# CONTENTS

## PART I.

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. Introduction.</td>
<td>1</td>
</tr>
<tr>
<td>II. Correlations between Mental Abilities.</td>
<td>4</td>
</tr>
<tr>
<td>III. Cause of correlations between mental abilities: mathematical expressions and geometrical illustration.</td>
<td>8</td>
</tr>
<tr>
<td>IV. Connections between correlations: two main lines of inquiry: the first of these, with an application of the geometrical method to some of its problems.</td>
<td>16</td>
</tr>
<tr>
<td>V. Special property of a set of correlations between mental abilities. The Hierarchy of Abilities and the Theory of Two Factors.</td>
<td>23</td>
</tr>
<tr>
<td>VI. Definition of Hierarchy: (a) Inter-columnar correlation; (b) The Tetrad difference. Statement of the special purpose of this Thesis.</td>
<td>27</td>
</tr>
</tbody>
</table>

## PART II.

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>VII. Alternative possible standpoints, if the tetrad differences are found to be zero within the limits of experimental error.</td>
<td>35</td>
</tr>
<tr>
<td>VIII. The Sampling Theory of Ability.</td>
<td>47</td>
</tr>
<tr>
<td>IX. Some Objections to the Sampling Theory.</td>
<td>52</td>
</tr>
<tr>
<td>X. Value of the tetrad difference on the Sampling Theory for 'all or none' factors.</td>
<td>58</td>
</tr>
<tr>
<td>XI. Value of the tetrad difference on the Sampling Theory for variable factors.</td>
<td>64</td>
</tr>
<tr>
<td>XII. Summary and Conclusion.</td>
<td>73</td>
</tr>
</tbody>
</table>

## PART III.

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Three published papers on the subject.</td>
<td></td>
</tr>
<tr>
<td>Appendices.</td>
<td></td>
</tr>
</tbody>
</table>
PART I.

SECTION I.

Introduction.

The subject we propose to consider in this thesis is an important question in one branch of psychology, and at the outset it is desirable to see how it is related to the science as a whole and what contribution it seeks to make to it.

It has long been recognised as one of the chief difficulties in psychology that no person has direct knowledge of any mind but his own; and for a long time, therefore, the truths about the mind discovered and expounded by students were, strictly speaking, truths about their own individual minds, which were assumed to be true for all normal minds on the principle that the students themselves were ordinary specimens of humanity, a principle by no means likely to lead anyone far astray. These methods of introspection were aided and supplemented by observations which could be made of other people, observations of the conditions and circumstances attending mental experiences, and observations of their outward expression. It is by these types of observation that the child, that sharp observer, learns that his experiences have their counterparts in other human beings; the infant is delighted by his father's pretended fright,
and at school —

'Well had the boding tremblers learned to trace
The day's disasters in his morning face.'

By a skilful description, also, of such accompaniments of mental experience the author conveys to the reader an idea of the state of mind of his characters. Who does not feel that he knows Dr Johnson as himself? and to whom is the mind of Mr Micawber not an open book?

Such were the methods of the older psychology, such were the facts its hypotheses were designed to explain. Its aim was essentially descriptive, to give a clear and consistent account of mental processes.

In comparatively recent times it has been recognised that further progress might be made by studying, not only those things wherein minds resembled each other, but also those wherein they differed. Formerly, the psychologist spoke of 'the mind' as the anatomist spoke of 'the body', or the engineer of 'the steam engine', the ideal mind, like the ideal body or engine. Not that individual minds were regarded as alike, but that their differences were believed to be of a subtle kind, not amenable to measurement, and rather in the province of the poet and novelist than in that of the scientific investigator. That individuals do differ in important quantitative respects is an old belief. The Parable of the Talents is founded upon it; as also is the principle that a man's
worth is not to be measured by his accomplishments except in relation to his powers, and that there must be taken into account

'All instincts immature,
All purposes unsure,
That weighed not as his work, yet swelled the man's amount.'

'There is scarcely one great writer,' says Professor Burt, 'of any age or nation, who has not seen that the proper study of mankind was not man but men - the passions and peculiarities of differing individuals.'

What part is this study of individuals to play in advancing the study of the mind? If the notion is so old, in what respect is it new? To quote Professor Burt again, 'What distinguishes the individual psychology of to-day from Plutarch's "Lives" or Johnson's "Poets" is simply its exactitude. It aims at an almost mathematical precision, and proposes nothing less than the measurement of mental powers.' A bold aim, and a proposal whose accomplishment may be thought impossible; yet it is the inevitable attempt to push the inquiry a stage further back, to seek to account for the mind's diversity in its manifestations, and to find out how the various mental abilities are produced.
SECTION II.

Correlations between Mental Abilities.

In pure psychology questions sometimes arose regarding the identity of certain mental functions. For example, some psychologists held the view that memory and habit were dependent on the same physiological quality; others maintained that they were entirely unconnected. The question can to a great degree be settled by the newer methods of measurement. Professor McDougall describes an experiment in which he tested a number of subjects in four tasks, two involving habit-formation and the other two pure memory. He found that, while the marks assigned for the four tasks corresponded to a high degree in each of the two types, they did not correspond at all for the tasks of different types; and he concludes that this is good evidence that memory and habit are essentially different in their nature.

This experiment exemplifies the method of individual psychology. In the first place, for each task a measurement was made of each individual's performance. It may be, and has been, doubted whether such measurements can really be made. McDougall for his part is cautious; in his words there was 'a figure assigned to each denoting his degree of excellence in each of the four tasks'. Certainly it is possible to assign a figure, just as it is to assign a figure denoting the strength of the wind.
on the Beaufort scale, or denoting the hardness of some material; but these figures are not measurements in the ordinary sense. Again, the temperature of a body as found by a thermometer is a measurement only in the sense that by means of it we can place the body in its proper relative position amongst other bodies with regard to their hotness, and can say, moreover, that A is hotter than B to a greater extent than C is hotter than D. The question of the validity of measurements of mental powers cannot be entered into fully here; but we may take it as agreed that such measurements are of the same kind as those of temperature; that by means of a good mental test the subjects can be arranged, not only in order, but at their proper intervals. The measurements are relative and not absolute; and, the zero point being thus at our disposal, it is usual, especially when the measurements are to be used in mathematical calculations, to fix it at the average, so that half of the subjects have a measurement or score above zero, and half below zero.

Having now, let us suppose, measured a group of subjects in two mental qualities, and arranged the measurements in parallel columns, we may notice that the numbers tend to go together, high with high, and low with low; or it may be the reverse, high going with low, and low with high; or again there may be no apparent
connection, each number in one column going with a mixture in the other. In the first case there is said to be a high degree of correlation between the qualities, in the second a high degree of negative correlation, and in the third no correlation. Perfect agreement is denoted by +1 and the opposite by -1, numbers between these limits signifying the various degrees to which a pair of qualities may correspond. This amount of correspondence, the 'correlation coefficient', is calculated thus: - If x and y are the measures of an individual in two qualities X and Y, the correlation coefficient, \( r_{xy} \), is equal to \( \frac{S(xy)}{\sqrt{S(x^2)} \sqrt{S(y^2)}} \), the S indicating summation for all the individuals measured. Correlation coefficients are thus quantities defined mathematically; but calculations and theories derived from them are ultimately dependent on the mental measurements themselves, and are as logically defensible as those dealt with in thermodynamics.

Primarily, correlations are a set of interesting facts; they may prove useful to the psychologist in confirming or refuting some conjecture, as in the example quoted above, or to the teacher in educational practice. No science, however, can stand still, especially when it has gathered together a mass of measurements. Attempts to coordinate the facts follow, and theories to account for them are invented, theories which must stand the
fiercest light of examination and criticism and be altered, remoulded, or replaced, until one stands out above its fellows as that most deserving of acceptance. In Chemistry we see the process at work in the theories which account for chemical action in general and not this or that particular one; in Physics facts about light, radiant heat, and electricity have been brought under a single theory. The discovery of any measurable quantities in a science invites the cooperation of mathematics, and, once the aid of that science is invoked, it insists on staying and extending its inquiries. Even the botanist now sees logarithmic spirals on the head of a sunflower, and the wheat-breeder uses the binomial theorem.

So in Psychology, having these correlations calculated, we cannot escape from the desire to ask why these two qualities are correlated so closely and why those so slightly. This study of correlations, of why they exist at all and why they are interrelated as they are, has led to hypotheses, in some degree complementary, in some degree antagonistic, which are still at the stage of being examined and their consequences traced and tested.
SECTION III.

Cause of correlations between mental abilities: mathematical expressions and geometrical illustration.

The search for an adequate explanation of the cause of the correlation between two mental abilities would involve the consideration of what is meant by the mind, what are its innate capabilities and its acquired characteristics; it would involve the examination of how the mind appears, as the result of other psychological methods of study, to function; it would involve the study of the nervous system and of the physiological processes which are recognised as having a bearing upon mental activity. The task would, in short, call for the cooperation of all branches of the science. But, as we shall be concerned mainly with the mathematical consequences of various theories, we shall (as mathematicians must) accept as a starting point the existence of certain fundamental data. We shall speak of the mind, its qualities and abilities, and of mental factors, without deeming it necessary to specify exactly of what nature we conceive these to be. It has been facetiously observed that mathematics is the subject in which we never know what we are talking about, nor whether what we are saying is true; an illuminating aphorism which expresses the privileges of the mathematician of which we shall avail ourselves.
We begin, then, by accepting the notion that underlying the qualities or activities of the mind is a number of mental factors, and that any one activity is due to the action of some of these factors while the others are in abeyance. Every normal individual will be supposed to possess the same set of elementary factors, and will, generally, use the same selection of them in performing a given mental task. The difference between individuals is due to the differences in the quality of the various factors they possess. This theory affords a very simple explanation of correlation. The sets of factors acting, say, in the learning of Latin and of mathematics are thought of as overlapping. We can now explain why Smith, who is very good at Latin, is also pretty good at mathematics; it is because the factors in his mind common to both

![Venn Diagram]

**Fig. 1.**

are good factors. We see too why he is only pretty good at mathematics; it is because his factors which act in mathematics and not in Latin are relatively
inferior. The situation can be clearly shown by a
diagram (fig. 1), in which the ovals represent the fac-
tors operating in the two activities. The second
diagram gives the case of Jones, who is quite good at
Latin but weak in mathematics.

It will be evident that the worth of the common
factors will have a large influence, and that to some
extent, determined by the proportion of factors that
are common, the abilities of any one person in the
two subjects will tend to correspond. It is by study-
ing this correspondence that we draw conclusions re-
garding the amount of overlapping, and hope ultimately
to arrive at a more precise knowledge of the parts
played by the factors whose existence we have assumed.

We now proceed to the expression of this theory
by means of mathematical equations. Suppose that
there are \( n \) independent factors, called \( X_1, X_2, \ldots, X_n \);
and that \( x_1, x_2, \ldots, x_n \) are the amounts of these, meas-
ured from the mean, possessed by a particular individual.
We shall suppose also that the factors are variable;
that is, any factor entering into some mental activity
need not act with all its force, but may act with any
fraction of its total value. On these assumptions
the measure of a quality \( Q \), as exhibited by one indivi-
dual is given by

\[
q = \sum_{i=1}^{n} l_i x_i \]

\[
q = l_1 x_1 + l_2 x_2 + l_3 x_3 + \ldots \ldots 
\]
where the \( l \)'s are numbers between 0 and 1 denoting the
degree to which the several factors take part in the
quality \( Q \). The values of the \( x \)'s vary from individu-
al to individual, while the \( l \)'s are fixed for a given
quality. The distribution of the values of any one
\( X \) among the population is assumed to be a Normal dis-
btribution, and that of any \( Q \) is also taken to be nor-
mal. On these premises it has been shown that \( q \)
must be a linear function of the \( x \)'s, as in the above
equation. There is a further assumption regarding
the measures of the \( x \)'s and \( q \)'s which is extremely
convenient for mathematical analysis and is easily
justifiable. It is that all the standard deviations
are made equal. This is merely a question of suit-
ably choosing the magnitudes of the units in which
the variables are measured. The effect on the equa-
tion for \( q \) is that, while the coefficients \( l \) retain
their relative values, they are each altered so that
now \( l_1^2 + l_2^2 + l_3^2 + \ldots = 1 \).

Considering the two qualities \( Q_1 \) and \( Q_2 \), we have
\( q_1 = l_1 x_1 + l_2 x_2 + l_3 x_3 + \ldots \)
\( q_2 = l_1 x_1 + l_2 x_2 + l_3 x_3 + \ldots \)
and the correlation coefficient \( r_{12} \) is equal to
\( r_{12} = l_1 l_2 l_1 + l_2 l_2 + l_3 l_3 + \ldots \)
This, then, is the basis on which have been built those
theories of the structure of the mind which have in
recent years received much attention from mathematical psychologists.

One great advantage of equalising the standard deviations is that we can then illustrate the relation between mental qualities geometrically; many of the problems facing us can thus be looked on as geometrical, and another pathway is opened up for the attack. Although the geometry is of many dimensions, and therefore purely analytical, we can nevertheless think in terms of lines, planes and angles, which is often of great utility in making things clearer to the mind.

Let us for simplicity consider first a mental quality $Q$ depending on two independent factors $X$ and $Y$, and let the equation for $q$ be $q = lx + my$, where $l^2 + m^2 = 1$.

Then a line $OQ$, making angles $\cos^{-1} l$, $\cos^{-1} m$, with $OX$. 

Fig. 2.
and OY, may be taken to represent the quality Q. Now suppose that for each person a point is marked showing the values of x and y in his case; the assemblage of points will be distributed symmetrically about 0, closely packed near 0 and falling off in number farther off, the density following the Normal law. If P is such a point, and Pp is drawn perpendicular to OQ, then Op = lx+my, so that Op is the measure of that person for the quality Q. If OQ' represents another quality Q', the correlation between them is 11'+mm', which is equal to \cos QOQ', a very simple relation between the two lines.

If the qualities are due to three factors, we have three axes, OX, OY, OZ, and any person's measures x, y, z, for the three factors is indicated by the point P (x, y, z). The whole population gives an assemblage of points surrounding 0 and falling off in every direction according to the normal law with the same standard deviation. A quality Q measured by q = lx+my+nz, where \( l^2 + y^2 + z^2 = 1 \), is represented by a line through 0 whose direction cosines are l, m, and n; and as before the projection of OP on OQ gives the measure in the quality Q for the person whose representative point is P. Also for two qualities we have \( r_{QQ'} = 11' + mm' + nn' = \cos QOQ' \). And likewise for any number of factors there is in n-dimensional space a directed line, whose 'direction cosines' are \( l_1, l_2, \ldots, l_n \), representing the quality Q; and the correlation
between two of them, viz. $1_1 l_1 + 1_2 l_2 + \cdots + 1_n l_n$, is represented by the cosine of the angle between $OQ_1$ and $OQ_2$.

The use of these geometrical conceptions may be indicated briefly. Garnett, to whom the method is due, makes effective use of it, especially in his paper on 'General Ability, Cleverness and Purpose'; it is also possible to illustrate partial correlation, and to establish Yule's formula; and we can show very clearly the coefficient of alienation $k = \sqrt{1 - r^2}$.

There is yet a further assumption sometimes made regarding the mental factors. It is imagined that they may not be variable at all, but that they enter into an activity either with their full force or not at all. Factors of this nature are termed 'all or none'. There follow certain modifications in the formula for a correlation coefficient. The coefficients $l$ will be initially either unity or zero, and to equalise the standard deviations they require to be divided by $\Sigma l^2$, which will be simply $\sqrt{a}$, where $a$ is the number of factors in $Q$, so that

$$q = \frac{1}{\sqrt{a}} (1_1 x_1 + 1_2 x_2 + 1_3 x_3 + \cdots)$$

If another quality $Q'$ depends on $b$ factors, its measure

$$q' = \frac{1}{\sqrt{b}} (1'_1 x_1' + 1'_2 x_2' + 1'_3 x_3' + \cdots)$$

and then

$$r = \frac{\Sigma ll'}{\sqrt{a}\sqrt{b}}$$

and since clearly $\Sigma ll'$ is equal to the number of common factors, say $c$, the formula for 'all
or none' factors is the very simple one \( r = \frac{\sigma}{\sqrt{ab}} \).

Any quality may still be represented by a directed line, which in this case will be equally inclined to all the axes concerned; otherwise the various geometrical illustrations still hold good.

**Note:** The geometrical representation described in this Section is given in Maxwell Garnett's articles, 'On Certain Independent Factors in Mental Measurement', Proc.Roy.Soc. 1919, xcvi.A., and 'General Ability, Cleverness and Purpose', Brit. Journ. Psychol. 1919, ix. The formula for \( r \) when the factors are 'all or none' is due to Professor Spearman and Professor Godfrey Thomson.
SECTION IV.

Connections between correlations: two main lines of inquiry: the first of these, with an application of the geometrical method to some of its problems.

In Section III we have examined theories which explain how mental abilities come to be correlated. A correlation, as we have seen, gives a quantitative measurement of the degree to which two abilities correspond. The question now arises, Is there any connection between the various correlation coefficients, and, if so, can we through a study of it gain any enlightenment regarding the mind as a whole? The course of the inquiry thus suggested divides into two main branches. On the one hand there is the mathematical development of the theory to find out what connections should exist, to interpret the results and apply them to observed facts. On the other there is the careful inspection of the experimental facts to see whether there are any other unsuspected empirical laws which require further explanation. Any scientific theory worth its name will be found to shed light on many facts other than those it was framed to explain; and few theories there are but sooner or later encounter a fact which bids the investigator pause and think, Is this after all the truth?

For a little let us look at the first line of inquiry. We may approach the question thus: - knowing
and the correlations between one quality and each of two others, are we prepared for any value for $r_{23}$? Or are we like the chemists, who acquainted with the results of two chemical actions will hazard a prediction regarding a third which they conceive as intermediate? This certainly appears reasonable. If there is a close connection between qualities A and B, and a close connection between A and C, we might think some connection likely between B and C. And this is indeed the case; for, on the theory of variable elements we can, given the correlations between A and B and between A and C, put limits to the value of the correlation between B and C. For instance, if the known correlations are .9 and .9, we should expect the third to lie between .62 and 1. The exact relation will be given later. Correlations, then, are not a set of haphazard numbers. If a pretended list of correlations were drawn up, and the numbers filled in at random, a psychologist who had studied the possible causes of correlations such as we have outlined would discover the fraud.

In the next place it is proper to ask, if the correlation between A and B is due to common elements, and so also for A and C, and if, as a result, we expect a correlation, and therefore common elements, in the case of B and C, are there then necessarily some elements common to all three? The answer to this is No, unless
the three correlation coefficients are on the average too large to account for without some elements common to all. The exact condition is that
\[ r_{12}^2 + r_{13}^2 + r_{23}^2 + 2r_{12}r_{13}r_{23} \]
shall be less than 1 if we are to be able to account for the correlations without factors common to all three. If this condition is fulfilled, then (apart from other considerations) we can suppose that there are no elementary factors entering into all three; although it is possible that in point of fact there are. On the other hand, if the condition is not fulfilled, we are forced to conclude that the qualities concerned have factors in common.

Pursuing this line of inquiry, and extending it to any number of qualities, we might arrive at conclusions regarding various groups of qualities, indicating the minimum amount of overlapping there must be. Applying the results to experimental data, it could be said that such and such a group of abilities certainly had factors common to all, and that such and such another set might have none. These would be conclusions we were forced to come to on the theory we have adopted, without modifying it in any way.

Having indicated what is the nature of this one line of inquiry, we conclude this section by giving a proof of the two relations referred to above. The
formula \( r_{12}^2 + r_{13}^2 + r_{23}^2 + 2r_{12}r_{13}r_{23} \leq 1 \) was proved by Professor Godfrey Thomson and Dr Ridley Thompson for 'all or none' factors some years ago, and by the latter for variable factors recently. The new geometrical proof given here at once illustrates the simplicity and elegance of the geometrical method and gives additional confidence in using it on a more ambitious scale.

We are given the three correlations among three qualities \( Q_1, Q_2, Q_3 \), and are to try to account for them without a common factor. This will be done if we can make \( Q_1 \) depend on two factors \( X \) and \( Y \), \( Q_2 \) on \( X \) and \( Z \), and \( Q_3 \) on \( Y \) and \( Z \). If the qualities are represented by directed lines, we have to find, if we can, a frame of reference such that each line is in one of the coordinate planes. Stated as a purely geometrical problem it is this:

To find the condition that three mutually perpendicular planes may be found such that three given concurrent straight lines may lie one in each of these planes.

The three lines are specified by the angles they make with one another. We may therefore with perfect generality take the lines to be

\[
\begin{align*}
OQ_1 &= OX = (1,0,0) \\
OQ_2 &= (1,m,0) \\
OQ_3 &= \left(1',m',n\right),
\end{align*}
\]

where \( 1^2 + m^2 = 1 = 1'^2 + m'^2 + n^2 \).
Take any plane through $OQ_1$, i.e., through $OX$. Its equation is
\[ y + az = 0 \quad \ldots \quad (1) \]

The equation of any plane through $OQ_2$ is
\[ mx - ly + bz = 0, \quad \ldots \quad (2) \]
and if we choose that one which is perpendicular to (1) we have $-1 + ab = 0$, whence $b = 1/a$

Thus the plane through $OQ_2$ perpendicular to (1) is
\[ mx - ly + \frac{1}{a}z = 0 \quad \ldots \quad (3) \]

The planes (1) and (3) intersect in a line, viz.
\[ \frac{x}{l(a+1/a)} = \frac{y}{am} = \frac{z}{-m}. \]

This is to be perpendicular to the line $(l',m',n)$.

Therefore $ll'(a+1/a) + amm' - mn = 0$, which becomes
\[ (ll' + mm')a^2 - mna + ll' = 0 \quad \ldots \quad (4) \]
This is an equation determining the value of $a$. 
On solving it, and substituting in the equations (1) and (3), we get two planes which along with the plane perpendicular to them form a set of mutually perpendicular planes each containing one of the given lines.

Now, equation (4) will not give any solution unless $m^2n^2 - 4(ll' + mm')ll'$ is positive. .... (5)

This is therefore the condition sought for.

Suppose now that we had been given the angles between the lines in the form $\cos\theta_1 \theta_2 = k$, $\cos\theta_1 \theta_3 = k'$, $\cos\theta_2 \theta_3 = k''$. Then $l = k; l' = k'; ll' + mm' = k''$

Also $m = \sqrt{(1-k^2)}$. Therefore $kk' + m'\sqrt{(1-k^2)} = k''$,

whence $m' = \frac{k'' - kk'}{\sqrt{(1-k^2)}}$. Again, since $l'^2 + m'^2 + n^2 = 1$,

$$n^2 = 1 - k'^2 - \frac{k''^2 - 2kk'k'' + k'^2 k''^2}{1 - k^2}$$

\[ m^2n^2 = 1 - k^2 - k'^2 + k'^2 k''^2 - k''^2 + 2kk'k'' - k'^2 k''^2 \]

\[ = 1 - k^2 - k'^2 - k''^2 + 2kk'k'' \]. .... (6)

Hence from (5)

\[ 1 - k^2 - k'^2 - k''^2 + 2kk'k'' - 4kk'k'' \] must be positive.

That is $k^2 + k'^2 + k''^2 + 2kk'k''$ must be less than 1. (7)

If the three lines represent mental qualities, and the cosines (the k's) the correlations, then (7) becomes

$$r_{12}^2 + r_{13}^2 + r_{23}^2 + 2r_{12}r_{13}r_{23} < 1\] , .... (7')

which is the condition already given as having been obtained by quite other methods by Dr Ridley Thompson.

Referring again to equation (6), which may be
looked on as determining \( n \) in terms of the \( k \)'s, we see that \( 1 - k^2 - k'^2 - k''^2 + 2kk'k'' \) must be positive; otherwise the lines could not be drawn at all. Expressing the \( k \)'s as correlations, this means that any set of correlations among three qualities must satisfy the inequality

\[
\frac{r_{12}^2 + r_{13}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23}}{1} > 1 \quad \cdots \quad (8)
\]

Hence, if we are given \( r_{12} \) and \( r_{13} \),

\[
r_{23}^2 - 2r_{12}r_{13}r_{23} + r_{12}^2 + r_{13}^2 - 1 \neq 0;
\]

i.e., \( r_{23} \) must lie between the limits

\[
r_{12}r_{13} \pm \sqrt{(1 - r_{12}^2)(1 - r_{13}^2)} \quad \cdots \quad (9)
\]

If \( r_{12}^2 + r_{13}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23} = 1 \), the lines are co-planar, and we have the case discussed by Garnett in his papers mentioned in the previous section.

This example will suffice to show the use of the geometrical conceptions and how they yield results in agreement with those otherwise obtained.

Note: A discussion of the explanation of intercorrelations and of the overlapping of factors is to be found in Brown and Thomson, 'The Essentials of Mental Measurement', chapter VII. The line of inquiry sketched out in this section has been pursued by Dr J. Ridley Thompson in several papers recently published.
SECTION V.

Special property of a set of correlations between mental abilities. The Hierarchy of abilities, and the Theory of Two Factors.

An explorer may take up one of two tasks, that of making his way across an unknown land and observing those natural features he comes upon, or that of finding a road to some known goal. Columbus, guided by the stars and fortified by faith in his theories, sailed into the West, and, thinking to reach the Indies, discovered America; the Elizabethan mariners, trying to make the North-west Passage, found that the route to Cathay was not by the Arctic seas. So the scientific theorist follows out his theories, while the experimenter informs him of the facts of experience his theories must lead to. In our present inquiry we have glanced at the road along which our theory will lead us; now we take a view of the landscape that that road, or, failing it, some other, must reach.

So far we have considered the consequences of the theory in its simplest form, that the various mental abilities are due to the operation of elementary mental factors, a selection of which contribute to the exercise of any particular ability; some of which certainly contribute to more than one; and some of which must certainly contribute to every one of a particular group of
abilities when the correlations among that group fulfil certain conditions. Beyond this, for aught we know to the contrary, there may be some factors which are concerned in every conceivable ability; on the other hand, there may be none such. It now behoves us to inquire whether the correlations among a set of abilities exhibit any special properties which make it necessary or desirable to modify our original theory. It is worth while noting carefully that the consequences of such special properties may be explainable only if we do make some modification; in that case we have no option but to make it. But it may be that either the theory or the suggested modification is capable of explaining the facts; the new theory must then be supported by its greater inherent simplicity, the neater and more attractive way in which the correct consequences flow from it, or its superiority from a psychological point of view. In that case there is scope for legitimate difference of opinion.

The special property shown by a set of correlation coefficients is what is known as 'hierarchical order'. If the correlations existing amongst a set of mental qualities are known, and they are tabulated in a square array, it is found that by suitably choosing the order in which the qualities are written down (otherwise a matter of indifference) the numbers in the array diminish steadily from the top lefthand corner to the bottom
righthand corner. This seems sufficiently remarkable to call for explanation. Are the correlations we might get, by chance as it were, subject only to mild restrictions of the type given in equation (9) of Section III, likely to show this order? Or must we modify the theory to account for it? An affirmative answer to this question has been given by Professor Spearman and other investigators. The modification proposed is as follows. Among the factors which operate in the various abilities there is one which acts in every ability; while each of the others acts in one ability only. Any one ability is thus the result of the operation of two factors, the general factor and the factor specific to itself. This
is the Theory of Two Factors. In its widest form it is thus expressed in Spearman's own words:— 'All branches of intellectual activity have in common one fundamental function (or group of functions), whereas the remaining or specific elements seem in every case to be wholly different from that in all the others'.

Upon this theory a vast amount of work has been done, and a full discussion of it in all its aspects is to be found in Spearman's book 'The Abilities of Man'. A theory attractive in its simplicity, and offering such a comprehensive generalisation as to make it comparable, as its author with not unjustifiable pride expresses it, with 'a Copernican revolution', it has commanded the assent of many psychologists of eminence. Yet for these very reasons, that its implications are so fundamental and its possible effects so far-reaching, not only does it demand the most careful consideration, but its foundations call for the closest scrutiny.
SECTION VI.

Definition of Hierarchy: (a) Inter-columnar correlation; (b) The Tetrad difference. Statement of the special purpose of this Thesis.

It is clear that before we can apply any mathematical treatment to the question we must have an exact mathematical definition of what shall be regarded as constituting a 'hierarchy'. Since the numbers in any two columns of the square array dwindle away side by side, the correlation between the numbers in the columns should be high. Accordingly, the definition of a hierarchy first adopted was that the columnar correlation between any two columns should be equal to unity. The Two Factor theory was then shown to lead to a hierarchy in this sense. The term 'hierarchy' being now defined, the next step is to see whether the correlations actually got in practice do form such a hierarchy. Now of course the values of correlation coefficients obtained in practice are mere approximations to the true values, being vitiated by sampling errors. Even when the most elaborate precautions are made in the methods of testing and known sources of error allowed for, there remains the fact that the subjects of the measurement form only a small fraction of the whole population whose characteristics are the object of investigation. The measurements, true values though
they be for the group of subjects measured, are probably in error when taken as universally applicable. It is necessary, therefore, to make allowance for these sampling errors, to find out what is the consequent error in the columnar correlation, and to see whether the actual correlations are, within the limits of error, consistent with the statement that the hierarchy exists as defined.

To obtain a formula for the sampling error of most quantities dealt with in psychology is not easy; and it becomes a task of extreme difficulty when, as here, the quantity is itself a fairly complicated function of other quantities each liable to an error of its own. In their paper 'General Ability, Its Existence and Nature' Spearman and Hart give a formula for the true columnar correlation as follows:

\[ R'_{ab} = \frac{S(\rho_{xa}^2 \rho_{xb}^2) - (n-1) \bar{\tau}_{ab} \bar{\sigma}_{xa} \bar{\sigma}_{xb}}{\sqrt{S(\rho_{xa}^2) - (n-1) \bar{\sigma}_{xa}^2} \sqrt{S(\rho_{xb}^2) - (n-1) \bar{\sigma}_{xb}^2}} \]

where \( R'_{ab} \) is the true columnar correlation between the columns headed a and b, \( \rho_{xa} \) and \( \rho_{xb} \) are the observed correlations between the qualities a and b respectively and the other qualities x in turn as we descend the column, \( \bar{\sigma}_{xa} \) and \( \bar{\sigma}_{xb} \) the mean standard deviations for the column of \( \rho_{xa} \) and \( \rho_{xb} \), and \( n \) the number of pairs of correlation coefficients in the two columns. As in most such cases the formula is an approximate one. In the first place an exact formula would bring in the
true values of the correlations $\rho$, which are not known; and secondly, partly owing to the desirability of obtaining for use in practice as simple a formula as possible, terms have been dropped as negligible in comparison with the others. On applying the formula to a large number of examples, Spearman and Hart found that the corrected columnar correlation was indeed pretty close to unity.

This criterion for a hierarchy was subjected to criticism on these grounds. First, the question was raised whether the approximations made in obtaining the formula for the corrected value of $R$ were justified; whether, for example, the terms rejected were really insignificant compared with those retained. A second, and a more damaging, objection was raised to the way in which the criterion was applied. All those columns wherein the mean square deviation was less than double the correction to be applied to it were excluded, on the ground that where corrections are too large compared with the quantity corrected they become untrustworthy. While the reason is sound, the standard for exclusion is arbitrary, and it was argued that it was the particular standard chosen which made the average of the columnar correlations calculated turn out so near to unity. Professor Spearman himself criticises the criterion of inter-columnar correlation, especially
its 'fault of being only applicable in a selective manner'; he explains that it was adopted as being the only one for which, at the time, it was possible to calculate the allowance to be made for sampling errors, and when later researches made it possible he abandoned the use of this definition of a hierarchy in favour of what he terms 'the true criterion', which now falls to be considered.

This new definition is expressed by means of the 'Tetrad equation'. From this point of view the numbers in a pair of columns are supposed to diminish proportionally down the column. Thus, if there are two pairs of numbers as shown,

\[
\frac{r_{pa}}{r_{qa}} = \frac{r_{pb}}{r_{qb}},
\]

or

\[
r_{pa}r_{qb} - r_{pb}r_{qa} = 0.
\]

This is the Tetrad equation, the quantity on the left being called the Tetrad difference. Clearly, if the equation holds good for successive rows p and q, it will hold for the correlations at the corners of any rectangle in the array. Now obviously this assumed proportional diminution in the values of the correlations will give rise to a hierarchy resembling what actually occurs. It may therefore be taken as the definition of a hierarchy that the tetrad differences
should be equal to zero throughout a whole set of correlations. This is a simpler notion than that of inter-columnar correlation, and in fact preceded it; its practical application, as already noticed, was precluded by absence of knowledge about its sampling error.

There are now four definite questions to be considered: (1) Does a hierarchy actually exist, or, as it may now be put, is the tetrad difference everywhere equal to zero? (2) Does the Two Factor Theory explain this? (3) Conversely, does the vanishing of the tetrad difference lead to the Two Factor Theory? (4) Can any other theory, the original unmodified theory already sketched out, for example, explain the facts? All these questions must be answered satisfactorily before any conclusions can be said to rest on a sound logical basis. If, for instance, it were found that, while the Two Factor theory did lead to the fulfilment of the tetrad equation, that equation was actually not fulfilled in practice, the Two Factor theory would be definitely disproved. On the other hand, if an affirmative answer be given to the first three questions and a negative to the fourth, the theory would be as definitely logically established, unless indeed the entire basis of the work with its ideas of mental factors were to be dismissed on other grounds as nonsense.
It is easily proved that, if four qualities are each due to the operation of a general factor $g$ and a specific factor $s$, the tetrad difference is zero. This follows from an application of Yule's formula for partial correlation, and is given in the appendix of 'The Abilities of Man'. The converse proposition is not so easy to establish; and as the proof is long and difficult and we shall have frequent occasion to refer to different aspects of it later on we shall here merely state that it has been proved that, if the values of the tetrad difference are everywhere zero, the qualities may be referred to the operation of a single uniquely-determined general factor and a number of specific factors.

Appeal must now be made to experience to find whether or not the tetrad differences are actually zero. This has meant an immense amount of labour, as the consideration of one example will show. On page 145 of 'The Abilities of Man' is given a set of correlations arranged in hierarchical order. From this table, dealing with only 14 qualities, there are obtained as many as 3003 tetrad differences, which, when calculated, give a distribution which is exhibited graphically. To test these values the probable error due to sampling should be found in each case; but an approximate theoretical distribution of the observed
values can be got on the assumption that the true values are all zero. When this is done, the theoretical and the actual distributions agree with remarkable closeness; in the words of the author, there is 'one of the most striking agreements between theory and observation ever recorded in psychology'. The formulae used in this connection have recently been criticised, but the consideration of their adequacy is foreign to our present purpose, and we accept it as, at any rate, approximately true that the correlations obtained from observation are consistent with the statement that the tetrad difference is always zero.

We have now answers to three of the questions we proposed; and summarise them to make the position clear. (1) The values of the tetrad difference in practice are, within the limits of sampling error, very nearly zero. (2) The Two Factor theory leads to such values. (3) The vanishing of the tetrad differences leads to the Two Factor theory.

The advocates of this theory give a more definite answer to the first question, and accept it as satisfactorily proved that the tetrad difference is actually zero; then, with regard to the last question, in view of the answers to the first three it is held that the answer is bound to be that no other theory can explain
the facts, a contention which, although there is much force in it, is open to argument. Accepting these definite conclusions, they consider that the Two Factor theory has been established on a firm basis. And in this they are quite right.

We have now arrived at a definite stage in our inquiry. The theory of mental factors underlying the work done in this line has been explained, and the reasons given for the adoption of the Two Factor theory by its advocates. Far from being merely an inspired guess, that theory has been subjected to rigorous examination logically and experimentally. Yet there seems room for further investigation, especially regarding the fourth question; and this is the special purpose of this thesis. In particular, then, we desire to follow out two lines of thought; first, what views we may legitimately hold about the true values of the tetrad differences; second, what are the consequences of other theories, and whether they lead to conclusions in agreement with the facts of experience.
PART II.

SECTION VII.

Alternative possible standpoints, if the tetrad differences are found to be zero within the limits of experimental error.

In formulating an experimental law of physical science the physicist is always careful to assert no more than that his observations within the limits of experimental error are consistent with it. The possibility is always recognised that the law may not be quite exactly fulfilled, and even that a different ultimate explanation of the phenomena may have to be sought. So the equation of Boyle gives place to that of Van der Waals; and while a falling apple and the revolving Moon display behaviour agreeable to Newton's laws the more distant planets require a wider theory to account for their movements. In the branch of psychological science under discussion a like caution must be preserved; and, after it has been taken as established that observation is consistent with the statement that tetrad differences are, within the limits of sampling error, equal to zero, there are two alternative standpoints to be considered. It may then be postulated either (a) that the tetrad differences are in fact zero, or (b) that they are, if not zero, small compared
with their probable error.

Consequences of the former alternative.

As already mentioned, it is a necessary consequence of the Two Factor theory that the tetrad difference must be zero. This may be proved geometrically as follows.

Suppose there are four qualities due to five independent factors, \(Q_1\) being due to \(G\) and \(X\), \(Q_2\) to \(G\) and \(Y\), \(Q_3\) to \(G\) and \(Z\), and \(Q_4\) to \(G\) and \(U\). If the factors are represented by a set of mutually perpendicular axes in 5-dimensional space, the representative lines \(OQ_1\), \(OQ_2\), \(OQ_3\), \(OQ_4\), will lie one in each of the planes \(GOX\), \(GOY\), \(GOZ\), \(GOU\). The correlation between \(Q_1\) and \(Q_3\) is

\[ r_{13} = \cos Q_1OQ_3 = \cos \theta \cos \chi, \]

and so for the other correlations; \(\theta\), \(\phi\), \(\chi\) and \(\psi\) being the angles between the
lines and $OG$. Hence

$$F = r_{13}r_{24} - r_{14}r_{23} = \cos \theta \cos \chi \cos \phi \cos \psi - \cos \theta \cos \psi \cos \phi \cos \chi = 0.$$  

Conversely, if the tetrad difference is always zero, it is possible to find factors, uniquely determined, which will give the correlations observed, of which factors one is general and all the others specific. In order fully to appreciate the arguments in the later sections it is necessary to consider here in some detail the proof of this, more particularly the ideas which appear as the basis of the proof given by Garnett.

---

**Fig. 7.**

All that is known from observation are the correlations, i.e. the angles between the lines representing the qualities, and the mathematical question is to what axes these lines can, or should be, referred. If only
two qualities be considered, clearly they can, and that in an infinite number of ways, be referred to three axes, OG, OX, OY, so that each of the lines lies in one of the coordinate planes, as in figure 7. In that case \( Q_1 \) depends on \( G \) and \( X \), and \( Q_2 \) on \( G \) and \( Y \). If three qualities whose correlations are known be next thought of, represented by three lines making the proper angles with each other, again it is always possible to find a system of four rectangular axes, OG, OX, OY, OZ, such that one line lies in the plane GOX, one in GOY, and one in GOZ; that is, if we please we may always suppose each of three qualities to depend on a general and on a specific factor. (This may be compared with the fact discussed in Section IV, that, provided a certain condition is fulfilled, the three qualities may alternatively be referred to three axes, each depending on a different pair of them.)

When, however, there are four qualities represented by lines in, of course, four-dimensional space, it is no longer possible generally to carry out a similar process.

Fig. 9.
It is only when \( F = \cos 13 \cdot \cos 24 - \cos 14 \cdot \cos 23 = 0 \) that a system of five axes of reference can be obtained such that each line lies in one of the planes GOX, GOY, GOZ, GOU; only when the tetrad difference is zero can the lines in figure 9 be made to appear as in figure 6. And if there are any number of lines with the relation \( F = 0 \) fulfilled by any four of them there can be obtained a system of axes such that each line depends on one of the axes OG and on another belonging to itself; which establishes the proposition.

Thus if it is to be possible to account for each quality by the operation of a single general factor and a specific factor the representative lines must not make angles with each other in a haphazard fashion. Three of them may make any angles, but after that there are only certain positions out of the totality of imaginable ones which the other lines may occupy. This is analogous to the theorem that any three points in a plane are concyclic, but that four or more points are so only when a set of conditions are fulfilled. The point immediately at issue, however, is that when the conditions \( F = 0 \) throughout are fulfilled it is possible to find a single general factor and a number of specific factors, to the exclusion of factors which take part in some activities but not in all. Provided, therefore, that we have no theory of the nature of the factors to begin with, and that we accept the view
that the tetrad differences are in fact exactly zero, it is a valid conclusion that observed facts are in accordance with the Two Factor theory.

The Two Factor theory, it is important to observe, is limited to qualities which are sufficiently varied. If, for instance, there were some quality closely allied to \( Q_2 \) in figure 6, but yet resembling \( Q_1 \) to some extent, it would be represented by a line near \( OQ_2 \), but lying in the space between \( GOX \) and \( GOY \); and the tetrad difference would no longer be zero. When therefore observed correlations were studied to see whether the tetrad equation were fulfilled, if among the qualities there were some known on other grounds to be closely allied, all save one were dropped. Thereafter, in the development of the study, if for a set of four qualities the tetrad difference was found to depart significantly from zero, this was taken as indicating that among the four there were similar qualities. So great is the confidence put in the theory that the failure of four qualities to conform with its crucial test is taken as proof that they are not all dissimilar. This is quite legitimate procedure in an experimental science, although it must be urged that unless this similarity is confirmed in every case by other considerations the argument as a whole loses a certain amount of cogency.

Another limitation to be noted is that the field of
experience is not exhausted by this theory. The investigations of Webb on estimates of character led him to the theory that there existed another factor, independent of the general factor traced in intellectual functions, which behaved like a single general factor among a large group of qualities; and Garnett, in a further study of Webb's data, discovered the necessity of postulating the existence of a third, which among a third group played the part of a single general factor. These three have been styled 'General Ability', 'Cleverness' and 'Purpose' by Garnett, who says that when a person's value in these three respects has been determined 'a surprisingly large proportion, but of course by no means the whole, of his moral as well as of his intellectual qualities have also been defined'.

To the objection that the Two Factor theory in its further development is only hypothetical a conclusive answer has been given by Spearman in 'The Abilities of Man' (pp.127,128). As he says in another place, 'this general factor g is primarily not any concrete thing but only a value or magnitude'. There can be no doubt that there is nothing unsound in discovering some important mathematical entity in any branch of science and subsequently seeking to find what fact corresponds to it.

It is with the mathematical aspects of the theory that this thesis is mainly concerned; and our position
may be summed up by stating that the mathematical foundations are sound; the parts of the theory are consistent; and if the small values of the tetrad differences could be explained in no other way the case for the Two Factor theory might be considered well made out.

Consequences of the second alternative.

The acceptance of the second alternative, that the values of the tetrad difference are not zero, but only closely grouped round it, involves the rejection of the Two Factor theory, at any rate in its exact form. Another theory must therefore be found which will account for the small values which have been accepted as facts of observation; and it is natural to return to the original theory described in Section III. There it was supposed that mental qualities were composed of overlapping sets of factors acting in varying proportions. Now it has just been pointed out that if the tetrad difference is always exactly zero the sets of factors acting, and the proportions in which they act cannot be any sets and any proportions. Suppose on the contrary that any imaginable combination of factors is regarded as a possibility; the tetrad differences will no longer be all zero, the important question now being whether the values will be small, small even compared with their sampling errors.

An answer to this question was given in general terms by Professor Godfrey Thomson in a paper published in 1919.
In that paper he showed that if mental qualities arose through the chance selection of the elementary factors the correlations would probably show some degree of hierarchical order. This was based on a theorem of Pearson and Filon regarding the sampling errors of correlation coefficients. The theorem states that such errors are correlated; and that, while 'errors in the correlations of a first organ with a second and a third have a correlation themselves of the first order', 'errors in the correlation of two organs and in the correlation of a second two have only correlation of the second order'. That is to say, there is a correlation between the errors in \( r_{12} \) and \( r_{13} \) which is probably greater than that between the errors in \( r_{12} \) and \( r_{34} \). Following out the consequences of this theorem in a set of correlations assumed to be truly equal, it was shown that the observed values, in error only through sampling, would probably exhibit hierarchical order, using the term in its descriptive sense. Next, it was argued that samples gave values differing from the true values because there was caused an apparent change in the factors taking part; that the change, though only apparent, was effective; and that were the change real, as when different mental tests are used, the same reasoning would be applicable. Professor Thomson holds that, viewed philosophically or treated mathematically, the sampling of the population and the sampling of factors
throughout the entire population are equivalent. Thus it was concluded that a hierarchy was normally to be expected. If then part of the array were

\[ r_{ap} \quad r_{bp} \]
\[ r_{aq} \quad r_{bq} \]

\( r_{ap} \) would be the greatest of the four correlations and \( r_{bq} \) the least; and the tetrad difference \( r_{ap} r_{bq} - r_{aq} r_{bp} \) would therefore, probably, be small. Thomson followed this up with a number of experiments, in which the measurements of the various qualities were obtained by dice-throws, a form of experiment possessing the advantage that the observed values for a sample of the population imitated by actual throws could be compared with the true correlations calculated for an infinite population. These experiments are described in complete detail in his published works, so that it is necessary here only to mention those features and results which are strictly germane to the present discussion. A set of imaginary qualities have the numbers of factors contributing to them decided by chance, by drawing numbers from a pack of cards; out of a given number of available factors the particular ones acting in the various qualities is again decided by drawing cards; and further to each quality is allotted a chance number of specific factors. There is thus obtained a set of qualities due to factors
which overlap in a manner that is completely known; and, if the factors are supposed to be 'all or none', the theoretical correlations for an infinite population can be calculated. To complete the experiment the value of the factors for any individual is obtained by throwing dice, and for any number of subjects, limited only by the amount of labour involved, there is obtained a set of 'observed scores' for the imaginary tests, and also the observed correlations. Applying the criterion of inter-columnar correlation, Thomson found that these imaginary correlations passed the test; that is, the inter-columnar correlation, as calculated by the Spearman-Hart formula turned out to be practically unity. But — and here lies the crux of the matter — the true inter-columnar correlation was not unity, the factors were not either general to all, or specific to single, qualities, neither could they be replaced by such. The test of inter-columnar correlation had therefore failed to distinguish between the alternatives, whether the Two Factor theory requirements were really exactly fulfilled or only nearly so. And, although one single instance to the contrary is sufficient to cause the downfall of a theory, these imaginary qualities were not specially contrived to produce that contrary instance, but were the outcome of some sort of random make-up. The result of these
experiments led Professor Thomson to formulate his Sampling Theory of Ability, which, along with the further tests to which he subjected it, will be taken up in the next section.

**Note:** The subject matter of this section is discussed in 'The Abilities of Man' and 'The Essentials of Mental Measurement', and in the original papers referred to in those works.
SECTION VIII.

The Sampling Theory of Ability.

The Sampling Theory states that there are 'a number of factors at play in the carrying out of any activity such as a mental test, these factors being samples of all those which the individual has at his command'.

All the factors are thought of, though not necessarily of the same nature, as being on the same footing, there being none standing out from its fellows like the Single General Factor. When therefore we measure a subject's standing by some mental test, we obtain an indication of the worth of the whole of his factors, just as by examining a handful of gems from Aladdin's cave we might judge of what value were the gems as a whole. Any factor must be thought of as capable of entering into any number of activities, and the factors belonging to several activities as overlapping in every sort of way. Thus, if there are four activities considered, there is the possibility of some factors occurring in all four, some in three, some in two, and some in only one. In the first statement of the theory, as also in the mathematical and experimental work which he has done upon it, Professor Thomson supposes the factors to be 'all or none' in their action, though on the other hand he regards this restriction as unnecessary. It is not therefore essential to the theory that the factors should be 'all or none'.
A little consideration of the statement of this theory and its implications shows that it is nothing else than the original theory discussed in Section III, the theory which is the basis of the work done in this subject, and the theory assumed by Garnett in proving that the several qualities could be expressed by two factors when the tetrad equation is fulfilled.

Now there is no direct means of knowing either what fraction of the total number of factors acts in any particular ability, nor how the factors acting in several abilities overlap. Taking a very simple instance, suppose a certain correlation is found to be 1/2. Then, taking the factors to be 'all or none', the grouping may be illustrated thus: a factors act in Q₁, b in Q₂, c being common. Since \( r = \frac{c}{\sqrt{ab}} \), any

![Diagram](image)

Fig. 10.

values of a, b and c will suit, so long as \( \sqrt{ab} = 2c \); e.g. the values might be a = b = 2, c = 1; or a = 18, b = 3, c = 6. When three or more qualities are dealt with, the actual correlations lead to the consideration
of the structure diagrams representing the various possibilities. Which of the many possible diagrams is the right one is a question left meanwhile unanswered, except that the knowledge of the nature of the mental activities may give the preference to some over others; while in certain groups of abilities definite conclusions as to the minimum amount of overlapping may be come to. All this is involved in the line of research referred to in Section IV.

From the fact that the elementary factors are, in this theory, considered capable of combining in any manner it follows that the tetrad differences are considered not to be truly zero in every case; and the important question to be faced is whether the tetrad differences following from the theory will be small enough to agree with the observed values. Professor Thomson therefore subjected his experimental results to a new test. The 'observed' correlations among the imaginary qualities, obtained by actual throws of dice, were tabulated and the tetrad differences calculated. When these were grouped according to their magnitude and a distribution graph drawn, the scatter of the distribution from zero was found to be within the limits of error as calculated by the formula of Spearman and Holzinger. The true tetrad differences were also found, so that the distribution of observed tetrad differences as in figure 11
Values of tetrad differences $F$.

Fig. 11.

Values of $F$.

Fig. 12

Fig. 13

arose from the true distribution shown in figure 12, and not from a set of values exactly zero as in figure 13.
These curves are smoothed (by eye) from the histograms in Professor Thomson's article (Brit. Journ. Psychol., 1927, xvii.), the dotted curve being the theoretical distribution. The test therefore had failed to distinguish between the supposition that the true values were exactly zero and the supposition that they were merely closely grouped round zero. A fortuitous arrangement among the factors had given results in accordance with observation; but the true values of the tetrad difference were not zero, the qualities were not due to two factors, nor could they be expressed in terms of two factors.

It will be convenient here to summarise the chief points of the Sampling Theory.

1. The factors are all on the same footing.
2. Any quality calls for the operation of a certain number of the factors available, and these are regarded as a sample of the whole.
3. All kinds of overlapping are regarded as possible.
4. Provided the arrangements of the factors among the qualities give the required correlations, there is no knowledge, obtainable from the study of the correlations, of what the exact arrangements are.
5. So far as the values of the tetrad difference are concerned, it is asserted that it does not matter what these arrangements are, and that they might even be supposed to occur by mere chance.
SECTION IX.

Some Objections to the Sampling Theory.

In the long course of the investigation of the theories now before us objections have from time to time been urged against the Sampling Theory just described. Some of these objections are on purely psychological grounds, and with these we are here not immediately concerned. Others are on mathematical grounds, and, as our present concern is with the mathematical aspects of the problem, we must not neglect considering these objections.

First, it was said that, while it was possible for a chance grouping of factors constituting mental qualities to yield tetrad differences equal to zero, such a grouping would be improbable to such an excessive extent as to be hardly worth considering; and further that, for exactness, in order to cover all cases the number of elements would need to be very large, if not infinite. Now it has already been stated above that the exact vanishing of the tetrad difference does make the case for the Two Factor theory complete. It is also true that the most probable arrangement of 'all or none' factors among a set of qualities leads to tetrad differences which vanish, but that the probability of this occurring by chance, though greater than the probability of any other arrangement, is extremely small. In such circumstances, though it would still be possible
to explain the qualities in terms of factors other than those of the Two Factor theory, the case for preferring the latter theory would become overwhelmingly strong. But the Sampling Theory is not supposed to give tetrad differences exactly equal to zero; it is in fact the belief that they may not be actually zero that gives the Sampling Theory any standing. What is in question is how near to zero the values may be expected to be, without such a special arrangement happening.

A second objection is that after all the Sampling Theory is but a variant of the Two Factor theory. This is discussed in the article published in 'The Journal of Educational Psychology' for December 1923, incorporated in Part III of this thesis, and accordingly the mathematical details will not be entered into here.

Let us take it as established (for the sake of the present argument) that the Sampling Theory gives as a probable result very small values of the tetrad difference. Let us further admit that, as we can express qualities by means of two factors if the tetrad differences are exactly zero, we can nearly or approximately so express them if the tetrad differences are nearly zero. We are not convinced that this last statement is sound, not at any rate without some precise estimate of what is to be regarded as 'nearly'; for it may well be that the word 'nearly' ought not to connote the same degree in both
cases. For the moment, however, let it be accepted. It then follows that we may either accept the Sampling Theory or, as approximately true, the Two Factor theory. But while this is so it is not the case that we can express each in terms of the conceptions underlying the other. The point of the argument in the article referred to is that in both cases there is a quantity 'g', which has one psychological description on the one theory, and another on the other; but that the attempt to interpret the specific factors in terms of the concepts underlying the Sampling Theory breaks down; and therefore that when an ultimate explanation — or, as perhaps it ought to be put, a more fundamental explanation — in psychological terms is desired, one or other of the two theories must be preferred. As it may seem curious to admit the existence of a quantity 'g' on the Sampling Theory, while denying existence to entities corresponding to the specific factors 's', it may be worth while to consider an example. Consider the case of a person for whom the quantity 'g' has been measured, according to the methods explained by Spearman, and for whom a value has been obtained for a performance in, say, geometry. Suppose for the sake of definiteness that his 'g' is high, while his geometry score g is below the average. These facts are illustrated in figure 14. Then according to the Two Factor theory the following is the explanation.
ON = \textit{g} for the man whose OA = \textit{q} representative OM = \textit{s} point is P.

Fig. 14.

The man possesses a high general ability, but the value of the specific factor operating in geometry is a low one; and this low value causes his ability in geometry to be low, owing to the degree in which the specific factor contributes to geometrical ability. On the other hand, all we can say on the Sampling Theory is that the value of the quantity $\frac{1}{\sqrt{N}} \sum x$ is high; i.e., that the general level, or average, of the factors he possesses is high. Why then is his ability in geometry low? The answer is that it is because it is not the factors operating in geometry that give him a high average, but that on the contrary they must be a relatively poor selection. The definite entity 's' of the Two Factor theory is replaced by a verbal explanation in terms of the Sampling Theory.

There is a third objection, that the consequence of the Sampling Theory would be that, when the number of elementary factors is very large, all individuals would tend to have the same total ability. This conclusion
is arrived at from the fact that the standard deviation of the mean value of a number of measurements is equal to that of one measurement divided by the square root of the number; so that, if \( e \) is the measure of one element for an individual, and \( \bar{e} \) the mean value of \( e \) for an individual, the mean value of \( \bar{e} \) for the population will be \( \eta \) with a standard deviation \( \frac{\sigma_{\bar{e}}}{\eta} = \frac{1}{\sqrt{n}} \sigma_e \). Hence, as \( \eta \) remains finite, \( \frac{\sigma_{\bar{e}}}{\eta} = \frac{\sigma_e}{\sqrt{n.\eta}} \) approaches zero as \( n \) becomes infinite; i.e., the standard deviation of the average value of an element is negligible in comparison with that average value. In other words, the total ability \( T \) tends to have the same value for all individuals. On this argument we offer three observations.

1. The conclusion is inconsistent with the second objection we have just considered. For the total ability \( T \) is (to a numerical factor) just the quantity 'g'. If it be true that the total ability on the Sampling Theory is practically the same for everybody, it must also be true that the value of 'g', the General Ability of the Two Factor theory, is practically the same for everybody.

2. The proof depends on the assumption that the \( e \)'s are positive. But the numerical measures we attach to mental abilities, and therefore also to the factors, are not absolute measurements, any more than degrees on a thermometer. If we measure the \( e \)'s from their mean, we then have \( \frac{\sigma_{\bar{e}}}{\eta} = \frac{0}{0} \) in the limit, an indeterminate quantity. We then have
still a distribution of $\bar{e}$ which has a standard deviation which is small compared with that of the e's: $\bar{e} = \frac{1}{\sqrt{n}} \sigma_e^e$.

But there still is a variance, and if $n$ is finite, however large, $\sigma_e^e$ is not actually zero. In any case, the standard deviation of the total ability $T$ would be $\sqrt{n} \sigma^e_e$, not a vanishing quantity.

(3) It may be true that, compared with their total mental ability, the differences among men are really slight. This is a view that has been entertained by some observers of human nature. In 'Crotchet Castle' Peacock makes his characters hold conversation regarding natural qualities and education. The Rev. Dr Folliott says: "I hold that there is every variety of natural capacity from the idiot to Newton and Shakspeare; the mass of mankind, midway between these extremes, being blockheads of different degrees; education leaving them pretty nearly as it found them," etc. Later, Mr MacQuedy says: "I say, cutting off idiots, who have no minds at all, all minds are by nature alike. Education (which begins from their birth) makes them what they are." It would not be unreasonable to think that, in the scale of evolution, with the zero at the lowest form of life, indicating the entire and absolute absence of mental power, the variations among the abilities of 'homo sapiens' are relatively not great.
SECTION X.

Value of the Tetrad Difference on the Sampling Theory for 'all or none' factors.

To return now to the values of the tetrad difference on the Sampling Theory, the next step is to estimate more exactly what these values may be expected to be. Since the factors are assumed to be on the same footing, any arrangement of them is regarded as a possibility, and all arrangements as equally likely; the calculations will therefore be calculations of probabilities. In what follows we consider a single group of four qualities, and the object is to predict a value for the resulting tetrad difference. By predicting a value is meant stating that it is an even chance that the tetrad difference will not exceed such and such a quantity.

Each quality will depend on a certain number of factors; suppose this number is known for each of the four. Then the tetrad difference will depend on how the four groups of factors overlap, and there will be as many possible cases as there are ways of selecting the four groups of factors out of the whole number available. All these being taken as equally probable, there will be a distribution of possible values for the tetrad difference. This distribution will have a mean value and a standard deviation, and the standard deviation will give a good estimate of the limits within which
the tetrad difference may be expected to lie.

This problem was taken up by Professor Thomson in his paper 'A Worked out Example of the Possible Linkages of Four Correlated Variables on the Sampling Theory.' He had previously proved the important theorem that the most probable value of the tetrad difference $F$ is zero, and that the distribution of possible values has therefore a mode at zero. He now worked out in detail two examples; one in which the number of factors available was six, and the qualities due to 2, 3, 4 and 5 of them respectively; and one in which these numbers were all doubled, viz., 12, 4, 6, 8, 10. In both examples the mean value of $F$ was found to be zero, and in the first example the variance $\sigma^2_F$ was found to be 0.040, while in the second it was 0.018. Taking the exact figures in that paper, if there are 6 factors in all, and one quality is due to 2 of them, a second to 3, a third to 4, and a fourth to 5, the prediction is that the tetrad difference $F$ has almost an even chance of not exceeding numerically $\frac{1}{\sqrt{120}}$, the exact probability being $\frac{216}{450}$; if there are 12 factors, and the qualities are due to the same proportions of them, $F$ has almost an even chance of not exceeding $\frac{3}{\sqrt{1920}}$, the exact probability being $\frac{490451}{1078110}$. Now $\frac{3}{\sqrt{1920}}$ is $\frac{3}{4}$ of $\frac{1}{\sqrt{120}}$, so that by taking the factors to be 12 instead of 6 the value of $F$ to be expected has been reduced. With so few factors the possible
values of $F$ jump by quite appreciable amounts, and the best way of comparing the two results is to calculate $\sigma_F^2$; and the result shows that with 12 factors the value of $\sigma_F^2$ is about half of what it is with 6 factors. From these examples Professor Thomson believed that with any number of factors the value of $\sigma_F^2$ would be found inversely proportional to the number of factors, a view subsequently verified.

The general problem of which these were two particular examples is to find the value of $\sigma_F$, when the qualities are due to known fractions $p_1$, $p_2$, $p_3$, $p_4$, of a universe of $N$ factors. This forms the subject of the paper 'The Probable Value of the Tetrad Difference on the Sampling Theory', which is included in Part III of this thesis. The principal results obtained in that paper are:

1. The Mean Value of $F$ is zero.
2. The Standard Deviation of the distribution of the possible values of $F$ is given by

$$\sigma_F^2 = \frac{1}{N-1} \left\{ (p_1 p_3 + p_2 p_4 + p_1 p_4 + p_2 p_3)^2 - 2 \sum p_1 p_2 p_3 + 4 p_1 p_2 p_3 p_4 \right\}$$

or approximately $4p^2(1-p)^2/N$, when $N$ is large and the $p$'s are replaced by their mean. The probable value is given as $0 \pm 6.745 \sigma_F$, which assumes a distribution at least approximately normal. The exact form of the
distribution is unknown, but the two numerical examples, in which it is known, show that it is one with a mode at zero, and actually in these examples \( \frac{2}{3} \sigma_F \) does give the limits within which lie about half of the values.

The general conclusion is that for four qualities due to randomly selected groups out of a universe of 'all or none' factors the tetrad difference is expected to be a very small quantity. Even if we do not know, nor assume, the values of the fractions \( p \), we can put each equal to 1/2, which gives the maximum scatter of the values of \( F \); \( \sigma_F \) is then approximately \( \frac{1}{2\sqrt{N}} \). Hence for any four qualities, knowing nothing about them except that we look on them as 'samples' of the available factors, we may put the expected value of \( F \) as being very small. Therefore the fact that an observed tetrad difference is small, even almost zero, does not demand a special explanation, but is in keeping with the Sampling Theory, if the factors are 'all or none'.

Two additional points in connection with the formula for \( \sigma_F^2 \) may be noticed here.

(1) The equation may be put into a shape that is rather suggestive, viz.,

\[
\sigma_F^2 = \frac{1}{N-1} \left\{ (p_1+p_2 - 2p_1p_2)(p_3+p_4 - 2p_3p_4) \right. \\
\left. + \frac{2(N-2)}{(N-1)^2} (1-p_1)(1-p_2)(1-p_3)(1-p_4) \right\}
\]

Consider a single factor. The probability that it
occurs in $Q_1$ is $p_1$, that it occurs in $Q_2$ is $p_2$; that it occurs in one but not both is $p_1(1-p_2) + p_2(1-p_1)$, i.e. $p_1 + p_2 - 2p_1p_2$. Hence

$$\sigma_F^2 = \frac{1}{N-1} \left\{ \text{Probability that any given factor occurs in one, but not both, of } Q_1 \text{ and } Q_2, \text{ and also in one, but not both, of } Q_3 \text{ and } Q_4 \right. $$

$$+ \frac{2(N-2)}{(N-1)^2} \left( \text{Probability that it occurs in no Quality at all} \right) \} .$$

What significance this may have we have not been able to find out; it might suggest a shorter way of proving the formula for $\sigma_F^2$.

(2) The formula in its approximate form is very similar to that of Spearman and Holzinger for the probable error of a tetrad difference due to sampling errors.

Taking the form

$$\sigma_F = \frac{1}{\sqrt{N}} p(1-p) = \frac{1}{\sqrt{N}} r(1-r) ,$$

we note that $\sigma_F$ is proportional to $p(1-p)$. If the fractions $p$ are small, $F$ will tend to have small values, because the amount of overlapping will most often be small or zero and consequently the correlations will be small or zero. On the other hand, values of $p$ almost unity will result in a high degree of overlapping and consequently high correlations, so that again $F$ will be small. In Spearman and Holzinger's formula

$$\sigma_F = \frac{1}{\sqrt{N}} r(1-r) ,$$

where $r$ is the average of the four correlations. If
this average is low, the tetrad difference will tend to be small merely because of the small values of \( r \); while if \( r \) is close to unity the error in \( F \) will be small because the observed correlations will be nearly always correct. Both formulae (in this approximate form) thus express what we should expect as a reasonable result; and their agreement points to the correctness of the view that sampling the factors in the entire population and taking the same factors for samples of the population are mathematically equivalent.
SECTION XI.

Value of the Tetrad Difference on the Sampling Theory for variable factors.

To complete the investigation of the expected value of the tetrad difference it is now necessary to suppose the factors variable. In this connection reference must be made to the discussion in Section VII, where the qualities were represented by directed lines in $N$-dimensional space. It was shown there that, in order to meet the requirements of the Two Factor theory, the lines could not be supposed to be drawn anywhere in a random fashion. This interpreted means that the factors cannot be supposed to operate in random proportions. We are now going to suppose that the factors can operate in random proportions, which means that we suppose the lines drawn in a random fashion; and, whereas for those lines which could be equally well referred to two factors apiece the tetrad difference is exactly zero, we propose to ask what the tetrad difference will be corresponding to any randomly drawn four lines. Again there will be a distribution of possible values, the standard deviation of which will indicate within what limits we may expect such a tetrad difference to lie.

This problem is solved in the article on 'Mathematical Consequences of Certain Theories of Mental Ability', included in Part III. The value of the standard
The geometrical interpretation of this is: If the tetrad difference is zero for a quartet of lines, these may be made to lie in the coordinate planes of a new system of reference GOX, GOY, GOZ, GOU; yet for any quartet of lines (if the space in which they are drawn be of a large number of dimensions), for which therefore in general this transformation cannot be carried out, the value of the tetrad difference is expected to be nearly zero. The psychological interpretation is that if four qualities are due to random proportions of a large number of elementary variable factors, the value of the tetrad difference is expected to be small.

It is pointed out in the article that this cannot be regarded as a complete solution of the problem, because the four qualities, depending as they do on the same set of N factors, should more properly be looked on as varieties of the same quality. The problem remaining is to find the expected value of F on the same assumptions as in Section X, except that the factors are taken to be variable instead of 'all or none'.

Suppose, then, as in the preceding section, that we assume known the size of the sample of factors operating in each of four qualities; that all possible groupings
of the factors are taken as equally likely; and that all proportions of the factors in any grouping are taken as equally likely; what, under these conditions, would $\sigma^F$ be?

The grouping of the factors may be represented as

![Diagram](image)

Fig. 15.

in figure 15. The process of finding $\sigma^F$ is as follows:

1. For any one arrangement calculate the mean value of $F^2$ by considering all possible variations in the factors. Geometrically, we have to consider four lines, one drawn in a certain space of $p_1N$ dimensions, a second drawn in a certain other space of $p_2N$ dimensions, a third in a certain third space of $p_3N$ dimensions, and a fourth in a certain fourth space of $p_4N$ dimensions, these spaces overlapping in the manner indicated in figure 15. As an example, take the total number of factors as 6, and the qualities as due to 5, 4, 3 and 2 of them, and arranged as in figure 16. The space
Fig. 16.

is then of six dimensions; and
Q₁ is in the 5-dimensional space OX, OY, OZ, OU, OV,
Q₂ is in the 4-dimensional space OY, OZ, OU, OV,
Q₃ is in the 3-dimensional space OU, OV, OW,
Q₄ is in the 2-dimensional space OZ, OU.

We now suppose the lines to take all possible positions, so long as they keep each to its own space, and by the method of the article referred to the mean value of $F^2$ could be got.

(2) This is the mean value for one arrangement, and its value would depend on what the particular arrangement was. The mean of these means for all arrangements would be $\sigma^2_F$, the quantity we are in search of.

When the work of finding $\sigma^2_F$ for variable factors when the spaces for the four lines are coextensive is considered, it will be believed that the calculation when they are overlapping as described above would be
one of extreme laboriousness. The amount of work would be so excessive as to preclude attempting it, for it would require the consideration of about 6500 calculations of the sort carried out in the article below. We therefore give the results for a few very simple instances, from which may be inferred what the result would be like in the general case.

Example 1.

\[ N = 2 ; \, p_1 = 1, \, p_2 = 1/2, \, p_3 = 1/2, \, p_4 = 1. \]

There are 2 possible arrangements:

(a)

(b)

Fig. 17.

Results.

<table>
<thead>
<tr>
<th>Arrangement</th>
<th>Factors 'all or none':</th>
<th>Factors variable:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( F )</td>
<td>( F^2 )</td>
</tr>
<tr>
<td>(a)</td>
<td>(- \frac{1}{2})</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>(b)</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>Mean Values</td>
<td>0</td>
<td>( \sigma^2_F = \frac{1}{4} )</td>
</tr>
</tbody>
</table>
Example 2.

\[ N = 3; \quad P_1 = 1, \quad P_2 = \frac{2}{3}, \quad P_3 = \frac{2}{3}, \quad P_4 = \frac{2}{3}. \]

There are 5 possible arrangements:

(a) \hspace{2cm} (b)

(c) \hspace{2cm} (d) \hspace{2cm} (e)

Fig. 18.

The relative frequencies of these arrangements are 1, 2, 2, 2, 2.

The calculation of \( \text{Mean } F^2 \) for each arrangement in this very simple example involved six subsidiary calculations.
Results.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Arrangement} & \text{Frequency} & \text{Factors 'all or none'} & \text{Factors variable} \\
\hline
(a) & 1 & 0 & 0 & \frac{1}{6} \times 0.3537 \\
(b) & 2 & -\frac{1}{\sqrt{6}} & \frac{1}{6} & -\frac{8}{\pi^2} \left\{1 + 0.0268\right\} \\
(c) & 2 & \frac{1}{\sqrt{6}} & \frac{1}{6} & \frac{8}{\pi^2} \left\{1 + 0.0268\right\} \\
(d) & 2 & 0 & 0 & \frac{1}{6} \times 0.8157 \\
(e) & 2 & 0 & 0 & \frac{1}{6} \times 1.206 \\
\hline
\text{Mean Values} & 0 & \frac{\sigma^2_F}{27} & 0 & \frac{\sigma^2_F}{27} \left\{1 + 0.583\right\} \\
\hline
\end{array}
\]

Example 3.

\(N = N\); \(p_1 = 1, p_2 = \frac{1}{2}, p_3 = \frac{1}{2}, p_4 = 1\).

One arrangement considered, corresponding to 1(b) above.

Ex. 3. Ex. 4.

\[\text{Fig. 19.}\]

Results.

Factors 'all or none':\( F = \frac{1}{2} \); \(F^2 = \frac{1}{4}\).

Factors variable:

Mean \(F = \frac{1}{2} \left(\frac{2}{\pi}\right)^2\), when \(N\) is large;

\[
\text{Mean} \ F^2 = \frac{1}{4} \left(\frac{2}{\pi}\right)^4 + \frac{1}{N} \left(\frac{2}{\pi}\right)^2 \left\{1 - \left(\frac{2}{\pi}\right)^2\right\} + \frac{1}{N^2} \left\{1 - \left(\frac{2}{\pi}\right)^2\right\}^2
\]
A comparison of these results with those for Example 1(b) shows the effect of making $N$ large while keeping to the same arrangement. Also, if we put $F_a$ for the value of $F$ when the factors are 'all or none', then for variable factors

\[
\text{Mean } F = F_a \cdot \left(\frac{2}{\pi}\right)^2, \text{ when } N \text{ is large, and}
\]

\[
\text{Mean } F^2 = F_a^2 \cdot \left(\frac{2}{\pi}\right)^4 + \text{a quantity of the order } \frac{1}{N}.
\]

**Example 4.**

$N = N$; $p_1 = 1$, $p_2 = p_2$, $p_3 = p_3$, $p_4 = 1$, where $p_2 + p_3 = 1$.

One arrangement considered, like that in Example 3.

**Results.**

Factors 'all or none' :- $F = \sqrt{p_2 p_3}$; $F^2 = p_2 p_3$.

Factors variable :-

\[
\text{Mean } F = \sqrt{p_2 p_3} \cdot \left(\frac{2}{\pi}\right)^2, \text{ when } N \text{ is large;}
\]

\[
\text{Mean } F^2 = p_2 p_3 \cdot \left(\frac{2}{\pi}\right)^4 + \frac{1}{N} \left(\frac{2}{\pi}\right)^2 \left\{1 - \left(\frac{2}{\pi}\right)^2\right\} + \frac{1}{N^2} \left\{1 - \left(\frac{2}{\pi}\right)^2\right\}^2
\]

These examples suggest that for one single arrangement the mean value of $F$ is the value when the factors are 'all or none' multiplied by $\left(\frac{2}{\pi}\right)^2$. Since the mean value of $F$ for all arrangements of 'all or none' factors is zero, it follows that the mean value for variable factors is also zero.
It further appears that for one single arrangement the mean value of $F^2$, obtained by varying the relative proportions of the factors, is

$$F^2 \text{ when the factors are 'all or none'} \times \left(\frac{2}{n}\right)^4$$

$\pm$ a function of the arrangement, of the order $\frac{1}{N}$.

For all possible arrangements, therefore, of variable factors

$$\sigma^2_F = \sigma^2_F \text{ when the factors are 'all or none'} \times \left(\frac{2}{n}\right)^4$$

$\pm$ a function of the order $\frac{1}{N}$.

This last function will be of the form

$$\phi \left( p_1, p_2, p_3, p_4, n, \frac{2}{n} \right),$$

and when the $p$'s are each unity will reduce to the expression already given at the beginning of this section.

Now $\sigma^2_F$ when the factors are 'all or none' is of the order $\frac{1}{N}$; so that the final conclusion is this:

If four qualities are regarded as being due to the activity of four samples of elementary factors, whether variable or 'all or none', and the total number of factors available is fairly large, the resulting tetrad difference is, according to the laws of probability, expected to be very small, in fact of the order $\frac{1}{\sqrt{N}}$. 
SECTION XII.
Summary and Conclusion.

After our long mathematical investigation it will be well to summarise what has been established, either in previous publications or in the present thesis. The mathematical work into which the subject leads is of necessity very much involved; yet the ideas underlying that work and the questions it seeks to answer are fairly simple.

First, then, let us set out the mathematical argument, stripped of all details, and without any symbols.

On the assumption that mental qualities are compounded of variable factors, so that they may be represented geometrically by directed lines in a space of many dimensions, then the totality of such lines represents all imaginable qualities.

(1) If the tetrad difference is everywhere exactly zero, then the lines which do in fact occur form a group or family, and are such that they may be represented as dependent on a single general factor and a number of specific factors.

(2) If from the whole set of lines quartets are taken in all possible ways, the resulting tetrad differences are grouped round zero, and that very closely if the space is of a high number of dimensions. Accordingly, if one quartet is taken at random, the probability
is that its tetrad difference will be small.

(3) If only those lines are supposed to exist which are equally inclined to the axes taken any number at a time, that is, if each factor is supposed simply either to act or not to act, then the possible tetrad differences are closely grouped round zero. Again, therefore, for one quartet taken at random the probability is that the tetrad difference will be small.

(4) If the lines considered in the previous paragraph are supposed to be able to occupy any position in the space to which they belong, then once more the possible tetrad differences are closely grouped round zero. In this case too the probability is that the tetrad difference for one quartet taken at random will be small.

Next we turn our minds to the psychological meaning of these statements.

Each person is assumed to have at his command, if we may use that flattering word, a number of factors, whose action gives rise to what appear to others as mental qualities or abilities. Regarding these factors and their nature we are in a double state of ignorance, for we know neither what they are like nor how they are formed into groups to do their work. The spectators in Plato's cave could see the shadows of the real objects, and could form a correct idea of what the real objects were saying
and trying to do, although ignorant of their true nature. The mental factors we are thinking of are not even shadows, for we know only the results of their labours. Here is evident the insufficiency of the mathematical method by itself, and the necessity of supplementing it by other methods. By methods of introspection men have tried as it were to catch the factors at work; to describe their functions they have given them names, while phrenologists and physiologists have sought to give them local habitations. The mathematical psychologist examines the quantitative values of mental abilities, not with the hope of discovering the nature of the factors, but expecting to find out a little about their manner of working, to determine how they club together to fulfil the various duties assigned to them.

(1) If the tetrad difference is zero throughout a set of qualities, then each appears as if it were the work of two factors, one of which is called upon on every occasion, while the others are specialists. The nature of the factors is still left to be discovered. It is likely that the general one will be of a different kind from the others, and they have been likened by Professor Spearman to a number of engines along with the energy which actuates them.

(2) We may prefer to picture the factors as a number of servants ready to be called on to discharge some duty,
and imagine that, while some of the tasks to be performed can be done by a few of them, others make a heavier demand on their resources. If this were a correct picture of the situation, and if four tasks are set, no matter what the size of the groups needed for each may be, and even supposing the groups to be formed at random, the chances are that the tetrad difference for the four qualities will be small. This is so whether we suppose each factor's contribution to be always of the same, or of varying, importance.

Whatever notion be preferred, there are, if the tetrad differences are nearly or exactly zero, certain quantities which exist, and may be interpreted.

(i) The quantity \( \frac{a_{11} a_{12} a_{21} a_{22}}{a_{12} a_{21}} \). On the Two Factor theory this is constant for a quality Q, and expresses the degree to which the general factor contributes to the quality Q. On the Sampling Theory it is not constant, and the different values give estimates of the size of the sample.

(ii) The quantity 'g'. On the Two Factor theory this is the single general factor. On the Sampling Theory it is an estimate of the value of the factors as a whole.

(iii) The quantities 's'. On the Two Factor theory these are the specific factors. On the Sampling Theory they correspond to nothing, but are replaced by verbal
explanations of how the values of the various qualities diverge from the average value of the whole mind.

The tetrad difference criterion is by itself not sufficient to establish the Two Factor theory. That theory is in full agreement with it, but its position as an indispensable theory depends on the tetrad differences being exactly zero. The Sampling Theory is also in full agreement with the observational fact that tetrad differences are very small. So far as the tetrad difference criterion is concerned, the observed facts allow us to adopt whichever of the theories we think fit, and the choice between them must be made from other considerations.
PART III.

Part III consists of the following published papers in the order in which they are referred to in Part II.

1. The Sampling Theory as a Variant of the Two Factor Theory. Published in 'The Journal of Educational Psychology', December, 1928.


The Sampling Theory as a Variant of the Two Factor Theory.

from the

Journal of Educational Psychology, December, 1928.
THE SAMPLING THEORY AS A VARIANT OF THE TWO FACTOR THEORY

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The Sampling Theory of Ability as propounded and developed by Professor Godfrey Thomson has been objected to on three main grounds.

In the first place, the probability that mental factors would be arranged among a set of abilities so as to make the resultant correlations conform with the laws which seem actually to govern these correlations is, it is said, so slight as to render the theory unworthy of acceptance. Secondly, even granting that the theory is mathematically possible, it is argued that by means of certain transformations it can be shown to be a variant of the Theory of Two Factors, that, in fact, the mathematical expressions for the two theories can be made to agree. And, thirdly, it is held that the consequences of such a theory leads to conclusions which are psychologically indefensible and contrary to the facts of experience.

It is the second of these that appears the most serious, and it is with it we propose here to deal. Were it the case, indeed, that the two theories could be made to agree in their mathematical expressions, and that these expressions could be reasonably interpreted in terms of each other, then the theories would really differ not in their basic ideas but almost solely in their verbal statement. That it is an important question is recognised by Professor Spearman, who includes the Sampling Theory in his “Sub-theories of the Two Factors;”¹ in a later article he refers to Garnett’s having “plainly set forth the general factor in Thomson’s own scores;”² while in his recent book³ he says: “As Garnett proceeded to show, the $V$ (i.e., the measure of a given quality) could equally well be divided instead as follows:

$$ V = g + s_v, $$

... It appears, then, that each of Thomson’s $v$'s had really introduced a little bit out of the $g$ together with a little bit of the $s_v$,” etc. Clearly


614
the reconciliation of the Sampling Theory with the Theory of Two Factors is regarded, in our opinion rightly so, as of cardinal importance.

In considering this question it is important to realise that there is a difference in the methods of approaching the problem in these two theories, both of which methods are scientifically sound. We shall first examine the Two Factor Theory as set forth by Garnett.¹

The assumption is that any mental quality is due to the operation of certain variable elementary factors, which for generality we must assume to be at least as numerous as the qualities under consideration. It is then shown that, if we take \( n \) mutually perpendicular axes in \( n \)-dimensional space to represent the elements, the \( n \) qualities may (if we have chosen suitable units in our measurements) be represented by directed lines, any one of which we may picture as leaning near to those axes upon which it depends most and farther from those of which it is more independent. The correlation between two qualities is then shown to be measured by the cosine of the angle between the two representative lines.

Now, we may assume that qualities are due to the operation of variable factors, and yet set out from the standpoint that we know nothing else about these factors, but shall take whatever factors seem to express our facts most simply. Pursuing the geometrical method, what we have to begin with is a set of lines in \( n \)-dimensional space making certain known angles with each other; we have no axes of coordinates, but are going to fit on any set which seems most suitable. This is analogous to setting out to discuss a plane figure by means of coordinate geometry and drawing the axes \( XOY \) in whatever position we like.

Garnett proceeds to show that if the set of correlations satisfy the tetrad equation there is a certain constant less than unity associated with each line which may be taken as the cosine of an angle, and that there is a uniquely determined line in \((n + 1)\)-dimensional space which makes these angles with the existing lines. Taking this uniquely determined line as one axis \( Og \), we can draw \( n \) other axes, each perpendicular to it and to each other. Lastly, keeping \( Og \) fixed, the other axes are whirled round, so to speak, to a certain position, in which it is found that the plane \( gOx_1 \) contains one of the lines we began with, \( gOx_2 \) a second, \( gOx_3 \) a third, and so on. It is clear that whatever

¹ On Certain Independent Factors in Mental Measurements. *Proceedings of the Royal Society*, Vol. XCVI, A, pp. 91 et seq. We have taken the liberty of paraphrasing the attractive analysis contained in that paper.
entities be represented by $Og, Ox_1, Ox_2, \ldots, Ox_n$, each quality depends upon $g$ and one of the $x$'s and upon nothing else. Hence, if we start by assuming ignorance of the nature of the underlying elements, we may, if we please, define whatever is represented by $g$ and the $x$'s as elementary factors, of which $g$ is general and all the rest specific. The task of the psychologist as opposed to the mathematician would now be to investigate whether this theory is a likely one, and what psychological meanings can be given to the factors thus mathematically defined; and this is indeed the next step carried out by Professor Spearman in his book.\footnote{"The Abilities of Man." Chap. VII.} It is to be observed however, and this has always been recognised, that the entities represented by these ($g$ and $x$'s) may not actually exist; and on the other hand only fair to admit that, in this respect, the theory is in no worse case than many accepted theories in the domain of physical science.

Turning now to the Sampling Theory, we note that it begins with a hypothesis which, a priori at least, is admitted to be psychologically plausible. There are assumed to be a number of elementary factors, each of which acts with its full force or not at all, and the various mental qualities are assumed to be samples of those elements. Can we form a geometrical picture of these so as to compare it with the other theory? We can, by regarding a set of $N$ axes as representing the $N$ elements. If a certain quality $Q_s$ depends on $s$ of the elements, then the $s$ corresponding axes will define an $s$-dimensional space, in which we may imagine a line representing the quality $Q_s$. This line will be equally inclined to all the $s$ axes; for, since each of the $s$ elements acts with its full force, the quality $Q_s$ will depend on the $s$ elements equally. Thus we have a set of $n$ lines lying in a space of $N$ dimensions, the cosine of the angle between any pair being the correlation between the two qualities represented. Now these correlations may, under certain conditions, satisfy exactly the tetrad equation. It would seem correct therefore to think, and Garnett proves\footnote{The Single General Factor in Dissimilar Mental Measurements. \textit{British Journal of Psychology}, Vol. X, 1920, pp. 242 et seq. What follows in the present article refers to the transformation effected by Garnett in the place mentioned.} that we can, in this event, replace our $N$ axes by a different set of $n + 1$ axes where, as before, one of these along with each of the others in turn delimit a plane containing one of the representative lines. On this ground the Sampling Theory is held to be but a variant of the Two Factor Theory. We note in passing that the validity of the Sampling
Theory does not depend on its exact fulfilment of the tetrad equation criterion. But if we admit that a *tendency* to fulfil that criterion indicates a *tendency* for the qualities to be due to a general factor and specific factors the discussion thereupon involves the whole theory.

Now there is no question that if the tetrad equation is satisfied exactly, the qualities can be regarded as due to a single general factor and specific factors. From one point of view, this is a geometrical theorem in \( n + 1 \) dimensions. And, if any other theory were advanced upon which the tetrad equation were exactly satisfied, there is no doubt that it could be forced into apparent agreement with the two factor theory. But having started from a hypothesis about our original axes we are entitled, before abandoning them, to examine the relation between them and the new set proposed to replace them. Garnett shows that when the factors are so distributed among the qualities that the tetrad differences are zero,

\[
g = \frac{1}{\sqrt{N}} \cdot (\Sigma x)\]

He offers the following psychological interpretation of this equation in terms of the hypothesis underlying the Sampling Theory: “The whole number of a subject’s neurones rendered active by an effort of his will would then be proportional to his \( g \).”¹ He does not, however, seek to interpret the resultant “specific factors.” It is this interpretation we proceed to find, and, as we shall see, the interpretation is a very unsatisfactory one.

Geometrically the equation \( g = \frac{1}{\sqrt{N}} (\Sigma x) \) means that the axis \( Og \) is drawn equally inclined to all the original axes. This means that \( g \) is a measure of some activity calling for the action of every elementary factor possessed by the mind. The factor \( \frac{1}{\sqrt{N}} \) does not signify that each element enters with only \( \frac{1}{\sqrt{N}} \) of its force; it is merely a factor which ensures that the measure of \( g \) will have the same standard deviation as the other measures, merely a factor defining units. The general factor \( g \) is thus the *whole mind*, not only those elements capable of entering into two or more activities. How then can we have other factors independent of such a factor as this?

¹ *Loc. cit.*, p. 257.
In the figure \( OX_1, \ OX_2, \ldots, \ OX_N \) are the positive directions of the axes representing (in \( N \)-dimensional space) the elementary factors; \( Og \) a line equally inclined to all of them. \( OQ_s \) is a line equally inclined to those axes concerned in \( Q_s \); \( OR_s \) a line equally inclined to all the rest. Now since \( OQ_s \) and \( OR_s \) are independent, they are perpendicular to each other. Also, \( \cos gQ_s = \frac{\sqrt{s}}{\sqrt{N}} \) and \( \cos gR_s = \frac{\sqrt{N-8}}{\sqrt{N}} \), whence
\[
\sin gR = \sqrt{1 - \frac{N - s}{N}} = \frac{\sqrt{s}}{\sqrt{N}} = \cos gQ_s.
\]
Therefore \( gQ_s \) is the complement of \( gR_s \), and since \( Q_sR_s \) is a right angle the lines \( OQ_s, OR_s \) and \( Og \) are co-planar. In this plane \( O\xi_s \) is drawn perpendicular to \( Og \). \( Q_s \) may now be looked upon as depending on \( g \) and \( \xi_s \). The same may be done with all the qualities in turn, whether the
tetrad equation is satisfied or not; and Garnett's analysis shows that, if the qualities do satisfy that equation exactly, the $\xi$-axes are perpendicular to each other also. Following the diagram we are able to interpret the factor $\xi$. It is something which is helped by the factors making up $Q_1$ and hindered by the factors which do not; while $g$ is helped by both sets. It would be possible, say by increasing the values of both sets by properly chosen amounts, to increase $g$ and leave $\xi$, unaltered, or by increasing the value of one set and diminishing that of the other to leave $g$ unaltered while altering $\xi$, so that $g$ and $\xi$ are independent. We are led then to describe the dependence of $Q_1$ upon $g$ and $\xi$, in the following language: $Q_1$ is a quality which depends: (1) partly upon the action of the whole set of mental factors possessed by the brain; and (2) partly upon the action of some composite power which in turn is aided by a certain set of those factors and hindered by all the others; the degrees to which these two contributions take place being such that the action of the certain set mentioned is made complete, while that of the others is completely nullified. This sounds somewhat absurd, for we are only saying again that $Q_1$ depends on the action of a certain set of factors and not on the others at all. This interpretation of the "specific" factors $\xi$, etc., differs from that usually implied. If we ask "What, in the Theory of Two Factors, is that which enables a boy to do Latin?" it is reasonable to be told: "Partly a general factor and partly a factor specifically devoted to Latin." If we start with the Sampling Theory, and allow the transformation of Garnett's (which is mathematically possible under the limiting conditions we are considering) to be made, and are then asked the same question, we must answer: "Partly his whole brain, and partly that part of his brain which is active in Latin hindered by the rest of his brain." The "specific" factor is thus partly due to those elements active in Latin hindered by all the rest.

The only way out of this absurdity is either to abandon the Sampling Theory altogether, *i.e.*, to scrap our first set of axes in favour of the new ones, or to say that, while the transformation is of course possible (for what a chance we have of drawing axes to our pleasure in $n$ dimensions!), the interpretation of the new variables in terms of the original ones is such that they cannot be accepted as representing any entities of which we can satisfactorily conceive. As we have stated above, it is scientifically sound to assume a hypothesis regarding the entities causing any set of phenomena; and if a mathematical transformation enables us to postulate a new set of entities which
are not satisfactorily expressible in terms of our hypothesis we are quite justified if we decline to make that transformation and adhere to our original hypothesis. In the familiar equation in physics, \( pv = RT \), it would be easy to replace the axes of \( p \) and \( v \) by two others in the same plane, and obtain a new equation \( T = \phi(q, w) \), where \( q \) would be something dependent on \( p \) and \( v \), and \( w \) dependent on \( p \) and hindered by \( v \). As such a transformation would be meaningless, we should not make it.

We may venture to make our argument clearer by an analogy. Suppose that some archaeologists find the ruins of three temples, one of which had been made of brick and stone, another of brick and wood, and the third of stone and wood. Some of the archaeologists might say that in the building of each temple we see the operation of two factors, one of which is common to all; this might be the religious urge of the nation which built them. The factors peculiar to each of the temples might be respectively: (1) Absence of trees in the locality, (2) want of stone, and (3) abundance of stone and wood. Others of the archaeologists say that the nation in question possessed skilled builders and that the three temples are samples of their work, the first being due to brickmakers and masons, the second to brickmakers and carpenters, and the third to masons and carpenters. The first school of archaeologists then claim that the second theory is but a variant of theirs, for, say they, we can make a mathematical transformation by means of which the brickmakers, masons and carpenters are replaced by four influences, one affecting all three temples and the other three one each. If it is inquired what those influences are, the reply must be: The general influence is the work of the whole set of builders; the specific influence in the first temple is the work of the brickmakers and masons hindered by the carpenters, and similarly for the others.\(^1\) A sort of mathematical equation would be:

\[
\text{Brick-and-stone Temple} = (\text{Work of brickmakers, masons and carpenters working with part of their might}) + (\text{Work of brickmakers and masons working with the rest of their might} - \text{Work of carpenters already done}).
\]

\(^1\) By taking three axes to represent the work of the three sets of builders and supposing their respective shares to be equivalent, we could actually carry out this transformation. The “correlation” between any two of the temples would be one-half; the new \( g \) axis would be equally inclined to the original axes. With only three things under consideration, as Garnett points out, we can always get a \( g \) axis, without any conditions.
It would then be open to the second set of archaeologists to point out that this was just what they said at first, except that the carpenters had been brought in to build and then knock down what they had built; and that they might justifiably prefer to think that the carpenters were never there at all. As long as they chose to keep to their own hypothesis about the groups of builders it would be true to say that their theory could not be regarded as variant of the other one.

We conclude then that it is only in the most formal mathematical sense that the Sampling Theory can be brought in under the Two Factor Theory; that if we adhere to the hypothesis underlying the Sampling Theory the interpretation we are compelled to put upon the specific factors obtained by the mathematical transformation is such as to show that these factors are mere mathematical fictions. Per contra, if, having arrived at the Two Factor Theory, we make the transformation from it, then the elements of the Sampling Theory expressed by means of the transformation are mere mathematical fictions. If either theory should be abandoned, it is not because they are equivalent.
THE PROBABLE VALUE OF THE TETRAD DIFFERENCE ON THE SAMPLING THEORY

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I. Introduction (p. 65).
II. Calculation of $\sigma^2_r$ (pp. 65-72).
III. Mean Value of $F_r$ and the Probable Value of a Correlation (pp. 72-73).
IV. Other Forms of the Formula for $\sigma^2_r$, and Approximations (pp. 73-76).
V. Summary (p. 76).

I. INTRODUCTION.

In a recent contribution to the British Journal of Psychology, Professor Godfrey Thomson has given the results in two numerical examples of the possible distribution of the values of the tetrad difference of four correlated variables on the Sampling Theory. In the present article this problem is generalized. We shall state the problem in its general form as follows.

Let the number of elementary factors, 'all or none' in nature, be $N$, and let four mental activities, $Q_1, Q_2, Q_3$ and $Q_4$, depend upon the fractions $p_1, p_2, p_3$ and $p_4$ respectively of the $N$ factors. In the absence of knowledge as to which factors contribute to which activities, we assume that, a priori, all distributions are equally likely. Any one distribution will give us values for the six correlations between the four qualities, and a value for the tetrad difference

$$F_1 = r_{13}r_{24} - r_{14}r_{23}.$$ 

To find the probable value† of $F_1$ we must find the standard deviation of $F_1$ from zero. We propose, then, to sum the values of $F_1$ for all possible distributions, and divide by their number.

II. CALCULATION OF $\sigma^2_r$.

Let the long oval represent the universe of $N$ factors, and the oval within it the $p_1N$ factors concerned in the first quality $Q_1$. The factors concerned in $Q_2$ will consist of $a_1$ factors not concerned in $Q_1$ and $a_2$ factors in common with $Q_1$, $a_1$ and $a_2$ being any numbers so long as $a_1 + a_2 = p_2N$.

* This Journal, xviii, 68, 1927.
† To avoid circumlocution, we use the term 'probable value' to signify a statement of the limits within which it is an even chance that $F$ (or whatever quantity we are dealing with) will lie.
The \( p_3N \) factors of \( Q_3 \) may be distributed among four sorts of factors, those already concerned in both \( Q_1 \) and \( Q_2 \), those in \( Q_1 \) alone, those in \( Q_2 \) alone, and those in neither. This is indicated in Fig. 2*, where again \( b, c, d_1, d_2 \) may be any numbers, provided their sum is \( p_3N \).

\[ \begin{align*}
&\quad - - a_2 - - N - - a_1 - - \quad \text{Fig. 1.}
&\quad - - d_2 - - d_1 - - c - - b - - \quad \text{Fig. 2.}
&\quad - - m_2 - - m_1 - - l - - k - - h - - g - - f - - e - - \quad \text{Fig. 3.}
\end{align*} \]

And similarly, Fig. 3 shows a distribution of the factors of \( Q_4 \), the letters \( e, f, g, h, k, l, m_1, m_2 \) being capable of any values, so long as \( e + f + g + h + k + l + m_1 + m_2 = p_4N \).

* In Figs. 2 and 3, the outline of the oval containing the \( N \) factors is deleted, and indicated by the dotted line at the end. Figure 3 is a form of the 'Structure diagram' used by Professor Thomson in his article referred to and elsewhere. Comparing it with Fig. 1 on p. 69 of vol. xviii, we see the same 16 possible sorts of factors with 4 qualities. The form of Fig. 3 was devised in order to follow the subsequent mathematical reasoning.
Now \[ r_{12} = \frac{\text{No. of factors common to } Q_1 \text{ and } Q_3}{\sqrt{\text{No. in } Q_1 \times \text{No. in } Q_3}}, \]
\[ = \frac{d_1 + d_2}{\sqrt{p_1N \cdot p_3N}}. \]

Similarly \[ r_{31} = \frac{g + h + m_1 + m_2}{\sqrt{p_2N \cdot p_4N}}, \]
\[ r_{14} = \frac{k + l + m_1 + m_2}{\sqrt{p_1N \cdot p_4N}}, \]
\[ r_{23} = \frac{c + d_2}{\sqrt{p_2N \cdot p_3N}}. \]

Hence
\[ F_1 = \frac{(d_1 + d_2) (g + h + m_1 + m_2) - (c + d_2) (k + l + m_1 + m_2)}{\sqrt{p_1p_2p_3p_4N^4}} \ldots (1), \]
and
\[ F_1^a = \frac{\{(d_1 + d_2) (g + h + m_1 + m_2) - (c + d_2) (k + l + m_1 + m_2)\}^2}{p_1p_2p_3p_4N^4} \ldots (2). \]

We have to find the sum of the expression on the right for all possible values of the letters involved in the numerator.

It will be convenient to let \( M \) stand for \( m_1 + m_2 \), \( L \) for \( l + m_1 + m_2 \), \( K \) for \( k + l + m_1 + m_2 \), etc. We then have
\[ p_1p_2p_3p_4N^4 \times F_1^a = \{(d_1 + d_2) (G - k - l) - (c + d_2) K\}^2 = \phi^a, \text{ say } \ldots (3). \]

In the subsequent calculation we make use of the following Lemmas:

(i) \[ \sum_{r=0}^{s} \binom{n}{r} \times \binom{m}{s-r} = \binom{m+n}{s}, \]
subject to the restrictions that \( r \geq n \), and \( s - r \geq m \). These are merely two ways of counting the same number.

(ii) \[ \sum_{r=0}^{s} r \binom{n}{r} \times \binom{m}{s-r} = n \cdot \binom{m+n-1}{s-1} \]
\[ = \frac{n \cdot s}{m+n} \cdot \binom{m+n}{s}. \]

(iii) \[ \sum_{r=0}^{s} r^2 \binom{n}{r} \times \binom{m}{s-r} = n \cdot \binom{m+n-1}{s-1} + n (n-1) \cdot \binom{m+n-2}{s-2} \]
\[ = \frac{ns}{m+n} + \frac{n (n-1) \cdot s (s-1)}{(m+n) (m+n-1)} \binom{m+n}{s}. \]

Returning to equation (3), we have
\[ \phi^a = \{(d_1 + d_2) (G - k - l) - (c + d_2) K\}^2 \]
\[ = (d_1 + d_2)^2 (G^2 + k^2 + l^2 - 2Gl - 2Gk + 2kl) \]
\[ + (c + d_2)^2 K^2 - 2 (d_1 + d_2) (c + d_2) (KG - Kk - Kl) \ldots (4). \]
Suppose now that \( a_1, a_2, b, c, \ldots \) up to \( l \) are kept constant, while \( m_1 \) and \( m_2 \) are given all their possible values; i.e. \( m_1 \) has the values \( 0, 1, 2, \ldots, M \), and \( m_2 = M - m_1 \). The number of ways in which any one pair of values for \( m_1 \) and \( m_2 \) occurs is, as may be seen by consulting Fig. 3, \( \phi^2_{a_2} C_{m_1} \times \phi^2_{a_2} C_{m_2} \), and hence the summation of \( \phi^2 \) is

\[
\sum_{m_1=0}^{M} \phi^2_{a_2} C_{m_1} \times \phi^2_{a_2} C_{M-m_1} = \phi^2_{a_2} C_M \quad \ldots \ldots (5),
\]

by Lemma (i) since \( \phi^2 \) does not contain \( m_1 \).

Next, while keeping the values of \( a_1, a_2, b, c, \ldots \) up to \( k \), constant, allow \( l \) to vary from 0 to \( L \) (i.e. \( p_{dN} - e - f - g - h - k \)). The number of times any one value of \( l \) may happen is \( a_2 C_l \), and the sum of \( \phi^2 \) for all possible values of \( l, m_1, m_2 \), corresponding to any given set of values for \( a_1, a_2, \ldots, h, k \), is therefore

\[
\sum_{l=0}^{L} \phi^2_{a_2} C_{m_1} \times a_1 C_l = \sum_{l=0}^{L} \phi^2_{a_2} C_{L-l} \times a_1 C_l \quad \ldots \ldots (6).
\]

This summation can be effected by the help of the three lemmas given above, as \( \phi^2 \) is of the form

\[
P + Ql + Rl^2,
\]

where \( P, Q \) and \( R \) are, for the time being, constants.

There results from (6) an expression of the form

\[
(S + Tk + Uk^2)_{a_2+di} C_k = (S + Tk + Uk^2)_{a_2+di} C_{K-k} \quad \ldots (7),
\]

where \( S, T, U \), are functions of the letters \( a_1, a_2, \ldots \) up to \( h \).

The process is now repeated for \( k \); viz., (7) is multiplied by \( p_{dN} - a_2 - di \), \( C_k \) and summed from \( k = 0 \) to \( k = K \); and so on, the letters disappearing one by one, until we have dealt with all the parts \( e, f, \ldots \), \( m_2 \), going to make up \( p_{dN} \).

In practice, it is best to carry the work right through for each term of (4) separately. As an example, we give the working for the term in \( K^2 \). For a given set of values of \( a_1, a_2, \ldots \) up to \( h \), the sum of the values of \( K^2 \) is \( K^2 \times p_{dN} C_K \).

Allowing \( h \) to vary, we get for the sum at the next stage

\[
\sum_{h=0}^{H} K^2 \times p_{dN} C_K \times C_h;
\]
i.e., remembering that $K = H - h$

\[
\sum_{h=0}^{h=H} (H^2 - 2Hh + h^2) \times _{p1N} C_{H-h} \times _e C_h.
\]

This, by the three lemmas, is equal to

\[
\left\{ H^2 - 2H \left[ \frac{Hc}{p1N + c} + \frac{Hc}{p1N + c} + \frac{c (c-1) H (H-1)}{(p1N + c) (p1N + c - 1)} \right] \right\} \times _{x1N+c} C_H,
\]

which reduces to

\[
\left\{ \frac{p1N (p1N - 1)}{(p1N + c) (p1N + c-1)} H^2 + \frac{c p1N}{(p1N + c) (p1N + c-1)} H \right\} \times _{x1N+c} C_H
\]

Allowing now $g$ to vary, and taking the term in $H^2$ in (8) by itself, we get as the next sum for $H^2$

\[
\sum_{g=0}^{G2} H^2 \times _{p1N+a1} C_G \times _{a1-c} C_g = \sum_{g=0}^{G2} (G^2 - 2Gg + g^2) \times _{p1N+a1} C_{G-g} \times _{a1-c} C_g,
\]

which is equal to

\[
\left\{ \frac{G2 - 2G}{p1N + a1} \frac{G (a1 - c)}{p1N + a1} + \frac{G (a1 - c)}{(p1N + a1) (p1N + a1 - 1)} \right\} \times _{p1N+a1} C_G,
\]

which reduces to

\[
\left\{ \frac{(p1N + c) (p1N + c - 1)}{(p1N + a1) (p1N + a1 - 1)} G^2 + \frac{(a1 - c) (p1N + c)}{(p1N + a1) (p1N + a1 - 1)} G \right\} \times _{p1N+a1} C_G.
\]

For $G^2$ at the next stage we get

\[
\sum_{f=0}^{F} G^2 \times _{p1N+a1+b} C_G \times _b C_f
\]

\[
= \left\{ \frac{(p1N + a1) (p1N + a1 - 1)}{(p1N + a1 + b) (p1N + a1 + b - 1)} F^2 + \frac{b (p1N + a1)}{(p1N + a1 + b) (p1N + a1 + b - 1)} F \right\}
\]

\[
\times _{p1N+a1+b} C_F
\]

And $F^2$ in turn gives

\[
\sum_{e=0}^{p1N} F^2 \times _{p1N+a1+b} C_F \times _{N-p1N-a1-b} C_e
\]

\[
= \left\{ \frac{(p1N + a1 + b) (p1N + a1 + b - 1)}{N (N-1)} (p1N)^2
\]

\[
+ \frac{(N - p1N - a1 - b) (p1N + a1 + b)}{N (N-1)} p1N \right\} \times _{N} C_{p1N}
\]

\[
....(11).
\]
Probable Value of Tetrad Difference on Sampling Theory

The term in \( F \) in (10) gives
\[
\sum_{e=0}^{e=p_4N} (p_4N - e) \times p_4N + a_1 + b \times C_{p_4N - e} \times N - p_4N - a_1 - b \times C_e
\]
\[
= \left\{ p_4N - N - p_4N - a_1 - b \times p_4N \right\} \times N \times C_{p_4N}
\]
\[
= p_4 \left( p_4N + a_1 + b \right) \times N \times C_{p_4N}
\]......(12).

The term in \( G \) in (9) gives at the next stage
\[
\sum_{F=-F}^{F} (F - f) \times p_4N + a_1 \times F - f \times C_F = \left\{ F - \frac{bF}{p_1N + a_1 + b} \right\} \times p_4N + a_1 + b \times C_F
\]
\[
= p_4 \left( p_1N + a_1 + b \right) \times F \times p_4N + a_1 + b \times C_F,
\]
which at the next stage, using the result in (12), becomes
\[
p_4 \left( p_1N + a_1 \right) \times N \times C_{p_4N}
\]......(13).

Similarly the term in \( H \) in (8) finally gives
\[
p_4 \left( p_1N + c \right) \times N \times C_{p_4N}
\]......(14).

Using (11) and (12) in the expression (10), then using (10) and (13)
in the expression (9), and then (9) and (14) in the expression (8), we get
for the sum of \( K^2 \) for all sets of values of \( e, f, \ldots, m_2, \)

Sum of \( K^2 = \frac{p_1p_4N}{N-1} \left\{ (p_1N - 1) p_4N + N - p_1N \right\} \)......(15).

In this fashion, expressions are obtained corresponding to each term
on the right of (4), on adding which together and simplifying we obtain

Sum of values of \( \phi^2 \)
\[
able \left[ (d_1 + d_2)^2 \times \frac{p_4p_1N}{N-1} \right] \times N \times C_{p_4N}
\]
\[
= 2 \left( d_1 + d_2 \right) (c + d_2) \times \frac{p_4}{N-1} \left\{ p_4N \left( p_4p_2N^2 - a_2 \right) + a_2N - p_1p_2N^2 \right\}
\]
\[
+ (c + d_2)^2 \times \frac{p_4p_1N}{N-1} \left\{ p_4N \left( p_1N - 1 \right) + N - p_1N \right\} \times N \times C_{p_4N}
\]......(16).

If we fix, then, values for \( a_1, a_2, b, c, d_1 \) and \( d_2 \), and let the \( p_4N \) factors
of the fourth quality be arranged in every possible way, the sum of the values of \( \phi^2 \) which result is given by the expression (16).

We now suppose the \( p_4N \) factors of the third quality to be selected
in every possible way, and by carrying out summations for \( d_2 \) and \( d_1, \)
\( c \) and \( b \) we get (omitting all the algebra) this result:
Sum of values of \( \phi^2 \) for given values of \( a_1 \) and \( a_2 \) and all possible values of the other letters

\[
\sum \text{values of } 412 \text{ for given values of } a_1 \text{ and } a_2 \text{ and all possible values of the other letters} = \left[ \frac{p_2 p_4 N}{N - 1} \{ p_4 N (p_2 N - 1) + N - p_2 N \} \times \frac{p_1 p_3 N}{N - 1} \{ p_3 N (p_1 N - 1) + N - p_1 N \} \right. \\
- \frac{2p_4}{N - 1} \{ p_4 N (p_1 p_4 N^2 - a_2) + a_2 N - p_1 p_4 N^2 \} \\
\times \frac{p_3}{N - 1} \{ p_3 N (p_1 p_3 N^2 - a_2) + a_2 N - p_1 p_3 N^2 \} \\
+ \frac{p_1 p_4 N}{N - 1} \{ p_4 N (p_1 N - 1) + N - p_1 N \} \\
\times \frac{p_2 p_3 N}{N - 1} \{ p_3 N (p_2 N - 1) + N - p_2 N \} \\
\times N C_{p_1 N} \times N C_{p_2 N},
\]

which becomes on collecting and arranging

\[
\left[ \frac{p_1 p_2 p_3 p_4 N^4}{(N - 1)^2} \{ 2 p_1 p_2 p_3 p_4 N^2 - 2 \Sigma p_1 p_2 p_3 p_4 N + (p_1 p_3 + p_1 p_4 + p_2 p_3 + p_3 p_4) N \right. \\
+ 2 (p_1 p_2 + p_3 p_4) + (p_1 p_3 + p_1 p_4 + p_2 p_3 + p_3 p_4) + 2 \Sigma p + 2 \} \\
- \frac{2 p_3 p_4}{(N - 1)^2} \{ p_1^2 p_2^2 N^4 (p_4 N - 1) (p_3 N - 1) \\
+ a_2 p_1 p_3 N^2 [(p_3 N - 1) (N - p_3 N) + (p_3 N - 1) (N - p_4 N)] \\
+ a_2^2 (N - p_4 N) (N - p_3 N) \} \\
\times N C_{p_1 N} \times N C_{p_2 N} \times N C_{p_3 N}
\]

\[ \cdots (17). \]

Lastly, we let \( a_1 \) and \( a_2 \) take all their possible values. The second term of (17) within the crooked brackets is the only one that varies, and the second part results after some simplification in the expression

\[
- \frac{2 p_1 p_2 p_3 p_4 N^4}{(N - 1)^2} \{ p_1 p_2 p_3 p_4 (N - 1)^2 + \Pi (1 - p) \} \times N C_{p_1 N} \times N C_{p_2 N} \times N C_{p_3 N}
\]

where \( \Pi (1 - p) \) stands for \( (1 - p_1)(1 - p_2)(1 - p_3)(1 - p_4) \).

Using this in (17) we get for the sum of \( \phi^2 \) for all possible values of all the letters \( a_1, a_2, b, \ldots, m \)

\[
\frac{p_1 p_2 p_3 p_4 N^4}{(N - 1)^2} \left[ 4 p_1 p_2 p_3 p_4 N - 2 p_1 p_2 p_3 p_4 - 2 \Sigma p_1 p_2 p_3 p_4 N \\
+ (p_1 p_3 + p_1 p_4 + p_2 p_3 + p_3 p_4) N + (p_1 p_2 + p_3 p_4) \\
+ \Sigma p_4 p_3 - 2 \Sigma p + 2 - \frac{2 \Pi (1 - p)}{N - 1} \right] \times N C_{p_1 N} \times N C_{p_2 N} \times N C_{p_3 N}
\]

\[ \cdots (18). \]
Since the \( p_1N \) factors concerned in the first quality may be selected in \( N \times C_{p_1N} \) ways, we get the final sum of the values of \( \phi^2 \) by multiplying (18) by \( N \times C_{p_1N} \).

Now \( F_1^2 = \phi^2 / p_1p_2p_3p_4N^4 \); also the total number of arrangements is \( N \times C_{p_1N} \times N \times C_{p_2N} \times N \times C_{p_4N} \). Hence we obtain finally from (18), after a little manipulation,

\[
\sigma_{F_1}^2 = \frac{1}{N - 1} \left\{ \left( p_1p_3 + p_2p_4 + p_1p_4 + p_2p_3 \right) - 2 \sum p_1p_2p_3 + 4p_1p_2p_3p_4 \right. \\
+ \left. \frac{2(N - 2)}{(N - 1)^2} (1 - p_1)(1 - p_2)(1 - p_3)(1 - p_4) \right\} \quad \ldots \ldots \text{(19)}.
\]

It will be noticed that the suffixes in \( F_1 = r_{12}r_{23} - r_{13}r_{22} \) are those which occur at the beginning of (19), all the rest of the expression being symmetrical in the \( p \)'s. It will thus be clear how to write down \( \sigma_{\phi}^2 \) for the other two tetrad differences.

In Professor Godfrey Thomson’s paper already referred to \( p_1 = \frac{8}{10}, \quad p_2 = \frac{3}{10}, \quad p_3 = \frac{4}{10}, \quad p_4 = \frac{3}{10}, \) while \( N = 6 \) in his first example and \( N = 12 \) in the second. The values of \( \sigma^2 \) for \( F_1, F_2 \) and \( F_3 \) given in that paper agree with those obtained by substituting these numbers in equation (19).

It will be well at this point to restate clearly what this formula signifies. We suppose that mental activities are due to a number \( N \) of ‘all or none’ factors, and that four of them are due to \( p_1N, p_2N, p_3N, p_4N \) of the factors respectively. Further we suppose that we obtain the intercorrelations of these four activities free from errors of sampling of the population. Under these conditions the \textit{a priori} probable value of \( F_1 = r_{12}r_{23} - r_{13}r_{22} \) is equal to 0 ± 6745 \( \sigma_{F_1} \), where \( \sigma_{F_1} \) is the quantity given by (19).

III. MEAN VALUE OF \( F \), AND THE PROBABLE VALUE OF A CORRELATION.

By similar methods to those used in the previous section, and with considerably less work, we obtain the following results, which we shall be content with stating.

(i) The mean value of \( F \) is zero* \( \ldots \ldots \text{(20)} \).

(ii) The mean value of \( r_{13} \) is \( \sqrt{p_1p_3} \) \( \ldots \ldots \text{(21)} \), and similar values for \( r_{12}, r_{14}, \) etc.

(iii) The mean square deviation of \( r_{13} \) about zero is

\[
\frac{1}{N - 1} \left\{ p_1p_3N - p_1 - p_3 + 1 \right\},
\]

* Prof. Thomson has already shown (this \textit{Journal}, 1927, xvii, 253) that these are the most probable values of \( F \) and \( r \).
and about its mean value $\sqrt{p_1p_3}$ we have therefore
\[
\sigma_{r_{13}, r_{24}}^2 = \frac{1}{N-1} \left( p_1p_3(N - p_1 - p_3 + 1) - \sqrt{p_1p_3} \right)^2
= \frac{1}{N-1} \left( (1 - p_1 - p_3 + p_1p_3) \right)
= \frac{1}{N-1} \left( (1 - p_1) (1 - p_3) \right)
\]

with similar expressions for $\sigma_{r_{13}, r_{34}}^2$, etc.

The probable value for a correlation, $r_{13}$, for example, is therefore
\[
\sqrt{p_1p_3} \pm 6.745 \sqrt{\frac{(1 - p_1)(1 - p_3)}{N-1}} \quad \text{.....(23)}
\]

IV. Other Forms of the Formula for $\sigma_p^2$, and Approximations.

It is to be observed that all the above formulae are exact, with no restrictions on any of the numbers involved.

If now we let $r_{13}$ stand for the mean value, viz. $\sqrt{p_1p_3}$, and so with the others, we have
\[
p_1p_2p_3 = \sqrt{p_1p_2} \cdot \sqrt{p_1p_3} \cdot \sqrt{p_2p_3}
\]
and similarly
\[
p_1p_2p_4 = r_{13}r_{14}r_{23};
p_1p_3p_4 = r_{13}r_{34}r_{14};
p_1p_4p_3 = r_{13}r_{14}r_{34}.
\]

Likewise $p_1p_2p_3p_4 = r_{13}r_{14}r_{23}r_{14}.

Also, by (22) we have
\[
\frac{2\Pi (1 - p)}{N - 1} = \frac{(1 - p_1)(1 - p_3)(1 - p_3) (1 - p_4)}{N - 1}
+ \frac{(1 - p_1)(1 - p_4)(1 - p_3)(1 - p_3)}{N - 1}
= \sigma_{r_{13}}^2 \sigma_{r_{24}}^2 + \sigma_{r_{14}}^2 \sigma_{r_{34}}^2.
\]

Substituting these values in (19) we get (dropping the suffix in $F_1$, since we shall now deal only with $F_1$)
\[
\sigma_p^2 = \frac{1}{N-1} \left[ r_{13}^2 + r_{24}^2 + r_{14}^2 + r_{34}^2 - 2 \left( r_{12}^2r_{13}r_{23} + r_{12}^2r_{14}r_{24} 
+ r_{23}^2r_{24}r_{34} + r_{23}^2r_{24}r_{34} + 4r_{13}^2r_{14}r_{24}r_{34} \right)
+ N - 2 \left( \sigma_{r_{13}}^2 \sigma_{r_{24}}^2 + \sigma_{r_{14}}^2 \sigma_{r_{34}}^2 \right) \right] \quad \text{.....(24)}
\]
74 Probable Value of Tetrad Difference on Sampling Theory

Unless $N$ is very small this becomes, without serious error,

$$\sigma_p^2 = \frac{1}{N} \left[ \tau_{13}^2 + \tau_{24}^2 + \tau_{14}^2 + \tau_{23}^2 - 2 \left( \tau_{12}^2 \tau_{13} \tau_{23} + \tau_{12}^2 \tau_{14} \tau_{24} + \tau_{13}^2 \tau_{14} \tau_{23} + \tau_{23}^2 \tau_{24} \right) + 4 \tau_{12}^2 \tau_{13}^2 \tau_{24} \tau_{34} \right]$$

(25).

This expresses the standard deviation of $F$ in terms of the mean values of the correlation coefficients and of their standard deviations.

A rougher approximation to (19) may be got by replacing each $p$ by their mean $\bar{p}$, say. From (21) we see that the mean of the $r$'s will then be $r = \bar{p}$. We get then

$$\sigma_p^2 = \frac{4 \bar{p}^2 (1 - \bar{p})^2}{N - 1} + \frac{2 (N - 2) (1 - \bar{p})^4}{(N - 1)^3}$$

(26),

and approximately, unless $N$ is small,

$$\sigma_p^2 = \frac{4 \bar{p}^2 (1 - \bar{p})^2}{N + 2 (1 - \bar{p})^4/N^2}$$

(27),

or, if we please,

$$\sigma_p^2 = \frac{4 \bar{r}^2 (1 - \bar{r})^2}{N + 2 (1 - \bar{r})^4/N^2}$$

(28).

If the $r$'s are not also small, the last term may be neglected, and then

$$\sigma_p^2 = \frac{4 \bar{r}^2 (1 - \bar{r})^2}{N} = \frac{4 \bar{r}^2 (1 - \bar{r})^2/N}{N}$$

(29).

There is a remarkable similarity between these results and the equations given by Spearman and Holzinger in their article on "The Sampling Error in the Theory of Two Factors." If in their equation (9) we replace the last term by its value in their equation (8), it becomes

$$\sigma_p^2 = \frac{1}{N} \left[ \tau_{13}^2 + \tau_{14}^2 + \tau_{23}^2 + \tau_{24}^2 - 2 \left( \tau_{12}^2 \tau_{13} \tau_{23} + \tau_{12}^2 \tau_{14} \tau_{24} + \tau_{13}^2 \tau_{14} \tau_{23} + \tau_{23}^2 \tau_{24} \right) + 4 \tau_{12}^2 \tau_{13}^2 \tau_{24} \tau_{34} \right]$$

which is identical with equation (25) above.

In the same article the authors proceed to give as approximations

$$\sigma_p^2 = \frac{4 \bar{r}^2 (1 - \bar{r})^2/N + 2 (1 - \bar{r})^4/N^2}{N},$$

and

$$\sigma_p = 2 \bar{r} (1 - \bar{r})/\sqrt{N},$$

which, if in the first we replace $(1 - \bar{r}^2)/\sqrt{N}$ by $\sigma_r$, are identical with (28) and (29).

The similarity of these formulae is the more remarkable when we

* The reader will observe that from (21) and (22), if the $p$'s and $r$'s are replaced by their means, we have approximately $\sigma_r = (1 - r)/\sqrt{N}$ under the conditions we are considering. When $\sigma_r$ refers to errors caused by sampling the population, $\sigma_r = (1 - r^2)/\sqrt{N}$. The distinction between the two must be borne in mind when we come to compare our results with certain equations of Spearman and Holzinger.

† This Journal, 1924, xv, 18, 19.
consider that the letters have different meanings. In Spearman and Holzinger's formula \( N \) is the number of cases in a sample, the \( r^\prime \)s are the true values, and the \( \sigma \)s are standard errors. In our formula \( N \) is the number of mental factors from which those involved in the four activities are drawn, the \( r^\prime \)s are the true values, while the \( \sigma \)s are the standard deviations from the mean values to be expected owing to our ignorance of the actual distributions of the four sets of factors among the total \( N \).

That there must be some underlying cause of the similarity of the two formulae is apparent. It may be that here we have proof that, as Thomson avers\(^*\), the effect of sampling the population and the effect of sampling mental elements are, mathematically at all events, equivalent. Or there may be some more general arithmetical theorem which includes both.

Returning to equation (19), and regarding \( \sigma_p^2 \) as a function of the \( p \)'s for a given value of \( N \), it is evident that the turning values occur when the \( p \)'s are all equal, say to \( p \); and by differentiating we find that a minimum value occurs when \( p = 1/N \), and the maximum when \( p = (N - 2)/(2N - 3) \). Since on our hypotheses \( pN \) must be integral, the value of \( p \) which gives the maximum is \( \frac{1}{2} \), in which case

\[
\sigma_p^2 = \frac{1}{N - 1} \left( \frac{1}{4} + \frac{N - 2}{8(N - 1)^2} \right)
\]

or approximately \( \frac{1}{4N} \).

The effect of disregarding the last term in (19) is shown by the graph in Fig. 4, in which the upper curve shows the values of \( \sigma_p^2 \) calculated

\[\text{Fig. 4.}\]

from the full formula, and the lower one the values calculated from
\[ \sigma_F^2 = 4p^2(1-p)^2/N. \] The value of \( N \) is 12, and if \( N \) were greater the
two curves would lie still closer together, save for small values of \( p \).

V. SUMMARY.

(i) On the Sampling Theory we have obtained a formula for the
probable value of the tetrad difference \( F \).
(ii) \( F \) is a function whose mean value is zero, and is distributed about
zero with the standard deviation whose value we have calculated.
(iii) This deviation is, if \( N \) is large, inversely proportional to the
square root of \( N \); and if we postulate the number of mental factors to be
at least, say, 100 the probable value of \( F \) is very small; actually, in the
most unfavourable case when the \( p \)'s each \( = \frac{1}{3} \), and \( N = 100 \), \( F = 0 \pm 0.034 \).
(iv) The formulae obtained are identical in form with certain formu-
lae of Spearman and Holzinger, although differing in meaning.

Note. The results in equations (19), (20) and (29) were published in a letter to
Nature of 27th August, 1927.
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Mathematical Consequences of Certain Theories of Mental Ability.

By John Mackie.
II.—Mathematical Consequences of Certain Theories of Mental Ability. By John Mackie, Department of Education, Edinburgh University. Communicated by Professor Godfrey Thomson.

(MS. received July 26, 1928. Read November 5, 1928.)

CONTENTS.

Introduction ........................................................ 16
I. Statement of Problem ............................................ 17
II. Distribution of the Lines representing the Mental Qualities .......... 18
III. Theorem regarding $F_{12,34}$ .................................. 20
IV. Calculation of $\sigma^2$ ............................................ 22
V. Mean Value and Standard Deviation of a Correlation .................. 27
VI. Application to Theories of Mental Structure .......................... 30
VII. (1) Comparison of Formulae for $\sigma$ ............................. 32
       (2) Mean Value of $r$ when the Qualities are assumed to be due to certain
           Fractions of the Factors. Suggested maximum probable value of $r$ for
           two diverse qualities ................................................. 32
       (3) Comparison of Formulae for $\sigma_r$ ............................ 34
VIII. (1) Distribution of the Coefficients $\lambda$ assumed in this paper .... 35
       (2) Distribution of $F$ when $N=2$ ................................... 38
       (3) Variation of $\sigma_r$ with $N$ ...................................... 37
       (4) Variation of Mean Value of $r$ with $N$ .......................... 37

INTRODUCTION.

Researches of recent years have shown that the intercorrelations which exist among the measures of mental qualities are connected in a particular way, from which it has been sought to deduce facts concerning the nature of the mental structure on which the various qualities depend. If we take any four such qualities, there are six correlation coefficients, $r_{12}, r_{13}, r_{14}$, etc., and it is found that the quantity $F=r_{12}r_{24}-r_{14}r_{23}$ is in practice approximately zero. Since the observed correlations are vitiated by errors due to measuring a sample of the population instead of the whole it is legitimate to think that the value of $F$ throughout a set of mental qualities is truly zero. Assuming this to be so, it has been shown that such a result would follow if we suppose each mental quality to be due to the operation of a general factor which takes part in every quality and of another factor peculiar to the quality itself; * and further, and conversely, that if the value of $F$ is everywhere zero, then the measure of

* Spearman, The Abilities of Man, Appendix, pp. ii, iii.
every quality can be divided into two components, one of which is general to all and the other specific to that quality.*

It is necessary, however, to inquire what values of F we might expect on other theories, for it is conceivable that they might on other suppositions be found to be so small as to be quite possibly the true values of which our observed values are the imperfect measures. Professor Godfrey Thomson,† reasoning from a theorem of Pearson and Filon, concludes that, if mental qualities are supposed due to chance selections of elementary factors, the resulting correlations will probably exhibit "hierarchical order"; which is equivalent to saying that the values of F will probably be small. Also, if the factors are "all or none" in their action, the mean value of F to be expected in any particular case is zero, with a standard deviation which is small if the number of factors is large.‡

The present article is a contribution to the solution of the problem for variable factors. The question we are faced with is shortly this. If mental qualities are due to the operation of a number of variable elementary factors, the Two Factor theory asserts that the proportions in which these factors contribute to the qualities are not random proportions, but certain proportions of such a kind that we may, if we please, replace the factors by others, of which one is general and the others specific.§ What we ask here is, if the proportions be at random, what sort of values for F are we to expect? Will they be significantly larger than the sampling error? or will they be well within its limits? Our problem, then, stated formally is as follows:—

I. STATEMENT OF PROBLEM.

Let there be N elementary factors, and let each quality involve the operation of all of them but in different proportions. On the supposition that all proportions are equally likely, it is required to find the probable value of the tetrad-difference F; that is, to find the standard deviation of F.

The measure $q_1$ of the quality $Q_1$ for a given individual will be

$$q_1 = l_{1x}^2 + l_{1y}^2 + l_{1z}^2 + l_{1w}^2 + \ldots$$

and the measure $q_2$ of the quality $Q_2$ will be

$$q_2 = l_{2x}^2 + l_{2y}^2 + l_{2z}^2 + l_{2w}^2 + \ldots$$

§ See Garnett's article referred to.
and so for \( Q_3 \) and \( Q_4 \). The quantities \( x, y, z, u, \) etc., have fixed values for an individual, and differ for different individuals; the coefficients \( l_x, l_y, l_z, \) etc., are fixed for the quality \( Q_1 \), and so for the others. The elementary factors will be supposed not to act in any activity as interference factors, so that all the \( l \)’s are positive, having values between 0 and 1. We shall assume that the measures \( x, y, z, \) etc., are distributed among the population according to the normal law, and that the units are so chosen that these measures have the same standard deviation. Further, for each quality we take the coefficients such that

\[
l_x^2 + l_y^2 + l_z^2 + \ldots = 1 \quad (2)
\]

This ensures that \( q_1, q_2, \) etc., have also the same standard deviations as \( x, y, z, \) etc. The correlation between two qualities \( Q_1 \) and \( Q_2 \) is then

\[
l_x^2 l_2^2 + l_y^2 l_2^2 + l_z^2 l_2^2 + \ldots \quad (3)
\]

and if the qualities are represented in \( N \)-dimensional space by directed lines whose “direction-cosines” are the \( l \)’s, the correlation is represented geometrically by the cosine of the angle between the two lines. Our problem then becomes equivalent to the following geometrical problem:—

Four lines, \( OQ_1, OQ_2, OQ_3, OQ_4 \), are drawn at random through the origin in the positive region of \( N \)-dimensional space; it is required to find the probable value of \( \cos \theta_3 \cos \theta_4 - \cos \theta_1 \cos \theta_3 \), where \( \cos \theta \) stands for \( \cos Q_1, OQ_2, \) etc.

II. THE DISTRIBUTION OF THE LINES.

In the first place it is necessary to determine what is meant by the expression “drawn at random.”

Consider first a space of two dimensions. We shall assume that the frequency of a line occurring within an angle \( \alpha \) is proportional to \( \alpha \), in other words, that the set of lines we consider as equally likely to occur is a set making equal angles each with its neighbours; or we may put it, the frequency of lines is proportional to the arc cut off by them on a circle of unit radius.

In fig. 1 the number of lines between \( \theta \) and \( \theta + d\theta \) is proportional to \( d\theta \). Reverting to the expression \( q = l_x x + l_y y \), this becomes \( q = \cos \theta \cdot x + \sin \theta \cdot y \). Since \( l_x^2 + l_y^2 = 1 \), we can choose only one of the \( l \)’s, say \( l_x \), as we please; and we must choose a set of values for \( l_x \) from 0 to 1 such that the resulting set of values for \( l_y \) from 0 to 1 is distributed in the same way.

of Certain Theories of Mental Ability.

otherwise we should not be treating \( x \) and \( y \) alike; we must ensure, for example, that the chance that \( l_x = \frac{1}{2} \) and \( l_y = \frac{\sqrt{3}}{2} \) is the same as that \( l_x = \frac{\sqrt{3}}{2} \) and \( l_y = \frac{1}{2} \). The distribution of lines at equal angular distances does ensure this. We might, it is true, imagine many other distributions wherein \( x \) and \( y \) were treated alike, but this one seems the simplest geometrical distribution.

If we have three dimensions, the condition that the distribution of the set of values we give to each coefficient in
\[
q = l_x x + l_y y + l_z z
\]
shall be the same is fulfilled if we take the number of lines in any pencil of lines to be proportional to the solid angle formed by the lines; or, we may put it, the frequency is proportional to the area cut off by the lines on unit sphere. Thus, in fig. 2, if the position of the line \( OQ \) is specified by longitude and latitude, \( \theta_1 \) and \( \theta_2 \), the number of lines between \((\theta_1, \theta_2)\) and \((\theta_1 + d\theta_1, \theta_2 + d\theta_2)\) is proportional to \( d\theta_1 \cos \theta_2 \, d\theta_2 \). The algebraic connections between the \( l \)'s and \( \theta \)'s are as follows:

\[
\begin{align*}
\text{for 2 dimensions:} & \quad l_x = \cos \theta_1, \quad l_y = \sin \theta_1; \quad \ldots \quad (4) \\
\text{for 3 dimensions:} & \quad l_x = \cos \theta_1 \cos \theta_2, \quad l_y = \sin \theta_1 \cos \theta_2, \quad l_z = \sin \theta_2; \quad \ldots \quad (5)
\end{align*}
\]

By considering a four-dimensional figure, it can be shown that if a line be specified by \( \theta_1, \theta_2, \theta_3 \), where
\[
\begin{align*}
l_x &= \cos \theta_1 \cos \theta_2 \cos \theta_3, \\
l_y &= \sin \theta_1 \cos \theta_2 \cos \theta_3, \\
l_z &= \quad \sin \theta_2 \cos \theta_3, \\
l_u &= \quad \sin \theta_3,
\end{align*}
\]
the number of lines between \((\theta_1, \theta_2, \theta_3)\) and \((\theta_1 + d\theta_1, \theta_2 + d\theta_2, \theta_3 + d\theta_3)\) is proportional to \(\cos \theta_2 \cos^2 \theta_3 d\theta_1 d\theta_2 d\theta_3\). And in general, for \(n+1\) dimensions, we have \(n\) angles determined by the relations,

\[
\begin{align*}
  l_x &= \cos \theta_1 \cos \theta_2 \cos \theta_3 \ldots \cos \theta_n, \\
  l_y &= \sin \theta_1 \cos \theta_2 \cos \theta_3 \ldots \cos \theta_n, \\
  l_z &= \sin \theta_2 \cos \theta_3 \ldots \cos \theta_n, \\
  l_u &= \sin \theta_3 \ldots \cos \theta_n,
\end{align*}
\]

etc.,

the number of lines between

\((\theta_1, \theta_2, \theta_3, \ldots, \theta_n)\) and \((\theta_1 + d\theta_1, \theta_2 + d\theta_2, \theta_3 + d\theta_3, \ldots, \theta_n + d\theta_n)\)

being proportional to

\[
\cos \theta_2 \cos^2 \theta_3 \cos^3 \theta_4 \ldots \cos^{n-1} \theta_n d\theta_1 d\theta_2 d\theta_3 \ldots d\theta_n^* \quad \ldots \quad (8)
\]

III. Theorem.

If \(F_{12, 34}\) stand for the tetrad-difference \(r_{13}^r r_{24}^r - r_{14}^r r_{23}^r\), where the qualities are due to the variable factors \(x, y, z, u, v, \ldots\), etc., then \(F_{12, 34}\) is equal to the sum of the products like \(F_{xy, 12} F_{av, 34}\).

We have, if \(Q_1, Q_2, Q_3, Q_4\) are four qualities expressed as in Section I,

\[
r_{13} = l_x x_2 l_x^2 x_3 l_x + l_2 x_3 l_3 x_4 l_4 + \ldots
\]

and

\[
r_{24} = l_2 x_4 l_2^2 x_5 l_5 + l_3 x_4 l_3^2 x_6 l_6 + \ldots
\]

\[
r_{13} r_{24} = \{ l_1 x_2 x_3 l_1 x_4 l_4 + l_2 x_3 x_4 l_2 l_3 x_3 l_4 + \ldots \} \\
+ \{ l_1 x_3 x_4 l_1 x_3 l_3 x_4 l_4 + l_2 x_4 x_5 l_2 l_3 x_4 l_5 x_4 l_6 + \ldots \} \\
+ \{ l_1 x_4 x_5 l_1 x_4 l_4 x_5 l_5 x_6 l_6 + \ldots \} + \{ \ldots \} + \ldots \ etc.
\]

* It has been pointed out to me that the same result may be obtained thus. The volume of an \(N\) dimensional sphere is \(V = \int \ldots \int dx_1 dx_2 \ldots dx_N\). Let \(N\) new variables be defined by equations (7), where the \(l_1\)'s are replaced by \(x_1\)'s, along with \(x_1^2 + x_2^2 + x_3^2 + \ldots + x_N^2\). The integral is transformed into

\[
V = \int \ldots \int J(x_1, \ldots, x_N) dr_1 d_2 \ldots d_8 - 1,
\]

where \(J\) is the Jacobian. The surface area is \(\frac{dV}{dr} = \int \ldots \int J d_1 \ldots d_8 - 1\), so that the surface element is \(\frac{\partial V}{\partial(r, \theta_1, \ldots, \theta_{n-1})} \ d\theta_2 \ldots d\theta_{n-1}\). When this is evaluated, and \(r\) put equal to 1, the result (8) is obtained.
Similarly, interchanging the suffixes 3 and 4,

\[ r_{14}^{r_{23}} = \{ 1^{x_3}x_2^{x_3}x_2 + 1^{x_2}x_3^{x_2}x_2 \} + \ldots \}
\[ + \{ 1^{x_3}x_2^{x_3}x_2 + 1^{x_2}x_3^{x_2}x_2 \} + \ldots \]
\[ + \{ 1^{x_3}x_2^{x_3}x_2 + 1^{x_2}x_3^{x_2}x_2 \} + \ldots \]
\[ + \{ \ldots \} + \ldots \text{ etc.} \]

\[ r_{14}^{r_{24}} - r_{14}^{r_{23}} = \{ 1^{x_3}x_2^{x_3}x_2 - 1^{x_2}x_3^{x_2}x_2 \} + \ldots \}
\[ + \{ 1^{x_3}x_2^{x_3}x_2 - 1^{x_2}x_3^{x_2}x_2 \} + \ldots \]
\[ + \{ 1^{x_3}x_2^{x_3}x_2 - 1^{x_2}x_3^{x_2}x_2 \} + \ldots \]
\[ + \{ \ldots \} + \ldots \text{ etc.} \]

There are \( n \) groups within crooked brackets, with \( n - 1 \) terms in each.

Now

\[ y_3^{x_2}x_2 - d_3^{x_2}x_2 = r_{23}^{r_{24}} x_2^{r_{23}} \]

and so for the other expressions in round brackets. Hence

\[ r_{14}^{r_{23}} - r_{14}^{r_{24}} = \{ 1^{x_2}x_3^{x_2}x_3 + 1^{x_3}x_2^{x_3}x_2 \} F_{x_2, x_3} + \ldots \]
\[ + \{ 1^{x_3}x_2^{x_3}x_2 + 1^{x_2}x_3^{x_2}x_2 \} F_{x_3, x_2} + \ldots \]
\[ + \{ 1^{x_2}x_3^{x_2}x_3 + 1^{x_3}x_2^{x_3}x_2 \} F_{x_2, x_3} + \ldots \]
\[ + \{ \ldots \} + \ldots \text{ etc.} \]

Observing that \( F_{x_2, x_3} = -F_{x_3, x_2} \), etc., and putting the terms in (10) together in pairs, we get

\[ \{ (1^{x_2}x_3 - 1^{x_3}x_2) \} F_{x_2, x_3} + \ldots \]
\[ + \{ (1^{x_3}x_2 - 1^{x_2}x_3) \} F_{x_3, x_2} + \ldots \]
\[ + \{ \ldots \} + \ldots \text{ etc.} \]

\[ = F_{x_2, x_3} + F_{x_2, x_3} + F_{x_2, x_3} + \ldots \]
\[ + F_{x_2, x_3} + F_{x_2, x_3} + F_{x_2, x_3} + \ldots \]
\[ + \ldots \text{ etc.} \]

there being \( \frac{n(n-1)}{2} \) terms in all.

Thus the proposition is proved.

The theorem postulates (i) that the measures \( q \) of the qualities \( Q \) are expressed as linear functions of \( x, y, z, u, \) etc., and (ii) that the distribution of \( x, y, z, \) etc., among the population is symmetrical about zero, of the same form with the same standard deviation for all, and, of course, that
the integral of the distribution from \(-\infty\) to \(+\infty\) is convergent. It is independent of the way in which a large number of qualities are supposed to be made up.*

IV. CALCULATION OF \(\sigma^2\).

We are now in a position to solve our main problem. For convenience we shall take \(n+1\) axes, so that a line \(OQ\) will depend on \(n\) angles \(\theta_1, \theta_2, \ldots, \theta_n\), as explained in Section II; and, as also explained there,

\[
\begin{align*}
&l_x = \cos \theta_1 \cos \theta_2 \cos \theta_3 \ldots \cos \theta_n \\
&l_y = \sin \theta_1 \cos \theta_2 \cos \theta_3 \ldots \cos \theta_n \\
&l_z = \sin \theta_2 \cos \theta_3 \ldots \cos \theta_n \\
&l_u = \sin \theta_3 \ldots \cos \theta_n
\end{align*}
\]

etc.

First we find the number of events.

Since the position of each of the four lines is independent of the position of the other three, the number of ways in which the four may be drawn is the fourth power of the number in which one may be drawn. Now the frequency of one is, by equation (8), proportional to

\[
\cos \theta_2 \cos^2 \theta_3 \cos^3 \theta_4 \ldots \cos^{n-1} \theta_n d\theta_1 d\theta_2 d\theta_3 \ldots d\theta_n.
\]

The number of ways in which one line can be drawn is therefore this quantity multiplied by some constant indicating a measure of frequency. As this constant will ultimately divide out, we shall omit it, and refer to any quantity which it should multiply as a "number." Each \(\theta\) can vary from 0 to \(\frac{\pi}{2}\) independently of all the others; because on our hypothesis all the \(\ell\)'s are positive, and the lines lie in that part of space corresponding to the first quadrant in a plane, or the eighth part of a three-dimensional sphere. We get the number for one line therefore by evaluating

\[
\int \int \ldots \int \cos \theta_2 \cos^2 \theta_3 \cos^3 \theta_4 \ldots \cos^{n-1} \theta_n d\theta_1 d\theta_2 d\theta_3 \ldots d\theta_n,
\]

the limits in each case being 0 and \(\frac{\pi}{2}\). Moreover, since the \(\theta\)'s are independent, we can carry out the integration with respect to each separately and multiply the results.

* In essence this is a geometrical theorem. E.g., if there are only two variables \(x\) and \(y\), and the lines \(OQ_1, OQ_2, OQ_3, OQ_4\) make the angles \(\alpha, \beta, \gamma, \delta\) with \(OX\), we have

\[
\cos (\gamma - \alpha) \cos (\delta - \beta) - \cos (\gamma - \beta) \cos (\delta - \alpha) = (\cos \alpha \sin \beta - \cos \beta \sin \alpha)(\cos \gamma \sin \delta - \cos \delta \sin \gamma),
\]

a result easily proved by elementary trigonometry.
of Certain Theories of Mental Ability. 23

Now

\[
\begin{align*}
\int_{0}^{\pi/2} d\theta_1 &= \frac{\pi}{2} \\
\int_{0}^{\pi/2} \cos \theta_2 d\theta_2 &= 1, \\
\int_{0}^{\pi/2} \cos^2 \theta_3 d\theta_3 &= \frac{1}{2} \frac{\pi}{2}, \\
\int_{0}^{\pi/2} \cos^3 \theta_4 d\theta_4 &= \frac{2}{3} \frac{\pi}{2}, \\
\int_{0}^{\pi/2} \cos^4 \theta_5 d\theta_5 &= \frac{3}{4} \frac{\pi}{2}, \\
\int_{0}^{\pi/2} \cos^n \theta_n d\theta_n &= \frac{(n-2)(n-4) \ldots 6.4.2}{(n-1)(n-3) \ldots 5.3.1} \\
\end{align*}
\]  \tag{13}

taking \( n \) in the meantime to be an even number. The value of the integral is therefore

\[
\frac{\pi}{2} \times 1 \times \frac{1}{2} \frac{\pi}{2} \times \frac{2}{3} \frac{\pi}{2} \times \frac{3}{4} \frac{\pi}{2} \times \ldots \times \frac{(n-2)(n-4) \ldots 6.4.2}{(n-1)(n-3) \ldots 5.3.1} = \left(\frac{\pi}{2}\right)^n \times \frac{1}{(n-1)(n-3) \ldots 5.3.1} \tag{14}
\]

The number of events, i.e. the number of ways in which the four lines may be drawn, is therefore, taking the fourth power,

\[
\left(\frac{\pi}{2}\right)^{2n} \times \frac{1}{(n-1)(n-3) \ldots 5.3.1} \tag{14'}
\]

Next we have to evaluate the sum total of \( F^2 \). To do this we must express \( F^2 \) in terms of the \( \theta \)'s, multiply by the frequency, and integrate. Now, by the theorem of Section III,

\[
F = \sum_{x, y, z, u} E_{xy, 12} F_{xy, 34}.
\]

\[
F^2 = \sum_{x, y, z, u} E_{xy, 12}^2 F_{xy, 34} + \sum_{x, y, z, u} 2E_{xy, 12} E_{xy, 34} F_{xy, 12} F_{xy, 34} + \sum_{x, y, z, u} 2E_{xy, 12} F_{xy, 34} F_{xy, 12} F_{xy, 34} \ldots \tag{15}
\]

Of the first kind of term there are \( \frac{(n+1)n}{2} \). Of the second kind, where
three letters are involved, one occurring in all four suffixes, there are \( \frac{(n+1)n(n-1)}{2} \). Of the third kind, where four letters are involved, there are \( \frac{(n+1)n(n-1)(n-2)}{8} \).

The total number is

\[
\frac{(n+1)n(n-1)}{2} + \frac{(n+1)n(n-1)(n-2)}{4} = \left( \frac{(n+1)n}{2} \right)^2.
\]

Since \( Q_1, Q_2, Q_3, \) and \( Q_4 \) depend on \( x \) and \( y \) in the same kind of way as on any other two variables, it follows that when we integrate the term \( F^2_{x_1, x_2} F^2_{y_1, y_2} \), we shall get the same result as from any other term of the same kind; the same applies to the second kind of term, and to the third. We have thus only to consider one of each kind, and multiply the result by the appropriate number.

Taking the third kind first, viz.,

\[
F_{x_1, x_2} F_{y_1, y_2} = (\cos X_1 \cos Y_1 - \cos X_2 \cos Y_2)(\cos Z_1 \cos U_1 - \cos Z_2 \cos U_2)
\]

The first term of this expression, since \( Q_1 \) and \( Q_2 \) are independent of \( Q_3 \) and \( Q_4 \), will give the square of the result of integrating \( \cos X_1 \cos Z_1 \); so also the second will give the square of the result of integrating \( \cos Y_1 \cos Z_1 \), which will be the same as with \( \cos X_1 \cos Z_1 \). Each of the four terms will thus give the same result, and therefore the third kind of term in (15) will, when integrated, vanish.

Consider now the first kind, viz., \( F^2_{x_1, x_2} F^2_{y_1, y_2} \). The considerations already adduced show that we have to integrate \( F^2_{x_1, x_2} F^2_{y_1, y_2} \) and square the result. If we let \( \theta \)'s specify the position of OQ\(_1\) and \( \phi \)'s that of OQ\(_2\), \( F^2_{x_1, x_2} \) is equal to

\[
\begin{align*}
&\{\cos \theta_1 \cos \theta_2 \cos \theta_3 \ldots \cos \theta_n \cdot \sin \phi_1 \cos \phi_2 \cos \phi_3 \ldots \cos \phi_n \}^2 \\
&\quad - \cos \phi_1 \cos \phi_2 \cos \phi_3 \ldots \cos \phi_n \cdot \sin \theta_1 \cos \theta_2 \cos \theta_3 \ldots \cos \theta_n \}^2
\end{align*}
\]

\[
= \cos^2 \theta_1 \cos^2 \theta_2 \ldots \cos^2 \theta_n \cdot \sin^2 \phi_1 \cos^2 \phi_2 \ldots \cos^2 \phi_n
+ \cos^2 \phi_1 \cos^2 \phi_2 \ldots \cos^2 \phi_n \cdot \sin^2 \theta_1 \cos^2 \theta_2 \ldots \cos^2 \theta_n
- 2 \sin \theta_1 \cos \theta_1 \cos^2 \theta_2 \ldots \cos^2 \theta_n \cdot \sin \phi_1 \cos \phi_1 \cos^2 \phi_2 \ldots \cos^2 \phi_n.
\]

(16)
of Certain Theories of Mental Ability.

This has to be multiplied by the frequency (equation 8),

\[ \cos \theta_2 \cos^2 \theta_3 \ldots \cos^{n-1} \theta_n \cos \phi_2 \cos^2 \phi_3 \ldots \cos^{n-1} \phi_n \]

\[ d \theta_1 d \theta_2 \ldots d \theta_n \ d \phi_1 d \phi_2 \ldots d \phi_n , \]

and integrated.

Evidently we can integrate with respect to each of the \( \theta \)'s and \( \phi \)'s separately. Taking the first term of (16) and considering \( \theta \)'s only, we have

\[
\int \ldots \int \cos^2 \theta_1 \cos^3 \theta_2 \ldots \cos^{n+1} \theta_n d \theta_1 d \theta_2 \ldots d \theta_n
= \frac{\pi}{2} \times \frac{2}{3} \times \frac{3}{1} \times \frac{\pi}{4} \times \frac{4}{2} \times \ldots \times \frac{n(n-2)}{(n+1)(n-1)} \ldots \frac{3}{1}
\]

(still supposing \( n \) an even number),

\[
= \left( \frac{\pi}{2} \right)^n \times \frac{1}{P(n+1)}
\]

The \( \phi \) part gives

\[
\int \ldots \int \sin^2 \phi_1 \cos^2 \phi_2 \ldots \cos^{n+1} \phi_n d \phi_1 d \phi_2 \ldots d \phi_n
= \left( \frac{\pi}{2} \right)^n \times \frac{1}{P(n+1)}
\]

Hence the first term of (16) leads to

\[
\left( \frac{\pi}{2} \right)^n \times \frac{1}{P^2(n+1)}
\]

The second term of (16) gives the same result.

For the third term, taking the \( \theta \) part only, we have

\[
\int \ldots \int \sin \theta_1 \cos \theta_1 \cos^3 \theta_2 \ldots \cos^{n+1} \theta_n d \theta_1 d \theta_2 \ldots d \theta_n
= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{1} \times \frac{\pi}{4} \times \frac{4}{2} \times \ldots \times \frac{n(n-2)}{(n+1)(n-1)} \ldots \frac{3}{1}
\]

\[
= \left( \frac{\pi}{2} \right)^{n-1} \times \frac{1}{P(n+1)}
\]

The \( \phi \) part gives the same, so that the third term leads to

\[-2 \left( \frac{\pi}{2} \right)^{n-2} \times \frac{1}{P^2(n+1)}
\]

Collecting these results, we have

\[
2 \left( \frac{\pi}{2} \right)^n \times \frac{1}{P^2(n+1)} -2 \left( \frac{\pi}{2} \right)^{n-2} \times \frac{1}{P^2(n+1)}
= \left( \frac{\pi}{2} \right)^{n-2} \times \frac{2}{P^2(n+1)} \times \left\{ \left( \frac{\pi}{2} \right)^2 -1 \right\}.
\]
The first type of term in (15) therefore gives

\[
\left( \frac{\pi}{2} \right)^{2n-1} \times \frac{4}{P^2(n+1)} \times \left\{ \left( \frac{\pi}{2} \right)^2 - 1 \right\}^2.
\]

and this has to be multiplied by \( \frac{(n+1)n}{2} \), the number of such terms.

The second kind of term in (15) is \( F_{xy, 12} F_{xy, 34} F_{xx, 12} F_{xx, 34} \). Again we take \( F_{xy, 12} F_{xy, 34} \), and square the result.

This is equal to

\[
(\cos \theta_1 \cos \theta_2 \cdots \cos \theta_n \sin \phi_1 \cos \phi_2 \cdots \cos \phi_n - \cos \phi_1 \cos \phi_2 \cdots \cos \phi_n \sin \theta_1 \cos \theta_2 \cdots \cos \theta_n) \\
\times (\cos \theta_1 \cos \theta_2 \cdots \cos \theta_n \sin \phi_2 \cos \phi_3 \cdots \cos \phi_n - \cos \phi_1 \cos \phi_2 \cdots \cos \phi_n \sin \theta_2 \cos \theta_3 \cdots \cos \theta_n)
\]

\[
= \cos^2 \theta_1 \cos^2 \theta_2 \cdots \cos^2 \theta_n \sin \phi_1 \sin \phi_2 \cos \phi_3 \cos^2 \phi_3 \cdots \cos^2 \phi_n \\
+ \text{a similar term with } \theta \text{ and } \phi \text{ interchanged}
\]

\[
- \sin \theta_1 \cos \theta_2 \cdots \cos^2 \theta_n \sin \phi_1 \sin \phi_2 \cos \phi_3 \sin \phi_3 \cdots \cos \phi_n \\
- \text{a similar term with } \theta \text{ and } \phi \text{ interchanged.}
\]

The first line leads to

\[
\int \int \cdots \int \cos^2 \theta_1 \cos^3 \theta_2 \cdots \cos^{n+1} \theta_n \sin \phi_1 \sin \phi_2 \cos \phi_3 \cos^4 \phi_3 \cdots \cos^{n+1} \phi_n \\
\times d\theta_2 \cdots \sin \theta_n \sin \phi_2 \sin \phi_3 \sin \phi_4 \cdots \sin \phi_n \\
= \frac{1}{2} \times \frac{\pi}{2} \times \frac{2}{3} \times \frac{3.1}{4} \times \frac{\pi}{2} \times \cdots \times \frac{P(n)}{P(n+1)} \times \frac{1}{2} \times \frac{3.1}{4} \times \frac{\pi}{2} \times \cdots \times \frac{P(n)}{P(n+1)}
\]

\[
= \left( \frac{\pi}{2} \right)^{n-1} \times \frac{1}{P^2(n+1)}.
\]

The third line leads to

\[
\int \int \cdots \int \sin \theta_1 \cos \theta_2 \sin \phi_1 \cos \phi_2 \cdots \sin \phi_n \\
\times \cos \phi_2 \cos \phi_3 \cdots \cos \phi_n \\
\times d\theta_2 \cdots \sin \theta_n \sin \phi_2 \sin \phi_3 \sin \phi_4 \cdots \sin \phi_n
\]

\[
= \frac{1}{2} \times \frac{3.1}{4} \times \frac{\pi}{2} \times \cdots \times \frac{P(n)}{P(n+1)} \times \frac{1}{2} \times \frac{3.1}{4} \times \frac{\pi}{2} \times \cdots \times \frac{P(n)}{P(n+1)}
\]

\[
= \left( \frac{\pi}{2} \right)^{n-1} \times \frac{1}{P(n+1)}.
\]

Collecting these, we get

\[
\frac{2}{P^2(n+1)} \cdot \left( \frac{\pi}{2} \right)^{n-2} \left( \frac{\pi}{2} - 1 \right).
\]
of Certain Theories of Mental Ability. 27

We have to square this to get the result of the second kind of term of (15), and get
\[
\left(\frac{\pi}{2}\right)^{2n-1} \times \frac{4}{P^4(n+1)} \times \left(\frac{\pi}{2} - 1\right)^2 \cdot \frac{(n+1)n}{2} \cdot \frac{1}{(n+1)n(n-1)} \cdot \frac{1}{P^4(n+1)}
\]

\[(18)\]

The number of terms like this is \((n+1)n(n-1)\).

From (17) and (18) we get the total sum or the values of \(F_{12,34}\), namely,
\[
\left(\frac{\pi}{2}\right)^{2n-1} \times \frac{4}{P^4(n+1)} \times \left(\frac{\pi}{2} - 1\right)^2 \cdot \frac{(n+1)n}{2} \cdot \frac{1}{(n+1)n(n-1)} \cdot \frac{1}{P^4(n+1)}
\]

\[(18)\]

\[
= \left(\frac{\pi}{2}\right)^{2n-1} \times \frac{1}{P^4(n+1)} \cdot \left[ \frac{4(n+1)n(n-1)}{(\pi/2)^2} \left(\frac{\pi}{2} - 1\right)^2 + 2(n+1)n \left(\frac{\pi}{2} - 1\right)^2 \right]
\]

\[
= \left(\frac{\pi}{2}\right)^{2n-1} \times \frac{1}{P^4(n-1)} \cdot \left(\frac{\pi}{2}\right)^2 \cdot \frac{1}{(n+1)^2} \cdot \left[ \frac{4(n+1)n(n-1)}{(\pi/2)^2} \left(\frac{\pi}{2} - 1\right)^2 + 2(n+1)n \left(\frac{\pi}{2} - 1\right)^2 \right]
\]

Now from (14) the number of events is \(\frac{\pi^{2n}}{2 \cdot P^4(n-1)}\). Hence, on dividing, and putting the number of variables, \(n+1\), as \(N\), we obtain
\[
\sigma_r^2 = \frac{1}{N^2} \left[ \frac{4(N-1)(N-2)\left(\frac{\pi}{2}\right)^2 \left(1 - \frac{\pi}{2}\right)^2}{(\pi/2)^4} + 2(N-1) \left(1 - \frac{\pi}{2}\right)^2 \right]
\]

\[(19)\]

Since the four lines we are considering will each occupy every position, it is clear that for every positive value of \(F\) there will be an equal negative one: e.g., when the third and fourth change places, while the first and second remain unchanged. The mean value of \(F\) is therefore zero, and the standard deviation we have calculated is thus the standard deviation about the mean.

V. MEAN VALUE AND STANDARD DEVIATION OF A CORRELATION.

In considering any hypothesis regarding mental activities we must not only find out what values we are to expect for the tetrad-differences, but also what values are to be expected for the correlations themselves. On our present assumptions the correlation between any two qualities \(Q_1\) and \(Q_2\) is measured by \(\cos Q_1 O Q_2\), and we now proceed to find the mean value and standard deviation of this quantity, \(\overline{O Q}_1\) and \(\overline{O Q}_2\) being two lines drawn from \(O\) at random, as discussed in Section II.

We have
\[
\cos Q_1 O Q_2 = 1^2 s^2 d^2 z + 1^2 s^2 d^2 y + 1^2 s^2 d^2 z + \ldots
\]

* The above analysis supposes that \(n\) is an even number. If \(n\) is odd, there will be throughout slight modifications in the various integrations, but the final result as expressed in equation (19) will be the same.
We have to give the \( l \)'s all possible values, sum, and divide by the number.

Now
\[
1_x \ell_x = \cos \theta_1 \cos \theta_2 \ldots \cos \theta_n \cos \phi_1 \cos \phi_2 \ldots \cos \phi_n,
\]
and the frequency is
\[
\cos \theta_2 \cos^2 \theta_3 \ldots \cos^{n-1} \theta_n \cos \phi_2 \cos^2 \phi_3 \ldots \cos^{n-1} \phi_n.
\]

Integrating with respect to the \( \theta \)'s,
\[
\int \ldots \int \cos \theta_1 \cos^2 \theta_2 \ldots \cos^n \theta_n \, d\theta_1 d\theta_2 \ldots d\theta_n
= \left( \frac{\pi}{2} \right)^n \frac{1}{P(n)}
\]

if \( n \) is even. Integrating now with respect to the \( \phi \)'s, we get the same; and by multiplication we have, for the integration of \( 1_x \ell_x \),
\[
\left( \frac{\pi}{2} \right)^n \frac{1}{P(n)}.
\]

Now there are \( n+1 \) axes, and the integration of each term \( 1_x \ell_x \) must give the same.* Hence we have, for the summation of the values of \( \cos \theta_1 \phi_1 \phi_2 \),
\[
(n+1) \left( \frac{\pi}{2} \right)^n \frac{1}{P(n)}.
\]

The number (cf. equation (14)) is
\[
\left( \frac{\pi}{2} \right)^n \frac{1}{P^2(n-1)}.
\]

Hence the mean value of \( \cos \theta_1 \phi_1 \phi_2 \) is
\[
(n+1) \frac{P(n-1)}{P^2(n)}.
\]

Writing this in full, it is
\[
(n+1) \cdot \left( \frac{(n-1)(n-3) \ldots 3 \cdot 1}{n(n-2) \ldots 4 \cdot 2} \right)^2.
\]

* This is because the distribution of every \( l \) is the same. If we take the \( n+1 \)th axis as our example, we have from equations (12) \( \ell_x = \sin \theta_n \sin \phi_n \). Multiplying by the frequency and integrating, we have
\[
\int \ldots \int \cos \theta_n \cos^2 \theta_n \ldots \cos^{n-2} \theta_n \, d\theta_1 d\theta_2 \ldots d\theta_n
= \left( \frac{\pi}{2} \right)^2 \frac{1}{P(n-2)} \frac{1}{n} = \left( \frac{\pi}{2} \right)^2 \frac{1}{P^2(n)},
\]
as before.
of Certain Theories of Mental Ability.

If \( n \) is an odd number, we get for the mean value

\[
(n+1) \cdot \left( \frac{(n-1)(n-3) \ldots 4 \cdot 2}{n(n-2) \ldots 3.1} \right)^2 \frac{\pi^2}{n}.
\]  

(20)

Now

\[
\lim_{n \to \infty} \left( (n+1) \cdot \left( \frac{(n-1)(n-3) \ldots}{n(n-2) \ldots} \right)^2 \right)
\]

is \( \frac{2}{\pi} \) if \( n \) is even, and \( \frac{\pi}{2} \) if \( n \) is odd. Hence in either case

\[
\lim_{n \to \infty} \left( \text{Mean Value of \( \cos Q_1 Q_2 \)} \right) = \frac{2}{\pi}.
\]

(21)

To obtain the standard deviation of \( \cos Q_1 Q_2 \), we first find the sum of the values of \( \cos^2 Q_1 Q_2 \).

\[
\cos^2 Q_1 Q_2 = \left\{ \sum x \cos^2 x \right\}^2 - \sum x^2 \cos^2 x = 2 \sum \sum x^2 \sin^2 y.
\]

There are \( n+1 \) terms like the first, and \( \frac{(n+1)n}{2} \) like the second. The first kind, multiplied by the frequency, is

\[
\cos^2 \theta_1 \cos^2 \theta_2 \ldots \cos^2 \theta_n \cos^2 \phi_1 \cos^2 \phi_2 \ldots \cos^2 \phi_n \times \text{frequency} \times \{ \text{frequency} \}.
\]

which on integration gives

\[
\frac{1}{2} \times \frac{2}{3.1} \times \ldots \times \frac{n(n-2) \ldots 4.2}{(n+1) \ldots 3.1} = \frac{\pi^n}{8} \times \frac{1}{P^2(n+1)}.
\]

The second kind

\[
\cos \theta_1 \cos \theta_2 \ldots \cos \theta_n \times \sin \theta_1 \cos \theta_2 \ldots \cos \theta_n \times \cos \phi_1 \cos \phi_2 \ldots \cos \phi_n
\]

leads to

\[
\int \ldots \int \cos \theta_1 \sin \theta_1 \cos^2 \theta_2 \ldots \cos^2 \theta_n \cos \phi_1 \sin \phi_1 \ldots \cos^2 \phi_n
\]

giving

\[
\frac{1}{8} \times \frac{2}{3.1} \times \frac{3.1}{4.2} \times \frac{\pi^2}{2} \times \ldots \times \frac{P(n)}{P(n+1)} = \frac{\pi^{n-2}}{8} \times \frac{1}{P^2(n+1)}.
\]

The total is

\[
(n+1) \left( \frac{\pi}{2} \right)^n \times \frac{1}{P^2(n+1)} + (n+1)P\left( \frac{\pi}{2} \right)^{n-2} \times \frac{1}{P^2(n+1)} = \frac{n+1}{P^2(n+1)} \left( \frac{\pi}{2} + n\left( \frac{\pi}{2} \right)^{n-2} \right).
\]

Dividing by the number, we get the mean square deviation about zero, viz.,

\[
\frac{n+1}{P^2(n+1)} \left( \frac{\pi}{2} + n\left( \frac{\pi}{2} \right)^{n-2} \right) \times \frac{1}{P^2(n-1)}
\]

\[
= (n+1) \left( 1 + n\left( \frac{\pi}{2} \right)^{n-2} \right) \times \frac{1}{(n+1)(n-1) \ldots 3.1} = \frac{1}{n+1} \left( 1 + n\left( \frac{\pi}{2} \right)^{n-2} \right)
\]

(22)
We have taken, as before, \( n \) to be even; but equation (22) is true also when \( n \) is odd. Subtracting (Mean Value)\(^2\), we get

\[
\text{Deviation about the mean} = \frac{1}{n+1} \left[ 1 + n \left( \frac{\theta}{\pi} \right)^2 - (n+1) \left( \frac{(n+1)P^2(n-1)}{P^2(n)} \right) \right], \text{ if } n \text{ is even,}
\]

or

\[
\frac{1}{N} \left[ 1 + (N-1) \left( \frac{\theta}{\pi} \right)^2 - N \left( \frac{N \cdot P^2(N-2)}{P^2(N-1)} \right) \right], \text{ if } n \text{ is odd.} \quad (23)
\]

In any case, \( N \) odd or even, as \( N \) approaches infinity we get

\[
\sigma^2_{\cos \theta, \phi_0} = \frac{1}{N} \left( 1 - \left( \frac{\theta}{\pi} \right)^2 \right). \quad (24)
\]

VI. Application to Theories of Mental Structure.

We must now consider what these results imply psychologically. The four qualities we are thinking of are all due to the same set of factors, which need not, however, be the whole set underlying mental activities. We have taken it to be possible for these factors to make their several contributions in every conceivable proportion, and have taken care that the distribution of these proportions is the same for each factor. Those cases in which one or more factors have contributed in zero degree have been included, and it might therefore seem that we have included every imaginable quality. So, in a way, we have. But the fraction of the total number of cases in which one or more factors enter with zero coefficients (or any other named coefficient) is infinitesimal. For instance, with three factors, the number we have supposed to be due to two only is the number of points on a line compared with the number on the surface of a sphere, an infinitely small proportion. Had we excluded them by the familiar device of drawing lines close to the quadrants (fig. 3) and integrated over the surface within these lines, our results would have remained unaltered. In effect, then, we have excluded the cases where some of the factors do not operate; and what we are really dealing with is a set of four qualities into which the same set of factors effectively enter. That is, we are dealing with, not four different qualities.
of Certain Theories of Mental Ability.

but four varieties of the same quality. With four such varieties, we have found that the values of the tetrad-difference are grouped round zero with a scatter given by equation (19), the standard deviation when the number of factors is large (say 100) being approximately inversely proportional to \(\sqrt{N}\), being in fact about \(0.63/\sqrt{N}\). It follows that by taking the number of factors to be large enough we get the standard deviation of \(F\) to be as small as we please.

From another point of view, we may say that the lines representing the mental qualities, of which, according to the Two Factor Theory, only certain ones actually exist, belong to a set of lines which give \(F = 0 \pm \) a small quantity as a probable result. Whereas, according to the Two Factor Theory, \(F\) is truly zero, and consequently the representative lines belong to a selected set, we conclude from our calculations that the whole set of lines of which these are a selection gives values of \(F\) which are so small as to be quite possibly the true values.

We cannot, however, regard this as being a complete solution of the problem. For, as we have said, the qualities we have considered have all involved the whole of the \(N\) factors, and may indeed be said to be varieties of the same quality. This is further shown by our results that the values of the correlations are closely grouped round a central value. If the factors were “all or none,” the varieties could not arise; every correlation would be 1, and the tetrad-differences zero. Thus making the factors variable has caused a scatter of the values of the tetrad-difference.

In the Sampling Theory* we suppose that one quality is due to the operation of a certain set of factors, another quality to another set, and so on; and that these sets may overlap in any manner. On the assumption that the factors are “all or none,” the values of the tetrad-difference have been shown to be grouped round zero with a small standard deviation.† If, having selected the factors which are to operate in the several qualities, we now take them in varying proportions, we shall obtain a different standard deviation for the tetrad-difference. The calculation of this appears somewhat difficult; but it seems reasonable, after our statement at the end of the last paragraph, to think that it would be greater than with “all or none” factors, and yet that, with a large total number of factors, it would be very small.

If, then, we suppose that mental qualities are made up in the following “chance” fashion: Any mental activity calls for the operation of a certain set of factors, which may enter into varieties of that activity in

* See Brown and Thomson, The Essentials of Mental Measurement (1925), chap. x.
† Mackie, loc. cit.
any proportions; the certain set of factors may be supposed drawn by
chance, and, when they have been drawn, to have their proportions
decided by chance—if this is how we suppose them to be made up,
then we should expect a resulting tetrad-difference to be very nearly
zero. If this contention be admitted, then the fact that the values of
the tetrad-difference are in practice approximately zero does not compel
us to accept the Two Factor Theory. For we could say that on the
Sampling Theory the values we are led to expect are so small as also
to be in agreement with those which actually occur.

VII. FURTHER POINTS OF INTEREST.

There are some points arising out of our results which are worthy
of remark.

(1) The mean value of the correlation between two qualities when
they are due to the same set of N variable factors is approximately \( \frac{2}{\pi} \);
and the standard deviation is \( \frac{1}{\sqrt{N}} \sqrt{1 - \left(\frac{2}{\pi}\right)^2} \). If we put \( r \) for \( \frac{2}{\pi} \); we
get \( \sigma_r \) expressed formally in terms of \( r \), viz.,

\[
\sigma_r = \frac{\sqrt{1 - r^2}}{\sqrt{N}}
\]  

(25)

When two qualities are due to the fractions \( p_1 \) and \( p_2 \) of N “all or
none” factors, the mean value of \( r \) to be expected is \( \sqrt{p_1 p_2} = p \), where
each \( p \) is replaced by their mean, and then

\[
\sigma_r = \frac{1 - r^2}{\sqrt{N}}
\]  

(26)

Here the scatter is due to the selection of the factors by chance out of
the N factors available.

It is interesting to compare these with \( \sigma_r \) when we are considering
errors in the ordinary sense, errors due to sampling. We then have, if
N is the size of the sample,

\[
\sigma_r = \frac{1 - r^2}{\sqrt{N}}
\]  

(27)

(2) If we have two qualities due to \( p_1 N \) and \( p_2 N \) factors respectively,
and the factors selected by chance out of the N factors available, and
also their respective proportions chosen at random, we can find the mean
expected value of \( r \) as follows.

* Mackie, loc. cit., equation (21).
of Certain Theories of Mental Ability.

Suppose that $Q_1$ is due to $a$ factors, $Q_2$ to $b$ factors, and that of these $c$ factors are common. Then in the expression $\cos Q_1 Q_2 = \sum_{l=1}^{x} \sum_{l'=1}^{x} c$ only $c$ terms will have values other than zero, for in the others either $l$ or $l'$ will be zero. Hence, as in Section V, we obtain for the sum of values of $\cos Q_1 Q_2$, for fixed values of $a$, $b$, and $c$, but varying proportions of the factors,

$$\left(\frac{\pi}{2}\right)^{\frac{c-1}{2}} \times \frac{1}{P(a-1)} \times \left(\frac{\pi}{2}\right)^{\frac{b-1}{2}} \times \frac{1}{P(b-1)} \times c$$

$$= c \times \left(\frac{\pi}{2}\right)^{\frac{a+b-2}{2}} \times \frac{1}{P(a-1)P(b-1)}.$$

Also the number of events, as in Section V, is

$$\left(\frac{\pi}{2}\right)^{\frac{a+b-2}{2}} \times \frac{1}{P(a-2)P(b-2)}.$$

Therefore the mean value is

$$c \times \frac{P(a-2)}{P(a-1)} \times \frac{P(b-2)}{P(b-1)} = \frac{c}{\sqrt{ab}} \times \frac{P^2(a-2)}{P^2(a-1)} \times \frac{P^2(b-2)}{P^2(b-1)},$$

and if $a$ and $b$ are large this is approximately equal to

$$\frac{c}{\sqrt{ab}} \times \frac{2^2}{\pi^2} = \frac{2}{\pi} \cdot \frac{c}{\sqrt{ab}}.$$

This, we may note, is the value of $r$ when the factors are "all or none" multiplied by $\frac{2}{\pi}$. If now $a=p_1 N$ and $b=p_2 N$, and the factors are chosen out of $N$ factors in every possible way, as well as being varied in every possible way, we get the mean value of $r$ by finding the mean of the means $\frac{2}{\pi} \cdot \frac{c}{\sqrt{ab}}$. This is so because the number of events due to varying the proportions is the same for each selection of the factors, depending as it does only on $a$ and $b$. Hence the mean value of $r$ is $\frac{2}{\pi} \times$ mean value of $\frac{c}{\sqrt{ab}} = \frac{2}{\pi} \sqrt{p_1 p_2}$.

Thus we find that, while we need only postulate small values of $p_1$ and $p_2$ to account for small correlations, $\frac{2}{\pi}$ appears as a maximum probable correlation. At first sight this is surprising, and indeed seems contrary to fact. Certainly, on the hypotheses we have assumed any correlation in excess of $\frac{2}{\pi}$ calls for some explanation. The following is
suggested as a possible explanation: When the correlation between two qualities significantly exceeds \( \frac{2}{\pi} \), the two qualities are varieties of the same quality. This gives a numerical test for qualities not sufficiently diverse, which are excluded from the mathematical treatment of Garnett and Spearman, and also from that of the Sampling Theory. The detailed examination of this question we leave to another paper.

(3) We now set out equation (19) so as to show the formal dependence of \( \sigma_r^2 \) on the mean value of a correlation; and put \( r = \frac{2}{\pi} \), and \( \sigma_r^2 = \frac{1 - \left( \frac{2}{\pi} \right)^2}{N} \). We then have

\[
\sigma_r^2 = \frac{(N - 1)(N - 2)}{N^3} \cdot 4r^2(1 - r)^2 + \frac{2(N - 1)}{N} \sigma_r^4.
\]

When we consider four qualities due to selections out of \( N \) “all or none” factors, we have as an approximation

\[
\sigma_r^2 = \frac{1}{N} \cdot 4r^2(1 - r)^2 + \frac{2}{N^2}(1 - r)^4
\]

\[= \frac{1}{N} \cdot 4r^2(1 - r)^2 + 2\sigma_r^4. \]

On the assumption that the values of \( F \) are truly zero, Spearman and Holzinger find for the scatter of values due to sampling errors, \( N \) being the number of individuals in the sample,

\[
\sigma_r^2 = \frac{1}{N} \cdot 4r^2(1 - r)^2 + \frac{2}{N^2}(1 - r^2)^4 + \frac{1}{N} \cdot 4r^2(1 - r)^2 + 2\sigma_r^4. \]

Here we have three expressions for \( \sigma_r^2 \), calculated on entirely different sets of assumptions. In all the mean value of \( F \) is zero. In one case \( N \) is the number of individuals in the sample measured for the data; in another \( N \) is the fund of “all or none” factors out of which those operating in the various qualities are drawn; and in the third \( N \) is the number of variable factors supposed to operate in all possible proportions in the several qualities. It is remarkable to find that in all these cases the formal dependence of \( \sigma_r \) on \( N \), \( r \), and \( \sigma_r \) is so nearly alike, and, if we take a rougher approximation, absolutely identical, viz. \( \sigma_r = 2r(1 - r/\sqrt{N}) \).

* Mackie, loc. cit.
VIII. DISTRIBUTION AND VARIATION OF SOME OF THE QUANTITIES UNDER CONSIDERATION.

(1) Distribution of values of $l$ for a factor.

We have for the $(n+1)$th axis (equations (12), p. 22) $l = \sin \theta_n$, and the number between $\theta_n$ and $\theta_n + d\theta_n$ proportional to $\cos^{n-1} \theta_n d\theta_n$. Now $dl = \cos \theta_n d\theta_n$; therefore $d\theta_n = dl/\sqrt{1-l^2}$. Hence frequency is propor-

$$f = (1 - l^2)^{n-2}/(n-2)!.$$  

The frequency curves for various values of $N(=n+1)$, where $N$ is the number of factors, are shown in fig. 4. It will be noticed that for large values of $N$ the values of $l$ are crowded near zero. This is because $\Sigma l^2 = 1$, and if the $N$ axes and therefore the $l$'s are numerous the latter will most often (whatever sort of distribution we assume, so long as we have it the same for all the $l$'s) be small, in fact of the order $1/\sqrt{N}$. If
the factors are "all or none," \( l \) is the same for all, viz. \( 1/\sqrt{N} \); for comparison, this is shown by the ordinates terminating at the various curves. The mean value of \( l \) in our distribution is \( \frac{P(n-1)}{P(n)} \) or \( \frac{P(n-1)}{P(n)} \cdot \frac{\pi}{2} \), according as \( n \) is even or odd; and for \( n \) great, this approximates to \( \frac{1}{\sqrt{N}} \sqrt{\frac{2}{\pi}} \), showing that the smallness of \( l \) is due to \( \Sigma l^2 = 1 \).

(2) Distribution of \( F \) when \( N = 2 \).

Here we have two axes \( OX, OY \), and if \( \theta, \phi, \chi, \psi \) are the angles made

with \( OX \) by \( OQ_1, OQ_2, OQ_3, OQ_4 \), \( F = \sin (\theta - \phi) \sin (\chi - \psi) \). The frequency of a given value of \( \sin (\theta - \phi) \) is proportional to \( \frac{\pi}{2} - (\theta - \phi) \)

if we put \( s \) for \( \sin (\theta - \phi) \), the frequency of \( s \) is \( \left( \frac{\pi}{2} - \sin^{-1} s \right) / \sqrt{1 - s^2} \). The frequency of \( \sin (\chi - \psi) \) follows the same law. We have to find the frequency of values of the product. The equation we ultimately get is

\[
\int_{-1}^{1} dv \cdot \frac{\left( \frac{\pi}{2} - \sin^{-1} v \right)}{\sqrt{1 - v^2}} \times \frac{\left( \frac{\pi}{2} - \sin^{-1} \frac{F}{v} \right)}{v \sqrt{1 - \left( \frac{F}{v} \right)^2}}.
\]

To get the frequency curve, the frequencies of \( F \) were calculated at intervals of .05 by the help of various graphs. This was a very laborious process; and was done in order to find the distribution of \( F \) for a case.
of Certain Theories of Mental Ability.

however simple. The curve is shown in fig. 5. It may not, however, be typical of those where \( N \) is greater than 2. (Cf. fig. 4, where the curve for \( N=2 \) differs from all the others.)

![Figure 7](image)

(3) Variation of \( \sigma_r \) with \( N \). In fig. 6 \( \sigma_r \) is plotted against \( N \).
(4) Variation of mean value of a correlation \( r \) with \( N \). This is shown in fig. 7. When \( N=41 \), the value of \( r \), \( 41 \cdot \sqrt{\frac{39.37.35.\ldots.3.1}{40.38.36.\ldots.4.2}} \), is \( .6415 \), which is not far from \( .6366 \ (2/\pi) \).

The author desires to express his thanks to Professor Godfrey Thomson for his valuable criticisms and kind encouragement in carrying out the study of which this article forms a part.

(Issued separately February 2, 1929.)
Referring to the condition in equation (7') on page 21, that three qualities may be accounted for without a general factor; when the expression
\[ r_{12}^2 + r_{13}^2 + r_{23}^2 + 2r_{12}r_{13}r_{23} \]
is equal to 1, we get for \( a \) from equation (4)
\[ a = \frac{mn}{2(11' + mm')} \]
\[ = 2\sqrt{kk'k''}/2k'' \]
\[ = \sqrt{\frac{kk'}{k''}} = \sqrt{\frac{r_{12}r_{13}}{r_{23}}} \]

The equations of the new axes are
\[
\begin{align*}
0\xi & : \frac{x}{1(a+1/a)} = \frac{y}{am} = \frac{z}{-m} \\
0\eta & : \frac{x}{am} = \frac{y}{-al} = \frac{z}{l} \\
0\zeta & : \frac{y}{1} = \frac{z}{a}
\end{align*}
\]

\( q_1 \) depends on \( \xi \) and \( \eta \), \( q_2 \) on \( \xi \) and \( \zeta \), \( q_3 \) on \( \eta \) and \( \zeta \).
\[
\begin{align*}
q_1 &= \sqrt{\frac{r_{12}(r_{23} + r_{23}r_{12})}{r_{12} + r_{23}r_{12}}} \cdot \xi + \sqrt{\frac{r_{13}(1 - r_{12}^2)}{r_{13} + r_{23}r_{12}}} \cdot \eta \\
q_2 &= \sqrt{\frac{r_{12}(r_{13} + r_{23}r_{12})}{r_{23} + r_{12}r_{13}}} \cdot \xi + \sqrt{\frac{r_{23}(1 - r_{12}^2)}{r_{23} + r_{12}r_{13}}} \cdot \zeta \\
q_3 &= \sqrt{\frac{r_{13}(r_{13} + r_{23}r_{12})}{1 - r_{12}^2}} \cdot \eta + \sqrt{\frac{r_{23}(r_{23} + r_{12}r_{13})}{1 - r_{12}^2}} \cdot \zeta
\end{align*}
\]

Now in this border case it may be shown that
\[ 1 - r_{12}^2 = (r_{23} + r_{12}r_{13})(r_{13} + r_{23}r_{12})/(r_{12} + r_{13}r_{23}) \]

Hence an alternative symmetrical expression : -
with similar expressions for \( q_2 \) and \( q_3 \).

For 'all or none' factors \( \xi, \eta, \) and \( \zeta \) enter into the \( q \)'s in the ratio

\[
\left( \frac{r_{12}^2 + \beta_{12} \beta_{13} \beta_{23}}{r_{12} + \beta_{12} \beta_{13} \beta_{23}} \right) : \left( \frac{r_{13}^2 + \beta_{12} \beta_{13} \beta_{23}}{r_{13} + \beta_{12} \beta_{13} \beta_{23}} \right) : \left( \frac{r_{23}^2 + \beta_{12} \beta_{13} \beta_{23}}{r_{23} + \beta_{12} \beta_{13} \beta_{23}} \right)
\]

e.g., if \( \beta_{12} = .5, \beta_{13} = .4, \beta_{23} = .594 \), the boundary condition is fulfilled; \( r_{12}^2 = .25, r_{13}^2 = .16, r_{23}^2 = .353, \beta_{12} \beta_{13} \beta_{23} = .119 \), and the ratio is 369:279:472, or, say, 37:23:47. Then

\[
\beta_{12} = \frac{37}{\sqrt{65}} = .50; \quad \beta_{13} = \frac{28}{\sqrt{65}} = .40; \quad \beta_{23} = \frac{47}{\sqrt{84}} = .593
\]

approximately as required.

![Fig. i.](image)

General formula for 'all or none' factors giving required \( \beta \)'s when they satisfy the boundary condition. The numbers of factors are proportional to the quantities shown.

![Fig. ii.](image)
APPENDIX B.

The distribution of $F$ in figure 5 (page 35) of the paper in the 'Proceedings of the Royal Society of Edinburgh' was obtained in the following way.

$$F = \sin(\theta - \phi) \cdot \sin(\chi - \gamma) = xy$$  

(see p. 36)

Consider $x$ by itself. The angles $\theta$ and $\phi$ are distributed uniformly and independently of each other.

For values of $x$ between $x$ and $x + dx$ there is a strip $DE$ parallel to $OB$, and the number is proportional to the area of the strip, $DE \times d(\theta - \phi)$. $DE \propto \frac{\pi}{2} - (\theta - \phi)$, and $d(\theta - \phi) = \frac{dx}{\sqrt{1-x^2}}$.

The frequency of $x$ is thus $\left(\frac{\pi}{2} - \sin^{-1}x\right)\sqrt{1-x^2}$. This function is plotted in figure iv. The frequency of $y$ follows the same law.

If $x$ and $y$ are plotted in the same diagram (figure v), then for an element of area $dx.dy$ near $(x,y)$ $F = xy$, and the frequency is

$$\frac{\pi}{2} - \sin^{-1}x \times \frac{\pi}{2} - \sin^{-1}y \quad dx.dy$$  

...(1)

We have to collect those elements for which $F = xy$ is a constant; i.e., to divide up the area by curves $F = xy = constant$, and evaluate the frequency between successive curves $F$ and $F + dF$. The expression (1) must be transformed into one with new variables, one of which must be $F = xy$. If we take the other to be $u = x$, we get the
expression at the foot of page 36. In figure v the
curves \( F = xy \) are drawn at intervals of \( 0.05 \); the straight
lines are \( u = \frac{y-1}{x-1} \). For the interval between two successive
values of \( F \), the frequency was calculated for each
elementary area by multiplying the area by the frequency
of \( xy \) at the middle of the area, the frequency of \( xy \)
being got from figure iv; e.g., between \( F = 0.25 \) and \( 0.30 \)
we have for the 10 elementary areas from the top to the
middle

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<td>1.164</td>
<td>1.492</td>
<td>368</td>
<td>550</td>
</tr>
<tr>
<td>405</td>
<td>555</td>
<td>1.262</td>
<td>1.182</td>
<td>1.496</td>
<td>349</td>
<td>522</td>
</tr>
<tr>
<td>434</td>
<td>518</td>
<td>1.246</td>
<td>1.200</td>
<td>1.496</td>
<td>320</td>
<td>479</td>
</tr>
<tr>
<td>460</td>
<td>490</td>
<td>1.229</td>
<td>1.214</td>
<td>1.492</td>
<td>292</td>
<td>435</td>
</tr>
</tbody>
</table>

This was done for each of the intervals. For that between
0.65 and 0.70 the frequency of \( xy \) was found to be practically
the same throughout, so that it was unnecessary to split
up the remaining intervals. Also, the first interval
was divided in two. The frequency of $F$ for the various intervals is

<table>
<thead>
<tr>
<th>Value of $F$</th>
<th>Frequency</th>
<th>Value of $F$</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to .025</td>
<td>.121</td>
<td>.5 to .55</td>
<td>.0196</td>
</tr>
<tr>
<td>.025</td>
<td>.0723</td>
<td>.55</td>
<td>.0162</td>
</tr>
<tr>
<td>.05</td>
<td>.1096</td>
<td>.6</td>
<td>.0135</td>
</tr>
<tr>
<td>.1</td>
<td>.0833</td>
<td>.65</td>
<td>.0111</td>
</tr>
<tr>
<td>.15</td>
<td>.0666</td>
<td>.7</td>
<td>.0089</td>
</tr>
<tr>
<td>.2</td>
<td>.0545</td>
<td>.75</td>
<td>.0069</td>
</tr>
<tr>
<td>.25</td>
<td>.0455</td>
<td>.8</td>
<td>.0051</td>
</tr>
<tr>
<td>.3</td>
<td>.0382</td>
<td>.85</td>
<td>.0035</td>
</tr>
<tr>
<td>.35</td>
<td>.0322</td>
<td>.9</td>
<td>.0020</td>
</tr>
<tr>
<td>.4</td>
<td>.0272</td>
<td>.95</td>
<td>.0006</td>
</tr>
<tr>
<td>.45</td>
<td>.0231</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From this table were drawn the histogram and the frequency curve. \( \sigma_F \) calculated from these numbers worked out to .298, the theoretical value being

\[
\frac{1}{2} \left\{ 1 - \left( \frac{2}{\pi} \right)^2 \right\} = .297, \text{ a sufficiently close agreement.}
\]

If the frequency of $xy$ were constant the frequency curve for $F$ would be $-\log F$, which runs to infinity at zero. Hence, a fortiori, the frequency curve as it is runs to infinity at zero, and resembles roughly the curve $-\log F$.
Graph of \( \frac{T}{2} - \sin^2 \theta \sqrt{1 - x^2} \)

**Fig. iv.**

**Fig. v.**
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