UNIVERSITY OF EDINBURGH

DEPARTMENT OF MECHANICAL ENGINEERING

COMBINATION INSTABILITIES AND NON-LINEAR VIBRATORY INTERACTIONS IN BEAM SYSTEMS

A Thesis submitted to the University of Edinburgh in fulfilment of the requirements for the degree of Doctor of Philosophy

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Declaration

I hereby certify that the research presented within this thesis was carried out by me alone, under the supervision of Dr J.W. Roberts of the Department of Mechanical Engineering, University of Edinburgh, and that all references to the work of others have been duly acknowledged.

It should also be noted that the paper "Forced Vibration of a beam system with autoparametric coupling effects" by J.W. Roberts and M.P. Cartmell is contained in its entirety in Appendix 9. This paper was submitted to the British Society for Strain Measurement and was accepted and published in "Strain": Vol 20, No. 3, August 1984 with the approval of Dr J.W. Roberts.

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ABSTRACT

As an extension of previously reported work on effects of internal resonance on non-linear vibration of beams, it has been shown that for blade-like beams excited parametrically by support motion in the plane of maximum stiffness, complex combination instabilities are observed. In addition to the well-known sum-type combination instability existing between the fundamental out-of-plane bending and the torsional modes of the beam, investigation has revealed the occurrence of higher order instabilities producing detectable bending and torsional vibrations which are not synchronous with external excitations. These effects are subsequently shown to exist in a related way in coupled beam configurations under forced vibration when specific internal resonance conditions exist between the natural frequencies of the various modes, and to produce visible patterns of non-linear energy flow between modes.

This study considers one such effect both experimentally and theoretically, consisting predominantly of a coupling between the fundamental and second nonplanar bending modes, and torsion mode. This combination resonance was modelled by taking the perturbation analysis to second order and including other contributory terms in the system governing equations. An expression for the transition curve for this resonance has been derived which shows the regions of stable and unstable solutions in a two parameter plane. Very close agreement is obtained between theoretical and experimental results for different beam lengths. It is also shown that if the geometry of the system is such that these two combination resonances can be excited simultaneously, very small alterations to the internal tuning of the system can generate noticeable intermodal energy exchange effects.

This system is then examined in the context of non-linear forced vibration and to this end an arrangement of coupled beams is studied. The vertical blade-like beam is coupled to the free end of a horizontal cantilever beam which is externally excited at a frequency in the region of its second bending mode frequency. This allows for the possibility of four mode interaction between the three nonplanar modes described above and also the second planar bending mode.
A four-degree-of-freedom model was formulated and perturbation analysis revealed that complex multimodal responses could occur for a single-frequency excitation. Steady-state solutions were derived by means of numerical integration techniques. A reasonable degree of agreement was observed between theoretical and experimental results.
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ACKNOWLEDGEMENTS
Principal Notation

\[ A(T_1), B(T_1), C_1(T_1), C_2(T_1), C_3(T_1) \]  \quad \text{Complex functions of 'slow time'}

\[ a(T_1), b(T_1), c_1(T_1), c_2(T_1), c_3(T_1) \]  \quad \text{Amplitudes of polar forms of } A_1B_1C_1, ...

B, B_1, B_2, B_3, B_4  \quad \text{Deflection form function integrals}

a, b  \quad \text{Scaling factors}

b_1, b_2  \quad \text{'Ordered' deflection form function integrals}

C_B  \quad \text{Coupling point reaction moment}

D_0, D_1, D_2  \quad \text{Differential Operators } D_i = \frac{\partial}{\partial T_i}

\[ d_1, \ldots, d_{13} \]  \quad \text{'Expansion' coefficients}

F_0  \quad \text{Raw excitation forcing term}

F_1, F_2  \quad \text{Excitation parameters}

f  \quad \text{'ordered' excitation parameter}

f(z), f_1(z), f_2(z)  \quad \text{Deflection functions}

G  \quad \text{Bulk modulus of Elasticity}

\[ H_1, \ldots, H_7 \]  \quad \text{Coefficients of slow-time equations, second order expansion}

h(z)  \quad \text{Function of 'twist' about z-axis}

\[ \bar{I}, \bar{J}, \bar{K} \]  \quad \text{Unit base vector (w.r.t X-Y-Z)}

\[ \bar{I}, \bar{J}, \bar{K} \]  \quad \text{Unit base vector (w.r.t x-y-z)}

\[ J_1, \ldots, J_6 \]  \quad \text{Coefficients of slow-time equation, second order expansion}

\[ K_1, \ldots, K_{24} \]  \quad \text{Coefficients of first order perturbation solutions}

\[ [K] \]  \quad \text{System stiffness matrix}

k_1, k_2  \quad \text{Local curvatures}

k_c  \quad \text{Torsion constant for noncircular section}

\[ L_1, \ldots, L_{78} \]  \quad \text{Coefficients of second order perturbation equations}
l
M
[M]
m_0
N
P_B
P_k
[P]
ρ_1, ρ_2, q_1, q_2, r_1, r_2, r_3
Q_1,..., Q_5
q_1, q_2
q_i
R
r_{ij}
r_{ij}, r_{2j}, r_{kj}
r_{12}, r_{22}
T_0, T_1, T_2
t
U_0, U_{u1}, U_{u2}, U_1, U_2
U_{10}, U_{11}, U_{12}, U_{20}, U_{21}, U_{22}
V
V_1, V_2, V_3
v, v_0
W
W_0
W(t)
W_0 S^2
w, w_0
X, Y

Length of secondary beam
Mass of planar system
System mass matrix
Lumped end mass
Number of degrees of freedom
Coupling point reaction force
Generalised excitation force
Generalised forces vector
Constituent parts of coefficients (Appendix 7)
Amplitude and frequency dependent coefficients
Specific Co-ordinates
Generalised Co-ordinate
Ratio of lumped mass to moment of Inertia
i^{th} element of j^{th} eigenvector
j^{th} eigenvectors at 1^{st}, 2^{nd} and k^{th} elements
Eigenvectors associated with q_1 and q_2
Successively 'slower' timescales
time
Bending displacements
Perturbations of U_1 and U_2 respectively
Strain Energy function
Coefficients of resonance 3.2.3-2 (Appendix 7)
Small in-plane displacements
Work function
Excitation amplitude (peak)
Harmonic excitation
Raw excitation acceleration
Axial displacements
Nondimensionalised variables
$x_0, x_1, y_0, y_1$  

Perturbations of $X$ and $Y$ respectively

$x_j$  

Planar displacement co-ordinate

$\alpha, \beta$  

Phase angles associated with $a, b$  
Also used to represent Euler angles

$\beta_1, \beta_2, \phi$  

Phase angles associated with $c_1, c_2, c_3$

$\Gamma_1, \Gamma_2, \Gamma_3$  

Coefficients of Solvability Equations

$\delta$  

Virtual change

$\epsilon$  

Small arbitrary parameter, perturbation parameter, Excitation parameter

$\xi_h, \xi_t, \xi_B$  

'Ordered' damping coefficients

$\gamma_1, \gamma_2$  

Internal detuning parameters

$\Theta$  

Autonomous system phase angle

$\Lambda_1, \ldots, \Lambda_{27}$  

Constituent parts of Coefficients (Appendix 5)

$\rho$  

Mass ratio

$\xi_b, \xi_t, \xi_B$  

Raw damping coefficients

$\rho, \rho_1$  

Internal detuning parameters

$\sigma$  

External detuning parameter

$\tau$  

Local twist rate

$\phi$  

Euler angle

$\phi_0, \phi_1$  

Torsional displacement variables

$\phi_{10}, \phi_{11}, \phi_{12}$  

Perturbations of $\phi_1$

$\Phi, \psi, \psi_1, \psi_2$  

Autonomous system phase angles

$\Omega$  

Excitation frequency

$\omega_b, \omega_t, \omega_{b1}, \omega_{t1}, \omega_{b2}, \omega_{t2}$  

Linear natural frequencies
CHAPTER 1

Introduction

1.1 General Remarks

The behaviour of certain vibrating physical systems cannot always be adequately represented by the classical theories of linear vibrations. In some cases a monofrequent excitation can generate responses over a range of frequencies when combination resonances are present. Similarly a fractional relationship between the excitation and system natural frequencies can produce subharmonic or superharmonic effects and regions of multivalued solutions where the responses depend on the initial conditions. Jump phenomena are often characteristic of these systems. Systems possessing limit cycles can display non-steady responses in which the amplitude varies with time, and in multidegree-of-freedom systems the responses can, in certain examples, be seen to exchange energy continuously.

The research to be described here is concerned with motions that are the result of time-dependent excitations on simple beam structures. However, the excitations do not appear merely as inhomogeneities, but as coefficients in the governing differential equations. Since excitations of this nature appear as parameters in the governing equations they are termed 'Parametric Excitations'. In contrast with the case of external excitations in which large responses cannot be generated by small excitations unless the excitation frequency is in close proximity to a system natural frequency, a small parametric excitation can generate a large response when a system natural frequency is in the region of twice the excitation frequency. This is the simplest form of parametric resonance and is called 'Principal Parametric Resonance'. In the present investigation the effects of parametric excitation in conjunction with terms additional to the inertia, damping and stiffness terms found in most linear systems are studied in two structures. The additional nonlinear terms are often quadratic or cubic and are usually regarded as being relatively inconsequential and thus ignored. Their inclusion in a parametrically excited system dictates a mathematical approach which is rather different to that encountered in linearised problems. Exact analytical solutions are rarely attainable and so various approximation techniques are used.
The first structure under investigation is a parametrically excited cantilever beam for which a three-degree-of-freedom model is ultimately postulated for certain forms of internal resonance. A logical extension of this is to incorporate such a system into one in which steady states may be observed, as in the later chapters where an autoparametric system of coupled cantilever beams with four-degrees-of-freedom is described and investigated.

1.2 Literature Review

For the case of a static lateral buckling loading condition imposed on a narrow beam of rectangular cross section it is possible that an instability will develop due to the transverse bending and torsional displacement of the beam perpendicular to the plane of the load when the load is in the plane of greatest stiffness. Hodges and Peters (1) investigated this static instability in an end loaded cantilever beam. The corresponding dynamical problem is one of parametric instability where the external periodic excitation appears as an explicit time-variant coefficient in the governing equations, the resulting instability occurring as lateral vibrations in both bending and torsion. Fundamental work into parametric vibration of structures is attributable to Bolotin (2) who discussed principal parametric resonance in a simply supported beam subjected to periodic end moments.

Bolotin also investigated the behaviour of a cantilever and end mass configuration on which a periodic load is imposed at the free end.

Research into the parametric combination resonance of a cantilever undergoing support motion has been carried out by Dugundji and Mukhopadhyay (3) and this is discussed and re-examined in this study as an introduction to the phenomena under investigation. More recently work has been reported by Dukumaci (4) on a similar problem. However, unlike Dugundji and Mukhopadhyay (3) who used the method of harmonic balance to obtain a stability determinant and hence a boundary expression, finitization by means of the Rayleigh-Ritz method and then the application of the small parameter stability criterion to the resulting periodic linear system was the chosen approach. Both these studies highlight the way in which the unstable regions are affected by the introduction of system damping. The work of Yamamoto and Saito (5) is fundamental to these papers. However their original work is mainly theoretical with some
experimental verification presented for the case of a spring-loaded double pendulum. Parametric excitation of a dynamical system with multiple degrees of freedom was treated to first order by Hsu (6). The analysis utilised the method of Struble (7) and displayed both sum and difference combination instabilities for a generalised set of ordinary differential equations. This technique combined the method of averaging and the asymptotic method. Recent work by Evan-Iwanowski (8) and Nayfah and Mook (9) and Ibrahim and Barr (10) has served to broaden and intensify the field of parametric vibration. The exhaustive work of Nayfah and Mook (9) covers a wide selection of engineering and physical problems and their analyses.

Many analytical techniques are available for the treatment of non-linear vibration problems, most of which are explored by the work of Nayfah (11), Nayfah and Mook (9), and Jordan and Smith (12). It is of interest to note that a recent more qualitative advance in analytical interpretation, in the form of adaptations of Bifurcation and Catastrophe theory, to problems in non-linear mechanics has been made by Holmes and Rand (13), Poston and Stewart (14) and Saunders (15). Holmes and Rand (13) discussed the bifurcations of Duffing's equation from a catastrophe theory viewpoint for both hardening and softening springs and indicated ways by which such an approach can be applied to engineering systems.

Of the quantitative analytical treatments applicable to non-linear vibration problems the Fourier series method as used by Bolotin (2) is one of the most well known, and this relies on the transformation of a set of temporal governing equations into a system of homogeneous algebraic equations according to the Fourier coefficients. The transition from stable to unstable solutions is represented by the system determinant being equated to zero.

The method of averaging discussed in the work of Bogoliubov and Mitropolsky (16) was applied by Evan-Iwanowsky (8) in an analysis of many non-linear resonances. Perturbation methods offer useful methods of solution for a variety of problems, and the definitive work by Nayfah (11) discussed most of these in considerable depth. This illustrates how straight forward perturbation techniques such as the Lindstedt-Poincaré method, and the amended version due to Lighthill, yield ununiformly valid expansions for weakly nonlinear oscillating systems
but only in terms of their limit cycles and points. These approaches
do not cater for systems in which damping or sharp changes occur.
The method of multiple time scales, popularised by Nayfeh (11) and
Nayfeh and Mook (9) may be applied to problems that can be treated
using strained-coordinate techniques such as the methods of Lindstedt-
Pointcaré and Lighthill, and also to damped systems. All the perturbation
methods mentioned can be used to derive approximate solutions to the
general vibration problem described mathematically by the weakly non-
linear system,

$$\ddot{q} + \omega^2 q = \epsilon f(q, \dot{q}, t)$$

where $f(q, \dot{q}, t)$ is a non-linear function and $\epsilon$ a small parameter.
Obviously $\epsilon = 0$ is the purely linear case for which exact solutions
are available.

Parametric excitation of multi-degree-of-freedom systems may
be characterised by the resulting combination resonance effects expressed
as,

$$m_i \Omega = \sum_{i=1}^{N} k_i \omega_i$$

where $\Omega$ and $\omega_i$ are excitation and natural frequencies respectively,
$m_i$ and $k_i$ are integers and $N$ denotes the number of degrees of freedom.
There is considerable recent work relating to the general analysis
of parametrically excited systems and combination instabilities, notably
the extension of the method of harmonic balance to a generalised system
in order to determine the regions of combination resonances, by
Szemplinska-Stupnicka (17). Takahashi (18) has applied Liapunov-Floquet
theory to systems for the numerical calculation of the characteristic
exponents and has been able to assess stability from direct inspection
of these. Variational principles as used by Hu (19) and Papastavridis
(20) can be used to determine zones of stability for parametrically
excited systems.

The equation of most fundamental importance in problems involving
parametric effects is the Mathieu-Hill equation which occurs in many
physical situations, and is discussed by Bolotin (2), Nayfeh (11),
Nayfeh and Mook (9), Jordan and Smith (12). New methods for determining
the stability zones for the equation are emerging, a Newtonian Iterative method devised by Asner (21) and the Bubnov-Galerkin method as used by Pederson (22) for the damped form are of this category. Damping of different forms has been incorporated into several studies because of its profound effects on real systems. Parametrically excited linear systems will theoretically display an unbounded response within the unstable region(s) and the presence of linear damping only modifies the zones of instability and not the response characteristics. Finite resonances occur in physical systems however, due to non-linear factors inherent in the system which modify the response behaviour accordingly.

Quadratic damping and cubic stiffness non-linearities have been investigated in a single Mathieu equation in a paper by Hsu (6), while quadratic damping alone was explored for a pair of coupled Mathieu equations by Mukhopadhyay (23).

Parametric Instabilities are prevalent in a diversity of physical systems and there is much published work relating to applications and occurrences in beam and plate problems. A concentrated mass under gravity on a parametrically excited horizontal beam has been considered by Sato, Saito and Otomi (24), after which Saito and Otomi went on to investigate viscoelastic beams with viscoelastic end supports.

Periodic variation of the length of a beam is another form of parametric excitation and Zajaczkowski, Lipinski and Yamada (25) analysed this theoretically in both beam and plate elements. Ostiguy and Evan-Iwanowski (26) discussed the influence of aspect ratio on the dynamic response of parametrically excited plates. Evan-Iwanowski and Evensen (27) studied the effects of longitudinal inertia upon the parametric response of elastic columns while the nonlinear analysis of the lateral response of periodically loaded columns has recently been presented by Tezak, Mook and Nayfeh (28). The nonlinear behaviour of rectangular plates with large but finite deflections is the title of work by Bauer, Bauer, Becker and Reiss (29), while the subject of the dynamic stability of cylindrical shells, taking into account in-plane inertia and in-plane disturbance, was analysed by Shirakawa (30) who used the method of Bolotin (2) to obtain the stability regions. Ali-Hasan and Barr (31) presented very comprehensive work on both Linear and Non-linear vibrations of thin walled beams of equal angle section, this being of particular relevance since such members are basic structural elements. Research
into a parametrically driven non-linear stretched string in which the length is time-variant is due to Tagata (32) who used the harmonic balance approach.

More generalised theoretical work has been carried out by Nayfeh (11) on the response of multidegree-of-freedom systems with quadratic non-linearities for a harmonic parametric resonance, and Tezak, Nayfeh and Muuk (28) considered parametrically excited non-linear multidegree-of-freedom systems displaying repeated natural frequencies, with particular reference to aerodynamic flutter. Asfar and Nayfeh (33) have taken a Van der Pol type self-excited oscillatory system into consideration in their paper on the response of self-excited two-degree-of-freedom systems to multifrequency excitations. Nayfeh (11) has also discussed multifrequency parametric excitations of two-degree-of-freedom systems with both sum and difference type combination resonances.

There are also several published works relating to parametric instabilities in liquid-structure systems whereby fluctuations in piped fluids can act parametrically on the conveying pipe itself. Principal parametric resonance was observed for such a system by Paidoussis, Issid and Tsui (34) both theoretically and experimentally. Khandelwal and Nigam (35) found both principal and secondary instabilities in a liquid filled flexible container under vertical periodic excitation.

A development of parametric instability into a steady state system in which the parametric excitation results from the response of an element undergoing forced excitation is generally described by the term 'Autoparametric'. This phenomenon is less well documented than the parametric field, however several researchers have contributed studies involving autoparametric interaction effects and the credit for both the descriptive term and initial work on quadratic inertia coupling in a two-degree-of-freedom system is due to Minorsky (36).

The main feature of autoparametric systems is that of internal resonance, which arises when the following linear relationship is satisfied,

$$\sum_{i=1}^{N} k_i \omega_i = 0$$
Where \( \omega_1 \) are the linear natural frequencies of a N-degree-of-freedom system and \( K_1 \) are integers.

The elastic pendulum is the classical example of an autoparametric system where both longitudinal extension and swinging pendulum motion can occur, and this has been researched in depth by Kane and Khan (37), Tsel'man (38), Van der Burgh (39), Srinivasan and Sankar (40), Roth and Kane (41), Breitenberger and Mueller (42), and, Hatwal, Mallik and Ghosh (43). When the frequency of longitudinal motion is approximately twice the natural frequency of the pendulum (internal resonance) complex behaviour occurs in which energy is continuously exchanged between the modes. Amplitude modulated responses are generated in this situation. Sethna (44) described a general analysis for the vibrations of a non-autonomous dynamical system with two-degrees-of-freedom and quadratic nonlinearities, while Ryland and Meirovitch (45) extended the elastic pendulum analysis to incorporate viscous damping. Cubic nonlinearities in systems with internal resonances were investigated generally by Gilchrist (46) by means of the Krylov-Bogoliubov-Mitropolsky asymptotic formulation. Specific structural problems involving beam elements of variable thickness undergoing harmonic excitation in which two modes interact were studied by Nayfah, Mook and Lobitz (47). An extension of this work to include superharmonic, subharmonic and combination resonances in hinged-clamped beams was presented by Sridhar, Nayfah and Mook (48). Multimodal responses in the forced vibrations of circular plates were the subject of a paper by Sridhar, Mook and Nayfah (49). Haxton and Barr (50) and Haxton (51) developed an autoparametric vibration absorber incorporating two mode interaction and Barr and Nelson (52) considered a coupled beam system with two and three mode interactions.

Autoparametric systems with broadband excitations were studied in depth by Roberts (53) and Ibrahim and Roberts (54) using Stochastic Calculus and a method based on the Markov vector approach to obtain theoretical results. Roberts and Bux (55) and Bux (56) examined simultaneous combination resonances to first order for both three and four mode (with zero damping) interactions in an autoparametrically coupled beam system, while Roberts and Cartmell (57) developed a first order analytical solution with experimental verifications for a similar structure with two interacting modes.
Nayfeh, Mook and Marshall (58) showed the possibility of autoparametric non-linear coupling in ship motion in which two mode interaction is generated by pitching and rolling of the vessel. Ibrahim and Barr (59) found both two and three mode interactions in a structure-liquid system and Barr and Ashworth (60) discussed multimodal interactions in an aeroplane fuselage-tail plane assembly model. Barr has also shown the possibility of cascading interactions such that one or more modes involved in one internal resonance condition can also satisfy another separate internal resonance condition which can propagate to result in multimodal responses. The most recent work on the subject of a coupled non-linear two-degree-of-freedom system is by Haddow (62) and this discussed the possibility of five external resonances and duly highlighted a variety of phenomena.
1.3 Scope of Investigation

A cantilever beam with an adjustable lumped end mass parametrically excited by a base motion at right angles to its axis in the plane of maximum stiffness is represented initially as a two-degree-of-freedom model. The kinematics are derived from Love (63) and a suitable Galerkin representation is taken to reduce the system to a pair of transformed governing equations. By means of the multiple time scales method an expression for the stability boundary of the sum type combination resonance (between the fundamental bending and torsion modes) is developed. Further theoretical and experimental work shows the existence of a new type of combination instability involving predominantly bending motion according to the resonance condition,

\[ \omega = \frac{1}{2} (\omega_{b1} + \omega_{b2}) + \varepsilon^2 \rho_1 \]

where \( \omega \) is the excitation frequency and \( \omega_{b1} \) and \( \omega_{b2} \) are the fundamental and second bending frequencies respectively and \( \rho_1 \) is a detuning parameter.

Revision of the first order two-degree-of-freedom model to include the second bending mode by means of an extended Galerkin approximation and including other small terms produces a set of three governing equations which are expanded to second order by the method of multiple scales. An analytical expression is developed for the stability boundary or Transition curve for this combination resonance and experimental work is carried out to support it. The 'slow-time' equations which result from the second order expansion are integrated numerically to show the unboundedness of the two bending responses with time for points within the unstable region. Points outwith this region are shown to produce stable, trivial, zero solutions. Experimental work shows that for certain beam lengths the two combination resonances can be generated simultaneously and that the 'weaker' second order effect can affect the purity and strength of the modal content of the first order resonance. This is investigated theoretically by considering a third combination resonance of the form,

\[ \omega = \omega_{b2} - \omega_{t1} + \varepsilon \eta_2 \]
Where $\omega_{t1}$ is the fundamental torsion frequency, and $\gamma_2$ is a detuning parameter. This resonance is also simultaneous with the bending/torsion combination and it is shown that the second order effect can be implied by considering this in conjunction with the bending/torsion combination. Thus it is possible to explore the bending/torsion and the bending/bending resonances by first order theory and the governing equations are, in this context, redefined and expanded to first order. An analytical expression emerges from which the modal response ratio of second bending to first torsion is derived for two cases of internal tuning over a range of excitation accelerations. Inherent solution stabilising effects are displayed in the 'steady' responses encountered in the experimental work, and theoretical and experimental results for the exchange of energy (principally between the second bending and first torsion modes) are presented.

This three-degree-of-freedom model is then incorporated into a structure of coupled beams with the following external and internal resonance conditions,

$$\Omega = \omega_{B2} + \epsilon \sigma$$

$$\omega_{B2} = \omega_{b1} + \omega_{t1} + \epsilon \gamma_1$$

$$\omega_{B2} = \omega_{b2} - \omega_{t1} + \epsilon \gamma_2$$

$$\omega_{B2} = \frac{1}{2} \left[ \omega_{b1} + \omega_{b2} \right] + \epsilon \left( \frac{\gamma_1 + \gamma_2}{2} \right)$$

Where, $\Omega$ is the external forcing frequency, $\omega_{B2}$ is the second bending frequency of the system in planar bending, $\sigma$ is the external detuning parameter, $\omega_{b1}, \omega_{b2}, \omega_{t1}$ are the fundamental and second nonplanar bending frequencies and fundamental torsion frequency respectively, and $\gamma_1, \gamma_2$ are internal detuning parameters.

The governing equations are expanded to first order after which stationary solutions are obtained by numerical integration of the slow-time equations resulting from the expansion. Various cases are investigated in which the effects of detuning and excitation acceleration levels are presented. Comparisons with the theoretical results are provided from measurements of the responses of the experimental model.
Where $\omega_{t1}$ is the fundamental torsion frequency, and $\eta_2$ is a detuning parameter. This resonance is also simultaneous with the bending/torsion combination and it is shown that the second order effect can be implied by considering this in conjunction with the bending/torsion combination. Thus it is possible to explore the bending/torsion and the bending/bending resonances by first order theory and the governing equations are, in this context, redefined and expanded to first order. An analytical expression emerges from which the modal response ratio of second bending to first torsion is derived for two cases of internal tuning over a range of excitation accelerations. Inherent solution stabilising effects are displayed in the 'steady' responses encountered in the experimental work, and theoretical and experimental results for the exchange of energy (principally between the second bending and first torsion modes) are presented.

This three-degree-of-freedom model is then incorporated into a structure of coupled beams with the following external and internal resonance conditions,

$$\Omega = \omega_{B2} + \epsilon \sigma$$

$$\omega_{B2} = \omega_{b1} + \omega_{t1} + \epsilon \eta_1$$

$$\omega_{B2} = \omega_{b2} - \omega_{t1} + \epsilon \eta_2$$

$$\omega_{B2} = \frac{1}{2} [\omega_{b1} + \omega_{b2}] + \epsilon (\frac{\eta_1 + \eta_2}{2})$$

Where, $\Omega$ is the external forcing frequency, $\omega_{B2}$ is the second bending frequency of the system in planar bending, $\sigma$ is the external detuning parameter, $\omega_{b1}, \omega_{b2}, \omega_{t1}$ are the fundamental and second nonplanar bending frequencies and fundamental torsion frequency respectively, and $\eta_1, \eta_2$ are internal detuning parameters.

The governing equations are expanded to first order after which stationary solutions are obtained by numerical integration of the slow-time equations resulting from the expansion. Various cases are investigated in which the effects of detuning and excitation acceleration levels are presented. Comparisons with the theoretical results are provided from measurements of the responses of the experimental model.
CHAPTER 2

Parametrically excited combination resonances in a thin cantilever beam

2.1 Kinematics

The geometry of a bent and twisted beam may be represented as in Figure 2.1-1 where the section centre A is displaced to A' by a lateral displacement $U$ due to bending, a small displacement $v$ arising from combined bending and torsion and a twist angle $\phi$. The small axial displacement $w$ is ignored.

The vibrational excitation is applied at the base in the plane of greatest stiffness and is a prescribed harmonic motion of the form $W(t) = W_0 \cos \Omega t$. Following previous work by Roberts and Bux, it may be shown that the curvatures about the local axes A'x, A'y to quadratic order are (Appendix 1):

$$k_1 = U''\phi - v'' \quad (2.1-1)$$

$$k_2 = v''\phi + U'' \quad (2.2-2)$$

$$\tau = \phi' + U''v \quad (2.1-3)$$

The deformed cross-section is shown in Figure 2.1-2.

Curvatures in this work are only taken up to quadratic order in an attempt to keep the resulting equations manageable yet reasonably representative of the system. As $k_1$ is the curvature in the plane of greatest stiffness then,

$$k_1 = 0 \text{ gives } v'' = U''\phi \quad (2.1-4)$$

thus,

$$k_2 = U''\phi^2 + U''xU'' \quad (2.1.5)$$

and,

$$\tau = \phi' + U''U'\phi \approx \phi' \quad (2.1-6)$$

Ignoring the cubic terms in $k_2$ and $\tau$. 

Fig. 2.1-1 Geometry of a bent and twisted beam
Fig. 2.1-2 Deformed beam at section AA'

Fig. 2.1-3 End mass Inertia Forces
2.2 Governing equations for 2 degrees of freedom to first order

The strain energy due to lateral twisting and bending, from Timoshenko, ref. 34 is,

\[ V = \int_0^l \frac{EI_y}{2} \left[ \theta'' \right]^2 dz + \int_0^l \frac{k_c GJ}{2} \left[ \phi' \right]^2 dz \]  \hspace{1cm} (2.2-1)

Where \( k_c \) is a constant relating to a noncircular section.

The lateral displacement at any point A is assumed to be related to that of B by a Galerkin representation,

\[ U(z,t) = f(z)U_0(t) \]  \hspace{1cm} (2.2-2)

\[ \phi(z,t) = h(z)\phi_0(t) \]  \hspace{1cm} (2.2-3)

hence,

\[ V = U_0^2 \int_0^l \frac{EI_y}{2} \left[ \theta'' \right]^2 dz + \phi_0^2 \int_0^l \frac{k_c GJ}{2} \left[ h' \right]^2 dz \]  \hspace{1cm} (2.2-4)

Considering virtual displacements \( \delta U_0 \) and \( \delta \phi_0 \) in \( U_0 \) and \( \phi_0 \) respectively the virtual change in strain energy is given by,

\[ \delta V = \left[ \int_0^l \frac{EI_y}{2} \left[ f'' \right]^2 dz \right] \delta U_0 + \left[ \int_0^l \frac{k_c GJ}{2} \left[ h' \right]^2 dz \right] \delta \phi_0 \]  \hspace{1cm} (2.2-5)

The inertia forces acting on the lumped end mass are shown in Figure 2.1-3 with the exception of axial components, and \( v_0 \) can be related to \( U_0 \) and \( \phi_0 \) from 2.1-4,

\[ v_0 = \phi_0 U_0 B \]  \hspace{1cm} (2.2-6)

Where \( B \) is an integration constant which depends on the choice of functions \( f \) and \( h \).

The corresponding virtual displacement is,

\[ \delta v_0 = B \delta \phi_0 \delta U_0 + B \delta U_0 \delta \phi_0 \]  \hspace{1cm} (2.2-7)
While the second derivative of $v_0(t)$ is given by,

$$\dddot{v}_0 = B \left[ \dot{\phi}_0 \dddot{u}_0 + 2 \dot{u}_0 \ddot{\phi}_0 + \dot{u}_0 \dot{\phi}_0 \right] \quad (2.2-8)$$

Using relationship (2.2-7) the virtual work of inertia forces becomes,

$$\delta W = -I_0 \dot{\phi}_0 \delta \phi_0 - m_0 \delta u_0 \delta u_0 - m_0 \left[ \delta w + \delta v_0 \right] \left[ \delta \phi_0 \delta u_0 + u_0 \delta \phi_0 \right] B \quad (2.2-9)$$

Applying the Principle of Virtual Work in the form $\delta V = \delta W$ for an arbitrary set of virtual displacements the following first order equations are obtained,

$$\dddot{u}_0 + \omega_b^2 u_0 + WB \dot{\phi}_0 = 0 \quad (2.2-10)$$

$$\dddot{\phi}_0 + \omega_t^2 \phi_0 + m_u \frac{WBU_o}{I_o} = 0 \quad (2.2-11)$$

The small inplane acceleration term in $\dddot{v}_0$ leads to cubic terms which are ignored in a first order development.

Equations (2.2-10, 2.2-11) can be made symmetric by a suitable coordinate transformation and written in terms of nondimensional amplitudes.

Also viscous damping terms are added to give:

$$\ddot{x} + 2 \xi_b \omega_b \dot{x} + \omega_b^2 x - \epsilon \cos \omega t \dot{y} = 0 \quad (2.2-12)$$

$$\ddot{y} + 2 \xi_t \omega_t \dot{y} + \omega_t^2 y - \epsilon \cos \omega t \dot{x} = 0 \quad (2.2-13)$$

$$x = \frac{u_0}{b}, \quad y = \frac{a}{b} \phi_0, \quad \alpha = \sqrt{\frac{I_0}{m_0}}$$

Where $\epsilon = \frac{W_0 \Omega^2 B}{\sqrt{\frac{m_0}{I_0}}} \quad (2.2-14)$
2.3 Governing equations for 3 degrees of freedom to second order

The additional degree of freedom taken is that represented by the second bending mode, for which the Galerkin approximation (2.2-2) is redefined as,

\[ U(z,t) = r_1(z)U_{o1}(t) + r_2(z)U_{o2}(t) \]  
(2.3-1)

Also \( v_o = \int_0^1 \phi_o U_{o1}B_1 + \int_0^1 \phi_o U_{o2}B_2 \)  
(2.3-2)

\( B_1 \) and \( B_2 \) are given in Appendix 2.

Including \( v_o \) and its associated small cubic terms in the second order development gives the following virtual work expression,

\[ \delta W = -\int_0^1 \phi_0 \dddot{\phi}_o \delta \phi_o - \int_0^1 \phi_0 U_{o1} \delta U_{o1} - \int_0^1 \phi_0 U_{o2} \delta U_{o2} \\
- \int_0^1 \phi_0 U_{o1}B_1 + 2 \int_0^1 \phi_0 U_{o1}B_2 + \int_0^1 \phi_0 U_{o1}B_1 \delta \phi_o \\
+ 2 \phi_0 U_{o2}B_2 + \phi_0 U_{o2}B_2 \delta \phi_o \right) \left[B_1 \phi_0 \delta U_{o1} + B_1 U_{o1} \delta \phi_o \\
+ B_2 \delta U_{o2} + B_2 U_{o2} \delta \phi_o \right] \]  
(2.3-3)

and the strain energy expression,

\[ \delta V = \int_0^l EI_y [r_1']^2 \, dz \int_0^l U_{o1} \delta U_{o1} + \int_0^l EI_y [r_2']^2 \, dz \int_0^l U_{o2} \delta U_{o2} \\
+ \int_0^l K_c GJ \, [h']^2 \, dz \int_0^1 \phi_0 \delta \phi_o \]  
(2.3-4)

Applying again the Principle of Virtual Work in the form \( \delta V = \delta W \) for an arbitrary set of virtual displacements, the following second order equations are produced,

\[ \dddot{U}_{o1} (1 + \phi_0^2 B_1^2) + 2\dot{U}_{o1} \phi_0 \dot{\phi}_o B_1 + \dddot{U}_{o1} (\phi_0^2 B_1^2 + \phi_0 \phi_o B_1^2) \\
+ 2\dot{U}_{o2} \phi_0 \phi_o B_2 + \dddot{U}_{o2} \phi_0 \phi_o B_2 \]
\[ \ddot{U}_{o2} \left( 1 + \dot{\phi}_{o}^2 B_{o}^2 \right) + 2\dot{U}_{o2} \dot{\phi}_{o} \dot{B}_{o}^2 + U_{o2} \left( \omega_{b2}^2 + \phi_{o} \ddot{\phi}_{o} B_{o}^2 \right) + 2U_{o1} \dot{\phi}_{o} \dot{B}_{o}^2 + U_{o1} \dot{\phi}_{o} \ddot{B}_{o}^2 + U_{o1} \phi_{o} \dddot{B}_{o}^2 + \dddot{W}_{B2} \phi_{o} = 0 \]  

(2.3-5)

\[ \dddot{\phi}_{o} \left[ \begin{array}{c} 1 + \frac{m_{o} B_{o}^2 U_{o1}^2}{I_{0}} + \frac{2m_{o} B_{1} B_{2} U_{o1} U_{o2}}{I_{0}} + \frac{m_{o} B_{o}^2 U_{o2}^2}{I_{0}} \\ \frac{2m_{o} B_{2}^2 U_{o1} U_{o2}}{I_{0}} + \frac{2m_{o} B_{1} B_{2} U_{o1} U_{o2}}{I_{0}} + \frac{2m_{o} B_{1} B_{2} U_{o1} U_{o2}}{I_{0}} \\ \frac{2m_{o} B_{2}^2 U_{o1} U_{o2}}{I_{0}} \end{array} \right] + \dddot{\phi}_{o} \left[ \begin{array}{c} 1 + \frac{m_{o} B_{o}^2 U_{o1} U_{o1}}{I_{0}} + \frac{m_{o} B_{1} B_{2} U_{o1} U_{o2}}{I_{0}} + \frac{m_{o} B_{o}^2 U_{o2}^2}{I_{0}} \\ \frac{2m_{o} B_{2}^2 U_{o1} U_{o2}}{I_{0}} \end{array} \right] + \dddot{\phi}_{o} \left[ \begin{array}{c} 1 + \frac{m_{o} B_{o}^2 U_{o1} U_{o2}}{I_{0}} + \frac{m_{o} B_{1} B_{2} U_{o1} U_{o2}}{I_{0}} + \frac{m_{o} B_{o}^2 U_{o2}^2}{I_{0}} \\ \frac{2m_{o} B_{2}^2 U_{o1} U_{o2}}{I_{0}} + \frac{2m_{o} B_{1} B_{2} U_{o1} U_{o2}}{I_{0}} + \frac{2m_{o} B_{1} B_{2} U_{o1} U_{o2}}{I_{0}} \end{array} \right] + \dddot{W}_{B1} U_{o1} m_{o} + \dddot{W}_{B2} U_{o2} m_{o} = 0 \]  

(2.3-6)

These equations may be rewritten in the following form,

\[ \dddot{U}_{1} \left( 1 + \epsilon \phi_{1}^2 B_{1} b_{1} \right) + 2 \epsilon \phi_{b1}^2 \omega_{b1} U_{1} + \omega_{b1}^2 U_{1} + 2 \epsilon \phi_{1} \phi_{1} B_{1} b_{1} + \epsilon \phi_{1} \phi_{1} B_{1} b_{1} + 2 \epsilon \phi_{1} \phi_{1} B_{1} b_{2} + \epsilon \phi_{1} \phi_{1} B_{1} b_{2} + \epsilon \phi_{1} \phi_{1} W_{o} n_{o}^2 \cos \Omega \right) t = 0 \]  

(2.3-8)
\begin{align*}
\ddot{U}_2 (1 + \varepsilon \phi_1^2 b_2) + 2 \varepsilon^2 \int_b \omega_2 \ddot{U}_2 + \omega_2^2 U_2 + 2 \varepsilon U_2 \phi_2 \phi_2 b_2 \\
+ \varepsilon U_2 \phi_2 \phi_2 b_2 + 2 \varepsilon \dot{U}_1 \phi_1 \phi_1 b_2 + \varepsilon U_1 \phi_1 \phi_1 b_2 \\
+ \varepsilon \ddot{U}_1 \phi_1^2 b_2 - \varepsilon \phi_1 b_2 w_0 n^2 \cos nt = 0
\end{align*}
(2.3-9)

\begin{align*}
\ddot{\phi}_1 (1 + \varepsilon R U_1^2 b_1) + 2 \varepsilon R U_1 U_2 \phi_1 b_2 + \varepsilon R U_2^2 b_2) + 2 \varepsilon^2 \int_t \omega b_1 \ddot{\phi}_1 \\
+ \omega_1^2 \phi_1 + 2 \varepsilon R \ddot{\phi}_1 (U_2 \ddot{U}_2 b_2 + U_2 \ddot{U}_2 b_2 + U_1 \ddot{U}_1 b_2 \\
+ U_1 \ddot{U}_1 b_2 + \varepsilon R \phi_1 (U_1 \ddot{U}_1 b_1 + U_1 \ddot{U}_1 b_2 \\
+ U_1 \ddot{U}_1 b_2 + U_2 \ddot{U}_2 b_2) \\
- \varepsilon F_1 U_1 \cos nt - \varepsilon F_2 U_2 \cos nt = 0
\end{align*}
(2.3-10)

Where \( U_{01}, U_{02}, \phi_0 \) are rewritten as \( U_1, U_2, \phi_1 \) respectively, and

\[
R = \frac{m_0}{\Gamma_0}, \quad F_1 = Rb_1 w_0 n^2, \quad F_2 = Rb_2 w_0 n^2,
\]

\[
\xi_b = \varepsilon^2 \int_b, \quad \xi_t = \varepsilon^2 \int_t
\]

The additional nonlinear cubic terms contained in the above are shown in Chapter 3 to produce many theoretical resonance conditions involving the three natural frequencies. One particular resonance of the form,

\[
\omega = \frac{1}{2} (\omega_{b1} + \omega_{b2}) + \varepsilon^2 \rho_1
\]

is studied in subsequent sections and is found to be generated specifically by the coupling in the parametric excitation terms when the equations are expanded to second order.
3.1 Expansion of the 2 degree of freedom equations to first order

The nontrivial nonzero solutions to equations (2.2-12, 2.2-13) are theoretically unbounded with time, however in the physical system itself such effects as non-linear system stiffness due to the large deflections involved tend to stabilise and limit the solutions. In this context the equations may be expanded by means of the method of multiple time scales (Nayfeh, 11) and an analytical expression derived for the boundary line or Transition Curve, depicting the change from stable zero solutions to unstable nonzero solutions in a two parameter plane.

The method of multiple scales has been utilised in the solution of many non-linear oscillatory problems (Nayfeh and Mook, 9) and considers the expansion representing the response as a function of multiple independent variables (scales). Its chief advantage over other perturbation techniques such as the Lindstedt-Poincaré method is that it is easily applicable to damped systems.

The time scales are introduced such that

\[ T_n = \varepsilon^n t \quad n = 0, 1, 2, \ldots \]  

(3.1-1)

Where successive \( T_n \) as \( n \) increases represent 'slower' time scales with respect to real time \( t \). Each time scale is treated as an independent variable and the required order of approximation to the solution dictates the number of time scales used.

Expressing variables \( x \) and \( y \) as formal asymptotic series' in terms of a small parameter \( \varepsilon \) up to and including first order,

\[ x = x_0(T_0 T_1) + \varepsilon x_1(T_0 T_1) + \ldots \]  

(3.1-2)

\[ y = y_0(T_0 T_1) + \varepsilon y_1(T_0 T_1) + \ldots \]  

(3.1-3)
with similar transformations to the derivatives with respect to $t$,

\[
\frac{d}{dt} = \frac{dT_u}{dt} \frac{\partial}{\partial T_u} + \frac{dT_1}{dt} \frac{\partial}{\partial T_1} + \ldots
\]

\[= D_0 + \epsilon D_1 + \ldots \tag{3.1-4}\]

\[
\frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \ldots \tag{3.1-5}\]

$D$ operator notation representing $D_0 = \frac{\partial}{\partial T_u}$, $D_1 = \frac{\partial}{\partial T_1}$

Substitution of the above forms into equations (2.2-12, 2.2-13), with coefficients equated to equal powers of $\epsilon$ gives the following equations to first order,

\[\epsilon^0 : D_0^2 X_0 + \omega_b^2 X_0 = 0 \tag{3.1-6}\]

\[\epsilon^1 : D_0^2 X_1 + 2D_0 D_1 X_0 + 2\xi_b \omega_b D_0 X_0 + \omega_b^2 X_1
- Y_0 \cos \Omega t = 0 \tag{3.1-7}\]

\[\epsilon^n : D_0^2 Y_0 + \omega_t^2 Y_0 = 0 \tag{3.1-8}\]

\[\epsilon^1 : D_0^2 Y_1 + 2D_0 D_1 Y_0 + 2\xi_t \omega_t D_0 Y_0 + \omega_t^2 Y_1
- X_0 \cos \Omega t = 0 \tag{3.1-9}\]

where $\xi_b = \epsilon \xi_b$, $\xi_t = \epsilon \xi_t$

Solutions to the homogeneous equations (3.1-6, 3.1-8) in complex form are,

\[X_0 = A(T_1) \exp (i\omega_b T_0) + \overline{A}(T_1) \exp (-i\omega_b T_0) \tag{3.1-10}\]

\[Y_0 = B(T_1) \exp (i\omega_t T_0) + \overline{B}(T_1) \exp (-i\omega_t T_0) \tag{3.1-11}\]
Hence, substitution of these zero-order solutions (3.1-10, 3.1-11) into equations (3.1-7, 3.1-9) gives,

\[
\begin{align*}
D_0^2 X_1 + \omega_b^2 X_1 &= \exp \left( i \omega_b T_0 \right) \left[ \frac{\mathcal{A}}{2} \exp i(\omega_b + \Omega - \omega_b)T_0 ight. \\
&\quad + \frac{\mathcal{B}}{2} \exp i(\Omega - \omega_b - \omega_b)T_0 - i2 \omega_b D_1 A \\
&\quad \left. - i2 \xi_b^2 A \right] + cc \tag{3.1-12}
\end{align*}
\]

\[
\begin{align*}
D_0^2 Y_1 + \omega_t^2 Y_1 &= \exp \left( i \omega_t T_0 \right) \left[ \frac{\mathcal{A}}{2} \exp i(\omega_b + \Omega - \omega_b)T_0 \\
&\quad + \frac{\mathcal{A}}{2} \exp i(\Omega - \omega_b - \omega_t)T_0 - i2 \omega_t D_1 B \\
&\quad \left. - i2 \xi_t^2 B \right] + cc \tag{3.1-13}
\end{align*}
\]

where cc denotes the complex conjugates of the preceding terms.

3.1.1 Resonance Conditions

Following the method of multiple scales it is apparent that there are certain frequency relationships which will reduce the exponents in equations (3.1-12, 3.1-13) to zero as required. Only one resonance condition is found to generate nonzero solutions, this is the 'sum' combination resonance condition,

\[
\Omega = \omega_b + \omega_t + \epsilon \rho \tag{3.1.1-1}
\]

where \( \epsilon \rho \) allows for detuning around this frequency.

Thus for nonzero solutions the right hand sides of equations (3.1-12, 3.1-13) are equated to zero and all 'fast' terms omitted to give,

\[
\frac{\mathcal{B}}{2} \exp \left( i \rho T_1 \right) - i2 \omega_b D_1 A - i2 \xi_b^2 \omega_b^2 A = 0 \tag{3.1.1-2}
\]

\[
\frac{\mathcal{A}}{2} \exp \left( i \rho T_1 \right) - i2 \omega_t D_1 B - i2 \xi_t^2 \omega_t^2 B = 0 \tag{3.1.1-3}
\]
Complex amplitudes $A$ and $B$ are written in their polar forms,

$$A = a \exp \left( \frac{i\alpha}{2} \right) \quad B = b \exp \left( \frac{i\beta}{2} \right)$$

$$\bar{A} = a \exp \left( -\frac{i\alpha}{2} \right) \quad \bar{B} = b \exp \left( -\frac{i\beta}{2} \right) \quad (3.1.1-4)$$

with derivatives,

$$\frac{\partial A}{\partial T^1} = \frac{a'}{2} \exp \left( \frac{i\alpha}{2} \right) + ai \frac{\alpha'}{2} \exp \left( \frac{i\alpha}{2} \right) \quad (3.1.1-5)$$

$$\frac{\partial B}{\partial T^1} = \frac{b'}{2} \exp \left( \frac{i\beta}{2} \right) + bi \frac{\beta'}{2} \exp \left( \frac{i\beta}{2} \right) \quad (3.1.1-6)$$

Making substitutions for $(3.1.1-4, 3.1.1-5, 3.1.1-6)$ in equations $(3.1.1-2$ and $3.1.1-3)$ and splitting the emergent equations into their real and imaginary parts, which equate to zero, gives,

$$\frac{b}{4} \cos \psi + \omega_b a \alpha' = 0 \quad (3.1.1-7)$$

$$\frac{b}{4} \sin \psi - \omega_b a' - \xi_b \omega^2_b a = 0 \quad (3.1.1-8)$$

$$\frac{a}{4} \cos \psi + \omega_t b \beta' = 0 \quad (3.1.1-9)$$

$$\frac{a}{4} \sin \psi - \omega_t b' - \xi_t \omega_t^2 b = 0 \quad (3.1.1-10)$$

Where, for an autonomous system it is necessary to put,

$$\rho T^1 - \beta - \alpha = \psi \quad (3.1.1-11)$$

3.1.2 Solvability Equations

Stipulating that $a$ and $b$ do not change with respect to slow time $T^1$, i.e. $a' = b' = 0$

the following solvability equations result,
\[ b \cos \frac{\psi}{4} \cdot \omega_b a' = 0 \]  
(3.1.2-1)

\[ b \sin \frac{\psi}{4} \cdot \omega_b^2 a = 0 \]  
(3.1.2-2)

\[ \frac{a}{4} \cos \psi + \omega_t b \beta' = 0 \]  
(3.1.2-3)

\[ \frac{a}{4} \sin \psi - \omega_t^2 b = 0 \]  
(3.1.2-4)

In the same way as \( a' = b' = 0 \), \( \psi' = 0 \) therefore from this substitution and some further manipulation the following relationships emerge,

\[ \beta' = \frac{\rho}{\left[ 1 + \frac{\omega_b \xi_b}{\omega_t \xi_t} \right]} \]  
(3.1.2-5)

\[ \alpha' = \frac{\rho}{\left[ 1 + \frac{\omega_t \xi_t}{\omega_b \xi_b} \right]} \]  
(3.1.2-6)

3.1.3 Transition curve expression for combined fundamental bending and torsion modes

With substitution of (3.1.2-5, 3.1.2-6) back into the solvability equations and further algebraic manipulation, a biquadratic expression in \( \rho \) results, for which two cases are considered,

(a) case when \( \xi_b = \xi_t = 0 \)

reduces to \( \rho^4 - \frac{1}{256 \omega_b^2 \omega_t^2} = 0 \)  
(3.1.3-1)

or, \( \rho = \pm \frac{1}{\sqrt[4]{\omega_b \omega_t}} \)  
(3.1.3-2)

substituting for \( \rho \) in (3.1.1-1) gives,
\[ \Omega = \omega_b + \omega_t \pm \frac{\epsilon}{4\omega_b \omega_t} \]  \hspace{1cm} (3.1.3-3)

This is the expression for the theoretical transition curve for zero system damping.

(b) case when \( \xi_b \neq 0, \xi_t \neq 0 \)

the biquadratic expression can be rewritten as,

\[
|\rho| = \left[ \frac{1 + \frac{\omega_t \xi_t}{\omega_b \xi_b}}{2 \frac{\omega_t \xi_t}{\sqrt{\omega_b \xi_b}}} \right] \sqrt{\frac{1}{4\omega_b \omega_t} - 4\omega_b \omega_t \xi_b \xi_t} \]  \hspace{1cm} (3.1.3-4)

Which, when substituted back into (3.1.1-1) gives the expression for the transition curve for this combination resonance with nonzero system damping. This is an identical result to that obtained by previous researchers (Dugundji and Mukhopadhyay, \(^3\)) who used the method of harmonic balance in a similar dynamical problem.

3.2 Expansion of the 3 degree of freedom equations to second order

In order to generate further combination resonances, which will produce nonzero solutions to the governing equations, a second order multiple scales expansion is required. The 3 degree of freedom governing equations include extra small terms as a result of not discounting the small inplane acceleration \( V_0 \) in the previous derivation. Because of the extra terms in the governing equations and the higher order of expansion a considerable number of intermediate calculations are performed.
The power series in $\varepsilon^2$ are now truncated at $O(\varepsilon^2)$, thus the system variables in the 3 d.o.f. equations are expressed as,

$$U_1 = U_{10} + \varepsilon U_{11} + \varepsilon^2 U_{12} + \ldots$$  \hspace{1cm} (3.2-1)

$$U_2 = U_{20} + \varepsilon U_{21} + \varepsilon^2 U_{22} + \ldots$$  \hspace{1cm} (3.2-2)

$$\dot{\phi}_1 = \dot{\phi}_{10} + \varepsilon \dot{\phi}_{11} + \varepsilon^2 \dot{\phi}_{12} + \ldots$$  \hspace{1cm} (3.2-3)

with derivatives,

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots$$  \hspace{1cm} (3.2-4)

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + 2\varepsilon^2 D_0 D_2 + \varepsilon^2 D_1^2 + \ldots$$  \hspace{1cm} (3.2-5)

Substitution of (3.2-1 thru 3.2-5) into equations (2.3-8, 2.3-9, 2.3-10) with coefficients equated to equal powers of $\varepsilon$ produces,

$$\varepsilon^0 : \quad D_0^2 U_{10} + \omega^2 U_{10} = 0$$  \hspace{1cm} (3.2-6)

$$\varepsilon^1 : \quad D_0^2 U_{11} + \omega^2 U_{11} = b_1 \omega_0 \sum \phi_{10} \cos \Omega t - 2D_0 D_1 U_{10}$$

$$- 2b_1 \phi_{10} D_0 U_{20} \phi_{10}$$

$$- 2D_0 \phi_{10} U_{20} D_0 \phi_{10}$$  \hspace{1cm} (3.2-7)

$$\varepsilon^2 : \quad D_0^2 U_{12} + \omega^2 U_{12} = b_1 \omega_0 \sum \phi_{11} \cos \Omega t - 2D_0 D_1 U_{11}$$

$$- 2D_0 D_2 U_{10} - D_1^2 U_{10}$$

$$- 2b_1 \phi_{10} D_0 U_{20} \phi_{11}$$

$$- 2b_1 \phi_{10} U_{20} D_0 \phi_{11}$$

$$- 2b_1 \phi_{10} D_0 U_{20} \phi_{10} - 2b_1 \phi_{10} U_{20} D_0 \phi_{10}$$
\[ 2B_1 b_1 \Phi_{10} D_0 U_{11} D_0 \Phi_{10} \]
\[ 2B_1 b_2 \Phi_{11} D_0 U_{10} D_0 \Phi_{10} - 2 b_6 \Omega_{b_1} D_0 U_{11} \]
\[ 2B_1 b_2 \Phi_{10} D_0 U_{20} D_0 \Phi_{11} \]
\[ 2B_1 b_2 \Phi_{10} D_0 U_{20} D_1 \Phi_{10} \]
\[ 2B_1 b_2 \Phi_{10} D_0 U_{21} D_0 \Phi_{10} \]
\[ 2B_1 b_2 \Phi_{10} D_0 U_{20} D_0 \Phi_{10} \]
\[ 2B_1 b_2 \Phi_{11} D_0 U_{20} D_0 \Phi_{10} \]

\[ \varepsilon^0 : D^2 U_{20} + \Omega^2_{b_2} U_{20} = 0 \quad (3.2-9) \]

\[ \varepsilon^1 : D^2 U_{21} + \Omega^2_{b_2} U_{21} = b_2 \Omega_0 \Phi_{10} \cos \Omega t - 2D_0 D_1 U_{20} \]
\[ - 2B_2 b_2 \Phi_{10} D_0 U_{20} D_0 \Phi_{10} \]
\[ - 2B_2 b_2 \Phi_{11} D_0 U_{10} D_0 \Phi_{10} \quad (3.2-10) \]

\[ \varepsilon^2 : D^2 U_{22} + \Omega^2_{b_2} U_{22} = b_2 \Omega_0 \Phi_{11} \cos \Omega t - 2D_0 D_1 U_{21} \]
\[ - 2D_0 D_2 U_{20} D^2_{120} \]
\[ - 2B_2 b_2 \Phi_{10} D_0 U_{20} D_0 \Phi_{11} \]
\[ 2B_2 b_2 \Phi_{10} D_0 U_{20} D_0 \Phi_{10} \]
\[ - 2B_2 b_2 \Phi_{11} D_0 U_{21} D_0 \Phi_{10} \]
\[ \epsilon^0 : D_t^2 \phi_{10} + \omega^2_{11} \phi_{10} = 0 \]  

\[ \epsilon^1 : D_t^2 \phi_{11} + \omega^2_{11} \phi_{11} = F_{1} U_{10} \cos \Omega t + F_{2} U_{20} \cos \Omega t \]
- \[ - 2D_0^0 \phi_{10} - 2R \cdot b \cdot U_{10} D_0^0 \phi_{10} \]
- \[ - 2R \cdot b \cdot U_{20} D_0^0 \phi_{10} \]
- \[ - 2RB \cdot b \cdot U_{10} D_0^0 \phi_{10} \]
- \[ - 2RB \cdot b \cdot U_{20} D_0^0 \phi_{10} \]

\[ \epsilon^2 : D_t^2 \phi_{12} + \omega^2_{12} \phi_{12} = F_{1} U_{11} \cos \Omega t + F_{2} U_{21} \cos \Omega t \]
- \[ - 2D_0^0 \phi_{11} - 2D_0^0 \phi_{21} + D_0^0 \phi_{10} \]
- \[ - 2RB \cdot b \cdot U_{20} D_0^0 \phi_{10} \]
- \[ - 2RB \cdot b \cdot U_{20} D_0^0 \phi_{10} \]
- \[ - 2RB \cdot b \cdot U_{20} D_0^0 \phi_{10} \]
- \[ - 2RB \cdot b \cdot U_{20} D_0^0 \phi_{10} \]
\[-2Rb b_1 U_{20} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{21} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{10} D_{10} U_{21} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{10} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{11} D_{10} U_{20} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{10} D_{10} U_{11} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{10} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{11} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{20} D_{10} U_{20} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{20} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{10} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{10} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{11} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{10} D_{10} U_{11} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{10} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{11} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{20} D_{10} U_{20} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{20} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{10} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{11} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{10} D_{10} U_{11} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{10} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{11} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{20} D_{10} U_{20} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{20} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{10} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]
\[-2Rb b_1 U_{11} D_{10} U_{10} D_{10} \dot{\phi}_{10}\]

where, \( R = \frac{M_o}{I_o} \), \( F_1 = Rb_1 w_0 \mathcal{J}^2 \)

\[ F_2 = RU_2 w_0 \mathcal{J}^2 \]
\[ \xi_b = \epsilon^2 \xi_b \]
\[ \xi_t = \epsilon^2 \xi_t \]
Solutions to equations (3.2-6, 3.2-9, 3.2-12) are,

\[
U_{10} = C_1 (T_1, T_2) \exp (i \omega b_1 T_0) + C_1 (T_1, T_2) \exp (-i \omega b_1 T_0)
\]

\[
U_{20} = C_2 (T_1, T_2) \exp (i \omega b_2 T_0) + C_2 (T_1, T_2) \exp (-i \omega b_2 T_0)
\]

\[
\dot{\phi}_{10} = C_3 (T_1, T_2) \exp (i \omega t_1 T_0) + C_3 (T_1, T_2) \exp (-i \omega t_1 T_0)
\]

(3.2-15, 3.2-16, 3.2-17)

These solutions are substituted back into equations (3.2-7, 3.2-10, 3.2-13) giving respectively,

\[
\frac{d^2 U_{11}}{d t_1^2} + \omega_{b1}^2 U_{11} = \frac{b_1}{2} W_0 \mathcal{S}^2 C_3 \exp (\mathcal{S} + \omega t_1) T_0
\]

\[+ \frac{b_1}{2} W_0 \mathcal{S}^2 C_3 \exp (-\mathcal{S} + \omega t_1) T_0
\]

\[+ \frac{b_1}{2} W_0 \mathcal{S}^2 C_3 \exp (\mathcal{S} - \omega t_1) T_0
\]

\[+ \frac{b_1}{2} W_0 \mathcal{S}^2 C_3 \exp (-\mathcal{S} - \omega t_1) T_0
\]

\[- i2D_1 C_1 \omega_{b1} \exp (i \omega_{b1} T_0)
\]

\[- i2D_1 C_1 \omega_{b1} \exp (i \omega_{b1} T_0)
\]

\[+ 2B_1 b_1 C_2^2 C_1 \omega_{b1} \omega_{t1} \exp (\omega_{b1} + 2 \omega_{t1}) T_0
\]

\[- 2B_1 b_1 C_2^2 C_1 \omega_{b1} \omega_{t1} \exp (2 \omega_{t1} - \omega_{b1}) T_0
\]

\[- 2B_1 b_1 C_2^2 C_1 \omega_{b1} \omega_{t1} \exp (\omega_{b1} - 2 \omega_{t1}) T_0
\]

\[+ 2B_1 b_1 C_2^2 C_1 \omega_{b1} \omega_{t1} \exp (-\omega_{b1} - 2 \omega_{t1}) T_0
\]

\[+ 2B_1 b_2 C_2^2 C_2 \omega_{b2} \omega_{t1} \exp (\omega_{b2} + 2 \omega_{t1}) T_0
\]

\[- 2B_1 b_2 C_2^2 C_2 \omega_{b2} \omega_{t1} \exp (2 \omega_{t1} - \omega_{b2}) T_0
\]
\[ \begin{align*}
\mathcal{D}_0^2 u_{21} + \omega_{b2}^2 u_{21} &= \frac{b_2}{2} W_0 \mathcal{N}^2 C_2 \exp \left( i \mathcal{N} + \mathcal{O}_{t1} \right) T_0 \\
&\quad + \frac{b_2}{2} W_0 \mathcal{N}^2 C_3 \exp \left( -i \mathcal{N} + \mathcal{O}_{t1} \right) T_0 \\
&\quad + \frac{b_2}{2} W_0 \mathcal{N}^2 \mathcal{C}_3 \exp \left( -i \mathcal{N} - \mathcal{O}_{t1} \right) T_0 \\
&\quad + \frac{b_2}{2} W_0 \mathcal{N}^2 \mathcal{C}_3 \exp \left( i \mathcal{N} + \mathcal{O}_{t1} \right) T_0 \\
&\quad + i \mathcal{D}_1 C_2 \omega_{b2} \exp \left( i \mathcal{O}_{b2} T_0 \right) \\
&\quad + i \mathcal{D}_1 \mathcal{C}_2 \omega_{b2} \exp \left( i \mathcal{O}_{b2} T_0 \right) \\
&\quad + 2B_2 b_2 \mathcal{C}_2 C_2 \omega_{b2} \omega_{t1} \exp \left( \omega_{b2} + 2 \mathcal{O}_{t1} \right) T_0 \\
&\quad - 2B_2 b_2 \mathcal{C}_2 \mathcal{C}_2 \omega_{b2} \omega_{t1} \exp \left( 2 \mathcal{O}_{t1} - \omega_{b2} \right) T_0 \\
&\quad - 2B_2 \mathcal{C}_2 \mathcal{C}_2 \omega_{b2} \omega_{t1} \exp \left( \omega_{b2} - 2 \mathcal{O}_{t1} \right) T_0 \\
&\quad + \mathcal{R}_2 b_2 \mathcal{C}_2 \mathcal{C}_2 \omega_{b2} \omega_{t1} \exp \left( - \omega_{b2} - 2 \mathcal{O}_{t1} \right) T_0 \\
&\quad + 2B_2 b_1 \mathcal{C}_1 \omega_{b1} \omega_{t1} \exp \left( \omega_{b1} + 2 \mathcal{O}_{t1} \right) T_0 \\
&\quad - 2B_2 b_1 \mathcal{C}_1 \omega_{b1} \omega_{t1} \exp \left( 2 \mathcal{O}_{t1} - \omega_{b1} \right) T_0 \\
&\quad + 2B_2 b_1 \mathcal{C}_1 \omega_{b1} \omega_{t1} \exp \left( \omega_{b1} - 2 \mathcal{O}_{t1} \right) T_0 \\
&\quad - 2B_2 b_1 \mathcal{C}_1 \omega_{b1} \omega_{t1} \exp \left( - \omega_{b1} - 2 \mathcal{O}_{t1} \right) T_0 \\
&\quad + 2B_2 b_1 \mathcal{C}_1 \omega_{b1} \omega_{t1} \exp \left( - \omega_{b1} - 2 \mathcal{O}_{t1} \right) T_0 \\
\end{align*} \]
\[ D_{\omega_{l1}}^2 \phi_{l1} + \omega_{l1}^2 \phi_{l1} = \frac{F_1}{2} C_1 \exp i (\mathcal{S} + \omega_{b1}) T_0 + \frac{F_1}{2} C_1 \exp i (\omega_{b1} - \mathcal{S}) T_0 \]

\[ + \frac{F_1}{2} C_1 \exp i (-\mathcal{S} - \omega_{b1}) T_0 + \frac{F_1}{2} C_1 \exp i (-\omega_{b1} - \mathcal{S}) T_0 \]

\[ + \frac{F_2}{2} C_2 \exp i (\mathcal{S} + \omega_{b2}) T_0 + \frac{F_2}{2} C_2 \exp i (\omega_{b2} - \mathcal{S}) T_0 \]

\[ + \frac{F_2}{2} C_2 \exp i (-\mathcal{S} - \omega_{b2}) T_0 + \frac{F_2}{2} C_2 \exp i (-\omega_{b2} - \mathcal{S}) T_0 \]

\[-i2D_1 C_3 \omega_{t1} \exp i (\omega_{t1} T_0) + i2D_1 C_3 \omega_{t1} \exp i (\omega_{t1} T_0) \]

\[ + 2RB_2 b_2 C_2 \omega_{b2} \omega_{t1} \exp i (2\omega_{b2} + \omega_{t1}) T_0 \]

\[ - 2RB_2 b_2 C_2 \omega_{b2} \omega_{t1} \exp i (2\omega_{b2} - \omega_{t1}) T_0 \]

\[ - 2RB_2 b_2 C_2 \omega_{b2} \omega_{t1} \exp i (\omega_{t1} - 2\omega_{b2}) T_0 \]

\[ + 2RB_2 b_2 C_2 \omega_{b2} \omega_{t1} \exp i (-\omega_{t1} - 2\omega_{b2}) T_0 \]

\[ + 2RB_2 b_1 C_2 C_2 \omega_{b1} \omega_{t1} \exp i (\omega_{b1} + \omega_{b2} + \omega_{t1}) T_0 \]

\[ - 2RB_2 b_1 C_2 C_2 \omega_{b1} \omega_{t1} \exp i (\omega_{b2} - \omega_{b1} + \omega_{t1}) T_0 \]

\[ + 2RB_2 b_1 C_2 C_2 \omega_{b1} \omega_{t1} \exp i (\omega_{b1} - \omega_{b2} + \omega_{t1}) T_0 \]

\[ - 2RB_2 b_1 C_2 C_2 \omega_{b1} \omega_{t1} \exp i (\omega_{t1} - \omega_{b1} - \omega_{b2}) T_0 \]

\[ - 2RB_2 b_1 C_2 C_2 \omega_{b1} \omega_{t1} \exp i (\omega_{b1} + \omega_{b2} - \omega_{t1}) T_0 \]

\[ + 2RB_2 b_1 C_2 C_2 \omega_{b1} \omega_{t1} \exp i (\omega_{b2} - \omega_{b1} - \omega_{t1}) T_0 \]

\[ - 2RB_2 b_1 C_2 C_2 \omega_{b1} \omega_{t1} \exp i (\omega_{b1} - \omega_{b2} - \omega_{t1}) T_0 \]

\[ + 2RB_2 b_1 C_2 C_2 \omega_{b1} \omega_{t1} \exp i (-\omega_{b1} - \omega_{b2} - \omega_{t1}) T_0 \]

\[ + 2RB_2 b_1 C_2 C_2 \omega_{b1} \omega_{t1} \exp i (\omega_{b1} + \omega_{b2} + \omega_{t1}) T_0 \]
\[-2R_2 b_1 c_2 \bar{c}_1 \omega_{t1} \exp i (\omega_{t1} - \omega_{b1} T_0) \]
\[+ 2R_2 b_1 c_2 \bar{c}_1 \omega_{b1} \exp i (\omega_{b1} + \omega_{b2} + \omega_{t1} T_0) \]
\[-7R_2 b_1 c_2 \omega_{b1} \exp i (\omega_{b1} - \omega_{b2} T_0) \]
\[-2R_2 b_1 c_2 \omega_{t1} \exp i (\omega_{b1} + \omega_{b2} - \omega_{t1} T_0) \]
\[-2R_2 b_1 \bar{c}_1 \omega_{t1} \exp i (\omega_{b1} - \omega_{b2} - \omega_{t1} T_0) \]
\[+ 2R_2 b_1 \bar{c}_1 \omega_{b2} \exp i (\omega_{b1} - \omega_{b2} - \omega_{t1} T_0) \]
\[+ 2R_2 b_1 \bar{c}_1 \omega_{t1} \exp i (-\omega_{b1} - \omega_{b2} - \omega_{t1} T_0) \]
\[+ 2R_2 b_1 c_2 \omega_{t1} \exp i (2\omega_{b1} + \omega_{t1} T_0) \]
\[-2R_1 b_1 \bar{c}_1 \omega_{b1} \omega_{t1} \exp i (\omega_{b1} - \omega_{t1} T_0) \]
\[-2R_1 b_1 c_2 \omega_{b1} \omega_{t1} \exp i (\omega_{b1} - 2\omega_{b1} T_0) \]
\[+ 2R_1 b_1 \bar{c}_1 \omega_{b1} \omega_{t1} \exp i (2\omega_{b1} - \omega_{t1} T_0) \]

Equations (3.2-18, 3.2-19, 3.2-20) represent the first order perturbations of \( U_1, U_2, \phi_1 \) and contain some resonant terms (secular terms) which must be eliminated before solutions can be sought. From the last three equations the secular terms are, respectively,*

\[i 2D_1 c_1 \omega_{b1} \exp i \omega_{b1} T_0 + c.c\]
\[i 2D_1 c_2 \omega_{b2} \exp i \omega_{b2} T_0 + c.c\]
\[i 2D_1 c_3 \omega_{t1} \exp i \omega_{t1} T_0 + c.c\]

* In order to analyse second order resonances without excessive complexity it is assumed at this stage that first order external and internal resonances are not realised.
The required conditions for finite solutions to equations (3.2-18, 3.2-19, 3.2-20) are that the above terms equal zero, and therefore it is obvious that,

$$D_1 C_1 = D_1 C_2 = D_1 C_3 = 0$$

Hence $C_1$, $C_2$, $C_3$ are only functions of slowtime scale $T_2$.

Thus, $C_1 = C_1 (T_2)$, $C_2 = C_2 (T_2)$, $C_3 = C_3 (T_2)$

Solutions to the first order perturbation equations

With $D_1 C_1 = 0$ the solution to equation (3.2-18) is of the form,

$$U_{11} = K_1 \exp (\mathcal{N} + \omega_{t1}) T_0 + K_2 \exp (\mathcal{N} - \omega_{t1}) T_0$$

$$+ K_3 \exp (\omega_{b1} + 2\omega_{t1}) T_0$$

$$- K_4 \exp (2\omega_{t1} - \omega_{b1}) T_0 + K_5 \exp (\omega_{b2} + 2\omega_{t1}) T_0$$

$$- K_6 \exp (\omega_{b2} - 2\omega_{t1}) T_0 + c.c$$  \hspace{1cm} (3.2-21)

Where the $K_j$ represent amplitude and frequency dependent coefficients given in full in Appendix 3,

With $D_1 C_2 = 0$ the solution to equation (3.2-19) is of the form,

$$U_{21} = K_7 \exp (\mathcal{N} + \omega_{t1}) T_0 + K_8 \exp (\mathcal{N} - \omega_{t1}) T_0$$

$$+ K_9 \exp (\omega_{b2} + 2\omega_{t1}) T_0$$

$$- K_{10} \exp (2\omega_{t1} - \omega_{b2}) T_0 + K_{11} \exp (\omega_{b1} + 2\omega_{t1}) T_0$$

$$- K_{12} \exp (\omega_{b1} - 2\omega_{t1}) T_0 + c.c$$  \hspace{1cm} (3.2-22)
and finally with $D_1C_3 = 0$ the solution to (3.2-20) is of the form,

\[ \phi_{11} = K_{13} \exp(i(2\omega_{b2} + \omega_{t1})) T_0 - K_{14} \exp(i(2\omega_{b2} - \omega_{t1})) T_0 + K_{15} \exp(i(\omega_{b1} + \omega_{b2} + \omega_{t1})) T_0 - K_{16} \exp(i(\omega_{b2} - \omega_{b1} + \omega_{t1})) T_0 - K_{17} \exp(i(\omega_{b1} + \omega_{h2} - \omega_{t1})) T_0 + K_{18} \exp(i(\omega_{b2} - \omega_{b1} - \omega_{t1})) T_0 + K_{19} \exp(i(2\omega_{b1} + \omega_{t1})) T_0 - K_{20} \exp(i(\omega_{t1} - 2\omega_{b1})) T_0 + K_{21} \exp(i(\omega_{b1} + \omega_{b2})) T_0 + K_{22} \exp(i(\omega_{b1} - \omega_{b2})) T_0 + K_{23} \exp(i(\omega_{b1} + \omega_{b2})) T_0 + K_{24} \exp(i(\omega_{b1} - \omega_{b2})) T_0 + \text{c.c} \]

(3.2-23)

**Second order perturbation equations**

Solutions (3.2-21, 3.2-22, 3.2-23) may be substituted into equations (3.2-8, 3.2-11, 3.2-14) to give the final forms of the second order perturbation equations. This process requires lengthy calculations and the ensuing amplitude and frequency dependent coefficients $L_1$ are very complicated and tedious to reproduce in their entirety. Therefore only those coefficients that pertain to the resonance conditions to be investigated are shown in Appendix 4.

The second order perturbation equations are, respectively,

\[ D_0^2U_{12} + \omega_{b1}^2U_{12} = \exp(i\omega_{b1} T_0) \left[ L_1 \exp(i(2\omega_{b2} + \omega_{t1} - \omega_{b1} + \omega_h)) T_0 + L_2 \exp(i(2\omega_{b2} - \omega_{b1} - \omega_{t1} + \omega_h)) T_0 + L_3 \exp(i(\omega_{b2} - \omega_{t1} + \omega_h)) T_0 + L_4 \exp(i(\omega_{b2} - 2\omega_{b1} + \omega_{t1} + \omega_h)) T_0 \right] \]
+ L_5 \exp (\omega_{b2} - \omega_{t1} + \Omega) T_0
+ L_6 \exp (\omega_{b2} - 2\omega_{b1} + \omega_{t1} + \Omega) T_0
+ L_7 \exp (\omega_{b1} + \omega_{t1} + \Omega) T_0
+ L_8 \exp (\omega_{t1} - 3\omega_{b1} + \Omega) T_0
+ L_9 \exp (i2\Omega) T_0 + L_{10} \exp (2\Omega - 2\omega_{b1}) T_0
+ L_{11} \exp (2\Omega + \omega_{b2} - \omega_{b1}) T_0
+ L_{12} \exp (2\Omega - \omega_{b2} - \omega_{b1}) T_0 + L_{13}
+ L_{14} \exp (\Omega + \omega_{t1} - \omega_{b1}) T_0
+ L_{15} \exp (\Omega - \omega_{t1} - \omega_{b1}) T_0
+ L_{16} \exp (\Omega - \omega_{b2} + \omega_{t1}) T_0
+ L_{17} \exp (\Omega - 2\omega_{b1} - \omega_{b2} + \omega_{t1}) T_0
+ L_{18} \exp (\Omega - \omega_{t1} + \omega_{b1}) T_0
+ L_{19} \exp (\Omega - \omega_{b2} - \omega_{t1}) T_0
+ L_{20} \exp (\Omega - 3\omega_{b1} - \omega_{t1}) T_0
+ L_{21} \exp (\Omega - 2\omega_{b1} - \omega_{t1} - \omega_{b2}) T_0
+ L_{22} \exp (\Omega + 3\omega_{t1} - \omega_{b1}) T_0
+ L_{23} \exp (\Omega - 3\omega_{t1} - \omega_{b1}) T_0
+ L_{24} \exp (i\Omega) T_0
+ L_{25} \exp (\Omega - 2\omega_{b2} + \omega_{t1} - \omega_{b1}) T_0
+ L_{26} \exp (\Omega - 2\omega_{b2} - \omega_{t1} - \omega_{b1}) T_0
+ c.c

(3.2-24)
\[ D^2_{0} u_{22} + \omega_{b2}^2 u_{22} = \exp \left( i \omega_{b2} T_0 \right) \left[ L_{27} \exp \left( \mathcal{N} + \omega_{b2} + \omega_{t1} \right) T_0 \right. \\
+ L_{28} \exp \left( \mathcal{N} + \omega_{b2} - \omega_{t1} \right) T_0 \\
+ L_{29} \exp \left( \mathcal{N} + \omega_{b1} + \omega_{t1} \right) T_0 \\
+ L_{30} \exp \left( \mathcal{N} - \omega_{b1} + \omega_{t1} \right) T_0 \\
+ L_{31} \exp \left( \mathcal{N} + \omega_{b1} - \omega_{t1} \right) T_0 \\
+ L_{32} \exp \left( \mathcal{N} - \omega_{b1} - \omega_{t1} \right) T_0 \\
+ L_{33} \exp \left( \mathcal{N} + 2 \omega_{b1} + \omega_{t1} - \omega_{b2} \right) T_0 \\
+ L_{34} \exp \left( \mathcal{N} + \omega_{t1} - 2 \omega_{b1} - \omega_{b2} \right) T_0 \\
+ L_{35} \exp \left( 2 \mathcal{N} + \omega_{b1} - \omega_{b2} \right) T_0 \\
+ L_{36} \exp \left( 2 \mathcal{N} - \omega_{b1} - \omega_{b2} \right) T_0 \\
+ L_{37} \exp \left( i 2 \mathcal{N} T_0 \right) + L_{38} \exp \left( 2 \mathcal{N} - 2 \omega_{b2} \right) T_0 \\
+ L_{39} \exp \left( \mathcal{N} + \omega_{t1} - \omega_{b2} \right) T_0 \\
+ L_{40} \exp \left( \mathcal{N} - \omega_{t1} + \omega_{b2} \right) T_0 + L_{41} \\
+ L_{42} \exp \left( \mathcal{N} + \omega_{b1} + \omega_{t1} - 2 \omega_{b2} \right) T_0 \\
+ L_{43} \exp \left( \mathcal{N} - \omega_{b1} + \omega_{t1} - 2 \omega_{b2} \right) T_0 \\
+ L_{44} \exp \left( \mathcal{N} - 3 \omega_{b2} + \omega_{t1} \right) T_0 \\
+ L_{45} \exp \left( \mathcal{N} + \omega_{b1} - \omega_{t1} - 2 \omega_{b2} \right) T_0 \\
+ L_{46} \exp \left( \mathcal{N} - \omega_{b1} - \omega_{t1} - 2 \omega_{b2} \right) T_0 \\
+ L_{47} \exp \left( \mathcal{N} - 3 \omega_{b2} - \omega_{t1} \right) T_0 \\
+ L_{48} \exp \left( \mathcal{N} + 3 \omega_{t1} - \omega_{b2} \right) T_0 \]
+ L_{49} \exp i (\Omega - 3\omega_{t1} - \omega_{b2}) T_0

+ L_{50} \exp i (\Omega + 2\omega_{b1} - \omega_{t1} - \omega_{b2}) T_0

+ L_{51} \exp i (\Omega - 2\omega_{b1} - \omega_{t1} - \omega_{b2}) T_0

+ c.c.] 

(3.2-25)

\partial_0^2 \phi_{t12} + \omega_{t1}^2 \phi_{t12} = \exp (i\omega_{t1} T_0) \left[ L_{52} \exp i (\Omega + \omega_{b1} - \omega_{t1}) T_0

+ L_{53} \exp i (\Omega - \omega_{b1} - \omega_{t1}) T_0

+ L_{54} \exp i (\Omega + \omega_{b2} - \omega_{t1}) T_0

+ L_{55} \exp i (\Omega - \omega_{b2} - \omega_{t1}) T_0 + L_{56}

+ L_{57} \exp i (\Omega + \omega_{t1} + \omega_{b2}) T_0

+ L_{58} \exp i (\Omega - \omega_{b2} + \omega_{t1}) T_0

+ L_{59} \exp i (\Omega + \omega_{b1} + \omega_{t1}) T_0

+ L_{60} \exp i (\Omega + \omega_{t1} - \omega_{b1}) T_0

+ L_{61} \exp i (\Omega - 3\omega_{t1} + \omega_{b2}) T_0

+ L_{62} \exp i (\Omega - 3\omega_{t1} - \omega_{b2}) T_0

+ L_{63} \exp i (\Omega - 3\omega_{t1} + \omega_{b1}) T_0

+ L_{64} \exp i (\Omega - 3\omega_{t1} - \omega_{b1}) T_0

+ L_{65} \exp i (\Omega + \omega_{b1} + 2\omega_{b2} - \omega_{t1}) T_0

+ L_{66} \exp i (\Omega - \omega_{b1} + 2\omega_{b2} - \omega_{t1}) T_0

+ L_{67} \exp i (\Omega + 3\omega_{b2} - \omega_{t1}) T_0
+ L_{68} \exp(i(\Omega - 2\omega_{b1} + \omega_{b2} - \omega_{t1}) T_{0})
+ L_{69} \exp(i(\Omega + 2\omega_{b1} + \omega_{b2} - \omega_{t1}) T_{0})
+ L_{70} \exp(i(\Omega + \omega_{b1} - 2\omega_{b2} - \omega_{t1}) T_{0})
+ L_{71} \exp(i(\Omega - \omega_{b1} - 2\omega_{b2} - \omega_{t1}) T_{0})
+ L_{72} \exp(i(\Omega - 3\omega_{b2} - \omega_{t1}) T_{0})
+ L_{73} \exp(i(\Omega + 2\omega_{b1} - \omega_{b2} - \omega_{t1}) T_{0})
+ L_{74} \exp(i(\Omega - 2\omega_{b1} - \omega_{b2} - \omega_{t1}) T_{0})
+ L_{75} \exp(i(\Omega + 3\omega_{b1} - \omega_{t1}) T_{0})
+ L_{76} \exp(i(\Omega - 3\omega_{b1} - \omega_{t1}) T_{0})
+ L_{77} \exp(2\Omega T_{0}) + L_{78} \exp(2\Omega - 2\omega_{t1}) T_{0}
+ \text{c.c.} \right) \quad (3.2-26)

3.2.1 Resonance Conditions

The exponents of equations (3.2-24, 3.2-25, 3.2-26) admit many theoretical resonance conditions, approximately a quarter of which are simultaneous between all three modes, and half between some pairs of modes (Table 3.2.1-1).

The first order combination resonance previously discussed is represented here as expected, however the effect to be studied in the subsequent sections of this chapter is only generated by the second order expansion and is simultaneous only between the fundamental and second bending modes equations.

This combination resonance is,

\[ \Omega = \frac{1}{2}\left[ \omega_{b1} + \omega_{b2} \right] + \epsilon^2 \rho_1 \] \quad (3.2.1-1)

where \( \epsilon^2 \rho_1 \) is a detuning parameter.
Table 3.2.1-1 Theoretical resonance conditions generated by the second order perturbation equations.

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<thead>
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a refers to resonance (3.1.1-1)

b refers to resonance (3.2.1-1)

c refers to resonance (3.5.1-1)
This combination resonance is of particular interest in that it excludes the presence of the torsion mode, even though its participation is kinematically necessary for any out of plane motion.

Experimental observations have shown that the level of input acceleration required to generate the resonance (3.2.1-1) is substantially higher than that for resonance (3.1.1-1) as shown in the transition curves in section 3.3.

It is interesting to note that there is an initial intermodal exchange of energy after which the system 'settles' into its unstable state for the resonance condition (3.2.1-1), the exchange occurring principally from the torsion mode to the second bending mode, effectively from a lower to a higher mode as,

$$\omega_{b2} > \omega_{t1}$$

for both short and long beams, thus there is a very close kinematic relationship between the two combination resonances.

The rest of this work is concerned with an investigation into the close proximity effects of these two combination resonances, both as instabilities and as steady states in a coupled system.

Table (3.2.1-1) shows the predicted resonance conditions to second order but is incomplete in that there would be again as many resonances resulting from the complex conjugate terms of equations (3.2-24, 3.2-25, 3.2-26). However only resonances which suggest $Q > 0$ are of any physical significance, and all conditions quoted which imply $Q < 0$ have their positive counterparts in the conjugate terms, hence a complete representation of all resonance conditions to second order is possible from studying the table.
Resonance

(a) is the previously explored first order effect
(b) is the second order effect of interest.
(c) is another first order resonance and is used in later sections.
3.2.2 Solvability Equations

Substituting for the resonance (3.2.1-1) across the three modes, the final order perturbation equations can be reduced to the following,

\[
D_0^2 U_{12} + \omega_{b1}^2 U_{12} = \exp \left( i \omega_{b1} T_0 \right) \left[ L_{12} \exp \left( i \epsilon^2 \rho_{1T_0} \right) + L_{13} + F.N.R.T + \text{c.c.} \right]
\]

\[
D_0^2 U_{22} + \omega_{b2}^2 U_{22} = \exp \left( i \omega_{b2} T_0 \right) \left[ L_{36} \exp \left( i \epsilon^2 \rho_{1T_0} \right) + L_{41} + F.N.R.T + \text{c.c.} \right]
\]

\[
D_0^2 \phi_{12} + \omega_{t1}^2 \phi_{12} = \exp \left( i \omega_{t1} T_0 \right) \left[ L_{56} + F.N.R.T + \text{c.c.} \right]
\]

Where F.N.R.T denotes 'fast nonresonant terms'.

The following forms are taken for the complex amplitudes and their derivatives,

\[
C_1 = \frac{c_1}{2} \exp \left( i \beta_1 \right) \quad C_2 = \frac{c_2}{2} \exp \left( i \beta_2 \right) \quad C_3 = \frac{c_3}{2} \exp \left( i \delta \right)
\]

\[
D_2 C_1 = \frac{c_1'}{2} \exp \left( i \beta_1 \right) + \frac{c_1}{2} i \beta_1' \exp \left( i \beta_1 \right)
\]

\[
D_2 C_2 = \frac{c_2'}{2} \exp \left( i \beta_2 \right) + \frac{c_2}{2} i \beta_2' \exp \left( i \beta_2 \right)
\]

\[
D_2 C_3 = \frac{c_3'}{2} \exp \left( i \delta \right) + \frac{c_3}{2} i \delta' \exp \left( i \delta \right)
\]

Re-substitution of the required \( T_1 \) coefficients and forms (3.2.2-4 through 3.2.2-9) and also,

\[
\Psi = \epsilon^2 \rho_{1T_0} - \beta_1 - \beta_2 = \rho_{1T_2} - \beta_1 - \beta_2
\]
with the necessary algebraic manipulation to group 'like' terms, the following slow time amplitude and phase equations emerge,

$$c_1' = c_2 H_1 \sin \Psi - J_b \omega_{b_1} c_1$$  \hspace{1cm} (3.2.2-11)

$$c_2' = c_1 H_2 \sin \Psi - J_b \omega_{b_2} c_2$$  \hspace{1cm} (3.2.2-12)

$$c_3' = -J_t \omega_{t_1} c_3$$  \hspace{1cm} (3.2.2-13)

$$\beta_1' = H_5 c_3^4 + H_6 c_3^2 c_2^2 - H_4 c_2^2 c_1^2 - H_3 - \frac{c_2}{c_1} H_1 \cos \Psi$$  \hspace{1cm} (3.2.2-14)

$$\beta_2' = H_7 c_3^4 - J_7 c_3^2 c_2^2 + J_8 c_3^2 c_1^2 - \frac{c_1}{c_2} H_2 \cos \Psi$$  \hspace{1cm} (3.2.2-15)

$$\gamma' = J_4 c_1^4 - J_4 c_3^2 c_2^2 - J_2 c_2^2 c_1^2 - J_3 c_1^2 c_2^2 + J_6 c_4^4 - J_6$$  \hspace{1cm} (3.2.2-16)

Where coefficients $H_j$ and $J_k$ are quoted in Appendix 5, and $'$ denotes differentiation with respect to $t_2$.

In order to develop an analytical expression for $\varepsilon^2 \rho_1$ the torsion amplitude and phase equations are ignored. The decaying nature of the solution to (3.2.2-13) seems to justify this approach.

Thus, $c_3 = 0$ ;  \hspace{1cm} $c_1' = c_2' = \Psi' = 0$

which reduces equations (3.2.11 to 3.2.2-16) to,

$$c_2 \Gamma_1 \cos \Psi + c_1 \left[ \Gamma_2 + \varepsilon^2 \omega_{b_1} \beta_1' \right] = 0$$  \hspace{1cm} (3.2.2-17)

$$c_2 \Gamma_1 \sin \Psi - J_b \varepsilon^2 \omega_{b_1}^2 c_1 = 0$$  \hspace{1cm} (3.2.2-18)

$$c_1 \Gamma_3 \cos \Psi + c_2 \omega_{b_2} \varepsilon^2 \beta_2' = 0$$  \hspace{1cm} (3.2.2.19)

$$c_1 \Gamma_3 \sin \Psi - J_b \varepsilon^2 \omega_{b_2}^2 c_2 = 0$$  \hspace{1cm} (3.2.2-20)

($\Gamma_1, \Gamma_2, \Gamma_3$ given in Appendix 6)
These are the solvability equations for resonance condition (3.2.1-1). As in the earlier case it is possible to find expressions for the slowly varying phase angles $\beta'_1$ and $\beta'_2$, these are,

$$\beta'_1 = \left[ \frac{\omega^2_{b1} \varepsilon^2 \rho_1 - \omega_{b2} \Gamma_2}{\varepsilon^2 \left( \omega^2_{b1} + \omega_{b1} \omega_{b2} \right)} \right]$$

(3.2.2-21)

and

$$\beta'_2 = \frac{\omega_{b2}}{\omega^2_{b1} \varepsilon^2} \left[ \Gamma_2 + \frac{(\omega^2_{b1} \varepsilon^2 \rho_1 - \omega_{b2} \Gamma_2)}{(\omega_{b1} + \omega_{b2})} \right]$$

(3.2.2-22)

3.2.3 Transition curve expression for combined fundamental and second bending modes

Substitution of expressions (3.2.2-21, 3.2.2-22) back into the solvability equations, with further trigonometrical and algebraic manipulations, produces,

$$\omega^2_{b1} \varepsilon^4 \rho^2_1 + 2 \Gamma_2 \omega_{b1} \varepsilon^2 \rho_1 + \Gamma^2_2 + (\omega_{b1} + \omega_{b2})^2 \left[ \omega^2_{b1} \delta^2_{b2} \varepsilon^4 - \frac{\Gamma_1 \Gamma^3_2}{\omega^2_{b2}} \right] - 0$$

(3.2.3-1)

Coefficients $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ given in Appendix 6 contain $\Omega$, for which (3.2.1-1) is substituted giving rise to very small terms of up to $O(\varepsilon^8)$. As $\varepsilon$ itself is defined as being small, terms of order greater than 4 can justifiably be ignored and thus expression (3.2.3-1) may be written as a quadratic in $\varepsilon^2 \rho_1$,

$$V_1 \varepsilon^4 \rho^2_1 + V_2 \varepsilon^2 \rho_1 + V_3 = 0$$

(3.2.3-2)

This is the equation for the transition curve for the combination resonance (3.2.1-1), with the frequency dependent coefficients $V_1$, $V_2$, $V_3$ given in Appendix 7.
3.3 Discussion of theoretical and experimental results

The transition curves for,

\[ \Omega = \omega_{b1} + \omega_{t1} + \epsilon \rho \]

and

\[ \Omega = \frac{1}{2} \left( \omega_{b1} + \omega_{b2} \right) + \epsilon^2 \rho_1 \]

both with nonzero system damping, described respectively by, (3.1.3-4, 3.2.3-2) are shown for two beam lengths in Figures 3.3-1 and 3.3-2. The experimental and theoretical results are in close agreement, however there are some qualifications to be made, particularly concerning the curve for the second order effect.

Figure 3.3-1 shows both transition curves on the \((\Omega, W_0 \Omega^2)\) plane for a 90mm beam with experimental points included. The curve for the second order effect is centred about a higher excitation frequency than that for the first order resonance, and unstable solutions are shown to have a threshold level in the region of \(W_0 \Omega^2 = 130\) to \(140\) MS\(^{-2}\). In practice a slightly higher level \(\approx 175\) MS\(^{-2}\) is required. The theoretical curve for the bending/torsion resonance is much wider than the other, and unstable solutions for this resonance condition are theoretically and experimentally realisable at quite low excitation levels, \(W_0 \Omega^2 \approx 10\) MS\(^{-2}\). The experimental results can be seen to deviate to some degree as the detuning increases for both effects, particularly the first order effect.

The more flexible 160mm beam (Figure 3.3-2) produces a much wider region of unstable solutions for the bending/torsion resonance, and all experimental points lie within the theoretical boundary. There is a very narrow theoretical region for the second order effect. It seems that equation (3.2.3-2) is less of a valid approximation as beam length increases, the experimental points lying well outside the theoretical boundary curve. As would be expected for a less stiff system, the onset of instability for the second order effect
Fig. 3.1-1 Transition curves for a 60 mm beam in the 2 parameter plane evaluated from equations (3.1.3.4, 3.2.3.2)

Peak Ex. Acceleration MS^2

Resonance (3.1.1-1)

Resonance (3.2.1-1)

Experimental

Theoretical

Ex. Frequency Hz
Fig. 3.3.2. Transition curve for a 160 mm beam in the 2 parameter plane evaluated from equations (3.1.3.4, 3.2.3.2)

Peak Ex. Acceleration, MS²

Resonance (3.2.1-1)

Resonance (3.1.1-1)

Experimental

Theoretical

Ex. Frequency Hz
Fig. 3.3-3 Frequencies of three combination resonances as functions of small detuning in the form of varying beam length over a small range.
occurs at a much lower level of excitation acceleration, \( W_0 \omega^2 \approx 55 \) to 60 MS\(^{-2}\).

Inevitably the question arises as to what happens when the two effects occur either exactly simultaneously or in very close proximity. Figure 3.3-3 shows the two resonances represented as functions of beam length against frequency and that the intersection point is around \( l = 120\)mm. For short stiff beams \( 80 \leq l \leq 115\)mm expression (3.2.3-2) holds well, however for \( 125 \leq l \leq 180\)mm the agreement between theory and experiment lessens for increasing \( l \). The transition curve expression (3.2.3-2) completely fails for the beam length region over which the combinations begin to meet and overlap, \( 115 \leq l \leq 125\)mm, producing a peculiar double hyperbola.

This simultaneous region is investigated in section 3.5 and later chapters.

3.4 Numerical Integration of the 3 mode slow time equations

Equations (3.2.2-11 through 3.2.2-16) can be integrated numerically by a fourth order Runge-Kutta-Gill routine and modal responons for the 90mm beam are shown in Figures 3.4-1, 3.4-2, 3.4-3 for three different input acceleration levels.

Although the coupling between the six variables is intricate the equations prove to be not too stiff and integration progresses quickly. The responses shown are for the central 'line of symmetry' of the 90mm beam's transition curve,

i.e., \( \psi = -\beta_1 - \beta_2 ; \rho_1 = 0 \)

making the trigonometrical function arguments time invariant. Integration proceeds at a much slower rate when \( \rho_1 \neq 0 \) and no results are presented for that case.
Fig. 3.4-1 Theoretical response amplitudes for a 90 mm beam as functions of slow timescale $T_2$.
Fig. J.4-2 Theoretical response amplitudes for a 90 mm beam as functions of slow timescale $T_2$. 

Ex. Accn. = $150 \text{ MS}^{-2}$

$\rho_1 = 0$

1st BENDING

2nd BENDING

1st TORSION

Slow Timescale $T_2$
3.4.1 Discussion of the theoretical responses

Figure 3.4-1 illustrates the case where the input acceleration is 0.85 of the theoretical threshold level, the modal responses are shown with slow time $T_2$. This input level is not quite sufficient to promote the instability and the responses never rise above their small initial perturbations.

Figure 3.4-2 is for the case in which the excitation level is just above the theoretical threshold and hence after some time the two bending modes begin to respond, quickly becoming unbounded with $T_2$.

The third case shows the response behaviour for an input well above the threshold, and clearly demonstrates the fast build up of the instability at such levels. The torsion mode does not respond in any of these cases.

3.5 Expansion of the 3 degree of freedom equations to first order

3.5.1 Simultaneous combination resonance conditions

This section considers the simultaneous resonances,

$$\Omega = \omega_{b1} + \omega_{t1} + \varepsilon \rho$$

$$\Omega = \frac{1}{2} \left[ \omega_{b1} + \omega_{b2} \right] + \varepsilon^2 \rho_1$$

Second order theory predicts both of these effects, however the algebra involved in calculating coefficients of the resonant terms in the solvability equations is considerable so an alternative approach is taken.

Considering the combination resonance of the form,

$$\Omega = \omega_{b2} - \omega_{t1} + \varepsilon \eta$$

(See Table 3.2.1-1, Figure 3.3-3)

This may exist in close proximity to the first, and is predicted by first order theory for 3 degrees of freedom.

Eliminating $\omega_{t1}$ in (3.1.1-1) and (3.5.1-1) gives,
\[ 2 \Omega = \omega_{b1} + \omega_{b2} + \epsilon (\rho + \gamma) \]

which can be rearranged to produce,

\[ \Omega = \frac{1}{2} \left[ \omega_{b1} + \omega_{b2} \right] + \epsilon (\rho + \gamma) \]  

(3.5.1-2)

It is therefore possible to examine resonance (3.2.1-1) using first order theory, the detuning parameter being redefined as in (3.5.1-2) above.

(Nayfeh and Mook, page 315).

The 3 degree of freedom governing equations (2.3-8, 2.3-9, 2.3-10) are rewritten to first order, with \( \nu_0 \) discounted so eliminating the small derivative product terms,

\[ \ddot{U}_1 + 2 \dot{U}_1 \epsilon \dot{\xi}_b \omega_{b1} + \omega_{b1}^2 U_1 - \Phi_{1b1} \epsilon W_0 \Omega^2 \cos \Omega t = 0 \]  

(3.5.1-3)

\[ \ddot{U}_2 + 2 \dot{U}_2 \epsilon \dot{\xi}_b \omega_{b1} + \omega_{b1}^2 U_2 - \Phi_{1b2} \epsilon W_0 \Omega^2 \cos \Omega t = 0 \]  

(3.5.1-4)

\[ \ddot{\phi}_1 + 2 \dot{\phi}_1 \epsilon \dot{\xi}_t \omega_{t1} + \omega_{t1}^2 \phi_1 - F_1 \epsilon U_1 \cos \Omega t - F_2 \epsilon U_2 \cos \Omega t = 0 \]  

(3.5.1-5)

(with damping defined to first order)

Where,

\[ F_1 = R b_1 (W_0 \Omega^2) ; \quad F_2 = R b_2 (W_0 \Omega^2) ; \quad B_1 = b_1 \epsilon \]

\[ B_2 = b_2 \epsilon \]

\[ \xi_b = \epsilon \xi_b \]

\[ \xi_t = \epsilon \xi_t \]

Taking the usual series forms for variables \( U_1, U_2, \phi_1 \) and their derivatives, yields to first order,

\[ \epsilon^0 : \quad D_0^2 U_{10} + \omega_{b1}^2 U_{10} = 0 \]  

(3.5.1-6)

\[ \epsilon^1 : \quad D_0^2 U_{11} + \omega_{b1}^2 U_{11} = \phi_{1b1} l b_1 W_0 \Omega^2 \cos \Omega t - 2D_0 D_1 U_{10} \]

\[ - 2 \xi_b \omega_{b1} D_0 U_{10} \]  

(3.5.1-7)
\[ \varepsilon^0 : D_o^2 U_{20} + \omega_{b2}^2 U_{20} = 0 \quad (3.5.1-8) \]

\[ \varepsilon^1 : D_o^2 U_{21} + \omega_{b2}^2 U_{21} = \phi_{10} b_o b_1 W_0 \Omega^2 \cos \Omega t - 2 D_o D_1 U_{20} - 2 \oint b \omega_{b2} b_o U_{20} \quad (3.5.1-9) \]

\[ \varepsilon^0 : D_o^2 \phi_{10} + \omega_{t1}^2 \phi_{10} = 0 \quad (3.5.1-10) \]

\[ \varepsilon^1 : D_o^2 \phi_{11} + \omega_{t1}^2 \phi_{11} = F_1 U_{10} \cos \Omega t + F_2 U_{20} \cos \Omega t \]

\[- 2 D_o D_1 \phi_{10} - 2 \oint t \omega_{t1} b_o \phi_{10} \quad (3.5.1-11) \]

Solutions to the homogeneous equations (3.5.1-6, 3.5.1-8, 3.5.1-10) are,

\[ U_{10} = C_1(T_1) \exp(i \omega_{b1} T_0) + \bar{C}_1(T_1) \exp(-i \omega_{b1} T_0) \quad (3.5.1-12) \]

\[ U_{20} = C_2(T_1) \exp(i \omega_{b2} T_0) + \bar{C}_2(T_1) \exp(-i \omega_{b2} T_0) \quad (3.5.1-13) \]

\[ \phi_{10} = C_3(T_1) \exp(i \omega_{t1} T_0) + \bar{C}_3(T_1) \exp(-i \omega_{t1} T_0) \quad (3.5.1-14) \]

These are substituted into equations (3.5.1-7, 3.5.1-9, 3.5.1-11) to give,

\[ D_o^2 U_{11} + \omega_{b1}^2 U_{11} = \exp(i \omega_{b1} T_0) \left[ \frac{C_3 b_1 W_0 \Omega^2}{2} \exp(i (\Omega + \omega_{t1} - \omega_{b1}) T_0} \right. \]

\[ + \frac{C_3 b_1 W_0 \Omega^2}{2} \exp(i (\omega_{t1} - \omega_{b1} - \Omega) T_0} \]

\[ + \frac{C_3 b_1 W_0 \Omega^2}{2} \exp(i (\Omega - \omega_{t1} - \omega_{b1}) T_0} \]

\[ - i 2 D_1 C_1 \omega_{b1} + i 2 D_1 \bar{C}_1 \omega_{b1} \exp(-i \omega_{b1} T_0) \]

\[ - i 2 \oint b \omega_{b1} c_1 + i 2 \oint b \omega_{b1} \bar{C}_1 \exp(-i \omega_{b1} T_0) \right] \quad (3.5.1-15) \]
\[
D_0^2 u_{21} + \omega_{b2}^2 u_{21} = \exp \left( i\omega_{b2} T_0 \right) \left[ \begin{array}{c}
C_3 b_2 W_0 \Omega^2 \exp \left( \omega_{t1} - \Omega - \omega_{b2} \right) T_0 \\
+ C_3 b_2 W_0 \Omega^2 \exp \left( \omega_{t1} - \Omega - \omega_{b2} \right) T_0 \\
+ \overline{C}_3 b_2 W_0 \Omega^2 \exp \left( \Omega - \omega_{t1} - \omega_{b2} \right) T_0 \\
+ \overline{C}_3 b_2 W_0 \Omega^2 \exp \left( \Omega - \omega_{t1} - \omega_{b2} \right) T_0 \\
- i2D_4 C_2 \omega_{b2} + i2D_4 \overline{C}_2 \omega_{b2} \exp \left( -i2\omega_{b2} T_0 \right) \\
- i2 \int b \omega_{b2} C_2 + i2 \int b \omega_{b2} \overline{C}_2 \exp \left( -i2\omega_{b2} T_0 \right) \end{array} \right] (3.5.1-16)
\]

\[
D_0^2 \phi_{11} + \omega_{t1}^2 \phi_{11} = \exp \left( i\omega_{t1} T_0 \right) \left[ \begin{array}{c}
F_1 C_1 \exp \left( \omega_{t1} + \Omega - \omega_{t1} \right) T_0 \\
+ F_1 \frac{C_1}{\overline{C}_1} \exp \left( \omega_{t1} - \Omega - \omega_{t1} \right) T_0 \\
+ F_1 \frac{C_1}{\overline{C}_1} \exp \left( \Omega - \omega_{t1} - \omega_{t1} \right) T_0 \\
+ F_1 \frac{C_1}{\overline{C}_1} \exp \left( -\Omega - \omega_{t1} - \omega_{t1} \right) T_0 \\
+ F_2 C_2 \exp \left( \omega_{t1} + \Omega - \omega_{t1} \right) T_0 \\
+ F_2 \frac{C_2}{\overline{C}_2} \exp \left( \omega_{t1} - \Omega - \omega_{t1} \right) T_0 \\
+ F_2 \frac{C_2}{\overline{C}_2} \exp \left( \Omega - \omega_{t1} - \omega_{t1} \right) T_0 \\
+ F_2 \frac{C_2}{\overline{C}_2} \exp \left( -\Omega - \omega_{t1} - \omega_{t1} \right) T_0 \\
+ F_2 \frac{C_2}{\overline{C}_2} \exp \left( \omega_{t1} - \Omega - \omega_{t1} \right) T_0 \end{array} \right]
\]
\[ -i2D_1 c_3 \omega_{t1} + i2D_1 \bar{c}_3 \omega_{t1} \exp \left(-i2 \omega_{t1} t_0 \right) \]
\[ -i2 \int_0^t \omega_{t1}^2 c_3 + i2 \int_0^t \omega_{t1}^2 \bar{c}_3 \exp \left(-i2 \omega_{t1} t_0 \right) \]

(3.5.1-17)

The same substitutions for the complex amplitudes and their derivatives are made here as in section 3.2.2, leading to the following slow time equations,

\[ \frac{c_3 b_1 \omega_0 T^2}{4} \exp \left( \rho_{t1} - \beta_1 - \chi \right) - i \omega_{b1} c_1' + \omega_{b1} \bar{c}_1 \beta_1' \]
\[ - i \int_0^t \omega_{b1}^2 c_1 = 0 \]

(3.5.1-18)

\[ \frac{c_3 b_2 \omega_0 T^2}{4} \exp \left( \eta_{t1} + \chi - \beta_2 \right) - i \omega_{b2} c_2' + \omega_{b2} \bar{c}_2 \beta_2' \]
\[ - i \int_0^t \omega_{b2}^2 c_2 = 0 \]

(3.5.1-19)

\[ \frac{f_{1c_1}}{4} \exp \left( \rho_{t1} - \beta_1 - \chi \right) + \frac{f_{2c_2}}{4} \exp \left( - \eta_{t1} - \chi + \beta_2 \right) \]
\[ - i \omega_{t1} c_3' + \omega_{t1} \bar{c}_3 \chi' - i \int_0^t \omega_{t1}^2 c_3 = 0 \]

(3.5.1-20)

For an autonomous system the following substitutions are made,

\[ \Phi = \rho_{t1} - \beta_1 - \chi \]
\[ \Theta = \eta_{t1} + \chi - \beta_2 \]

(3.5.1-21, 3.5.1-22)

3.5.2 Solvability Equations

Taking, \( c_1' = c_2' = c_3' = 0 \), and separating the above equations into their real and imaginary parts yields,
\[
\begin{align*}
\frac{c_3b_1w_0\Omega^2}{4} \cos \Phi + \omega b_1c_1 \beta_1' &= 0 \\ (3.5.2-1) \\
\frac{c_3b_1w_0\Omega^2}{4} \sin \Phi - \zeta_b \omega b_1c_1 &= 0 \\ (3.5.2-2) \\
\frac{c_3b_2w_0\Omega^2}{4} \cos \Theta + \omega b_2c_2 \beta_2' &= 0 \\ (3.5.2-3) \\
\frac{c_3b_2w_0\Omega^2}{4} \sin \Theta - \zeta_b \omega b_2c_2 &= 0 \\ (3.5.2-4) \\
\frac{c_1f_1}{4} \cos \Phi + \frac{c_2f_2}{4} \cos \Theta + \omega t_1c_3 \chi' &= 0 \\ (3.5.2-5) \\
\frac{c_1f_1}{4} \sin \Phi - \frac{c_2f_2}{4} \sin \Theta - \zeta_t \omega t_1c_3 &= U \\ (3.5.2-6) \\
\end{align*}
\]

Also, \( \Phi' = \Theta' = 0 \) is a required condition for solvability, thus from (3.5.1-21, 3.5.1-22) the following expressions are used to eliminate the explicit presence of \( \beta_1', \beta_2', \chi' \) in the above,

\[ \rho - \beta_1' - \chi' = \gamma + \chi' - \beta_2' = 0 \]

where suitable algebraic manipulation produces,

\[
\begin{align*}
\beta_1' &= \frac{\left[ R \omega b_2c_2^2 \left( \rho + \gamma \right) - \omega t_1c_3^2 \rho \right]}{\left[ R \left( \omega b_2c_2^2 - \omega b_1c_1^2 \right) - \omega t_1c_3^2 \right]} \\
\beta_2' &= \frac{\left[ R \omega b_1c_1^2 \left( \rho + \gamma \right) + \omega t_1c_3^2 \gamma \right]}{\left[ R \left( \omega b_1c_1^2 - \omega b_2c_2^2 \right) + \omega t_1c_3^2 \right]} \\
\chi' &= \frac{\left[ R \omega b_1c_1^2 \rho + \omega b_2c_2^2 \gamma \right]}{\left[ R \left( \omega b_1c_1^2 - \omega b_2c_2^2 \right) + \omega t_1c_3^2 \right]} \\
\end{align*}
\]
3.5.3 Effects of internal detuning on modal responses and the derivation of an expression relating to modal energy exchange

Equations (3.5.2-1 to 3.5.2-6) can be arranged, together with expressions (3.5.2-7 to 3.5.2-9) to yield an analytical expression relating to torsion and second bending mode responses as functions of variable detuning parameters. Experimental observations have shown little or no exchange of energy involving the fundamental bending mode therefore substitutions for this, in terms of the other modes, from equations (3.5.2-2, 3.5.2-4, 3.5.2-6) are made as required.

The resulting 'energy exchange' expression for \( c_2 \) and \( c_3 \) is,

\[
\left[ Q_1 Q_5 R^2 \omega_{b2}^2 \left( \varepsilon (\phi + \eta) \right)^2 + Q_4 R^2 (\omega_{b2} - \omega_{b1} Q_4)^2 \right]_{c_2}^{\frac{1}{2}} \nonumber
\]

\[
- \left[ (Q_3 - Q_2 Q_4) (R \omega_{b1} Q_2 + \omega_{t1})^2 - Q_2 Q_5 \omega_{t1}^2 \left[ \varepsilon \phi \right]^2 \right]_{c_3}^{\frac{1}{2}} = 0
\]

(3.5.2-10)

Where, \( Q_1 = \frac{\omega_{b2}^2}{\omega_{b1}^2} \) ; \( Q_2 = \frac{\xi_b \omega_{t1}^2}{\xi_b \omega_{b1} R} \)

\[ Q_3 = B_1^4 (W_0 \Omega_1^2) \] ; \[ Q_4 = 16 \xi_b^2 \omega_{b1}^2 B_1^2 (W_0 \Omega_1^2)^2 \]

\[ Q_5 = 16 \omega_{b1}^2 B_1^2 (W_0 \Omega_1^2)^2 \]

The theoretical modal responses based on (3.5.2-10) are shown in Figure 3.5.3-1 for two different pairs of values for the detuning parameters \( \varepsilon \phi \) and \( \varepsilon (\phi + \eta) \) as a response ratio \( c_2/c_3 \) against peak excitation acceleration,

Case (i) \( \varepsilon \phi = 0.615 \) ; \( \varepsilon (\phi + \eta) = 0 \)

Case (ii) \( \varepsilon \phi = 0 \) ; \( \varepsilon (\phi + \eta) = -0.615 \)
From equation (2.5.2-10)

determining cases as functions of excitation acceleration.

The $c_2/c_3$ ratio for second bending to first torsion response for two

**Figure 2.5.2-1**: Ratio of second bending to first torsion response for two

**Response Ratio $c_2/c_3$**

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<th>0.00</th>
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**Peak Ex. Acceleration MS-2**

- Case (1)
- Case (2)

**Regions**

- Region A
- Region B
- Region C

Expt. Region

Expt. Region

Expt. Region
The fact that the tuning condition favours the bending/bending resonance of case (i) promotes a very strong second bending mode content in the resulting 'stable' behaviour compared to the converse situation of case (ii) in which the torsional presence reasserts itself. This emphasises the intriguing situation whereby a second order resonance may radically affect the modal content of a stronger first order resonance under certain tuning conditions.
3.6 Experimental Investigation

3.6.1 Experimental Apparatus

The apparatus comprises a short flexible spring steel beam securely fixed at its base to a support which is capable of being vibrated in the stiffest plane of the beam. A rectangular lumped mass attached at the free end may be moved along the axis of the beam in order to adjust the natural frequencies and consequently the tuning of the system. This is illustrated in Figure 2.1-1. The vibrational responses of the beam are monitored by two inductive probes mounted on the support bracket. The whole configuration is rigidly bolted to the face plate of a large horizontally orientated electrodynamic shaker which is capable of high acceleration levels. The mass of the beam system along with the associated support and probes is negligible compared to that of the armature of the shaker which in turn helps to reduce any undesirable effects caused by the system acting back on the input.

The arrangement of the probes relative to the beam is shown in plan in Figure 3.6-1. Each probe forms part of a radio frequency oscillator tuned circuit: any ferrous material in close proximity to the probe tip detunes the oscillator in such a manner that the output voltage varies proportionally with the gap between the probe tip and the beam. The probes are offset so that torsional motion will produce two signals in phase and, conversely, two antiphase signals for bending motion. The phases of the outputs are accordingly used to route the signals by means of a "phase separator" circuit which also provides a degree of amplification. It is possible to achieve minimum imaging of one channel upon another by careful setting of the probe tip gaps 'a' and fine adjustment of feedback elements within the separator circuit. The resulting signals are displayed on a storage oscilloscope and a two-channel spectrum analyser in order to obtain complete monitoring of the behaviour of the system. The shaker head acceleration is monitored by a calibrated accelerometer and charge amplifier.

3.6.2 Calibration

For calibration purposes the beam and support are turned through 90° so that the excitation acts in the plane of minimum bending stiffness and therefore generates linear steady state bending responses of the beam.
Fig. 3.6-1a Plan view of beam and probes

Fig. 3.6-1b Beam in bending, showing signals in antiphase

Fig. 3.6-1c Beam in torsion, showing signals in phase
The bending responses of the system are then calibrated by means of an optical method utilizing an x-y traversing microscope which enables the precise peak displacement of the beam tip to be recorded for a variety of different input levels. The peak output voltages displayed on the spectrum analyser for both the fundamental and second bending modes respectively are recorded against corresponding tip displacements.

The torsional response calibration is rather more difficult to achieve and only one method is found to be both realistic and reasonably accurate. The endmass comprises two similar clamping blocks of mild steel but because of small dimensional dissimilarities between them the centre of mass is slightly offset. This gives rise to a small torque when the system is linearly excited which promotes torsional motion of the beam when the excitation frequency is coincident with the fundamental torsion mode frequency. The twist angle is detected by directing a 3.5 mW laser at the front face of the end mass, which is finely ground and polished. A deflection of the incident light occurs when the beam responds in torsion. By using a reflective graduated scale the twist angle can be calculated by simple trigonometry. The laser equipment is shown in Figure 3.6-4. As in the previous case the spectrum analyser, response peak level (for the torsion mode here) is plotted against observed twist angle. Linearity is maintained for small angles (<10°).

It is assumed that dissipation within the system takes the form of classical linear viscous damping for which numerical values are obtained from captured transient decays. The linear natural frequencies of the participating modes are derived by examining the frequency spectra of the beam in free vibration.

3.6.3 Procedure

Experimental points for the transition curves are shown in Figures 3.3-1 and 3.3-2 and are obtained by selecting several frequency points on a sine wave oscillator (which drives the electrodynamic shaker via a power amplifier), and at each point slowly increasing the excitation level until observable nonplanar motion commences. This is a slow process as the exact point at which the instability occurs is difficult to locate accurately. The excitation acceleration and frequency are recorded for each of the two combination resonances and for each selected beam length.
The second experimental test series applies to the theory of section 3.5.3 in which the modal content of the combination instabilities previously discussed is shown to change due to small degrees of internal detuning in the simultaneous region. At the beginning of this chapter it is stated that there are inherent solution stabilising effects in systems of this sort and it is upon this premise that the following experimental work is founded. Therefore it is not merely the existence of physical responses generated by these combination resonances that is of interest but the relative changes in stabilised modal responses that are shown to occur for small shifts in internal tuning. The ratios of second bending to first torsion amplitudes obtained from the spectrum analyser are plotted against peak input acceleration for the two tuning cases discussed in section 3.5.3 over a substantial working range.

3.6.4 Discussion of results

The results of the experimental tests for the transition curves are discussed together with the theoretical curves in section 3.3.

The experimental results for modal response ratios against input acceleration are plotted in Figure 3.6-2. As can be seen from the theoretical results in Figure 3.5.3-1 the correlation is generally quite good, particularly in region 'B'. There is, however, less conformity in the outer regions 'A' and 'C'. In the low excitation region 'A' the modal responses are nonstationary for both cases up to a threshold point after which, in region 'B', some stabilising effect, possibly of nonlinear stiffness, influences the responses. As the excitation level is increased another threshold point occurs, which is slightly different for the two cases, and in region 'C' the nonstationary behaviour is once again apparent.

It would seem that in the very high excitation region 'C' the stabilising effects are insufficient to keep the system responding steadily and the resulting high bending and torsional responses will quickly promote the onset of fatigue failure.

Typical spectrum analyser response traces for both cases in their stationary regions are shown in Figure 3.6-3. The excitation frequency is superimposed onto the modal responses in order to depict its position in the frequency
Fig. 3.6-2 Experimental results for the ratio of second bending to first torsion responses for two detuning cases as functions of excitation acceleration. System constants and parameters identical to the theoretical case presented in Fig. 3.5.3-1.
Fig. 3.6-3 Spectrum analyser traces for both tuning cases, showing the exchange effect between the fundamental torsion and second bending modes respectively.
Fig. 3.6.4 Calibration of beam motion in torsion using a low power laser
spectrum. No significance should be attached to the actual value of excitation amplitude shown on the figures. The exchange of energy between the fundamental torsion mode and second bending mode is clearly shown for the two cases, the torsional response increases and the second bending response decreases as the internal tuning changes from case (1) to case (2) and vice versa.
CHAPTER 4

Simultaneous combination resonances and non-linear interactions in an autoparametrically coupled beam system with 4-degrees-of-freedom

4.1 Description of the system

The mechanical system comprises two beams of uniform cross-section coupled as shown in Figure 4.1-1. The horizontal, or primary, beam is clamped at one end to a heavy support block with the free end coupled to the smaller secondary beam. The secondary beam is free to vibrate in its least stiff plane which is perpendicular to the plane of vibration of the primary beam. The plane of vibration of the primary beam is the reference plane and is henceforth referred to as 'planar' and therefore the motion of the secondary beam is deemed 'nonplanar'.

In this study the form of external harmonic excitation considered is seismic and is at a frequency in the region of the second planar bending mode frequency. The effect of this is such that the coupling point experiences a mainly rotational motion that is manifested as an angular acceleration, and hence as an inertial load in the stiff plane of the secondary beam. Two simultaneous internal resonance conditions, giving rise to a third, are shown to generate complicated responses and intermodal energy exchange effects for small changes in external and internal tuning. This occurs because of the nature of the nonlinear coupling which gives rise to interactive motions of the planar and nonplanar modes. The nonplanar responses act back on the planar system through the interactive force and moment at the coupling point.

4.2 Kinematics and Constraints

The coupled beam system in Figure 4.1-1 consists of a horizontal primary beam BC of length L and a vertical secondary beam AC of length l.

The coupling point is at C and the support, which experiences a sinusoidal acceleration, is at B. The reference frame is supplied by the Cartesian co-ordinate system X-Y-Z with its origin chosen at B. The positive Y-axis is taken to be along the undeformed elastic axis of the primary beam, also unit base vectors \( \hat{I}, \hat{J}, \hat{K} \) are orientated along the X,Y,Z axes respectively. This constitutes the reference plane against which the motion of the nonplanar system can be separately defined. The primary beam is assumed to have negligible flexibility in bending in the XY plane. The secondary
Fig. 4.1-1 Co-ordinate systems of coupled beam configuration
system is prescribed by a second co-ordinate system, x-y-z, with its origin at the coupling point C and its positive z-axis directed along the undeformed elastic axis of the secondary beam. A set of unit base vectors $\vec{i}, \vec{j}, \vec{k}$ is stipulated along the x, y, and z axes respectively, and the secondary beam is allowed any displacement relative to the x-y-z frame. This is discussed in detail in Appendix 1.

The PAFEC finite element beam package provides the modelling of the linearised planar system and Euler-Bernoulli beam theory is used to model the secondary beam (Appendix 2). As in the work of Roberts and Bux the effects of shear deformation are neglected.

4.3 Governing Equations for the coupled system with four interacting modes

Figure 4.3-1 shows how the planar motion of the primary beam relative to the $XYZ$ axes is specified by the co-ordinates $q_i$, $i = 1, 2, \ldots, n$, with $n$ being the number of degrees of freedom. The specific transverse and rotational displacements of the coupling point are denoted by $q_1$ and $q_2$ respectively, and thus the virtual work equation for the secondary beam may be written in the following form,

$$\delta W = - I_0 \delta \phi \phi_0 - m_0 (\dddot{\bar{w}}_0 + \dddot{q}_1 + \dddot{q}_2 + \dddot{q}_2^2) \delta w_0$$

$$- m_0 (\dddot{\bar{v}}_0 + \dddot{q}_2) \delta v_0 - m_0 \dddot{\bar{u}}_0 \delta u_0 - m_0 \dddot{\bar{u}}_0 \delta u_0$$

(4.3-1)

Where $\phi_0$, $w_0$, $v_0$ are, respectively, the twist angle, axial displacement, and in-plane displacement of the secondary beam as it undergoes nonplanar bending and twisting. This is shown in Figure 2.1-1. The support acceleration $\dddot{w}_0$ is a relatively small term compared to $\dddot{q}_1$ and for this reason is henceforth to be neglected in the development of the governing equations for the secondary beam.

The Galerkin representations (2.2-3) and (2.3-1) are reapplied here and from these an expression for strain energy in the secondary beam results,
Fig. 4.3-1 Coupled beam system showing co-ordinate displacements and reaction force and moment at the coupling point
\[ \delta V = \left[ \int_0^1 EI_y (f_1'')^2 \, dz \right] U_{01} \delta U_{01} + \left[ \int_0^1 EI_y (f_2'')^2 \, dz \right] U_{02} \delta U_{02} \\
+ \left[ \int_0^1 K_{c,6,1} (h_1')^2 \, dz \right] \phi \delta \phi \\
\]

This is expression (2.3-4) from Chapter 2.

The axial displacement \( w_0 \) can be written in terms of \( U_0 \) and \( v_0 \), hence,

\[ w_0 = \frac{1}{2} \int_0^1 \left[ \left( \frac{\partial U}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \right] \, dz \]  

(4.3-2)

For small \( v \), \( \left( \frac{\partial v}{\partial z} \right)^2 \to 0 \) thus,

\[ w_0 \approx \frac{1}{2} \int_0^1 \left( \frac{\partial U}{\partial z} \right)^2 \, dz \text{ and therefore, } w_0 \approx \frac{1}{2} \int_0^1 f_1'^2 U_{01}^2 \, dz \\
+ \frac{1}{2} \int_0^1 f_2'^2 U_{02}^2 \, dz \]  

(4.3-3)

From this the axial acceleration \( \ddot{w}_0 \) of the lumped mass is,

\[ \ddot{w}_0 \approx (U_{01} \dddot{U}_{01} + \dddot{U}_{01}^2) \int_0^1 f_1'^2 \, dz + (U_{02} \dddot{U}_{02} + \dddot{U}_{02}^2) \int_0^1 f_2'^2 \, dz \]  

(4.3-4)

The parametric term \( \dddot{U}_{01} \) arising from the rotation of the coupling point is very much larger than the small in-plane acceleration term \( \dddot{V}_0 \), therefore \( \dddot{V}_0 \) is neglected in this first order development and its associated cubic terms do not appear in the final equations. The virtual displacements \( \delta v_0 \) and \( \delta w_0 \) are, respectively,
\[
\delta v = B_1 \phi_1 \delta u_1 + B_1 u_1 \delta \phi_1 + B_2 \phi_2 \delta u_2 + B_2 u_2 \delta \phi_2
\] (4.3-5)
\[
\delta w = U_1 \delta u_1 \int_0^1 f_1^2 dz + U_2 \delta u_2 \int_0^1 f_2^2 dz
\] (4.3-6)

By substituting expressions (4.3-4, 4.3-5, 4.3-6) into (4.3-1) and applying the Principle of Virtual Work for an arbitrary set of virtual displacements the following equations of non-planar motion are obtained,
\[
m_0 \ddot{u}_1 + U_1 \int_0^1 EI_y (f_1')^2 dz + m_0 \ddot{q}_1 U_1 B_1 + m_0 \ddot{q}_1 U_1 B_3 + m_0 \ddot{q}_2 U_1 B_1 = 0
\] (4.3-7)
\[
m_0 \ddot{u}_2 + U_2 \int_0^1 EI_y (f_2')^2 dz + m_0 \ddot{q}_1 U_2 B_4 + m_0 \ddot{q}_2 U_2 B_4 + m_0 \ddot{q}_2 \phi_2 B_2 = 0
\] (4.3-8)
\[
I_0 \ddot{\phi} + \int_0^1 K_{cGJ(h')} \phi_2 dz + m_0 \ddot{q}_2 U_1 B_1 + m_0 \ddot{q}_2 U_2 B_2 = 0
\] (4.3-9)

Where the deflection form function integrals \( B_i \), \( i = 1, \ldots, 4 \) are given in Appendix 2.

The generalised equations for the planar system are,
\[
[M] \{\ddot{q}\} + [K] \{\ddot{q}\} = \{\ddot{p}\}
\] (4.3-10)

Where \([M]\) and \([K]\) are the mass and stiffness matrices respectively with \( \{\ddot{q}\} \) and \( \{\ddot{q}\} \) being the respective acceleration and displacement vectors of \( \{q\} \). The term \( \{\ddot{p}\} \) is a generalised forces vector containing non-linear inertia forces due to centripetal and tangential accelerations at the coupling point and also on axial acceleration arising from nonplanar bending and twisting of the secondary beam. Thus \( \{\ddot{p}\} \) comprises the following,
\[ P_1 = P_{10} + P_B = -m_0 \left[ B_3 \left( U_{o1} \ddot{U}_{o1} + U_{o1} \dot{U}_{o1} \right) + B_4 \left( U_{o2} \ddot{U}_{o2} + U_{o2} \dot{U}_{o2} \right) \right. \\
\left. + 1q_2 + q_1 \right] + P_{10} \tag{4.3-11} \]

\[ P_2 = P_{20} + C_B = -m_0 \left[ B_1 \left( U_{o1} \ddot{U}_{o1} + 2U_{o1} \dot{U}_{o1} \right) + B_2 \left( U_{o2} \ddot{U}_{o2} + 2U_{o2} \dot{U}_{o2} \right) + U_{o1} \dot{U}_{o1} \right] + P_{20} \tag{4.3-12} \]

\[ P_k = P_{k0} \tag{4.3-13} \]

where \( P_{k0} = F_k \cos \Omega t \quad k = 1, \ldots, N \)

\( F_k \cos \Omega t \) is a prescribed external excitation acting at the \( q_k \) th co-ordinate and \( P_B \) is the reaction force on the primary beam due to the bending motion of the secondary beam and the centripetal and tangential accelerations of the lumped end mass, with \( C_B \) representing the reaction moment on the primary beam due to the coupled beam bending-torsional motion.

The planar motion is assumed to be represented by a single mode Galerkin approximation,

\[ q_i = r_{ij} X_j \tag{4.3-14} \]

Where \( r_{ij} \) is the \( i \)th element of the \( j \)th eigenvector of the linearised problem. This is substituted into equation (4.3-10) giving,

\[ [M] r_{ij} \ddot{X}_j + [K] r_{ij} X_j = \{P\} \tag{4.3-15} \]

This is premultiplied by the transpose \( r_{ij}^T \),

\[ r_{ij}^T [M] r_{ij} \ddot{X}_j + r_{ij}^T [K] r_{ij} X_j = r_{ij}^T \{P\} \tag{4.3-16} \]

Where the transpose \( r_{ij}^T = [r_{ij} \ r_{2j} \ldots \ r_{kj}] \)
The planar system is vibrating in its second bending mode so the coordinates $q_1$ can be specified as,

$$ q_1 = r_{12} \ddot{x} $$

$$ q_2 = r_{22} \ddot{x} $$

(4.3-18)

Thus the equation for planar motion can be written by making the required substitutions in (4.3-16),

$$ x + 2 \xi_b \omega_b \dot{x} + \omega_b^2 x = F_2 \cos \Omega t - r_{12} \rho B_3 (U_{\dot{1}} \dddot{U}_1 + U_1) $$

$$ - r_{12} \rho B_4 (U_{\dot{2}} \dddot{U}_2 + U_2) - r_{12} r_{22} \rho \beta_1 \dddot{x} $$

$$ - r_{22} \rho B_{11} (\dddot{U}_1 + 3 \ddot{U}_1 + \ddot{U}_1) $$

$$ - r_{22} \rho B_{21} (\dddot{U}_2 + 3 \ddot{U}_2 + \ddot{U}_2) $$

(4.3-19)

The nonplanar equations (4.3-7, 4.3-8, 4.3-9) can be rewritten in a similar manner,

$$ \dddot{U}_1 + 2 \xi_b \omega_b \dot{U}_1 + \omega_b^2 U_1 + r_{12} B_3 \dddot{U}_1 + r_{22} B_3 \dddot{x} U_1 + r_{22} B_1 \dot{x} \dddot{U}_1 = 0 $$

(4.3-20)

$$ \dddot{U}_2 + 2 \xi_b \omega_b \dot{U}_2 + \omega_b^2 U_2 + r_{12} B_4 \dddot{U}_2 + r_{22} B_4 \dddot{x} U_2 + r_{22} B_2 \dot{x} \dddot{U}_2 = 0 $$

(4.3-21)

$$ \dddot{\phi}_1 + 2 \xi_t \omega_t \dot{\phi}_1 + \omega_t^2 \phi_1 + R_{1r} r_{22} D_{1x} \dddot{U}_1 + R_{1r} r_{22} D_{2x} \dddot{U}_2 = 0 $$

(4.3-22)

Where $U_{\dot{1}}, U_{\dot{2}}, \phi_1$ are rewritten as $U_1, U_2, \phi_1$ respectively, and linear viscous damping terms are added to each equation. Secondary system modal frequencies are defined as,
\[ \omega_{b1}^2 = \int_0^1 \frac{EI_y(f_1')^2}{m_0} \, dz ; \quad \omega_{b2}^2 = \int_0^1 \frac{EI_y(f_2')^2}{m_0} \, dz \]

\[ \omega_{t1}^2 = \int_0^1 \frac{k_c GJ(h')^2}{I_0} \, dz \]

For the case of vertical seismic excitation of the support, the external excitation parameter becomes,

\[ f_2 = \frac{w_0 \Omega^2 \left[ \sum r_{k_2 m_k} \right]}{(M + r_{12 m_0}^2)} \]

and \[ \rho = \frac{m_0}{(M + r_{12 m_0}^2)} \quad ; \quad R = \frac{m_0}{I_0} \]
CHAPTER 5
Theoretical Analysis

5.1 Expansion of the 4-degree-of-freedom equations to first order

The equations derived in Chapter 4 are rewritten in the following form,

\[ \ddot{u}_1 + 2 \epsilon \int_b \omega_{b1} \dot{u}_1 + \omega_{b1}^2 u_1 + \epsilon d_1 \ddot{x} u_1 + \epsilon d_2 x u_1 + \epsilon d_3 x \dot{\phi}_1 = 0 \]  
(5.1-1)

\[ \ddot{u}_2 + 2 \epsilon \int_b \omega_{b2} \dot{u}_2 + \omega_{b2}^2 u_2 + \epsilon d_4 \ddot{x} u_2 + \epsilon d_5 x u_2 + \epsilon d_6 x \dot{\phi}_1 = 0 \]  
(5.1-2)

\[ \ddot{\phi}_1 + 2 \epsilon \int_t \omega t \dot{\phi}_1 + \omega_{t1}^2 \phi_1 + \epsilon d_7 \ddot{x} u_1 + \epsilon d_8 x u_2 = 0 \]  
(5.1-3)

\[ \ddot{x} + 2 \epsilon \int_B \omega_{B2} \dot{x} + \omega_{B2}^2 x = \epsilon f \cos \Omega t - \epsilon d_9 (u_1 \ddot{u}_1 + u_1^2) \]  

\[ - \epsilon d_{10} (u_2 \ddot{u}_2 + u_2^2) - \epsilon d_{11} \dot{x}^2 \]  

\[ - \epsilon d_{12} (\ddot{\phi}_1 u_1 + 2 \dot{\phi}_1 u_1 + \dot{\phi}_1^2 u_1) \]  

\[ - \epsilon d_{13} (\ddot{\phi}_2 u_2 + 2 \dot{\phi}_2 u_2 + \dot{\phi}_2^2 u_2) \]  
(5.1-4)

Where, \( \epsilon \int_b = \xi_b \); \( \epsilon \int_t = \xi_t \); \( \epsilon \int_B = \xi_B \); \( \epsilon d_1 = r_{12} B_3 \); \( \epsilon f = F_2 \);

\( \epsilon d_2 = r_{22}^{2} B_3 \); \( \epsilon d_3 = r_{22}^{1} B_1 \); \( \epsilon d_4 = r_{12} B_4 \)

\( \epsilon d_5 = r_{22}^{2} B_4 \); \( \epsilon d_6 = r_{22}^{1} B_2 \); \( \epsilon d_7 = r_{22}^{1} B_1 R \)

\( \epsilon d_8 = r_{22}^{1} B_2 R \); \( \epsilon d_9 = r_{12} B_3 \); \( \epsilon d_{10} = r_{12} B_4 \)

\( \epsilon d_{11} = r_{12} r_{22}^{2} \mu_1 \); \( \epsilon d_{12} = r_{22} \mu B_1 \); \( \epsilon d_{13} = r_{22} \mu B_2 \)
Following the method of multiple scales, the variables and their derivatives are expanded to first order,

\[ U_1 = U_{10} + \mathcal{E} U_{11} + \ldots \]
\[ U_2 = U_{20} + \mathcal{E} U_{21} + \ldots \]
\[ \phi_1 = \phi_{10} + \mathcal{E} \phi_{11} + \ldots \]
\[ x = x_0 + \mathcal{E} x_1 + \ldots \]

\[ \frac{d}{dt} = D_0 + \mathcal{E} D_1 + \ldots \]
\[ \frac{d^2}{dt^2} = D_0^2 + 2\mathcal{E} D_0 D_1 + \ldots \]

These forms are substituted into equations (5.1-1, 5.1-2, 5.1-3, 5.1-4) respectively. Upto and including first order the resulting perturbation equations are,

\[ \mathcal{E}^0 : \quad D_0^2 U_{10} + \omega_{b1}^2 U_{10} = 0 \quad (5.1-5) \]

\[ \mathcal{E}^1 : \quad D_0^2 U_{11} + \omega_{b1}^2 U_{11} = -2 D_0 D_1 U_{10} - 2 \int_b \omega_{b1} D_0 U_{10} \]
\[ - d_1 U_{10}^2 x_0 - d_2 U_{10} (D_0 x_0)^2 \]
\[ - d_3 \phi_{10} D_0^2 x_0 \quad (5.1-6) \]

\[ \mathcal{E}^0 : \quad D_0^2 U_{20} + \omega_{b2}^2 U_{20} = 0 \quad (5.1-7) \]
\[ \varepsilon^1 : D_{0}^2 u_{21} + \omega_{02}^2 u_{21} = -2D_{0}D_{1}u_{20} - 2 \oint_{b} \omega_{b2}D_{0}u_{20} \]

\[-d_{4}u_{20}D_{0}^{2}x_{0} - d_{5}u_{20}(D_{0}x_{0})^{2} \]

\[-d_{6}c_{10}D_{0}^{2}x_{0} \]  
\hspace*{1cm} (5.1-8)

\[ \varepsilon^{0} : D_{0}^{2}c_{10} + \omega_{0}^{2}c_{10} = 0 \]  
\hspace*{1cm} (5.1-9)

\[ \varepsilon^1 : D_{0}^{2}c_{11} + \omega_{t1}^{2}c_{11} = -2D_{0}D_{1}c_{10} - 2 \oint_{t} \omega_{t1}D_{0}c_{10} \]

\[-d_{7}u_{10}D_{0}^{2}x_{0} - d_{8}u_{20}D_{0}^{2}x_{0} \]  
\hspace*{1cm} (5.1-10)

\[ \varepsilon^{0} : D_{0}^{2}x_{0} + \omega_{20}^{2}x_{0} = 0 \]  
\hspace*{1cm} (5.1-11)

\[ \varepsilon^1 : D_{0}^{2}x_{1} + \omega_{20}^{2}x_{1} = \frac{f}{2} \exp i \Omega_{0} + \frac{f}{2} \exp (-i \Omega_{0} \tau_{0}) \]

\[-2D_{0}D_{1}x_{0} - 2 \oint_{b} \omega_{b2}D_{0}x_{0} \]

\[-d_{9}u_{10}D_{0}^{2}x_{10} - d_{g}(D_{0}u_{10})^{2} \]

\[-d_{10}u_{20}D_{0}^{2}u_{20} - d_{10}(D_{0}u_{20})^{2} \]

\[-d_{11}(D_{0}x_{0})^{2} - d_{12}u_{10}D_{0}^{2}c_{10} \]

\[-2d_{12}D_{0}c_{10}D_{0}u_{10} - d_{12}c_{10}u_{20} \]

\[-d_{13}u_{10}D_{0}^{2}c_{10} - 2d_{13}D_{0}c_{10}D_{0}u_{20} \]

\[-d_{13}c_{10}D_{0}^{2}u_{20} \]  
\hspace*{1cm} (5.1-12)
The solutions to the zero order equations (5.1-5, 5.1-7, 5.1-9, 5.1-11) are, respectively,

\[ U_{10} = C_1(T_1) \exp \left( i \omega_{b1} T_0 \right) + \overline{C}_1(T_1) \exp \left( -i \omega_{b1} T_0 \right) \]  \hspace{2cm} (5.1-13)

\[ U_{20} = C_2(T_1) \exp \left( i \omega_{b2} T_0 \right) + \overline{C}_2(T_1) \exp \left( -i \omega_{b2} T_0 \right) \]  \hspace{2cm} (5.1-14)

\[ \phi_{10} = C_3(T_1) \exp \left( i \omega_{t1} T_0 \right) + \overline{C}_3(T_1) \exp \left( -i \omega_{t1} T_0 \right) \]  \hspace{2cm} (5.1-15)

\[ X_o = A(T_1) \exp \left( i \omega_{B2} T_0 \right) + \overline{A}(T_1) \exp \left( -i \omega_{B2} T_0 \right) \]  \hspace{2cm} (5.1-16)

Making substitutions for solutions (5.1-13 through 5.1-16) in the first order perturbation equations gives, respectively,

\[ D_0^2 U_{11} + \omega_{b1}^2 U_{11} = \exp \omega_{b1} T_0 \left[ -i2 \omega_{b1} D_1 C_1 + i2 \omega_{b1} D_1 \overline{C}_1 \exp \left( -i2 \omega_{b1} T_0 \right) \right. \]

\[ - i2 \int_b \omega_{b1}^2 C_1 + i2 \int_b \omega_{b1}^2 \overline{C}_1 \exp \left( -i2 \omega_{b1} T_0 \right) \]

\[ - d_1 \omega_{R2} \left[ -AC_1 \exp i \omega_{R2} T_0 - \overline{A}C_1 \exp \left( -i \omega_{R2} T_0 \right) \right] \]

\[ - AC_1 \exp \left( \omega_{B2} - 2 \omega_{b1} \right) T_0 - \overline{A}C_1 \exp \left( -2 \omega_{b1} - \omega_{B2} \right) T_0 \]

\[ - d_2 \omega_{B2} \left[ -A^2 C_1 \exp i2 \omega_{B2} T_0 + 2A\overline{A}C_1 \right. \]

\[ - \overline{A} C_1 \exp \left( -i2 \omega_{b1} T_0 \right) - A^2 \overline{C}_1 \exp \left( 2 \omega_{B2} - 2 \omega_{b1} \right) T_0 \]

\[ + 2A\overline{A}C_1 \exp \left( -i2 \omega_{b1} T_0 \right) - \overline{A} \overline{C}_1 \exp \left( -2 \omega_{B2} - 2 \omega_{b1} \right) T_0 \]

\[ - d_3 \omega_{B2} \left[ -AC_3 \exp \left( \omega_{B2} + \omega_{t1} - \omega_{b1} \right) T_0 \right. \]

\[ - \overline{A}C_3 \exp \left( \omega_{t1} - \omega_{B2} - \omega_{b1} \right) T_0 \]

\[ - \overline{A}C_3 \exp \left( \omega_{b1} - \omega_{B2} - \omega_{b1} \right) T_0 \]

\[ \left. - \overline{A} \overline{C}_3 \exp \left( \omega_{b1} - \omega_{B2} - \omega_{b1} \right) T_0 \right] \]  \hspace{2cm} (5.1-17)
\[ \begin{align*}
D_0^{2}u_{21} + \omega_{b_2}^{2}u_{21} &= \exp (\omega_{b_2}T_0) \left[ -i2\omega_{b_2}D_4C_2 + i2\omega_{b_2}D_1\bar{C}_2 \right. \\
&\left. - i2\int_{b} \omega_{b_2}^{2}C_2 + i2\int_{b} \omega_{b_2}^{2}\bar{C}_2 \right. \\
&\left. \exp (-i2\omega_{b_2}T_0) \\
&- d_4\omega_{b_2}^{2} \left[ -AC_2 \exp (\omega_{b_2}T_0) - \bar{A}\bar{C}_2 \exp (-\omega_{b_2}T_0) \right] \\
&- AC_2 \exp (\omega_{B_2} - 2\omega_{b_2})T_0 - \bar{A}\bar{C}_2 \exp (-2\omega_{b_2} - \omega_{B_2})T_0 \\
&- d_5\omega_{b_2}^{2} \left[ -A^2C_2 \exp (\omega_{b_2}T_0) + 2A\bar{A}\bar{C}_2 \right] \\
&\left. - \frac{2}{A}\bar{C}_2 \exp (-i2\omega_{b_2}T_0) - \frac{2}{A}\bar{C}_2 \exp (2\omega_{b_2} - 2\omega_{b_2})T_0 \\
&+ 2A\bar{A}\bar{C}_2 \exp (-i2\omega_{b_2}T_0) - \frac{2}{A}\bar{C}_2 \exp (-2\omega_{b_2} - \omega_{b_2})T_0 \right] \\
&- d_6\omega_{b_2}^{2} \left[ A\bar{C}_0 \exp (\omega_{b_2} - \omega_{b_2} - \omega_{B_2})T_0 \\
&- AC_3 \exp (\omega_{B_2} - \omega_{b_2} - \omega_{b_2})T_0 \\
&- \bar{A}\bar{C}_3 \exp (\omega_{B_2} - \omega_{b_2} - \omega_{B_2})T_0 \right] \\
&\left. - \bar{A}\bar{C}_3 \exp (-\omega_{b_2} - \omega_{B_2} - \omega_{b_2})T_0 \right] \\
&\right. \\
(5.1-18)
\end{align*} \]
\[- d_8 \omega_{B2}^{2} \left[ -\tilde{A}C_2 \exp \left( \omega_{b2} + \omega_{B2} - \omega_{t1} \right) \right] \]
\[- \tilde{A}C_2 \exp \left( \omega_{B2} - \omega_{b2} - \omega_{t1} \right) \cdot \]
\[- \tilde{A}C_2 \exp \left( \omega_{b2} - \omega_{B2} - \omega_{t1} \right) \cdot \]
\[ + \bar{c}_1 c_3 \exp i \left( -\omega_{b1} - \omega_{t1} - \omega_{B2} \right) T_0 \]
\[ + d_{12} \omega_{b1}^2 \left[ c_1 \bar{c}_3 \exp i \left( \omega_{b1} + \omega_{t1} - \omega_{B2} \right) T_0 \right. \]
\[ + \bar{c}_1 c_3 \exp i \left( \omega_{t1} - \omega_{b1} - \omega_{B2} \right) T_0 \]
\[ + \bar{c}_1 c_3 \exp i \left( -\omega_{b1} - \omega_{t1} - \omega_{B2} \right) T_0 \]
\[ + d_{13} \omega_{t1}^2 \left[ c_2 \bar{c}_3 \exp i \left( \omega_{b2} + \omega_{t1} - \omega_{B2} \right) T_0 \right. \]
\[ + c_2 \bar{c}_3 \exp i \left( \omega_{b2} - \omega_{t1} - \omega_{B2} \right) T_0 \]
\[ + \bar{c}_2 \bar{c}_3 \exp i \left( -\omega_{b2} - \omega_{t1} - \omega_{B2} \right) T_0 \]
\[ + 2d_{13} \omega_{b2} \omega_{t1} \left[ c_2 \bar{c}_3 \exp i \left( \omega_{t1} + \omega_{b2} - \omega_{B2} \right) T_0 \right. \]
\[ - \bar{c}_2 \bar{c}_3 \exp i \left( \omega_{t1} - \omega_{b2} - \omega_{B2} \right) T_0 \]
\[ + \bar{c}_2 \bar{c}_3 \exp i \left( -\omega_{b2} - \omega_{t1} - \omega_{B2} \right) T_0 \]
\[ + d_{13} \omega_{b2}^2 \left[ c_2 \bar{c}_3 \exp i \left( \omega_{b2} + \omega_{t1} - \omega_{B2} \right) T_0 \right. \]
\[ + \bar{c}_2 \bar{c}_3 \exp i \left( \omega_{t1} - \omega_{b2} - \omega_{B2} \right) T_0 \]
\[ + \bar{c}_2 \bar{c}_3 \exp i \left( -\omega_{b2} - \omega_{t1} - \omega_{B2} \right) T_0 \]

(5.1-20)
5.1.1 Resonance Conditions and Slow time equations

The required resonance conditions to first order are,

\[ \Omega = \omega_{B2} + \epsilon \sigma \]  
(5.1.1-1)

\[ \omega_{B2} = \omega_{b1} + \omega_{t1} + \epsilon \eta_1 \]  
(5.1.1-2)

\[ \omega_{B2} = \omega_{b2} - \omega_{t1} + \epsilon \eta_2 \]  
(5.1.1-3)

Over a small range of \( \Omega \) the internal resonances (5.1.1-2, 5.1.1-3) occur simultaneously, whereupon the following is implied,

\[ \omega_{B2} = \frac{1}{2} [\omega_{b1} + \omega_{b2}] + \frac{\epsilon (\eta_1 + \eta_2)}{2} \]  
(5.1.1-4)

Resonances (5.1.1-1, 5.1.1-2, 5.1.1-3) are substituted into equations (5.1-17 through 5.1-20) and all 'fast' terms are omitted so that by equating the right hand sides to zero the conditions for nonzero solutions are established. Also complex amplitudes \( A, C_1, C_2, C_3 \) are written in their polar forms,

\[ A = \frac{a}{2} \exp (i\alpha) \quad C_1 = \frac{c_1}{2} \exp (i\beta_1) \quad C_2 = \frac{c_2}{2} \exp (i\beta_2) \]
\[ C_3 = \frac{c_3}{2} \exp (i\theta) \]  
(5.1.1-5)

With derivatives,

\[ D_1 A = \frac{a'}{2} \exp (i\alpha) + \frac{ai}{2} \omega' \exp (i\alpha) \]

\[ D_1 C_1 = \frac{c_1'}{2} \exp (i\beta_1) + \frac{c_1 i}{2} \beta'_1 \exp (i\beta_1) \]

\[ D_1 C_2 = \frac{c_2'}{2} \exp (i\beta_2) + \frac{c_2 i}{2} \beta'_2 \exp (i\beta_2) \]

\[ D_1 C_3 = \frac{c_3'}{2} \exp (i\theta) + \frac{c_3 i}{2} \theta' \exp (i\theta) \]  
(5.1.1-6)
By splitting the emergent equations into their real and imaginary parts, and by using the required autonomy expressions,

\[
\sigma^T_1 - \alpha = \Theta 
\]
\[
\eta^T_1 - \chi + \alpha - \beta_1 = \psi_1 
\]
\[
\eta^T_2 + \chi + \alpha - \beta_2 = \psi_2 
\]

the 'slowtime' solvability equations may be written in their autonomous form, as follows,

\[
c'_1 = -\int_{B} \omega B_1 c_1 - \frac{d_3}{4} \frac{\omega^2}{\omega B_1} \sin \psi_1 
\]
\[
c'_2 = -\int_{B} \omega B_2 c_2 - \frac{d_6}{4} \frac{\omega^2}{\omega B_1} \sin \psi_2 
\]
\[
c'_3 = -\int_{T} \omega T_1 c_3 - \frac{d_7}{4} \frac{\omega^2}{\omega T_1} \sin \psi_1 + \frac{d_8}{4} \frac{\omega^2}{\omega T_1} \sin \psi_2 
\]
\[
a' = -\int_{B} \frac{\omega B_2}{\omega B_2} a + \frac{f}{2} \sin \Theta + \frac{d_{12}}{4} \left( \frac{\omega B_1 + \omega T_1}{\omega B_2} \right)^2 c_1 c_3 \sin \psi_1 
\]
\[
\quad + \frac{d_{13}}{4} \left( \frac{\omega B_2 - \omega T_1}{\omega B_2} \right)^2 c_2 c_3 \sin \psi_2 
\]
\[
\Theta' = \sigma + \frac{f}{2 \omega B_2} \cos \Theta - \frac{d_{12}}{4} \left( \frac{\omega B_1 + \omega T_1}{\omega B_2} \right)^2 \frac{c_1 c_3}{a} \cos \psi_1 
\]
\[
\quad - \frac{d_{13}}{4} \left( \frac{\omega B_2 - \omega T_1}{\omega B_2} \right)^2 \frac{c_2 c_3}{a} \cos \psi_2 
\]
\( \psi_1 = \gamma_1 - \frac{d_7}{4} \frac{\omega^2_{B2}}{\omega_{t1}} \frac{ac_1}{c_3} \cos \psi_1 - \frac{d_8}{4} \frac{\omega^2_{B2}}{\omega_{t1}} \frac{ac_2}{c_3} \cos \psi_2 \)

\[ + \frac{d_{12}}{4} \left( \frac{\omega_{b1} + \omega_{t1}}{\omega_{B2}} \right)^2 \frac{c_1 c_3}{a} \cos \psi_1 \]

\[ + \frac{d_{13}}{4} \left( \frac{\omega_{b2} - \omega_{t1}}{\omega_{B2}} \right)^2 \frac{c_2 c_3}{a} \cos \psi_2 \]

\[ - \frac{f \cos \Theta}{2 \omega_{B2} a} - \frac{d_3}{4} \omega_{B2} \frac{ac_3}{c_1} \cos \psi_1 \]

\[ + \frac{d_2}{4} \left( \frac{\omega^2_{B2}}{\omega_{b1}} \right)^2 \]

(5.1.1-15)

\( \psi_2 = \gamma_2 \cdot \frac{d_7}{4} \frac{\omega^2_{B2}}{\omega_{t1}} \frac{ac_1}{c_3} \cos \psi_1 + \frac{d_8}{4} \frac{\omega^2_{B2}}{\omega_{t1}} \frac{ac_2}{c_3} \cos \psi_2 \)

\[ + \frac{d_{12}}{4} \left( \frac{\omega_{b1} + \omega_{t1}}{\omega_{B2}} \right)^2 \frac{c_1 c_3}{a} \cos \psi_1 \]

\[ + \frac{d_{13}}{4} \left( \frac{\omega_{b2} - \omega_{t1}}{\omega_{B2}} \right)^2 \frac{c_2 c_3}{a} \cos \psi_2 \]

\[ - \frac{f \cos \Theta}{2 \omega_{B2} a} - \frac{d_6}{4} \frac{\omega^2_{B2}}{\omega_{b2}} \frac{ac_3}{c_2} \cos \psi_2 \]

\[ + \frac{d_5}{4} \left( \frac{\omega^2_{B2}}{\omega_{b2}} \right)^2 \]

(5.1.1-16)
5.2 Numerical Integration of the 4-mode slow-time equations

Considerable investigative work has been carried out in an attempt to obtain quantitatively meaningful solutions to equations (5.1.1-10 to 5.1.1-16), however although steady-state solutions have been forthcoming (for this system with damping) there are certain dissimilarities between these results and their experimental counterparts of Chapter 6. A comparison is made between theoretical and experimental results in Chapter 7.

A Predictor-Corrector library routine is used to integrate the above equations starting from small initial perturbations. The three 'phase' equations (5.1.2-14, 5.1.2-15, 5.1.2-16) are not in standard form, this is because certain terms contained therein have response variables located in their denominators. This form of equation precludes the start-up case of zero initial conditions, thus exact zero-valued solutions cannot theoretically occur. Various small initial perturbations have been tried and all lead to the same steady-state solutions for a given set of system parameters.

5.2.1 Integration control parameters

The D02EAF Predictor-Corrector routine utilises two special controlling parameters, the first of which is 'TOL', an accuracy control which is also closely involved in the speed of the integration process. A value for this is specified which is found to maintain accuracy but which doesn't adversely affect the running speed of the program. The other control parameter is 'XEND' which specifies the range of the independent variable (T1) and it is required to keep 'XEND' as low as possible, but consistent with the satisfactory establishment of steady-state conditions, and hence solutions. This is discussed in section 5.2.4.
5.2.2 Theoretical 4-mode responses as functions of external detuning

In this section the following cases are considered,

Fig.5.2.2-1: \( \omega_0 \Omega^2 = 3 \text{ MS}^{-2}; \ \eta_1 = \eta_2 = 0 \)
Fig.5.2.2-2: \( \omega_0 \Omega^2 = 10 \text{ MS}^{-2}; \ \eta_1 = \eta_2 = 0 \)
Fig.5.2.2-3: \( \omega_0 \Omega^2 = 40 \text{ MS}^{-2}; \ \eta_1 = \eta_2 = 0 \)
Fig.5.2.2-4: \( \omega_0 \Omega^2 = 3 \text{ MS}^{-2}; \ \eta_1 = 6.15; \ \eta_2 = -6.15 \)
Fig.5.2.2-5: \( \omega_0 \Omega^2 = 10 \text{ MS}^{-2}; \ \eta_1 = 6.15; \ \eta_2 = -6.15 \)
Fig.5.2.2-6: \( \omega_0 \Omega^2 = 10 \text{ MS}^{-2}; \ \eta_1 = 0; \ \eta_2 = -12.3 \)

For all the above cases the system damping coefficients are,

\[ J_B = 0.28; \quad J_b = 0.013; \quad J_t = 0.003 \]

The first case considers the theoretical conditions in which all three internal resonances are perfectly tuned. For an excitation acceleration level of 3 \( \text{ MS}^{-2} \) the system is seen to respond in a purely linear fashion over the range of external detuning. The nonplanar modes do not respond, and remain at their small initial perturbations, thus it is seen that theoretically, for this level of excitation acceleration the linear response of the planar mode is stable as are the trivial 'zero' solutions for the nonplanar system. Increasing the excitation acceleration level to 10 \( \text{ MS}^{-2} \) introduces nonlinear interaction (Figure 5.2.2-2), however the region is suppressed and is only seen to occur at, and just around, the externally resonant point. There is a very sharp trough in the response of the planar mode at external resonance (\( \Omega/\omega_0 = 1.0; \ \sigma = 0 \)) at which point the linear solution becomes unstable and 'jumps' to the stable nonlinear solution.

The previously stable 'zero' solution of the three nonplanar modes at this point become unstable and jump to new stable non-zero solutions. The quantitative significance (or otherwise) of the theoretical solutions is discussed, in context with the experimental results, in chapter 7.
$w_0^2 = 3.85 \times 10^{-2}$

Nonplanar Bending Response $c_2$

Planar Bending Response $a$

Nonplanar Torsion Response $c_3$

Nonplanar Bending Response $c_1$

$c_i = 0.28$; $I = 0.013$; $I_t = 0.003$

Fig. 5.2-2: Theoretical 4-mode responses
An additional case of very high excitation acceleration is presented in Figure 5.2.2-3, where \( \omega_0 \Omega^2 = 40 \text{ MS}^{-2} \). This is too high a level to realise experimentally, however it is included here because the theoretical responses generated by such an acceleration level display interesting qualitative characteristics. It can be seen that the nonlinear region in the planar response curve has widened and that smaller 'troughs' occur at approximately the same point either side of the externally resonant point.

Stable non-zero solutions of the nonplanar modes are generated which correspond with the nonlinear regions of the planar response curve. The principal region of autoparametric interaction, occuring around external resonance, displays nonplanar responses more typical of such systems than the first two cases in that \( c_1 \) and \( c_2 \) exhibit 'troughs' (points A and B on Figure 5.2.2-3, respectively). The torsion mode \( c_3 \) does not show this effect and peaks even more sharply at external resonance. The two additional regions of interaction are very narrow, but with sizeable response amplitudes. There would seem to be no apparent explanation for these additional regions.

Figure 5.2.2-4 illustrates the case in which the bending/bending resonance is favoured, ie. \( \eta_1 = 6.15; \eta_2 = -6.15 \). The system responds exactly as in Figure 5.2.2-1 with the planar mode behaving in a linear manner together with 'zero' response of the nonplanar modes. The linear solutions are identical to those of Figure 5.2.2-1. Nonlinear interaction is seen to occur for the higher excitation level (Figure 5.2.2-5) however the principal effect of this internal tuning case is the shift in the externally resonant point from \( \Omega/\omega_{B2} = 1.0 \) to \( \Omega/\omega_{B2} = 0.992 \). There are relative shifts in the modal response amplitudes between the case of Figure 5.2.2-2 and this case, but the width of the region over the frequency axis is identical to that in Figure 5.2.2-2.

An interesting feature of this internal tuning condition is in the case of excitation at a frequency exactly equal to that of the planar mode \( \omega = \omega_{B2}(\sigma = 0; \Omega/\omega_{B2} = 1.0) \) for which the system responses would effectively be linearised. The planar mode would in this case respond
\[ \omega^2 = 40 \text{ m/s}^2 ; \eta_1 = 0 ; \eta_2 = 0 ; \delta_B = 0.28 ; \delta_t = 0.013 ; \delta_t = 0.003 \]

Fig. 5.2.2-3 Theoretical mode responses
$|w_0|^2 = 10 \text{ MS}^{-2}$; $\eta_1 = 6.15$; $\eta_2 = -5.15$; $\bar{g}_B = 0.28$; $g_b = 0.013$; $g_t = 0.003$

Fig. 5.2.2-5 Theoretical 4 mode responses
\[ \frac{\Omega}{\omega_{B2}} \]

Planar Bending Response a

Nonplanar Bending Response c

Nonplanar Bending Response c2

Nonplanar Torsion Response c3

\[ \frac{\Omega}{\omega_{B2}} \]

\[ \frac{\Omega}{\omega_{B2}} \]

\[ \left| w_0 \right| \Omega^2 = 10 \text{ MS}^{-2} ; \; \gamma_1 = 0 ; \; \gamma_2 = -12.3 ; \; \int_B = 0.28 ; \; \int_b = 0.013 ; \; \int_t = 0.003 \]

Fig. 5.2.2-6 Theoretical 4 mode responses
at its maximum amplitude with no predicted nonlinear interaction effects, (point C on Figure 5.2.2-5a).

The final results in this section are for an internal tuning condition specifically favouring the bending/torsion resonance ($\gamma_1 = 0$; $\gamma_2 = -12.3$) and the predicted behaviour is as might be expected in such circumstances with symmetrical response curves and a much reduced response amplitude of the second nonplanar bending mode $c_2$. This example is shown in Figure 5.2.2-6.

5.2.3 Theoretical 4-mode responses as functions of excitation acceleration

Three conditions are investigated in this section,

Fig.5.2.2-7: $\sigma = 0$ ; $\gamma_1 = \gamma_2 = 0$

Fig.5.2.2-8: $\sigma = -33.57$; $\gamma_1 = 6.15$; $\gamma_2 = -6.15$

Fig.5.2.2-9: $\sigma = 0$ ; $\gamma_1 = 0$ ; $\gamma_2 = -12.3$

For all the above cases the system damping coefficients are,

$$\int_B = 0.28; \int_b = 0.013; \int_t = 0.003$$

Figure 5.2.2-7 shows the response curves for perfect external and internal tuning. For very low levels of excitation acceleration the system response is linear (from D to E) with no response of the nonplanar modes. The transition from linear to nonlinear behaviour is in the region of E to F (Figure 5.2.2-7a).

As the excitation acceleration is increased the planar mode begins to saturate (point G), after which point all energy put into the system is manifested only in increasing nonplanar responses. Decreasing the excitation acceleration level generates responses which follow exactly the same paths in the opposite direction. It is not possible to theoretically predict regions of 'overhang' and/or hysteresis effects using this process of numerical integration.
\[ W_0 \Omega^2 \]

\[ \sigma = 0 \; ; \; \eta_1 = 0 \; ; \; \eta_2 = 0 \; ; \; \int_B = 0.28 \; ; \; \int_b = 0.013 \; ; \; \int_t = 0.003 \]

Fig. 5.2.2-7 Theoretical 4 mode responses.
\( \sigma = -33.57 \); \( \eta_1 = 6.15 \); \( \eta_2 = -6.15 \); \( J_B = 0.28 \); \( J_B = 0.013 \); \( J_t = 0.003 \)

Fig. 5.2.2-8 Theoretical 4 mode responses
Fig. 5.2.2-9 Theoretical 4 mode responses

\[ \sigma = 0 ; \ \eta_1 = 0 ; \ \eta_2 = -12.3 ; \ \int_B = 0.28 ; \ \int_b = 0.017 ; \ \int_t = 0.003 \]
The second case considers the condition in which the bending/bending resonance is perfectly tuned but the excitation frequency is removed from \( \omega_{B2} \). Linear system responses occur at higher levels of excitation acceleration than in the previous case with autoparametric interaction not commencing until the excitation level is approximately 5.2 MS\(^{-2}\) (point H on Figure 5.2.2-8). The dotted lines at \( \omega_o \omega^2 = 14.0 \) MS\(^{-2}\) show how the respective response amplitudes vary between the two cases for that particular level of excitation acceleration.

The third tuning condition is given in Figure 5.2.2-9 and it can be seen that the only significant change in the response characteristics from those of Figure 5.2.2-7 is the general, large reduction in the amplitude of the nonplanar mode \( \eta_2 \). This would seem to be a reasonable outcome because \( \eta_2 = -12.3 \) (away from \( \eta_1 \) and zero).

5.2.4 Theoretical 4-mode responses as functions of slow time scale \( T_1 \)

One condition is presented here,

\[
\text{Fig.5.2.2-10: } \omega_o \omega^2 = 10 \text{ MS}^{-2}; \quad \sigma = 0; \quad \eta_1 = \eta_2 = 0
\]

(Damping coefficients as in sections 5.2.2 and 5.2.3)

The numerical integration programs require an end point to be specified for the integration independent variable, which in this case is \( T_1 \), and is assigned the variable name XEND.

It can be seen in Figure 5.2.2-10 that after initial fluctuations the four modal response amplitudes settle down to their steady-state values as calculated for the numerical conditions given above. Steady-state conditions are established by \( T_1 = 2.0 \) however further 'time' is allowed to elapse in order to confirm that such conditions are present.

Results given in Figures 5.2.2-1 to 5.2.2-9 are derived from a specified end-point of XEND = 10.0.
\[ W \Omega^2 = 10 \text{ MS}^{-2} ; \sigma = 0 ; \eta_1 = 0 ; \eta_2 = 0 ; L_B = 0.28 ; \delta_b = 0.013 ; \delta_t = 0.003 \]

Fig. 5.2.2-10 Theoretical 4 mode responses
CHAPTER 6

Experimental Investigation, coupled system with four mode interaction

6.1 Experimental Apparatus

The physical model has been briefly described in the introductory section to Chapter 4, however the mechanical details are to be given here. The apparatus consists of two subsystems, the first comprising a heavy support block and the horizontal primary beam to which is coupled the secondary system consisting of a vertical beam with a movable end mass. A side elevation layout of the complete system is given in Figure 6.1-1. The primary beam is of mild steel with cross-sectional dimensions 37.5 x 3.28mm and the effective length of the beam is adjustable, this being achieved by slackening off the top plate clamping bolts on the support block and sliding the beam along in its machined out groove in the block until the desired length is achieved. An accelerometer for the purpose of input excitation monitoring is mounted on the support block and the response of the primary beam is detected by another accelerometer mounted at a convenient point along the length of the beam. The excitation is provided by a large electrodynamic shaker with its excitation axis vertical, and the support block is securely bolted to the shaker force plate.

The secondary beam and inductive probes are utilised in exactly the same mechanical format as described in section 3.6 of Chapter 3. Further elaboration is not considered necessary as no modifications have been made to this part of the structure.

The instrumentation is identical to that described in Chapter 3 with an extension in the form of an additional oscilloscope and a switchable AC voltmeter connected to the charge amplifiers of the support and primary beam accelerometers.

6.2 Procedure

6.2.1 Calibration

For the purposes of Calibration the secondary system is restrained from nonplanar motion and for this case the whole system may be interpreted dynamically as a linear system comprising a seismically excited horizontal beam with a lumped mass (the secondary system) at its free end.
Fig. 6.1-1 Experimental model of coupled system used in investigation of 4 interacting modes
The bending responses of the linearised system are calibrated by means of the optical method outlined for the vertical beam in Chapter 3. In this instance the x-y traversing microscope is arranged so that it is focused on a scribed line at the coupling point. The peak output voltages displayed on the AC voltmeter are recorded against corresponding coupling point displacements for a variety of excitation acceleration levels. The measured displacements of the coupling point when the system is vibrating in its second bending mode are not, strictly speaking absolute, however the displacement of the support is seen to be negligible compared to the size of the response measured at the coupling point, therefore no adjustments in the recorded responses are made.

Several linear natural frequencies of forced vibration as functions of primary beam length are plotted, so that desired system frequencies can be quickly attained by interpolating the resulting graph and altering the length \( L \) accordingly. As in the secondary system classical linear viscous damping is assumed and a numerical value for this is calculated from captured transient decays.

The response calibrations described in Chapter 3 for the secondary system are used again as no internal changes to this part of the system have been made, and therefore no description of this procedure is given here.

6.2.2 System Damping Coefficients

These are calculated from the transient decays for both beams, however for either beam vibrating in its second bending mode this was only possible under forced vibration conditions. The small change in length of the primary beam required when the natural frequency is to be changed does not significantly alter the damping value derived in this way. Similarly the slight discrepancies between damping levels obtained under free and forced conditions are not considered to be significant. The damping coefficient for the primary system is found to be,

\[ \xi_B = 0.028 \]
The damping coefficients for vibration of the secondary beam in its fundamental bending and torsion modes are found to be relatively low, and are, respectively,

\[ \xi_b = 0.0013 \quad \xi_t = 0.0003 \]

The damping experienced by the secondary beam when vibrating in its second bending mode is calculated from the transient decay which results from a forced vibration situation, and the resulting numerical value is virtually equal to that of the fundamental bending mode. For this reason no distinction is made between damping coefficients of fundamental and second nonplanar bending, and they are both described by, \( \xi_b \).

6.2.3 Tests

Experimental points for modal responses as functions of external detuning are presented for different cases of internal tuning and excitation acceleration levels. The excitation acceleration is set to the required level, with the secondary system restrained, and then the modal responses are noted over a range of excitation frequencies with the secondary system restraint removed. This procedure is adopted for both increasing and decreasing excitation frequency over the chosen range with the sweep rate kept as slow as possible so that the 'entry' and 'exit' points of the nonplanar modes are clearly identifiable. Regions of 'overhang' which arise from the direction of approach of the frequency sweep are clearly evident.

The other series of tests involves maintaining a fixed excitation frequency and slowly sweeping up and down over a selected range of excitation acceleration levels. As in the previous case this procedure is adopted because the direction of the sweep dictates at which point interaction is seen to commence.

Records of modal responses with time and also spectral responses are presented.

6.2.4 Experimental results

Figures 6.2.4-1 to 6.2.4-8, 6.2.4-13 to 6.2.4-20, 6.2.4-25 to 6.2.4-32 show the system responses as functions of the external detuning parameter \( \sigma \). Graphs of system responses as functions of excitation acceleration are shown in Figures 6.2.4-9 to 6.2.4-12, 6.2.4-21 to 6.2.4-24, 6.2.4-33 to 6.2.4-36.
The experimental results are shown as discrete points in which $\Delta$ represents values obtained for increasing excitation frequency and $\square$ for decreasing frequency. The onset of autoparametric resonance is illustrated by the 'jump' arrows. The system parameters for the figures are as follows,

- Fig 6.2.4-1 to 6.2.4-4: $w_0 \Omega^2 = 3 MS^{-2}, \eta_1 = 0, \eta_2 = 0$
- Fig 6.2.4-5 to 6.2.4-8: $w_0 \Omega^2 = 10 MS^{-2}, \eta_1 = 0, \eta_2 = 0$
- Fig 6.2.4-9 to 6.2.4-12: $\sigma = 0, \eta_1 = 0, \eta_2 = 0$
- Fig 6.2.4-13 to 6.2.4-16: $w_0 \Omega^2 = 3 MS^{-2}, \eta_1 = 6.15, \eta_2 = -6.15$
- Fig 6.2.4-17 to 6.2.4-20: $w_0 \Omega^2 = 10 MS^{-2}, \eta_1 = 6.15, \eta_2 = -6.15$
- Fig 6.2.4-21 to 6.2.4-24: $\sigma = 0, \eta_1 = 6.15, \eta_2 = -6.15$
- Fig 6.2.4-25 to 6.2.4-28: $w_0 \Omega^2 = 3 MS^{-2}, \eta_1 = 0, \eta_2 = -12.3$
- Fig 6.2.4-29 to 6.2.4-32: $w_0 \Omega^2 = 10 MS^{-2}, \eta_1 = 0, \eta_2 = -12.3$
- Fig 6.2.4-33 to 6.2.4-36: $\sigma = 0, \eta_1 = 0, \eta_2 = -12.3$

(1) **Response amplitudes as functions of excitation frequency**

(external detuning)

The first series of results are given in Figures 6.2.4-1 to 6.2.4-4. The response of the externally excited planar bending mode displays the usual characteristics of autoparametric interaction in which there is a linear response with increasing frequency up to a threshold point at which the autoparametric resonance region occurs. The suppression effect experienced by this mode is evident, and continues through exact external resonance until the excitation frequency arrives at a similar point on the other side of exact external resonance.
\[ \frac{\Omega}{\omega_{b2}} \]

\[ W \quad \Omega^2 = 3 \text{ MS}^{-2}; \quad \eta_1 = 0; \quad \eta_2 = 0; \quad \gamma_B = 0.28; \quad \gamma_b = 0.013; \quad \gamma_t = 0.003 \]

Fig. 6.2.4-2 Experimental \( \gamma \) mode response
$\frac{\Omega}{\omega_{B2}}$

$W \sum_{0}^{2} Z^2_{MS} - 2 \; \eta_1 = 0 ; \; \eta_2 = 0 ; \; J_B = 0.28 ; \; J_b = 0.013 ; \; J_l = 0.003$

Fig. 6.2.4-3 Experimental 4 mode response
\[ \frac{\Omega}{\omega_{B2}} \]

\[ \omega = 3 \text{ MS}^{-2} \quad \eta_1 = 0 \quad \eta_2 = 0 \quad I_B = 0.28 \quad I_b = 0.013 \quad I_t = 0.003 \]

Fig. 6.2.4-4 Experimental 4 mode response
The mode then resumes its original linear response as the excitation frequency increases. At the point of exact external resonance the planar mode is suppressed to its minimum value. Similar overall behaviour is displayed for decreasing external frequency, however the 'jumps' occur at slightly different frequencies. The nonplanar modes do not respond at frequencies outside the autoparametric region, however at the changeover points from linear to nonlinear responses of the planar mode, large 'jumps' occur and significant steady vibrations in all three nonplanar modes are generated. The two nonplanar bending modes respond in a qualitatively similar fashion and the response curves both display reasonable symmetry about the externally resonant point. The torsion mode $c_3$ shows less symmetry however its minimum point coincides on the frequency axis with those of the other modes.

Increased excitation level for the same tuning conditions widens the region of autoparametric resonance and there is a corresponding increase in the overall amplitudes of $a$, $c_1$, $c_2$, and $c_3$. This case is presented in Figures 6.2.4-5 to 6.2.4-8. An additional feature of the response of $c_2$ is the sharp narrowing of the curve just around the externally resonant point and also the appearance of small 'sub-regions' within the main region. Both 'sub-regions' are fairly clearly defined and occur at frequencies close to those at which modal interaction commence. The torsion mode $c_3$ displays even less symmetry than previously and there is some evidence here too of small 'sub-regions' occurring at the same points as in $c_2$.

Figures 6.2.4-13 to 6.2.4-16 depict a case in which the bending/bending resonance is perfectly tuned ($\eta_1 = -\eta_2$) for the lower excitation level of $W_0 \Omega^2 = 3 \text{MS}^{-2}$. The predominant feature here is that all four modal responses are offset with respect to the external detuning parameter, the minimum point in the nonlinear region of the planar mode is at $\Omega / \Omega_{B2} \approx 0.98$. Other than this the response of the planar mode has the characteristics of the first case ($\eta_1 = \eta_2 = 0$). The nonplanar mode $c_1$ exhibits a symmetrical response curve, however mode $c_2$ again shows some evidence of small 'sub-regions' on either side of the externally resonant point which correspond with those of the torsion mode $c_3$. The torsion mode reaches a lower minimum point than in the first case (c.f Figure 6.2.4-4).
$\Omega / \omega_{B2}$

$W \int_0^2 = 10 \text{ MS}^{-2} ; \eta_1 = 0 ; \eta_2 = 0 ; \int_8 = .28 ; \int_b = 0.013 ; \int_t = 0.003$

*Fig. 6.2.4-5 Experimental 4 mode response*
$\frac{\Omega}{\omega_{B2}}$

$w_0 \frac{2}{\eta_1} = 10 \text{ m/s}^2$; $\eta_1 = 0$; $\eta_2 = 0$; $R = 0.28$; $b = 0.013$; $t = 0.003$

Fig. 6.2.4-6 Experimental 4 mode response
$\frac{\Omega}{\omega_{B2}}$

$w = 10 \text{ MS}^{-2}$ ; $\eta_1 = 0$ ; $\eta_2 = 0$ ; $\int_R = 0.28$ ; $\int_h = 0.013$ ; $\int_t = 0.003$

Fig. 6.2.4-7 Experimental 4 mode response
$\Omega/\omega_{B2}$

$W = 10 \text{ MS}^{-2}$; $\eta_1 = 0$; $\eta_2 = 0$; $B = 0.28$; $b = 0.013$; $t = 0.003$

Fig. 6.2.4-8 Experimental 4 mode response
\[ \sigma = 0; \quad \eta_1 = 0; \quad \eta_2 = 0; \quad I_B = 0.28; \quad \bar{I}_b = 0.013; \quad \bar{I}_t = 0.003 \]

Fig. 6.2.4-9 Experimental 4 mode response
$\sigma = 0 ; \, \eta_1 = 0 ; \, \eta_2 = 0 ; \, I_B = 0.28 ; \, I_B = 0.013 ; \, I_t = 0.003$

Fig. 6.2.4-10 Experimental 4 mode response
Fig. 6.2.4-11 Experimental 4 mode response
\[ \sigma = 0; \quad \eta_1 = 0; \quad \eta_2 = 0; \quad I_B = 0.28; \quad I_h = 0.013; \quad I_t = 0.003 \]

*Fig. 6.2.4-12 Experimental \( m \) mode response*
$\frac{\Omega}{\omega_{B2}}$

$W \int \frac{1}{\omega_{B2}}^2 = 3 \text{ MS}^{-2}$; $\eta_1 = 6.15$; $\eta_2 = -6.15$; $\gamma_B = 0.28$; $\gamma_b = 0.013$; $\gamma_l = 0.003$

*Fig. 6.2.4-13 Experimental $h$ mode response*
Nonplanar Bending Response $c_i$

$\tilde{\Omega}/\omega_{B2}$

$w \sum_{0}^{2} \frac{2}{MS}$; $\eta_1 = 6.15$; $\eta_2 = -6.15$; $\tilde{f}_B = 0.28$; $\tilde{f}_b = 0.013$; $\tilde{f}_t = 0.003$

Fig. 5.2.4-14 Experimental 4 mode response
\[ \frac{\Omega}{\omega_{B2}} \]

\[ w_n^2 = 3 \text{ MS}^{-2}; \eta_1 = 6.15; \eta_2 = 0.15; \beta_B = 0.28; \beta_b = 0.013; \beta_t = 0.003 \]

Fig. 6.2.4-15 Experimental 4 mode response
Nonplanar Torsion Response $c_3$ vs $\Omega/\omega_{b2}$

$\mathbf{w}_0 \nabla^2 = 3 \text{ MS}^{-2}$; $\eta_1 = 6.15$; $\eta_2 = -6.15$; $\gamma_B = 0.28$; $\gamma_b = 0.013$; $\gamma_t = 0.003$

Fig. 6.2.4-16 Experimental $h$ mode response
An increase in the excitation level for the same internal tuning case produces responses similar to those described above except that the nonlinear region is wider, as expected, and also the 'sub-regions' are much more clearly represented in the responses of modes $c_2$ and $c_3$. This case is illustrated in Figures 6.2.4-17 to 6.2.4-20.

In comparison with the previous two cases, Figures 6.2.4-25 to 6.2.4-28 consider the condition in which the bending/torsion resonance is perfectly tuned ($\gamma_1 = 0; \gamma_2$ away from $\gamma_1$).

Although the bending/torsion resonance is dominant the results obtained are not as strikingly different from those of Figures 6.2.4-1 to 6.2.4-4 as might be expected. The relative amplitude levels of modes $c_1$ and $c_2$ and their qualitative characteristics are virtually unchanged. The planar mode however, does not reach a particularly low minimum amplitude level at its most suppressed point and the nonlinear region, although quite apparent, is rather confused. The response amplitudes of the torsion mode $c_3$ are all uniformly higher than those of the previous cases for this excitation level, as is to be expected for this case of internal tuning.

The final case in this series of tests considers the dominant bending/torsion resonance with a higher excitation level, and the results are given in Figures 6.2.4-29 to 6.2.4-32. The response of the planar mode is less confused than the preceding case and displays a fairly symmetrical absorption curve in the nonlinear region, however in the upward frequency sweep the downward jump from the nonlinear to linear solution occurs from a very high amplitude relative to the other levels apparent for this case. The three nonplanar modes respond vigorously in the nonlinear region and here too the 'sub-regions' are manifested in the responses of $c_2$ and $c_3$.

(2) Response amplitudes as functions of excitation acceleration

The three internal tuning cases discussed in series (1) are presented here for varying excitation accelerations. Figures 6.2.4-9 to 6.2.4-12 show the modal responses for the internal tuning case in which $\gamma_1 = \gamma_2 = 0$. The 'saturation' effect, in which
\[ w \Omega^2 = 10 \, \text{ms}^{-2}; \eta_1 = 6.15; \eta_2 = -6.15; \int_B = 0.28; \int_b = 0.013; \int_t = 0.003 \]

Fig. 6.2. Experimental \( \eta \) mode response
\[ \frac{\Omega}{\omega_{b2}} \]

\[ w^{2} = 10 \text{ ms}^{-2}; \quad \eta^{1} = 6.15; \quad \eta^{2} = -6.15; \quad \sum_{b} = 0.20; \quad \sum_{b} - 0.013; \quad \sum_{t} = 0.003 \]

Fig. 6.2.4-18 Experimental 4 mode response
Fig. 6.2.4-19 Experimental \( \beta \) mode response
\[ \Omega / \omega_{B2} \]

\[ w \Omega^2 = 10 \text{ MS}^{-2} ; \eta_1 = 6.15 ; \eta_2 = -5.15 ; \gamma = 0.28 ; \gamma_b = 0.013 ; \gamma_t = 0.003 \]

Fig. 6.2.4-20 Experimental 4 mode response
the directly excited mode responds at a constant level (once nonlinear interaction has commenced) with increasing excitation acceleration is typical of systems involving autoparametric coupling and is clearly shown to exist in this system. The transition from linear to nonlinear responses occurs at \( \omega_0 \Omega^2 \approx 1.5 \text{MS}^{-2} \) and after this point the planar mode responds at a fairly constant level until the excitation acceleration reaches \( \omega_0 \Omega^2 \approx 15 \text{MS}^{-2} \) at which point the response shows a tendency to increase with \( \omega_0 \Omega^2 \). The system is still behaving in a nonlinear fashion at and above this level of acceleration as modes \( c_1 \) and \( c_3 \) continue to absorb energy. The nonplanar modes do not respond until the threshold level of excitation acceleration is attained but after this point the nonplanar response amplitudes increase with acceleration. There is an exception to this in that the second nonplanar bending mode shows a marked tendency to decrease after the threshold of \( \omega_0 \Omega^2 \approx 15 \text{MS}^{-2} \) as the planar mode increases. This suggests that at this point an unexplored internal mechanism for the transfer of energy between the planar mode and the second nonplanar bending mode could be influencing the behaviour of the system.

Figures 6.2.4-21 to 6.2.4-24 are for the internal tuning case whereby \( \gamma_1 = -\gamma_2 \), and although nonlinear interaction occurs at approximately the same acceleration level the planar mode 'jumps' to a much higher amplitude than for the first case. The 'saturation' effect is evident although the steady responses of the planar mode fluctuate between higher and lower values for increasing excitation acceleration. The amplitudes of \( c_1 \) and \( c_3 \) increase with excitation once the threshold level has been crossed, but \( c_2 \) again starts to decrease at a high value threshold of excitation acceleration (\( \omega_0 \Omega^2 \approx 10 \text{MS}^{-2} \)).

This does not, in this case, seem to be coincident with a radical change in the behaviour of the planar mode, although it does commence a slight upward climb at and beyond this excitation level.

The final case considers \( \gamma_1 = 0 \), \( \gamma_2 \) away from \( \gamma_1 \), in which the bending/torsion resonance is dominant.
\[ \sigma = 0; \quad \eta_1 = 6.15; \quad \eta_2 = -6.15; \quad I_B = 0.28; \quad I_b = 0.013; \quad I_t = 0.003 \]

Fig. 6.2.4-21 Experimental mode response
Non-planar Bending Response $c_1$

$W_0 \Omega^2$

$\sigma = 0 ; \eta_1 = 5.15 ; \eta_2 = -6.15 ; I_B = 0.28 ; I_b = 0.013 ; I_t = 0.003$

Fig. 6.2.4-22 Experimental $h$ mode response
\[ \sigma = \eta : \eta_1 = 6.15 ; \eta_2 = -6.15 ; \int_\alpha = 0.28 ; \int_b = 0.013 ; \int_t = 0.003 \]

Fig. 6.2.4-23 Experimental \( \mu \) mode response
\[ W_0 \Omega^2 \]

\[ \sigma = 0; \quad \eta_1 = 6.15; \quad \eta_2 = -6.15; \quad B = 0.28; \quad b = 0.013; \quad t = 0.003 \]

Fig. 6.2.4-24 Experimental 4 mode response
$w_0 \Omega^2 = 3 \text{ MS}^{-2}; \eta_1 = 0; \eta_2 = -12.3; \int_B = 0.28; \int_b = 0.013; \int_t = 0.003$

Fig. 6.2.4-25 Experimental $\delta$ mode response
\[ \frac{\Omega}{\omega_{B2}} \]

\[ W \sum_{i=0}^{2} \eta_{i} = 0 \; ; \eta_{1} = -12.3 \; ; \int_{b}^{B} = 0.28 \; ; \int_{b} = 0.013 \; ; \int_{t} = 0.003 \]

Fig. 6.2.4-26 Experimental \( \eta \) mode response
\[ \Omega / \omega_{B_2} \]

\[ W_0 \Omega^2 = 3 \text{ MS}^{-2} ; \eta_1 = 0 ; \eta_2 = -12.3 ; \int_B = 0.28 ; \int_b = 0.013 ; \int_t = 0.003 \]

Fig. 6.2.4-27 Experimental \( n \) mode response
\[ \frac{\Omega}{\omega_{B2}} \]

\[ W \left( \sum_{n=2}^{\infty} \frac{1}{n} \right)^{-2} ; \eta_1 = 0 ; \eta_2 = -12.3 ; \int_{b}^{r} = 0.28 ; \int_{b} = 0.013 ; \int_{t} = 0.003 \]

Fig. 6.2.4-28 Experimental \( h \) mode response
$\frac{\Omega}{\omega_{B2}}$

$W \frac{\Omega^2}{\omega^2} = 10 \text{ ms}^{-2}$

$\gamma_1 = 0$; $\gamma_2 = -12.3$; $\overline{f}_b = 0.28$; $\overline{g}_b = 0.013$; $\overline{f}_t = 0.003$

Fig. 6.2.4-29 Experimental 4 mode response
$\Omega/\omega_{B2}$

$W_0 \omega^2 = 10 \text{ Ms}^{-2}$; $\eta_1 = 0$; $\eta_2 = -12.3$; $I_B = 0.28$; $I_b = 0.013$; $I_t = 0.003$

Fig. 6.2.4-30 Experimental $4$ mode response
\[ \Omega \sim 10 \text{ ms}^{-2} \quad \gamma_1 = 0 \quad \gamma_2 = -12.3 \quad \tilde{f}_B = 0.28 \quad \tilde{f}_b = 0.013 \quad \tilde{f}_t = 0.003 \]

*Fig. 6.2.1-31 Experimental 4 mode response*
\[ \Omega / \omega_{32} \]

\[ W \Omega^2 = 10 M_s^{-2} ; \eta_1 = 0 ; \eta_2 = -12.3 ; J_B = 0.28 ; J_b = 0.013 ; J_t = 0.003 \]

Fig. 6.2.4-32 Experimental \( \eta \) mode response
\[ \sigma = 0; \quad \eta_1 = 0; \quad \eta_2 = -12.3; \quad J_B = 0.28; \quad J_b = 0.013; \quad J_t = 0.003 \]

Fig. 6.2, p. 39 Experimental \( \mu \) mode response
The experimental results and model response are shown in the graph. The equation provided is:

\[ W = \frac{1}{0.003} \int_{0}^{0.13} \left( \int_{0}^{0.23} \left( \int_{-0.12}^{y} 0 \cdot 0 \right) \right) dy dx \]
\[ \sigma = 0 ; \; \eta_1 = 0 ; \; \eta_2 = -12.3 ; \; I_B = 0.28 ; \; I_b = 0.013 ; \; I_t = 0.003 \]

Fig. 6.2.4-35 Experimental 4 mode response
σ = 0; \eta_1 = 0; \eta_2 = -12.3; I_B = 0.28; I_b = 0.013; I_t = 0.003

Fig. 6.2.4-36 Experimental 4 mode response
The interesting feature of the results for this tuning case (Figures 6.2.4-33 to 6.2.4-36) is that the planar mode does not saturate, even at the relatively lower levels of excitation acceleration for which nonlinear interaction occurs. The threshold point is at \( w_0 \Omega^2 \approx 1.0 \text{MS}^{-2} \), slightly lower than in previous cases. Although the planar mode does not seem to saturate, all three nonplanar modes respond entirely predictably and continue to absorb energy as the level of excitation is increased.

(3) Time responses and spectral responses

Experimental time and spectral responses are given in Figures 6.2.4-37 to 6.2.4-42 for three different cases of internal tuning. The first tuning condition is \( \gamma_1 = 0; \gamma_2 = -12.3 \), with an excitation acceleration of \( 3 \text{MS}^{-2} \), and this is depicted in Figure 6.2.4-37. The traces of the first and second nonplanar bending modes, \( c_1 \) and \( c_2 \), are electronically separated by means of suitable filtering circuits, although a degree of modulation of \( c_2 \) and \( c_1 \) is still evident in the responses. The responses of modes \( c_3 \) and \( a \) are similarly modulated by the low frequency \( c_1 \) mode, this is particularly evident in \( c_3 \).

This tuning condition gives rise to a strong response of \( c_1 \) and \( c_3 \) and a weaker response of \( c_2 \) as can be seen in Figure 6.2.4-38, in which the spectral responses are presented. An interesting feature of the results in Figure 6.2.4-38 is that a higher level of excitation acceleration is seen to suppress \( c_2 \) even more, with a corresponding increase in \( c_3 \), although the response of \( c_1 \) does not increase at all.

Figures 6.2.4-39 and 6.2.4-40 show time and spectral responses respectively for the condition \( \gamma_1 = \gamma_2 = 0 \). There is a large relative increase in the response of the nonplanar torsion mode \( c_3 \) and a noticeable reduction in the responses of \( c_2 \) and the planar mode \( a \). Increasing the excitation acceleration does not reintroduce a stronger response of \( c_2 \) but it does result in stronger responses of \( c_1 \) and \( c_3 \) (the bending/torsion resonance is dominant). As in the previous case an increase in excitation acceleration from \( 3 \text{MS}^{-2} \) to \( 10 \text{MS}^{-2} \) also produces a response at twice the frequency of \( c_1 \).

Results for the third tuning condition, \( \gamma_1 = 6.15; \gamma_2 = -6.15 \), are given in Figures 6.2.2-41 and 6.2.2-42.
The fundamental nonplanar bending mode $c_1$ responds less vigorously for this condition and there is a strong representation of the other modes including the planar mode $a$. The torsion mode $c_3$ is seen to be highly modulated by $c_1$. The spectral responses show that for an increased excitation level the response of the torsion mode does not increase as dramatically as in the two earlier cases however there is a very large increase in $c_1$ and $c_2$. This reaffirms the statements made in preceding sections that the bending/bending resonance, when perfectly tuned or very near to this condition, will significantly alter the otherwise dominant effects of the bending/torsion resonance.
Figure 6.24-37: Sample Experimental Time Responses, Undamped Interaction

\( \omega^2 = 3 \text{MS}^2 \)
\( \omega_1 = \omega_2 = -12.3 \)
\( I_b = 0.28 \)
\( I_f = 0.013 \)
\( I_t = 0.003 \)
Fig. 6.2.4-36 Sample spectrum analyser traces of nonplanar responses for two levels of excitation acceleration.

\[ \eta_1 = 0 \; ; \; \eta_2 = -12.3 \; ; \; \tilde{F}_B = 0.28 \; ; \; \tilde{F}_b = 0.013 \; ; \; \tilde{F}_t = 0.003 \]

\[ \omega_1 \quad \omega_2 \quad \omega_3 \]

For the first level:
- \( \Omega^2 = 3 \text{ MS}^{-2} \)
- \( \omega = \omega_B \)

For the second level:
- \( \Omega^2 = 10 \text{ MS}^{-2} \)
- \( \omega = \omega_B \)

Frequency (Hz)
Time (seconds)

![Graph showing oscillatory responses over time with different labels and values: w, \( \omega \), \( \eta \), \( \eta' \), \( \eta'' \), \( \delta \), \( \delta' \), \( \delta'' \), \( \sigma \), \( \sigma' \), \( \sigma'' \), \( \nu \), \( \nu' \), \( \nu'' \), \( \tau \), \( \tau' \), \( \tau'' \).]
\[ \eta_1 = 0 \quad ; \quad \eta_2 = 0 \quad ; \quad \bar{\lambda}_B = 0.28 \quad ; \quad \bar{\lambda}_t = 0.003 \quad ; \quad \bar{\lambda}_b = 0.013 \]

Fig. 6.2.4-40 Sample spectrum analyser traces of non-planar responses for two levels of excitation acceleration.

First trace:
- Amplitude graph with peaks at \( \omega_{b1} \), \( \omega_{t1} \), and \( \omega_{b2} \).
- States change at \( \Omega \) and \( \omega_{b2} \).
- Frequency range from 0 to 250 Hz.

Second trace:
- Amplitude graph with peaks at \( \omega_{b1} \), and \( \omega_{t1} \).
- States change at \( \Omega \) and \( \omega_{b2} \).
- Frequency range from 0 to 250 Hz.
\[ W \Omega_0^2 = 3 \text{ MS}^{-2} \; ; \; \eta_1 = 6.15 \; ; \; \eta_2 = -6.15 \; ; \; \xi_B = 0.28 \; ; \; \xi_b = 0.013 \; ; \; \xi_t = 0.003 \]
Fig. 6.2.4-42 Sample spectrum analyser traces of nonplanar responses for two levels of excitation acceleration
CHAPTER 7

Conclusions

7.1 General discussion

The research described within this thesis has been concerned with the effects of specific forms of parametric excitation in beam systems. Initially, the particular structure under investigation was a single cantilever beam for which the system kinematics were treated via the Euler-Kirchoff theory of rods. Two important kinematical constraints were imposed on the motion of the beam, namely the inextensibility of the elastic axis of the beam and the assumption of negligible curvature in the most rigid plane of the beam.

Both constraints relied upon the stipulation that the beam should be thin and slender. The system equations of motion were derived by the Principle of Virtual Work for a two-degree-of-freedom system with a two-mode Galerkin approximation assumed for motion in bending and torsion. The resulting set of two coupled second order "Mathieu-type" differential equations was then analysed by the Multiple Scales perturbation technique.

The bending - torsion combination resonance was shown to generate non-zero solutions from which it was then possible to develop an expression describing a transition curve for the solutions. The transition curve delineated a two parameter space into stable and unstable regions with the transition from one state to another defined by co-ordinates on the curve itself. Solutions defined by points situated within the transition curve were known to be unstable and conversely for solutions defined by points outwith the curve. The analytical expression for the transition curve relating to this resonance confirmed that derived by Dugundji and Mukhopadhyay (3) for cases of zero and nonzero system damping.

Experimental tests were conducted on a mechanical model and very close agreement was obtained between the experimental transition points and those predicted by the theory. Two different beam lengths were investigated.
Further theoretical and experimental work showed the existence of higher order combination instabilities which also involved the second nonplanar bending mode. The theoretical model was redefined with the Galerkin approximation now extended to include this mode. Terms in the equations due to a small in-plane displacement previously discounted, were included, and the resulting set of three coupled equations was re-analysed by means of the Multiple Scales method. Many other theoretical resonance conditions were shown to be potentially capable of producing nonzero solutions, and these resonances were seen to be composed of various combinations of up to three modal frequencies. Experiments indicated that a large number of these predicted resonances either required very large excitation accelerations or were not physically realisable. One resonance which was easily excited involved excitation at half the sum of the fundamental and second bending mode frequencies, and a transition curve expression was obtained for this resonance. Comparisons between experimental transition points and the theoretically derived curve showed that close agreement was possible for short stiff beams and less so for more flexible beams.

The solvability equations for the second order resonance were integrated numerically and the two bending mode responses were shown to be unbounded with time provided the excitation level was above a certain threshold value.

Observations of the experimental model demonstrated that the bending/torsion and the bending/bending resonances could occur either simultaneously or at excitation frequencies very close to each other and that small degrees of detuning were achievable by slight alterations to the end-mass position on the beam. Theoretical analysis of simultaneous first and second order combination resonances was undertaken by resorting back to first order theory introducing an additional first order resonance also shown to occur simultaneously with the resonances under discussion. A redefinition of the detuning parameter for the second order resonance resulted from this and the three-degree-of-freedom equations were subsequently rewritten to first order and expanded by the method of Multiple Scales. Laboratory tests demonstrated that solution stabilising effects physically limited the unstable responses and that the system behaviour, although theoretically unstable, was stabilised
due to these other, unaccounted for, effects. An expression was derived from the analysis relating the response amplitude of the second nonplanar bending mode to the response amplitude of the fundamental torsion mode. Experiments showed that intermodal energy exchanges did not noticeably involve the fundamental bending mode. Two detuning cases were considered, each in turn favouring the bending/bending and bending/torsion resonances, and the ratio of second nonplanar to fundamental torsion mode was plotted as a function of excitation acceleration. The theoretical curves showed that there was a distinct change in the modal content of the responses due to whichever resonance was dominant. A particularly interesting characteristic of these curves was the apparent two-way nature of the energy exchange. Energy was shown to flow either from the fundamental torsion mode to the second bending mode or in the other direction, as dictated by the system tuning conditions. The results of the experimental work confirmed the theoretical predictions, however the stabilising effects mentioned above were only shown to occur for a specific range of excitation acceleration levels. The responses of the model were nonstationary for excitation levels above and below this range.

In a further stage of the investigation the single vertical cantilever beam was coupled to a horizontal beam which was subjected to a harmonic excitation at its support. This enabled the parametric resonance effects to be considered as internally generated phenomena within a system undergoing forced vibration. A single mode Galerkin approximation was assumed for the primary beam motion, while the coupled beam was considered as a three-degree-of-freedom system. A set of four coupled ordinary non-linear second order differential equations resulted and these were expanded to first order by the method of Multiple Scales. The resonance conditions under study considered the external resonance of the primary beam second bending mode together with the internal resonances; i.e. (a) the primary beam motion and the secondary beam combination of fundamental bending and torsion modes, and (b) the primary beam motion and the secondary beam combination of second bending and fundamental torsion modes. These two secondary system combination resonances,
when in close proximity, allowed for a first order representation of the bending/bending resonance in conjunction with the bending/torsion resonance. A Numerical Integration procedure was used to determine first order stationary solutions.

7.2 **Comparison of Theoretical and Experimental results for the coupled beam system**

Direct comparisons may be made between the following,

1) Fig.5.2.2-1(a-d) and Fig.6.2.4-1 to Fig.6.2.4-4
2) Fig.5.2.2-2(a-d) and Fig.6.2.4-5 to Fig.6.2.4-8
3) Fig.5.2.2-4(a-d) and Fig.6.2.4-13 to Fig.6.2.4-16
4) Fig.5.2.2-5(a-d) and Fig.6.2.4-17 to Fig.6.2.4-20
5) Fig.5.2.2-6(a-d) and Fig.6.2.4-29 to Fig.6.2.4-32
6) Fig.5.2.2-7(a-d) and Fig.6.2.4-9 to Fig.6.2.4-12
7) Fig.5.2.2-9(a-d) and Fig.6.2.4-33 to Fig.6.2.4-36

1) For the tuning and excitation parameters of this case the linear solution for the planar mode and the zero solutions for the nonplanar modes are shown to be theoretically stable over a range of excitation frequencies. Experimental evidence suggests otherwise, with non-linear interaction between the planar and nonplanar modes occurring over a large portion of the excitation frequency range. Very close qualitative agreement is apparent between the linear portions of the experimental planar response curve (Figure 6.2.4-1) and the theoretical curve in Figure 5.2.2-1(a).

2) The higher level of excitation investigated in this case generates non-linear theoretical solutions; however the frequency range over which interaction occurs is very small ($\Omega/\omega_{B2} = 0.992$ to 1.008) in comparison with the range of experimental interaction ($\Omega/\omega_{B2} = 0.964$ to 1.05 for increasing frequency, and $\Omega/\omega_{B2} = 0.92$ to 1.05 for decreasing frequency). Again there is close agreement between
theoretical and experimental linear responses, and the minimum points in the non-linear parts of the planar response curves occur at similar amplitudes. The 'peaks' of the theoretical non-linear nonplanar modal responses are coincident with the 'troughs' of the experimental nonplanar responses at the point of external resonance, however the amplitudes of the 'peaks' are very much higher than the observed experimental 'troughs'.

3) The theoretical solutions are virtually identical to those of (1) above, with the exception of the maximum amplitude attained by the planar mode, which is slightly reduced. Agreement between the linear responses is again very close.

4) The theoretical responses are shown to be quasi-symmetrical about $\Omega/\omega_{B2} = 0.992$ and the offset of the experimental curves is to $\Omega/\omega_{B2} = 0.98$, with the only quantitative agreement between the two being for the case of the linear solutions. There is however a recognisable similarity in the characteristic shapes of the theoretical and experimental planar response curves.

5) Theoretical non-linear solutions are presented for this case and the qualitative aspects of the discussion in (2) are relevant here. By comparing the theoretical response curves of Figure 5.2.2-2 and Figure 5.2.2-6 the effect of detuning of the bending/bending resonance is strikingly apparent (the detuned case is shown in Figure 5.2.2-6), with a reduction in the nonplanar second bending amplitude at external
resonance to approximately one tenth of that of the former case. There is a small but corresponding change in the other direction in the theoretical response of the torsion mode. The experimental non-linear region occurs over a similar frequency range to that in (2);
\[ \Omega / \omega_{B2} = 0.96 \text{ to } 1.06 \] as the excitation frequency is increased and \[ \Omega / \omega_{B2} = 0.914 \text{ to } 1.036 \] as the frequency is decreased. Figures 6.2.4-7 and 6.2.4-31 show the experimental responses of mode \( c_2 \), in which there is an obvious reduction in amplitude for this tuning case and a corresponding small increase in the torsion mode response (Figures 6.2.4-8 and 6.2.4-32). Although the theory and experimental work show the same trends, when comparing the results for this tuning case with those of (2), there is no quantitative agreement between theoretical and experimental results for either case.

6) Experimentally, non-linear interaction is shown to occur at a lower excitation acceleration level than predicted theoretically (\( \omega_0 \Omega^2 = 1.0 \text{ to } 1.5 \text{ MS}^{-2} \) as opposed to \( \omega_0 \Omega^2 \propto 3.5 \text{ MS}^{-2} \); Figures 6.2.4-9 to 6.2.4-12). The theoretical 'saturation amplitude' of the planar mode is approximately 0.4 of that found by experiment, whereas the experimental nonplanar response amplitudes are all very much lower, throughout the excitation range, than the theoretical predictions.

7) There is no experimental evidence of the saturation phenomenon in this case, however the amplitude at the non-linear 'entry point' is not too far removed from the theoretical amplitude. There is also a reasonable degree of quantitative agreement between the theoretical and experimental
results for $c_2$. Modes $c_1$ and $c_3$ are shown to respond generally at significantly lower amplitudes in practice than those proposed by the theory.

7.3 Conclusions

i) The previously reported work of Dugundji and Mukhopadhyay (3) on the bending/torsion combination instability encountered in a parametrically excited beam has been re-examined, and an identical expression for the transition curve obtained by an alternative method of analysis. Experimental tests on a laboratory model confirmed the theoretical result.

ii) A second order perturbational expansion of extended governing equations, with an additional degree of freedom, showed the possibility of higher order combination instabilities. An expression for the transition curve for a second order bending/bending instability has been derived. Experiments revealed that the expression holds for short and intermediate length beams but correlation was less good for more flexible structures.

iii) Numerical integration of the second order resonance 'slow-time' equations proved that the bending responses were under certain circumstances unbounded with time thus generating a dynamic instability within the system.
iv) Experiments verified that for specific beam geometries both the first order bending/torsion resonance and the second order bending/bending resonance could be simultaneously excited.

v) Energy exchanges, principally between the second bending mode and the fundamental torsion mode were noticed in the experimental model and an analytical expression was derived for this. A close agreement was obtained between experimental and theoretical results.

vi) The coupled beam structure, studied by Bux and Roberts (55), was considered in relation to the simultaneous generation of these resonances in the secondary beam, and a formal first order theoretical model was postulated. Stationary solutions were obtained by an Analytical-Numerical process for various cases of external and internal detuning. Both the theoretical and experimental results demonstrated that complex non-linear vibratory interactions could occur although there were, in some cases, considerable discrepancies between these results. For all the comparative theoretical and experimental cases presented, the linear responses were in very close agreement.

7.4 Suggestions for further work

There is scope for further investigations into higher order combination resonances (or secondary resonances) in the single beam structure. Some exploratory experiments showed the possibility of generating nonsynchronous responses for the following cases,

\[ \Omega = \omega_{t1} - 2\omega_{b1} + \omega_{b2} \]

\[ \Omega = 2\omega_{b1} + \omega_{t1} + \omega_{b2} \]

(see Table 3.2.1-1)
A study of the underlying mechanisms which govern the response stabilising effects encountered in the 'unstable' parametrically excited cantilever beam would also be advantageous. The possibility of treating the bending/bending resonance alone in the coupled structure should not be overlooked, however this would involve a second order multiple scales expansion of a revised set of four differential equations of motion and the level of calculations required could be prohibitive. Direct solution of the governing equations by a numerical method would probably be the most feasible approach.

The theoretical solutions to the first order expansion analysis require further consideration particularly in the light of their comparative inconsistencies with the experimental responses. The kinematic alliance between the bending/torsion and the bending/bending resonances has not been exhaustively researched and further work in this area could provide a clearer picture of the means by which energy is seen to flow between adjacent modes of vibration.

Non-stationary response behaviour has not been discussed in this study principally because such phenomena were not observed for the specific geometry of this coupled beam system. It is expected that the resonance conditions investigated could produce such effects in other systems and for this reason further advances in this area could be made by considering alternative methods of analysis.


NOTE, REFERENCE (57)

Strain, Vol. 20, No. 3, 1984
Appendix 1: Kinematics of a thin beam under combined bending and torsion

The following theory is based on the Euler-Kirchoff-Love treatment of rods, and considers an undeformed beam with one end fixed at O (the same point as C in Figure 4.2-1) and lying along the Oz axis, as shown in Figure A1-1. The orthogonal axis Oxz is taken as a reference frame and can therefore be seen to be fixed in space, with the unit base vectors \( \bar{I}, \bar{J}, \bar{K} \), corresponding to the x, y, z directions respectively.

A beam element at point Q, distance \( d \) from the origin O is, after an arbitrary displacement at \( Q' \), moved through \( u, v \) and \( w \) in the x, y and z directions respectively. An orthogonal axis system \( x'-y'-z' \) is defined for the displaced element with the \( x'-y' \) plane in the plane of the cross-section of the beam, and the positive \( z' \) axis perpendicular to the cross-section in the direction of increasing beam length. This is described by the term 'the body-fixed frame' in the work of Roberts and Bux and has the unit base vectors \( \bar{I}', \bar{J}', \bar{K}' \) in the \( x', y', z' \) directions respectively. These axes are termed 'principal torsion-flexure' axes of the beam at that point, thus each elemental cross-section will possess such axes, and these will undertake different rotations with respect to the elastic axis. The change in position of the element is interpreted as three successive rotations about some suitably defined intermediate axes through what are described as 'Euler angles'; this is shown in Figure A1-2. The first rotation is about the Oy axis, through angle \( \alpha \), which takes the \( x-y-z \) axis to an intermediate stage defined as \( x,-y,-z, \), with unit vectors \( \bar{I},\bar{J},\bar{K} \), respectively. This is represented mathematically by the direction cosine matrix equation,

\[
\begin{bmatrix}
\bar{I} \\
\bar{J} \\
\bar{K}
\end{bmatrix}
= \begin{bmatrix}
\cos\alpha & 0 & -\sin\alpha \\
0 & 1 & 0 \\
\sin\alpha & 0 & \cos\alpha
\end{bmatrix}
\begin{bmatrix}
\bar{I} \\
\bar{J} \\
\bar{K}
\end{bmatrix}
\quad \text{(A1-1)}
\]
Fig. A1-1 Geometrical representation of displaced beam element
Rotations of axes through three Eulerian angles from $x$-$y$-$z$ to $x'$-$y'$-$z'$. 
The next rotation is through angle $\beta$, about the Ox, axis, taking the $x,y,z$, axes to another intermediate stage $x_2y_2z_2$, with associated unit vectors $\overrightarrow{i}_2, \overrightarrow{j}_2, \overrightarrow{k}_2$, such that,

$$
\begin{bmatrix}
\overrightarrow{i}_2 \\
\overrightarrow{j}_2 \\
\overrightarrow{k}_2 
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos\beta & -\sin\beta \\
0 & \sin\beta & \cos\beta 
\end{bmatrix}
\begin{bmatrix}
\overrightarrow{i} \\
\overrightarrow{j} \\
\overrightarrow{k} 
\end{bmatrix}
$$

(A1-2)

Finally a rotation is made through angle $\phi$ about the Oz, axis which brings the $x_2y_2z_2$ axes to the 'Body-fixed' frame $x'y'z'$, thus,

$$
\begin{bmatrix}
\overrightarrow{i}' \\
\overrightarrow{j}' \\
\overrightarrow{k}' 
\end{bmatrix}
= \begin{bmatrix}
\cos\phi & \sin\phi & 0 \\
-\sin\phi & \cos\phi & 0 \\
0 & 0 & 1 
\end{bmatrix}
\begin{bmatrix}
\overrightarrow{i}_2 \\
\overrightarrow{j}_2 \\
\overrightarrow{k}_2 
\end{bmatrix}
$$

(A1-3)

The rate of rotation vector of the frame from Oxyz to Ox'y'z' is expressed by,

$$
\overrightarrow{\omega} = \alpha' \overrightarrow{j}' - \beta' \overrightarrow{i}' + \phi' \overrightarrow{k}',
$$

(A1-4)

By considering the direction cosine matrix equations the following emerge.

$$
\overrightarrow{i}_2 = \overrightarrow{i}' \cos\phi + \overrightarrow{j}' \sin\phi
$$

(A1-5)

$$
\overrightarrow{j}_2 = \overrightarrow{i}' \sin\phi \cos\beta + \overrightarrow{j}' \cos\phi \cos\beta + \overrightarrow{k}' \sin\beta
$$

(A1-6)

$$
\overrightarrow{k}_2 = \overrightarrow{k}'
$$

(A1-7)

which, when substituted into (A1-10) give,

$$
\overrightarrow{\omega} = \overrightarrow{i}' [\alpha' \sin\phi \cos\beta - \beta' \cos\phi] \\
+ \overrightarrow{j}' [\alpha' \cos\phi \cos\beta - \beta' \sin\phi] \\
+ \overrightarrow{k}' [\alpha' \sin\beta + \phi']
$$

(A1-8)

NOTE

Where ' denotes differentiation w.r.t. s for angles $\alpha, \beta, \phi$, but does not imply differentiation of $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$; in this case it is used merely to distinguish these vectors from $\overrightarrow{i}_1, \overrightarrow{i}_2, \overrightarrow{j}_1, \overrightarrow{j}_2, \overrightarrow{k}_1, \overrightarrow{k}_2$. 
By considering the geometry of Figure A1-3 the angles $\alpha$ and $\beta$ are obtained,

$$\sin \alpha = \frac{u'}{\sqrt{1 - (v')^2}}$$  \hspace{1cm} (A1-9)

$$\cos \alpha = \frac{\sqrt{1 - (v')^2 - (u')^2}}{\sqrt{1 - (v')^2}}$$  \hspace{1cm} (A1-10)

$$\sin \beta = v'$$  \hspace{1cm} (A1-11)

$$\cos \beta = \sqrt{1 - (v')^2}$$  \hspace{1cm} (A1-12)

From which the derivatives w.r.t. $s$ (or $z$, since it can be shown that to second order $\frac{\partial}{\partial s} = \frac{\partial}{\partial z}$) are calculated,

$$\alpha' = \frac{1}{\sqrt{1 - (u')^2 - (v')^2}} \left[ u'' + \frac{u'v''v'}{(1-(v')^2)} \right]$$  \hspace{1cm} (A1-13)

$$\beta' = \frac{v''}{\sqrt{1 - (v')^2}}$$  \hspace{1cm} (A1-14)

Thus equation (A1-4) may be rewritten in its component form,

$$\omega_i' = \left[ u'' + \frac{u'v''v'}{(1-(v')^2)} \right] \frac{\sqrt{1 - (v')^2}}{\sqrt{1 - (u')^2 - (v')^2}} \sin \phi$$

$$- \left[ \frac{v''}{1 - (v')^2} \right] \cos \phi$$  \hspace{1cm} (A1-15)
Fig. A1-3 Geometrical relationship between Euler angles and displacement variables.
\[ \omega_j = \left[ u'' + \frac{u'v'v''}{1 - (v')^2} \right] \frac{\sqrt{1 - (v')^2}}{\sqrt{1 - (u')^2 - (v')^2}} \cos \phi \]

\[ -\left[ \frac{v''}{\sqrt{1 - (v')^2}} \right] \sin \phi \]  \hspace{1cm} (A1-16)

\[ \omega_k = v' \left[ u'' + \frac{u'v'v''}{1 - (v')^2} \right] \frac{\sqrt{1 - (v')^2}}{\sqrt{1 - (u')^2 - (v')^2}} + \dot{\phi}' \]  \hspace{1cm} (A1-17)

An arbitrary displacement of the beam element is, therefore, described by three Euler angles and three deflection components, with principal bending curvatures \( k_1, k_2, \) and torsion rate \( \tau \) acting about the \( x', y', \) and \( z' \) axes respectively, for which the rotational vector forms are given above. For small \( \phi \) and small \((u')^2, (v')^2, u'v'v''\), the following quantities, to second order, result,

\[ k_1 = u'' \phi - v'' \]  \hspace{1cm} (A1-18)

\[ k_2 = u'' + v'' \phi \]  \hspace{1cm} (A1-19)

\[ \tau = \dot{\phi}' + v'u'' \]  \hspace{1cm} (A1-20)

**NOTE**

\( u \) should be read as \( U \), thus conforming to the notation used in the main text.
Appendix 2: Deflection form function Integrals

From expressions (2.2-2 and 2.33) the deflection form functions are represented by the Galerkin representations,

\[ U(z,t) = f(z)U_0(t) \]
\[ \phi(z,t) = h(z)\phi_0(t) \]

The constraint of negligible local curvature in the stiff plane,

\[ K_1 = 0 \]

gives,

\[ v'' - U''\phi \]

Thus the following expression is derived,

\[ v_0 = \int_0^1 (1 - z)\phi U''dz \quad (A2-1) \]

This may be written as,

\[ v_0 = \int_0^1 (1 - z)h\phi_0 f''U_0 dz \quad (A2-2) \]

From (2.2-6) it is seen that,

\[ B = \int_0^1 (1 - z)hf''dz \quad (A2-3) \]

By using Euler-Bernoulli beam theory the following expressions are obtained,

\[ f = 2.675 \left[ \sin \lambda - \sinh \lambda - 0.976 \left( \cos \lambda - \cosh \lambda \right) \right] \quad (A2-4) \]

where \[ \lambda = \frac{0.77z}{l} \quad (A2-5) \]

and also,

\[ h = \frac{z}{l} \quad (A2-6) \]
Thus, by substituting (A2-4 and A2-6) into (A2-3) and differentiating and integrating as required it is found that,

\[ B = 0.275 \]  \hspace{1cm} (A2-7)

Returning to expression (2.2-2), this is redefined for an additional degree of freedom (2.3-1),

\[ U(\zeta, t) = f_1(z)u_{01}(t) + f_2(z)u_{02}(t) \]

Therefore (A2-2) becomes,

\[ v_0 = \int_0^1 (1 - z)h \phi_{\theta} \left[ f_1''u_{01} + f_2''u_{02} \right] dz \]  \hspace{1cm} (A2-8)

And,

\[ B_1 = \int_0^1 (1 - z)hf_1''dz \hspace{1cm} ( = B) \]  \hspace{1cm} (A2-9)

\[ B_2 = \int_0^1 (1 - z)hf_2''dz \]  \hspace{1cm} (A2-10)

Where,

\[ f_2 = 0.939 \left[ \sin \rho - \sinh \rho - .905 \left( \cos \rho - \cosh \rho \right) \right] \]  \hspace{1cm} (A2-11)

and \( \rho = \frac{2.81\zeta}{l} \)  \hspace{1cm} (A2-12)

Calculations give,

\[ B_2 = -0.781 \]  \hspace{1cm} (A2-13)

The relationship between \( w_0, u_{01}, u_{02}, v_0 \) in Chapter 4 is stated as, \( (4.3-3) \),

\[ w_0 \propto \frac{1}{2} \int_0^1 f_1''^2 u_{01}^2 dz + \frac{1}{2} \int_0^1 f_2''^2 u_{02}^2 dz \]
Where numerical values are required for the functions,

\[ B_3 = \int_0^1 f_1'^2 \, dz \] \hspace{1cm} (A2-14)

\[ B_4 = \int_0^1 f_2'^2 \, dz \] \hspace{1cm} (A2-15)

By substituting (A2-4 and A2-11) into (A2-14 and A2-15) respectively, and differentiating and integrating as required, the values are obtained,

\[ B_3 = \frac{1.2}{1} \quad ; \quad B_4 = \frac{3.2}{1} \] \hspace{1cm} (A2-16)
Appendix 3: Coefficients of the solutions to the first order perturbation equations of Chapter 3, section 3.2

\[ K_1 = \frac{b_1 w_0 \omega_1^2 c_3}{2[\omega_1^2 - (\omega_1 + \omega_1)^2]} \quad ; \quad K_2 = \frac{b_1 w_0 \omega_1^2 c_3}{2[\omega_1^2 - (\omega_1 - \omega_1)^2]} \]

\[ K_3 = \frac{2b_1 b_1 c_1^2 c_1 \omega_1 \omega_1}{[\omega_1^2 - (\omega_1 + 2\omega_1)^2]} \quad ; \quad K_4 = \frac{2b_1 b_1 c_1^2 c_1 \omega_1 \omega_1}{[\omega_1^2 - (2\omega_1 - \omega_1)^2]} \]

\[ K_5 = \frac{2b_2 b_2 c_2^2 c_2 \omega_2 \omega_1}{[\omega_2^2 - (\omega_2 + 2\omega_1)^2]} \quad ; \quad K_6 = \frac{2b_2 b_2 c_2^2 c_2 \omega_2 \omega_1}{[\omega_2^2 - (\omega_2 - 2\omega_1)^2]} \]

\[ K_7 = \frac{b_2 w_0 \omega_2^2 c_3}{2[\omega_2^2 - (\omega_2 + \omega_1)^2]} \quad ; \quad K_8 = \frac{b_2 w_0 \omega_2^2 c_3}{2[\omega_2^2 - (\omega_2 - \omega_1)^2]} \]

\[ K_9 = \frac{2b_2 b_2 c_2^2 c_2 \omega_2 \omega_1}{[\omega_2^2 - (\omega_2 + 2\omega_1)^2]} \quad ; \quad K_{10} = \frac{2b_2 b_2 c_2^2 c_2 \omega_2 \omega_1}{[\omega_2^2 - (2\omega_1 - \omega_2)^2]} \]

\[ K_{11} = \frac{2b_1 b_1 c_1^2 c_1 \omega_1 \omega_1}{[\omega_1^2 - (\omega_1 + 2\omega_1)^2]} \quad ; \quad K_{12} = \frac{2b_1 b_1 c_1^2 c_1 \omega_1 \omega_1}{[\omega_1^2 - (\omega_1 - 2\omega_1)^2]} \]

\[ K_{13} = \frac{2b_2 b_2 c_2^2 c_2 \omega_2 \omega_1}{[\omega_2^2 - (2\omega_2 + \omega_1)^2]} \quad ; \quad K_{14} = \frac{2b_2 b_2 c_2^2 c_2 \omega_2 \omega_1}{[\omega_2^2 - (2\omega_2 - \omega_1)^2]} \]

\[ K_{15} = \frac{2b_2 b_2 c_2^2 c_2 \omega_1 \omega_1}{[\omega_1^2 - (\omega_1 + \omega_2 + \omega_1)^2]} \]
\[ K_{16} = \frac{2Rb_1\overline{C}_1C_2C_3\omega_{t1}(\omega_{b1} - \omega_{b2})}{[\omega_{t1}^2 - (\omega_{b2} - \omega_{b1} + \omega_{t1})^2]} \]

\[ K_{17} = \frac{2Rb_1\overline{C}_1C_2\omega_{t1}(\omega_{b1} + \omega_{b2})}{[\omega_{t1}^2 - (\omega_{b1} + \omega_{b2} - \omega_{t1})^2]} \]

\[ K_{18} = \frac{2Rb_1\overline{C}_1C_2\omega_{t1}(\omega_{b1} - \omega_{b2})}{[\omega_{t1}^2 - (\omega_{b2} - \omega_{b1} - \omega_{t1})^2]} \]

\[ K_{19} = \frac{2Rb_1\overline{C}_1C_2C_3\omega_{b1}\omega_{t1}}{[\omega_{t1}^2 - (2\omega_{b1} + \omega_{t1})^2]} \]

\[ K_{20} = \frac{2Rb_1\overline{C}_1C_2\omega_{b1}\omega_{t1}}{[\omega_{t1}^2 - (\omega_{t1} - 2\omega_{b1})^2]} \]

\[ K_{21} = \frac{Rb_1\omega_0\overline{\Omega}^2C_1}{2[\omega_{t1}^2 - (\overline{\Omega} + \omega_{b1})^2]} \]

\[ K_{22} = \frac{Rb_1\omega_0\overline{\Omega}^2C_1}{2[\omega_{t1}^2 - (\overline{\Omega} - \omega_{b1})^2]} \]

\[ K_{23} = \frac{Rb_2\omega_0\overline{\Omega}^2C_2}{2[\omega_{t1}^2 - (\overline{\Omega} + \omega_{b2})^2]} \]

\[ K_{24} = \frac{Rb_2\omega_0\overline{\Omega}^2C_2}{2[\omega_{t1}^2 - (\overline{\Omega} - \omega_{b2})^2]} \]
Appendix 4: Required coefficients of the second order perturbation equations of Chapter 3, section 3.2

\[ L_{12} = \frac{R_b \alpha b C_1 W \Omega^2}{4[\Omega_{t1}^2 - (\Omega - \Omega_{b2})^2]} \]

\[ L_{13} = \frac{b_1 f C_1 W \Omega^2}{4[\Omega_{t1}^2 - (\Omega + \Omega_{b1})^2]} - 12D_2 C_1 \omega_{b1} - i2 \int b \omega_{b1} C_1 \]

\[ + \frac{4RB_{21}^2 C_2^2 C_3^2 C_1 \omega_{b1} \omega_{t1}^2}{[\omega_{t1}^2 - (2\omega_{b1} + \omega_{t1})^2]} \]

\[ - \frac{4b_2^2 C_2^2 C_3^2 C_1 \omega_{b1} \omega_{t1}^2}{[\omega_{b1}^2 - (\omega_{b1} + 2\omega_{t1})^2]} \]

\[ - \frac{4RB_{12}^2 b_1 b_2 C_1 C_2 C_3 \omega_{b2} \omega_{t1}^2}{[\omega_{t1}^2 - (\omega_{b1} + \omega_{b2} + \omega_{t1})^2]} \]

\[ + \frac{4RB_{12}^2 b_1 b_2 C_1 C_2 C_3 \omega_{b2} \omega_{t1}^2}{[\omega_{t1}^2 - (\omega_{b1} + \omega_{b2} - \omega_{t1})^2]} \]

\[ - \frac{4B_1 b_2 C_2 C_3 \omega_{b1} \omega_{t1}^2}{[\omega_{b2}^2 - (\omega_{b1} - 2\omega_{t1})^2]} \]

\[ + \frac{\omega_{b1} + 2\omega_{t1}}{[\omega_{b2}^2 - (\omega_{b1} + 2\omega_{t1})^2]} \]
\[ L_{36} = \frac{R_{b} b_{2} c_{1} w_{0} \omega_{b2}^{2}}{4(\omega_{t1}^{2} - (\omega_{t} - \omega_{b1})^{2})} \]

\[ L_{41} = \frac{8R_{b} b_{2} c_{1} c_{2} c_{3} c_{4} \omega_{b1} \omega_{b2}^{3} - i2D_{2} \omega_{b2}}{[\omega_{t1}^{2} - (2\omega_{b2} - \omega_{t1})^{2}]^{2}} \]

\[ -i2 \int_{b} \omega_{b2}^{2} c_{2} + \frac{4R_{b} b_{2} c_{1} c_{2} c_{3} \omega_{b2}^{2} \omega_{t1}(\omega_{b1} - 2\omega_{b2} - \omega_{t1})}{[\omega_{t1}^{2} - (2\omega_{b2} + \omega_{t1})^{2}]^{2}} \]

\[ - 4R_{b} b_{2} c_{1} c_{2} c_{3} \omega_{b1} \omega_{b2}^{2} \frac{\omega_{t1}(\omega_{b2} + 2\omega_{t1})}{[\omega_{b2}^{2} - (\omega_{b2} + 2\omega_{t1})^{2}]^{2}} \]

\[ - 4R_{b} b_{2} c_{1} c_{2} c_{3} \omega_{b1} \omega_{b2}^{2} \frac{\omega_{t1}(\omega_{b1} - \omega_{b2})^{2}}{[\omega_{t1}^{2} - (\omega_{b2} - \omega_{b1} - \omega_{t1})^{2}]^{2}} \]

\[ + 4R_{b} b_{2} c_{1} c_{2} c_{3} \omega_{b1} \omega_{b2}^{2} \frac{\omega_{t1}(\omega_{b1} + \omega_{b2})(\omega_{b1} + \omega_{b2} + 2\omega_{t1})}{[\omega_{t1}^{2} - (\omega_{b1} + \omega_{b2} - \omega_{t1})^{2}]^{2}} \]

\[ + 4R_{b} b_{2} c_{1} c_{2} c_{3} \omega_{b1} \omega_{b2}^{2} \frac{(\omega_{b1} - \omega_{b2})^{2}}{[\omega_{t1}^{2} - (\omega_{b2} - \omega_{b1} + \omega_{t1})^{2}]^{2}} \]

\[ - 4R_{b} b_{2} c_{1} c_{2} c_{3} \omega_{b1} \omega_{b2}^{2} \frac{(\omega_{b1} + \omega_{b2})^{2}}{[\omega_{t1}^{2} - (\omega_{b1} + \omega_{b2} + \omega_{t1})^{2}]^{2}} \]
\[-4R^2_{\omega_2, \omega_1} \left( \frac{(\omega_{b_2} - 2\omega_{t_1})}{[\omega_{b_1} - (\omega_{b_2} - 2\omega_{t_1})^2]} \right)
+ \frac{(\omega_{b_2} + 2\omega_{t_1})}{[\omega_{b_1} - (\omega_{b_2} + 2\omega_{t_1})^2]} \right] \]

\[L_{56} = i2D_2c_3(\omega_{t_1} - i2\int_0^t \omega_{t_1}^{2}c_3 \]
\[+ \frac{4R^2_{\omega_2, \omega_1}c_2^2c_3^2(\omega_{b_2}^2 - (2\omega_{t_1} - \omega_{b_2}^2)^2)}{[\omega_{b_2} - (2\omega_{t_1} - \omega_{b_2}^2)^2]} \]
\[- \frac{4R^2_{\omega_2, \omega_1}c_2^2c_3^2(\omega_{b_2}^2 + 2\omega_{t_1})}{[\omega_{b_2} - (\omega_{b_2} + 2\omega_{t_1})^2]} \]
\[- \frac{4R^2_{\omega_2, \omega_1}c_2^2c_3^2(\omega_{b_2}^2 + 2\omega_{t_1})}{[\omega_{b_2} - (\omega_{b_1} + 2\omega_{t_1})^2]} + \frac{4R^2_{\omega_2, \omega_1}c_2^2c_3^2(\omega_{b_2}^2 + 2\omega_{t_1})}{[\omega_{b_2} - (2\omega_{t_1} - \omega_{b_2})^2]} \]
\[+ \frac{4R^2_{\omega_2, \omega_1}c_2^2c_3^2(\omega_{b_2}^2 + 2\omega_{t_1})}{[\omega_{b_2} - (\omega_{b_2} + 2\omega_{t_1})^2]} + \frac{4R^2_{\omega_2, \omega_1}c_2^2c_3^2(\omega_{b_2}^2 + 2\omega_{t_1})}{[\omega_{b_1} - (\omega_{b_2} + 2\omega_{t_1})^2]} \]
\[- \frac{4R^2_{\omega_2, \omega_1}c_2^2c_3^2(\omega_{b_2}^2 + 2\omega_{t_1})}{[\omega_{b_1} - (\omega_{b_2} + 2\omega_{t_1})^2]} \]
\[ + \frac{4R^2b^2c_1c_1c_3^2 \omega b_1\omega^2 t_1 (2\omega_{t1} - \omega_{b1})}{[\omega_{b1} - (2\omega_{t1} - \omega_{b1})^2]} \]

\[ - \frac{4R^2b^2c_1c_1c_3^2 \omega b_1\omega^2 t_1 (\omega_{b1} + 2\omega_{t1})}{[\omega_{b1} - (\omega_{b1} + 2\omega_{t1})^2]} + \frac{4R^2b^2c_1c_1c_3^2 \omega b_1\omega^2 t_1}{[\omega_{b1} - (2\omega_{t1} - \omega_{b1})^2]} \]

\[ + \frac{4R^2b^2c_1c_1c_3^2 \omega b_1\omega^2 t_1}{[\omega_{b1} - (\omega_{b1} + 2\omega_{t1})^2]} \]

\[ - \frac{4R^2b^2c_2c_2c_3^2 \omega b_1\omega^2 t_1 (2\omega_{b2} + \omega_{t1})}{[\omega_{t1} - (2\omega_{b2} + \omega_{t1})^2]} \]

\[ + \frac{4R^2b^2c_1c_1c_2c_3 \omega b_1\omega^2 t_1 (\omega_{b1} - \omega_{b2}) (\omega_{b2} - \omega_{b1} + \omega_{t1})}{[\omega_{t1} - (\omega_{b2} - \omega_{b1} + \omega_{t1})^2]} \]

\[ - \frac{4R^2b^2c_1c_1c_2c_3 \omega t_1 (\omega_{b1} + \omega_{b2}) (\omega_{b1} + \omega_{b2} + \omega_{t1})}{[\omega_{t1} - (\omega_{b1} + \omega_{b2} + \omega_{t1})^2]} \]

\[ - \frac{4R^2b^2c_1c_1c_3 \omega b_1\omega^2 t_1 (\omega_{t1} - 2\omega_{b1})}{[\omega_{t1} - (\omega_{t1} - 2\omega_{b1})^2]} \]

\[ - \frac{4R^2b^2c_1c_1c_3 \omega b_1\omega^2 t_1 (\omega_{b1} + \omega_{b2}) (\omega_{b1} + \omega_{b2} + \omega_{t1})}{[\omega_{t1} - (\omega_{b1} + \omega_{b2} + \omega_{t1})^2]} \]

\[ - \frac{4R^2b^2c_2c_2c_3 \omega b_1\omega^2 t_1 (2\omega_{b1} + \omega_{t1})}{[\omega_{t1} - (2\omega_{b1} + \omega_{t1})^2]} \]
\[
\frac{Rb_2^2(W_1, \Omega^2) C_3}{4[\omega_{b2}^2 - (\Omega + \omega_t)^2]} + \frac{Rb_1^2(W_o, \Omega^2) C_3}{4[\omega_{b1}^2 - (\Omega + \omega_t^1)^2]}
\]
Appendix 5: Coefficients of the slow time solvability equations of Chapter 3, section 3.2.2

\[
H_1 = \frac{Rb_1 b_2 (W_0 \Omega^2)^2}{8 \omega_{b1} [\omega_{t1}^2 - (\Omega - \omega_{b2})^2]} ; \\
H_2 = \frac{Rb_1 b_2 (W_0 \Omega^2)^2}{8 \omega_{b2} [\omega_{t1}^2 - (\Omega - \omega_{b1})^2]} \\
H_3 = \frac{Rb_1^2 (W_0 \Omega^2)^2}{8 \omega_{b1} [\omega_{t1}^2 - (\Omega + \omega_{b1})^2]} ; \\
H_4 = \frac{Rb_1^2 \omega_{b1} \omega_{t1}}{8 [\omega_{t1}^2 - (2\omega_{b1} + \omega_{t1})^2]} \\
H_5 = \frac{B_1^2 b_1^2 \omega_{t1}^2 (\omega_{b1} + 2\omega_{t1})}{8 [\omega_{b1}^2 - (\omega_{b1} + 2\omega_{t1})^2]} + \frac{B_1^2 b_2^2 \omega_{t1}^2}{8} \left[ \frac{(\omega_{b1} - 2\omega_{t1})}{[\omega_{b2} - (\omega_{b1} - 2\omega_{t1})^2]} \right. \\
+ \left. \frac{(\omega_{b1} + 2\omega_{t1})}{[\omega_{b2}^2 - (\omega_{b1} + 2\omega_{t1})^2]} \right] \\
H_6 = \frac{Rb_1^2 b_2 \omega_{b2} \omega_{t1} (\omega_{b1} + \omega_{b2})^2}{8 \omega_{b1}} \left[ \frac{1}{[\omega_{t1}^2 - (\omega_{b1} + \omega_{b2} + \omega_{t1})^2]} \\
- \frac{1}{[\omega_{t1}^2 - (\omega_{b1} + \omega_{b2} - \omega_{t1})^2]} \right] \\
H_7 = \frac{B_2^2 b_2^2 \omega_{t1}^2 (\omega_{b2} + 2\omega_{t1})}{8 [\omega_{b2}^2 - (\omega_{b2} + 2\omega_{t1})^2]} + \frac{B_2^2 b_2^2 \omega_{t1}^2}{8} \left[ \frac{(\omega_{b2} - 2\omega_{t1})}{[\omega_{b1}^2 - (\omega_{b2} - 2\omega_{t1})^2]} \right. \\
+ \left. \frac{(\omega_{b2} + 2\omega_{t1})}{[\omega_{b1}^2 - (\omega_{b2} + 2\omega_{t1})^2]} \right] \\
J_1 = \frac{[\Lambda_1 - \Lambda_2 + \Lambda_4 + \Lambda_5 + \Lambda_6 - \Lambda_7]}{\omega_{t1}}
\[ J_2 = \frac{[\Lambda_8 - \Lambda_3 - \Lambda_q + \Lambda_{10} + \Lambda_{11} + \Lambda_{12}]}{\omega_{t1}} \]

\[ J_3 = \frac{[\Lambda_{14} - \Lambda_{17} - \Lambda_{18} - \Lambda_{20}]}{\omega_{t1}} ; \quad J_4 = \frac{[\Lambda_{19} + \Lambda_{21}]}{\omega_{t1}} \]

\[ J_5 = \frac{\Lambda_{13}}{\omega_{t1}} ; \quad J_6 = \frac{[\Lambda_{15} + \Lambda_{16}]}{\omega_{t1}} \]

\[ J_7 = \frac{[\Lambda_{22} + \Lambda_{23}]}{\omega_{b2}} ; \quad J_8 = \frac{[\Lambda_{24} - \Lambda_{25} - \Lambda_{26} + \Lambda_{27}]}{\omega_{b2}} \]

Where the \( \Lambda \) are:

\[ \Lambda_1 = \frac{R B_{z2}^2 b_2^2 \omega_{t1}^2 (2 \omega_{t1} - \omega_{b2})}{8 [\omega_{b2}^2 - (2 \omega_{t1} - \omega_{b2})^2]} \]

\[ \Lambda_2 = \frac{R B_{z2}^2 b_2^2 \omega_{b2}^2 \omega_{t1}^2 (2 \omega_{t1} + \omega_{b2})}{8 [\omega_{b2}^2 - (2 \omega_{t1} + \omega_{b2})^2]} \]

\[ \Lambda_3 = \frac{R B_{z1} b_2^2 \omega_{t1} \omega_{b1}^2 (\omega_{b1} + 2 \omega_{t1})}{8 [\omega_{b2}^2 - (\omega_{b1} + 2 \omega_{t1})^2]} \]

\[ \Lambda_4 = \frac{R B_{z2}^2 b_2^2 \omega_{b2}^2 \omega_{t1}^2}{8 [\omega_{b2}^2 - (2 \omega_{t1} - \omega_{b2})^2]} \]

\[ \Lambda_5 = \frac{R B_{z2}^2 b_2^2 \omega_{t1}^2}{8 [\omega_{b2}^2 - (\omega_{b2} + 2 \omega_{t1})^2]} \]
\[ \Lambda_6 = \frac{R^2b_2^2 \omega^2 \omega_{b1}}{8 \left( \omega_{b1}^2 - (\omega_{b2} + 2\omega_{t1})^2 \right)} \]
\[ \Lambda_7 = \frac{R^2b_2^2 \omega_2 \omega_{t1}(\omega_{b2} + 2\omega_{t1})}{8 \left( \omega_{b1}^2 - (\omega_{b2} + 2\omega_{t1})^2 \right)} \]
\[ \Lambda_8 = \frac{R^2b_2^2 \omega_{b1} \omega_2 \omega_{t1}(2\omega_{t1} - \omega_{b1})}{8 \left( \omega_{b1}^2 - (2\omega_{t1} - \omega_{b1})^2 \right)} \]
\[ \Lambda_9 = \frac{R^2b_2^2 \omega_{b1} \omega_{t1}(\omega_{b2} + 2\omega_{t1})}{8 \left( \omega_{b1}^2 - (\omega_{b2} + 2\omega_{t1})^2 \right)} \]
\[ \Lambda_{10} = \frac{R^2b_2^2 \omega_2 \omega_{b1} \omega_{t1}}{8 \left( \omega_{b1}^2 - (2\omega_{t1} - \omega_{b1})^2 \right)} \]
\[ \Lambda_{11} = \frac{R^2b_2^2 \omega_2 \omega_{b1} \omega_{b2}}{8 \left( \omega_{b2}^2 - (\omega_{b1} + 2\omega_{t1})^2 \right)} \]
\[ \Lambda_{12} = \frac{R^2b_2^2 \omega_2 \omega_{b1} \omega_{t1}}{8 \left( \omega_{b1}^2 - (\omega_{b1} + 2\omega_{t1})^2 \right)} \]
\[ \Lambda_{13} = \frac{R^2b_2^2 \omega_2 \omega_{b1} \omega_{t1}(2\omega_{b2} + \omega_{t1})}{8 \left( \omega_{t1}^2 - (2\omega_{b1} + \omega_{t1})^2 \right)} \]
\[ \Lambda_{14} = \frac{R^2b_2^2 \omega_2 \omega_{b1} \omega_{t1}(\omega_{b1} - \omega_{b2})(\omega_{b2} - \omega_{b1} + \omega_{t1})}{8 \left( \omega_{t1}^2 - (\omega_{b2} - \omega_{b1} + \omega_{t1})^2 \right)} \]
\[ \Lambda_{15} = \frac{Rb_2^2 (\omega_0 \Omega)^2}{8 \left( \omega_{b2}^2 - (\omega_0 + \omega_{t1})^2 \right)} \]
\[ \Lambda_{16} = \frac{Rb_2^2 (\omega_0 \Omega)^2}{8 \left( \omega_{b1}^2 - (\omega_0 + \omega_{t1})^2 \right)} \]
\[ \Lambda_{17} = \frac{R^2b_2^2 \omega_2 \omega_{b1} \omega_{t1}(\omega_{b1} + \omega_{b2} + \omega_{t1})(\omega_{b1} + \omega_{b2})}{8 \left( \omega_{t1}^2 - (\omega_{b1} + \omega_{b2} + \omega_{t1})^2 \right)} \]
\[ \Lambda_{18} = \frac{R^2b_2^2 \omega_2 \omega_{b1} \omega_{t1}(\omega_{b1} - \omega_{b2})(\omega_{b2} - \omega_{b1} + \omega_{t1})}{8 \left( \omega_{t1}^2 - (\omega_{b2} - \omega_{b1} + \omega_{t1})^2 \right)} \]
\[
\Lambda_{19} = \frac{R^2 B_{b1}^2 \omega_{b1}^2 \omega_{t1} (\omega_{t1} - 2 \omega_{b1})}{8 [\omega_{t1}^2 - (\omega_{t1} - 2 \omega_{b1})^2]}
\]

\[
\Lambda_{20} = \frac{R^2 B_{b1}^2 \omega_{b1}^2 \omega_{t1} (\omega_{b1} + \omega_{b2}) (\omega_{b1} + \omega_{b2} + \omega_{t1})}{8 [\omega_{t1}^2 - (\omega_{b1} + \omega_{b2} + \omega_{t1})^2]}
\]

\[
\Lambda_{21} = \frac{R^2 B_{b1}^2 \omega_{b1}^2 \omega_{t1} (2 \omega_{b1} + \omega_{t1})}{8 [\omega_{t1}^2 - (2 \omega_{b1} + \omega_{t1})^2]}
\]

\[
\Lambda_{22} = \frac{R^2 B_{b2}^2 \omega_{b2}^2 \omega_{t1} (\omega_{b1} - 2 \omega_{b2} - \omega_{t1})}{8 [\omega_{t1}^2 - (2 \omega_{b2} + \omega_{t1})^2]}
\]

\[
\Lambda_{23} = \frac{R^2 B_{b2} \omega_{b2}^3 \omega_{t1}}{4 [\omega_{t1}^2 - (2 \omega_{b2} - \omega_{t1})^2]}
\]

\[
\Lambda_{24} = \frac{R^2 B_{b1}^2 \omega_{b1} \omega_{t1} (\omega_{b1} - \omega_{b2})^2}{8 [\omega_{t1}^2 - (\omega_{b2} - \omega_{b1} - \omega_{t1})^2]}
\]

\[
\Lambda_{25} = \frac{R^2 B_{b1}^2 \omega_{b1} \omega_{t1} (\omega_{b1} + \omega_{b2}) (\omega_{b1} + \omega_{b2} + \omega_{t1})}{8 [\omega_{t1}^2 - (\omega_{b1} + \omega_{b2} + \omega_{t1})^2]}
\]

\[
\Lambda_{26} = \frac{R^2 B_{b2}^2 \omega_{b1} \omega_{t1} (\omega_{b1} - \omega_{b2})^2}{8 [\omega_{t1}^2 - (\omega_{b2} - \omega_{b1} + \omega_{t1})^2]}
\]

\[
\Lambda_{27} = \frac{R^2 B_{b2}^2 \omega_{b1} \omega_{t1} (\omega_{b1} + \omega_{b2})^2}{8 [\omega_{t1}^2 - (\omega_{b1} + \omega_{b2} + \omega_{t1})^2]}
\]
Appendix 6: Coefficients of solvability equations, Chapter 3, section 3.2.2

\[ \Gamma_1 = \varepsilon_1^{2H_1} \omega_{b1} \quad ; \quad \Gamma_2 = \varepsilon_2^{2H_2} \omega_{b1} \]

\[ \Gamma_3 = \varepsilon_3^{2H_2} \omega_{b2} \]
Appendix 7: Coefficients of the biquadratic expression for the transition curve of resonance (3.2.1-1), Chapter 3 section 3.2.3

\[ V_1 = \left[ \frac{64q_1\Gamma_1\omega_{b1}^2}{R^2B_1^2(W_o\Omega)^2} \right] + \frac{16p_2q_1\omega_{b1}}{R(W_o\Omega)^2} + q_2 - \frac{B_2r_3(\omega_{b1} + \omega_{b2})^2}{\omega_{b2}} \]

\[ + \frac{64\xi_b^2\omega_{b1}^2(\omega_{b1} + \omega_{b2})^2(r_1q_2 + r_3q_4)}{R^2B_1^2(W_o\Omega)^2} \]

\[ V_2 = \left[ \frac{16n_1q_1\omega_{b1}}{R(W_o\Omega)^2} \right] + \frac{64\xi_b^2\omega_{b1}^2r_2q_1(\omega_{b1} + \omega_{b2})^2}{R^2B_1^2(W_o\Omega)^2} - \frac{B_2r_2(\omega_{b1} + \omega_{b2})^2}{\omega_{b2}} \]

\[ V_3 = \left[ \frac{4q_1^2}{R^2B_1^2(W_o\Omega)^2} \right] + \frac{64\xi_b^2\omega_{b1}^2q_1(\omega_{b1} + \omega_{b2})^2}{R^2B_1^2(W_o\Omega)^2} - \frac{B_2r_4(\omega_{b1} + \omega_{b2})^2}{\omega_{b2}} \]

Where, \( p_1 = [\omega_{t1}^2 - \frac{9}{4}\omega_{b1}^2 - \frac{1}{4}\omega_{b2}^2 - \frac{3}{2}\omega_{b1}\omega_{b2}] \)

\( p_2 = [-3\omega_{b1} + \omega_{b2}] \)

\( q_1 = [\omega_{t1}^4 - \frac{1}{2}\omega_{t1}\omega_{b1}^2 - \frac{1}{2}\omega_{t1}\omega_{b2}^2 + \omega_{b1}\omega_{b2}\omega_{t1}] + \frac{9}{16}\omega_{b1}\omega_{b2}^2 + \frac{1}{16}\omega_{b1}^4 - \frac{1}{4}\omega_{b1}^3\omega_{b2} + \frac{1}{16}\omega_{b2}^4 \]

\( - \frac{1}{4}\omega_{b1}\omega_{b2}^3 \]

\( q_2 = [\omega_{b1}\omega_{b2} - 2\omega_{t1}^2 - \frac{1}{2}\omega_{b1}^2 - \frac{1}{2}\omega_{b2}^2] \)
\[
\begin{align*}
\Gamma_1 &= \left[ \omega_{t1}^4 - \frac{9}{2} \omega_{b1}^2 \omega_{t1}^2 - \frac{1}{2} \omega_{b2}^2 \omega_{t1}^2 - 3 \omega_{b1} \omega_{b2} \omega_{t1}^2 \\
&\quad + \frac{81}{16} \omega_{b1}^4 + \frac{27}{8} \omega_{b1}^2 \omega_{b2}^2 + \frac{27}{4} \omega_{b1} \omega_{b2}^3 + \frac{3}{4} \omega_{b1} \omega_{b2}^4 \\
&\quad + \frac{1}{16} \omega_{b2}^4 \right] \\
\Gamma_2 &= \left[ \frac{27}{2} \omega_{b1}^2 + \frac{27}{2} \omega_{b1} \omega_{b2}^2 + \frac{9}{2} \omega_{b1} \omega_{b2}^2 + \frac{1}{2} \omega_{b1}^2 - 6 \omega_{b1} \omega_{t1}^2 - 2 \omega_{b2} \omega_{t1}^2 \right] \\
\Gamma_3 &= \left[ \frac{27}{2} \omega_{b1}^2 + \frac{3}{2} \omega_{b2}^2 + 9 \omega_{b1} \omega_{b2} - 2 \omega_{t1}^2 \right]
\end{align*}
\]

Resonance (3.2.1-1) does not contain \( \omega_{t1} \), as it deals exclusively with the coupling between the first two bending modes. However, \( \omega_{t1} \) occurs explicitly in the above substitutions because of its presence in the coefficients \( H_1, H_2, H_3 \) which are generated via the coefficients of the intermediary first order solutions, \( K_{24}, K_{22}, K_{21} \) respectively. The fractional terms in \( \rho_1, \rho_2, \rho_3 \) result from the substitution of \( \frac{1}{2} [\omega_{b1} + \omega_{b2}] + \epsilon^2 \rho_1 \) for \( \Omega \), as required, in the denominators of \( H_1, H_2, H_3 \) and hence \( \Gamma_1, \Gamma_2, \Gamma_3 \).

**Numerical calculation of \( \epsilon^2 \rho_1 \) for varying \( W_0 \Omega^2 \)**

This is performed by a computer program into which the necessary system constants, natural frequencies and incremental values of excitation acceleration are fed. The constants \( \rho_1, \rho_2, \rho_3, \rho_1, \rho_2, \rho_3 \) are evaluated and then coefficients \( V_1, V_2, V_3 \). The biquadratic expression is solved in the usual way by means of the well known quadratic solution formula. The solution(s) for \( \epsilon^2 \rho_1 \) are presented in the output in the form of \( \Omega \) vs \( W_0 \Omega^2 \) thus enabling the transition curve to be directly plotted.
From expression (4.3-19) in Chapter 4 the excitation parameter is defined as,

\[ F_2 = \frac{W_0 \Omega^2 \left( \sum_{k=1}^{n} r_{k2}^2 m_k \right)}{(M + \frac{r_m^2}{12} m_o)} \]

Where \( W_0 \Omega^2 \) is the peak acceleration in MS\(^{-2} \), experienced by the support, and \( \sum_{k=1}^{n} r_{k2}^2 m_k \) is the sum of the product of the normalised eigenvector and the mass of \( n \) successive discretised points along the length of the primary beam, \( k \) being the chosen station locations along this length. The PAFEC finite element beam package is used to calculate a numerical value for this product sum,

\[ \sum_{k=1}^{n} r_{k2}^2 m_k = 0.75 \]

From experimental observations, \( W_0 \Omega^2 = 3.0 \) MS\(^{-2} \) is seen to produce interesting nonlinear interactions with acceptable system responses. A higher excitation level of \( W_0 \Omega^2 = 10.0 \) MS\(^{-2} \) is also used and its effects investigated.
Listing of Computer Program for the numerical integration of
the 3-D.O.F. Equations (3.2.2-11 to 3.2.2-16, Chapter 3)

DIMENSION X(30),Y(30),Z(30)
COMMON/ONE/CK(10)
COMMON/TWO/CJ(10)
READ 100,N,NOVAR
READ 110,(CK(I),I=1,9)
READ 120,(CJ(L),L=1,9)
100 FORMAT(2I6)
110 FORMAT(9E13.6)
120 FORMAT(9E13.6)
   DO 10 I=1,NOVAR
10   Z(I)=0
    X(1)=0.000001
    X(2)=0.000001
    X(3)=0.000001
    X(4)=0.000001
    X(5)=0.000001
    X(6)=0.000001
    X(7)=0.000001
    STEP=720.0/N
   DO 20 J=1,N
      CALL GILL(X,Y,Z,STEP,NOVAR)
      WRITE(14,200) X(1)
      WRITE(15,200) X(2)
      WRITE(16,200) X(3)
      WRITE(17,200) X(4)
200   FORMAT(E12.4)
20   CONTINUE
      ENDFILE 14
      REWIND 14
      ENDFILE 15
      REWIND 15
      ENDFILE 16
      REWIND 16
      ENDFILE 17
      REWIND 17
      CALL EZGRAF ("A1_A1",5)
      STOP
      SUBROUTINE GILL(X,Y,Z,STEP,NGIL)
      DIMENSION X(NGIL),Y(NGIL),Z(NGIL),TZ(4),SZ(4),RZ(4)
      DATA TZ/0.5,0.292893,1.707107,0.166666/,SZ/2.0,1.0,1.0,2.0/
      C, RZ/0.5,0.292893,1.707107,0.5/
      DO 1 ITRN=1,4
         CALL DIFEQN(X,Y,NGIL)
         DO 1 J=1,NGIL
            HK=(Y(J)-SZ(ITRN)*Z(J))*TZ(ITRN)
            Z(J)=Z(J)+3*HK-RZ(ITRN)*Y(J)
            X(J)=X(J)+HK*STEP
1   CONTINUE
      RETURN
      END
      SUBROUTINE DIFEQN(X,Y,NGIL)
      DIMENSION X(NGIL),Y(NGIL)
COMMON/ONE/CK(10)
COMMON/TWO/CJ(10)
G=X(2)/X(3)
H=X(3)/X(2)
Y(1)=1
Y(2)=CK(1)*X(3)*SIN(-X(5)-X(6))-CK(2)*X(2)
Y(3)=CK(3)*X(2)*SIN(-X(5)-X(6))-CK(4)*X(3)
Y(4)=-CJ(7)*X(4)
Y(5)=CK(7)*(X(4)**4)+CK(8)*(X(4)**2)*(X(3)**2)-CK(6)*(X(4)**2)
C*(X(2)**2)-CK(5)-CK(1)*H*COS(-X(5)-X(6))
Y(6)=CK(9)*(X(4)**2)-CJ(8)*(X(4)**2)*(X(3)**2)+CJ(9)*(X(4)**2)
C*(X(2)**2)-CK(3)*G*COS(-X(5)-X(6))
Y(7)=CJ(4)*(X(2)**4)-CJ(1)*(X(4)**2)*(X(3)**2)-CJ(2)*
C*(X(4)**2)*(X(2)**2)-CJ(3)*(X(2)**2)*(X(3)**2)+CJ(5)*(X(3)**2)
C4)-CJ(6)
RETURN
END
DOUBLE PRECISION TOL, X, XEND, P, ES, EE1, EE2, REF, L, FB1, FB2 
C, FT1, PF2, OHM, XIB, XIT, PXI, MO, M2, R12, R22, B1, B2, B3, B4, R 
C, SUM, DIFF, MU, W(7, 25), Y(7) 
INTEGER I, IFAIL, IW, J, N, NOUT 
COMMON/THREE/ES 

ES=-17.5D0 
DO 20 J=1, 140 
ES=ES+0.25D0 

COMMON/ONE/CK(10) 
COMMON/TWO/CJ(5) 
COMMON/FOUR/EE1 
COMMON/FIVE/EE2 
EXTERNAL FCN 
DATA NOUT/6/ 
N=7 
TOL=0.001D0 
X=0.0D0 
XEND=1.0D1 
IW=25 

P=1.0D1 
EE1=0.0D0 
EE2=0.0D0 
FB1=7.54D0 
FB2=1.2380D2 
FT1=5.8130D1 
PF2=6.5670D1 
L=1.200D2 
OHM=PF2*ES 
EPS=1.0D-1 
XIB=1.3D-3 
XIT=3.0D-4 
PXI=2.8D-2 
MO=8.10D-2 
M2=1.14D0 
R12=6.5D-1 
R22=-1.0D0 
B1=2.75 D-1 
B2=-7.81D-1 
B3=(1.2D3)/L 
B4=(3.2D3)/L 
R=-5.7D3 
L=L/1.0D3 
PF2=PF2*6.284D0 
OHM=OHM*6.284D0 
FB1=FB1*6.284D0 
FB2=FB2*6.284D0 
FT1=FT1*6.284D0 
EE1=EE1*6.284D0 
EE2=EE2*6.284D0 
SUM=(FB1+FT1)**2 
DIFF=(FB2-FT1)**2 
MU=(MO/(M2+(R12**2)*MO))
REF=1.0D0/EPS

CK(1)=(P*MU/(2.0D0*PF2*MO*REF))
CK(2)=PXI*PF2
CK(3)=(R22*L*B1*SUM*MU*REF)/(4.0D0*PF2)
CK(4)=(R22*L*B2*DIFF*MU*REF)/(4.0D0*PF2)
CK(5)=XIB*FB1
CK(6)=-(R22*B1*L*(PF2**2)*REF)/(4.0D0*FB1)
CK(7)=XIB*FB2
CK(8)=(R22*B2*L*(PF2**2)*REF)/(4.0D0*FB2)
CK(9)=XIT*FT1
CJ(1)=(R22*B1*L*R*(PF2**2)*REF)/(4.0D0*FT1)
CJ(2)=(R22*B2*L*R*(PF2**2)*REF)/(4.0D0*FT1)
CJ(3)=-(R22**2)*L*B3*(PF2**2)*(REF**2))/(4*FB1)
CJ(4)=-(R22**2)*L*B4*(PF2**2)*(REF**2))/(4*FB2)

Y(1)=.001D0
Y(2)=.001D0
Y(3)=.001D0
Y(4)=.001D0
Y(5)=.001D0
Y(6)=.001D0
Y(7)=.001D0
IFAIL=0

CALL DO2EAF (X,XEND,N,Y,TOL,FCN,W,IW,IFAIL)
WRITE(30,200) ES
WRITE(22,200) Y(1)
WRITE(23,200) Y(2)
WRITE(24,200) Y(3)
WRITE(25,200) Y(4)
IF(TOL.LT.0.0D0) WRITE(NOUT,99994)
IF(IFAIL.GT.0.0D0) WRITE(NOUT,99996) IFAIL
200 FORMAT(1D15.7) 99996 FORMAT(8H,IFAIL=,I1) 99994
FORMAT(24H RANGE TOO SHORT FOR TOL)
20 CONTINUE
ENDFILE 30
REWIND 30
ENDFILE 22
REWIND 22
ENDFILE 23
REWIND 23
ENDFILE 24
REWIND 24
ENDFILE 25
REWIND 25
CALL EZGRAF ('NPB1',4)
CALL EZGRAF ('PLAN',4)
CALL EZGRAF ('NPB2',4)
CALL EZGRAF ('NPT1',4)
STOP
END

SUBROUTINE FCN(T,Y,F)
DOUBLE PRECISION T,F(7),Y(7),DSIN,DCOS,X
COMMON/ONE/CK(10)
COMMON/TWO/CJ(5)
COMMON/THREE/ES
COMMON/FOUR/EE1
COMMON/FIVE/EE2
F(1) = -CK(1) * SIN(Y(7)) - CK(2) * Y(1) + CK(3) * Y(2) * Y(4) * SIN(Y(5))
C + CK(4) * Y(3) * Y(4) * SIN(Y(6))
F(2) = -CK(6) * Y(1) * Y(4) * SIN(Y(5)) - CK(5) * Y(2)
F(3) = -CK(8) * Y(1) * Y(4) * SIN(Y(6)) - CK(7) * Y(3)
F(4) = -CJ(1) * Y(2) * Y(1) * SIN(Y(5))
C - CK(9) * Y(4) + CJ(2) * Y(3) * Y(1) * SIN(Y(6))
F(5) = EE1 - ((CJ(1) * Y(2) * Y(1) * COS(Y(5))) / Y(4))
C - ((CJ(2) * Y(3) * Y(1) * COS(Y(6))) / Y(4))
C - ((CK(1) * COS(Y(7))) / Y(1))
C + ((CK(3) * Y(2) * Y(4) * COS(Y(5))) / Y(1)) + CJ(3) * (((Y(1)) ** 2)
C + ((CK(4) * Y(3) * Y(4) * COS(Y(6))) / Y(1))
C - ((CJ(6) * Y(1) * Y(4) * COS(Y(5))) / Y(2))
F(6) = EE2 + ((CJ(1) * Y(2) * Y(1) * COS(Y(5))) / Y(4))
C + ((CJ(2) * Y(3) * Y(1) * COS(Y(6))) / Y(4))
C - ((CJ(1) * COS(Y(7))) / Y(1))
C + ((CK(3) * Y(2) * Y(4) * COS(Y(5))) / Y(1))
C + ((CK(4) * Y(3) * Y(4) * COS(Y(6))) / Y(1))
C + CJ(4) * (((Y(1)) ** 2) - (((CK(8) * Y(1) * Y(4) * COS(Y(6))) / Y(3))
F(7) = ES + ((CK(1) * COS(Y(7))) / Y(1))
C - ((CK(3) * Y(2) * Y(4) * COS(Y(5))) / Y(1))
C - ((CK(4) * Y(3) * Y(4) * COS(Y(6))) / Y(1))
RETURN
END
Appendix 9: Published work; paper entitled

"Forced vibration of a beam system with autoparametric coupling effects"

By J.W. Roberts and M.P. Cartmell
Forced vibration of a beam system with autoparametric coupling effects

by J. W. Roberts and M. P. Cartmell, Department of Mechanical Engineering, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JL.

Small non-linear interactions of the type referred to as autoparametric may have a considerable effect on the pattern of forced vibration response of certain categories of structures exhibiting low damping characteristics. Under conditions of internal resonance these may bring about a complete neutralisation of the normal forced vibration resonance curve with a transfer of energy to an indirectly excited mode. The paper examines non-linear interactions between a pair of beam modes in forced vibration of a simple structure. Steady state solutions are obtained for the non-linear system equations. Measurements of vibratory acceleration and strain on a laboratory model are shown to give good confirmation of the analytical results and show clearly the substantial effects of internal resonance.

Key words: Forced vibrations, non-linear beam systems, autoparametric coupling.

Nomenclature

\( A(T_i), B(T_j) \) Complex functions of 'slow' time.
\( a, b \) Amplitudes of polar forms of \( A, B \).
\( \varphi \) Scaling factor.
\( \zeta_b \) Coupling point interactive moment.
\( D_o, D_i \) Differential Operators \( D_i = \frac{\partial}{\partial T_i} \).
\( D \) Viscous Damping Matrix.
\( F_0 \) In-plane excitation parameter.
\( G(x) \) Displacement Interpolation Function.
\( K, K \) Stiffness matrices.
\( M, m \) Generalised Mass. Mass matrices.
\( P \) Force vector.
\( p, q \) Vectors of generalised co-ordinates.
\( \rho_1, \rho_2 \) Generalised co-ordinates.
\( r \) Eigenvector.
\( r^{(j)} \) Eigenvector components.
\( T_0 \) 'Fast' time scale. \( T_p = \frac{t}{T_0} \).
\( T_i \) 'Slow' time scale. \( T_i = \frac{t}{T_i} \).
\( \omega(x, t) \) Axial displacement function.
\( \omega_0 \) Axial displacement co-ordinate. \( \omega_0 = \omega(x, t) \).
\( V \) Non-dimensional planar response amplitude.
\( W \) Non-dimensional non-planar response amplitude.
\( X, Y \) Non-dimensional responses.
\( X_0, Y_0 \) Principal parts of series solution.
\( X_1, Y_1 \) First perturbations in series solution.
\( x, x_0 \) Axial position co-ordinates.
\( x_1 \) Equivalent static deflection.
\( x(x, t) \) Transverse displacement of secondary beam.
\( z \) Interactive force at coupling point.
\( \alpha, \beta \) Phase angles of polar forms of \( A \) and \( B \).
\( \gamma \) Coefficient of centripetal acceleration term.
\( \delta_i \) Modal viscous damping coefficient.
\( \epsilon \) Small coupling parameter.
\( \zeta \) Excitation parameter.
\( \zeta_1 \) Axial position co-ordinate.
\( \theta \) Slope angle of secondary beam.
\( \mu \) Effective mass ratio.
\( \xi_1, \xi_2 \) Modal co-ordinates.
\( \rho_1, \rho_2 \) Detuning parameters.
\( \phi \) Coupling point rotation.
\( \psi \) Axial displacement parameter. See equation (8).
\( \Omega \) Excitation frequency (rads/sec).
\( \omega_1, \omega_2 \) Natural frequencies of undamped free vibration.

Introduction

In many structural and machine vibration problems specific contributions to the system kinetic energy arising from the geometry and deformation of the system lead to small quadratic and higher order non-linear coupling terms in the system equations of motion. Such terms are usually ignored in comparison with the dominant linear stiffness, damping and inertia terms. When an additional factor of internal resonance is present i.e. critical relationships between values of the natural frequencies of the system are realised, these terms even when small, may have significant effects on the system forced vibration response, such that gross departures from the normal patterns of linear forced vibration may be observed. It is possible for the linear resonance peak of a directly excited mode to be absorbed, with an accompanying steady vibration in one or more indirectly excited modes at frequencies remote from the external excitation frequency. Additionally the responses can be highly non-linear with respect to excitation amplitude and will show a complex pattern of amplitude 'jumps' thresholds and hysteresis effects which depend on direction of excitation frequency changes. Regions of unsteady 'beating' response are also observed.

Quadratic coupling terms in particular are frequently referred to as 'autoparametric' because the form of these terms represents a situation where response in one mode may appear as a parametric excitation term in another mode and the response analysis has much in common with the parametric excitation case. In general the subject is one of some mathematical complexity and has developed rapidly in recent years. A recent concise and illustrative survey paper has been given by Barr and the series of review articles contains a detailed

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bibliography. Coupled vibratory responses which involve up to four modes responding simultaneously at their individual resonant frequencies to a single frequency excitation have been reported recently by one of the authors.7 Such patterns of vibratory response are likely to occur in many structural configurations under conditions of low damping and high levels of force or seismic excitation and it is of some value to the structural engineer to be aware of the consequences of internal resonance, and of the value of avoiding it.

The present paper describes the effects of autoparametric interaction between two isolated modes of bending vibration, the main purpose being to illustrate the theoretical and experimental responses obtained for a single internal resonance condition. It extends the work reported by Haxton and Barr8 on autoparametric effects in a discrete two degree of freedom system. A more general system of distributed mass and stiffness is considered and reduced to a non-linear two degree of freedom model in terms of generalised parameters. Steady-state solutions are obtained using the Multiple Scales Perturbation scheme.9 Measurements of vibratory accelerations and strains on a laboratory model give very good confirmation of the theoretical results and show the substantial effects of internal resonance.

System equations of motion

The system consists of a pair of beams coupled together in an L configuration as shown in Fig. 1. For the purposes of the present paper, consideration is given only to interactive response of pairs of bending modes of vibration due to transverse response of the primary beam AB causing predominantly axial stimulus of the coupled beam BD at the coupling point. More complex forms of coupled motion involving torsional and bending modes are possible with this configuration, and are the subject of Ref. 10. It is assumed here that the excitation is at a single frequency, and acts transversely on the primary beam AB. The coupled beam element BD is presumed to have a high flexural stiffness ratio as indicated in Fig. 1 so that bending deformation of BD in the plane ABD may be neglected over a range of low frequencies. The following section summarises the derivation of the system equations of motion. This is discussed fully in Ref. 11.

Figure 1 indicates an adequate lumped parameter model of the primary beam plane motion in terms of an arbitrary number of generalised co-ordinates $p_i$, $i = 1, 2, \ldots, n$. In particular the rotation $\phi$ and displacement $z$ at the coupling point B are designated $p_{\phi}$ and $p_z$ respectively. Motion of the primary beam is considered to be in the linear range with system equation:

$$M\ddot{p} + D\dot{p} + Kp = P(t)$$   (1)

where $M$, $D$, $K$ are mass, viscous damping and stiffness matrices. Support conditions are embodied in $K$. $P$ is a vector of nodal dynamic forces and moments, where for $j = 1$ to $n - 2$:

$$P_j = P_{j0} \cos \Omega t$$   (2)

representing an external monofrequent excitation, and:

$$P_{p_{\phi}} = C_{\phi}$$
$$P_{p_z} = Z_n$$   (3)

where $C_{\phi}$ and $Z_n$ are the interaction moment and transverse force at the coupling point.

The system end elevation Fig. 2(b) shows the assumed transverse deformation of the coupled beam BD, denoted $y(x, t)$. A typical particle at $E$ on the undeformed neutral axis, deflects to $E'$ where the deformation includes a small axial component as a consequence of inextensibility of the neutral axis. The axial displacement is denoted $u(x, t)$ where to second order:

$$u(x, t) = \frac{x}{\phi} \frac{\partial^2 y}{\partial \xi^2} dx$$   (4)

Clearly $z(t)$ represents an imposed axial motion at the coupling point with $z(t) = p_z$. Figure 2(a) represents a general lumped parameter model of the coupled beam motion in terms of a set of generalised co-ordinates $q_j$, $j = 1, \ldots, N$. Figure 2(b) indicates an alternative terminology for the same set of generalised co-ordinates. $x_k$ represents the initial axial position of the $k$th discretised inertia element of the beam, with $y_k$, $\theta_k$ and $u_k$ its component deformations. It will be seen to be convenient to make use of both representations to simplify the expression of summations. Figure 2(c) shows the system forces and moments, and inertia forces.

A general representation of the system deformation is:

$$y(x, t) = \sum_{i=1}^{N} G_i(x)q_i(t)$$   (5)

where $G_i(x)$ etc. are assumed displacement interpolation functions, each giving beam displacement for a unit displacement in a single generalised co-ordinate; for example, static deflection functions could be used. From (4)

$$u_k = \frac{1}{\phi} \left( \frac{\partial^2 y}{\partial \xi^2} dx \right)$$
$$= \frac{1}{\phi} \sum_{j=1}^{N} q_j \psi_{jk}$$   (6)

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where
\[ \psi_{ijk} = \frac{\partial^2 G_i}{\partial x^i} \frac{\partial G_j}{\partial x^j} \frac{\partial G_k}{\partial x^k} \]

Hence
\[ \ddot{u}_k = \frac{1}{\sum_{i} m_i} \sum_{i} (q_i \dot{q}_j + \dot{q}_i q_j) \psi_{ijk} \]
\[ \ddot{u}_k = \frac{1}{\sum_{i} m_i} \sum_{i} (q_i \ddot{q}_j + 2\dot{q}_i \dot{q}_j) \psi_{ijk} \]

The combination of coupling point velocity \( \dot{z} \) and axial contraction velocity \( u_k \) leads to an addition to the system kinetic energy given by \[ \sum_{i} \frac{1}{2} m_i (\dot{z} + 2 \dot{u}_k + u_k^2) \] where the sum is over the set of discretised masses of the beam BD. Referring to (9) it is seen that the term in \( \ddot{u}_k \) will be effectively of cubic order in the system generalised co-ordinates and will lead to a quadratic term in the system equations. This will be retained. Applying Lagrange's equations to the system energy functions leads to system equations for the beam BD, neglecting cubic and higher order terms:

\[ m_0 + d_0 + k_0 - \frac{1}{2} \sum_{i} m_i \psi_{ijk} q = 0 \] (11)

where \( m, d, \) and \( k \) are appropriate system matrices. Applying D'Alembert's principle to the system of Fig. 2 one obtains an expression for the interactive force:

\[ Z_n = \sum_{i} m_i (u_n - \dot{z} + x_i \phi_i^2) \]
\[ = \sum_{i} \sum_{j} \sum_{k} (q_i \dot{q}_j + \dot{q}_i q_j) m_i \psi_{ijk} \]
\[ + \phi_i^2 \sum_{k} m_k - \sum_{k} m_k \] (12)

The in-plane moment interaction at the coupling point \( C_0 \) is given to second order by

\[ C_0 = -\phi \sum_{i} m_i x_i^2 \] (13)

The results (12) and (13) may be applied to the primary beam equations (1) to (3) to obtain a complete equation for in-plane motion of the whole system, in which case \( C_0 \) and the final term in the expression for \( Z_n \) may be subsumed into the primary beam mass matrix \( M_0 \).

Finally a two degree of freedom representation of the system equations may be derived by applying Galerkin's method using a single selected in-plane mode together with an out-of-plane bending mode, each chosen with advantage to be mode shapes of linear undamped free vibration of the system. Define modal co-ordinates \( \xi_1, \xi_2 \) such that

\[ p = r_1 \xi_1, \quad q = r_2 \xi_2 \] (14)

\( r_1 \) and \( r_2 \) are the mode shapes (eigenvectors) of the selected in-plane and out-of-plane modes respectively with generalised masses:

\[ M_1 = r_1^T M r_1; \quad M_2 = r_2^T M r_2 \] (15)

and corresponding natural frequencies of undamped free vibration \( \omega_1, \omega_2 \) such that:

\[ \omega_1^2 M_1 = r_1^T K r_1; \quad \omega_2^2 M_2 = r_2^T K r_2 \] (16)

This information is presumed available from measurements or from (say) finite element analysis of the system.

Introducing the transformation (14) into (1) and (11) and premultiplying both sides by \( r_1^T \) and \( r_2^T \) respectively leads to a pair of coupled modal equations in \( \xi_1 \) and \( \xi_2 \) which may be reduced to the
Forced vibration analysis

For the case where the in-plane dynamic forces are harmonic with frequency $\Omega$, the system forced vibration response may be investigated using the Multiple Time Scales perturbation approach. Introducing the following substitutions:

\begin{equation}
\epsilon f_0 \cos \Omega t = F_0(t), \quad \delta_i = \epsilon \xi_i
\end{equation}

(17) and (18) may be written:

\begin{equation}
X + \omega_1^2 X = \epsilon \{ f_0 \cos \Omega t + \mu (Y^2 + YY) 
- 2\xi_1 \omega_1 X + \gamma X^2 \} 
\end{equation}

(27)

\begin{equation}
Y + \omega_2^2 Y = \epsilon X Y - 2\xi_2 \omega_2 Y 
\end{equation}

(28)

In this form the terms on the right side represent small perturbations of the linear undamped system response. Following the method, two time scales are introduced by the substitution:

\begin{equation}
T_0 = t; \quad T_1 = \epsilon t \\
\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} = D_0 + \epsilon D_1 
\end{equation}

(29)

$T_1$ represents a 'slow' time variable. A solution is then sought in the form of a power series in the small parameter $\epsilon$, here curtailed at the term in $\epsilon^k$. Assume

\begin{align*}
X(t) &= X_0(T_0, T_1) + \epsilon X_1(T_0, T_1) \\
Y(t) &= Y_0(T_0, T_1) + \epsilon Y_1(T_0, T_1)
\end{align*}

(30)

$X_0, Y_0$ represent the principal part of the solution, with $X_1, Y_1$ the first perturbations. Substituting (30) in (27) and (28) and equating like powers of $\epsilon$ leads to for terms in $\epsilon^0$

\begin{equation}
D_0^2 X_0 + \omega_1^2 X_0 = 0 
\end{equation}

(31)

\begin{equation}
D_0^2 Y_0 + \omega_2^2 Y_0 = 0 
\end{equation}

(32)

and for terms in $\epsilon^1$

\begin{align*}
D_0^2 X_1 + \omega_1^2 X_1 = & f_0 \cos \Omega T_0 - 2D_0 D_1 X_0 \\
- 2\xi_1 \omega_1 D_0 Y_0 + \mu (Y_0 D_0^2 Y_0 + (D_0 Y_0)^2) \\
+ \gamma (D_0 X_0)^2
\end{align*}

(33)

\begin{align*}
D_0^2 Y_1 + \omega_2^2 Y_1 = & (D_0^2 X_0) Y_0 - 2D_0 D_1 Y_0 \\
- 2\xi_2 \omega_2 D_0 Y_0
\end{align*}

(34)

General solutions to (31) and (32) may be written

\begin{align*}
X_0 = & A(T_1) \exp(i\omega_1 T_0) + \bar{A}(T_1) \exp(-i\omega_1 T_0) \\
Y_0 = & B(T_1) \exp(i\omega_2 T_0) + \bar{B}(T_1) \exp(-i\omega_2 T_0)
\end{align*}

(35)

(36)

where $A$ and $B$ are (complex) functions of slow time. The bar denotes complex conjugate.
(35) and (36) into (33) and (34) leads to non-homogeneous linear equations for the first order perturbation terms $X_1$, $Y_1$. These equations are not solved explicitly, but a condition for validity of the power series solution (30) is that $X_1$ and $Y_1$ must remain bounded and consequently there must be no resonant terms on the right side of (33) and (34) i.e terms in $\exp(i\omega_1 T_0)$ or $\exp(i\omega_2 T_0)$ with constant or slowly varying coefficients. Such terms are generated if $\Omega = \omega_1$, and $\omega_2 = \omega_1/2$, representing external and internal resonance, in which case necessary conditions for existence of the solution form (30) are:

$$\frac{f_0}{2} \exp(i\Omega T_0 - i\omega_1 T_0) - i\omega_1 D_1 A - i2\omega_1^2 B_2 = 0$$

$$\omega_2^2 A B \exp(i\omega_2 T_0 - i\omega_1 T_0) + i2\omega_1 D_1 B + i2\omega_2^2 B_2 = 0$$

Introducing polar forms of the complex amplitudes:

$$A = \frac{a}{2} \exp(i\alpha) \quad B = \frac{b}{2} \exp(i\beta)$$

with $a$, $b$, $\alpha$ and $\beta$ functions of slow time $T_1$, and also detuning parameters $\rho_1$, $\rho_2$ for the resonance conditions defined as:

$$\epsilon \rho_1 = \Omega - \omega_1$$

$$\epsilon \rho_2 = \omega_2 - \omega_1$$

(37) and (38) may be written as a set of four first order differential equations in $T_1$

$$\omega_1 a' = \omega_2^2 b + \frac{b^2}{2} \cos(2\rho_2 T_1 + 2\beta - \alpha) + \frac{f_0}{2} \cos(\rho_1 T_1 - \alpha)$$

$$\omega_1 a' = -\zeta_1 \omega_1^2 a - \omega_2^2 b + \frac{b^2}{2} \sin(2\rho_2 T_1 + 2\beta - \alpha) + \frac{f_0}{2} \sin(\rho_1 T_1 - \alpha)$$

$$\omega_2 b' = \omega_2^2 a + \frac{a b}{2} \cos(2\rho_2 T_1 + 2\beta - \alpha)$$

$$\omega_2 b' = -\zeta_2 \omega_2^2 b + \omega_2 a \cos(2\rho_2 T_1 + 2\beta - \alpha) + \frac{f_0}{2} \sin(\rho_1 T_1 - \alpha)$$

where $'$ denotes $d/dT_1$. Solutions representing steady state vibrations may be investigated by imposing the conditions $a'' = b'' = 0$ and further that the arguments of the trigonometric functions in (42-45) must be independent of $T_1$. This leads to

$$a' = \rho_1; \quad b' = \frac{f_0}{2} \rho_1$$

and a set of algebraic equations which may be solved for the steady vibration amplitudes.

Two types of solution are found possible. First of all it is seen that $\epsilon = 0$ is a solution, with

$$\frac{a}{2\omega_1} \left( \frac{\epsilon_1}{\omega_1^2} + \frac{\epsilon_2^2}{\omega_2^2} \right)^{1/2}$$

This represents purely in-plane motion and is simply the normal linear forced vibration resonance peak approximated for small detuning of the excitation frequency. In order to compare theoretical and measured vibration amplitudes it is convenient to introduce the following response parameters. Let $x$ and $y$ denote the steady response amplitudes at the selected in-plane and out-of-plane locations. Let $x_r$ denote the $x$ response as $\Omega \to 0$ i.e. the equivalent static deflection of the force system. ($x_r$ is thus an external excitation parameter). Further introduce response ratios:

$$V = \frac{x}{x_r}; \quad W = \frac{y}{x_r}$$

and let:

$$\epsilon_1 = \frac{\epsilon}{\alpha_0}$$

where $\epsilon$ is given by (21).

The in-plane solution is then given by

$$V = \frac{\omega_1}{2(\Omega - \omega_1)^2 + \delta_1^2 \omega_1^2}$$

$$W = 0$$

The second solution is obtained for the case $\epsilon \neq 0$ in (42-45), which yields after some manipulation:

$$V = \frac{4 \left( \frac{\omega_2}{\omega_1} \right)^2 \left( \left( \frac{\Omega}{\omega_2} - 1 \right)^2 + \delta_2^2 \right)^{1/2}}{\epsilon_1 \left( \frac{\omega_1}{\omega_1} - 1 \right)}$$

The solution for the out-of-plane motion is given by the real roots of:

$$W^4 - \frac{16}{\mu} \epsilon_2^2 \left[ \left( \frac{\Omega}{\omega_1} - 1 \right)^2 + \delta_1^2 \right] W^2$$

$$+ \frac{64}{\mu} \epsilon_4^2 \left[ \left( \frac{\Omega}{\omega_1} - 1 \right)^2 + \delta_2^2 \right]$$

$$\times \left[ \left( \frac{\Omega}{\omega_2} - 1 \right)^2 + \delta_2^2 \right] - \frac{\epsilon_1^2 \omega_1^2}{64 \omega_2^2} = 0$$

(53) yields zero, one or two real values of $W^2$ and these together with (52) represent steady vibrations due to the non-linear interactions. The amplitudes depend in a complex way on the system natural frequency and forcing frequency detuning, damping ratios, mass ratio and non-linear coupling coefficient $\epsilon_1$, which contains the excitation parameter $x_r$. The essential features of the system response behaviour are related to the stability or otherwise of these solutions, discussed in the next section.

At this point it is worth emphasising the non-
synchronous nature of the steady forced vibration response of the secondary beam, which is of course clear from the assumed form of solution in (36). A set of steady forced vibration responses is presented in Fig. 3 and plainly shows a two to one relationship between the response frequencies under the critical internal resonance condition.

![Fig. 3 Model responses for forcing frequency \( \Omega = 9.5 \) Hz. (a) Secondary beam dynamic strain. (b) Primary beam acceleration.](image)

**Theoretical response regions for forced vibration**

The theoretical results for the stationary forced vibration responses (50-53) may be plotted conveniently as a graph of response ratio \( V \) or \( W \) versus excitation frequency ratio \( \Omega / \omega_1 \). Consideration of the stability of these solutions is necessary to determine the patterns of system behaviour. The stability of the solutions has been investigated by the standard method of introducing a small perturbation of the steady solutions and determining the nature of the characteristic roots of the linearised perturbational equations. The complexity of the coefficients of the characteristic equation made it necessary to perform this numerically for specific sets of curves.

Figures 4 (a and b) show a typical set of forced vibration response curves for the case of exact internal resonance \( \Omega / \omega_1 = 0.5 \). Corresponding solution curves are designated I and II in the figure and dashed portions of the curves indicate solutions which have been determined to be unstable and will not be realised in practice.

In Fig. 4(b) the \( W = 0 \) axis is solution 1 of which the segment \( AE \) is unstable. The principle feature is that in the region of external resonance the normal linear resonance peak of the in-plane mode labelled \( R \) in Fig. 4(a) is absorbed by non-linear energy transfer and replaced by an inverted \( V \) curve giving a response minimum at resonance. As shown in Fig. 4(b) this is accompanied by an instability of the normal zero motion solution and a region of large steady vibration of the out-of-plane mode. On each side of external resonance there is then a region where both solution types are stable. The system will vibrate in either condition depending on initial conditions, or may transfer between them as a consequence of a shock load. For example if the system is performing in-plane vibration at a frequency corresponding to points \( K \) and \( K' \) in Figs. 4(a) and b, it may be altered to the alternative stable condition \( L \) and \( L' \) involving large coupled out-of-plane motions by means of an impact.

Remote from the external resonance condition (beyond the ‘horns’ of the solution II curve in Fig. 4(b) the normal in-plane linear response curve is the only stable solution. The form of these curves will give rise to a complex pattern of forced vibration response for a slowly changing excitation frequency, with amplitude jumps and hysteresis effects. For a slowly increasing frequency, the response will follow the normal linear resonance curve with \( W = 0 \) until points \( A \) and \( A' \) are reached in Fig. 4. At this point the \( W = 0 \) solution is unstable and the system will jump to curve II at point \( B \). With further increase of frequency the system will hold to the non-linear solution until condition \( C \) and \( C' \) is reached, at which point a jump will be made to the linear solution at \( D \) and \( D' \). With frequency decreasing through the resonance region from above, the response paths \( EGH \) and \( E'GH' \) will be followed with amplitude jumps now occurring at frequencies corresponding to points \( F \) and \( G \).

![Fig. 4 Theoretical forced vibration response for exact internal resonance.](image)

(a) In-plane response \( \omega_2 = 0.5 \), \( t_2 = 0.0027 \)  
(b) Out-of-plane response \( \omega_1 \), \( \delta_1 = 0.012 \), \( \delta_2 = 0.001 \).

Figures 6 and 7 show typical steady-state response curves obtained for the case of small detuning below and above the exact internal resonance. Once more dashed portions of the curves represent unrealizable solutions. In these cases the symmetry of the response curves is lost but the basic feature of a loss of stability of the simple linear in-plane motion together with the occurrence of a region of stable coupled response of the in-plane and out-of-plane
modes is clearly visible. The size of the region of non-linear coupled response increases with the magnitude of the coupling parameter \( \varepsilon_1 \), but decreases with increase of damping coefficients. From (33) it may be shown that the following

\[
\varepsilon_1 > 2\delta_1 \delta_2
\]  

(54)
is a necessary condition for the existence of the non-linear responses.

**Experimental investigation**

Tests were carried out on a laboratory model to corroborate the response results. Details of the model are given in Fig. 5 which clearly shows the arrangement of beams coupled in an L configuration. The secondary beam was of thin spring steel with an adjustable mass so that tuning of the system in the vicinity of the internal resonance condition was possible. The primary beam was driven by an electromagnetic shaker from a decade oscillator and current drive amplifier. In-plane response was obtained from an accelerometer and charge amplifier. Response of the coupled beam was obtained from a pair of strain gauges attached near the coupling point as shown and used in half-bridge arrangement with a Bridge amplifier. The bridge output was calibrated in terms of the tip mass amplitude. Damping coefficients were measured from transient decays.

A storage oscilloscope was used on a very slow sweep rate to capture a large number of cycles of decaying vibration from which damping coefficients were evaluated. The low levels obtained are typical of those found in related work on lightly damped structural models. The mass ratio parameter was obtained by a mass perturbation method. The non-linear coupling coefficient \( \varepsilon_1 \) was evaluated on a basis of a one degree of freedom modelling of the coupled beam using the cantilever static deflection curve as a deflection function. In this case (49) simplifies to:

\[
\varepsilon_1 = \frac{r_{1x}}{r_{1x}^6} x_1
\]  

(55)

---

Fig. 5 Experimental apparatus. (1) primary beam, (2) secondary beam, (3) adjustable end mass, (4) light damping blocks, (5) strain gauges, (6) accelerometer, (7) actuator rod, (8) shaker, (9) main clamps, (10) solid baseplate.

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Fig. 8 Forced vibration response for exact internal resonance.

\[ w(t) = 0.5 \]

- (a) In-plane response
- (b) Out-of-plane response

Theoretical response

(1) Experimental response, frequency decreasing
(2) Experimental response, frequency increasing

\[ r_1 = 0.0018 \quad \delta_1 = 0.012 \quad \delta_2 = 0.001 \]

The ratio of the primary beam eigenvector components was obtained from a PAFEC finite element dynamic analysis. The static deflection factor \( x \) was deduced from a calibration of the in-plane response against vibrator current.

The system was initially set up with perfect internal tuning, the adjustable end mass provided a fine tuning facility, the input force level was fixed and the excitation frequency was slowly swept over the region of interest stopping at points where amplitude responses were to be noted. Measurements were made only when the system had obviously settled down into steady-state conditions. This procedure was followed for a higher force level and then the cases of detuning were examined in a similar manner. System natural frequencies were quickly obtained from a spectrum analyser for the various states of internal detuning.

Figure 3 shows a section of typical dynamic strain signal representing out-of-plane response and a portion of the in-plane accelerometer signal. These show clearly the two to one frequency ratio of the forced responses characteristic of the internal resonance condition.

Figures 6-8 show a selection of measured vibration levels for increasing and decreasing frequency and comparison with results from the analysis for a number of tuning conditions at, and in the vicinity of internal resonance. By and large the measured vibration amplitudes show very good correlation with the theory and give a clear indication of the strong effects of small non-linear coupling in the presence of internal resonance, namely the replacement of the normal linear response peak by a trough representing non-linear energy transfer, the existence of regions of large out-of-plane response due to non-linear coupling with amplitude jumps at entry and exit frequencies, these being dependent upon the direction of frequency change. Discrepancies between measured and predicted values are small and may be reasonably attributed to measurement errors.

CONCLUSIONS

The effects of non-linear coupling of autoparametric type between bending modes of vibration of a pair of coupled beams have been investigated theoretically and experimentally. System equations have been derived and shown to be a generalisation of previous work. These were then solved to obtain stationary forced vibration solutions by the method of multiple scales. It was shown that the small non-linear interaction terms have a significant effect under conditions of internal resonance i.e. when the bending modal frequencies are in a ratio which is close to 2/1. The normal linear forced vibration resonance peak is absorbed by non-linear energy transfer and large responses of the indirectly excited mode are driven at half the excitation frequency. The non-linear responses show a complex pattern of amplitude jumps at entry and exit frequencies. Very good correlation was obtained between measured dynamic strain and acceleration levels and predicted values in a series of forced vibration tests on a laboratory model.

It must be expected that similar response phenomena will occur in more complicated structural configurations under conditions of high vibratory excitation and light damping when the coincidence of natural frequencies leads to internal resonance of the type discussed. Work is proceeding on the investigation of more complex forms of non-linear modal interaction and these are proving to be abundant in simple structural models. Generally these interaction effects may be neutralised by damping levels. However modern manufacturing techniques lead to structures with reduced intrinsic damping and hence an increased susceptibility.

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