic.

Putting, for brevity,
\[ \Lambda \left( i\xi, s, \nu \right) = P_i(\xi) \]
\[ \Lambda_i \left( i\xi, s, \nu \right) = P_2(\xi) \]
we have
\[ \omega_2 = Q(\eta) \left\{ C P_i(\xi) + D P_2(\xi) \right\} . \]
Hence we obtain finally
\[ \omega = \left[ G(\eta) \left\{ \alpha F_1(\xi) + \beta F_2(\xi) \right\} \right] \\
+ A(\eta) \left\{ c P_1(\xi) + D P_2(\xi) \right\} \cdot \sin(b x + c) \quad \ldots (20) \]

§3. THE CASE OF THE ELLIPTIC DISC.

Let \( \xi = \xi_o \) be the boundary of the ellipse, so that we we have, for a rigidly clamped boundary, \( \xi = \xi_o \), \( \omega = \frac{d\omega}{d\xi} = 0 \).

Hence from (20) we have
\[ G(\eta) \left\{ \alpha F_1'(\xi_o) + \beta F_2'(\xi_o) \right\} + A(\eta) \left\{ c P_1'(\xi_o) + D P_2'(\xi_o) \right\} = 0 \quad \ldots (21) \]
where \( F_1'(\xi_o), \ldots \) are the values of \( \frac{dF_i(\xi)}{d\xi} \), \ldots when \( \xi = \xi_o \).

Since the equations (21) must be true for all values of \( \gamma \), the coefficients of \( G(\eta) \) and \( A(\eta) \) must separately vanish. We therefore obtain, after the elimination of \( A, B, C, D \), the two equations
\[ \begin{vmatrix} F_1(\xi_o) & F_2(\xi_o) \\ F_1'(\xi_o) & F_2'(\xi_o) \end{vmatrix} = 0 \quad \begin{vmatrix} P_1(\xi_o) & P_2(\xi_o) \\ P_1'(\xi_o) & P_2'(\xi_o) \end{vmatrix} = 0 \quad \ldots (22) \]

These equations give two types of vibration.

§4. THE CASE OF THE ELLIPTIC ANNULUS.

If the boundaries are the confocal ellipses \( \xi = \xi_o \) and \( \xi = \xi_i \), we must have \( \omega = \frac{d\omega}{d\xi} = 0 \) for these values of \( \xi \). Hence we get, in addition to (22), the following
relations:

\[
G(\eta) \left\{ \alpha F_1(\xi) + BF_2(\xi) \right\} + A(\eta) \left\{ \alpha P_1(\xi) + DP_2(\xi) \right\} = 0,
\]

\[
G'_e(\eta) \left\{ \alpha F'_1(\xi) + BF'_2(\xi) \right\} + A(\eta) \left\{ \alpha P'_1(\xi) + DP'_2(\xi) \right\} = 0.
\]

These four equations must hold all values of \( \gamma \).

Hence the eight coefficients of \( G(\eta) \) and \( A(\eta) \) should all vanish. Thus, on eliminating \( A, B, C, D \), we get

\[
\begin{bmatrix}
F_1(\xi_0) & F_1(\xi_1) & F_1(\xi_2) & F_1(\xi_3) \\
F'_1(\xi_0) & F'_1(\xi_1) & F'_1(\xi_2) & F'_1(\xi_3)
\end{bmatrix} = 0
\]

and

\[
\begin{bmatrix}
P_1(\xi_0) & P_1(\xi_1) & P_1(\xi_2) & P_1(\xi_3) \\
P'_1(\xi_0) & P'_1(\xi_1) & P'_1(\xi_2) & P'_1(\xi_3)
\end{bmatrix} = 0 - - - (23)
\]

These give us the frequency-equations. It appears from the form of these equations that they are fairly simple and numerical calculations can be effected by means of (12), (13), (14), and (15).

§5. AN IMPORTANT PARTICULAR CASE.

There is an infinite number of values of \( s \) (i.e., the values of \( \xi \)) which make \( \mu = 0 \). And corresponding to every value of \( s \), we shall obtain a set of frequency-equations of the type (22) or (23).

Since

\[
\phi = \pm \sqrt{\frac{2\alpha (1-\xi^2)}{E \alpha^2}} \cdot \frac{h^2}{l^2} \cdot \phi, - - - - - (24)
\]
where $\sigma$ is the Poisson's ratio, it is evident that $\varphi$ is a small quantity for all ordinary substances, and hence the powers of $\varphi$ beyond the second may be omitted in obtaining an approximate solution. In this case, we have

$$\mu = \varphi \sin \pi \sigma,$$

and for $\mu = 0$, we have $s = o$ or $\frac{D}{2}$. In either case, the solutions $\Lambda (x, s, \varphi)$ and $\Lambda_1 (x, s, \varphi)$ assume simple forms.

In fact, we get, up to the second power of $\varphi$

$$\Lambda (x, 0, \varphi) = \sin x + (\varphi + \varphi^2) \sin \pi x + \frac{1}{2} \varphi^2 \sin 2 \pi x$$

$$\Lambda_1 (x, 0, \varphi) = -8 \varphi \cos \pi x + \varphi \cos 2 \pi x + 2 (\frac{1}{3} \varphi \cos 3 \pi x - \frac{1}{3} \varphi \cos 3 \pi x)$$

As a first approximation, we may take

$$\Lambda (x, 0, \varphi) = \sin x + \varphi \sin 3 \pi x$$

$$\Lambda_1 (x, 0, \varphi) = -8 \varphi \cos \pi x + \varphi \cos 3 \pi x$$

$$\Lambda (i x, 0, \varphi) = -i (\sinh \pi x + \varphi \sinh 3 \pi x)$$

$$\Lambda_1 (i x, 0, \varphi) = -8 \varphi \cosh \pi x + \varphi \cosh 3 \pi x$$

§6. THE FREQUENCY-EQUATION.

The simplest type for the complete elliptic disc $\xi = \xi_0$, is given by the equation
\[ P_1 \left( \xi_0 \right) P_2' \left( \xi_0 \right) - P_1' \left( \xi_0 \right) P_2 \left( \xi_0 \right) = \alpha \]

Substituting for \( P_1(\xi_0) \), \( P_1'(\xi_0) \), \( P_2(\xi_0) \), \( P_2'(\xi_0) \), from above, we get, retaining the first powers of \( \gamma \) only,

\[ \gamma = \frac{1}{4 \left( 1 - 2 \cos \xi_0 \right)} \quad \cdots \cdots \cdots \quad (2.5') \]

Hence if \( \nu \) be the frequency, we get from (24) and (25),

\[ \nu = \frac{4 \alpha}{\pi \xi^2} \sqrt{\frac{E}{3\rho \left( 1 - \sigma^2 \right)}} \cdot \frac{1}{\left( 4 \cos^2 \xi_0 - 3 \right)} \quad \cdots \cdots \cdots \quad (2.6) \]

§7. NUMERICAL RESULTS.

A few numerical results are added below. In the different cases considered, the material and the eccentricity of the ellipse have been varied, but the thickness and the focal distance are unaltered. The frequency is obtained as the number of vibrations per second. The units employed are the usual c.g.s. units.

The table is given on the next page.
<table>
<thead>
<tr>
<th>Material</th>
<th>Density</th>
<th>$E$ (Young's Modulus)</th>
<th>Poisson's Ratio</th>
<th>$\sigma$</th>
<th>$2\delta$</th>
<th>$2\sigma$</th>
<th>$E$</th>
<th>$\delta$</th>
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<td>.02</td>
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<td>.3</td>
<td>1.24</td>
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<tr>
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<td>8.90</td>
<td>$1.23 \times 10^{12}$</td>
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<td>Glass</td>
<td>2.60</td>
<td>$6.77 \times 10^{12}$</td>
<td>.26</td>
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<td>.3</td>
<td>1.18</td>
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</tbody>
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§1.

When a light thin rod is clamped vertically at the lower end and a weight is attached rigidly to the upper end, there is a certain length of the rod called the critical length, which if exceeded, will not allow the rod to stand upright; in other words, the equilibrium is unstable if the length of the rod exceeds this critical length. This has been verified by comparing the potential energies of a strut, whose length is slightly greater than the critical length, in the state when it is simply contracted and in the state in which it is contracted as well as bent. The comparison was effected by Love\(^1\) on the assumption that the average contraction of the bent state

\(^1\)Elasticity, 2nd. Ed., Art. 265.
is negligible. But this contraction of the rod does actually modify the calculation of energy, though the general conclusion remains practically the same, and in accordance with a suggestion of S. Timoshenko\textsuperscript{1} Love revised his article\textsuperscript{2} and obtained a fresh verification of the problem. The object of the present paper is to arrive at a verification by a different and much easier method.

\section{2}

Adopting Love's notation, let $t$ be the length of the rod (cross-section = $a$) in the unstrssed state which slightly exceeds the critical length, $\frac{\pi}{2} \sqrt{\frac{E}{K}}$, and $t'$ the length of the rod when it is bent. Let $s$ be the length measured from the point of clamping and let $x$ be the value of $\theta$ at the point $s = t'$. Let $x$ be the projection of $t'$ on the vertical.

The equation of equilibrium is

$$\frac{1}{2} B \left( \frac{d\theta}{ds} \right)^2 + K (\cos x - \cos \theta) = 0 \quad \text{(1)}$$

If $\kappa = s \sin \frac{\theta}{2}$ and $s \sin \frac{\theta}{2} = \kappa e^{i\varphi}$, this equation reduces to

\textsuperscript{1} "Sur la stabilite des systemes elastiques", S. Timoshenko, Paris, Ann. des ponts et chaussées, 1213.

\textsuperscript{2} Elasticity, Art. 265 (3rd. Edition)
\[ ds = \sqrt{\frac{B}{R}} \cdot \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \quad (2) \]

Hence
\[
\ell' = \sqrt{\frac{B}{R}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \sqrt{\frac{B}{R}} \cdot K,
\]

where \( K \) is the real quarter-period of the elliptic functions;
\[
= \sqrt{\frac{B}{R}} \int_0^{\frac{\pi}{2}} \left( 1 + \frac{1}{2} k^2 \sin^2 \phi + \frac{2}{5} k^4 \sin^4 \phi \right) d\phi
\]
\[
= \frac{\pi}{2} \sqrt{\frac{B}{R}} \left( 1 + \frac{1}{4} k^2 + \frac{7}{30} k^4 \right) \quad ; \quad (3)
\]
\[
\chi = \int_0^\theta \cos \theta \, ds
\]
\[
= \sqrt{\frac{B}{R}} \int_0^{\frac{\pi}{2}} \frac{\left( 1 - 2 k^2 \sin^2 \phi \right)}{\sqrt{1 - k^2 \sin^2 \phi}} \, d\phi
\]
\[
= \sqrt{\frac{B}{R}} \left\{ \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} - \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \right\}
\]
\[
= \sqrt{\frac{B}{R}} \left\{ 2 E \left( \frac{m}{2} \right) - K \right\}
\]
\[
= \ell' \left\{ \frac{2 E \left( \frac{m}{2} \right)}{K} - 1 \right\} \quad ; \quad (4)
\]
\[
\ell = \ell' \left( 1 + \frac{R \chi}{E \omega} \right)
\]
\[ = \mathcal{L}' \left[ 1 + \frac{R}{E \omega} \left\{ \frac{2E(\frac{\pi}{2})}{\kappa} - 1 \right\} \right] \]  

(5)

We also have
\[ \frac{2E(\frac{\pi}{2})}{\kappa} - \kappa = 1 - \kappa^2 - \frac{1}{8} \mu' \]  

(6)

§3.

We now proceed to calculate the potential energies in the two cases: (1) when the rod is simply contracted; and (2) when it is both bent and contracted. Now, the potential energy lost due to the descent of weight is

(1) in the first case \[ = \frac{R^2 \mathcal{L}}{E \omega} \] 

(2) in the second case \[ = R (\ell - x) \] 

(7)

To calculate the potential energy of deformation, we apply the formula*

\[ \iiint W \, dx \, dy \, dz = \frac{1}{2} \iiint \left( \mu \left( X + v Y + \omega Z \right) \right) dx \, dy \, dz \]

\[ + \frac{1}{2} \iiint \left( \mu X_n + v Y_n + \omega Z_n \right) \, ds \]  

(8)

which expresses the fact that the potential energy of de-

* Love, Elasticity, Art. 120.
formation, is equal to half the work done by the body forces and the surface tractions acting through the displacements from the unstressed state to the state of equilibrium.

In the first case, the strut is in equilibrium under two external forces, viz., the reaction at the base and the downward vertical thrust of the load. Hence the right-hand side of equation (8) is in this case equal to \( \frac{1}{2} \frac{R^2l}{E\omega} \)

In the second case, the external forces on the rod are (1) the reaction of the base which is equivalent to a force \( R \) and a couple \( R\gamma \) (\( \gamma \) being the projection of \( \ell' \) on the horizontal) and (2) the vertical thrust of the load. Hence the right-hand side of (8) is in this case equal to \( \frac{1}{2} R (\ell - x) \)

Hence it follows that the potential energy in the bent state will be less than that in the simply-contracted state if

\[ \frac{1}{2} R (\ell - x) > \frac{1}{2} \frac{R^2l}{E\omega} \]

i.e., if

\[ \ell - x > \frac{Rl}{E\omega} \]

or,

\[ x < \ell \left(1 - \frac{R}{E\omega}\right) \quad (7) \]

Substituting from (4), (6), and (5), this condition becomes
\[ \mathcal{L}' \left\{ 1 - k^2 - \frac{1}{8} k' \right\} < \mathcal{L}' \left\{ 1 + \frac{R}{E \omega} \left( 1 - k^2 - \frac{1}{8} k' \right) \right\} \left( 1 - \frac{R}{E \omega} \right) \]

or,

\[ 1 - k^2 - \frac{1}{8} k' < 1 - \frac{R}{E \omega} \left( k^2 + \frac{1}{8} k' \right) \]

or,

\[ \left( 1 - \frac{R}{E \omega} \right) \left( k^2 + \frac{1}{8} k' \right) > 0 \]

which is of course true.

§4.

This method can also be applied to shew that in the case when there are more points of inflexion than one, the form with the least number of inflexions is the most stable.

To shew this, let \( x_n, x_s \) be the values of \( x \) corresponding to the cases in which there are \( n \) and \( s \) inflexions respectively. Let \( (k_n, k_s), (E_n, E_s) \) be the corresponding values of \( k, k \) and \( E \). Let \( s > n \).

The condition that the potential energy in the case of \( s \) inflexions should be greater than that in the case of \( n \) inflexions is

\[ \frac{R (\ell - x_n)}{2} > \frac{R (\ell - x_s)}{2} \]

or,

\[ x_s > x_n \]
The condition (11) may be stated in the form:

The greater the vertical height of the load, the greater is the potential energy.

Now, since

\[ x_n = \ell' \left\{ 2 E_n \left( \frac{\pi}{n} \right) / k_n - 1 \right\} \]
\[ x_s = \ell' \left\{ 2 E_s \left( \frac{\pi}{n} \right) / k_n - 1 \right\} \]

the condition (11) becomes

\[ \frac{E_s \left( \frac{\pi}{n} \right)}{k_s} > \frac{E_n \left( \frac{\pi}{n} \right)}{k_n} \]

Since

\[ \ell' = (2n-1) \sqrt{\frac{b}{r}} \cdot k_n \]
\[ = (2s-1) \sqrt{\frac{b}{r}} \cdot k_s \]

the condition is

\[ (2s-1) E_s \left( \frac{\pi}{n} \right) > (2n-1) E_n \left( \frac{\pi}{n} \right) \quad \ldots \ldots \quad (12) \]

We have

\[ E \left( \frac{\pi}{n} \right) = \int_{0}^{\pi} \frac{E}{1 - \kappa^2 \omega_i \omega_n} \, d\phi \; ; \]

\[ \frac{d}{dk} \left\{ \frac{E \left( \frac{\pi}{n} \right)}{k} \right\} = -\frac{1}{k^2} \int_{0}^{\pi} \frac{\omega_i \omega_n}{\sqrt{1 - \kappa^2 \omega_i \omega_n}} \, d\phi \; = \frac{\kappa}{k^2} \]

\[ \therefore \frac{1}{k} \frac{d}{dk} \left[ E \left( \frac{\pi}{n} \right) \right] = -\frac{E \left( \frac{\pi}{n} \right)}{k^2} = \frac{\kappa}{k^2} \]
Thus $E(\frac{\pi}{2})$ decreases as $k$ increases.

Since $s > n$, we have $\zeta_s < \zeta_n$, and $\kappa_s < \kappa_n$.

Hence $E_s(\frac{\pi}{2}) > E_n(\frac{\pi}{2})$

Therefore the inequality (12) is true and it follows immediately that the form with the least number of inflexions corresponds to the minimum potential energy and is therefore most stable.
LONGITUDINAL VIBRATIONS OF A HOLLOW CYLINDER.

(Published in the Bulletin of the Calcutta Mathematical Society.)

§1.

The longitudinal vibrations of a thin circular cylinder have been discussed at great length by Lord Rayleigh.\(^1\) A second approximation (retaining terms up to the square of the radius of the cylinder), generally known as Pochhammer's solution, has also been obtained by C. Cheee.\(^2\) The frequency equation for a solid cylinder of any radius is given in Love's Elasticity. The object of this paper is to obtain the general frequency equation for hollow solid bounded by two co-axial circular cylinders.

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\(^1\) Theory of Sound, Vol.1, Chap. VII.
§2.

We take the axis of the cylinder as the axis of \( z \) and \((r, \theta, z)\) the cylindrical coordinates of any point. Denoting the displacements by \( u, u_\theta, u_z \), we may assume, as usual,
\[
\begin{align*}
  u &= U e^{i(\omega t + \mathbf{p} t)} \\
  u_\theta &= V e^{i(\omega t + \mathbf{p} t)} \\
  u_z &= W e^{i(\omega t + \mathbf{p} t)}
\end{align*}
\]  

In the case of longitudinal vibrations, we may put \( U = 0 \) and take \( V \) and \( W \) independent of \( \theta \). We then have
\[
\begin{align*}
  \Delta &= \left( \frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} + i \kappa W \right) e^{i(\omega t + \mathbf{p} t)} \\
  \mathbf{\omega} &= (\mathbf{\omega} - \mathbf{\omega}) = \omega \left( i \kappa U - \frac{\partial W}{\partial r} \right) e^{i(\omega t + \mathbf{p} t)}
\end{align*}
\]  

where \( \omega \) has been put for \( \omega_\theta \).

The equations of motion in terms of \( \Delta \) and \( \mathbf{\omega} \) are
\[
\begin{align*}
  \frac{\partial^2 \Delta}{\partial r^2} + \frac{1}{r} \frac{\partial \Delta}{\partial r} + \kappa^2 \Delta &= 0, \\
  \frac{\partial^2 \mathbf{\omega}}{\partial r^2} + \frac{1}{r} \frac{\partial \mathbf{\omega}}{\partial r} - \frac{\mathbf{\omega}}{\partial r} + \kappa^2 \mathbf{\omega} &= 0,
\end{align*}
\]  

where \( \kappa^2 = \frac{\beta^2 \rho}{\lambda + \mu} - \alpha^2 \), \( \kappa^2 = \frac{\beta^2 \rho}{\mu} - \alpha^2 \).

\(^{1}\) Love's Elasticity, Art. 201.
The solutions of the equations (3) may be written

\[
\Delta = \left\{ A' J_0(k_M) + B' Y_0(k_M) \right\} e^{i(\alpha z + \beta t)}
\]

\[
\omega = \left\{ C' J_1, (k_M) + D' Y_1, (k_M) \right\} e^{i(\beta z + \gamma t)}
\]

From (2) and (5), we have

\[
\frac{\partial U}{\partial n} + \frac{U}{z} + i \alpha W = A' J_0(k_M) + B' Y_0(k_M)
\]

\[
i \alpha U - \frac{\partial W}{\partial n} = C' J_1(k_M) + D' Y_1(k_M)
\]

These are satisfied by

\[
U = A \frac{\partial}{\partial n} J_0(k_M) + B \frac{\partial}{\partial n} Y_0(k_M) + C \alpha J_1(k_M) + D \alpha Y_1(k_M)
\]

\[
W = A i \alpha J_0(k_M) + B i \alpha Y_0(k_M)
\]

\[
+ \frac{i C}{n} \frac{\partial}{\partial n} \left\{ n J_1(k_M) \right\} + \frac{i D}{n} \frac{\partial}{\partial n} \left\{ n Y_1(k_M) \right\}
\]

where

\[
A = -\frac{A'}{k^2 + \alpha^2} \quad B = -\frac{B'}{k^2 + \alpha^2}
\]

\[
C = \frac{2C'}{(k^2 + \alpha^2)^2} \quad D = -\frac{2D'}{(k^2 + \alpha^2)^2}
\]

The tractions across any surface \( z = n \) are given by

\[
\gamma = \lambda A + \mu \frac{\partial a_n}{\partial n}
\]

\[
\alpha = \lambda \left[ A' J_0(k_M) + B' Y_0(k_M) \right] + 2 \mu \frac{\partial}{\partial n} \left[ A \frac{\partial}{\partial n} J_0(k_M) + C \alpha J_1(k_M) + B \frac{\partial}{\partial n} Y_0(k_M) + D \alpha Y_1(k_M) \right]
\]
which, after simplification,

\[= A \left\{ \lambda' J_0(k \alpha) + \frac{\mu'}{\alpha} J_1(k \alpha) \right\} + B \left\{ \lambda' Y_0(k \alpha) + \frac{\mu'}{\alpha} Y_1(k \alpha) \right\} + \frac{C \alpha}{\alpha} \left\{ k \alpha J_0(k \alpha) - J_1(k \alpha) \right\} + \frac{D \alpha}{\alpha} \left\{ k \alpha Y_0(k \alpha) - Y_1(k \alpha) \right\},\]

\[\hat{\gamma} = 0,\]

\[\gamma = \mu \left\{ 2 C + 2 \frac{\partial^2 \psi}{\partial \alpha^2} \right\} + \mathcal{C} \mu \left\{ 2 C J_1(k \alpha) + 2 D Y_1(k \alpha) + 2 \frac{\partial \psi}{\partial \alpha} \right\},\]

which reduces to

\[i \mu \left\{ A \frac{2 \alpha k}{\alpha} J_0(k \alpha) + B \frac{2 \alpha k}{\alpha} Y_1(k \alpha) + C (k^2 - \alpha^2) J_1(k \alpha) + D (k^2 - \alpha^2) Y_1(k \alpha) \right\},\]

in which we have put

\[\lambda' = \lambda (\alpha^2 - \mu k^2),\]

\[\mu' = 2 \mu k.\]

§3.

The notations for Bessel Functions used in this paper are those of Gray and Mathews and in the simplifications involved in the following processes, use will be made of the ordinary recurrence-formulae\(^1\) for the functions \(J_n(x), Y_n(x), J_n'(x), Y_n'(x),\) and some others derived from them. Use will also be made of the two theorems\(^1\):

Case I. Both Boundaries Free.

If the boundaries \( n = a \) and \( n = \ell \) are both free from tractions, we have the following conditions:

\[
A \left[ \chi' J_0 (ka) + \frac{\mu'}{\alpha} J_1 (ka) \right] + B \left[ \chi' \gamma_0 (ka) + \frac{\mu^I}{\alpha} \gamma_1 (ka) \right] \\
+ \frac{c}{\ell} \left[ \frac{\alpha}{k} J_0 (ka) - J_1 (ka) \right] + \frac{d}{\ell} \left[ \frac{\alpha}{k} \gamma_0 (ka) - \gamma_1 (ka) \right] = 0,
\]

\[
A \left[ \chi' J_0 (\ell \ell) + \frac{\mu'}{\alpha} J_1 (\ell \ell) \right] + B \left[ \chi' \gamma_0 (\ell \ell) + \frac{\mu^I}{\alpha} \gamma_1 (\ell \ell) \right] \\
+ \frac{c}{\ell} \left[ \frac{\alpha}{k} J_0 (\ell \ell) - J_1 (\ell \ell) \right] + \frac{d}{\ell} \left[ \frac{\alpha}{k} \gamma_0 (\ell \ell) - \gamma_1 (\ell \ell) \right] = 0
\]

\[
A \cdot 2 \alpha k J_0 (ka) + B \cdot 2 \alpha k \gamma_0 (ka) + C \left( k^2 \alpha^2 \right) J_1 (ka) \\
+ D \left( k^2 \alpha^2 \right) \gamma_1 (ka) = 0,
\]

\[
A \cdot 2 \alpha k J_0 (\ell \ell) + B \cdot 2 \alpha k \gamma_0 (\ell \ell) + C \left( k^2 \alpha^2 \right) J_1 (\ell \ell) \\
+ D \left( k^2 \alpha^2 \right) \gamma_1 (\ell \ell) = 0.
\]

Eliminating the constants \( A, B, C, D \), we get the frequency equation: --
\[
\begin{align*}
\rho^2 \left[ j^\alpha (ka) + \frac{\lambda^2}{a} j^\alpha (ka) \right] &+ \frac{\lambda}{6} \left[ j^\alpha (ka) \right] = \frac{\lambda}{6} \left[ j^\alpha (ka) \right], \\
\rho^2 \left[ \frac{\lambda}{6} Y^\alpha (ka) + \frac{\lambda^2}{a} Y^\alpha (ka) \right] &+ \frac{\lambda}{6} \left[ \frac{\lambda}{6} Y^\alpha (ka) \right] = \frac{\lambda}{6} \left[ \frac{\lambda}{6} Y^\alpha (ka) \right], \\
\frac{\alpha}{a} \left\{ \rho^2 \left[ j^\alpha (ka) \right] - \frac{\lambda}{6} \left[ j^\alpha (ka) \right] \right\} &+ \frac{\alpha}{a} \left\{ \frac{\lambda}{6} \left[ j^\alpha (ka) \right] - \frac{\lambda}{6} \left[ j^\alpha (ka) \right] \right\} = \frac{\alpha}{a} \left\{ \frac{\lambda}{6} \left[ j^\alpha (ka) \right] - \frac{\lambda}{6} \left[ j^\alpha (ka) \right] \right\}, \\
\frac{\alpha}{a} \left\{ \rho^2 \left[ Y^\alpha (ka) \right] - Y^\alpha (ka) \right\} &+ \frac{\alpha}{a} \left\{ \frac{\lambda}{6} \left[ Y^\alpha (ka) \right] - Y^\alpha (ka) \right\} = \frac{\alpha}{a} \left\{ \frac{\lambda}{6} \left[ Y^\alpha (ka) \right] - Y^\alpha (ka) \right\}, \\
2 \rho \cdot J^\alpha (ka) &+ 2 \rho \cdot j^\alpha (ka), \\
2 \rho \cdot Y^\alpha (ka) &+ 2 \rho \cdot y^\alpha (ka), \\
(k^2 - \alpha^2) J^\alpha (ka) &+ (k^2 - \alpha^2) j^\alpha (ka), \\
(k^2 - \alpha^2) Y^\alpha (ka) &+ (k^2 - \alpha^2) y^\alpha (ka), \\
= 0. &
\end{align*}
\]

or,

\[
\begin{align*}
\left\{ \rho \left[ j^\alpha (ka) + \frac{\lambda}{6} j^\alpha (ka) \right] \right\} I &- \left\{ \frac{\lambda}{6} \left[ j^\alpha (ka) \right] + \frac{\lambda}{6} \left[ j^\alpha (ka) \right] \right\} II \\
+ 2 \rho \cdot J^\alpha (ka) III &- 2 \rho \cdot j^\alpha (ka) IV = 0
\end{align*}
\]

where

\[
\begin{align*}
I & = \left\{ \rho \left[ j^\alpha (ka) + \frac{\lambda}{6} j^\alpha (ka) \right] \right\} \left( k^2 - \alpha^2 \right)^2 F_{\alpha \beta}, \\
&+ \frac{2 \alpha^2 k}{6} \left( k^2 - \alpha^2 \right) Y^\alpha (ka) \left\{ - k \cdot F_{\alpha \beta} + F_{\alpha \beta} \right\} , \\
\end{align*}
\]

\[
\begin{align*}
II & = \left\{ \frac{\lambda}{6} \left[ j^\alpha (ka) \right] + \frac{\lambda}{6} \left[ j^\alpha (ka) \right] \right\} \left( k^2 - \alpha^2 \right)^2 F_{\alpha \beta}, \\
&- 2 \rho \cdot y^\alpha (ka) \frac{\alpha}{a} \left( k^2 - \alpha^2 \right) \left\{ k \cdot F_{\alpha \beta} + F_{\alpha \beta} \right\} - 2 \rho \cdot y^\alpha (ka) \frac{\alpha}{a} \left( k^2 - \alpha^2 \right) ,
\end{align*}
\]
\[ III = \left\{ \alpha' Y_0(ka) + \frac{\mu_\alpha^l}{a^l} Y_1(ka) \right\} \frac{1}{a^l} (k^2 - \alpha^2) \]
\[ + \left\{ \alpha' Y_0(ka) + \frac{\mu_\alpha^l}{a^l} Y_1(ka) \right\} \frac{1}{a^l} (k^2 - \alpha^2) \left\{ \frac{\mu_\alpha^l}{a^l} \left\{ \frac{1}{a^l} F_{a_0 b_0} - \frac{1}{a^l} F_{a_0 b_0} \right\} \right\} \]
\[ + \frac{2a^2 \mu_\alpha^l}{a^l} \left\{ \frac{1}{a^l} F_{a_0 b_0} - \frac{1}{a^l} F_{a_0 b_0} \right\} \left\{ \alpha' Y_0(ka) + \frac{\mu_\alpha^l}{a^l} Y_1(ka) \right\} \]
\[ + \frac{4a^2 \mu_\alpha^l}{a^l} \left\{ \frac{1}{a^l} F_{a_0 b_0} - \frac{1}{a^l} F_{a_0 b_0} \right\} \left\{ \alpha' Y_0(ka) + \frac{\mu_\alpha^l}{a^l} Y_1(ka) \right\} \]
\[ + \frac{4a^2 \mu_\alpha^l}{a^l} \left\{ \frac{1}{a^l} F_{a_0 b_0} - \frac{1}{a^l} F_{a_0 b_0} \right\} \left\{ \alpha' Y_0(ka) + \frac{\mu_\alpha^l}{a^l} Y_1(ka) \right\} \]
\[ = \left\{ \alpha' Y_0(ka) + \frac{\mu_\alpha^l}{a^l} Y_1(ka) \right\} \frac{1}{a^l} (k^2 - \alpha^2) \left\{ \frac{1}{a^l} F_{a_0 b_0} - \frac{1}{a^l} F_{a_0 b_0} \right\} \]

\[ IV = \left\{ \alpha' Y_0(ka) + \frac{\mu_\alpha^l}{a^l} Y_1(ka) \right\} \frac{1}{a^l} (k^2 - \alpha^2) \left\{ \frac{1}{a^l} F_{a_0 b_0} - \frac{1}{a^l} F_{a_0 b_0} \right\} \]
\[ + \frac{2a^2 \mu_\alpha^l}{a^l} \left\{ \frac{1}{a^l} F_{a_0 b_0} - \frac{1}{a^l} F_{a_0 b_0} \right\} \left\{ \alpha' Y_0(ka) + \frac{\mu_\alpha^l}{a^l} Y_1(ka) \right\} \]
\[ + \frac{4a^2 \mu_\alpha^l}{a^l} \left\{ \frac{1}{a^l} F_{a_0 b_0} - \frac{1}{a^l} F_{a_0 b_0} \right\} \left\{ \alpha' Y_0(ka) + \frac{\mu_\alpha^l}{a^l} Y_1(ka) \right\} \]
\[ + \frac{4a^2 \mu_\alpha^l}{a^l} \left\{ \frac{1}{a^l} F_{a_0 b_0} - \frac{1}{a^l} F_{a_0 b_0} \right\} \left\{ \alpha' Y_0(ka) + \frac{\mu_\alpha^l}{a^l} Y_1(ka) \right\} \]
\[ = \left\{ \alpha' Y_0(ka) + \frac{\mu_\alpha^l}{a^l} Y_1(ka) \right\} \frac{1}{a^l} (k^2 - \alpha^2) \left\{ \frac{1}{a^l} F_{a_0 b_0} - \frac{1}{a^l} F_{a_0 b_0} \right\} \]

where

\[ F_{a_0 b_5} = - F_{b_5 a_2} = J_n(ka) Y_0(ka) - J_n(ka) Y_0(ka) \]

The frequency equation may be put, after a rather long simplification, into the form:

\[ (k^2 - \alpha^2)^l F_{a_0 a_3} = \lambda^l \mu^l \left\{ \frac{1}{a^l} C_{a_0 b_0} + \frac{1}{a^l} (C_{a_0 b_0}) \right\} + \frac{\mu_\alpha^l}{a^l} \left\{ \frac{1}{a^l} C_{a_0 b_0} \right\} \]
\[ + \frac{2a^2 \mu_\alpha^l}{a^l} \left\{ \frac{1}{a^l} C_{a_0 b_0} \right\} \left\{ \alpha' Y_0(ka) + \frac{\mu_\alpha^l}{a^l} Y_1(ka) \right\} \]
\[ + \frac{4a^2 \mu_\alpha^l}{a^l} \left\{ \frac{1}{a^l} C_{a_0 b_0} \right\} \left\{ \alpha' Y_0(ka) + \frac{\mu_\alpha^l}{a^l} Y_1(ka) \right\} \]
\[ - \frac{4a^2 (k^2 - \alpha^2)}{a^l} = 0 \quad \ldots \quad (8) \]

where

\[ C_{a_0 b_5} \equiv - C_{a_0 b_5} \equiv J_n(ka) Y_0(ka) - J_n(ka) Y_0(ka) \]
Case II. One Boundary Rigid.

If the cylinder be free at the surface and clamped at , the boundary conditions are:

\[
A \left[ A^l J_0 (ka) + \frac{\mu}{a} J_1 (ka) \right] + B \left[ A^l Y_0 (ka) + \frac{\mu}{a} Y_1 (ka) \right] + \frac{\alpha}{a} \left[ k a J_0 (ka) - J_1 (ka) \right] + \frac{D}{a} \left[ k a Y_0 (ka) - Y_1 (ka) \right] = 0
\]

\[
A \cdot 2\alpha h J_0 (ha) + B \cdot 2\alpha h Y_0 (ha) + C (k^2 \alpha^2) J_1 (ka)
+ D (k^2 \alpha^2) Y_1 (ka) = 0
\]

\[
A \alpha J_1 (hb) + B \alpha Y_1 (hb) = C \alpha J_0 (kb) - D \alpha Y_0 (kb) = 0
\]

\[
A \alpha J_0 (hb) + B \alpha Y_0 (hb) + C k J_1 (kb) + D k Y_0 (kb) = 0
\]

Eliminating A, B, C, D, we have the frequency equation:

\[
\left| \begin{array}{c}
A^l J_0 (ha) + \frac{\mu}{a} J_1 (ha) \\
\frac{\alpha}{a} \left[ k a J_0 (ka) - J_1 (ka) \right] \\
\frac{k}{a} \left[ k a Y_0 (ka) - Y_1 (ka) \right]
\end{array} \right| = 0
\]

After simplification, this equation will reduce to the following:
The equations (8) and (9) do not admit of exact solutions. Approximate solutions by trial may be obtained for assumed values of the ratio \( a : b \), by making use of the tables for the values of \( J_0(x) \), \( J_1(x) \), \( Y_0(x) \), \( Y_1(x) \). The actual work of calculation will of course be very complicated. The tables of \( J_0(x) \) and \( J_1(x) \) are given by Meissel\(^1\) and those of \( Y_0(x) \), \( Y_1(x) \) by Airy\(^2\). This method has been adopted by Mr. Southwell\(^3\) in the numerical calculation of some of the approximate values of the period in the case of the transverse vibrations of an annular disc, where, in addition to the ordinary Bessel and Neumann Functions, the corresponding functions with imaginary arguments also appear in the frequency equation.

---

1 Reproduced in Gray and Mathew's Bessel Functions.
§7.

Case III. Thin Shell.

When the thickness of the shell is very small, we may write \( a + da \) for \( b \), expand the functions containing \((a+da)\) in ascending powers of \( da \), and to a first approximation, neglect all powers of \( da \) beyond the first.

Performing these operations in the equation (7), we obtain the frequency-equation for a thin shell of radius \( a \) in the form:

\[
\begin{align*}
\lambda' J_0(ka) + \frac{\mu}{\alpha} J_1(ka) &= -\lambda' J_0(ka) - \frac{\mu}{\alpha^2} J_2(ka), \\
\lambda' Y_0(ka) + \frac{\mu}{\alpha} Y_1(ka) &= -\lambda' Y_0(ka) - \frac{\mu}{\alpha^2} Y_2(ka), \\
\alpha k J_1'(ka) &= \alpha k^2 J_1''(ka), \\
\alpha k Y_1'(ka) &= \alpha k^2 Y_1''(ka), \\
2\alpha k J_1(ka) &= 2\alpha k^2 J_1'(ka), \\
2\alpha k Y_1(ka) &= 2\alpha k^2 Y_1'(ka), \\
(k^2 - \alpha^2) J_1(ka) &= (k^2 - \alpha^2) J_1'(ka), \\
(k^2 - \alpha^2) Y_1(ka) &= (k^2 - \alpha^2) Y_1'(ka)
\end{align*}
\]

which gives, on expansion,
\[
\frac{(k^2 - \alpha^2)^2 \lambda^2 \kappa}{\kappa a} \left\{ J_I (ka) Y_0 (ka) - J_0 (ka) Y_I (ka) \right\} \\
+ \frac{\lambda^4 u^5 (k^2 - \alpha^2)^2}{\kappa a^3} \left\{ J_2 (ka) Y_0 (ka) - J_0 (ka) Y_2 (ka) \right\} \\
+ \frac{\mu^2 \lambda^2 (k^2 - \alpha^2)^2}{\kappa a^4} \left\{ J_3 (ka) Y_1 (ka) - J_1 (ka) Y_2 (ka) \right\} \\
+ 2 k \alpha^2 k^2 \lambda' (k^2 - \alpha^2) H \left\{ J_0 (ka) Y_1 (ka) - J_1 (ka) Y_0 (ka) \right\} \\
+ \frac{2 \lambda^4 \alpha^2 k^2 (k^2 - \alpha^2)}{\alpha^2} \left\{ J_0 (ka) Y_1 (ka) - J_1 (ka) Y_0 (ka) \right\} \\
+ \frac{2 \mu^2 \alpha^4 k^1 (k^2 - \alpha^2)}{\alpha^3} \left\{ J_1 (ka) Y_1 (ka) - J_0 (ka) J_1 (ka) \right\} \\
+ \frac{2 \lambda^2 \alpha^2 \lambda^3 (k^2 - \alpha^2)}{\kappa a} \left\{ - J_1 (ka) Y_1 (ka) + J_0 (ka) Y_1 (ka) \right\} \\
+ \frac{2 \mu^4 \alpha^2 k^2 (k^2 - \alpha^2)}{\alpha^2} \left\{ J_1 (ka) Y_2 (ka) - J_2 (ka) Y_1 (ka) \right\} \\
+ 4 a^4 \alpha^4 k^3 H \left\{ J_1 (ka) Y_1 (ka) - J_0 (ka) J_1 (ka) \right\} \\
= 0,
\]

where

\[
H = J_1 (ka) Y_1'' (ka) - J_0'' (ka) Y_1 (ka)
\]

\[
= \frac{1}{4} \left[ \left\{ J_0 (ka) - \frac{1}{\kappa a} J_1 (ka) \right\} \left\{ -3 Y_1 (ka) + Y_3 (ka) \right\} \\
- \left\{ Y_0 (ka) - \frac{1}{\kappa a} Y_1 (ka) \right\} \left\{ -3 J_1 (ka) + J_3 (ka) \right\} \right]
\]

\[
= \frac{1}{4} \left[ 3 \left\{ J_1 (ka) Y_0 (ka) - J_0 (ka) Y_0 (ka) \right\} + \frac{1}{\kappa a} \left\{ J_0 (ka) Y_3 (ka) - Y_0 (ka) J_3 (ka) \right\} \right] = \frac{1}{\kappa a} - \frac{1}{\kappa^3 a^3}.
\]
This can be further simplified into the form:

\[- \frac{(k^2 \omega^2)^2 \lambda^2}{ka^3} + 2 \lambda' \mu' (k^2 \omega^2) + \frac{\mu^2 (k^2 \omega^2)}{ka} \]

\[- 2 \alpha^2 k^2 \lambda^2 \left( \frac{1}{ka} - \frac{1}{k^2 a^2} \right) + 2 \lambda' \omega^2 \left( \frac{1}{ka} - \frac{1}{k^2 a^2} \right) + 2 \mu^2 \alpha^2 \lambda^2 \left( k^2 \omega^2 \right) \]

\[- 2 \lambda' \omega^2 \left( k^2 \omega^2 \right) + 4 \alpha^2 k^2 \omega^2 \left( \frac{1}{ka} - \frac{1}{k^2 a^2} \right) - 2 \alpha \omega^2 \left( k^2 \omega^2 \right) \]

or, multiplying throughout by $\alpha^5$, we get the equation

\[
a^3 \left\{ 4 \alpha^4 k^2 \lambda^2 - 2 \lambda' \omega^2 \left( k^2 \omega^2 \right) - 2 \lambda' \omega^2 \left( k^2 \omega^2 \right) - \frac{1}{k^2} \lambda^2 \left( k^2 \omega^2 \right) \right\} \\
+ a^3 \left\{ \frac{2}{k^2} \lambda' \omega^2 \left( k^2 \omega^2 \right) + 2 \mu^2 \alpha^2 \left( k^2 \omega^2 \right) \right\} \\
+ a \left\{ \frac{1}{k^2} \mu' \omega^2 \left( k^2 \omega^2 \right) \right\} \\
+ 2 \mu' \omega^2 \left( k^2 \omega^2 \right) = 0 \] (10)

If the tube is of very small bore, and we may neglect all powers of $\alpha$ beyond the first, the frequency equation is:

\[
\alpha \left\{ \frac{1}{k^2} \mu'^2 \left( k^2 \omega^2 \right) + \frac{2}{k} \lambda' \omega^2 \left( k^2 \omega^2 \right) + 2 \lambda' \omega^2 - 4 \alpha^2 k^2 \right\} + 2 \mu' \omega^2 \left( k^2 \omega^2 \right) = 0. \] (II)

**Conclusion.**

The results may be summarised briefly as follows:

The frequency equation for the longitudinal vibrations of a thick shell is given by (9) or (8) according as one or both of the boundaries are free. The equations are very complicated and do not admit of exact solutions. The equa-
tion (10) for a thin shell is comparatively simple and 
the equation (11) for a tube of small bore is still 
simpler. The conditions at the plane ends of the tube 
give the value of \( \alpha \), and equations (4) give the values 
of \( h \) and \( k \) in terms of \( \rho \), so that the equations (8), 
(9), (10) and (11) become equations in \( \rho \) alone.

§ 8.

A Numerical Example.

We shall work out the numerical result in the case 
of a long steel tube of comparatively small bore, whose 
plane ends are clamped.

The solutions (1) may be written

\[
\begin{align*}
\nu_\rho &= V \left( P \cos \alpha z + Q \sin \alpha z \right) \cos (\beta t + \xi) \\
\nu_\alpha &= W \left( P' \cos \alpha z + Q' \sin \alpha z \right) \cos (\beta t + \xi)
\end{align*}
\]

Since the tube is clamped at \( z = 0 \) and \( z = L \), we 
must have \( \nu_\rho = \nu_\alpha = 0 \) for \( z = 0 \) and \( z = L \). This gives

\[\rho = \rho' = 0 \quad \text{and} \quad \alpha = \frac{i \pi}{L}, \]  
where \( i \) is any integer. We 
shall take \( i = 1 \).

Suppose that the length \( L \) of the tube = 200 cm, and 
its radius \( a = 1 \) cm. Then \( \alpha = \frac{\pi}{200} = 0.016 \); so that \( \alpha^2 = 0.00256 \).

For steel, we may take \( \lambda = 1.33 \times 10^{12}, \; \mu = 819 \times 10^{12}, \)
\[ \rho = 7.85; \; \text{so that} \; \frac{\rho}{\mu} = 9.6 \times 10^{-12}, \; \frac{\rho}{\lambda + 4\mu} = 2.64 \times 10^{-12}. \]
Now, the equation (11) may be written
\[ 4\mu^2 \frac{k}{E} (k^2 - \kappa^2) + \frac{2}{k} \kappa^2 (k^2 - \kappa^2) \left\{ \lambda \left( \kappa^2 + \kappa^2 \right) - 2 \mu \kappa^2 \right\} - 4 \kappa^2 \kappa^2 = 0. \]

Substituting the above values and omitting the terms which are insignificant compared with those retained, we get the approximate equation
\[ \beta^2 \left\{ 4 \left( \frac{819 \times 10^{12}}{3} \right)^2 \times \frac{5}{2} \times 9.6 \times 10^{-12} \right\} \]
\[ = 4 \left( \frac{819 \times 10^{12}}{3} \right)^2 \times \left( \frac{1000256}{3} \right) \]
whence
\[ \beta^2 = \frac{16 \times 10^7}{3} \]
and
\[ \beta = 13703 \]

Hence, if \( n \) be the frequency, we have
\[ n = \frac{\beta}{2n} = 11.59. \]

My thanks are due to Prof. S.N. Basu, at whose suggestion I took up the work.
I. INTRODUCTORY.

§1. When a body rotates about an axis with constant angular velocity and is in relative equilibrium, every point of the body may be considered as being acted on by a force which varies as the distance from the axis of rotation. The discussion of the vibrations of elastic solids acted on by such body-forces generally involves equations which cannot be solved in finite terms or in any convergent infinite series. It is probably due to this cause that very few problems relating to the vibrations of rotationg bodies have hitherto been solved. But an indirect method has often been applied in such cases to obtain the
frequencies of vibrations which are very approximate for all practical purposes. This approximate method is due to Lord Rayleigh and one very interesting problem has been dealt with by Prof. Lamb and R.V. Southwell.* They have investigated the transverse vibrations of a thin homogeneous circular disc rotating about its axis with constant angular velocity. They observe that "the problem has a practical bearing, as throwing light on the occasional failure of turbine discs", causing the blades which are fitted to them, to come in contact with the adjacent parts of the machine. This problem of the laminar wheel suggests the case of a wheel with straight spokes and a circular rim, which is by no means a less common thing in mechanical contrivances.

§2. It is clear that the discussion of the problem naturally resolves into two distinct parts, viz., (1) the vibrations of the straight spokes, and (2) the vibrations

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of the circular rim. Both the spokes and the rim will be assumed to have small cross-sections, so that the effects of what is known as 'rotatory inertia' will be negligible. A spoke can vibrate transversally in two ways, either in the plane of the wheel or in a plane perpendicular to it. The mathematical solution is identical in the two cases; the rim may vibrate in the same two ways, but the equations of motion are different, though it is known that the frequencies of the gravest modes of free vibration are very nearly the same. * When the spokes and the rim are taken as forming one body, the solutions become very complicated on account of the points of junction. In the work of the present paper, they are considered as separate bodies and independent solutions have been obtained for a thin rotating rod and a rotating circular ring.

II. THIN ROTATING ROD.

§3. Suppose that a rod (AB) of length $\alpha$ is rotating about A with constant angular velocity $\omega$. Since the rod is thin, we assume the stress-system to consist of a longitudinal Tension ($T_x$) only. If A be taken as origin and the axis of $x$ along AB, we have

---

\[ \frac{dT_x}{dx} + \rho \omega^2 x = 0 \]

whence

\[ T_x = \frac{1}{2} \rho \omega^2 (A - x^2) \]

§ 4. Case A. Let the end of the rod B be free, so that

\[ T_x = 0 \quad \text{when } x = a \text{ and we have} \]

\[ T_x = \frac{1}{2} \rho \omega^2 (a^2 - x^2) \quad (1) \]

Case B. Let a mass \( m \) e.g., \([\text{(mass of the rim)}/(\text{number of spokes})]\) be attached to B, so that when \( x = a \), we have

\[ T_x = m \omega^2 a \]

Hence, in this case

\[ T_x = \frac{1}{2} \rho \omega^2 \left\{ a \left( a + \frac{2m}{r} \right) - x^2 \right\} \quad (2) \]

§ 5. Both the forms (1) and (2) may be included in the formula

\[ T_x = \frac{1}{2} \rho \omega^2 \left( e^2 - x^2 \right) \quad (3) \]

where

\[ e^2 = a^2 \quad \text{or} \quad a \left( a + \frac{2m}{r} \right), \]

according as the end B is free or carries a mass \( m \).

When \( \omega \) is very large and the flexural forces are negligible compared with the longitudinal tension, the equation of transverse vibration is

\[ \rho \alpha dx \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x} \left[ T_x \alpha \frac{\partial^2 v}{\partial x^2} \right] dx, \]

where \( \alpha \) is the small cross-section, and \( v \), the lateral displacement of an element of the bar at a distance \( x \) from
the origin.

Substituting from (3) the value of \( T_\alpha \), we have

\[
\frac{\partial^2 \nu}{\partial t^2} = \frac{1}{2} \omega^2 \frac{\partial^2 \nu}{\partial \alpha} \left( \frac{\partial^2 - \alpha^2}{\partial \alpha} \right)
\]

Assuming the solution

\[
u = f(x) e^{i(\beta t + \epsilon)}
\]

we have

\[
(c^2 - x^2) \frac{\partial^2 f}{\partial x^2} - 2 \alpha \frac{\partial f}{\partial x} + \nu^2 f = 0,
\]

where

\[
\nu^2 = \frac{2 \beta^2}{\omega^2}
\]

To solve this, it will be convenient to assume a series in ascending powers of \( \frac{x}{\alpha} \); a quantity which is never greater than unity. Let us assume

\[
f(x) = A_0 + A_1 \frac{x}{\alpha} + A_2 \left( \frac{x}{\alpha} \right)^2 + \cdots + A_k \left( \frac{x}{\alpha} \right)^k + \cdots, \cdots \quad (6)
\]

Substituting this in (4), we get

\[
(c^2 - x^2) \left[ \cdots + \frac{k(k-1)}{\alpha^k} A_k x^{k-2} + \cdots + \frac{(k+1)(k+2)}{\alpha^{k+2}} A_{k+2} x^k + \cdots \right]
\]

\[-2x \left[ \cdots + \frac{k}{\alpha^k} A_k x^{k-1} + \cdots \right] + \nu^2 \left[ \cdots + \frac{A_k}{\alpha^k} x^k \right] = 0
\]

Equating the coefficients of \( x^k \) to zero, we have

\[
(k+2)(k+1) A_{k+2} = \frac{k(k+1)}{2} \nu^2 A_k
\]
Calculating the coefficients of (6) by this formula, we obtain

\[ f(x) = A_0 S_0(x) + A_1 S_1(x), \]

where \( A_0 \) and \( A_1 \) are constants and \( S_0(x) \) and \( S_1(x) \) stand for the following series:

\[
S_0(x) = \left( \frac{a^2}{2!} \left( \frac{\alpha}{x} \right)^2 + \frac{a^2(3\lambda - 6\beta)}{4!} \left( \frac{\alpha}{x} \right)^4 + \frac{a^2(3\lambda - 6\beta)(5\mu - 6\nu)}{6!} \left( \frac{\alpha}{x} \right)^6 \right) + \ldots
\]

\[
S_1(x) = \frac{x}{c} + \frac{2\lambda + \lambda^2}{2!} \left( \frac{\alpha}{x} \right)^2 - \frac{(2\lambda - \lambda^2)(4\mu - \lambda^2)}{4!} \left( \frac{\alpha}{x} \right)^4 + \ldots
\]

The complete solution is therefore

\[ \psi = [A_0 S_0(x) + A_1 S_1(x)] e^{(\beta t + \varepsilon)} \]

§6. We have assumed the end \( A \) (i.e., \( x = 0 \)) to be fixed, so that we must have \( \psi = 0 \) when \( x = 0 \). This shews that we must put \( A_0 = 0 \), and the appropriate solution is

\[ \psi = A_1 S_1(x) e^{(\beta t + \varepsilon)} \]

The series \( S_1(x) \) is convergent when \( x < c \), but it is divergent when \( x = c \) or when \( x > c \). We have now to distinguish between the two cases indicated in Art. 4 above.
In case A., we have $x = c(x - a)$ at the edge, the series $s_i(x)$ is divergent and the solution is meaningless unless the series consists of a finite number of terms. Hence we see from the form of $s_i(x)$ that, in order that the series may terminate, $\ell^2$ must be of the form $2n (2n - 1)$, where $n$ is any positive integer. We therefore have

$$\ell^2 = 2n (2n - 1),$$
onumber

or, by (5),

$$p_r^2 = n (2n - 1) \omega^2,$$

$n$ being any positive integer.

In case (B), we have (from Art. 5)

$$c^2 = a (a + \frac{2n}{\rho})$$

and $x$ is always less than $c$. The series $s_i(x)$ is therefore always convergent. The condition at the end $x = a$ may be expressed by

$$[\frac{m \omega^2 v}{\partial c}]_{x = a} = [\frac{-2a}{\partial a} \cdot m \frac{\partial v}{\partial a}]_{x = a}$$

Substituting for $v$, this becomes

$$p_r^2 S_i(a) - \frac{\omega^2 a}{c} S_i'(a) = 0,$$

or,

$$\ell^2 S_i(a) - \frac{2a}{c} S_i'(a) = 0 \quad \cdots \cdots \cdots \cdots (10)$$

which is an equation in $p_r^2$. 
§7. When, on the other hand, the influence of rotation is small compared with the flexural forces, we know that, the rotatory inertia of the cross-section of the rod being neglected, the equation of motion is

\[ \frac{\partial^2 u}{\partial t^2} + \frac{E k^2}{\rho} \frac{\partial^4 u}{\partial x^4} = 0, \]

where \( k \) is the radius of gyration of the cross-section about a diameter perpendicular to the plane of vibration. If \( \beta_2 \) be the frequency, it is given by

\[ \beta_2^2 = \frac{m^4 k^2 E}{4 \pi^2 a^4 \rho}. \tag{11} \]

where \( m \) is given, in the case of a free-free bar, by

\[ \cosh m \cos m = 1, \]

and in the case of a clamped-free bar, by

\[ \cosh m \cos m = -1. \]

Both the flexural

§8. When the centrifugal forces are taken into account, the equation of motion becomes

\[ \frac{\partial^2 u}{\partial t^2} = \frac{1}{2} \omega^2 \frac{\partial^2}{\partial x^2} \left[ (c^2 - x^2) \frac{\partial^4 u}{\partial x^4} \right] - \frac{E k^2}{\rho} \frac{\partial^4 u}{\partial x^4}. \]

If we assume

\[ u = f(x) \cos (\beta t + \epsilon), \]

we have

\[ \frac{2E}{\omega^2 \rho} \frac{\partial^4 f}{\partial x^4} - (c^2 - x^2) \frac{\partial^2 f}{\partial x^2} + 2 \chi \frac{\partial f}{\partial x} - \beta^2 f = 0. \]

If a series analogous to (6) be substituted in this equation, the relation between the successive coefficients consists of three terms (e.g., \( A_{k+4}, A_{k+2}, A_k \)), so that a general solution in finite terms or in a convergent infa-
nite series is not obtainable.

We may, however, obtain approximate solutions by a method,\(^1\) indicated by Rayleigh.\(^2\) According to this method, we may assume a given form for the displacement \(v\), calculate the kinetic energy and equate it to the sum of the potential energies due to the angular motion and the flexural forces considered separately. The equation thus obtained yields the frequency of vibration.

We proceed to apply this method to the case A of Art. 4. The potential energy \(V\) of the centrifugal forces is given by

\[
V = \frac{1}{2} \int \alpha T_{\kappa} \left( \frac{\partial v}{\partial x} \right)^2 dx,
\]

where \(\alpha\) = cross-section of the rod and \(T_{\kappa} = \frac{1}{2} \rho \omega^2 (a^2 - x^2)\).

The potential energy of the flexural forces is given by

\[
V' = \frac{1}{2} \int E k^2 \alpha \left( \frac{\partial^2 v}{\partial x^2} \right)^2 dx.
\]

The kinetic energy is given by

\[
T = \frac{1}{2} \int \left( \frac{\partial v}{\partial t} \right)^2 dx.
\]

---

1 This method has been adopted by Prof. Lamb and Mr. R.V. Southwell in the papers cited.

2 Theory of Sound, Vol. 1, Chap IV, Arts. 88 et seq.
Let us assume the form

$$v^* = f(x)c_0(b + \varepsilon)$$

Then

$$V = \frac{1}{a} f \omega^2 \alpha \int (a^2 x^2) \left( \frac{\partial f}{\partial x} \right)^2 c_0 \omega^2 (b + \varepsilon) \, dx$$

$$V' = \frac{1}{b} E k^2 \alpha \int \left( \frac{\partial f}{\partial x} \right)^2 c_0 \omega^2 (b + \varepsilon) \, dx$$

and

$$T = \frac{1}{2} \beta \alpha \int f^2 \sin^2 (b + \varepsilon) \, dx$$

$$= \beta^2 \int f^2 \sin^2 (b + \varepsilon) \, dx$$

If $V_f, V'_f$ and $T_f$ denote the mean values of $V, V'$ and $T$ respectively, we have

$$\beta^2 = \frac{V_f + V'_f}{T_f} \quad \cdots \quad (12)$$

The closer the assumed function $f(x)$ agrees with the actual form of the vibrating bar, the more will the value of $\beta^2$ approach \( \frac{V_f + V'_f}{T_f} \).

Moreover, the frequency remains stationary for small deviations from the actual type. Hence, if $\beta_1$ and $\beta_2$ be the two values of the frequency, obtained from the equation (9) or (10) and (11) respectively,
we have very approximately, \(^1\)

\[ \rho_1^2 = \frac{V_f}{l_f}, \quad \rho_2^2 = \frac{V_f^1}{l_f}, \]

and

\[ \rho^2 = \rho_1^2 + \rho_2^2. \tag{13} \]

§9. Assume as an example that

\[ f(x) = \beta \left\{ \frac{x}{a} + m \left( \frac{x}{a} \right)^3 \right\} \tag{14} \]

where \(m\) is a variable parameter whose value is to be determined from the fact that the value of the period given by equation (12) should be a minimum. \(^1\) Let us now calculate the values of \(V_f, V_f',\) and \(T_f.\)

Since the mean value of \(c_2 \beta^2 (\beta t + \epsilon)\) or \(\sigma \beta^2 (\beta t + \epsilon)\) is \(\frac{1}{2},\) we find

\[ V_f = \frac{1}{8} \int_0^a \left( a^2 x^2 \right) \left\{ \frac{1}{a} + \frac{3m}{a^3} x^2 \right\}^2 dx \]

\[ = \frac{1}{8} \int_0^a \frac{\alpha^2 A}{a^6} \left\{ a^6 + (6m-1) a^4 x^2 + (6a^2 - 6m) a^2 x^4 \right\} \frac{dx}{a} \]

\[ = \frac{\omega^2 \alpha A_1}{4 \cdot 10^5} \left( 27m^2 + 42m + 35 \right) \]

\[ V_f' = \frac{1}{4} E k \frac{2}{a^2} \left( \frac{6m x}{a^3} \right)^2 dx \]

\[ = \frac{2E k^2 \alpha A_1}{a^5} m^2 \]

\(^1\) See remarks by Mr. R.V. Southwell in the paper "Vibrations of a spinning disc."
\[ T_f = \frac{1}{2} \varepsilon \alpha A_1 \int_0^\alpha \left\{ \frac{x}{a} + m \left( \frac{x}{a} \right)^3 \right\}^2 dx \]

\[ = \frac{\varepsilon \alpha A_1}{4 \cdot 10^5} \left( \frac{1}{5} \frac{x^2}{a^2} + \frac{4}{7} \frac{x^2}{a^2} + \frac{3}{5} \right). \]

As a partial verification of the above results, let us make \( n = 2 \) in equation (9), so that

\[ \beta_2 = 6 \omega^2 \]

The corresponding value of \( \beta_2 \) is 12, the series \( S_i(x) \) terminates at the second term and

\[ S_i(x) = A_1 \left\{ \frac{x}{a} - \frac{2}{3} \left( \frac{x}{a} \right)^3 \right\}, \]

so that

\[ m = -\frac{2}{3} \]

Substituting this value of \( m \) in the expressions for \( V_f \) and \( T_f \) found above, we see that

\[ V_f = \frac{2A_1}{2!} \varepsilon \omega^2 \alpha \alpha, \]

and

\[ T_f = \frac{A_1}{6} \varepsilon \alpha \alpha, \]

so that

\[ \beta_2 = \frac{V_f}{T_f} = 6 \omega^2 \]

which is the same as that obtained from equation (9) by putting \( n = 2 \).

To return to our general case, we have
\[ V_f + V'_f = \frac{3\omega^2 a u}{4 r_0} \left( 27 m^2 + 42 m + 35 \right) + \frac{3E k^2 a}{a^3} m^2 \]

and

\[ \beta^2 = \frac{\frac{3\omega^2 a u}{4 r_0} \left( 27 m^2 + 42 m + 35 \right) + \frac{3E k^2 a}{a^3} m^2}{\frac{3a u}{4 r_0} \left( 15 m^2 + 42 m + 35 \right)} \]

If, for brevity, we put

\[
\begin{align*}
A &= \frac{\omega}{\omega_0} + \omega^2 a + \frac{3E k^2}{a^3} \\
B &= \frac{1}{10} \omega^2 a, \\
C &= \frac{1}{12} \omega^2 a \\
A' &= \frac{1}{2g} a, \\
B' &= \frac{1}{10} \omega a, \\
C' &= \frac{1}{12} \omega a
\end{align*}
\]

we get

\[ \beta^2 = \frac{A m^2 + B m + C}{A' m^2 + B' m + C'} \]

We have now to find \( \beta \) in order that the values of \( \beta^2 \) may be stationary. The corresponding values of \( \beta \) are given by

\[ (AB' - A'B) m^2 - 2 (\omega^2 - \omega^4) m + B'c' - B'c = 0 \quad \quad (16) \]

and the value of \( \beta^2 \) by

\[ \left( \omega^2 - \frac{4}{3} B' \right) \beta^4 - (C' - cA') \frac{1}{2} B' \beta^2 + A' c - \frac{1}{4} B^2 = 0 \quad \quad (17) \]

The values of \( m \) and \( \beta^2 \) may be calculated when the values of the constants (15) are known, and the true value of the frequency will then be obtained.
III. ROTATING CIRCULAR RING.

§10. We assume that a circular ring of radius $a$ and small cross-section, rotating in its plane with constant angular velocity $\omega$, is vibrating transversally, the displacements being perpendicular to the plane of the ring. If $\beta_1$ and $\beta_2$ be the values of the frequency in the two extreme cases, viz., (1) when the flexural forces are negligible and (2) when the angular motion is negligible, then, according to our observations in Art. 8, we have very approximately

$$\beta^2 = \beta_1^2 + \beta_2^2$$

It is known that, when the rotatory inertia is neglected, the value of $\beta_2$ is given by

$$\beta_2^2 = \frac{Emc^4}{4na^6} \frac{n^2(n^2-1)}{n^2+1+\sigma}$$

where $c$ is the radius of the cross-section, $m$ is the mass per unit length and $n$ is any integer.

We proceed to find $\beta_1$.

§11. Taking the centre of the ring as origin and $(\alpha, \theta)$ the polar coordinates of any point on the circumference, we

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1 Love, Elasticity, Art. 293(b); or, Michell, Messenger of Mathematics, XIX, 1889.
have, assuming the stress-system to consist of a longitudinal tension only,

\[ -\frac{l}{a} \frac{d^2 \theta}{d \tau^2} + \rho \omega^2 \alpha = 0 \]

whence

\[ T_{\theta}(= \theta \theta) = \rho \omega^2 a^2 \]

The equation of motion is accordingly

\[ \rho \alpha \frac{\partial^2 \psi}{\partial \tau^2} \alpha \partial \phi = \frac{\partial}{\partial \theta} \left[ \alpha T_{\theta} \frac{\partial \psi}{\partial \phi} \right] d \theta \]

or,

\[ \frac{\partial^2 \psi}{\partial \tau^2} = \omega^2 \frac{\partial^2 \psi}{\partial \phi^2} \]

The solution of this equation is

\[ \psi = A \cos (\mu \phi + \beta) \cos (\beta \phi + \epsilon) \]

where

\[ \mu^2 = \frac{\beta r^2}{\omega^2} \]

(1) If the point \( \theta = 0 \) of the ring is relatively fixed, we have

\[ \psi = A s \omega \mu \theta \cos (\beta \phi + \epsilon) \]

Since, in this case, \( \psi = 0 \) when \( \theta = 2s\pi \), \( s \) being any integer, we have

\[ s \omega \mu 2s\pi = 0 \]

so that \( 2\mu = \frac{k}{s} \), \( k \) and \( s \) being any integers and

\[ \beta r = \frac{k}{2s} \omega \]

\[ \beta = \frac{k}{2s} \omega \]
(ii) If two diametrically opposite points, \( \theta = 0 \) and \( \theta = \pi \) be fixed, we have
\[ s \omega \mu \pi = 0 \]
so that
\[ \mu = s, \text{ any integer}, \]
and
\[ \beta_1 = s \omega \]

(iii) If the ends of a quadrant, \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \) be fixed, we have
\[ s \omega \mu \frac{\pi}{2} = 0 \]
and
\[ \beta_1 = 2 s \omega, \]
where \( s \) is any integer.

(iv) Generally, if the ends of the arc, \( \theta = 0 \) and \( \theta = \frac{2\pi}{n} \) are fixed, then
\[ s \omega \mu \frac{2\pi}{n} = 0 \]
whence
\[ \beta_1 = \frac{1}{2} n s \omega, \]
\( s \) being any integer.

The solution (18) for \( \beta_2 \) refers to a complete ring. Hence the corresponding solution for \( \beta_1 \), should taken from (22), and the period, when both the angular velocity and the flexural forces are taken into account, will then be given by the equation, \( \beta^2 = \beta_1^2 + \beta_2^2 \)
The results in (ii), (iii), (iv) give very simple relations between the angular velocities and the periods of free transverse vibrations of thin flexible rotating arcs of any angle clamped at the extremities.
TORSIONAL VIBRATIONS OF A CIRCULAR TUBE.

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§1. The problem of the vibrations of cylinders has been discussed at great length by Raleigh. A particular solution in the case of the torsional vibrations of a solid cylinder has also been obtained. It is proposed in this paper to find the frequency equation in a more general case, viz., when the solid is bounded by two co-axial cylinders and also when the thickness of the shell is small enough to be regarded as an infinitesimal of the first order.

§2. Taking the axis of the cylinder as the axis of \( z \) and \((\zeta, \theta, z)\) as the cylindrical coordinates of a point, the components of displacement of any point may be denoted by \( u_{\zeta}, u_\theta, u_z \), which are usually assumed to be of the forms

\[
\begin{align*}
    u_{\zeta} &= u e^{i(\gamma_{\zeta} + \beta t)} \\
    u_\theta &= v e^{i(\gamma_\theta + \beta t)} \\
    u_z &= w e^{i(\gamma_z + \beta t)}
\end{align*}
\]

(1)

1Theory of Sound, Vol.1, Chaps. VII, VIII.
2Love's Elasticity, Art. 200.
where \( U, V, W \) are independent of \( z \) and \( t \).

§3. In our present problem, we have

\[
U_e = U_z = 0, \quad \text{and} \quad U_\phi = V e^{i(yz + \beta t)},
\]

where \( V \) is a function of \( \alpha \) only.

The equation of motion gives

\[
\frac{\partial^2 V}{\partial \alpha^2} + \frac{1}{\alpha} \frac{\partial V}{\partial \alpha} - \frac{1}{\alpha^2} V + \kappa^2 V = 0
\]

where

\[
\kappa^2 = \frac{\beta^2 \rho}{\mu} - \gamma^2
\]

The solution of (2) is evidently

\[
V = AJ_i(\kappa \alpha) + BY_i(\kappa \alpha)
\]

where \( J_i \) is a Bessel Function of the first kind and \( Y_i \), the Bessel Function of the second kind, both of the first order.

The tractions across any surface \( \alpha = a \) are given by

\[
\hat{\tau} = \hat{\tau}_z = 0,
\]

and

\[
\tau_\theta = \mu \left[ \frac{\partial U_\phi}{\partial \alpha} - \frac{U_\phi}{\alpha} \right]
\]

Hence, if the surface \( \alpha = a \) is free from tractions, we have

\[
\beta \frac{\partial J_i(\kappa \alpha)}{\partial (\kappa \alpha)} + B \frac{\partial Y_i(\kappa \alpha)}{\partial (\kappa \alpha)} - \frac{1}{\kappa^2} \left\{ \beta J_i(\kappa \alpha) + BY_i(\kappa \alpha) \right\} = 0,
\]

which, by means of the identities

\[
\frac{\partial J_n(\alpha)}{\partial \alpha} = J_{n-1}(\alpha) - \frac{n}{\alpha} J_n(\alpha),
\]

\[
\frac{\partial Y_n(\alpha)}{\partial \alpha} = Y_{n-1}(\alpha) - \frac{n}{\alpha} Y_n(\alpha)
\]

reduces to
Writing the conditions at \( z = a \) and \( z = \ell \) and eliminating \( A \) and \( B \) from these conditions, we get

\[
\frac{ka \, J_0(ka) - 2 J_1(ka)}{ka \, Y_0(ka) - 2 Y_1(ka)} = \frac{k \, J_0(kb) - 2 J_1(kb)}{k \, Y_0(kb) - 2 Y_1(kb)} \quad \ldots \ldots \quad (6)
\]

This equation determines \( \kappa \). We may get the value of \( \gamma \) from the conditions at the plane ends of the cylinder and the period \( \beta \) is then obtained by means of (3).

One solution of (6) is found to be \( \kappa = 0 \), and the corresponding solution for a cylinder clamped at \( z = 0 \) and \( z = \ell \) is

\[
\kappa_\theta = \left( C_n + \frac{D_n}{\alpha} \right) i \omega n^2 \frac{\pi R^2}{\ell} \cos \left( \frac{n \pi \ell}{\ell} \sqrt{\frac{k}{\rho}} + \alpha \right).
\]

For a solid cylinder, we must put \( D = 0 \).

§4. We shall consider two particular cases.

(i) One of the boundaries rigidly fixed. If \( z = \alpha \) be a rigid boundary, we have from (4) and (5) the frequency-equation

\[
\frac{ka \, J_0(ka) - 2 J_1(ka)}{ka \, Y_0(ka) - 2 Y_1(ka)} = \frac{J_1(kb)}{Y_1(kb)}
\]

(ii) Thickness of the shell very small. For a shell of radius \( a \) and thickness an infinitesimal of the first order the frequency equation (6) may be replaced by the equation

\[
\frac{\partial}{\partial \alpha} \left[ \frac{ka \, J_0(ka) - 2 J_1(ka)}{ka \, Y_0(ka) - 2 Y_1(ka)} \right] = 0 \quad \ldots \ldots \quad (7)
\]

or,
\[
\left[ k \alpha \gamma_0 (k\alpha) - 2 \gamma_i (k\alpha) \right] \left[ J_0 (k\alpha) + k \alpha J_0' (k\alpha) + 2 J_r' (k\alpha) \right] \\
- \left[ k \alpha J_0 (k\alpha) - 2 J_r (k\alpha) \right] \left[ \gamma_0 (k\alpha) + k \alpha \gamma_0' (k\alpha) - 2 \gamma_i' (k\alpha) \right] = 0.
\]

By means of the identities
\[
\frac{\partial J_0 (x)}{\partial x} = -J_r (x), \quad \frac{\partial \gamma_0 (x)}{\partial x} = -\gamma_i (x),
\]
\[
\frac{\partial J_r (x)}{\partial x} = J_0 (x) - \frac{1}{x} J_r (x),
\]
\[
\frac{\partial \gamma_r (x)}{\partial x} = \gamma_0 (x) - \frac{1}{x} \gamma_r (x),
\]
the above equation reduces to
\[
k^2 \alpha^2 \left[ J_0 (k\alpha) \gamma_i (k\alpha) - J_r (k\alpha) \gamma_0 (k\alpha) \right] = 0.
\]

Since the value of the expression within brackets is of the form \(\frac{c}{k\alpha}\), where \(c\) is independent of \(k\alpha\), we get \(k = 0\), and this is the only solution of equation (7). It is noteworthy that the relation between \(\beta\) and \(\gamma\) and hence the value of \(\beta\) itself, is, in the case of a thin shell, independent of the radius of the shell.

\(\S 5\). Numerical determination of the frequency.

In the case of a tube of length, \(\ell\), clamped at \(x=0\) and free at \(x=\ell\), the solution is obviously
\[
K = \frac{1}{2} \sum_{n} A J_r (k_n \alpha) + B \gamma_i (k_n \alpha) \gamma_0 (k_n \alpha + \frac{2\pi}{\ell} \cos (\beta t + \phi)), \ldots \ldots \ldots \ldots \quad (8)
\]
where
\[
k^2 = \frac{\beta^2 \rho}{\mu} - \frac{i^2 m^2}{4 \ell^2},
\]
i being any integer, or, if \(n\) be the frequency, we have
\[ n = \sqrt{\frac{\mu}{\rho}} \left\{ \frac{\kappa^2}{4 \pi^2} + \frac{i^2}{\kappa^2} \right\} \]  \hspace{1cm} (7)

Now, by means of the identities
\[ V_2(x) = \frac{2}{x} V_1(x) - Y_0(x), \]
\[ J_2(x) = \frac{2}{x} J_1(x) - J_0(x), \]
the equation (6) becomes
\[ \frac{J_2(x)}{V_2(x)} = \frac{J_2(mx)}{V_2(mx)} \]  \hspace{1cm} (10)
where
\[ x = k\alpha \]
\[ b = ma \]  \hspace{1cm} (11)

If \( x_s \) be the \( S^{th} \) root of the equation (10) in order of magnitude, we have
\[ x_s = \delta + \frac{\beta}{\delta} + \frac{\beta^2}{\delta^3} + \frac{\beta^3}{\delta^5} + \ldots \]
where
\[ \delta = \frac{s\pi}{m-1}, \quad \beta = \frac{15}{8m}, \quad \gamma = \frac{-540(m-1)}{3(m^3)(m-1)^2} \]
\[ \lambda = \frac{-237600(m-1)}{5(8m)^3(m-1)} \]

Excluding the solution \( k=0 \), let \( x_r \) and \( k_r \) be the values of \( x \) and \( k \) corresponding to \( s=1 \). We have then
\[ k_r a = x_r = \delta + \frac{\beta}{\delta}, \quad \text{approximately}, \]
or,
\[ k_r = \frac{\pi}{a(m-1)} + \frac{15(m-1)}{8\pi a m} = \frac{\pi}{6-a} + \frac{15(6-a)}{8\pi ab} \]  \hspace{1cm} (12)

*Gray and Mathews, Bessel Functions, Chap. XV.*
When the radii of the tube are given, we obtain \( \kappa \), from (12) and substituting this value in (9), we obtain the frequency.

In particular, if \( a = 1, \ b = 2, \ c = 10 \), and if we take \( i = 7 \), the frequency is

\[
\nu = \sqrt{\frac{\kappa}{P} \left\{ \frac{(6 \pi^2 + 15^2) - \frac{1}{1024 \pi^2} + \frac{1}{1600}}{1024 \pi^2} \right\}}.
\]