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**Transformation of Statistical Variate.**

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### CHAPTER 11
**Polynomial transformation of normal variate.**

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CHAPTER 1.

Transformation of Statistical Variate.

1.1 The two approaches to the problem of Skewness.

It has been shown by Laplace (1), Crofton (2), Edgeworth (3), etc. that when a large number of independent causes, each of them producing a small additive effect, are operating on the members of a population, then under certain wide conditions their resulting distribution is the normal distribution. As often in nature an individual measure may be regarded as the cumulation of a large number of small independent effects, a large number of distributions in nature may be expected to be normal. Though the tendency to assume normality because it is expected sometimes still exists among statisticians, it has been well known for some time past that the normal distribution is found far less in practice than this derivation of it might lead us to expect; hence the problem of finding other mathematical forms that can fit the various skew distributions that occur in Statistics has engaged the attention of many statisticians.

The approach to this problem has been either: (i) from a priori considerations of the possible causes that can produce skewness; or (ii) from empirical considerations of the shapes of frequency curves, their mathematical forms and tractability etc.

The first, the a priori approach, has led the statisticians to genuine probability distributions.
Though a large number of such distributions can be derived by imagining a variety of possible causes, the field has to be limited to causes that can actually and frequently arise in practice and to those distributions for which a method of fitting can be developed and which are found to occur in practice. The normal distribution as derived by Laplace and others, its generalization by Edgeworth into "The generalized law of error", the Bernoullian binomial distribution, together with its extensions to the binomial distribution of Poisson and the Lexian distribution, Coolidge's extension of the Lexian Scheme, the Poissonian function of rare statistical frequency: these are some results of this approach which have been found to occur, throwing important light on the causes of skewness.

The result of the second, the empirical approach, have been more varied and fruitful, giving us a large number of curves which have been found useful in Statistics, Economics, Actuarial Science and many other branches of Science. The most important system of curves in Statistics is the Pearsonian system of curves (4), though we have also the Gram-Charlier Type A series, Type B series and many other curves. As early as 1895, Karl Pearson not only gave us this system of curves but also a complete technique of choosing the curve and fitting it by the first four moments. At the basis of the system is the differential equation

\[ \frac{1}{y} \frac{dy}{dx} = \frac{A + Bx}{C + Dx + Ex^2 + \cdots} \]
An attempt has been made to justify this differential equation from probability considerations by deriving it from hypergeometric series; but it is difficult to conceive how such a situation as drawing of balls from an urn without re-placement, which is imagined in deriving hypergeometric series, can be the common basis of many statistical data. The form of this differential equation clearly ensures in general unimodality and high contact at the two ends, the two important and prominent features of many frequency distributions; and the truncating of terms in the denominator on the right hand side to terms in $\chi^2$, with the consequent limitation of the number of constants, avoids the use of higher moments than the fourth which, as Karl Pearson knew, have large sampling errors. It seems more natural and straightforward to accept such practical considerations for the choice of this differential equation than the far-fetched a priori considerations. It is true that many curves of this system, e.g. Types I, II, III, VI, IX, XI and the normal curve, are probability curves, but the others can be given an interpretation in forced and artificial ways only. However the system has very great flexibility and the system, together with its method of fitting by moments, has met with great success in graduating a variety of statistical material. Though the method of fitting has been criticized and R.A. Fisher (5) has shown that moments are not always the best parameters, the usefulness of the system of curves has never been questioned. What we point out here is that it makes no attempt to explain skewness of normality.

It may be that the causes operating behind many statistical distributions are so varied, multiple, complex and correlated that any attempt to
graduate them by the probability distributions that we can derive from a priori considerations of a few simple causes being operative, will generally fail. Moreover there is always a certain heterogeneity in statistical data and the pure homogeneous material which is conceived behind probability distributions is very difficult to find actual fact. It may be that the only course for us is to choose any suitable curve and condense the information before us into the values of a few parameters; and this itself, as we know, is a difficult task. In Specification, the restriction to probability curves only, might make our subsequent tasks of Estimation and Sampling Distributions formidable.

Though conscious of these difficulties, we take up the first, the a priori approach, to the problem of skewness; for we feel that skewness can arise in nature in certain cases, in a simple way by a non-linear transformation of the latent normal variate.

1.2 Transformation of Statistical Variate.

It is generally believed now that, the pre-eminent position occupied by the normal distribution in Statistics is because of its importance in sampling distributions but as a frequency curve it has an extremely limited applicability. In the Pearsonian system of curves, the normal curve is a member, a particular case and its frequent non-occurrence has not to be explained. Very recently R.C. Geary (6) has gone to the extent of writing that "Normality is a myth. There never was and never will be a normal distribution". If that is so, we are tempted to doubt the Laplacian derivation of it.
Yet normality can be not only generative, as it is in Type A series, but basic and fundamental in nature and yet generating a variety of skew curves in the following way.

If a variate \( x \) is normally distributed, the variates \( x^2, x^3, \log x, e^x, \sin x \) etc. will not be so distributed. But it may be that these are the functions which we may be encountering or measuring. In many branches of Science it is quite possible that owing to the inherent difficulty of measuring a certain quantity, or owing to our ignorance about its nature, what we may be measuring is a certain paraphrase of the original effect or mathematically speaking, a certain function of it. Though the original quantity may be normally distributed, its non-linear transformation or effect will not be so distributed. Or again, the latent normal variates may be growing under a non-linear law of growth giving us skew distributions. Thus we can conceive normality to be basic and generative of a variety of skew curves by a non-linear transformation of the latent normal variate.

To illustrate: (i) If height is normally distributed, weight cannot be expected to be so distributed, as weight may be taken to be proportional to the cube of height: (ii) If several balls have their diameters normally distributed, their surfaces, their volumes, the number contained per cubic inch (say) will not be so distributed: (iii) Velocity may be normally distributed but in such a case energy is not.

To enter a controversial subject not strictly within our sphere: we do not know what intelligence is; it is supposed to be constant from birth to
death, but it may be an organic phenomena subject to fluctuations, growth and decay; we do not know. What intelligence tests measure is certainly not intelligence itself, whatever it may be, but products or effects of intelligence which may be non-linear functions of intelligence. Wechsler in his Bellevue Tests of Adult Intelligence gets a negatively skew distribution for the results of his tests. If intelligence is normally distributed, either the tests interpret intelligence non-linearly or the latent normal intelligence may be subject to a non-linear law of growth. In fact, Wechsler asserts from other results that in adults, intelligence declines with age (7).

Again the distribution of visual photometric observations of stellar brightness was always found to be unsymmetrical and negatively skew. A convincing reason is that the actual brightness may be normally distributed, but according to Weber-Fechner's law of reaction to stimuli (8), the eye interprets it logarithmically. Instead of the normal variate \( x \) - the actual brightness, what is measured is a transformed variate \( u = \log x \). If the normal distribution is

\[
d\mu = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}} dx,
\]

the transformed distribution for \( u = \log x \) will be

\[
d\mu = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \frac{(e^u-\mu)^2}{\sigma^2}} e^u du
\]

(1.1)
a distribution which has its median at $\log \mu$ and
mode at $\log \left\{ \frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} + \sigma^2} \right\}$ and is distinctly
negatively skew.

Thus we can generate a variety of skew curves
by non-linear transformations of the normal variate. In fact, any probability distribution can be made to
generate a variety of other distributions by trans-
formations, but in this paper we have attempted only
the non-linear transformation of the normal variate
not only because of the greater complexity of the more
general problem, but also because we imagine it to be
possible and occurring frequently in nature.

1.3 Macalister's Curve for Geometric Mean.

At the suggestion of Galton (9) that as in
vital and social Statistics the geometric mean is more
probable of the true measure than the arithmetic mean,
the law of error for geometric mean needs investigation,
Donald Macalister in 1879, following the methods of
Gauss and Quetelet, obtained the distribution

$$
d\mu = \frac{h}{x\sqrt{\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{\log \frac{\mu}{a}}{a} \right)^2 \right\} dx
$$

(1.2)

where $h$ is the "measure of precision" (10). We shall
consider this curve in greater detail in Chapter III, but at this stage we shall only note that this curve
can be derived very easily from the normal curve by
transforming the normal variate exponentially.

Let the normal variate $\kappa$ have the probability
distribution \[ dp = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}} \ dx. \]

Let \[ a e^{x-\mu} = u. \]

We then have for \( u \) the distribution

\[ dp = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left\{ \log \frac{u}{a} \right\}^2 / \sigma^2} \frac{du}{u}, \]

and, using \( \sqrt{2} \sigma = 1/\lambda \), the relation between the standard deviation \( \sigma \) and the old "measure of precision" \( \lambda \) and writing \( x \) for \( u \) we get Macalister's form (1'2).

1'4 Edgeworth's Method of Translation.

Edgeworth gave the genesis of a new variety of curves by his "method of translation" of the normal curve. Let \( u \) be a skew variate, and \( x \) a normal variate in standard measure and with mean as origin -- \( u \) being related to \( x \) by the relation

\[ u = a (x + k x^2 + \lambda x^3) \]  

(1'3)

where \( k \) and \( \lambda \) are small and \( a \) is finite.

Edgeworth gave two methods for obtaining the estimated values of the parameters \( a, k \) and \( \lambda \) one by method of moments and the second by "Percentiles".

It is important for us to know how Edgeworth arrived at (1'3). Bowley writes (11, p.65), "In the earlier papers he was inclined to contrast his method of translation with Prof. Pearson's well known system, claiming that his was based on a presumed vera causa while the latter was purely empirical. More recently he was inclined to make that claim for the generalized law of error only, and to admit that translation had the same kind of validity as had Prof. Pearson's hypothesis and to depend rather on the criterion of
convenience of working and success in application". Yet it is quite evident from his writings that he knew he was dealing here with a functional transformation, viz \( u = f(x) \), where \( f(x) \) was expanded in a Taylor series as far as terms up to \( x^3 \) only. He clearly saw the possibility of such a transformation in nature, but called it "Translation" instead of "Transformation" because of the visual effect of, the transformation on each elementary rectangle of the normal curve. He first introduced this method of translation in 1898 in a very significant manner (12), which we can put in the following way.

It is known that the height \( h \) of adults is normally distributed with a mean \( \mu \) which is large in comparison with its standard deviation \( \sigma \). Thus in mens' height, \( \mu = 67.5 \) inches approx., \( \sigma = 2.5 \) inches approximately. Let \( h = \mu + \chi \sigma \), where \( \chi \) is the deviation of \( h \) from the mean in standard measure. Let us now take any function of \( h \), say \( \omega = f(h) \). Then expanding \( f(h) \) i.e., \( f(\mu + \chi \sigma) \) as a Taylor series, we get a first approximation, if \( \chi \) is small in comparison with \( \mu \),

\[
 \omega = f(\mu) + \chi \sigma f'(\mu)
\]

i.e. \( \omega \) is approximately linear in \( \chi \) and so the distribution of \( \omega \) will be approximately normal with mean \( f(\mu) \) and standard deviation \( \sigma f'(\mu) \). This explains why we get approximate normality in nature. To a third approximation however we get:

\[
 \omega = f(\mu) + \chi \sigma f'(\mu) + \frac{\chi^2 \sigma^2}{2!} f''(\mu) + \frac{\chi^3 \sigma^3}{3!} f'''(\mu)
\]

Referring \( \omega \) to the first approximate mean and writing \( a \) for \( \sigma f'(\mu) \) we get

\[
 \omega = a \left( \chi + \frac{\sigma f''(\mu)}{2! f'(\mu)} \chi^2 + \frac{\sigma^2 f'''(\mu)}{3! f'(\mu)} \chi^3 \right);
\]
hence we get (referring to the approximate mean)

\[
\omega = a (x + k x^2 + \lambda x^3)
\]

As Bowley points out, Edgeworth makes somewhat conflicting statements about his method of translation; but throughout he consistently adheres to such an explanation of its derivation. In his last paper on the subject in 1924, "Untried methods of Representing Frequency", he very clearly and lucidly elaborates this explanation (13). "The case to which translation primarily applies is where the observations grouped are known to be simple functions of quantities conforming to the law of error". He also states that "We are not embarrassed by the question what is \( x \) (the abscissa of the generating normal curve), of which the observed character \( X \) is a function?" And here he not only mentions the Taylorian expansion of \( f(x) \), but gives a justification of it from practical considerations. He quotes Karl Pearson that "In non-linear regression the second and third order parabolas amply suffice to describe the skewness of regression line" and then argues that this is the very form which the regression curve assumes when we translate a normal surface.

It appears therefore that Edgeworth probably had the idea of the transformation of the normal variate in his mind as the basis of his method of translation.

1.5 Kapteyn's Theory and Curves.

Kapteyn, not satisfied with Pearson's empirical approach in deriving his system of frequency curves, put forward his own theory in 1903 and gave the genesis of a variety of skew curves together with a technique of fitting (14).
He raises the question "How do skew curves originate" and gives a long explanation with examples summing up: "We find that causes independent of the size of the individuals produce normal curves, causes dependent on the size produce skew curves". This we may call Kapteyn's Theory. He continues, "The latter case must be the general one. There seems every reason to expect however that the skewness will be exceedingly small in many cases". He also describes a "Skew-curve machine" which he devised on the same lines as Galton's apparatus for the normal curve. He then arrives at his general frequency curve in the following way:

On certain quantities \( u \) which at starting are equal, there come to operate certain causes of deviation the effects of which depend in a given way on the values of \( u \). Let us imagine certain other quantities \( x \) depending on \( u \) in the way given by

\[
x = F(u).
\]

Then,

\[
\Delta x = F'(u) \Delta u,
\]

or,

\[
\Delta u = \frac{1}{F'(u)} \Delta x.
\]

If now we suppose that on these quantities \( u \) the causes of deviations operating are such that the effects of them are proportional to \( 1/F'(u) \), then it is clear that the deviations in \( x \) will be dependent of \( x \), and according to Kapteyn's Theory, \( x \) must be normally distributed, say as

\[
d\phi = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2} dx.
\]

But \( x = F(u) \), and the distribution of \( u \) will be

\[
d\phi = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{F(u)-\mu}{\sigma}\right)^2} F'(u) du.
\]
Thus we get Kapteyn's general frequency curve

\[
y = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left\{ \frac{F(u) - \mu}{\sigma} \right\}^2} F'(u).
\]

(1.4)

According to Kapteyn our aim should be to find this latent function \( F(u) \), which in the general case was found to be difficult. But he gives complete treatment for \( F(u) = (u+k)^q \) and for \( q = 0 \) and \( q = \pm \infty \), he obtains exceptional forms.

For \( F(u) = (u+k)^q \),

\[
y = \pm \frac{A q}{\sigma \sqrt{2\pi}} (u+k)^{q-1} e^{-\frac{1}{2} \left\{ \frac{(u+k)^q - \mu}{\sigma} \right\}^2}.
\]

(1.5)

For \( q = 0 \),

\[
y = \frac{1}{\sigma \sqrt{2\pi}} \cdot \frac{1}{u+k} e^{-\frac{1}{2} \left\{ \log (u+k) - \mu \right\}^2},
\]

(1.6)

and for \( q = \pm \infty \),

\[
y = \pm \frac{A \cdot a}{\sigma \sqrt{2\pi}} \cdot a(u-l) e^{-\frac{1}{2} \left\{ \frac{a(u-l)}{\sigma} - \mu \right\}^2}.
\]

(1.7)

Kapteyn gives a method of fitting and applies it to five problems with some success. One of his problems contains the results of experiments on the threshold of sensation which are fitted by equation (1.6), and he claims that "If Fechner's law had not been proposed long ago, we would thus be very naturally led to its existence". 
We can obtain Kapteyn's curves very easily in the following way: Let \( u \) be equal to \( f(x) \), assumed monotonic over the range of \( x \), so that the inverse function \( x = F(u) \) is single-valued, where \( x \) is normally distributed with probability differential

\[
dp = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left\{ \frac{x - \mu}{\sigma} \right\}^2} \, dx.
\]

Then the distribution of \( u \) will be

\[
dp = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left\{ \frac{F(u) - \mu}{\sigma} \right\}^2} F'(u) \, du
\]

which gives Kapteyn's general frequency curve. His curves (1.6) and (1.7) can be obtained by the transformations \( x = \log(u + k) \) and \( x = e^{a(u - b)} \). The transformation for (1.5) will be \( x = (u + k)^q \). Thus we can arrive at Kapteyn's curves very easily by transformation of the normal variate.

It is true that Kapteyn arrives at his general frequency curve in a form and manner quite different from those of Edgeworth in his method of translation. But the underlying basic idea in the two methods, of the transformation of the normal variate, could not possibly have escaped the notice of Kapteyn. It also does not seem possible that he had failed to come across the work of Edgeworth, for he mentions the works of Galton, Macalister and Pearson. It is therefore surprising that he makes no mention of Edgeworth's important contribution. Karl Pearson (15) holds that Kapteyn has merely extended the idea of Edgeworth and criticizes him for the omission. On the other hand we find in 1906 Edgeworth (16) giving credit to Kapteyn of "independently excogitating and applying a method equivalent to that of translation".
1.6 A basis for Kapteyn's general curve in Symbolic Generating Functions for non-additive variates.

Let us try to arrive at Kapteyn's general curve by principles of probability.

Consider first a case comparable with Poisson's extension of Bernoullian binomial. Imagine that in \( n \) intervals of time \( \Delta \tau \), from \( \tau = 0 \) to \( \tau \), there is a function \( f = f(\tau) \), the resultant of many positive increments \( \Delta f \) liable to occur in each interval \( \Delta \tau \) under some law of probability. We shall assume that the \( \Delta f \) are not necessarily equal nor even of equal orders of magnitude; in the \( i \)th. interval suppose they have probability \( \varphi_2 \) of occurring, \( \varphi_2 \) of not occurring and suppose also that when \( i \) such increments of \( \Delta f \) have accrued to \( f \), the value of \( f \) is \( f_2 \); so that there are \( n+1 \) possible values \( f_0, f_1, \ldots \ldots, f_n \). We desire the probability distribution of these values.

We may construct a Symbolic generating function

\[
G(t) = (\varphi_1 t^{\Delta f} + q_1) (\varphi_2 t^{\Delta f} + q_2) \ldots \ldots (\varphi_n t^{\Delta f} + q_n)
\]

provided that in the exponent of \( t \) in any term of the expansion, the sum of \( \Delta f \) increments shall be written \( f(x) \) or \( f(x) \).

Now if the \( \Delta f \)'s were constant, and all equal to \( \Delta x \), the generating function above would in the limit tend to that of a normal curve

\[
y = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}
\]

or, in the case \( \varphi_2 = O(n^{-1}) \), a Poisson curve
We should then have
\[ G(t) \Delta x = \sum \phi(x) t^x \Delta x \]
where \( \phi(x) \) tends to either type of function just mentioned.

The more general generating function that is being considered differs from this only in the exponents of \( t \), which are \( f(x) \) instead of \( x \). We have indeed
\[ G(t) \Delta x = \sum \phi(x) t^{f(x)} \Delta x \]
tending to
\[ \int \phi(x) \left( \frac{g(\xi)}{\sigma} \right)^x \frac{d\xi}{\sigma} \frac{d\xi}{\sigma} \]
which is the generating function of Kapteyn's curve
\[ y = \frac{g'(\xi)}{\sigma \sqrt{2\pi}} \left( \frac{g(\xi) - \mu}{\sigma} \right)^2 \]
except when probabilities \( h_r \) are \( O(\eta^{-1}) \) in which case
\[ y = \frac{e^{-\mu} \mu g(\xi)}{\left( g(\xi) + 1 \right)} \],
a case not noticed by Kapteyn.

When \( \Delta f \) is not constant, or \( df = h(\gamma) \, d\gamma \),
indicating a non-linear law of growth, we see that a non-normal (though possibly symmetrical ) distribution arises.

The conclusion is that the form of a probability distribution of a variate is intimately related to the law of growth by which the variate has reached its observed value at the time of measurement. One may infer that a fruitful field of investigation would be the study of this relationship( e.g. growth curves of height, weight etc. at certain ages of life).

1.7 Conclusion.

There is no doubt that skewness in distribution so frequently encountered has many reasons for it notably (i) limitation in the number of sources of variation, (ii) non-homogeneity or "Lexian" variability, (iii) Poissonian statistical rarity, and (iv) non-linear laws of growth or, equivalent in effect though not in derivation with the last named, non linear transformation of the variate. The idea of the transformation of the normal variate provides a common basis, for the works of Macalister, Edgeworth and Kapteyn mentioned earlier, from which their curves can be easily derived and opens before us a fruitful field of research in problems of form and growth.
Polynomial Transformation of Normal Variate.

2.1 Polynomial Transformation of Normal Variate.

We have seen that a new distribution can be generated by the transformation \( u = f(x) \) where \( x \) is the normal variate in standard measure with mean as origin, and that we may expand \( f(x) \) in a Taylor series, giving

\[
    u = a + bx + cx^2 + \ldots
\]  

(2.1)

We suppose that, in the effective region \( 0 < |x| < 3 \) say, \( u \) is monotonic in \( x \) so that the inverse function \( x = \phi(u) \) is single-valued in this region. We shall further suppose that the series is rapidly convergent, so that its first few terms provide a fairly good approximation of \( f(x) \). We might also, in various ways, represent \( u \) by a series

\[
    u = a_0 q_0(x) + a_1 q_1(x) + a_2 q_2(x) + \ldots
\]  

(2.2)

where \( q_n(x) \) is a polynomial of the \( n \)th degree in \( x \). These polynomials can be chosen to satisfy certain conditions. The problem is to develop a method of estimating the parameters \( a, b, c, \ldots \) or \( a_n \) and determining the polynomials \( q_n(x) \).
Fitting by Moments.

Edgeworth has given a method of fitting by moments the cubic transformation he considers (11, pp. 68-70). In this method the values of the parameters $a, b, c, \ldots$ in terms of the moments are functions so complicated that in the extended case we are considering we arrive at highly involved equations; it is difficult to solve these, and much more difficult to find the variances or standard errors of the solutions. Also the use of higher moments would introduce very large sampling errors. As an alternative, Edgeworth uses "Percentiles", to which we turn.

Fitting by Quantiles.

The "Percentiles" are the quantiles of the partition values. They have received some attention of statisticians. For example, regarding the median Kendall (17, p.36) says, "It is less dependent on the scale and the form of the frequency distribution than the mean and it seems to deserve more consideration in the advanced theory than it has received". Yet the quantiles have not been much used in Statistics and their properties have not been deeply investigated. It is believed that they can be easily determined, but, neither in the theoretical distributions, excepting a few, nor in the numerically specified distributions is it easy to determine them. The common method of obtaining them by a linear interpolation demands considerable improvement.

An important property of the quantiles is this: let $\nu = f(x)$ be a monotonic increasing function of $x$, and let the probability differential $\phi(x) dx$
be transformed to \( \psi(u) \, du \) by the transformation \( u = f(x) \). Since a quantile divides the probability integral in a certain ratio, then \( f(X) \) is a certain quantile of \( x \)-distribution, \( f(X) \) is the corresponding quantile of the \( u \)-distribution. Thus the quantiles of a transformed distribution are easy to calculate from the corresponding quantiles of the original distribution; but this is not the case with the moments.

The quantiles are correlated, and approximate values of their variances and co-variances have been obtained by Kendall (17, pp. 211-215).

Let \( u_1, u_2, \ldots, u_n \) be any \( n \) quantiles of the transformed distribution, the corresponding normal quantiles being \( x_1, x_2, \ldots, x_n \). Then supposing that the first \( h+1 \) terms of \( u \) are sufficient to represent \( f(x) \), by the property of quantiles just mentioned, we get from equation (2.2),

\[
\begin{align*}
u_x &= a_0 \varphi_0(x_1) + a_1 \varphi_1(x_2) + \cdots + a_k \varphi_k(x_n), \\
&= a_0 + a_1 t_2 + \cdots + a_k t_h + 1, \\
&= 1, 2, \ldots, n; n \geq h+1. \tag{2.3}
\end{align*}
\]

Let us put equation (2.3) in matrix notation. Let \( u \) and \( a \) be column vectors given by

\[
\begin{align*}
u &= \{ u_1, u_2, \ldots, u_n \}, \tag{2.4} \\
a &= \{ a_0, a_1, \ldots, a_k \}. \tag{2.5}
\end{align*}
\]
and let

\[
Q = \begin{bmatrix}
q_0(x_1) & q_1(x_1) & \cdots & q_K(x_1) \\
q_0(x_2) & q_1(x_2) & \cdots & q_K(x_2) \\
\vdots & \vdots & \ddots & \vdots \\
q_0(x_n) & q_1(x_n) & \cdots & q_K(x_n)
\end{bmatrix}
\]

(2.6)

Then we have equation (2.3) as

\[
Q \cdot a = u 
\]

(2.7)

Equation (2.2) shows that our problem reduces to the problem of fitting polynomials to correlated data, or as shown by equation (2.7) the problem is the solution of a set of inconsistent linear equations of which the fitting of polynomials is a particular case. This problem has been completely solved by Prof. Aitken. We give in details his solution of inconsistent linear equations (18), and his method of fitting polynomials to correlated data (19), as they form the basis of our subsequent work.

2.4 Prof. Aitken's Solution of a Set of Inconsistent Linear Equations.

The Problem: Suppose we have \( n \) inconsistent linear equations, in the form

\[
u_i = a_0 p_0(x_i) + a_1 p_1(x_i) + \cdots + a_K p_K(x_i),
\]

\( i = 1, 2, \ldots, n; \quad n > K+1. \) (2.8)

where \( u_i \) are a set of correlated data, and \( p_k(x_i) \) are values of \( K+1 \) prescribed linearly independent
functions, and the problem is to find \( a_r \). In matrix notation similar to that in the previous section we have

\[
P \cdot a = u \tag{2.9}
\]

and the problem is: to find the vector \( a \).

**Variance Matrix:** Let \( x \) be a column vector having variates \( x_i, \)

\[ i = 1, 2, \ldots, n \]

as elements. Let the mean value of each \( x_i \) be taken as origin, so that \( x_i \) are deviations from means. Then clearly

\[
E(\mathbf{x} \mathbf{x}') = [\rho_{ij} \sigma_i \sigma_j] = \mathbf{V}
\]

and \( \mathbf{V} \) is the Variance Matrix of the vector \( \mathbf{x} \). In general \( \mathbf{V} \) is positive definite but in exceptional cases it may be non-negative of less than full rank. Let \( \mathbf{V}^{\dagger} \) be written also as \( \mathbf{W} \).

**Lemma 1:** Let \( x_i \) be linearly transformed to a new variate \( u_i \) by \( u = H x \) where \( H \) is in general a rectangular matrix. Then the variance matrix of \( u \) is

\[
E(\mathbf{H} \mathbf{x} \mathbf{x}' \mathbf{H}^\prime) = \mathbf{H} \mathbf{V} \mathbf{H}^\prime.
\]

Thus \( \mathbf{V} \) is transformed like the matrix of a quadratic form, except that \( H \) and \( H' \) appear in reversed order.

**Lemma 2:** Let \( \mathbf{B} \) be a variable rectangular matrix with fewer rows than columns. Let \( \mathbf{A} \) and \( \mathbf{P} \) be given matrices such that \( \mathbf{B} \mathbf{P} = \mathbf{A} \) and \( \mathbf{V} \) be a positive definite symmetric matrix of such an order that \( \mathbf{P}' \mathbf{V}^{-1} \mathbf{P} \) can be constructed. Then the trace (sum of diagonal elements) of \( \mathbf{B} \mathbf{V} \mathbf{B}' \) is a minimum (or alternatively the diagonal elements of \( \mathbf{B} \mathbf{V} \mathbf{B}' \) attain independently the minimum values) provided that

\[
\mathbf{B} = \mathbf{A} (\mathbf{P}' \mathbf{V}^{-1} \mathbf{P})^{-1} \mathbf{P}' \mathbf{V}^{-1}.
\]
This is proved by Prof. Aitken (20, pp. 42-48) in the following way:

Let us introduce a Lagrange multiplier \( \lambda_{ij} \) for every element in the condition \( BP - A = 0 \). The diagonal elements of \( BVB' \) are quadratic forms \( b_i V b'_i \) where \( b_i \) is the \( i \)th row of \( B \) and the elements of \( BP - A \) are \( b_i p_j - a_{ij} \) where \( p_j \) is the \( j \)th column of \( P \). So we consider the minimum of

\[
f = \frac{1}{2} b_i V b'_i - \sum \lambda_{ij} (b_i p_j - a_{ij}).
\]

Minimal conditions \( \frac{\partial f}{\partial b'_i} = 0 \) in vector-matrix form are

\[
b_i V = \sum \lambda_{ij} p'_j = \lambda'_i P'
\]

where \( \lambda'_i \) is the \( i \)th row of \( [\lambda_{ij}] = \Lambda' \).

The ensemble of such conditions is:

\[
BV = \Lambda' P'
\]

Hence

\[
BVV^{-1}P = A = \Lambda' P' V^{-1} P
\]

i.e.

\[
\Lambda' = A(P' V^{-1} P)^{-1}
\]

and so

\[
B = A(P' V^{-1} P)^{-1} P' V^{-1}.
\]

**Solution:** Now we can turn to the problem of finding \( a \) from \( P.a = u \). We postulate that the optimal value of each \( a_j \) is a consistent linear function of \( u_i \) of minimum variance. \( u_i \) may be correlated and have a variance matrix \( V \). We then have to express \( a \) in the form \( Bu \) and so the variance matrix of \( a \) is, by Lemma 1, \( BVB' \).
Since \( P \cdot a = u \) and hence \( B \cdot u = B \cdot P \cdot a \), the condition of consistency gives \( B \cdot P = I \). Hence by Lemma 2,

\[
B = (P'V^{-1}p)^{-1}P'V^{-1}
\]

and the solution is

\[
a = (P'V^{-1}p)^{-1}P'V^{-1}u . \quad (2.10)
\]

The Variance Matrix of Solution: By Lemma, this is equal to

\[
(P'V^{-1}p)^{-1}P'V^{-1}V\cdot V^{-1}p\cdot (P'V^{-1}p)^{-1}
\]

\[
= (P'V^{-1}p)^{-1} . \quad (2.11)
\]

The Residual Quadratic: This is equal to

\[
(u - P \cdot a)' V^{-1} (u - P \cdot a)
\]

which can be expressed in a variety of ways using

\[
P'V^{-1}p \cdot a = P'V^{-1}u \quad \text{namely}
\]

or

\[
u'V^{-1}(u - P \cdot a)
\]

or

\[
u'V^{-1}u - a'P'V^{-1}p\cdot a
\]

or

\[
u'V^{-1}u - u'V^{-1}p\cdot (P'V^{-1}p)^{-1}P'V^{-1}u .
\]

Since a quadratic form involving a reciprocal matrix can be written as the quotient of a bordered determinant by the cofactor of its leading element (21, p. 75), the last form for the residual quadratic can be written as a quotient of two determinants,
where \( p_0, p_1, p_2, \ldots, p_K \) are the successive columns of \( P \).

Such a quotient of determinants can be expanded as a Schweinsian series (21, p. 107) giving the sum of squared residuals as

\[
S^2 = u'wu - \frac{|u'wp_0|^2}{p_0'wp_0} \quad \text{and} \quad \frac{|u'wp_0 u'wp_1|^2}{p_0'wp_0 p_0'wp_1} \quad \text{and} \quad \frac{|u'wp_0 u'wp_1 u'wp_2|^2}{p_0'wp_0 p_0'wp_1 p_0'wp_2} \\
- \frac{|p_0'wp_0 p_0'wp_1|^2}{p_0'wp_0 p_0'wp_1} \quad \frac{|p_0'wp_0 p_0'wp_1 p_0'wp_2|^2}{p_0'wp_0 p_0'wp_1 p_0'wp_2} \\
\]

(2.12)
The Graduating Matrix $G$:

$$G = P(P'V^{-1}P)^{-1}P'V^{-1}$$  \hspace{1cm} (2.13)

the graduated vector being given by $Gu$ as against the observed vector $u$.

These are perfectly general results, where $p_k(x)$ are $k+1$ linearly independent functions. There are other interesting results and properties given by Prof. Aitken, but these will be sufficient for our purpose of fitting polynomials to correlated data, to which we now return.

2.5 Prof. Aitken's Method of Fitting Polynomials to Correlated Data.

Let our functions $p_k(x)$ be the polynomials $g_0(x), g_1(x), \ldots, g_k(x)$ where $g_k(x)$ is a polynomial of the $k$th degree. Let these polynomials have the quasi-orthogonal properties

$$g_r^t W g_s = 0, \quad r \neq s;$$

$$\neq 0, \quad r = s,$$  \hspace{1cm} (2.14)

where $g_r$ is the column vector of the $r$ values $g_r(x)$, $r = 1, 2, \ldots, n$ or it is the $r$th column of the matrix $Q$. In this case $P'V^{-1}P$ will be a diagonal matrix, with elements $g_r^t W g_r$ along the diagonal; and so also will be $(P'V^{-1}P)^{-1}$ with elements $\frac{1}{g_r^t W g_r}$ along the diagonal.

The results (2.10) to (2.12) reduce to:

$$a_r = \frac{g_r^t W u}{g_r^t W g_r},$$  \hspace{1cm} (2.15)
Variance of 

\[ a_r = \frac{1}{q_r w r} \]  

(2.16)

and the sum of squared residuals

\[ S^2 = u' w u - \frac{(u' w q_0)^2}{q_0 w q_0} - \frac{(u' w q_1)^2}{q_1 w q_1} - \ldots \]  

(2.17)

and hence,

\[ S^2 = u' w u - \sum_{r=0}^{r=k} a_r q_r w q_r \]  

(2.18)

These results when applied to the problem of fitting polynomials to correlated data give an extremely smooth method. \( u, W \) and \( x \) are given in the problem. We shall develop a method of finding \( q_k(x) \) at a later stage. Supposing then that these quantities are known, we get the following method:

Calculate first \( u' w u \). Then find \( a_0 \), the best constant that fits the data, calculate \( a_0^2 q_0' w q_0 \) and \( u' w u - a_0^2 q_0' w q_0 \), the latter giving the residual quadratic error. Next find \( a_1 \), calculate \( a_1^2 q_1' w q_1 \) and \( u' w u - \frac{a_1^2}{a_0^2} q_1' w q_1 \), the latter giving the residual quadratic error left by fitting the straight line \( a_0 + a_1 q_1(x) \). In this way we go on building up our polynomial

\[ a_0 + a_1 q_1(x) + a_2 q_2(x) + \ldots \]

till the residual quadratic error \( u' w u - \sum a_r^2 q_r' w q_r \) is sufficiently small for our purposes. The standard errors of the solution are easily obtained using (2.16). It now remains only to give Prof. Aitken's method of finding \( q_k(x) \) (19):
\[ x = \{ x_1, x_2, \ldots, x_n \}. \]

\[ q_2 = \{ q_2(x_1), q_2(x_2), \ldots, q_2(x_n) \}. \]

By convention let \( q_0(x) = 1, q_0 = \{ 1, 1, \ldots, 1 \}. \)

Next if \( q_1(x) = x - c_1, \)

then since \( q_1'wq_0 = 0, \)

we have \( c_1 = \frac{x'wq_0}{q_0'wq_0}. \)

Next assume \( q_2(x) = x^2 - c_{12}q_1(x) - c_{22}, \)

Then since \( q_2'wq_1 = 0, q_2'wq_0 = 0, \)

we have \( c_{12} = \frac{x^2'wq_1}{q_1'wq_1}, c_{22} = \frac{x^2'wq_0}{q_0'wq_0}. \)

Again assume \( q_3(x) = x^3 - c_{23}q_2(x) - c_{33}q_1(x) - c_{32}, \)

Then since \( q_3'wq_2 = 0, q_3'wq_1 = 0, q_3'wq_0 = 0, \)

we have \( c_{23} = \frac{x^3'wq_2}{q_2'wq_2}. \)
Proceeding in this way we find the polynomials in the form,

\[ q_2(x) = x^2 - c_{2-1,2} q_{2-1}(x) - c_{2-2,2} q_{2-2}(x) - \cdots - c_{02} \]  

(2.19)

where

\[ c_{02} = \frac{x^{2'} W q_0}{q_0' W q_0} \]  

(2.20)

Instead of \( x^2 \) we can have any suitable polynomial of the \( z \) th degree, and Prof. Aitken uses factorials for equidistant data; but for our present purpose the simple form \( x^2 \) would be better.

2.6 The Reciprocal of the Variance Matrix for Quantiles.

Before we can take up the application of Prof. Aitken's method of polynomial fitting to our problem, we have to obtain the reciprocal of the variance matrix for quantiles.

Kendall (17, p. 214) has obtained the following result: Let \( n \) be the total number of sub-divisions giving \( n-1 \) quantiles (e.g. for sextiles \( n=6 \), giving five quantiles); let \( x_e \) and \( x_m \) be the \( \ell \) th and \( m \) th quantile, \( \ell > m \) so that \( x_e \) is a higher quantile than \( x_m \); let \( f_e \) and \( f_m \) be the ordinates at these quantiles in the parent population; and let \( N \) be
the number of observations in the sample. Then
\[
\text{cov} (x_c, x_m) = \frac{(n-1) \cdot m}{n^2 \cdot N \cdot f_c \cdot f_m}
\]
for example,
\[
\text{cov. of 2nd and 5th sextiles} = \frac{1 \times 2}{36 \cdot N \cdot f_2 \cdot f_5}
\]
or Variance of 3rd octile
\[
= \frac{3 \times 5}{64 \cdot N \cdot f_3^2}
\]
Let \( F \) be the diagonal matrix of the ordinates at the \( n-1 \) quantiles in the parent population, namely
\[
F = \begin{bmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_{n-1}
\end{bmatrix}
\]
Then the variance matrix \( V \) of the \( n-1 \) quantiles is
\[
V = \frac{1}{n^2 \cdot N} \cdot F^{-1} \begin{bmatrix}
  (n-1) & \cdots & \cdots & \cdots \\
  (n-2) & 2(n-2) & \cdots & \cdots \\
  (n-3) & 2(n-3) & 3(n-3) & \cdots \\
  \vdots & \vdots & \vdots & \ddots \\
  3 & 6 & 9 & \cdots & 3(n-3) \\
  2 & 4 & 6 & \cdots & 2(n-2) \\
  1 & 2 & 3 & \cdots & (n-1)
\end{bmatrix} F^{-1}
\]
The middle matrix is symmetrical (in fact, centro-symmetrical) and we shall call it \( A \). Then
\[
V^{-1} = n^2 \cdot N \cdot F \cdot A^{-1} \cdot F
\]
To find $A^{-1}$, pre-multiply $A$ by the following matrix $T$, this being equivalent to certain elementary operations on rows, e.g. row $1 - \frac{1}{2}$ row, row $2 - \frac{2}{3}$ row, row $3 - \frac{3}{4}$ row, 

$$T = \begin{bmatrix} 1 & -\frac{1}{2} & \cdots & \cdots & \cdots \\ & 1 & -\frac{2}{3} & \cdots & \cdots \\ & & 1 & -\frac{3}{4} & \cdots \\ & & & \ddots & \ddots \\ & & & & 1 & -\frac{n-2}{n-1} \\ & & & & & 1 \\ \end{bmatrix}$$

in which there are units along the principal diagonal and $-\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}, \ldots, -\frac{n-2}{n-1}$ on the diagonal above it, all the rest of the elements being zeros. Then $TA$ is a triangular matrix,

$$TA = \begin{bmatrix} \frac{n}{2} & \cdots & \cdots & \cdots \\ \frac{n}{3} & \frac{2n}{3} & \cdots & \cdots \\ \frac{n}{4} & \frac{2n}{4} & \frac{3n}{4} & \cdots \\ \ddots & \ddots & \ddots & \ddots \\ \frac{n}{n} & \frac{2n}{n} & \frac{3n}{n} & \cdots & (n-1)n \\ \end{bmatrix}$$

$$= n \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \vdots \\ \frac{1}{n} \\ \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n-1 \\ \end{bmatrix}$$
The middle matrix, a triangular matrix which has units in the lower half and zeros in the upper half, can be reduced to the unit matrix $I$ by certain operations on either rows or columns and its reciprocal is easy to get. In fact we have,

$$(TA)^{-1} = \frac{1}{n} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & n-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \cdots & 0 \\ -1 & 1 & \cdots \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \ddots & 1 \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ 2 & 3 & \cdots \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \ddots & n \end{bmatrix}$$

$$= \frac{1}{n} \begin{bmatrix} 2 & \cdots & 0 \\ -1 & 3/2 & \cdots \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \ddots & \frac{4}{3} \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ -1 & 4/3 & \cdots \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \ddots & \frac{5}{4} \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \ddots & \frac{n}{n-1} \end{bmatrix}.$$
We get therefore,

\[ A^{-1} = \frac{1}{n} \begin{bmatrix} 2 & \cdots & \cdots & \cdots \\ -1 & \frac{3}{2} & \cdots & \cdots \\ \cdots & -1 & \frac{4}{3} & \cdots \\ \cdots & \cdots & -1 & \frac{5}{4} \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \cdots & \cdots \\ \cdots & 1 & -\frac{2}{3} & \cdots \\ \cdots & \cdots & 1 & -\frac{3}{4} \\ \cdots & \cdots & \cdots & 1 -\frac{4}{5} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1 -\frac{n-2}{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \]

\[ = \frac{1}{n} \begin{bmatrix} 2 & -1 & \cdots & \cdots & \cdots \\ -1 & 2 & -1 & \cdots & \cdots \\ \cdots & -1 & 2 & -1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \]
Hence

\[ W = \mathbf{V}^{-1} = n \cdot \mathbf{N} \cdot \begin{bmatrix} 2 & -1 & \cdots & \cdots & \cdots \\ -1 & 2 & -1 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \cdots \\ \vdots & & \ddots & \ddots & \ddots \\ \vdots & & & \ddots & -1 & 2 \\ \vdots & \cdots & \cdots & \cdots & -1 & 2 \end{bmatrix} F \] (2.22)

The middle matrix is thus a continuant matrix with elements 2 along the principal diagonal and -1 along the superdiagonal and the subdiagonal, the rest of the elements being zeros. The determinant \(|A|\) may be found to be equal to \(n^{-2}\).

2.7 Standard Matrices for \(W\) and Solutions:

Our method of finding the first few terms of the polynomial expansion of \(f(x)\), the transformation of the normal variate, is now complete—at least theoretically. We shall use sextiles or octiles only, but there is no compelling reason to confine ourselves to these, and we might equally well use deciles or higher quantiles, or even quintiles etc. in which the median does not appear. They may be determined from the data, generally given in the form of frequencies per equal class-intervals, by a suitable method of interpolation, to which question we shall come in due course. We presume here that they can be determined with sufficient accuracy from the data. So \(\mathbf{u}\) is known. The weight matrix \(W\) can be found using equation (2.22), provided \(F\) is known. For determining \(F\), we have to find the ordinates at the various quantiles in the parent population; but we
may calculate them from the sample. We presume here too that they can be determined with sufficient accuracy. So $W$ may be presumed to be determinable.

The next step is to find the vector $x$ of the normal quantiles. This can be done to the required degree of accuracy by a suitable method of interpolation from Kondo and Elderton's "Table of the Normal Curve Functions to each Permille of Frequency" (22, pp. 368-376), a Table which can also be found in Tables for Statisticians and Biometricians, Part II. (23). This can be done very rapidly on a machine by Prof. Aitken's method of quadratic extrapolation (24). For sextiles and octiles we have determined them and also calculated the vectors $x^2, x^3,$ and $x^4$. The results are given in Appendix I.

Now we can follow the method of computation given by Prof. Aitken in the illustrative example at the end of his paper on fitting polynomials to correlated data (19) and the transformation can be obtained. But the task even for a single problem would be very heavy.

The $W$ matrix is really different for different data. But the effect of small changes in the weights, in the final transformation is likely to be rather small. We may therefore substitute for the ordinates at the quantiles in the parent population the ordinates at these quantiles in any known skew curve having approximately the same level of skewness, the measure of which we may take Pearson's $\sqrt{\beta_1}$. We can take Type A series or Pearson's Type III curve. Salvosa (25) has given the Tables of areas, ordinates and their derivatives of the latter curve for the values of skewness from $0.1$ to $1.1$. We can get the abscissae at sextiles and octiles from the Table of areas by
have gone up to quartic transformations. Similar matrices for deciles or higher quantiles can be prepared if necessary, once and for all. The results are given in Appendix III. The details of calculations are reproduced for one case in Appendix V. It may be noted that the omission of scalar multiplier \( n \) from \( W \) does not affect \( a_2 \).

The preparation of these matrices of solutions also yields the required polynomials \( g_0(x), g_1(x), g_2(x), \ldots \) and also \( g_0'(w_0), g_1'(w_1), g_2'(w_2), \ldots \) which are required for the errors of the solutions and for other purposes. These results are therefore included in the Standard Results.

Our task of finding the transformation is now much lighter. We make our choice between the use of sextiles or octiles. We find from the sample the estimate of \( \sqrt{\beta_2} \) which is required very approximately only correct to the first place of decimal only, and \( u \). We then refer to the Standard Results for these quantiles for that level of skewness. We calculate in order \( u'w_0', a_0, a_0^2 g_0'w_0', u'w_0 - a_0^2 g_0'w_0' \); then

\[
a_1, a_1^2 g_1'w_1, u'w_0 - a_1^2 g_1'w_1, \ldots \text{ and again } a_2, a_2^2 g_2'w_2, \]

\[
u'w_0 - \sum_{k=0}^{L} a_k^2 g_k'w_k'; \text{and so on till the residual quadratic error } u'w_0 - \sum_{k=0}^{L} a_k^2 g_k'w_k \text{ is reduced to minimum.} \]

The required transformation then is

\[
a_0 + a_1 g_1'(x) + a_2 g_2'(x) + \ldots \]

(2.23)

which can be simplified to

\[
a + bx + cx^2 + \ldots \]

(2.24)

Thus we get the transformation of the normal variate which produces the given skew distribution.
2.8 Variances of Solutions.

Not only do we get the transformation, but we also get the variances or standard errors of the solution. Equation (2.16) gives us at once the variance of \( a_r \), which is \( \frac{1}{\mathbf{Z}_r' \mathbf{W} \mathbf{Z}_r} \) and this result will be found directly in the Standard Results. But it should be noted that we had omitted \( \frac{n}{N} \) from our \( \mathbf{W} \) matrix and hence the given value of \( \frac{1}{\mathbf{Z}_r' \mathbf{W} \mathbf{Z}_r} \) should be divided by \( n/N \) where \( n \) is the number of sub-divisions (e.g. for sextiles, \( n = 6 \)), and \( N \) is the number of observations in the sample. The standard error for each \( a_r \) is then clearly

\[
\sqrt{\frac{1}{n N \mathbf{Z}_r' \mathbf{W} \mathbf{Z}_r}}
\]

(2.25)

The coefficients \( a, b, c, \ldots \) of equation (2.24) are certain linear functions of the \( a_r \) and as the \( a_r \) are linearly independent and their variances are known, it is easy to calculate the variances or the standard errors of these coefficients. Except for \( N \) in the denominator, the variances of \( a, b, c, \ldots \) can be calculated once and for all. We have calculated these and the results are given in Appendix IV.

2.9 Changes in the Standard Results for Negative Skewness.

Pearson's Type III curve, which we have selected, is always positively skew and hence all the results in our Tables are applicable only to positively-skew distributions. A little consideration will show that the results are also applicable to...
negatively-skew distributions after making the following changes:

(i) The rows and columns of \( W \) matrix should be reversed thus making \( JWJ \) transformation (21, p. 25) of it.

(ii) The rows only of the matrices of solutions of obtaining \( a_r \) should be reversed thus changing \( S \) to \( ST \).

(iii) In the polynomials \( q(x) \), \( x \) should be replaced by \(-x\).

2.10 Fitting the Transformation to the Data.

The data will be generally in the form of frequencies per equal class-intervals. Let \( u = 1, 2, 3, \ldots \) be the end values of the first, the second, the third class, \ldots \ldots \ldots \ldots \ldots . \) To get the frequencies in these classes according to the hypothesis of the transformation of the normal variate, we put in the transformation obtained, viz.

\[ u = a + b \cdot x + c \cdot x^2 + \ldots \ldots \ldots \ldots \ldots , \quad u = 1, 2, 3, \ldots \ldots \ldots \]

and solve these equations for \( x \). These equations will be quadratic or cubic or still higher degree in \( x \) and hence will have more than one root. But we have assumed that the inverse function \( x = \phi(u) \) of \( u = f(x) \) is single-valued in the region \( 0 < |x| < 3 \) and so the root of this equation for a certain value of \( u \), lying in this region can be obtained to the required degree of approximation by any of the various methods of solving such equations. We have found Prof. Aitken's method (24, p. 71) of solving such equations, based on his method of interpolation by iteration of proportional parts, very convenient on a machine. Knowing \( x \), we can find the area or the cumulated frequency from any Table of Normal Function, e.g. Sheppard's well-known Tables of the
Probability Integral [(27, p. 182) also (23)]. This must be also the area or the cumulated frequency of the \( u \)-distribution up to that value of \( u \). Hence the class-frequencies according to our hypothesis can be easily obtained and compared with the given observed frequencies. The goodness of fit can be tested by Pearson's \( \chi^2 \) test.

2.11 Method of Finding the Quantiles.

It is not easy to obtain the required quantiles from the numerical data before us. We may try to obtain them by using a divided difference formula or a method of inverse interpolation from the table of cumulated frequencies and the end values of the classes. But there is no convergence of the differences and such polynomial interpolation is not found to be adequate. Based on our hypothesis, we propose the following method:

Prepare first a preliminary table of the relative cumulated frequencies and the end values of various classes which we shall generally take as \( u = 1, 2, 3, \ldots \). Now using Kondo and Elderton's Table (22), obtain the corresponding normal quantiles and add to the table, the third column of these \( \chi \). According to our hypothesis, \( u \) is a certain polynomial in \( \chi \), say a cubic one. Then the third divided differences of \( u \) must be constant. We may therefore calculate the divided differences of the first, the second, the third, \( \ldots \) order, in Prof. Aitken's notation (26, p. 169) \( \Delta, \Delta^2, \Delta^3, \ldots \) and add the columns of these differences to our preliminary table. Some idea of the degree of the polynomial of transformation can be obtained from
this table, and the table will facilitate the use of a divided difference formula. Lidstone (23, p.25) has given a form of Newton's divided difference formula corresponding to Everett's central difference formula, which can be used. But the same results are obtained by Prof. Aitken's iteration method (24, p.71) where no differences are to be calculated, with greater ease and speed on a machine. Hence for sextiles we obtain $u_5$ for $x = \pm 0.9674, \pm 0.4307$ and 0; and for octiles $x = \pm 1.1503, \pm 0.6745, \pm 0.3186$ and 0.

We have found that still the necessary convergence of the differences is not attained in many problems. It may be that the particular problem does not satisfy our hypothesis, or it may be that the errors in $u$ introduce larger errors in the differences and hence in the final estimation of the quantiles. However we have used this method in our applications.

2.12 An Alternative Basis for Standardizing $W$

Matrices.

For the purpose of standardizing $W$ matrices we have taken the ordinates at the quantiles from Pearson's Type III curve. As an alternative, we could have proceeded in the following way:

Let the transformation be $u = a + bx + cx^2 + \ldots$ as before. Let $\eta$ and $y$ be the ordinates at a certain quantile in the $u$ and the $x$ distributions respectively. The constancy of any elementary area gives $\eta du = y dx$, where
\[ y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} \]

\[ \Rightarrow \eta = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} \frac{\eta}{b + 2c\eta + 3d\eta^2 + \ldots} \] (2.26)

We may take the denominator on the right hand side approximately as \( b + 2c\eta \) where \( \eta \) nearly lies between \(-\frac{3}{4} + 3\). Now \( \eta \) cannot be negative, and hence \( b + 2c\eta \) cannot be negative. Hence the values of \( \frac{c}{b} \) can range from 0.00 to 0.17 say, only.

We could have prepared our \( \mathbf{W} \) matrices and consequently our standard matrices of solutions for values of \( \frac{c}{b} \) from 0.01 to 0.17 for positive skewness and from -0.01 to -0.17 for negative skewness at suitable intervals, calculating the ordinates from equation (2.26) after truncating the denominator to \( b + 2c\eta \).

For quadratic transformation or where the coefficients \( d, e, \ldots \) are small in comparison with \( c \) and \( b \), this would be better. But this cannot generally be the case. Also this would require a preliminary determination, though approximate, of the values of \( b \) and \( c \) in order to determine what level of the ratio \( \frac{c}{b} \) is to be adopted. We therefore preferred our former basis and prepared our standard results accordingly.

Of course, the restriction of the denominator of (2.26) to \( b + 2c\eta \) implies a quadratic transformation of the normal variate and this is known to induce a Type III transformation of the normal curve. So had we adopted this basis, there would not have been any material difference in the final results that are obtained, or in the labour of application, for instead of \( \sqrt{b_1} \) which we have to calculate, it would have been necessary to calculate \( \frac{c}{b} \) approximately.
Exponential Transformation of Normal Variate.

3.1 Exponential Transformation of Normal Variate.

Let the normal variate \( x \), in standard measure and with mean \( \mu \) as origin, be transformed to a variate \( Z \) by the transformation

\[
Z = a e^b x + c.
\]  

(3.1)

This we shall call the exponential transformation of the normal variate. Then the distribution of \( Z \) will have the probability differential

\[
df = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{Z-c}{\sigma} \right)^2} \frac{1}{\sigma (Z-c)} dz,
\]

\[c < Z < \infty\]  

(3.2)

giving us the so-called "Logarithmically transformed curve" (29, p. 146).

3.2 Macalister's Curve: Estimation of the Parameters and Properties.

We shall first consider the particular case of (3.2) when \( c = 0 \), which gives Macalister's curve for the geometric mean (1.2). We shall write it as:

\[
f(Z; \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\theta_2^2} \left( \log Z - \theta_1 \right)^2} \frac{1}{\theta_2 Z},
\]

\[0 < Z < \infty\]  

(3.3)

We want to find the estimates for the parameters \( \theta_1 \) and \( \theta_2 \).
Method of Maximum Likelihood:- Let $L = \sum \log f(z; \theta_1, \theta_2)$ summed over all the $n$ values of $z$ of the sample. Then from (3.3) we have:

$$L = -n \log \theta_2 - \sum \log Z - \frac{1}{2} \sum \frac{1}{\theta_2^2} \{\log Z - \theta_1\}^2 + \text{const.}$$

(3.4)

then

$$\frac{\partial L}{\partial \theta_1} = \sum \frac{(\log Z - \theta_1)}{\theta_2^2},$$

$$\frac{\partial L}{\partial \theta_2} = -n \frac{1}{\theta_2} + \sum \frac{(\log Z - \theta_1)^2}{\theta_2^3}.$$

Hence we get the statistics for $\hat{\theta}_1$ and $\hat{\theta}_2$ as

$$\hat{\theta}_1 = \frac{1}{n} \sum \log Z = \log Z,$$

$$\hat{\theta}_2^2 = \frac{1}{n} \sum (\log Z - \hat{\theta}_1)^2.$$  

(3.5)  

(3.6)

Further

$$E \frac{\partial^2 L}{\partial \theta_1^2} = E \left( -\frac{n}{\theta_2^2} \right) = -\frac{n}{\theta_2^2},$$

$$E \frac{\partial^2 L}{\partial \theta_1 \partial \theta_2} = E \frac{\partial^2 L}{\partial \theta_2 \partial \theta_1} = E \left\{ -\frac{2}{\theta_2^3} \sum (\log Z - \hat{\theta}_1) \right\} = 0,$$

$$E \frac{\partial^2 L}{\partial \theta_2^2} = E \left\{ \frac{n}{\theta_2^4} - 3 \frac{1}{\theta_2^4} \sum (\log Z - \hat{\theta}_1)^2 \right\} = -\frac{2n}{\theta_2^4}.$$

Hence the variances of $\hat{\theta}_1$ and $\hat{\theta}_2$ are $\frac{\theta_2^2}{n}$ and

$$\frac{\theta_2^2}{2n}, \text{ or}$$

Var. $\hat{\theta}_1 = \frac{\theta_2^2}{n}$,  

(3.7)

Var. $\hat{\theta}_2 = \frac{2 \theta_2^4}{n}$  

(3.8)

It may be noted that $\hat{\theta}_1$ and $\hat{\theta}_2$ are jointly sufficient but not completely sufficient estimates.
and that \( \hat{\theta}_1 \) is a biased estimator.

**Method of Minimum Variance (30):**

Let \( \gamma(\theta) \) be the function of \( \theta \) the parameter to be estimated, for which it is possible to obtain the solution by this method. It is proved that there exists an estimator \( \hat{\theta} \) for \( \gamma(\theta) \) satisfying the condition of minimum variance if we can express

\[
\frac{\partial \log L}{\partial \tau} = \frac{t - \gamma}{\lambda}
\]

where \( \lambda \) may depend on \( \theta \) but not on \( z \), and

\[
L = f(z_1; \theta) \cdot f(z_2; \theta) \cdot \ldots \cdot f(z_n; \theta).
\]

Further, that in this case

\[
\log f(z; \theta) = \beta(\theta) t(z) + q(\theta) + r(z),
\]

and then \( \tau = -\frac{2q}{\partial \theta} \) and \( \text{Var.} t = \lambda = \frac{1}{n} \frac{2p}{\partial \theta} \).

First we shall estimate for \( \theta_1 \) only. Let

\[
f(z; \theta_1) = \frac{1}{\theta_2 \cdot z \cdot \sqrt{2\pi}} \exp -\frac{1}{2 \theta_2^2} \left\{ \log z - \theta_1 \right\}^2;
\]

\[
\therefore \log f(z; \theta_1) = -\frac{1}{2} \log 2\pi - \log \theta_2 - \log z
\]

\[-\frac{1}{2 \theta_2^2} (\log z)^2 + \frac{1}{\theta_2^2} \theta_1 \log z - \frac{1}{2 \theta_2^2} \theta_1^2; \]

hence

\[
\beta(\theta_1) = \frac{\theta_1}{\theta_2^2}; \quad t(z) = \log z; \quad q(\theta_1) = -\frac{\theta_1^2}{2 \theta_2^2};
\]

\[
\therefore \tau(\theta_1) = -\frac{2q}{\partial \theta_1} = \theta_1; \quad \lambda = \frac{1}{n} \frac{2p}{\partial \theta_1} = \frac{\theta_2^2}{\theta_1};
\]

\[
\therefore \frac{\partial \log L}{\partial \tau} = \frac{\sum \log z - n \theta_1}{\theta_2^2} = \frac{1}{n} \frac{\sum \log z - \theta_1}{\theta_2^2/n};
\]

\[
\therefore t = \hat{\theta}_1 = \frac{1}{n} \sum \log z; \quad \text{Var.} \hat{\theta}_1 = \frac{\theta_2^2}{n}. \]
Let us now estimate for $\theta_2$. Let

$$f(z; \theta_2) = \frac{1}{\theta_2 z \sqrt{2\pi}} e^{-\frac{1}{2\theta_2^2} \{\log z\}^2};$$

$$\therefore \log f(z; \theta_2) = -\frac{1}{2} \log 2\pi - \log \theta_2 - \log z - \frac{1}{2\theta_2^2} \{\log z\}^2.$$ 

Let $\mathcal{N}(\theta_2)$ be the function of $\theta_2$ which would give the solution. In this case

$$p(\theta_2) = -\frac{1}{2\theta_2^2}; \quad t(z) = \{\log z\}^2; \quad \varphi = -\log \theta_2;$$

$$\therefore \mathcal{N}(\theta_2) = -\frac{\partial \varphi}{\partial \theta_2} \frac{1}{\partial \theta_2} = \theta_2^2; \quad \lambda = \frac{1}{n \frac{\partial p}{\partial \varphi}} = \frac{2\theta_2^4}{n};$$

$$\therefore \frac{\partial \log L}{\partial \gamma} = -\frac{n}{2\gamma} + \frac{\sum \{\log z\}^2}{2\gamma^2} = \frac{1}{n} \frac{\sum (\log z)^2 - \gamma}{2\gamma^2/n}.$$ 

Thus $\theta_2^2 = \frac{1}{n} \sum (\log z)^2$ and Var. $\theta_2^2 = \frac{2\theta_2^4}{n}$. 

We thus get the same results by the method of Minimum Variance as those by the method of Maximum Likelihood and the results are similar to those in the normal distribution, there being $\log z$ instead of the normal $x$. We also get into the usual difficulty that the estimators are not completely sufficient and that $\hat{\theta}_2$ is biased. Supposing that $\hat{\theta}_2 = \frac{1}{n} \sum \log z = \bar{\log z}$ as fixed, we can get

$$\theta_2^2 = \frac{\sum (\log z - \bar{\log z})^2}{n-1},$$
which will be an unbiased estimator with variance
\[ \frac{2 \sigma^4}{n-1}. \]

**Method of Moments:**

The moments of the distribution (3.3) are found to be:

\[ \mu_0 = 1, \]
\[ \mu'_1 = e^{\theta_1} e^{\frac{1}{2} \sigma^2}, \]
\[ \mu'_2 = e^{2 \theta_1} e^{\sigma^2}, \]
\[ \mu'_k = e^{\theta_k} e^{k \sigma^2 / 2}. \]

Hence we get:

\[ \hat{\theta}_1 = \log \frac{m'_1}{\sqrt{m'_1}} = 2 \log m'_1 - \frac{1}{2} \log m'_2, \]
\[ \hat{\theta}_2 = \log \frac{m'_2}{m'_1^2} = \log m'_2 - 2 \log m'_1. \]

where \( m'_1 \) and \( m'_2 \) are the first two moments of the sample.

\[ d\hat{\theta}_1 = \frac{2dm'_1}{m'_1} - \frac{1}{2} \frac{dm'_2}{m'_2}. \]

Squaring and taking the mean values we get:

\[ \text{Var}(\hat{\theta}_1) = \frac{4 \text{Var}(m'_1)}{m'_1^2} - \frac{2 \text{Cov}(m'_1, m'_2)}{m'_1 m'_2} + \frac{\text{Var}(m'_2)}{4 m'_2^2} \]
\[ = \frac{4}{n m'_1^2} (\mu'_2 - \mu'_1^2) - \frac{2}{n m'_1 m'_2} (\mu'_3 - \mu'_1 \mu'_2) + \frac{1}{4 n \mu'_2^2} (\mu'_4 - \mu'_2^2) \]
\[ = \frac{1}{4n} \cdot \frac{16 \mu'_2^3 - 8 \mu'_1 \mu'_2 \mu'_3 + \mu'_1 \mu'_2^2}{\mu'_1^2 \mu'_2^2} - \frac{9}{4n} \]
\[ = \frac{1}{4n} \cdot \frac{16 e^{6 \theta_1} e^{6 \sigma^2} - 8 e^{6 \theta_1} e^{\sigma^2} + 6 e^{6 \theta_1} e^{2 \sigma^2}}{e^{6 \theta_1} e^{5 \sigma^2}} - \frac{9}{4n} \]
\[ = e^{4 \theta_2} - 8 e^{2 \sigma^2} + 16 e^{\sigma^2} - q \]
\[ = \frac{4 \theta_2^2 - 8 e^{2 \sigma^2} + 16 e^{\sigma^2} - q}{4n} \]
\[ -50 - \]

\[ \frac{1}{4n} \left\{ 4 \theta_2^2 + \frac{16}{6} \theta_2^6 + \cdots \right\} \]

\[ = \frac{\theta_2^2}{n} \left\{ 1 + \frac{2}{3} \theta_2^4 + \cdots \right\} \]

approximately if \( \theta_2^6 \) and higher powers of \( \theta_2 \) can be neglected in comparison with \( \theta_2^2 \).

Hence Var. \( \theta_1^2 = \frac{\theta_2^2}{n} \) approx.

\[ \tau = \theta_2^2 = \log m_1' - 2 \log m_1' \]

\[ \therefore d\tau = \frac{dm_2'}{m_2'} - 2 \frac{dm_1'}{m_1'} \]

Squaring and taking means we get:

\[ \text{Var}(\tau) = \frac{\text{var}(m_2')}{{m_2'}^2} - \frac{4 \text{cov}(m_1', m_2')}{{m_1'}{m_2'}^2} + \frac{4 \text{var}(m_1')}{{m_1'}^2} \]

\[ = \frac{\mu_4 - \mu_2^2}{n \mu_2^2} - \frac{4 (\mu_3^2 - \mu_2^3 \mu_1')}{n \mu_1' \mu_3^2} + \frac{4 (\mu_1^3 - \mu_4^2)}{n \mu_1'^2} \]

\[ = \frac{1}{n} \left( \mu_4 \mu_1^2 - 4 \mu_3^2 \mu_1' + 4 \mu_1^3 \right) - \frac{1}{n} \]

\[ = \frac{1}{n} \left( e^{6 \theta_2} - 4 e^{2 \theta_2} - 1 \right) \]

\[ = \frac{1}{n} \left\{ \left( e^{2 \theta_2^2} - 1 \right)^2 - 2 \left( e^{\theta_2^2} - 1 \right)^2 \right\} \]

\[ \frac{1}{n} \left\{ \left( 2 \theta_2^4 + 2 \theta_2^4 + \frac{8 \theta_2^6}{6} + \cdots \right)^2 - 2 \left( \theta_2^4 + \frac{\theta_2^4}{2} + \frac{\theta_2^6}{6} + \cdots \right)^2 \right\} \]

\[ = \frac{1}{n} \left\{ 4 \theta_2^4 + \theta_2^6 + \cdots - 2 \theta_2^4 - 2 \theta_2^6 \cdots \right\} \]

\[ = \frac{2 \theta_2^4}{n} \left\{ 1 + 3 \theta_2^2 + \cdots \right\} \]
approximately if \( \theta_2 \) and higher powers of \( \theta_2 \) can be neglected in comparison with \( \theta_2^4 \).

We shall show in the next section that the maximum value of \( \theta_2 \) that we can expect for moderately skew curves having \( \beta_1 \) not greater than 2 is about 0.42 and hence for large samples the above approximations for \( \text{Var. } \theta_1 \) and \( \text{Var. } \theta_2^2 \) are good. These approximate results are the same as those for Maximum Likelihood and Minimum Variance Statistics. For this curve then fitting by moments is nearly as efficient as by other methods.

We could have taken equation (3.3) as

\[
f(z; \theta_1, \theta_2) = \frac{1}{\theta_1 \sqrt{2\pi}} e^{-\frac{1}{2} \frac{z^2}{\theta_1}} \left\{ \log \frac{z}{\theta_1} \right\}^2.\]

Then the method of Maximum Likelihood would have given

\[\log \theta_1 = \frac{1}{n} \sum \log z.\]

In this case \( \theta_1 \) would not be normally distributed even for large samples but will be distributed like \( z \). In this case, the method of Minimum Variance would have given the estimating function as \( \log \theta_1 \) which of course is normally distributed though not \( \theta_1 \).

Macalister noticed a very important property of this distribution that the product of any number of such variates is similarly distributed.

Macalister's curve as given in equation (3.3) has its mode at \( z = e^{\theta_1} e^{\theta_2^2} \) median at \( z = e^{\theta_1} \) and mean at \( e^{\theta_1 + \frac{1}{2} \theta_2^2} \) and is always positively skew.

3.3 The General Case.

For the general case the methods of Maximum Likelihood or Minimum Variance do not yield solutions. The method of moments gives the following solution.

The moments of the general distribution (3.2) are found to be:
$$\mu_1' = ae^{\frac{b^2}{2}} + c,$$
$$\mu_2 = a^2 e^{2b^2} - a^2 e^{b^2},$$
$$\mu_3 = a^3 e^{3b^2} - 3a^3 e^{5b^2} + 2a^3 e^{3b^2}.$$ Let then
$$e^{b^2} = \rho = 1 + \alpha$$
(3.9)
$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{a^6 \rho^3 (\rho^3 - 3\rho + 2)^2}{a^6 \rho^3 (\rho - 1)^3}$$
$$= (\rho - 1)(\rho + 2)^2$$
$$\therefore \beta_1 = \alpha (3 + \alpha)^2$$
(3.10)
\(\alpha\) being positive.

Now we can find \(b\) (with an ambiguity of sign which we shall fix later on) if we know \(\alpha\). To find \(\alpha\), we have to solve equation (3.10) which has in general three roots, say \(\alpha_1, \alpha_2\) and \(\alpha_3\). We have really
$$\alpha_1^3 + 6\alpha_2^2 + 9\alpha_3 - \beta_1 = 0 ;$$
$$\therefore \alpha_1 + \alpha_2 + \alpha_3 = -6 \; ,$$
$$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = 9 \; ,$$
$$\alpha_1\alpha_2\alpha_3 = \beta_1 \; .$$

Now \(\beta_1\) is always positive. All the three roots cannot be real and positive; nor one real and negative and the other two real and positive; nor also all the three real and negative. Hence there is only one root of this equation which is real and positive and that is the one we want. A table of values of \(\alpha\) for given values of \(\beta_1\) can be constructed at suitable intervals from which the required value of \(\alpha\) for a given value of \(\beta_1\) can be obtained with sufficient accuracy. If the maximum value of \(\beta_1\) is taken as about 2, \(\alpha\) has the maximum value of about 0.2 and then \(b^2 = 0.18\), giving the maximum value of \(b\) as about 0.42.
Knowing \( \alpha \), we find \( a \) and \( c \) from

\[
a = \pm \frac{\mu_2}{\sqrt{\alpha(1+\alpha)}},
\]

(3.11)

\[
c = \mu'_1 - a\sqrt{1+\alpha}.
\]

(3.12)

There is an indeterminacy in the signs of \( a \), \( b \) and \( c \), which can be settled thus:

\[
\mu_3 = a^3 e^{3b^2/2} (P-1)^2 (P+2).
\]

Now \( P \) being equal to \( e^{b^2} \) is positive and greater than 1. The sign of \( a \) is therefore the sign of \( \mu_3 \), the third moment about the mean. Then \( c \) is given without ambiguity of sign by (3.12). In order that the probability ordinate be always positive, the sign of \( b \) should be so taken that \( b(z-c) \) may be positive for all values of \( z \).

We can therefore estimate the parameters \( a \), \( b \) and \( c \) from the sample moments \( m'_4 \) and \( m'_2 \) and the sample \( b_1 \). Pretorius has given a similar method (29, p. 120). But the task is not yet complete, for we have to find the sampling errors of these statistics which is a complicated task.

We have found that when \( \beta_1 = 2 \) the value of \( b \) is about 0.42. For moderate skewness \( b \) will be much smaller. Hence the first few terms of the expansion of \( e^{b^2} \) will be a good approximation of \( e^{b^2} \) in the region \( 0 < |x| < 3 \). We thus get the exponential transformation as

\[
z = a (1 + b^2 x + \frac{b^4 x^2}{2!} + \ldots + \frac{b^{2k} x^{2k}}{k!}) + c
\]

which is a form of the polynomial transformation of chapter II. Our assumption in chapter II, that the transformation of the normal variate \( x \) can be expanded in a Taylor series is justified in this case, and, for the exponential transformation considered in
this chapter, the method of polynomial transformation can be expected to give good results.

If \( C \), the "Starting point" can be determined by some method, the results of the previous section give the most efficient statistics and their errors. But it is not possible to determine \( C \) first, and hence the results of the previous section may be only of theoretical interest.

A method based on Fitting by quantiles and Least Squares by the use of the result (2.10) can be evolved for this case also: but the computational work would be very heavy and the method would not be of much use in practice.

<table>
<thead>
<tr>
<th>Weight in lbs</th>
<th>Frequency</th>
<th>Weight in lbs</th>
<th>Frequency</th>
</tr>
</thead>
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</tbody>
</table>
CHAPTER IV.

Numerical Examples.

4.1 Frequency-distribution of weights for 7749 Adult Males born in the United Kingdom (17, p. 10).

Table 1. The Data.

<table>
<thead>
<tr>
<th>Weight in lbs</th>
<th>Frequency</th>
<th>Weight in lbs</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>90 -</td>
<td>2</td>
<td>190 -</td>
<td>263</td>
</tr>
<tr>
<td>100 -</td>
<td>34</td>
<td>200 -</td>
<td>107</td>
</tr>
<tr>
<td>110 -</td>
<td>152</td>
<td>210 -</td>
<td>85</td>
</tr>
<tr>
<td>120 -</td>
<td>390</td>
<td>220 -</td>
<td>41</td>
</tr>
<tr>
<td>130 -</td>
<td>867</td>
<td>230 -</td>
<td>16</td>
</tr>
<tr>
<td>140 -</td>
<td>1623</td>
<td>240 -</td>
<td>11</td>
</tr>
<tr>
<td>150 -</td>
<td>1559</td>
<td>250 -</td>
<td>8</td>
</tr>
<tr>
<td>160 -</td>
<td>1326</td>
<td>260 -</td>
<td>1</td>
</tr>
<tr>
<td>170 -</td>
<td>787</td>
<td>270 -</td>
<td>1</td>
</tr>
<tr>
<td>180 -</td>
<td>476</td>
<td>280 -</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 2. The Preliminary Table.

<table>
<thead>
<tr>
<th>Provisional end values of the classes</th>
<th>Relative Cumulated Frequency</th>
<th>Normal quantiles</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>0.0243</td>
<td>-1.97208</td>
</tr>
<tr>
<td>-4</td>
<td>0.0746</td>
<td>-1.44236</td>
</tr>
<tr>
<td>-5</td>
<td>0.1865</td>
<td>-0.89087</td>
</tr>
<tr>
<td>-6</td>
<td>0.3959</td>
<td>-0.26397</td>
</tr>
<tr>
<td>-7</td>
<td>0.5971</td>
<td>0.2458</td>
</tr>
<tr>
<td>-8</td>
<td>0.7682</td>
<td>0.73293</td>
</tr>
<tr>
<td>-9</td>
<td>0.8698</td>
<td>1.12545</td>
</tr>
<tr>
<td>-10</td>
<td>0.9312</td>
<td>1.48479</td>
</tr>
<tr>
<td>-11</td>
<td>0.9652</td>
<td>1.81451</td>
</tr>
<tr>
<td>-12</td>
<td>0.9790</td>
<td>2.03352</td>
</tr>
</tbody>
</table>

We shall use octiles and hence we have to calculate the octiles of the data. We should therefore obtain the values of $u's$ -- the end values of the classes, corresponding to the normal quantiles $\mathbf{X} = \pm 1.15035$, $\pm 0.67449$, $\pm 0.31864$ and zero, by interpolation in Table 2. We presume that a cubic transformation is sufficient though it is found that the third order divided differences of $u$ are not constant. We use, therefore, four entries -- the two above and the two below the one we are seeking, and by Prof. Aitken's method (24, p. 71) we obtain the octiles in order:


Referring to the median 6.50507 as origin we get the vector \( \mathbf{u} \) of the octiles as:

\[
\mathbf{u} = \{-1.96613, -1.16397, -0.59706, 0.00000, 0.63565, 1.36469, 2.56146\}
\]

The second moment referred to the mean, \( m_2 \), with Sheppard's correction for grouping is equal to 4.459, \( m_3 \) is equal to 6.745, and hence the estimate of \( \sqrt{\beta_1} \) is equal to +0.72. Taking the Standard Results for octiles: level of skewness = 0.7 from Appendix III. and taking \( \mathbf{u} \) as given in (4.1), we obtain:

\[
\mathbf{u}'\mathbf{W}\mathbf{u} = 0.68057;
\]

\[
\begin{align*}
a_0 &= -0.72785; & a_0^2 \beta_0' \mathbf{W} \beta_0 &= 0.07501, \\
a_1 &= 1.87022; & a_1^2 \beta_1' \mathbf{W} \beta_1 &= 0.59672, \\
a_2 &= 0.20663; & a_2^2 \beta_2' \mathbf{W} \beta_2 &= 0.00744, \\
a_3 &= 0.08665; & a_3^2 \beta_3' \mathbf{W} \beta_3 &= 0.00085, \\
a_4 &= -0.01723; & a_4^2 \beta_4' \mathbf{W} \beta_4 &= 0.00001.
\end{align*}
\]

Now the standard error of \( a_4 \) can be easily found with the aid of Tables in Appendix IV to be 0.019 nearly, and hence the value of \( a_4 \) obtained is not significant. Also, the diminution in the residual quadratic error, made by the addition of \( a_4 \beta_4(x) \) to the cubic transformation, is negligible. Hence we shall accept the cubic transformation as the best one, though,
\[ z = 3 \sum a_{z_2} q_{z_2} W q_{z_2} \]

being equal to 0.68002, the residual quadratic error left is still 0.00055. The transformation \( q(x) \) is then,

\[
q(x) = -0.72785 + 1.87022 (x + 0.452873) + 0.20663 (x^2 + 0.36805 x - 0.53089) + 0.08665 (x^3 + 0.23617 x^2 - 1.07891 x - 0.14041)
\]

i.e. \( q(x) = 0.08665 x^3 + 0.22709 x^2 + 1.85278 x + 0.00274 \)

\[(4.2)\]

This is referred to the median 6.50507 as origin. Hence in the provisional scale and origin,

\[
q(x) = 0.08665 x^3 + 0.22709 x^2 + 1.85278 x + 6.50781
\]

\[(4.3)\]

The standard errors are easily found with the aid of the tables in Appendix IV to be:

- Standard error of the coefficient of \( x^3 \) = 0.012
- " " " " " " " " \( x^2 \) = 0.010
- " " " " " " " " \( x \) = 0.017
- " " " " " " " " constant term .... = 0.013

The weights in the data were taken to the nearest pound; hence the end values of the classes are really 99.5 lbs., 109.5 lbs., - - - - - - - - - -.
If \( \omega \) is the weight of an adult in lbs., the transformation is:

\[
\frac{\omega - 89.5}{10} = 0.08665 \, x^3 + 0.22709 \, x^2 + 1.85278 \, x + 6.50781
\]

i.e.

\[
\omega = 154.58 + 18.53 \, x + 2.27 \, x^2 + 0.87 \, x^3
\]

\( (4.4) \)

According to the transformation (4.2), the octiles are found to be,

\[-1.96000, \, -1.17022, \, -0.56738, \, 0.00274, \, 0.61897, \, 1.38232, \, 2.56650\]

which can be compared with the octiles of the data given in (4.1).

Putting \( \beta(x) \) equal to 1, 2, 3, \ldots \ldots \ldots \ldots \ldots in equation (4.3), solving these equations approximately for the proper values of \( x \) and using Sheppard's Table (27), we obtained the hypothetical cumulated frequencies. The following table compares the hypothetical class-frequencies with the observed class-frequencies and the hypothetical cumulated frequencies with the observed cumulated frequencies.
Table 3.

<table>
<thead>
<tr>
<th>End Values of the classes</th>
<th>Class-frequency</th>
<th>Cumulated-frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Hypothetical</td>
<td>Observed</td>
</tr>
<tr>
<td></td>
<td>Hypothetical</td>
<td>Observed</td>
</tr>
<tr>
<td>-2</td>
<td>52</td>
<td>36</td>
</tr>
<tr>
<td>-3</td>
<td>121</td>
<td>152</td>
</tr>
<tr>
<td>-4</td>
<td>376</td>
<td>390</td>
</tr>
<tr>
<td>-5</td>
<td>926</td>
<td>867</td>
</tr>
<tr>
<td>-6</td>
<td>1537</td>
<td>1623</td>
</tr>
<tr>
<td>-7</td>
<td>1648</td>
<td>1559</td>
</tr>
<tr>
<td>-8</td>
<td>1270</td>
<td>1326</td>
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<tr>
<td>-9</td>
<td>808</td>
<td>787</td>
</tr>
<tr>
<td>-10</td>
<td>463</td>
<td>476</td>
</tr>
<tr>
<td>-11</td>
<td>255</td>
<td>263</td>
</tr>
<tr>
<td>-12</td>
<td>136</td>
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<td>-13</td>
<td>73</td>
<td>85</td>
</tr>
<tr>
<td>-14</td>
<td>38</td>
<td>41</td>
</tr>
<tr>
<td>-15</td>
<td>30</td>
<td>16</td>
</tr>
<tr>
<td>-16</td>
<td>16</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>52</td>
<td>36</td>
</tr>
<tr>
<td></td>
<td>173</td>
<td>188</td>
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<tr>
<td></td>
<td>549</td>
<td>578</td>
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<tr>
<td></td>
<td>1475</td>
<td>1445</td>
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<td>3068</td>
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<tr>
<td></td>
<td>4660</td>
<td>4627</td>
</tr>
<tr>
<td></td>
<td>5930</td>
<td>5953</td>
</tr>
<tr>
<td></td>
<td>6738</td>
<td>6740</td>
</tr>
<tr>
<td></td>
<td>7201</td>
<td>7216</td>
</tr>
<tr>
<td></td>
<td>7456</td>
<td>7479</td>
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<tr>
<td></td>
<td>7592</td>
<td>7586</td>
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<td></td>
<td>7665</td>
<td>7671</td>
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<tr>
<td></td>
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<td></td>
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<td>7728</td>
</tr>
<tr>
<td></td>
<td>7749</td>
<td>7749</td>
</tr>
</tbody>
</table>
4.2 Frequency-distribution of Weights of 11382 Glasgow School Children. (31).

Table 4. The Data.

<table>
<thead>
<tr>
<th>Weight in lbs.</th>
<th>Frequency</th>
<th>Weight in lbs.</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>24 - 28</td>
<td>16</td>
<td>74 - 78</td>
<td>525</td>
</tr>
<tr>
<td>29 - 33</td>
<td>136</td>
<td>79 - 83</td>
<td>242</td>
</tr>
<tr>
<td>34 - 38</td>
<td>728</td>
<td>84 - 88</td>
<td>138</td>
</tr>
<tr>
<td>39 - 43</td>
<td>1476</td>
<td>89 - 93</td>
<td>66</td>
</tr>
<tr>
<td>44 - 48</td>
<td>1655</td>
<td>94 - 98</td>
<td>37</td>
</tr>
<tr>
<td>49 - 53</td>
<td>1706</td>
<td>99 - 103</td>
<td>19</td>
</tr>
<tr>
<td>54 - 58</td>
<td>1552</td>
<td>104 - 108</td>
<td>7</td>
</tr>
<tr>
<td>59 - 63</td>
<td>1297</td>
<td>109 - 113</td>
<td>1</td>
</tr>
<tr>
<td>64 - 68</td>
<td>1004</td>
<td>114 - 118</td>
<td></td>
</tr>
<tr>
<td>69 - 73</td>
<td>776</td>
<td>119 - 123</td>
<td>1</td>
</tr>
</tbody>
</table>
We shall use octiles. We use here also four entries - the two above and the two below the one we are seeking, and as in 4.1 obtain the octiles in order: 3.40753, 4.30443, 5.14924, 5.98433, 6.89289, 7.97451 and 9.46174. Referring to the median 5.98433 as origin, we get the vector $u$ of the octiles as,

$$u = \{2.57680, -1.67990, -0.83509, 0.00000, 0.90856, 1.99018, 3.47741\}.$$  (4.5)
The second moment referred to the mean, \( m_2 \), with Sheppard's correction for grouping is 6.647. \( m_3 \) is equal to 10.135 and hence the estimate of \( \sqrt{\beta_1} \) is +0.59. Taking the Standard Results for octiles: level of skewness = 0.6 from Appendix III, and taking \( u \) as given above in (4.5), we obtain:

\[
\begin{align*}
  u'Wu &= 1.19311; \\
  a_0 &= -0.80125; \quad a^2_0 q'Wq_0 = 0.08650, \\
  a_1 &= 2.54180; \quad a^2_1 q'Wq_1 = 1.08332, \\
  a_2 &= 0.35908; \quad a^2_2 q'Wq_2 = 0.02219, \\
  a_3 &= -0.09832; \quad a^2_3 q'Wq_3 = 0.00109,
\end{align*}
\]

\[
\sum_{i=0}^{3} a^2_i q'Wq_i = 1.19310.
\]

Hence we accept the cubic transformation as the best one. The transformation is:

\[
q(x) = -0.80125 + 2.54180 (x + 0.388131) \\
+ 0.35908 (x^2 + 0.31475 x - 0.54535) \\
- 0.09832 (x^3 + 0.20163 x^2 - 1.08666 x - 0.12010)
\]

i.e. \( q(x) = -0.09832 x^3 + 0.33925 x^2 + 2.76166 x \\
+ 0.00128 \).

(4.6)

This is referred to the median 5.98433 as origin. Hence in the provisional scale and origin,
\[ q(x) = -0.09832x^3 + 0.33925x^2 + 2.76166x + 5.98561 \]  

(4.7)

The standard errors are easily found with the aid of Tables in Appendix IV to be:

- Standard error of the coefficient of \( x^3 \) = 0.0099,
- \( x^2 \) = 0.0082,
- \( x \) = 0.0137,
- Constant term = 0.0106.

The end values of the various classes are clearly 28.5 lbs., 33.5 lbs. Hence if \( \omega \) is the weight of a child in lbs., the transformation is:

\[ \omega - 23.5 = -0.09832x^3 + 0.33925x^2 + 2.76166x + 5.98561 \]

\[ 5 \]

i.e. \( \omega = 53.43 + 13.81x + 1.70x^2 - 0.49x^3 \)  

(4.8)

There is a mode at \( x = -2.12 \) also when according to equation (4.7) \( q(x) \) i.e. \( \omega \) is equal to 2.6 nearly.

According to the transformation (4.6) the octiles are found to be,

- 2.57699, -1.67692, -0.84106, 0.00128, 0.91252,
- 1.98816, 3.47742

which can be compared with the octiles of the data given in (4.5).
Table 6.

<table>
<thead>
<tr>
<th>End Values of the classes</th>
<th>Class-frequency</th>
<th>Cumulated frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Hypothetical</td>
<td>Observed</td>
</tr>
<tr>
<td></td>
<td>845</td>
<td>880</td>
</tr>
<tr>
<td>-3</td>
<td>1495</td>
<td>1476</td>
</tr>
<tr>
<td>-4</td>
<td>1682</td>
<td>1655</td>
</tr>
<tr>
<td>-5</td>
<td>1693</td>
<td>1706</td>
</tr>
<tr>
<td>-6</td>
<td>1548</td>
<td>1552</td>
</tr>
<tr>
<td>-7</td>
<td>1306</td>
<td>1297</td>
</tr>
<tr>
<td>-8</td>
<td>1017</td>
<td>1004</td>
</tr>
<tr>
<td>-9</td>
<td>735</td>
<td>776</td>
</tr>
<tr>
<td>-10</td>
<td>489</td>
<td>525</td>
</tr>
<tr>
<td>-11</td>
<td>296</td>
<td>242</td>
</tr>
<tr>
<td>-12</td>
<td>161</td>
<td>138</td>
</tr>
<tr>
<td>-13</td>
<td>76</td>
<td>66</td>
</tr>
<tr>
<td>-14</td>
<td>39</td>
<td>65</td>
</tr>
</tbody>
</table>
Appendix I.

Sextiles and Octiles of Normal Curve with their powers.

(1) Sextiles

\[ \begin{align*}
\mathbf{\chi} &= \{ -0.9674216, -0.4307273, 0.0000000, \\
& \quad 0.4307273, \quad 0.9674216 \}.
\end{align*} \]

\[ \begin{align*}
\mathbf{\chi}^2 &= \{ 0.9359046, 0.1855260, 0.0000000, 0.1855260, \\
& \quad 0.9359046 \}.
\end{align*} \]

\[ \begin{align*}
\mathbf{\chi}^3 &= \{ -0.9054143, -0.0799111, 0.0000000, \\
& \quad 0.0799111, \quad 0.9054143 \}.
\end{align*} \]

(2) Octiles

\[ \begin{align*}
\mathbf{\chi} &= \{ -1.1503494, -0.6744898, -0.3186394, \\
& \quad 0.0000000, 0.3186394, 0.6744898, 1.1503494 \}.
\end{align*} \]

\[ \begin{align*}
\mathbf{\chi}^2 &= \{ 1.3233037, 0.4549364, 0.1015310, 0.0000000, \\
& \quad 0.1015310, \quad 0.4549364, \quad 1.3233037 \}.
\end{align*} \]

\[ \begin{align*}
\mathbf{\chi}^3 &= \{ -1.5222616, -0.3068500, -0.0323518, \\
& \quad 0.0000000, 0.0323518, 0.3068500, 1.5222616 \}.
\end{align*} \]

\[ \begin{align*}
\mathbf{\chi}^4 &= \{ 1.7511327, 0.2069671, 0.0103086, 0.0000000, \\
& \quad 0.0103086, \quad 0.2069671, \quad 1.7511327 \}.
\end{align*} \]
### Skewness at Sextiles

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.968081</td>
<td>-0.444096</td>
<td>0.416953</td>
<td>0.965945</td>
</tr>
<tr>
<td>0.2</td>
<td>0.967918</td>
<td>-0.457044</td>
<td>0.402781</td>
<td>0.963661</td>
</tr>
<tr>
<td>0.3</td>
<td>0.966928</td>
<td>-0.469554</td>
<td>0.388230</td>
<td>0.960573</td>
</tr>
<tr>
<td>0.4</td>
<td>0.965101</td>
<td>-0.481613</td>
<td>0.373307</td>
<td>0.956689</td>
</tr>
<tr>
<td>0.5</td>
<td>0.962425</td>
<td>-0.493202</td>
<td>0.358031</td>
<td>0.952011</td>
</tr>
<tr>
<td>0.6</td>
<td>0.958896</td>
<td>-0.504302</td>
<td>0.342411</td>
<td>0.946541</td>
</tr>
<tr>
<td>0.7</td>
<td>0.954502</td>
<td>-0.514894</td>
<td>0.326462</td>
<td>0.940286</td>
</tr>
<tr>
<td>0.8</td>
<td>0.949233</td>
<td>-0.524957</td>
<td>0.310200</td>
<td>0.933255</td>
</tr>
<tr>
<td>0.9</td>
<td>0.943082</td>
<td>-0.534465</td>
<td>0.293641</td>
<td>0.925448</td>
</tr>
<tr>
<td>1.0</td>
<td>0.936035</td>
<td>-0.543392</td>
<td>0.276801</td>
<td>0.916876</td>
</tr>
<tr>
<td>1.1</td>
<td>0.928093</td>
<td>-0.551712</td>
<td>0.259703</td>
<td>0.907545</td>
</tr>
</tbody>
</table>

### Ordinates at Sextiles

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.258253</td>
<td>0.369063</td>
<td>0.358611</td>
<td>0.242115</td>
</tr>
<tr>
<td>0.2</td>
<td>0.267400</td>
<td>0.375037</td>
<td>0.354070</td>
<td>0.234977</td>
</tr>
<tr>
<td>0.3</td>
<td>0.277383</td>
<td>0.381562</td>
<td>0.349947</td>
<td>0.228379</td>
</tr>
<tr>
<td>0.4</td>
<td>0.288308</td>
<td>0.388681</td>
<td>0.346223</td>
<td>0.222268</td>
</tr>
<tr>
<td>0.5</td>
<td>0.300301</td>
<td>0.396451</td>
<td>0.342878</td>
<td>0.216601</td>
</tr>
<tr>
<td>0.6</td>
<td>0.313508</td>
<td>0.404930</td>
<td>0.339897</td>
<td>0.211339</td>
</tr>
<tr>
<td>0.7</td>
<td>0.328103</td>
<td>0.414189</td>
<td>0.337266</td>
<td>0.206448</td>
</tr>
<tr>
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<td>0.344291</td>
<td>0.424308</td>
<td>0.334977</td>
<td>0.201896</td>
</tr>
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<td>0.362316</td>
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<td>0.333020</td>
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### Appendix II.

#### ORDINATES AT OCTILES.

OF PEARSON'S TYPE III CURVE

<table>
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<tbody>
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<td>0.388160</td>
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### Appendix II.

**ABSCISSA AT OCTILES OF PEARSON'S TYPE III CURVE.**

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<td>0.347959</td>
<td>0.033313</td>
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<tr>
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<td>1.130094</td>
<td>0.698930</td>
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<td>1.121523</td>
<td>0.705790</td>
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<td>1.101631</td>
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<table>
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<th>6th Octile</th>
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</tr>
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<tbody>
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</table>
Appendix III. Standard Results.

Sextiles: Skewness = 0.1

\[
W = \begin{bmatrix}
0.1334 & -0.0953 & . & . \\
0.0953 & 0.2724 & -0.1473 & . \\
. & 0.1473 & 0.3186 & 0.1431 \\
. & . & 0.1431 & 0.2572 \\
. & . & . & 0.0868 & 0.1172
\end{bmatrix}
\]

\[
q_0 \cdot W \cdot q_0 = 0.1537 ; \quad \text{Reciprocal} = 6.5058
\]
\[
q_1 \cdot W \cdot q_1 = 0.1806 ; \quad " = 5.5383
\]
\[
q_2 \cdot W \cdot q_2 = 0.1380 ; \quad " = 7.2480
\]
\[
q_3 \cdot W \cdot q_3 = 0.0537 ; \quad " = 18.616
\]

\[
q_1(x) = x + 0.0554 \\
q_2(x) = x^2 + 0.0374 x - 0.4838 \\
q_3(x) = x^3 + 0.0194 x^2 - 0.8425 x - 0.0096
\]

\[
a = \begin{bmatrix}
0.2477 & 0.1938 & 0.1833 & 0.1772 & 0.1979 \\
-0.4756 & -0.1300 & 0.0186 & 0.1567 & 0.4304 \\
0.6193 & -0.3915 & -0.4889 & -0.3314 & 0.5925 \\
-0.6945 & 1.5760 & -0.0465 & -1.5180 & 0.6830
\end{bmatrix}
\]

\[
u = \begin{bmatrix}
\end{bmatrix}
\]
**Sextiles** : Skewness = 0.2

\[
W = \begin{bmatrix}
0.1430 & 0.1003 & & \\
-0.1003 & 0.2813 & -0.1499 & \\
& 0.1499 & -0.3196 & -0.1415 \\
& & 0.1415 & 0.2507 & -0.0832 \\
& & & -0.0832 & 0.1104 \\
\end{bmatrix}
\]

\[
q'_0 W \beta_0 = 0.1552 ; \quad \text{Reciprocal} = 6.4442
\]

\[
q'_1 W \beta_1 = 0.1811 ; \quad " = 5.5224
\]

\[
q'_2 W \beta_2 = 0.1383 ; \quad " = 7.2316
\]

\[
q'_3 W \beta_3 = 0.0538 ; \quad " = 18.572
\]

\[
q'_1 (x) = x + 0.1107
\]

\[
q'_2 (x) = x^2 + 0.0749 x - 0.4819
\]

\[
q'_3 (x) = x^3 + 0.0389 x^2 - 0.8418 x - 0.0193
\]

\[
a = \begin{bmatrix}
0.2753 & 0.2005 & 0.1812 & 0.1676 & 0.1755 \\
-0.4993 & -0.1143 & 0.0371 & 0.1678 & 0.4087 \\
0.6329 & -0.4228 & -0.4870 & -0.3024 & 0.5793 \\
-0.7003 & 1.6051 & -0.0931 & -1.4889 & 0.6773 \\
\end{bmatrix}
\]

\[
u = -0.0931\]
Sextiles: Skewness = 0.3

\[ W = \begin{bmatrix} \begin{array}{cccc} 0.1539 & -0.1058 \\ -0.1058 & 0.2912 & -0.1529 \\ & -0.1529 & 0.3211 & -0.1402 \\ & & -0.1402 & 0.2449 & -0.0799 \\ & & & -0.0799 & 0.1043 \end{array} \end{bmatrix} \]

\[
q_0 \ W q_0 = 0.1577 ; \quad \text{Reciprocal} = 6.3423
\]
\[
q_1 \ W q_1 = 0.1820 ; \quad " = 5.4960
\]
\[
q_2 \ W q_2 = 0.1388 ; \quad " = 7.2042
\]
\[
q_3 \ W q_3 = 0.0541 ; \quad " = 18.498
\]

\[
q_1(x) = x + 0.1661
\]
\[
q_2(x) = x^2 + 0.1124 \ x - 0.4786
\]
\[
q_3(x) = x^3 + 0.0584 \ x^2 - 0.8406 \ x - 0.0290
\]

\[ a = \begin{bmatrix} \begin{array}{cccc} 0.3047 & 0.2058 & 0.1776 & 0.1572 & 0.1547 \\ -0.5238 & -0.0969 & 0.0555 & 0.1775 & 0.3877 \\ 0.6468 & -0.4550 & -0.4839 & -0.2741 & 0.5663 \\ -0.7061 & 1.6343 & -0.1399 & -1.4597 & 0.6715 \end{array} \end{bmatrix} \]

\[ x = 0.7061 \]
SEXTILES : SKEWNESS = 0.4

\[
W = \begin{bmatrix}
0.1662 & -0.1121 \\
-0.1121 & 0.3021 & -0.1563 \\
0.1563 & 0.3233 & -0.1392 \\
0.1392 & 0.2397 & -0.0770 \\
-0.0770 & 0.0988 & 0.1613
\end{bmatrix}
\]

\[q_0 W q_0 = 0.1613; \quad \text{Reciprocal} = 6.2010\]
\[q_1 W q_1 = 0.1832; \quad " = 5.4592\]
\[q_2 W q_2 = 0.1395; \quad " = 7.1660\]
\[q_3 W q_3 = 0.0544; \quad " = 18.396\]

\[q_1(x) = x + 0.2213\]
\[q_2(x) = x^2 + 0.1501 x - 0.4741\]
\[q_3(x) = x^3 + 0.0781 x^2 - 0.8390 x - 0.0387\]

\[
\begin{bmatrix}
0.3360 & 0.2096 & 0.1727 & 0.1462 & 0.1355 \\
-0.5490 & -0.0773 & 0.0738 & 0.1858 & 0.3673 \\
0.6608 & -0.4831 & -0.4795 & -0.2465 & 0.5533 \\
-0.7119 & 1.6636 & -0.1869 & 1.4304 & 0.6657
\end{bmatrix}
\]
SEXTILES : SKEWNESS = 0.5

\[ W = \begin{bmatrix}
0.1804 & -0.1191 & . & . & . \\
-0.1191 & 0.3143 & -0.1601 & . & . \\
. & -0.1601 & 0.3262 & -0.1385 & . \\
. & . & -0.1385 & 0.2351 & -0.0743 \\
. & . & . & -0.0743 & 0.0938
\end{bmatrix} \]

\[ q_0 W q_0 = 0.1661 ; \quad \text{Reciprocal} = 6.0218 \]
\[ q_1 W q_1 = 0.1848 ; \quad " = 5.4123 \]
\[ q_2 W q_2 = 0.1405 ; \quad " = 7.1170 \]
\[ q_3 W q_3 = 0.0547 ; \quad " = 18.265 \]

\[ q_1(x) = x + 0.2764 \]
\[ q_2(x) = x^2 + 0.1879 x - 0.4682 \]
\[ q_3(x) = x^3 + 0.0978 x^2 - 0.8369 x - 0.0484 \]

\[ a = \begin{bmatrix}
0.3692 & 0.2118 & 0.1664 & 0.1348 & 0.1178 \\
-0.5751 & -0.0568 & 0.0918 & 0.1923 & 0.3474 \\
0.6751 & -0.5222 & -0.4738 & -0.2195 & 0.5404 \\
-0.7177 & 1.6931 & -0.2343 & -1.4009 & 0.6598
\end{bmatrix} \]
Sextiles : Skewness = 0.6

\[ W = \begin{bmatrix} 0.1966 & -0.1269 & \cdot & \cdot & \cdot \\ -0.1269 & 0.3279 & -0.1644 & \cdot & \cdot \\ \cdot & -0.1644 & 0.3298 & -0.1380 & \cdot \\ \cdot & \cdot & -0.1380 & 0.2311 -0.0718 & \cdot \\ \cdot & \cdot & \cdot & -0.0718 & 0.0893 \end{bmatrix} \]

\[ q_0' W q_0 = 0.1722 ; \quad \text{Reciprocal} = 5.8066 \]
\[ q_1' W q_1 = 0.1867 ; \quad " = 5.3554 \]
\[ q_2' W q_2 = 0.1417 ; \quad " = 7.0574 \]
\[ q_3' W q_3 = 0.0552 ; \quad " = 18.105 \]

\[ q_1(x) = x + 0.3312 \]
\[ q_2(x) = x^2 + 0.2259 x - 0.4608 \]
\[ q_3(x) = x^3 + 0.1178 x^2 - 0.8342 x - 0.0582 \]

\[ a = \begin{bmatrix} 0.4043 & 0.2122 & 0.1588 & 0.1231 & 0.1016 \\ -0.6021 & -0.0339 & 0.1094 & 0.1984 & 0.3281 \\ 0.6897 & -0.5574 & -0.4668 & -0.1930 & 0.5275 \\ -0.7236 & 1.7229 & -0.2821 & -1.3711 & 0.6539 \end{bmatrix} \]
SEXTILES : SKEWNESS = 0.7

\[ W = \begin{bmatrix}
0.2153 & -0.1359 \\
-0.1359 & 0.3431 & -0.1693 \\
-0.1693 & 0.3341 & -0.1379 \\
-0.1379 & 0.2275 & -0.0696 \\
0.0696 & 0.0852 \\
\end{bmatrix} \]

\[ q_0 W q_0 = 0.1799 ; \quad \text{Reciprocal} = 5.5576 \]
\[ q_1 W q_1 = 0.1891 ; \quad " = 5.2889 \]
\[ q_2 W q_2 = 0.1431 ; \quad " = 6.9873 \]
\[ q_3 W q_3 = 0.0558 ; \quad " = 17.917 \]

\[ q_1(x) = x + 0.3858 \]
\[ q_2(x) = x^2 + 0.2641 x - 0.4521 \]
\[ q_3(x) = x^3 + 0.1379 x^2 - 0.3310 x - 0.0681 \]

\[ a = \begin{bmatrix}
0.4413 & 0.2107 & 0.1500 & 0.1112 & 0.0868 \\
0.6300 & -0.0089 & 0.1267 & 0.2028 & 0.3094 \\
0.7046 & -0.5938 & -0.4584 & -0.1671 & 0.5147 \\
0.7296 & 1.7530 & -0.3303 & -1.3410 & 0.6479 \\
\end{bmatrix} \]
\[
W = \begin{bmatrix}
0.2371 & -0.1461 & & & \\
-0.1461 & 0.3601 & -0.1748 & & \\
& -0.1748 & 0.3393 & -0.1380 & \\
& & -0.1380 & 0.2244 & -0.0676 \\
& & & -0.0676 & 0.0815
\end{bmatrix}
\]

\[
q_0'Wq_0 = 0.1895 \quad \text{Reciprocal} = 5.2776
\]

\[
q_1'Wq_1 = 0.1918 \quad " = 5.2132
\]

\[
q_2'Wq_2 = 0.1448 \quad " = 6.9069
\]

\[
q_3'Wq_3 = 0.0565 \quad " = 17.700
\]

\[
q_1(x) = x + 0.4400
\]

\[
q_2(x) = x^2 + 0.3026 x - 0.4418
\]

\[
q_3(x) = x^3 + 0.1583 x^2 - 0.8273 x - 0.0780
\]
SEXTILES

\[
W = \begin{bmatrix}
0.2625 & -0.1577 & \cdot & \cdot & \cdot \\
-0.1577 & 0.3791 & -0.1809 & \cdot & \cdot \\
\cdot & -0.1809 & 0.3453 & -0.1384 & \cdot \\
\cdot & \cdot & -0.1384 & 0.2218 & -0.0658 \\
\cdot & \cdot & \cdot & -0.0658 & 0.0781
\end{bmatrix}
\]

\[
q_0'Wg_0 = 0.2041 ; \quad \text{Reciprocal} = 4.9702
\]

\[
q_1'Wg_1 = 0.1950 ; \quad " = 5.1286
\]

\[
q_2'Wg_2 = 0.1467 ; \quad " = 6.8163
\]

\[
q_3'Wg_3 = 0.0573 ; \quad " = 17.457
\]

\[
ge_1(x) = xe + 0.4936
\]

\[
ge_2(x) = xe^2 + 0.3413 xe - 0.4298
\]

\[
ge_3(x) = xe^3 + 0.1789 xe^2 - 0.8230 xe - 0.0830
\]

\[
a = \begin{bmatrix}
0.5209 & 0.2012 & 0.1292 & 0.0876 & 0.0612 \\
-0.6888 & 0.0477 & 0.1593 & 0.2030 & 0.2734 \\
0.7355 & -0.6703 & -0.4373 & -0.1169 & 0.4891 \\
-0.7418 & 1.8144 & -0.4287 & -1.2796 & 0.6358
\end{bmatrix}
\]
Sextiles: Skewness = 1.0

\[ W = \begin{bmatrix}
0.2926 & -0.1712 & . & . & . \\
-0.1712 & 0.4005 & -0.1878 & . & . \\
. & -0.1878 & 0.3522 & -0.1391 & . \\
. & . & -0.1391 & 0.2196 & -0.0642 \\
. & . & . & -0.0642 & 0.0750
\end{bmatrix} \]

\[ q'_0 W q'_0 = 0.2156 \quad \text{Reciprocal} = 4.6391 \]
\[ q'_1 W q'_1 = 0.1986 \quad " = 5.0356 \]
\[ q'_2 W q'_2 = 0.1489 \quad " = 6.7160 \]
\[ q'_3 W q'_3 = 0.0582 \quad " = 17.186 \]

\[ q'_1(x) = x + 0.5465 \]
\[ q'_2(x) = x^2 + 0.3805 x - 0.4162 \]
\[ q'_3(x) = x^3 + 0.1998 x^2 - 0.8181 x - 0.0981 \]

\[ a = \begin{bmatrix}
0.5632 & 0.1929 & 0.1174 & 0.0760 & 0.0504 \\
-0.7199 & 0.0795 & 0.1754 & 0.2088 & 0.2562 \\
0.7515 & -0.7108 & -0.4244 & -0.0926 & 0.4763 \\
-0.7480 & 1.8457 & -0.4791 & 1.2482 & 0.6295
\end{bmatrix} \]
**SEXTILES : SKEWNESS = 1.1**

\[ W = \begin{bmatrix}
0.3282 & -0.1867 & \\
-0.1867 & 0.4247 & -0.1955 \\
  & -0.1955 & 0.3601 -0.1401 \\
  &  & -0.1401 0.2179 -0.0627 \\
  &  &  & -0.0627 0.0722
\end{bmatrix} \]

\[ q_0 W q_0 = 0.2332 ; \text{ Reciprocal} = 4.2889 \]
\[ q_1 W q_1 = 0.2026 ; \quad " = 4.9346 \]
\[ q_2 W q_2 = 0.1514 ; \quad " = 6.6062 \]
\[ q_3 W q_3 = 0.0592 ; \quad " = 16.889 \]

\[ q_1(x) = x^3 + 0.5985 \]
\[ q_2(x) = x^2 + 0.4199 x - 0.4008 \]
\[ q_3(x) = x^3 + 0.2211 x^2 - 0.8126 x - 0.1083 \]

\[ \mathbf{a} = \begin{bmatrix}
0.6071 & 0.1823 & 0.1050 & 0.0649 & 0.0407 \\
-0.7521 & 0.1140 & 0.1902 & 0.2084 & 0.2395 \\
0.7679 & -0.7527 & -0.4098 & -0.0688 & 0.4634 \\
-0.7543 & 1.8777 & -0.5303 & -1.2163 & 0.6232
\end{bmatrix} \]

\[ \mathbf{u} \]
OCTILES : SKEWNESS = 0.1

\[ W = \begin{bmatrix}
0.0917 & -0.0696 & . & . & . & . \\
-0.0696 & 0.2115 & -0.1247 & . & . & . \\
. & -0.1247 & 0.2941 & -0.1530 & . & . \\
. & . & -0.1530 & 0.3186 & -0.1498 & . \\
. & . & . & -0.1498 & 0.2918 & -0.1167 \\
. & . & . & . & -0.1167 & 0.1933 & -0.0617 \\
. & . & . & . & . & -0.0617 & 0.0786
\end{bmatrix} \]

\[ g_0 W g_0 = 0.1185 \] ; Reciprocal = 8.4354
\[ g_1 W g_1 = 0.1601 \] ; " = 6.2477
\[ g_2 W g_2 = 0.1684 \] ; " = 6.0085
\[ g_3 W g_3 = 0.1085 \] ; " = 9.2178
\[ g_4 W g_4 = 0.0432 \] ; " = 23.152

\[ g_1 (x) = x + 0.0646 \]
\[ g_2 (x) = x^2 + 0.0521 x - 0.5829 \]
\[ g_3 (x) = x^3 + 0.0332 x^2 - 1.1068 x - 0.0199 \]
\[ g_4 (x) = x^4 + 0.0208 x^3 - 1.4525 x^2 - 0.0233 x + 0.2160 \]

\[ a = \begin{bmatrix}
0.1360 & 0.1451 & 0.1376 & 0.1328 & 0.1289 & 0.1263 & 0.1433 \\
-0.3566 & -0.1358 & -0.0533 & 0.0128 & 0.0754 & 0.1453 & 0.3122 \\
0.4431 & -0.1189 & -0.216 & -0.2396 & -0.1972 & -0.0784 & 0.4126 \\
-0.4692 & 0.6432 & 0.3516 & -0.0218 & -0.3694 & -0.5849 & 0.4506 \\
0.4536 & -1.4309 & 0.4770 & 1.0400 & 0.3889 & -1.3723 & 0.4437
\end{bmatrix} \]
W=
\[
\begin{bmatrix}
0.0996 & -0.0744 & . & . & . & . & .
\end{bmatrix}
\]
\[
\begin{bmatrix}
-0.0744 & 0.2222 & -0.1294 & . & . & . & .
\end{bmatrix}
\]
\[
\begin{bmatrix}
-0.1294 & 0.3013 & -0.1552 & . & . & . & .
\end{bmatrix}
\]
\[
\begin{bmatrix}
-0.1552 & 0.3196 & -0.1487 & . & . & . & .
\end{bmatrix}
\]
\[
\begin{bmatrix}
-0.1487 & 0.2767 & -0.1133 & . & . & . & .
\end{bmatrix}
\]
\[
\begin{bmatrix}
-0.1133 & 0.1855 & -0.05 & . & . & . & .
\end{bmatrix}
\]
\[
\begin{bmatrix}
-0.05 & 0.07 & . & . & . & . & .
\end{bmatrix}
\]
\[
\begin{bmatrix}
0.0732 & . & . & . & . & . & .
\end{bmatrix}
\]

$g_0 W g_0 = 0.1198$; Reciprocal = 8.3461

$g_1 W g_1 = 0.1607$; 

$g_2 W g_2 = 0.1607$; 

$g_3 W g_3 = 0.1088$; 

$g_4 W g_4 = 0.0433$;

$g_1(x) = x + 0.1292$

$g_2(x) = x^2 + 0.1043 x - 0.5798$

$g_3(x) = x^3 + 0.0666 x^2 - 1.1052 x - 0.0398$

$g_4(x) = x^4 + 0.0417 x^3 - 1.4518 x^2 - 0.0467 x + 0.2156$

\[
\begin{bmatrix}
0.2105 & 0.1537 & 0.1403 & 0.1311 & 0.1232 & 0.1164 & 0.1248
\end{bmatrix}
\]
\[
\begin{bmatrix}
0.3805 & -0.1287 & -0.0410 & 0.0254 & 0.0851 & 0.1481 & 0.2916
\end{bmatrix}
\]
\[
\begin{bmatrix}
0.4590 & -0.1412 & -0.2332 & -0.2381 & -0.1844 & -0.0600 & 0.3979
\end{bmatrix}
\]
\[
\begin{bmatrix}
0.4788 & 0.6736 & 0.3408 & -0.0436 & -0.3765 & -0.5569 & 0.4414
\end{bmatrix}
\]
\[
\begin{bmatrix}
0.4586 & -1.4607 & 0.5227 & 1.0379 & 0.3462 & -1.3434 & 0.4387
\end{bmatrix}
\]
OCTILES: SKEWNESS = 0.3

\[
W = \begin{bmatrix}
0.1086 & -0.0797 \\
0.0797 & 0.2340 & -0.1345 \\
0.1345 & 0.3095 & -0.1576 \\
0.1576 & 0.3211 & -0.1479 \\
0.1479 & 0.2723 & -0.1102 \\
0.1102 & 0.1784 & -0.0552 \\
0.0552 & 0.0684
\end{bmatrix}
\]

\[
\begin{align*}
g'_0 W g'_0 &= 0.1220 ; & \text{Reciprocal} &= 8.1983 \\
g'_1 W g'_1 &= 0.1618 ; & \\
g'_2 W g'_2 &= 0.1677 ; & \\
g'_3 W g'_3 &= 0.1093 ; & \\
g'_4 W g'_4 &= 0.0435 ; & \\
\end{align*}
\]

\[
\begin{align*}
g_1(x) &= x + 0.1938 \\
g_2(x) &= x^2 + 0.1566 x - 0.5745 \\
g_3(x) &= x^3 + 0.1000 x^2 - 1.1023 x - 0.0598 \\
g_4(x) &= x^4 + 0.0626 x^3 - 1.4507 x^2 - 0.0701 x + 0.2149
\end{align*}
\]

\[
a = \begin{bmatrix}
1 \\
0.2378 & 0.1616 & 0.1419 & 0.1282 & 0.1167 & 0.1064 & 0.1080 \\
0.4055 & -0.1200 & -0.0279 & 0.0380 & 0.0939 & 0.1496 & 0.2719 \\
0.4754 & -0.1650 & -0.2443 & -0.2356 & -0.1713 & -0.0428 & 0.3836 \\
0.4885 & 0.7050 & 0.3286 & -0.0654 & -0.3825 & -0.5296 & 0.4324 \\
0.4637 & -1.4910 & 0.5696 & 1.0342 & 0.3045 & -1.3148 & 0.4338
\end{bmatrix}
\]
OCTILES : SKEWNESS = 0.4

\[ W = \begin{bmatrix}
  0.1191 & -0.0858 & & & \\
  -0.0858 & 0.2472 & -0.1403 & & \\
  -0.1403 & 0.3186 & -0.1605 & & \\
  -0.1605 & 0.3233 & -0.1473 & & \\
  -0.1473 & 0.2685 & -0.107 & & \\
  -0.107 & 0.1720 & -0.0525 & & \\
  & & & & \\
\end{bmatrix} \]

\[ q_0' W q_0 = 0.1251 \quad \text{Reciprocal} = 7.9935 \]
\[ q_1' W q_1 = 0.1633 \quad " = 6.1245 \]
\[ q_2' W q_2 = 0.1688 \quad " = 5.9235 \]
\[ q_3' W q_3 = 0.1101 \quad " = 9.0861 \]
\[ q_4' W q_4 = 0.0438 \quad " = 22.828 \]

\[ g_1(x) = x + 0.2586 \]
\[ g_2(x) = x^2 + 0.2091 x - 0.5671 \]
\[ g_3(x) = x^3 + 0.1336 x^2 - 1.0983 x - 0.0798 \]
\[ g_4(x) = x^4 + 0.0837 x^3 - 1.4491 x^2 - 0.0936 x + 0.2139 \]

\[ a = \begin{bmatrix}
  0.2662 & 0.1685 & 0.1423 & 0.1243 & 0.1095 & 0.0964 & 0.0928 \\
  -0.4319 & -0.1095 & -0.0139 & 0.0503 & 0.1017 & 0.1501 & 0.2532 \\
  0.4924 & -0.1904 & -0.2548 & -0.2321 & -0.1579 & -0.0267 & 0.3696 \\
  -0.4985 & 0.7375 & 0.3150 & -0.0872 & -0.3873 & -0.5030 & 0.4234 \\
  0.4688 & -1.5217 & 0.6177 & 1.0291 & 0.2635 & -1.2863 & 0.4289 \\
\end{bmatrix} \]
**OCTILES**:  \text{SKWENESS} = 0.5

\[
W = \begin{bmatrix}
0.1312 & -0.0927 & & & & & \\
-0.0927 & 0.2620 & -0.1467 & & & & \\
 & -0.1467 & 0.3287 & -0.1637 & & & \\
 & & -0.1637 & 0.3262 & -0.1471 & & \\
 & & & -0.1471 & 0.2652 & -0.1050 & \\
 & & & & -0.1050 & 0.1662 & -0.0500 \\
 & & & & & -0.0500 & 0.0603
\end{bmatrix}
\]

\[
\begin{align*}
&\bar{W}_0 = 0.1293 \quad \text{Reciprocal} = 7.7339 \\
&\bar{W}_1 = 0.1652 \\
&\bar{W}_2 = 0.1703 \\
&\bar{W}_3 = 0.1110 \\
&\bar{W}_4 = 0.0442
\end{align*}
\]

\[
\begin{align*}
&\bar{g}_1(x) = x + 0.3233 \\
&\bar{g}_2(x) = x^2 + 0.2618 x - 0.5574 \\
&\bar{g}_3(x) = x^3 + 0.1675 x^2 - 1.0931 x - 0.0999 \\
&\bar{g}_4(x) = x^4 + 0.1049 x^3 - 1.4470 x^2 - 0.1173 x + 0.2126
\end{align*}
\]

\[
\alpha = \begin{bmatrix}
0.2977 & 0.1744 & 0.1412 & 0.1193 & 0.1018 & 0.0865 & 0.0791 \\
0.4596 & -0.0970 & 0.0008 & 0.0623 & 0.1087 & 0.1495 & 0.2353 \\
0.5100 & -0.2176 & -0.2647 & -0.2276 & -0.1443 & -0.0116 & 0.3559 \\
0.5087 & 0.7712 & 0.2999 & -0.1089 & -0.3910 & -0.4770 & 0.4145 \\
0.4741 & -1.5531 & 0.6674 & 1.0224 & 0.2232 & -1.2578 & 0.4240
\end{bmatrix}
\]
\[ W = \begin{bmatrix} 0.1453 & -0.1006 \\ -0.1006 & 0.2786 & -0.1539 \\ & -0.1539 & 0.3400 & -0.1674 \\ & & -0.1674 & 0.3298 & -0.1471 \\ & & & -0.1471 & 0.2625 & -0.1028 \\ & & & & -0.1028 & 0.1609 & -0.0478 \\ & & & & & -0.0478 & 0.0568 \end{bmatrix} \]

\[ g_0 \, w_0 = 0.1347 ; \quad \text{Reciprocal} = 7.4223 \]
\[ g_1 \, w_1 = 0.1677 ; \quad \text{Reciprocal} = 5.9638 \]
\[ g_2 \, w_2 = 0.1721 ; \quad \text{Reciprocal} = 5.8113 \]
\[ g_3 \, w_3 = 0.1122 ; \quad \text{Reciprocal} = 8.9117 \]
\[ g_4 \, w_4 = 0.0446 ; \quad \text{Reciprocal} = 22.398 \]

\[ g_1(x) = x + 0.3881 \]
\[ g_2(x) = x^2 + 0.3148 x - 0.5454 \]
\[ g_3(x) = x^3 + 0.2016 x^2 - 1.0867 x - 0.1201 \]
\[ g_4(x) = x^4 + 0.1264 x^3 - 1.4445 x^2 - 0.1413 x + 0.2110 \]

\[ a = \begin{bmatrix} 0.3319 & 0.1788 & 0.1389 & 0.1133 & 0.0935 & 0.0768 & 0.0668 \\ -0.4383 & -0.0322 & 0.0161 & 0.0740 & 0.1146 & 0.1480 & 0.2183 \\ 0.5282 & -0.2466 & -0.2738 & -0.2221 & -0.1305 & 0.0024 & 0.3425 \\ -0.5191 & 0.8060 & 0.2831 & -0.1306 & -0.3936 & -0.4515 & 0.4057 \\ 0.4794 & -1.5851 & 0.7185 & 1.0141 & 0.1835 & -1.2295 & 0.4190 \end{bmatrix} \]
\begin{align*}
\mathbf{W} &= \\
\begin{bmatrix}
0.1619 & -0.1097 & 0.2973 & -0.1619 & 0.3527 & -0.1716 & -0.1619 & 0.3527 & -0.1716 & 0.1475 \\
-0.1097 & 0.2973 & -0.1619 & 0.3527 & -0.1716 & 0.3341 & -0.1716 & 0.3341 & -0.1716 & 0.2603 \\
0.2973 & -0.1619 & 0.3527 & -0.1716 & 0.3341 & -0.1475 & 0.3341 & -0.1475 & 0.2603 & -0.1008 \\
-0.1619 & 0.3527 & -0.1716 & 0.3341 & -0.1475 & 0.2603 & -0.1008 & 0.1561 & -0.0458 & 0.0537 \\
0.1475 & -0.1716 & 0.2603 & -0.1008 & 0.1561 & -0.0458 & 0.0537 & 0.0458 & 0.1416 \\
\end{bmatrix}
\end{align*}

\begin{align*}
\mathbf{a} &= \\
\begin{bmatrix}
0.3688 & 0.1816 & 0.1350 & 0.1063 & 0.0850 & 0.0674 & 0.0559 \\
-0.5195 & -0.0651 & 0.0322 & 0.0851 & 0.1196 & 0.1456 & 0.2020 \\
0.5471 & -0.2776 & -0.2822 & -0.2155 & -0.1166 & 0.0155 & 0.3293 \\
-0.5293 & 0.8423 & 0.2646 & -0.1522 & -0.3952 & -0.4265 & -0.3969 \\
0.4843 & -1.6178 & 0.7714 & 1.0042 & 0.1445 & -1.2011 & 0.4141 \\
\end{bmatrix}
\end{align*}
\[ W = \begin{bmatrix}
-0.1816 & -0.1203 \\
-0.1203 & 0.3186 & -0.1709 \\
& -0.1709 & 0.3668 & -0.1764 \\
& & -0.1764 & 0.3393 & -0.1481 \\
& & & -0.1481 & 0.2586 & -0.0990 \\
& & & & -0.0990 & 0.1517 & -0.0439 \\
& & & & & -0.0439 & 0.0509
\end{bmatrix} \]

\[ g_0 W g_0 = 0.1502 ; \quad \text{Reciprocal} = 6.6589 \]
\[ g_1 W g_1 = 0.1740 ; \quad " = 5.7457 \]
\[ g_2 W g_2 = 0.1768 ; \quad " = 5.6565 \]
\[ g_3 W g_3 = 0.1153 ; \quad " = 8.6702 \]
\[ g_4 W g_4 = 0.0459 ; \quad " = 21.802 \]

\[ g_1(x) = x + 0.5175 \]
\[ g_2(x) = x^2 + 0.4217 x - 0.5139 \]
\[ g_3(x) = x^3 + 0.2711 x^2 - 1.0698 x - 0.1608 \]
\[ g_4(x) = x^4 + 0.1701 x^3 - 1.4377 x^2 - 0.1899 x + 0.2068 \]

\[ a = \begin{bmatrix}
0.4084 & 0.1825 & 0.1297 & 0.0985 & 0.0762 & 0.0584 & 0.0462 \\
0.5519 & -0.0454 & 0.0483 & 0.0958 & 0.1236 & 0.1425 & 0.1866 \\
0.5667 & -0.3109 & -0.2896 & -0.2079 & -0.1024 & 0.0276 & 0.3164 \\
-0.5408 & 0.8800 & 0.2441 & -0.1737 & -0.3957 & -0.4020 & 0.3881 \\
0.4903 & -0.1651 & 0.8261 & 0.9925 & 0.1060 & -1.1727 & 0.4091
\end{bmatrix} \]
\[
W = \begin{bmatrix}
0.2051 & -0.1326 & & & & \\
-0.1326 & 0.3428 & -0.1811 & & & \\
& -0.1811 & 0.3826 & -0.1817 & & \\
& & -0.1817 & 0.3453 & -0.1491 & \\
& & & -0.1491 & 0.2574 & -0.0975 \\
& & & & -0.0975 & 0.1478 & -0.0422 \\
& & & & & -0.0422 & 0.0483 \\
\end{bmatrix}
\]

\[
\begin{align*}
\frac{9}{10}W_{fo} &= 0.1608; \\
\frac{9}{2}W_{fo} &= 0.1780; \\
\frac{9}{2}W_{fo} &= 0.1797; \\
\frac{9}{3}W_{fo} &= 0.1173; \\
\frac{9}{4}W_{fo} &= 0.0466;
\end{align*}
\]

Reciprocal = 6.2170

\[
\begin{align*}
\frac{9}{1}(x) &= x + 0.5817 \\
\frac{9}{2}(x) &= x^2 + 0.4758x - 0.4941 \\
\frac{9}{3}(x) &= x^3 + 0.3066x^2 - 1.0592x - 0.1314 \\
\frac{9}{4}(x) &= x^4 + 0.1924x^3 - 1.4335x^2 - 0.2416x + 0.2042 \\
\end{align*}
\]

\[
\mathbf{a} = \begin{bmatrix}
0.4508 & 0.1813 & 0.1230 & 0.0300 & 0.0673 & 0.0499 & 0.0376 \\
-0.5860 & -0.0228 & 0.0659 & 0.1058 & 0.1266 & 0.1386 & 0.1718 \\
0.5372 & 0.3465 & -0.2960 & -0.1991 & -0.0882 & 0.0389 & 0.3037 \\
-0.5522 & 0.9194 & 0.2217 & -0.1951 & -0.3952 & -0.3779 & 0.3793 \\
0.4959 & -1.6857 & 0.3829 & 0.9791 & 0.0681 & -1.1442 & 0.4040 \\
\end{bmatrix}
\]
OCTILES : SKEWNESS = 1.0

\[
\begin{bmatrix}
0.2334 & -0.1470 \\
-0.1470 & 0.3705 & -0.1926 \\
& -0.1926 & 0.4003 & -0.1877 \\
& & -0.1877 & 0.3522 & -0.1503 \\
& & & +0.1503 & 0.2567 & -0.0962 \\
& & & & -0.0962 & 0.1442 & -0.0407 \\
& & & & & -0.0407 & 0.0431
\end{bmatrix}
\]

\[
\begin{align*}
q_0 \ Wq_0 &= 0.1741 & \quad \text{Reciprocal} &= 5.7431 \\
q_1 \ Wq_1 &= 0.1826 & \quad " &= 5.4766 \\
q_2 \ Wq_2 &= 0.1831 & \quad " &= 5.4612 \\
q_3 \ Wq_3 &= 0.1196 & \quad " &= 5.3644 \\
q_4 \ Wq_4 &= 0.0475 & \quad " &= 21.045
\end{align*}
\]

\[
\begin{align*}
q_1 (x) &= x + 0.6453 \\
q_2 (x) &= x^2 + 0.5304 x - 0.4715 \\
q_3 (x) &= x^3 + 0.3427 x^2 - 1.0472 x - 0.2021 \\
q_4 (x) &= x^4 + 0.2151 x^3 - 1.4286 x^2 - 0.2398 x + 0.2011
\end{align*}
\]

\[
a = \begin{bmatrix}
0.4960 & 0.1777 & 0.1148 & 0.0809 & 0.0585 & 0.0419 & 0.0302 \\
-0.6221 & 0.0030 & 0.0834 & 0.1150 & 0.1286 & 0.1341 & 0.1578 \\
0.6086 & -0.3847 & -0.3012 & -0.192 & -0.0739 & 0.0493 & 0.2912 \\
-0.5639 & 0.9605 & 0.1970 & -0.2164 & -0.3936 & -0.3542 & 0.3706 \\
0.5017 & -1.7211 & 0.9418 & 0.9637 & 0.0306 & -1.1155 & 0.3939
\end{bmatrix}
\]

\[
u = \text{Reciprocal}
\]
### OCTILES : SKEWNESS = 1.1

<table>
<thead>
<tr>
<th>W</th>
<th></th>
</tr>
</thead>
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<tr>
<td>0.2678</td>
<td>-0.1642</td>
</tr>
<tr>
<td>-0.1642</td>
<td>0.4024</td>
</tr>
<tr>
<td></td>
<td>-0.2056</td>
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</tbody>
</table>

\[\begin{align*}
q_0' & = 0.1907 \\
q_1' & = 0.1878 \\
q_2' & = 0.1869 \\
q_3' & = 0.1221 \\
q_4' & = 0.0485 \\
\end{align*}\]

\[\begin{align*}
q_1(x) & = x + 0.7081 \\
q_2(x) & = x^2 + 0.5855x - 0.4458 \\
q_3(x) & = x^3 + 0.3793x^2 - 1.0334x - 0.2229 \\
q_4(x) & = x^4 + 0.2383x^3 - 1.4231x^2 - 0.2653x + 0.1977 \\
\end{align*}\]

\[\begin{align*}
\mathbf{a} & = \begin{bmatrix}
0.5437 & 0.1714 & 0.1053 & 0.0714 & 0.0499 & 0.0345 & 0.0238 \\
-0.6601 & 0.0322 & 0.1013 & 0.1234 & 0.1297 & 0.1290 & 0.1445 \\
0.6310 & -0.4258 & -0.3050 & -0.1782 & -0.0596 & 0.0588 & 0.2789 \\
-0.5761 & 1.0036 & 0.1700 & -0.2374 & 0.3909 & -0.3310 & 0.3618 \\
0.5076 & -1.7576 & 1.0031 & 0.9462 & -0.0064 & -1.0867 & 0.3938 \\
\end{bmatrix}
\end{align*}\]
APPENDIX IV.

Variance of Solutions
Sextiles; Quadratic Transformation.

The transformation is \( a + b \, x + c \, x^2 \). The result is to be divided by \( N \), the number of observations in the sample.

| Skewness Constant term; Coeff. of \( x \); Coeff. of \( x^2 \) |
|-----------------|-----------------|-----------------|
| 0.1  | 1.3699 | 0.9247 | 1.2080 |
| 0.2  | 1.3652 | 0.9272 | 1.2053 |
| 0.3  | 1.3574 | 0.9312 | 1.2007 |
| 0.4  | 1.3465 | 0.9368 | 1.1943 |
| 0.5  | 1.3325 | 0.9439 | 1.1862 |
| 0.6  | 1.3155 | 0.9526 | 1.1762 |
| 0.7  | 1.2955 | 0.9627 | 1.1645 |
| 0.8  | 1.2725 | 0.9742 | 1.1511 |
| 0.9  | 1.2466 | 0.9871 | 1.1361 |
| 1.0  | 1.2178 | 1.0013 | 1.1193 |
| 1.1  | 1.1863 | 1.0166 | 1.1010 |
Variances of Solutions.

Sextiles: Cubic Transformation

The transformation is \( a + b \cdot x + c \cdot x^2 + d \cdot x^3 \).

The result is to be divided by \( N \), the number of observations in the sample.

| Skewness | Constant term; Coeff. of \( x \); Coeff. of \( x^2 \); Coeff. of \( x^3 \) |
|----------|---------------------------------|-------------------------------------------------|-------------------------------------------------|
| 0.1      | 1.3702                          | 3.1270                                          | 1.2092                                          | 3.1026                                          |
| 0.2      | 1.3663                          | 3.1205                                          | 1.2099                                          | 3.0953                                          |
| 0.3      | 1.3600                          | 3.1099                                          | 1.2112                                          | 3.0831                                          |
| 0.4      | 1.3511                          | 3.0949                                          | 1.2130                                          | 3.0660                                          |
| 0.5      | 1.3397                          | 3.0758                                          | 1.2153                                          | 3.0441                                          |
| 0.6      | 1.3257                          | 3.0525                                          | 1.2181                                          | 3.0175                                          |
| 0.7      | 1.3093                          | 3.0250                                          | 1.2213                                          | 2.9861                                          |
| 0.8      | 1.2904                          | 2.9935                                          | 1.2250                                          | 2.9501                                          |
| 0.9      | 1.2691                          | 2.9579                                          | 1.2291                                          | 2.9094                                          |
| 1.0      | 1.2454                          | 2.9184                                          | 1.2337                                          | 2.8643                                          |
| 1.1      | 1.2193                          | 2.8752                                          | 1.2386                                          | 2.8148                                          |
Variances of Solutions.

Octiles: Quadratic Transformation

The transformation is $a + b \cdot x + c \cdot x^2$. The result is to be divided by $N$, the number of observations in the sample.

<table>
<thead>
<tr>
<th>Skewness</th>
<th>Constant term</th>
<th>Cof.f. of $x$</th>
<th>Cof.f. of $x^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.3129</td>
<td>0.7830</td>
<td>0.7511</td>
</tr>
<tr>
<td>0.2</td>
<td>1.3080</td>
<td>0.7860</td>
<td>0.7489</td>
</tr>
<tr>
<td>0.3</td>
<td>1.2999</td>
<td>0.7910</td>
<td>0.7454</td>
</tr>
<tr>
<td>0.4</td>
<td>1.2885</td>
<td>0.7979</td>
<td>0.7404</td>
</tr>
<tr>
<td>0.5</td>
<td>1.2739</td>
<td>0.8068</td>
<td>0.7341</td>
</tr>
<tr>
<td>0.6</td>
<td>1.2561</td>
<td>0.8174</td>
<td>0.7264</td>
</tr>
<tr>
<td>0.7</td>
<td>1.2353</td>
<td>0.8299</td>
<td>0.7174</td>
</tr>
<tr>
<td>0.8</td>
<td>1.2114</td>
<td>0.8440</td>
<td>0.7071</td>
</tr>
<tr>
<td>0.9</td>
<td>1.1845</td>
<td>0.8596</td>
<td>0.6955</td>
</tr>
<tr>
<td>1.0</td>
<td>1.1548</td>
<td>0.8756</td>
<td>0.6826</td>
</tr>
<tr>
<td>1.1</td>
<td>1.1223</td>
<td>0.8949</td>
<td>0.6687</td>
</tr>
</tbody>
</table>
Variance of Solutions.

Octiles: Cubic Transformation.

The transformation is \( a + bx + cx^2 + dx^3 \).

The result is to be divided by \( N \), the number of observations in the sample.

<table>
<thead>
<tr>
<th>Skewness</th>
<th>Constant term ( a )</th>
<th>Coeff. of ( x ) ( b )</th>
<th>Coeff. of ( x^2 ) ( c )</th>
<th>Coeff. of ( x^3 ) ( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.3134</td>
<td>2.1946</td>
<td>0.7523</td>
<td>1.1522</td>
</tr>
<tr>
<td>0.2</td>
<td>1.3098</td>
<td>2.1892</td>
<td>0.7540</td>
<td>1.1489</td>
</tr>
<tr>
<td>0.3</td>
<td>1.3039</td>
<td>2.1804</td>
<td>0.7568</td>
<td>1.1434</td>
</tr>
<tr>
<td>0.4</td>
<td>1.2957</td>
<td>2.1680</td>
<td>0.7607</td>
<td>1.1358</td>
</tr>
<tr>
<td>0.5</td>
<td>1.2851</td>
<td>2.1521</td>
<td>0.7657</td>
<td>1.1259</td>
</tr>
<tr>
<td>0.6</td>
<td>1.2722</td>
<td>2.1329</td>
<td>0.7717</td>
<td>1.1140</td>
</tr>
<tr>
<td>0.7</td>
<td>1.2570</td>
<td>2.1102</td>
<td>0.7787</td>
<td>1.0999</td>
</tr>
<tr>
<td>0.8</td>
<td>1.2394</td>
<td>2.0843</td>
<td>0.7867</td>
<td>1.0838</td>
</tr>
<tr>
<td>0.9</td>
<td>1.2196</td>
<td>2.0552</td>
<td>0.7956</td>
<td>1.0656</td>
</tr>
<tr>
<td>1.0</td>
<td>1.1975</td>
<td>2.0231</td>
<td>0.8054</td>
<td>1.0455</td>
</tr>
<tr>
<td>1.1</td>
<td>1.1731</td>
<td>1.9381</td>
<td>0.8160</td>
<td>1.0236</td>
</tr>
</tbody>
</table>
Variances of Solutions.

Octiles : Quartic Transformation.

The transformation is \( a + b x + c x^2 + d x^3 + e x^4 \).

The result is to be divided by \( N \), the number of observations in the sample.

<table>
<thead>
<tr>
<th>Skewness</th>
<th>Const. term; Coeff. of ( x ); Coeff. of ( x^2 ); Coeff. of ( x^3 ); Coeff. of ( x^4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.4483, 2.1962, 6.8579, 1.1535, 2.8940</td>
</tr>
<tr>
<td>0.2</td>
<td>1.4439, 2.1955, 6.8368, 1.1539, 2.8837</td>
</tr>
<tr>
<td>0.3</td>
<td>1.4365, 2.1945, 6.8018, 1.1547, 2.8724</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4262, 2.1930, 6.7528, 1.1557, 2.8535</td>
</tr>
<tr>
<td>0.5</td>
<td>1.4130, 2.1911, 6.6899, 1.1571, 2.8293</td>
</tr>
<tr>
<td>0.6</td>
<td>1.3968, 2.1887, 6.6134, 1.1587, 2.7998</td>
</tr>
<tr>
<td>0.7</td>
<td>1.3778, 2.1859, 6.5233, 1.1605, 2.7651</td>
</tr>
<tr>
<td>0.8</td>
<td>1.3560, 2.1826, 6.4200, 1.1626, 2.7253</td>
</tr>
<tr>
<td>0.9</td>
<td>1.3313, 2.1787, 6.3036, 1.1649, 2.6804</td>
</tr>
<tr>
<td>1.0</td>
<td>1.3039, 2.1743, 6.1746, 1.1673, 2.6306</td>
</tr>
<tr>
<td>1.1</td>
<td>1.2738, 2.1694, 6.0333, 1.1699, 2.5761</td>
</tr>
</tbody>
</table>
APPENDIX V

Octiles: Level of Skewness = 0.5

\[ F = \begin{bmatrix}
0.256112 \\
0.361910 \\
0.405413 \\
0.403867 \\
0.364138 \\
0.288268 \\
0.173591
\end{bmatrix} \]

\[ W = \begin{bmatrix}
0.1311867 & -0.0926895 \\
-0.0926895 & 0.2619577 & -0.1467230 \\
-0.1467230 & 0.3287194 & -0.1637329 \\
-0.1637329 & 0.3262171 & -0.1470633 \\
-0.1470633 & 0.2651930 & -0.1049693 \\
-0.1049693 & 0.1661969 & -0.0500407 \\
-0.0500407 & 0.0602677
\end{bmatrix} \]

\[ q_0 = \{1, 1, 1, 1, 1, 1, 1\} \]

\[ W q_0 = \{0.0384972, 0.0225452, 0.0182635, 0.0154209, 0.0131604, 0.0111869, 0.0102270\} \]

\[ q' W q_0 = 0.1293011 ; \quad \frac{1}{q' W q_0} = 7.7338863 \]
\[
\begin{align*}
x'Wq_0 &= -0.04180771; \quad x^2Wq_0 = 0.08301337 \\
x^3Wq_0 &= -0.04668503; \quad x^4W_0 = 0.09262791 \\
\frac{Wq_0}{q_0Wq_0} &= \begin{bmatrix}
0.297733, 0.174362, 0.141248, 0.119263 \\
0.101781, 0.086518, 0.079094
\end{bmatrix} \\
x'Wq_0 &= -0.3233361; \quad x^2Wq_0 = 0.6420160 \\
x^3Wq_0 &= -0.3610567; \quad x^4Wq_0 = 0.7163737 \\
q_1(x) &= x + 0.3233361 \\
q_1 &= \{-0.8270133, -0.3511537, 0.0046967, \\
&0.3233361, 0.6419755, 0.9978259, 1.4736855\} \\
Wq_1 &= \{-0.0759449, -0.0160211, 0.0001255, \\
&0.0102977, 0.0179555, 0.0247036, 0.0388837\} \\
q_1'Wq_1 &= 0.16524274; \quad \frac{1}{q_1'Wq_1} = 6.0517031 \\
x^2Wq_1 &= -0.04325739; \quad x^3Wq_1 = 0.18787238 \\
x^4Wq_1 &= -0.06291561
\end{align*}
\]
\[
\frac{W_{q_1}}{q_1 \cdot W_{q_1}} = \begin{bmatrix}
-0.459596, & -0.096955, & 0.000759 \\
0.062319, & 0.108661, & 0.149499, & 0.235313
\end{bmatrix}
\]

\[
\frac{\chi^{2} W_{q_1}}{q_1 \cdot W_{q_1}} = -0.2617809; \quad \frac{\chi^{3} W_{q_1}}{q_1 \cdot W_{q_1}} = 1.1369479
\]

\[
\frac{\chi^{4} W_{q_1}}{q_1 \cdot W_{q_1}} = -0.3807466
\]

\[
q_2(\chi) = \chi^{2} + 0.2617809 \cdot q_1(\chi) - 0.6420160
\]

\[
q_2 = \{0.4647914, -0.2790049, -0.5392555,
-0.5573728, -0.3724281, 0.0741321, 1.0670704\}
\]

\[
W_{q_2} = \{0.0868353, -0.0370476, -0.0450670,
-0.0387602, -0.0245779, -0.019829, 0.0606003\}
\]

\[
\frac{q_1^* W_{q_2}}{q_2^* W_{q_2}} = 0.17027448; \quad \frac{1}{q_2^* W_{q_2}} = 5.8728707
\]

\[
\chi^{3} W_{q_2} = -0.02851409; \quad \chi^{4} W_{q_2} = 0.24938321
\]

\[
\frac{W_{q_2}}{q_2^* W_{q_2}} = \begin{bmatrix}
0.509972, & -0.217576, & -0.264673,
-0.227633, & -0.144342, & -0.011645, & 0.355897
\end{bmatrix}
\]
\[
\frac{x^3 w_2}{q^2 w_2} = -0.1674596 \quad \frac{x^4 w_2}{q^3 w_2} = 1.4645953
\]

\[
q_3(x) = x^3 + 0.1674596 q_2(x) - 1.1369479 q_1(x) + 0.3610567 = x^3 + 0.1674596 x^2 - 1.0931102 x - 0.0998970
\]

\[
q_3 = \{ -0.1431000, 0.4067282, 0.2330615, -0.0998970, -0.3988509, -0.4541552, 0.3865058 \}
\]

\[
w_{q_3} = \{ -0.0564723, 0.0856140, 0.0332919, -0.0120916, -0.0434089, -0.0529531, 0.0462000 \}
\]

\[
q_3 w_{q_3} = 0.11101941 \quad \frac{1}{q_3 w_{q_3}} = 9.0074339
\]

\[
x^4 w_{q_3} = -0.0116474
\]

\[
w_{q_3} = \begin{bmatrix}
-0.508670 & 0.771162 & 0.299874 \\
-0.108914 & -0.391003 & -0.476972 & 0.414523
\end{bmatrix}
\]

\[
\frac{x^4 w_{q_3}}{q_3 w_{q_3}} = -0.1049162
\]
\[ q_4(x) = x^4 + 0.1049162 q_3(x) - 1.4645953 q_2(x) + 0.3807466 q_1(x) - 0.7163737 \\
= x^4 + 0.1049162 x^3 - 1.4470261 x^2 - 0.1173414 x + 0.2125802 \]

\[ q_4 = \{ 0.0241316, -0.1918054, 0.1099662, 0.2125802, 0.0419754, -0.2857096, 0.0735842 \} \]

\[ w_{q_4} = \{ 0.0209441, -0.0686162, 0.0294839, 0.0451692, 0.0098596, -0.0555724, 0.0187318 \} \]

\[ q_4 w_{q_4} = 0.04418047 \]

\[ \frac{1}{q_4' w_{q_4}} = 22.634435 \]

\[ \frac{w_{q_4}}{q_4' w_{q_4}} = \begin{bmatrix} 0.474058, -1.553089, 0.667352, 1.022379, 0.223166, -1.257349, 0.423985 \end{bmatrix} \]
A note on the Distribution of the \( n \)-iles in large samples.

If the frequency distribution of a sample of a population is divided into \( n \) equal sub-divisions giving \( n-1 \) quantiles, then the joint probability distribution of these \( n-1 \) " \( n \)-iles", as \( N \) the number in the sample becomes indefinitely large, tends to the compound normal correlated distribution, namely

\[
\phi(x) = \left( \frac{2 \pi}{n} \right)^{-\frac{n-1}{2}} |V|^{-\frac{1}{2}} e^{\frac{1}{2} x' \left( \frac{1}{V} \right) x} ;
\]

where \( V^{-1} \) is the reciprocal of the Variance Matrix of the \( n \)-iles obtained in (2.22) and \( x \) is the column vector of the means of the \( n-1 \) \( n \)-iles.

In the particular case when \( n=2 \), this gives us the known distribution of the median, namely

\[
\phi(x) = \frac{1}{2f\sqrt{2\pi N}} e^{-\frac{f^2}{2N}x^2},
\]

where \( f \) is the ordinate at the median in the parent population.
REFERENCES.

(1) Laplace P.S., (1814), "Theorie analytique des probabilites", Paris


(22) Kondo T. and Elderton E.M., (1931), "Table of Normal Curve Functions to each Permille of Frequency", Biometrika, Vol. 22 pp. 368-376


