CHAPTER 3

The relation between the elliptic function and series solutions.
The convergence of the series solution.
In Chapter 2 we have shown that a solution of the problem can be given in terms of elliptic functions and that this may be used for calculating the orbit whenever a real solution of the problem exists, except for values of \( s \) and \( q \) such that \( s = 2q \).

On the other hand it was shown in Chapter 1 that the solution in the form of trigonometric series is limited in its application and that these series will certainly be divergent for certain values of \( s \) and \( q \) for which real solutions of the problem exist.

We shall therefore seek to find the cause of the divergence of the series solution and for this purpose we shall attempt to reduce the elliptic function solution to the series solution.

It is convenient to concentrate attention on one coordinate, \( y = x \), the work being obviously similar for the other coordinates.

From equation (23) we have

\[
q_x = \frac{m k \text{sn}(u, k) + l}{1 + k \text{sn}^2 u}
\]

Since \( u \) is real, we can when \( k < 1 \) write this in the form

\[
q_x = (l + m k \text{sn} u)[1 - k \text{sn} u + k^2 \text{sn}^2 u - k^3 \text{sn}^3 u + k^4 \text{sn}^4 u - \cdots]
\]

\[
= l + (m - l)\left[k \text{sn} u - k^2 \text{sn}^2 u + k^3 \text{sn}^3 u - \cdots\right]
\]

Put \( u = \frac{2K x}{\pi} \), then

\[
k \text{sn} u = k^{1/2} \left[2^{1/4} \sin x - 2^{3/4} \sin 3x + 2^{7/4} \sin 5x - \cdots\right]
\]

\[
\left[1 - 2^{1/2} \cos 2x + 2^{3/2} \cos 4x - 2^{7/2} \cos 6x + \cdots\right]
\]

For our purposes it is sufficient to retain only powers of \( q \) up to the first.

\[
k \text{sn} u = 2 k^{1/2} q^{1/4} \sin x
\]

\[
k^2 \text{sn}^2 u = 2 k q^{3/2} (1 - \cos 2x)
\]

\[
k^3 \text{sn}^3 u = 2 k^{1/2} q^{5/4} (3 \sin x - \sin 3x)
\]

\[
k^4 \text{sn}^4 u = 2 k^2 q (3 - 4 \cos 2x + 4 \cos 4x)
\]
\[ q_2 = l + (m-k) \left[ 2 k^4 \sin x \cos^2 x - 2 k^2 (1 - \cos 2x) y \right]^{1/2} + 2 k^3 (3 \sin x - \sin 3x) y^{3/4} \]

This expansion for \( q_2 \) is valid throughout the whole region in which a real solution exists except in the immediate neighbourhood of the part of the double line, \( s = 2q \), for which \( k = 1 \), i.e. between the points of contact with the curved branches of the discriminant curve.

Further \( m, l, \) and \( k \) are given in terms of the roots of the cubic (16) by equations (24) viz

\[ m = y + \sqrt{(y-\mu)(y-\nu)} \]
\[ l = y - \sqrt{(y-\mu)(y-\nu)} \]
\[ k = \frac{2y - \mu - \nu - \lambda - 2 \sqrt{(y-\mu)(y-\nu)}}{\mu - \nu} \]

The series (25) reduces to the particular forms given in Chapter 2 on all boundaries of the permissible areas.

For on the part of the double line, \( s = 2q \), which is a boundary of the permissible areas we have

\[ \lambda = \mu = \frac{1}{2} < y \]

and therefore \( l = \frac{1}{2} \) and \( k = 0 \).

\[ q_2 \text{ reduces to } \frac{1}{2} \text{ which agrees with the previous result.} \]

On the curved part of the discriminant curve which is a boundary of the permissible area we have

\[ \lambda = \mu < \frac{1}{2} < y \]

and therefore \( l = \lambda \) and \( k = 0 \).

\[ q_2 = \lambda \text{ which agrees with the previous result.} \]
To reduce \( g \) to the form of the series solution we require to obtain \( \lambda, \mu, \) and \( \nu \) in the form of series of powers of \( \frac{x}{s} \).

The cubic of which \( \lambda, \mu, \nu \) are the roots is

\[
4x^2 \alpha^3 - (4x^2 + 8x) \alpha^2 + (x^2 + 2ag) \alpha - g^2 = 0 \tag{26}
\]

Writing \( g = k_2 s \), so that \( k_2^2 = \frac{1 - \lambda}{s} \) and is therefore the same as the \( k_2 \) of the series solution (see p. 6), this cubic becomes

\[
\alpha \left( 1 - 2x \alpha \right) = \frac{s^2}{k_2^2} \left( k_2^2 - \alpha \right)^2 \tag{27}
\]

Assuming \( x \) to be small compared with \( s \), the two smallest roots of the cubic are given by

\[
\alpha = k_2 \pm \frac{s}{5} \sqrt{5} \left( 1 - 2x \right) \tag{28}
\]

We proceed to solve this equation by successive approximations and find

\[
\begin{align*}
\mu &= A_0 + A_1 \frac{x}{s} + A_2 \frac{x^2}{s^2} + A_3 \frac{x^3}{s^3} + A_4 \frac{x^4}{s^4} + A_5 \frac{x^5}{s^5} + A_6 \frac{x^6}{s^6} + \cdots \\
\lambda &= A_0 - A_1 \frac{x}{s} + A_2 \frac{x^2}{s^2} - A_3 \frac{x^3}{s^3} + A_4 \frac{x^4}{s^4} - A_5 \frac{x^5}{s^5} + A_6 \frac{x^6}{s^6} - \cdots
\end{align*}
\tag{29}
\]

where

\[
A_0 = k_2^2
\]

\[
A_1 = k_2 \left( 1 - 2k_2^2 \right)
\]

\[
A_2 = \frac{1}{2} \left( 1 - 2k_2^2 \right) \left( 1 - 6k_2^2 \right)
\]

\[
A_3 = \frac{1}{8k_2^2} \left( 1 - 2k_2^2 \right) \left( 1 - 28k_2^2 + 84k_2^4 \right)
\]

\[
A_4 = -2 \left( 1 - 2k_2^2 \right) \left( 1 - 10k_2^2 + 20k_2^4 \right)
\]

\[
A_5 = \left( 1 - 2k_2^2 \right) \left( 1 + 72k_2^2 - 2376k_2^4 + 14028k_2^6 - 20592k_2^8 \right)
\]

\[
A_6 = 2 \left( 1 - 2k_2^2 \right) \left( 5 - 70k_2^2 + 220k_2^4 - 336k_2^6 \right)
\]

Also, since \( \lambda + \mu + \nu = 1 + \frac{s^2}{4x^2} \), we have
\[ y = \frac{s^2}{4d^2} + \left( -2A_0 \right) - 2 \left[ \frac{A_2}{s^2} + \frac{A_3}{s^4} + \frac{A_4}{s^4} + \frac{A_6}{s^6} \right] \] - - - (31)

We therefore obtain

\[ \mu + \lambda = 2 \left[ \frac{A_0 + A_2}{s^2} + \frac{A_3}{s^4} + \frac{A_4}{s^4} + \frac{A_6}{s^6} \right] \]

\[ 2 \left( \frac{y - \mu - \lambda}{2d^2} \right) \left[ 1 + \left( -3A_0 \right) \right] \left[ \frac{1}{s^2} - 12A_2 \frac{y}{s^4} - 12A_4 \frac{y}{s^6} - 12A_6 \frac{y^2}{s^8} \right] \]

\[ \mu - \lambda = 2 \left[ \frac{A_1}{s} + \frac{A_3}{s^3} + \frac{A_5}{s^5} \right] \]

\[ y - \mu = \frac{s^2}{4d^2} + 1 - 3 \left[ \frac{A_0 + A_2}{s^2} + \frac{A_3}{s^4} + \frac{A_4}{s^4} + \frac{A_6}{s^6} \right] - \left[ \frac{A_1}{s} + \frac{A_3}{s^3} + \frac{A_5}{s^5} \right] \]

\[ y - \lambda = \frac{s^2}{4d^2} + 1 - 3 \left[ \frac{A_0 + A_2}{s^2} + \frac{A_3}{s^4} + \frac{A_4}{s^4} + \frac{A_6}{s^6} \right] + \left[ \frac{A_1}{s} + \frac{A_3}{s^3} + \frac{A_5}{s^5} \right] \]

\[ \left( y - \mu \right) \left( y - \lambda \right) = \frac{s^2}{4d^2} + 1 - 3 \left[ \frac{A_0 + A_2}{s^2} + \frac{A_3}{s^4} + \frac{A_4}{s^4} + \frac{A_6}{s^6} \right] - \left[ \frac{A_1}{s} + \frac{A_3}{s^3} + \frac{A_5}{s^5} \right] \]

\[ \left( y - \mu \right) \left( y - \lambda \right) = \left[ \frac{s^2}{4d^2} + \left( -3A_0 \right) - 3 \frac{A_2}{s^2} - 3 \frac{A_4}{s^4} - 3 \frac{A_6}{s^6} \right] \times \]

\[ \left[ 1 - \left\{ \frac{A_1}{s} + \frac{A_3}{s^3} + \frac{A_5}{s^5} \right\} \right] \]

\[ \frac{s^2}{4d^2} \left[ 1 + \left( -3A_0 \right) \frac{y}{s^2} - 12A_2 \frac{y}{s^4} - 12A_4 \frac{y}{s^6} - 12A_6 \frac{y^2}{s^8} \right] \times \]

\[ \left[ 1 - \frac{y}{s^2} \left\{ \frac{A_1}{s} + \frac{A_3}{s^3} + \frac{A_5}{s^5} \right\} \right] \]

\[ 2 \left( y - \mu - \lambda - 2 \left( \mu - \lambda \right) \right) \left( y - \mu \right) \left( y - \lambda \right) = \frac{y}{s^2} \]
\[
\begin{align*}
\dot{r} &= 2 \gamma - \lambda - 2 \sqrt{(y-\mu)(y-\lambda)} \\
&= 2 A_1 \frac{x^3}{s^3} \left[ 1 + \left\{ \frac{2 A_2}{A_1} - 4 (1-3 A_0) \right\} \frac{x^2}{s^2} + \left\{ 12 A_2 + 16 (1-3 A_0)^2 + \frac{A_3}{A_1^2} + 2 A_5 - \frac{8 A_3}{A_1} (1-3 A_0) \right\} \frac{x^3}{s^4} \right] \times \\
&\quad \left[ 1 + \frac{A_3}{A_1} \frac{x^2}{s^2} + \frac{A_5}{A_1} \frac{x^3}{s^4} \right]^{-1} \\
&= 2 A_1 \frac{x^3}{s^3} \left[ 1 + \left\{ \frac{2 A_2}{A_1} - 4 (1-3 A_0) \right\} \frac{x^2}{s^2} + \left\{ 12 A_2 + 16 (1-3 A_0)^2 + \frac{A_3}{A_1^2} + 2 A_5 - \frac{8 A_3}{A_1} (1-3 A_0) \right\} \frac{x^3}{s^4} \right] \times \\
&\quad \left[ 1 - \frac{A_3}{A_1} \frac{x^2}{s^2} - \left( \frac{A_5}{A_1} \frac{x^3}{s^4} \right) \right] \\
&= 2 A_1 \frac{x^3}{s^3} \left[ 1 + \left\{ \frac{2 A_2}{A_1} - 4 (1-3 A_0) \right\} \frac{x^2}{s^2} + \left\{ 12 A_2 + 16 (1-3 A_0)^2 + \frac{A_3}{A_1^2} + 4 A_5 - 4 A_3 (1-3 A_0) \right\} \frac{x^3}{s^4} \right]
\end{align*}
\]

Putting in the values for \( A_1, A_2, \text{etc.} \) from equations (30) this gives

\[
\dot{r} = 2 k_e (1-2 k_e^2) \frac{x^3}{s^3} \left[ 1 + \frac{1}{8 k_e^2} (1-60 k_e^2 + 180 k_e^4) \frac{x^2}{s^2} - \frac{1}{128 k_e^4} \left( 1+136 k_e^2 - 1710 k_e^4 + 320 k_e^6 \right) \frac{x^4}{s^4} \right]
\]

\[
\dot{k}^2 = 4 k_e^2 (1-2 k_e^2) \frac{x^2}{s^2} + \left[ 1 + \frac{1}{4 k_e^2} (1-60 k_e^2 + 180 k_e^4) \frac{x^2}{s^2} \right]
\]

\[
- \frac{1}{16 k_e^2} \left( 64 - 2784 k_e^2 + 16203 k_e^4 - 24192 k_e^6 \right) \frac{x^4}{s^4}
\]

\[
\dot{k}_{12} = \dot{k}^2 - 1 = 1 - 4 k_e^2 (1-2 k_e^2) \frac{x^2}{s^2} - (1-2 k_e^2) (1-60 k_e^2 + 180 k_e^4) \frac{x^2}{s^2} \]

\[
+ \frac{1}{4} (1-2 k_e^2)^2 (64 - 2784 k_e^2 + 16203 k_e^4 - 24192 k_e^6) \frac{x^{10}}{s^{10}}
\]

\[
\dot{k}' = 1 - k_{12} \left( 1-2 k_e^2 \right) \frac{x^2}{s^2} - \frac{1}{4} (1-2 k_e^2) (1-60 k_e^2 + 180 k_e^4) \frac{x^2}{s^2} \\
+ \frac{1}{16} (1-2 k_e^2)^2 (64 - 2784 k_e^2 + 16203 k_e^4 - 24192 k_e^6) \frac{x^{10}}{s^{10}}
\]

To this approximation \( \gamma = \frac{1}{\dot{r}} = \frac{1 - \dot{k}^2}{1 + \dot{k}^2} \)

\[
\dot{\gamma} = \frac{1}{4} k_e^2 (1-2 k_e^2) \frac{x^2}{s^2} \left[ 1 + \frac{1}{8 k_e^2} (1-60 k_e^2 + 180 k_e^4) \frac{x^2}{s^2} \right]
\]

\[
- \frac{1}{16 k_e^2} \left( 64 - 2784 k_e^2 + 16203 k_e^4 - 24192 k_e^6 \right) \frac{x^4}{s^4}
\]
\[
\frac{d^2}{dx^2} - \left( \sqrt{\frac{1}{\kappa_2} (1 - 2 \kappa_2^2)} x^3 \right) \left[ \frac{1 + \frac{1}{16 \kappa_2^2} \left( 1 - 60 \kappa_2^2 + 180 \kappa_2^4 \right) \kappa_2^6}{S^2} \right] \left[ \frac{L}{128 \kappa_2^4} \left( 3 - 232 \kappa_2^2 + 6312 \kappa_2^4 - 32394 \kappa_2^6 + 4888 \kappa_2^8 \right) \frac{d^4}{dx^4} \right]
\]

\[
\frac{d^3}{dx^3} \left( \sqrt{\frac{1}{\kappa_2} (1 - 2 \kappa_2^2)} x^3 \right) \left[ 1 + \frac{1}{16 \kappa_2^2} \left( 1 - 60 \kappa_2^2 + 180 \kappa_2^4 \right) \kappa_2^6 \right] \left[ 1 + \frac{1}{16 \kappa_2^2} \left( 1 - 60 \kappa_2^2 + 180 \kappa_2^4 \right) \kappa_2^6 \right] \left[ 1 + \frac{1}{16 \kappa_2^2} \left( 1 - 60 \kappa_2^2 + 180 \kappa_2^4 \right) \kappa_2^6 \right]
\]

Using the values of \( \kappa_2 \) given by equation (32) these give

\[
k_{\frac{1}{2}} = \frac{1}{2} \kappa_2 \left( 1 - 2 \kappa_2^2 \right) \frac{d^3}{dx^3} \left( \sqrt{\frac{1}{\kappa_2} (1 - 2 \kappa_2^2)} x^3 \right)
\]

Also \( \frac{\sqrt{y - \mu}}{\sqrt{\mu - \gamma}} = \frac{S^2}{4 \mu^2} \left[ 1 + 4 (1 - 3 A_0) \frac{d^2}{dx^2} - 12 A_2 \frac{d^4}{dx^4} - (12 A_4 + 8 A_1^2) \frac{d^6}{dx^6} \right] \)

\[m = D + \frac{\sqrt{y - \mu}}{\sqrt{\mu - \gamma}} \]

\[= \frac{S^2}{2 \mu^2} \left[ 1 + 2 (2 - 5 A_0) \frac{d^2}{dx^2} - 10 A_2 \frac{d^4}{dx^4} - (10 A_4 + 4 A_1^2) \frac{d^6}{dx^6} \right] \]

\[l = \gamma - \frac{\sqrt{y - \mu}}{\sqrt{\mu - \gamma}} \]

\[= A_0 + A_2 \frac{d^2}{dx^2} + (A_4 + 2 A_1^2) \frac{d^4}{dx^4} \]

\[l = \frac{\kappa_2^2}{S^2} \left[ 1 - \frac{1}{2} (1 - 2 \kappa_2^2 + 16 \kappa_2^4) \frac{d^2}{dx^2} - 2 (1 - 2 \kappa_2^2) (1 - 11 \kappa_2^2 + 22 \kappa_2^4) \frac{d^4}{dx^4} \right] = - \frac{S^2}{\mu^2} \]

Now substitute the values for \( l, m, k_{\frac{1}{2}} ^{\frac{1}{4}} \) and \( k_{\frac{1}{2}} ^{\frac{1}{2}} \) in the first few terms of the expansion (26) for \( q_z \), giving
\[ q_2 = k_x^2 + \frac{1}{2} (1 - 2 k_x^2) (1 - 6 k_x^2) \frac{x^2}{s^2} - 2 (1 - 2 k_x^2) (1 - 10 k_x^2 + 22 k_x^4) \frac{x^4}{s^4} \]

\[ + \frac{s^2}{2 k_x^2} \left[ 1 + 2 (2 - 5 k_x^2) \frac{x^2}{s^2} - 5 (1 - 2 k_x^2) (1 - 6 k_x^2) \frac{x^4}{s^4} \right] \times \]

\[ \left[ 2 k_x (1 - 2 k_x^2) \sin x \frac{x^3}{s^3} + \frac{1}{4 k_x} (1 - 2 k_x^2) (1 - 60 k_x^2 + 180 k_x^4) \sin x \frac{x^5}{s^5} \right] \]

\[ - 2 k_x^2 (1 - 2 k_x^2)^2 (1 - \cos 2x) \frac{x^6}{s^6} \]

\[ = k_x^2 + k_x (1 - 2 k_x^2) \sin x \frac{x^2}{s^2} + \frac{1}{2} (1 - 2 k_x^2) (1 - 6 k_x^2) \frac{x^2}{s^2} \]

\[ + \frac{x^3}{s^3} \sin x \left\{ \frac{1}{8 k_x} (1 - 2 k_x^2) (1 - 60 k_x^2 + 180 k_x^4) + 2 k_x (1 - 2 k_x^2) (2 - 6 k_x^2) \right\} \]

\[ - \frac{x^4}{s^4} \left\{ 2 (1 - 2 k_x^2) (1 - 10 k_x^2 + 22 k_x^4) + k_x^2 (1 - 2 k_x^2)^2 (1 - \cos 2x) \right\} \]

We finally have

\[ q_2 = k_x^2 + k_x (1 - 2 k_x^2) \sin x \frac{x^2}{s^2} + \frac{1}{2} (1 - 2 k_x^2) (1 - 6 k_x^2) \frac{x^2}{s^2} \]

\[ + \frac{x^3}{s^3} \sin x \left\{ \frac{1}{8 k_x} (1 - 2 k_x^2) (1 - 60 k_x^2 + 180 k_x^4) + 2 k_x (1 - 2 k_x^2) (2 - 6 k_x^2) \right\} \]

\[ - \frac{x^4}{s^4} \left\{ 2 (1 - 2 k_x^2) (1 - 10 k_x^2 + 22 k_x^4) + k_x^2 (1 - 2 k_x^2)^2 (1 - \cos 2x) \right\} \]

We have now to identify this expansion for \( q_2 \) in powers of \( s \) with the expansion given by the series solution.

\( q_2 \) was given by the series solution in the form

\[ q_2 = a_1 + a_1 a_2 \frac{x^2}{s} \cos M + \frac{1}{4} a_1 (a_1 - 4 a_2) \frac{x^2}{s} - 2 a_1 a_2^2 (a_2 - a_3) \frac{x^3}{s^3} \cos M \]

\[ + \left\{ \frac{3}{4} a_1 (a_1^2 - 8 a_2 a_1 + 4 a_2^2) + a_2 a_3 \cos 2M \right\} \frac{x^4}{s^4} \]

We proceed to replace \( a_1 \) and \( a_2 \) by their values in terms of \( k_1^2 \) and \( k_2^2 \) using the relation

\[ k_1^2 = 1 - 2 k_x^2 \]

Then

\[ a_1 = k_1^2 - \frac{1}{2} k_1^2 (k_1^2 - 4 k_x^2) \frac{x^2}{s^2} + \frac{3}{2} k_1^2 (k_1^2 - 6 k_x^2 + 4 k_x^4) \frac{x^4}{s^4} \]

\[ = (1 - 2 k_x^2) - \frac{1}{2} (1 - 2 k_x^2) (1 - 6 k_x^2) \frac{x^2}{s^2} + \frac{3}{2} (1 - 2 k_x^2) (1 - 10 k_x^2 + 20 k_x^4) \frac{x^4}{s^4} \]
\[ a_z = k_z \frac{1}{4} \left( k_z^2 - 6 k_z^2 \right) \frac{\Delta^2 z}{S^2} - \frac{3}{4} k_z \left( k_z^4 - 6 k_z^2 k_z^2 + 4 k_z^4 \right) \frac{\Delta^4 z}{S^4} \]

\[ = k_z \left[ \frac{1}{8} k_z \left( -2 k_z^2 \right) \right] \frac{\Delta^2 z}{S^2} - \frac{3}{4} k_z \left( k_z^4 - 6 k_z^2 k_z^2 + 4 k_z^4 \right) \frac{\Delta^4 z}{S^4} \]

\[ a_z \frac{\Delta z}{z} = k_z \left[ \frac{1}{8} k_z \left( 1 - 10 k_z^2 + 20 k_z^4 \right) \frac{\Delta^2 z}{S^2} \right] - \frac{1}{32} k_z \left( 1 - 10 k_z^2 + 20 k_z^4 \right) \frac{\Delta^4 z}{S^4} \]

\[ a_{z,1} \frac{\Delta z}{z} = \frac{1}{2} \left( 1 - 2 k_z^2 \right) \frac{\Delta^2 z}{S^2} \]

\[ a_{z,1} = 4 a_z = \left( 1 - 10 k_z^2 \right) \frac{\Delta^2 z}{S^2} - \frac{3}{2} \left( 1 - 2 k_z^2 \right) \frac{\Delta^4 z}{S^4} \]

\[ a_{z,1} \frac{\Delta z}{z} \left( a_{z,1} - 4 a_z \right) = \left( 1 - 2 k_z^2 \right) \frac{\Delta^2 z}{S^2} - \frac{3}{2} \left( 1 - 2 k_z^2 \right) \frac{\Delta^4 z}{S^4} \]

\[ a_z \frac{\Delta z}{z} \left( a_z - 4 a_z \right) = \left( 1 - 2 k_z^2 \right) \frac{\Delta^2 z}{S^2} - \frac{3}{2} \left( 1 - 2 k_z^2 \right) \frac{\Delta^4 z}{S^4} \]

\[ a_z \frac{\Delta z}{z} \left( a_z - 4 a_z \right) = \left( 1 - 2 k_z^2 \right) \frac{\Delta^2 z}{S^2} - \frac{3}{2} \left( 1 - 2 k_z^2 \right) \frac{\Delta^4 z}{S^4} \]

\[ \frac{3}{4} a_z \left( a_z^2 - 8 a_z + 4 a_z^2 \right) + a_z^2 a_z \cos 2M \]

\[ = \frac{3}{4} \left( 1 - 2 k_z^2 \right) \left( 1 - 4 k_z^2 + 4 k_z^2 - 8 k_z^2 + 16 k_z^2 + 4 k_z^4 \right) \]

\[ = \frac{3}{4} \left( 1 - 2 k_z^2 \right) \left( 1 - 12 k_z^2 + 24 k_z^4 \right) + \left( 1 - 2 k_z^2 \right)^2 k_z^2 \cos 2M \]

\[ \text{Substituting these values} \]

\[ \frac{\Delta z}{z} = k_z + k_z \left( 1 - 2 k_z^2 \right) \cos 2M \frac{\Delta^2 z}{S^2} + \frac{\Delta^2 z}{S^2} \left\{ \frac{1}{4} \left( 1 - 2 k_z^2 \right) \left( 1 - 6 k_z^2 \right) + \frac{1}{4} \left( 1 - 2 k_z^2 \right) \left( 1 - 6 k_z^2 \right) \right\} \]

\[ + \frac{3}{S^3} \left\{ \frac{1}{8} k_z \left( 1 - 2 k_z^2 \right) \left( 1 - 6 k_z^2 \right) \cos 2M \right\} \]

\[ + \frac{3}{S^3} \left\{ \frac{1}{8} k_z \left( 1 - 2 k_z^2 \right) \left( 1 - 6 k_z^2 \right) \cos 2M \right\} \]

\[ + \frac{3}{S^3} \left\{ \frac{1}{8} k_z \left( 1 - 2 k_z^2 \right) \left( 1 - 6 k_z^2 \right) \cos 2M \right\} \]

\[ + \frac{3}{S^3} \left\{ \frac{1}{8} k_z \left( 1 - 2 k_z^2 \right) \left( 1 - 6 k_z^2 \right) \cos 2M \right\} \]

\[ + \frac{3}{S^3} \left\{ \frac{1}{8} k_z \left( 1 - 2 k_z^2 \right) \left( 1 - 6 k_z^2 \right) \cos 2M \right\} \]
Finally,

\[ \psi_L = k_2^2 + k_2 (1 - 2 k_2^2) \cos M x + \frac{1}{2} (1 - 2 k_2^2) (1 - 6 k_2^2) \psi_{H_2}^2 \]

\[ + \frac{1}{2} \cos M (1 - 2 k_2^2) (1 - 2 \gamma^2 k_2^2 + \delta^2 \gamma^2 k_2^4) \frac{L^2}{S_3} \]

\[ - \left[ (1 - 2 k_2^2) (2 - 21 k_2^2 + 42 k_2^4) + k_2^2 (1 - 2 k_2^2) \cos 2 M \right] \frac{L^2}{S_4} \]  

\[ (39) \]

Comparing equations (38) and (39) we see that these are identical provided we can identify \( M \) with \( \frac{x + \pi}{2} \).

Now \( x = \frac{\pi}{2} \),

and \( \frac{2k}{\pi} = 1 + 2q + 2q^2 + \cdots \)

and \( q \) is of the order \( x^b \), so that to the approximation considered we may take \( x = x \).

Also \( \mu = -\frac{2k}{\pi} \frac{(x - \lambda)(x - \lambda)(y - \lambda)}{A(m - \lambda)} \)

\[ \mu = -\frac{2k}{\pi} \frac{(x - \lambda)(x - \lambda)(y - \lambda)}{A(m - \lambda)}, \text{ } C + C \text{ where } C \text{ is an arbitrary constant.} \]

Now \( (x - \lambda) = A_1 \frac{x}{S_3} + A_3 \frac{x^3}{S_3} + 2 A_2 \frac{x^2}{S_4} + A_5 \frac{x^5}{S_5} \), neglecting higher powers of \( x \).

\( (x - \lambda) = A_1 \frac{x}{S_3} + A_3 \frac{x^3}{S_3} - 2 A_2 \frac{x^2}{S_4} + A_5 \frac{x^5}{S_5} \)

\[ (x - \lambda)(x - \lambda) = \left( A_1 \frac{x}{S_3} + A_3 \frac{x^3}{S_3} + A_5 \frac{x^5}{S_5} \right)^2 \]

\[ = \frac{x^2}{S_3^2} \left[ A_1^2 + 2 A_1 A_3 \frac{x^2}{S_3} + \left( A_3^2 + 2 A_1 A_5 \right) \frac{x^4}{S_5} \right] \]

Also \( (y - \lambda) = \frac{x^2}{S_3} \left[ 1 + 4 (1 - 3 A_0) \frac{x^2}{S_3} - 12 A_2 \frac{x^4}{S_4} \right] \)

\[ (x - \lambda)(y - \lambda) = \frac{A_1}{4} \left[ 1 + \left\{ \frac{2 A_3}{A_1} + 4 (1 - 3 A_0) \right\} \frac{x^2}{S_3} + \left\{ \frac{8 A_3}{A_1} \left( 1 - 3 A_0 \right) - 12 A_2 + \frac{2 A_5}{A_1} \right\} \frac{x^4}{S_4} \right] \]

\[ - \frac{1}{8} \left\{ \frac{2 A_3}{A_1} + 4 (1 - 3 A_0) \right\} \frac{x^2}{S_3} \]
\[
\frac{(\xi-\lambda)(\mu-\lambda)(\nu-\lambda)}{R(m-l)} = \frac{A_1}{A_2} \left[ 1 + \left\{ \frac{A_3}{A_1} + 2(1-3A_0) \right\} \frac{d^2}{s^2} + \left\{ \frac{2A_3}{A_1} (1-3A_0) - 6A_z + \frac{A_5}{A_1} - 2(1-3A_0) \right\} \frac{d^4}{s^4} \right]
\]

Also \[
m-l = 2 \frac{(\nu-\mu)(\nu-\lambda)}{2s} \left[ 1 + 4 \left( 1-3A_0 \right) \frac{d^2}{s^2} - 12A_z \frac{d^4}{s^4} \right]
\]

\[
k(m-l) = \frac{A_1}{s} \frac{d^2}{s^2} \left[ 1 + 4 \left( 1-3A_0 \right) \frac{d^2}{s^2} - 12A_z \frac{d^4}{s^4} \right]
\]

\[
= \frac{A_1}{s} \frac{d^2}{s^2} \left[ 1 + \left\{ \frac{A_3}{A_1} - 4 \left( 1-3A_0 \right) \right\} \frac{d^2}{s^2} + \left\{ \frac{12A_z}{A_1} + 16 \left( 1-3A_0 \right)^2 + A_5 - 4 \frac{A_3}{A_1} \right\} \frac{d^4}{s^4} \right]
\]

\[
= \frac{A_1}{s} \frac{d^2}{s^2} \left[ 1 + \left\{ \frac{A_3}{A_1} + 2(1-3A_0) \right\} \frac{d^2}{s^2} + \left\{ \frac{2A_3}{A_1} (1-3A_0) - 6A_z + \frac{A_5}{A_1} - 2(1-3A_0) \right\} \frac{d^4}{s^4} \right]
\]

\[
= \frac{A_1}{s} \frac{d^2}{s^2} \left[ 1 + 2 \left( 1-3A_0 \right) \frac{d^2}{s^2} - 2 \left\{ (1-3A_0)^2 + 3A_z \right\} \frac{d^4}{s^4} \right]
\]

\[
= \frac{s}{2A_1} \left[ 1 + 2 \left( 1-3A_0 \right) \frac{d^2}{s^2} - \left( 5-36k_z^2 + 54k_z^4 \right) \frac{d^4}{s^4} \right]
\]

We get therefore

\[
x = \mu = c - st \left[ 1 + 2 \left( 1-3k_z^2 \right) \frac{d^2}{s^2} - \left( 5-36k_z^2 + 54k_z^4 \right) \frac{d^4}{s^4} \right] \tag{40}
\]

Now from equation (14),

\[
M = 2E_1 - E_z - st \left[ 1 + 2 \left( \alpha_1 - \alpha_2 \right) \frac{d^2}{s^2} - \frac{1}{2} \left( \gamma \alpha_1^2 - 20\alpha_1 \alpha_2 + 4\alpha_2^2 \right) \frac{d^4}{s^4} \right]
\]

and \[
\alpha_1 - \alpha_2 = (1-3k_z^2) - \frac{3}{4} (1-2k_z^2)(1-6k_z^2) \frac{d^2}{s^2}
\]

\[
\gamma \alpha_1^2 - 20\alpha_1 \alpha_2 + 4\alpha_2^2 = \gamma \left( 1-4k_z^2 + 4k_z^4 \right) - 20k_z^2 + 40k_z^4 + 4k_z^4
\]

\[
= \gamma - 48k_z^2 + \gamma 2k_z^4
\]

\[
M = 2E_1 - E_z - st \left[ 1 + 2 \left( 1-3k_z^2 \right) \frac{d^2}{s^2} - \left\{ \frac{3}{2} (1-2k_z^2)(1-6k_z^2) + \frac{1}{2} \left( \gamma - 48k_z^2 + \gamma 2k_z^4 \right) \right\} \frac{d^4}{s^4} \right]
\]

\[
M = 2E_1 - E_z - st \left[ 1 + 2 \left( 1-3k_z^2 \right) \frac{d^2}{s^2} - \left( 5-36k_z^2 + 54k_z^4 \right) \frac{d^4}{s^4} \right] \tag{41}
\]
Comparing equations (40) and (41) we see that if we take the arbitrary constants $C$, $\varepsilon_1$, and $\varepsilon_2$ so that
\[ C + \frac{\Pi}{2} = 2\varepsilon_1 - \varepsilon_2 \]
we have, as required $M = \alpha + \frac{\Pi}{2}$.

The series solution has thus been completely identified with a series solution obtained from the elliptic function solution provided that $\varepsilon$ is sufficiently small compared with $s$.

In obtaining the result given in the preceding article it has been assumed that $\varepsilon$ is small compared with $s$. Now let us suppose that $s$ is small compared with $\varepsilon$.

In this case two of the roots of the cubic (2) are given by the equation
\[ \alpha = \frac{1}{2} + \frac{1}{2} \frac{s}{2} \left( \frac{k^2 - \alpha}{\alpha} \right) \]  

These two roots are evidently the two largest roots of the cubic. We proceed to solve equation (42) by successive approximations and find

\[ \gamma = \frac{1}{2} + \frac{1}{2} \frac{1}{2} \left( 1 - 2k^2 \right) \frac{s}{\alpha} + \frac{1}{8} \left( 1 - 2k^2 \right) \left( 1 + 2k^2 \right) \frac{s^2}{\alpha^2} \]

\[ + \frac{1}{32} \left( 1 - 2k^2 \right) \left( 1 + 4k^2 + 20k^4 \right) \frac{s^3}{\alpha^3} + \frac{1}{4} \left( 1 - 2k^2 \right) \frac{k^2 s^4}{\alpha^4} \]  

\[ \mu = \frac{1}{2} - \frac{1}{2} \frac{1}{2} \left( 1 - 2k^2 \right) \frac{s}{\alpha} + \frac{1}{8} \left( 1 - 2k^2 \right) \left( 1 + 2k^2 \right) \frac{s^2}{\alpha^2} \]

\[ - \frac{1}{32} \left( 1 - 2k^2 \right) \left( 1 + 4k^2 + 20k^4 \right) \frac{s^3}{\alpha^3} + \frac{1}{4} \left( 1 - 2k^2 \right) \frac{k^2 s^4}{\alpha^4} \]  

The remaining root is therefore
\[ \lambda = \frac{k^2 s^2}{\alpha^2} - \frac{1}{2} \left( 1 - 2k^2 \right) \frac{k^2 s^4}{\alpha^4} \]  

while
\[ \gamma = \gamma_0 + \gamma_1 \frac{s}{\alpha} + \gamma_2 \frac{s^2}{\alpha^2} + \gamma_3 \frac{s^3}{\alpha^3} + \gamma_4 \frac{s^4}{\alpha^4} \]  

\[ \mu = \mu_0 + \mu_1 \frac{s}{\alpha} + \mu_2 \frac{s^2}{\alpha^2} - \mu_3 \frac{s^3}{\alpha^3} + \mu_4 \frac{s^4}{\alpha^4} \]  

\[ \lambda = \frac{k^2 s^2}{\alpha^2} - 2 \frac{\mu_4}{\alpha^4} \]
where

\[
\begin{align*}
\beta_0 &= \frac{1}{2} \\
\beta_1 &= \frac{1}{a_0} (1 - 2 k_e^2) \\
\beta_2 &= \frac{1}{6} (1 - 2 k_e^2) (1 + 2 k_e^2) \\
\beta_3 &= \frac{1}{3 a_0} (1 - 2 k_e^2) (1 + 4 k_e^2 + 20 k_e^4) \\
\beta_4 &= \frac{1}{4} k_e^4 (1 - 2 k_e^2)
\end{align*}
\]

Then

\[
\begin{align*}
\rho - \mu &= 2 \left[ \beta_1 \frac{s}{\rho} + \beta_3 \frac{s^3}{\rho^3} \right] \\
\rho - \lambda &= \beta_0 + \beta_1 \frac{s}{\rho} + \left( \beta_2 - k_e^4 \right) \frac{s^2}{\rho^2} + \beta_3 \frac{s^3}{\rho^3} + 3 \beta_4 \frac{s^4}{\rho^4} \\
\rho - k_e &= \frac{2 s}{\rho} \left[ \left( \beta_1 + \beta_3 \right) \frac{s}{\rho} \right] \left[ \beta_0 + \beta_1 \frac{s}{\rho} + \left( \beta_2 - k_e^4 \right) \frac{s^2}{\rho^2} + \beta_3 \frac{s^3}{\rho^3} + 3 \beta_4 \frac{s^4}{\rho^4} \right] \\
&= 2 \beta_0 \beta_1 \frac{s}{\rho} \left[ \frac{1}{2} \frac{\beta_1}{\beta_0} \frac{s}{\rho} + \left( \frac{\beta_3}{\beta_1} + \frac{\beta_2}{\beta_0} - \frac{k_e^4}{\beta_0} \right) \frac{s^2}{\rho^2} + \frac{\beta_3}{\beta_0} \frac{s^3}{\rho^3} \right]
\end{align*}
\]

\[
\begin{align*}
\rho - \alpha &= \frac{2 \beta_0 \beta_1 \frac{s}{\rho}}{2 \beta_0 \beta_1 \frac{s}{\rho}} \left[ \frac{1}{2} \frac{\beta_1}{\beta_0} \frac{s}{\rho} + \left( \frac{\beta_3}{\beta_1} + \frac{\beta_2}{\beta_0} - \frac{k_e^4}{\beta_0} \right) \frac{s^2}{\rho^2} + \frac{\beta_3}{\beta_0} \frac{s^3}{\rho^3} \right] \\
&= \frac{2 \beta_0 \beta_1}{2 \beta_0 \beta_1} \left[ \frac{1}{2} \frac{\beta_1}{\beta_0} \frac{s}{\rho} + \left( \frac{\beta_3}{\beta_1} + \frac{\beta_2}{\beta_0} - \frac{k_e^4}{\beta_0} \right) \frac{s^2}{\rho^2} + \frac{\beta_3}{\beta_0} \frac{s^3}{\rho^3} \right]
\end{align*}
\]

\[
\begin{align*}
2 \beta_0 \beta_1 &= \frac{1}{2 a_0} (1 - 2 k_e^2) \\
\frac{1}{2} \frac{\beta_1}{\beta_0} &= \frac{1}{2 a_0} (1 - 2 k_e^2) \\
4 \frac{\beta_3}{\beta_1} + \frac{4 \beta_2}{\beta_0} - 4 k_e^4 - \frac{\beta_1}{\beta_0} &= \frac{1}{4} \left( 1 + 4 k_e^2 + 20 k_e^4 \right) + \left( 1 - 2 k_e^2 \right) \left( 1 + 2 k_e^2 \right) - 8 k_e^4 \frac{1}{8} (1 - 2 k_e^2)
\end{align*}
\]

\[
\begin{align*}
&= \frac{3}{4} (1 - 2 k_e^2) (1 + 6 k_e^2)
\end{align*}
\]
\[
\frac{B_3^3}{B_0^3} + 12 \frac{B_3}{B_0} - 4 \frac{B_3 B_1}{B_0^2} + 4 \frac{B_3}{B_0} B_1 = \left(1 - 2 \frac{k^2}{k_0^2}\right) \left(\frac{B_0^2}{B_0^2} \right) \left(1 + \frac{1}{2} \frac{k^2}{k_0^2} + \frac{3}{4} \left(1 + 4 \frac{k^2}{k_0^2} + 116 \frac{k^2}{k_0^2}\right)ight)
\]

\[
= \left(1 - 2 \frac{k^2}{k_0^2}\right) \left(1 + \frac{1}{2} \frac{k^2}{k_0^2} + \frac{3}{4} \left(1 + 4 \frac{k^2}{k_0^2} + 116 \frac{k^2}{k_0^2}\right)\right)
\]

\[
\therefore (y - y') (y - y) = C_1 \left(\frac{y}{y_0}\right)^{\frac{3}{2}} + C_2 \left(\frac{y}{y_0}\right)^{\frac{3}{2}} + C_3 \left(\frac{y}{y_0}\right)^{\frac{5}{2}} + C_4 \left(\frac{y}{y_0}\right)^{\frac{7}{2}} - - - (4.7)
\]

\[
\text{where}
\]

\[
C_1 = \frac{1}{\sqrt{v}} \left(1 - 2 \frac{k^2}{k_0^2}\right) + \frac{C_2}{\sqrt{v}}
\]

\[
C_2 = \frac{1}{2 \sqrt{v} \sqrt{y_0}} \left(1 - 2 \frac{k^2}{k_0^2}\right)^{\frac{3}{2}}
\]

\[
C_3 = \frac{3}{2 \sqrt{v} \sqrt{y_0}^3} \left(1 - 2 \frac{k^2}{k_0^2}\right)^{\frac{5}{2}} (1 + 6 \frac{k^2}{k_0^2})
\]

\[
C_4 = \frac{1}{6 \sqrt{v} \sqrt{y_0}^4} \left(1 - 2 \frac{k^2}{k_0^2}\right)^{\frac{7}{2}} (1 + 4 \frac{k^2}{k_0^2} + 116 \frac{k^2}{k_0^2})
\]

Also

\[
2y - \mu - \lambda = B_3 + 3 B_1 \frac{S}{d} + (B_3 - k_0^2 y_0) \frac{S^2}{d^2} + 3 B_3 \frac{S^3}{d^3} + 3 B_4 \frac{S^4}{d^4}
\]

\[
\therefore (y - y') (y - y) = B_3 - 2 C_1 \left(\frac{y}{y_0}\right)^{\frac{3}{2}} + 3 B_1 \frac{S}{d} - 2 C_2 \left(\frac{y}{y_0}\right)^{\frac{3}{2}} + (B_3 - k_0^2 y_0) \frac{S^2}{d^2} - 2 C_3 \left(\frac{y}{y_0}\right)^{\frac{5}{2}} + 3 B_3 \frac{S^3}{d^3} - 2 C_4 \left(\frac{y}{y_0}\right)^{\frac{7}{2}} + 3 B_4 \frac{S^4}{d^4}
\]

Also

\[
\mu - \lambda = B_3 - B_1 \frac{S}{d} + (B_3 - k_0^2 y_0) \frac{S^2}{d^2} - B_3 \frac{S^3}{d^3} + 3 B_4 \frac{S^4}{d^4}
\]

\[\text{naglating powers of \( \frac{S}{d} \) beyond the second.}\]

\[
K = \left[1 - 4 C_1 \left(\frac{y}{y_0}\right)^{\frac{3}{2}} + 6 B_1 \frac{S}{d} - 4 C_2 \left(\frac{y}{y_0}\right)^{\frac{3}{2}} + 2 (B_3 - k_0^2 y_0) \frac{S^2}{d^2} \right] \left[1 + 2 B_1 \frac{S}{d} + (4 B_3 - 2 B_1 + 2 k_0^2 y_0) \frac{S^2}{d^2}\right]
\]

\[
= 1 - 4 C_1 \left(\frac{y}{y_0}\right)^{\frac{3}{2}} + 8 B_1 \frac{S}{d} - (4 C_2 + 6 B_3) \left(\frac{y}{y_0}\right)^{\frac{3}{2}} + 16 B_3 \frac{S^2}{d^2}
\]

\[
k = 1 - 8 C_1 \left(\frac{y}{y_0}\right)^{\frac{3}{2}} + (16 B_1 + 16 C_2) \frac{S}{d} - (8 C_2 + 4 B_3) \left(\frac{y}{y_0}\right)^{\frac{3}{2}} + 32 C_2 \frac{S^2}{d^2} + 64 B_3 \frac{S^2}{d^2}
\]

\[
k = 1 - k^2 = 8 C_1 \left(\frac{y}{y_0}\right)^{\frac{3}{2}} - (16 B_1 + 16 C_2) \frac{S}{d} + (8 C_2 + 4 B_3) \left(\frac{y}{y_0}\right)^{\frac{3}{2}} - (96 B_1^2 + 32 C_2 \frac{S^2}{d^2} + 64 B_3 \frac{S^2}{d^2})
\]

\[
k = 8 C_1 \left(\frac{y}{y_0}\right)^{\frac{3}{2}} \left[1 - \left(\frac{2 B_1 + 2 C_1}{C_1}\right) \left(\frac{y}{y_0}\right)^{\frac{3}{2}} + \left(\frac{C_2 + 6 B_1}{C_1}\right) \frac{S}{d} - (4 B_1 + 4 C_2 + 8 B_3) \left(\frac{y}{y_0}\right)^{\frac{3}{2}}\right]
\]

\[
k' = 8 C_1 \left(\frac{y}{y_0}\right)^{\frac{3}{2}} \left[1 - \left(\frac{2 B_1 + 2 C_1}{C_1}\right) \left(\frac{y}{y_0}\right)^{\frac{3}{2}} + \frac{1}{8} \left(\frac{2 C_2 + 6 B_1 - 3 B_3 - 3 C_2}{C_1}\right) \frac{S}{d}\right]
\]
\[ q^2 = \frac{1}{2} \left[ 1 - \frac{4}{5} \delta C_1 \left( \frac{s}{2} \right)^{1/8} + \frac{1}{2} \delta C_1 \left( \frac{16}{5} C_1 + C_2 \right) \right] \left[ 1 - \frac{1}{5} \delta C_1 \left( \frac{s}{2} \right)^{1/8} + \frac{1}{2} \delta C_1 \left( \frac{s}{2} \right)^{1/4} \right] \]

\[ q^{1/4} = \frac{1}{\sqrt[4]{2}} \left[ 1 - \frac{1}{2} \delta C_1 \left( \frac{s}{2} \right)^{1/8} + \frac{1}{8} \delta C_1 \left( \frac{s}{2} \right)^{1/4} \right] \]

\[ q^{1/2} = \frac{1}{\sqrt[2]{2}} \left[ 1 - \frac{1}{2} \delta C_1 \left( \frac{s}{2} \right)^{1/8} + \frac{1}{2} \delta C_1 \left( \frac{s}{2} \right)^{1/4} \right] \]

Since \( k = 1 - \frac{1}{4} C_1 \left( \frac{s}{2} \right)^{1/2} + 8 P_1 \frac{s}{2} \),

we have the approximation considered

\[ k^{1/2} q^{1/4} = \frac{1}{\sqrt[4]{2}} \left[ 1 - \frac{1}{2} \delta C_1 \left( \frac{s}{2} \right)^{1/8} + \frac{1}{8} \delta C_1 \left( \frac{s}{2} \right)^{1/4} \right] \]

\[ k q^{1/2} = \frac{1}{\sqrt[2]{2}} \left[ 1 - \frac{1}{2} \delta C_1 \left( \frac{s}{2} \right)^{1/8} + \frac{1}{2} \delta C_1 \left( \frac{s}{2} \right)^{1/4} \right] \]

Also,

\[ m = \frac{1}{2} \left( y - \frac{1}{2} \left( y - \bar{y} \right) \left( y - \bar{y} \right) \right) = P_0 + C_1 \left( \frac{s}{2} \right)^{1/2} + P_1 \frac{s}{2} + C_2 \left( \frac{s}{2} \right)^{3/2} + P_2 \frac{s}{2}^2 + C_3 \left( \frac{s}{2} \right)^{5/2} + P_3 \frac{s}{2}^3 + C_4 \left( \frac{s}{2} \right)^{7/2} + P_4 \frac{s}{2}^4 \]

\[ l = y - \frac{1}{2} \left( y - \bar{y} \right) \left( y - \bar{y} \right) = P_0 - C_1 \left( \frac{s}{2} \right)^{1/2} + P_1 \frac{s}{2} - C_2 \left( \frac{s}{2} \right)^{3/2} + P_2 \frac{s}{2}^2 - C_3 \left( \frac{s}{2} \right)^{5/2} + P_3 \frac{s}{2}^3 - C_4 \left( \frac{s}{2} \right)^{7/2} + P_4 \frac{s}{2}^4 \]

\[ m - l = 2 \frac{1}{2} \left( y - \bar{y} \right) \left( y - \bar{y} \right) = 2 C_1 \left( \frac{s}{2} \right)^{1/2} + 2 C_2 \left( \frac{s}{2} \right)^{3/2} + 2 C_3 \left( \frac{s}{2} \right)^{5/2} + 2 C_4 \left( \frac{s}{2} \right)^{7/2} \]

Finally, substituting these values in equation (25) we obtain

\[ q^2 = P_0 - C_1 \left( \frac{s}{2} \right)^{1/2} + P_1 \frac{s}{2} - C_2 \left( \frac{s}{2} \right)^{3/2} + P_2 \frac{s}{2}^2 - C_3 \left( \frac{s}{2} \right)^{5/2} + P_3 \frac{s}{2}^3 - C_4 \left( \frac{s}{2} \right)^{7/2} + P_4 \frac{s}{2}^4 \]

\[ + 2 \left[ C_1 \left( \frac{s}{2} \right)^{1/2} + C_2 \left( \frac{s}{2} \right)^{3/2} + \ldots \right] \left[ \frac{1}{2} \left( \frac{1}{2} \delta C_1 \left( \frac{s}{2} \right)^{1/8} + \frac{1}{8} \delta C_1 \left( \frac{s}{2} \right)^{1/4} \right) \right] \]

\[ + \frac{1}{2} \left[ 1 - \frac{1}{2} \delta C_1 \left( \frac{s}{2} \right)^{1/8} + \frac{1}{2} \delta C_1 \left( \frac{s}{2} \right)^{1/4} \right] \left( 1 - \cos 2 \alpha \right) \]

\[ q^2 = \frac{1}{2} + D_1 \left( \frac{s}{2} \right)^{1/2} + D_2 \left( \frac{s}{2} \right)^{3/2} + D_3 \left( \frac{s}{2} \right)^{5/2} + D_4 \left( \frac{s}{2} \right)^{7/2} + D_5 \frac{s}{2}^3 + \ldots \]

where \( D_1, D_2, D_3, \ldots \) are certain coefficients consisting of infinite series of terms depending on \( k \) and \( x \).

We have in fact:

\[ \ldots \]

\[ D_1 = -C_1 + \frac{4}{3} C_1 \sin x - 2 \sqrt[3]{2} C_1 (1 - \cos 2x) + \ldots \]
\[ D_2 = -2 \sqrt[3]{2} C_1^{5/4} \left[ \sin x - \sqrt[3]{2} (1 - \cos 2x) + \ldots \right] \]

and so on.

A series of the form (54) will therefore represent \( g_1 \) in that part of the permissible area in which the series (39) is divergent.

Whether the series for \( D_1, D_2 \) are convergent is of great immediate importance. The interesting point is that when the roots of equation (27) are expressible in series of positive powers of \( x \), we obtain from the elliptic function solution series for the coordinates of an entirely different form from the original series solution.

12. To summarize the results we have obtained:

(i) We have shown that the elliptic function solution may be expressed in the form of series of powers of \( \sqrt{\frac{x}{A}} \), the coefficients in this series being functions of the roots of the cubic (26) and trigonometric functions of the quantity \( \xi \).

(ii) The roots of the cubic (26) may be expressed in the form of series of positive integral powers of \( \xi \), provided that \( \xi \) is sufficiently large in comparison with \( \sqrt{\frac{x}{A}} \); in this case the series (39) obtained from the elliptic function solution reduces to the original series solution.

(iii) On the other hand, if \( \xi \) is small compared with \( \sqrt{\frac{x}{A}} \), the roots of the cubic may be expressed in the form of series of positive powers of \( \xi \) and the series obtained from the elliptic function solution reduces to the form (54).

It appears therefore that the convergence or divergence of the series solution depends directly on the convergence or divergence of the series of positive powers of \( \xi \) which express the roots of the cubic (26). This suggests that the divergence of the series solution does not imply any discontinuity in the system but merely expresses the fact that this form of series is not capable of representing the coordinates over the whole range of permissible values of \( S \) and \( \xi \).

This particular point is treated latter in some detail (see Chapter 4).

But I may here anticipate the results here obtained by mentioning that for values of \( S \) and \( \xi \) for which the series solution just ceases to be convergent no characteristic features of the orbit can be discovered. This therefore confirms the suggestion here put forward.
that there is no discontinuity in the system in this region.

A fundamental difference is thus to be observed between the series solution and the series (25) obtained from the elliptic function solution. This latter series is divergent when $k = 1$, i.e., when $e = 0$ and this divergence does imply a discontinuity in the system. The orbit for such points is, as we have shown in Sec. 8, Case 1, p. 42, of the form known as asymptotic, i.e., an orbit which after an infinite lapse of time becomes a simple harmonic motion in one of the principal vibrations.

This result, together with the fact that the line $e = 2g$ appears doubled in the form of the discriminant (17), suggests that the double line may be the limiting case of a belt across the region of real solutions, such a belt occurring when other smaller terms are taken in the original form of $H$. It is suggested that for values of $e$ and $g$ in such a belt the solutions may be real but unstable. It would therefore be of great interest to try to choose another form of $H$ where such a region may occur and to determine the form of the orbits in this region.


To determine the limiting values of $e$ and $g$ for which the series solution is convergent, we observe first that the form (56) will be derived from the expansion (25) whenever the roots of the cubic (26) are expressible in the form of infinite series of positive powers of $\lambda$; and that the form (54) will be derived when the roots are expressible in the form of infinite series of positive powers of $\zeta$.

Furthermore, it is apparent that at least one of the roots of the cubic (26) is an even function of $\lambda$, this root being obtained from equation (27) when it is put in the form

$$x = \frac{s^2}{4 \alpha^2} \left( \frac{1 - \beta^2}{1 - \lambda^2} \right)^2$$

by solving it by successive approximations.

Suppose first that $x > \frac{1}{2}$; equation (55) can then be written

$$x = \frac{s^2}{4 \alpha^2} \left( \frac{1 - \frac{\beta^2}{x^2}}{1 - \frac{1}{x^2}} \right)^2.$$
\[ x = \frac{5}{4x^2} \left(1 - \frac{2k^2 + k_4}{4x^2} \right) \left(1 + \frac{1}{2x^2} + \frac{3}{4x^2} + \cdots \right) \quad (56) \]

We therefore obtain for this root an infinite series of positive powers of \( \frac{1}{x} \) and \( q_2 \) will therefore reduce to the form (38).

On the other hand suppose \( x < \frac{1}{2} \), then equation (56) will take the form

\[ x = \frac{5}{4x^2} \frac{(k_4 - x)^2}{(1 - 2x)^2} \]

or

\[ x = \frac{5}{4x^2} (k_4 - 2k_2 x + x^2) \left(1 + 4x + 12x^2 + \cdots \right) \quad (57) \]

On solving equation (57) by successive approximations we shall obtain an infinite series of positive powers of \( x \), and \( q_2 \) will reduce to the form (54).

Now it has been shown (p. 23) that, in the permissible area, one and only one root of the cubic (26) is greater than \( \frac{1}{2} \); so that when equation (58) reduces to the form (56) the root so obtained must be the greatest root of the cubic.

Thus a necessary condition that \( q_2 \) shall be represented by a series of the form (38) is that the greatest root of the cubic shall be an even function of \( x \).

The condition is also sufficient; for suppose the greatest root of the cubic to be an even function of \( x \).

This root must be given either by equation (55) or by equation (43) or by equation (28).

If it is given by equation (55), since it is \( x \), it will be given by a series of positive powers of \( x \), and therefore \( q_2 \) will reduce to the form (38).

If however the greatest root is given by either of equations (42) or (28), another root of the cubic must be obtained by changing the sign of \( x \). But we have supposed this root to be an even function of \( x \) and therefore in this case we must have two roots equal and greater than \( \frac{1}{2} \), which is impossible.

Thus if the greatest root is an even function of \( x \) it must be given by equation (55).

A necessary and sufficient condition for the convergence of the series solution is therefore that the greatest of the roots of the
cubic (26) shall be an even function of $\theta$.

This condition, though of interest theoretically, does not lend itself to the determination of the boundary of the region of convergence of the series solution. It will be seen however, that it does determine a certain range of values of $\delta$ and $\gamma$ for which the series (35) will be convergent.

In certain simple cases, this range of values of $\delta$ and $\gamma$ may be readily determined.

(i) On the double line, $\delta = 2\theta$ outside the points of contact with the discriminant curve, the roots of the cubic are

$$\lambda = \frac{1}{2}, \quad \mu = \frac{1}{2}, \quad \nu = \frac{S^2}{4\delta^2} > \frac{1}{2}$$

Thus for all points along this part of the boundary of the permissible area, the condition given in § 13 is satisfied.

Thus the series solution is convergent for all values of $\delta$ and $\gamma$ on the double line outside the points of contact with the discriminant curve.

(ii) When $\gamma = 0$, i.e. $k^2 = 0$, the cubic (26) reduces to

$$\lambda \left\{ 4x^2 \lambda^2 - (4\delta^2 + 3x^2) \lambda + 2x^2 \right\} = 0$$

The roots are therefore given by

$$\lambda = \frac{4x^2 + 3x^2 + 5 \sqrt{8x^2 + 8^2}}{8x^2}$$

$$\mu = \frac{4x^2 + 3x^2 - 5 \sqrt{8x^2 + 8^2}}{8x^2}$$

$$\lambda = 0$$

If therefore $S^2 > 8x^2$ the series obtained will be

$$\nu = \frac{S^2}{4x^2} \left\{ 1 + \frac{4x^2}{5^2} - \frac{4x^4}{5^4} + \frac{16x^6}{5^6} - \frac{80x^8}{5^8} + \frac{448x^{10}}{5^{10}} - \frac{2680x^{12}}{5^{12}} + \ldots \right\}$$

$$\mu = \frac{S^2}{5^2} \left\{ 1 - \frac{4x^2}{5^2} + \frac{20x^4}{5^4} - \frac{112x^6}{5^6} + \frac{672x^8}{5^8} - \ldots \right\}$$

$$\lambda = 0$$
It may be verified that, just as in the more general case treated previously, the expression (25) then reduces to

\[ q = \left( \frac{x^2}{25} - \frac{2x^4}{54} \right) (1 + \cos \theta) + \cdots \]

while the series solution in this case reduces to

\[ q = \left( \frac{x^2}{25} - \frac{2x^4}{54} \right) \left( 1 + \cos \frac{2\theta}{2} \right) \]

and it may be verified as before that \( \lambda = \sqrt{\frac{x^2}{25}} + \frac{2\theta}{2} \). It is to be noted that in this case \( x \) is an even function of \( \frac{x}{2} \).

If however, \( x^2 < 8\pi^2 \), \( x \) will be an odd function of \( \frac{x}{2} \), and we obtain

\[ \frac{x^2}{2} + \frac{x^4}{5} \frac{5^2}{2^2} + \frac{x^6}{16} \frac{5^3}{3^2} - \frac{x^8}{512} \frac{5^5}{5!} + \cdots \]

\[ x = \frac{x^2}{2} - \frac{x^4}{5} \frac{5^2}{2^2} + \frac{x^6}{16} \frac{5^3}{3^2} - \frac{x^8}{512} \frac{5^5}{5!} + \cdots \]

\[ \lambda = 0. \]

In this case it may be verified that we reach (25) reduces to the form (53).

Thus when \( \theta = 0 \), the demarcation between the two forms of series solution to which (25) may be reduced, occurs when

\[ x^2 = 8\pi^2 \text{ or } x = \pm 2\sqrt{2} \pi. \]
To determine the boundary of the region of convergence in the general case, we proceed as follows:

Suppose the roots of the cubic (26) to be diminished by $\lambda$ the smallest root. The cubic then becomes

$$x^3 - (\nu + \mu - 2\lambda)x^2 + (\nu - \lambda)(\mu - \lambda) = 0$$

having roots

$$x = 0$$

$$x = \nu + \mu - 2\lambda \pm 2\sqrt{(\nu + \mu - 2\lambda)^2 - 4(\nu - \lambda)(\mu - \lambda)}$$

We thus get the following relations between the roots of the original cubic (26)

$$y = \frac{x^3}{8x^2} + \frac{\nu - \lambda}{2} + \frac{x}{2}, \quad \frac{x^4}{16x^4} + \frac{x^2}{2} \left( \frac{1 + \lambda}{2} - 2x^2 \right) + 2\lambda - 3\lambda^2$$

These will develop into series of positive powers of $x$ if, and only if $x^4/16x^4$ is the dominant term under the radical, i.e. if

$$\frac{x^4}{16x^4} > \left. \frac{x^2}{2} \left( \frac{1 + \lambda}{2} - 2x^2 \right) + 2\lambda - 3\lambda^2 \right|$$

or

$$\left. \frac{x^2}{2} \left( 1 + \lambda - 4x^2 \right) + \frac{16x^4}{16x^4} (2\lambda - 3\lambda^2) \right| < 1$$

On the other hand if

$$\left. \frac{x^2}{2} \left( 1 + \lambda - 4x^2 \right) + \frac{16x^4}{16x^4} (2\lambda - 3\lambda^2) \right| > 1$$

the roots will be given as series of positive powers of $x$.

These conditions will therefore define a certain range of values of $\nu$ and $\mu$ for which the series solution will be convergent. The boundary between the two forms (58) and (54) to which the series (25) will reduce occurs when
\[ \left[ \frac{8x^2}{4} \left( 1 + 4x^2 \right) + 16x^4 \left( 2x^2 - 3x^2 \right) \right] = 1 \quad (61) \]

By eliminating \( \lambda \) between equation (61) and the equation of the cubic (26), which is satisfied by \( \lambda \), the equation of the boundary curve in terms of \( s \) and \( q \) may be obtained, as is done below, see §16, pages 73-76.

Owing to the complicated form of the equation of this boundary curve it is simpler in practice however to take trial values of \( s \) and \( q \) and solve the resulting cubic equation and test whether condition (57) is satisfied.

Proceeding in this way the boundary curve is found to consist of that part of the double line which lies between its points of contact with the curved branches of the discriminant curve together with a curved part. This curved part approaches very closely indeed to parts of the two parabolas

\[ \frac{S^2}{x^2} - 2S \frac{q_s}{x} + 4 \frac{q_s^2}{x^2} = 0 \]

and

\[ \frac{S^2}{x^2} + 2S \frac{q_s}{x} - 4 \frac{q_s^2}{x^2} = 0 \]

The approximate values so obtained are as follows:

<table>
<thead>
<tr>
<th>( s )</th>
<th>( q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.042</td>
<td>0.017</td>
</tr>
<tr>
<td>0.102</td>
<td>0.032</td>
</tr>
<tr>
<td>0.164</td>
<td>0.034</td>
</tr>
<tr>
<td>0.200</td>
<td>0.030</td>
</tr>
<tr>
<td>0.236</td>
<td>0.020</td>
</tr>
<tr>
<td>0.261</td>
<td>0.010</td>
</tr>
<tr>
<td>0.283</td>
<td>0.000</td>
</tr>
<tr>
<td>0.291</td>
<td>-0.050</td>
</tr>
</tbody>
</table>

The corresponding part of the curve for negative values of \( s \) is obtained by changing the signs of both \( s \) and \( q \).

The curve so defined is shown in figure 2 (page 22).

For values of \( s \) and \( q \) corresponding to points on that side of this curve which are situated from the origin, and which lie in the permissible area, the series solution (50) will be convergent.
for other points in the permissible area, the series solution will be divergent and the elliptic function solution will reduce to the form (54).

The discussion has been concerned only with the coordinate \( q_1 \)
but it is seen also to apply to the coordinate \( q_1 \), since it is related to \( q_2 \) by the equation

\[ q_1^2 + 2q_2 = 1. \]

The reduction of the coordinates \( p_1 \) and \( p_2 \) would be more difficult on account of the complicated form of their expression in terms of elliptic functions. There is however little doubt that agreement could be obtained in the same manner and that the convergence of the series for \( p_1 \) and \( p_2 \) will depend on the convergence of series representing the roots of the cubic (26).
CHAPTER 4

Orbits at the boundary of the region of convergence.
We have already seen in Chapter 2 that the series in terms of
elliptic functions represents the solution of the problem throughout
the region of values of \( S \) and \( G \) where a real solution exists except
for values of \( S \) and \( G \) such that \( S = 2G \) and \( n = 1 \). The series solution
however only represents the solution so long as the inequality \( (5) \) is
satisfied. It is of importance therefore to investigate what discontinuities,
if any, of the system correspond to values of \( S \) and \( G \) which lie on
the boundary of the region of convergence. The present Chapter is
concerned primarily with showing that for such values of \( S \) and \( G \)
no such discontinuity of the system exists, thus confirming the
suggestions made in §12, pp 64–65.

The first step is to obtain the analytical relation between \( S \) and \( G \)
concerning to fixed points on the boundary of the region of
convergence. For such points the quantity under the modulus on
the left-hand side of equation 6 must be equal either to +1
or to −1.

\[ \frac{S}{S^2} (1 + \lambda - 2k^2) + 16k^4 (2\lambda - 3\lambda^2) = -1 \]

\[ \frac{S}{16k^4} + \frac{S}{2k^2} (1 + \lambda - 2k^2) + 2\lambda - 3\lambda^2 = 0 \]

Thus the expression under the radical in the expression \((5)A\)
for \( V \) and \( u \) is zero

\[ V = \mu = \frac{S}{2k^2} + \lambda - \lambda^2 \]

But since \( V = \mu \) and \( V \) is the greatest root, they must both equal \( z \)
for any real solution

\[ V = \mu = \frac{z}{2} \]

and therefore \( \lambda = \frac{S}{4k^2} \)

The locus corresponding to these values of \( \lambda \), \( \mu \), \( V \) is as we have seen
(§ 27) the part of the double line, \( S = 2G \), between the points of
contact with the curves branches of the discriminant curve.

This alternative therefore gives the part of the double line
as a boundary of the region of convergence.
(ii) Suppose \[ \frac{8k^2}{s^2} (1 + \lambda - 4k^2) + 16\lambda^2 (2\lambda - 3k^2) = +1 \]

\[ \therefore \frac{s^4}{16k^4} = \frac{s^2}{\lambda^2} \left( \frac{1 + \lambda - 2k^2}{2} + 2\lambda - 3k^2 \right) \quad (62) \]

Putting this value in equations (58A) we obtain

\[ \gamma = \frac{s^2}{\lambda^2} \left( 1 + \lambda - \frac{1}{2} \right) + \frac{s^2}{4\sqrt[2]{x^2}} \]

\[ \mu = \frac{s^2}{\lambda^2} \left( 1 + \lambda - \frac{1}{2} \right) - \frac{s^2}{4\sqrt[2]{x^2}} \]

\[ \therefore \gamma - \mu = \frac{s^2}{2\sqrt[2]{x^2}} \quad (63) \]

To obtain the equation of the boundary it is simplest to eliminate \( \lambda \) from equation (62) and the equation of the cubic (26) which is satisfied by \( \lambda \).

Writing

\[ A = 1 + \frac{s^2}{4\lambda^2} \]

\[ B = \frac{s^4}{16\lambda^4} - \frac{s^2}{2\lambda^2} + \frac{2\lambda^3}{\lambda^2} \]

\[ C = \frac{1}{4} + \frac{\lambda^2}{\lambda^2} \]

\[ D = \frac{9\lambda^2}{4\lambda^2} \]

these equations may be written

\[ \begin{align*}
3\lambda^2 - 2A\lambda + B &= 0 \\
\lambda^3 - A\lambda^2 + C\lambda - D &= 0
\end{align*} \]

Using Sylvester's Analytic method of elimination these equations give

\[
\begin{vmatrix}
1 & -A & C & -D & 0 \\
0 & 1 & -A & C & -D \\
3 & -2A & B & 0 & 0 \\
0 & 3 & -2A & B & 0 \\
0 & 0 & 3 & -2A & B \\
\end{vmatrix} = 0
\]

\[
\begin{vmatrix}
1 & -A & C & -D \\
A & B - 3C & 3D & 0 \\
3 & -2A & B & 0 \\
0 & 3 & -2A & B \\
\end{vmatrix} = 0
\]
\[ (B + A^2 - 3C, 3D - AC, AD) = 0 \]

on expanding
\[ (B + A^2 - 3C)(B^2 - 3BC + 6AD) + A(-2A^2D - 3BD + ABC) + 27D^2 - 3ABD = 0 \]

which may be arranged as
\[ 27D^2 + 2AD(2A^2 - 9C) + B^2(18 + A^2 - 4B) + 13C(9C - 2A^2) = 0 \]

From equations (64) we have
\[ B + A^2 - 4C = \frac{s^4}{\delta^2 \alpha^4} \]

and our equation may be arranged as
\[ 27D^2 - 2AD(2B + C) + B^2C + (B + A^2 - 4C)(B^2 - 2BC + 4AD) = 0 \]

which on substituting from equations (64) gives
\[
\begin{align*}
27 & \left[ \frac{9}{16} \frac{s^4}{\alpha^4} - \frac{9}{2} \frac{s^4}{\delta^2 \alpha^4} \right] + \frac{9}{4} \left[ \frac{1}{64} \frac{s^6}{\alpha^4} - \frac{1}{8} \frac{s^4}{\delta^2 \alpha^4} + \frac{1}{16} \frac{s^4}{\delta^2 \alpha^4} \right] + \frac{9}{2} \left[ \frac{1}{16} \frac{s^6}{\alpha^4} - \frac{1}{4} \frac{s^4}{\delta^2 \alpha^4} + \frac{1}{4} \frac{s^4}{\delta^2 \alpha^4} \right] + \frac{9}{2} \left[ \frac{1}{16} \frac{s^6}{\alpha^4} - \frac{1}{4} \frac{s^4}{\delta^2 \alpha^4} + \frac{1}{4} \frac{s^4}{\delta^2 \alpha^4} \right] = 0
\end{align*}
\]

or rearranging and multiplying through by 16 gives
\[
\begin{align*}
\frac{1}{128} \frac{s^{12}}{\alpha^{12}} & \left[ \frac{s^8}{\alpha^8} - \frac{1}{8} \frac{s^{10}}{\alpha^{10}} + \frac{1}{16} \frac{s^8}{\alpha^8} + \frac{1}{2} \frac{s^6}{\alpha^6} + \frac{1}{16} \frac{s^4}{\alpha^4} - \frac{1}{2} \frac{s^2}{\alpha^2} \right] \\
+ \frac{9}{2} \left[ \frac{3}{8} \frac{s^6}{\alpha^6} - \frac{3}{8} \frac{s^6}{\alpha^6} - \frac{7}{4} \frac{s^8}{\alpha^8} - \frac{2}{3} \frac{s^6}{\alpha^6} + \frac{2}{3} \frac{s^4}{\alpha^4} \right] + \frac{9}{2} \left[ \frac{3}{2} \frac{s^6}{\alpha^6} - \frac{3}{2} \frac{s^6}{\alpha^6} - \frac{3}{2} \frac{s^4}{\alpha^4} + \frac{3}{2} \frac{s^4}{\alpha^4} + \frac{3}{2} \frac{s^4}{\alpha^4} \right] = 0
\end{align*}
\]

(65)

When \(q = 0\) this may be written
\[
\frac{1}{128} \frac{s^2}{\alpha^2} \left( s^8 - s^8 - s^8 + s^6 + s^4 + s^2 \right) = 0
\]

and in this case we therefore get the two real values
\[ s = 0 \] and \( s = \pm 2 \frac{s^2}{\alpha^2} \) in agreement with previous results.
In the more general case no simple factors of this expression have been discovered.

Owing to the complicated form of equation (6.5) which gives the relation between \( q \) and \( p \) for points on the curved part of the boundary of the region of convergence, it seems almost impossible to discuss analytically the particular characteristics of any of such orbits in the general case. 

To proceed, however, to investigate such special cases another general result may be given.

**Envelope of the Orbit.**

In the numerical cases which have been considered already it will be observed that the whole orbit is contained in a certain bounded area of the plane of \( q_1 \) and \( q_2 \), and moreover, from an examination of the plotted curves it is seen that the boundary curve consists approximately of three parabolas (see figures 1 and 3). It seems worth while to obtain the analytical expressions for these boundaries of the orbit. The determination of these boundaries will provide a check on the numerical work and moreover it is conceivable that for points on the boundary of the region of convergence, these boundary curves of the orbit may exhibit some peculiarities. We may refer to these boundary curves, for want of a better name, as the "envelope of the orbit."

It will be shown below that in the general case the "envelope of the orbit" consists exactly of three parabolas, the coefficients in the equations of these parabolas being functions of the roots of the cubic (26). It is further shown that the envelopes of the orbits possess no essential peculiarities for values of \( q \) and \( p \) on the boundary of the region of convergence.

To find the equation of the envelope of the orbit we proceed as follows.\(^*\)

The position of the particle at any moment is given by equations (1) viz.:

\[ q_1 = \frac{\sqrt{2}}{s_1} \cos \beta_1; \quad q_2 = \frac{\sqrt{2}}{s_2} \cos \beta_2. \]

\(^*\) Note. The method employed is essentially the same as that used by Betti (Archives Scientifiques de la Soc. Roy. de Nat. 3 (1910) 23-24).
At any particular instant $p_1, p_2, q_1, q_2$ have a certain value which may be determined. Suppose we write

$$2p_1 - p_2 = \varphi$$

then these equations become

$$q_1 = \sqrt{\frac{2p_1}{s_1}} \cos p_1, \quad q_2 = \sqrt{\frac{2p_2}{s_2}} \cos (2p_1 - \varphi) \quad (66)$$

where $q_1, q_2, p_1,$ and $p_2$ are determined at any instant.

Now suppose we keep $q_1, q_2,$ and $\varphi$ constant and vary $p_1,$ then the particle would describe a certain Lissajous figure which would be the curve instantaneously described by $p_1$ and $p_2$ at that moment. We may call such a curve an "osculatory curve," using a term employed by Poncelet in a similar connection (Deux de l'Art. 51 p. 90).

If we find the envelope of all such osculatory curves, we shall obtain the envelope of the orbit which we require.

Suppose we write $q_1 = \varphi$. By equation (6), $q_2 = \frac{1 - \varphi}{2}$

From equations (66)

$$\sqrt{\frac{2}{s_1}} \cos p_1 = \frac{q_1}{s_1} \quad (say) \quad (67)$$

$$\frac{1 - \varphi}{2} \cos (2p_1 - \varphi) = \frac{q_2}{s_2} \quad \sqrt{\frac{2}{s_2}} \quad (say) \quad (68)$$

Also from equation (5)

$$h = 1 - \varphi = s_1 q_1 + s_2 q_2 + l q_1 q_2 \cos \varphi$$

$$\therefore \quad \sqrt{\frac{2}{s_2}} \cos \varphi = \frac{s - 2\varphi}{s_2 \sqrt{2}} \quad \frac{s}{s_2 \sqrt{2}} \quad \delta$$

Write $k = \frac{s - 2\varphi}{s_2 \sqrt{2}}$ and $\delta = \frac{s}{s_2 \sqrt{2}} \quad (69)$

Then

$$\sqrt{\frac{2}{s_2}} \cos \varphi = k - \delta \quad (70)$$

We have then to eliminate $\varphi$ and $p_1$ from equations (67) (68) and (69).

Write $k = k - \delta \delta$ in equation (70) so that
\[ \int \frac{dt}{t-3} \cos y = K \]

From equation (68)

\[ \int \frac{dt}{t-3} \left( \cos 2z \cos y + \cos 2z \sin y \right) = 2y \]

\[ \int \frac{dt}{t-3} \left[ \frac{2 \cos^2 \frac{t}{2} - 1}{(t-3)^2} + \frac{2 \cos \frac{t}{2}}{(t-3)} \right] = 2y \]

\[ 2 \cos \left( \frac{t}{2} - \frac{\pi}{2} \right) \left( \frac{t}{2} - \frac{\pi}{2} \right) - K^2 = 2y \frac{t}{2} - K \left( 2 \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) \]

(i) squaring \[ 4 \cos^2 \left( \frac{t}{2} - \frac{\pi}{2} \right) \left( \frac{t}{2} - \frac{\pi}{2} \right) - K^2 = 4y^2 + K^2 \left( 2 \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) - 4y^2 \cos^2 \left( \frac{t}{2} - \frac{\pi}{2} \right) \]

or simplifying \[ 4 \left( \cos^2 \frac{t}{2} + \cos^2 \left( \frac{t}{2} - \frac{\pi}{2} \right) \right) + 5 \left( y - K \cos^2 \left( \frac{t}{2} - \frac{\pi}{2} \right) \right) + \left( \frac{K^2}{4} - 2y^2 \cos \left( \frac{t}{2} - \frac{\pi}{2} \right) \right) = 0 \]

and putting in the value of \( K \) this becomes

\[ 5 \left( \cos^2 \frac{t}{2} + \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) + 5 \left( y - K \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) + \left( \frac{K^2}{4} - 2y^2 \cos \frac{t}{2} - \frac{\pi}{2} \right) = 0 \]

Differentiating with respect to \( t \) this gives

\[ \frac{d}{dt} \left( \cos^2 \frac{t}{2} + \cos^2 \left( \frac{t}{2} - \frac{\pi}{2} \right) \right) + \left( y - K \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) + \left( \frac{K^2}{4} - 2y^2 \cos \frac{t}{2} - \frac{\pi}{2} \right) = 0 \]

\[ \frac{d}{dt} \left( \cos^2 \frac{t}{2} + \cos^2 \left( \frac{t}{2} - \frac{\pi}{2} \right) \right) + \left( y - K \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) + \left( \frac{K^2}{4} - 2y^2 \cos \frac{t}{2} - \frac{\pi}{2} \right) = 0 \]

\[ \frac{d}{dt} \left( \cos^2 \frac{t}{2} + \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) + \left( y - K \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) + \left( \frac{K^2}{4} - 2y^2 \cos \frac{t}{2} - \frac{\pi}{2} \right) = 0 \]

\[ \frac{d}{dt} \left( \cos^2 \frac{t}{2} + \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) + \left( y - K \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) + \left( \frac{K^2}{4} - 2y^2 \cos \frac{t}{2} - \frac{\pi}{2} \right) = 0 \]

\[ \frac{d}{dt} \left( \cos^2 \frac{t}{2} + \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) + \left( y - K \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) + \left( \frac{K^2}{4} - 2y^2 \cos \frac{t}{2} - \frac{\pi}{2} \right) = 0 \]

Substituting in (71)

\[ \frac{d}{dt} \left( \cos^2 \frac{t}{2} + \cos^2 \left( \frac{t}{2} - \frac{\pi}{2} \right) \right) + \left( y - K \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) + \left( \frac{K^2}{4} - 2y^2 \cos \frac{t}{2} - \frac{\pi}{2} \right) = 0 \]

On expansion this gives, after rearrangement and division by \( \cos^2 \frac{t}{2} \)

\[ \frac{d}{dt} \left( \cos^2 \frac{t}{2} + \cos^2 \left( \frac{t}{2} - \frac{\pi}{2} \right) \right) + \left( y - K \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) + \left( \frac{K^2}{4} - 2y^2 \cos \frac{t}{2} - \frac{\pi}{2} \right) = 0 \]

\[ \frac{d}{dt} \left( \cos^2 \frac{t}{2} + \cos^2 \left( \frac{t}{2} - \frac{\pi}{2} \right) \right) + \left( y - K \cos^2 \frac{t}{2} - \frac{\pi}{2} \right) + \left( \frac{K^2}{4} - 2y^2 \cos \frac{t}{2} - \frac{\pi}{2} \right) = 0 \]

Let us try to identify this with the product

\[ (\cos^2 + A_1 y + B_1) (\cos^2 + A_2 y + B_2) (\cos^2 + A_3 y + B_3) = 0 \]
\[ A_1 + A_2 + A_3 = -4k \]  
\[ (75) \]

\[ B_1 + B_2 + B_3 = -2 \]  
\[ (76) \]

\[ A_1A_2 + A_2A_3 + A_3A_1 = 4k^2 - 4 \]  
\[ (77) \]

\[ A_1(B_2 + B_3) + A_2(B_3 + B_1) + A_3(B_1 + B_2) = 6k \]  
\[ (78) \]

\[ B_1B_2 + B_2B_3 + B_3B_1 = 1 + kl - k^2 \]  
\[ (79) \]

\[ A_1A_2A_3 = 8k \]  
\[ (80) \]

\[ B_1B_2B_3 = kl - k^2 \]  
\[ (81) \]

From equations \((75)\) and \((77)\) it follows that \(A_1, A_2, A_3\) are the roots of the cubic

\[ \xi^3 + 4k\xi^2 + 4(k^2 - 1)\xi - 8k = 0 \]  
\[ (82) \]

Now using equations \((79)\) the cubic \((82)\) may be written

\[ x^3 - \left(1 + \frac{k^2}{2}\right)x^2 + \left[\frac{1}{4}k + k(l-k)\right]x - \left(\frac{k^2}{2}\right) \]  
\[ = 0 \]  
\[ (83) \]

This is then the cubic for \(y\) having roots \(\lambda, \mu, \nu, \nu\).

Transform this into the cubic for \(y\) by putting \(y = 1 - 2k\epsilon\) it gives

\[ \left(\frac{1-2y}{2}\right)^3 - \left(1 + \frac{k^2}{2}\right)\left(1-y\right)^2 + \left[\frac{1}{4} + k(l-k)\right]\left(1-y\right) - \left(\frac{k^2}{2}\right) \]  
\[ = 0 \]  
\[ (84) \]

or

\[ y^3 - (1-l^2)y^2 - 2ky + k^2 = 0 \]  
\[ (85) \]

Put now \(y = -2k\) \((84)\) becomes

\[ k^3 + 4k^2l^2 + 4(l^2-1)k - 8k = 0 \]  
\[ (86) \]
which is identical with (84).

Now the roots of (86) are 1-2λ, 1-2μ, 1-2ν and therefore the roots of (84) or (87) are

\[ A_1 = \frac{-2k}{1-2λ}, \quad A_2 = \frac{-2k}{1-2μ}, \quad A_3 = \frac{-2k}{1-2ν}. \] — (88)

From (88) we therefore obtain

\[ \frac{13}{1-2λ} \frac{1-μ-ν}{1-2μ} + \frac{13}{1-2μ} \frac{1-λ-ν}{1-2μ} + \frac{3}{1-2μ} = 0 \] — (89)

also from (81)

\[ k (1-2λ) B_1 + k (1-2μ) B_2 + k (1-2ν) B_3 = -l (1-2λ)(1-2μ)(1-2ν) \] — (90)

We may therefore solve equations (86) (89) and (90) for B1, B2, B3, giving

\[ B_1 = \frac{(1-2λ)(1-2μ)}{4k(1-λ)(1-μ)} \left[ k(1-6λ) - l(1-2λ)^2 \right] \]

and similar expressions for B2 and B3. It may readily be verified that these values satisfy the remaining equations (89) (82) and (83).

We find therefore that the envelope of the orbit breaks up into the three parabolas

\[ x^2 - \frac{2k}{1-2λ} y + \frac{(1-2λ)(1-2μ)}{4k(1-λ)(1-μ)} \left[ k(1-6λ) - l(1-2λ)^2 \right] = 0 \] — (91)

\[ x^2 - \frac{2k}{1-2μ} y + \frac{(1-2λ)(1-2μ)}{4k(1-λ)(1-μ)} \left[ k(1-6μ) - l(1-2μ)^2 \right] = 0 \] — (92)

\[ x^2 - \frac{2k}{1-2ν} y + \frac{(1-2λ)(1-2μ)}{4k(1-λ)(1-μ)} \left[ k(1-6ν) - l(1-2ν)^2 \right] = 0 \] — (93)

Now in all cases of course there is a real solution we have \( ν > \frac{1}{2}, \ λ > \frac{1}{2} \) and \( μ < \frac{1}{2} \).

1. when \( k > 0 \) i.e. \( s > 2g \) the parabolas (91) and (92) are both curved upwards whereas (93) is curved downwards.
when \( k < 0 \) i.e. \( s < 2g \) the two parabolas \((q_1)\) and \((q_2)\) are both curved downwards and \((q_3)\) is curved upwards.

As examples we may consider the orbits already plotted:

10. Consider the case which was solved by means of the series solution (see pages 13-16).

\[
S = 0.25, \quad g = 0.06466, \quad \lambda = 0.1
\]

\[
\lambda = 0.1502, \quad \mu = 0.3350, \quad \nu = 2.0773
\]

\[
s - 2g = 0.12068, \quad k = 0.8534, \quad l = 1.7678
\]

\[
1 - 2\lambda = 0.6996, \quad 1 - 2\mu = 0.3300, \quad 1 - 2\nu = -3.1546
\]

\[
k_1 = \frac{2k}{1 - 2\lambda} = 2.4397, \quad k_2 = \frac{2\lambda}{1 - 2\mu} = 5.1721, \quad k_3 = \frac{2\mu}{1 - 2\nu} = -0.5411
\]

\[
6\lambda = 0.9012, \quad 6\mu = 2.0100, \quad 6\nu = 12.4638
\]

\[
1 - 6\lambda = 0.0988, \quad 1 - 6\mu = 1.0100, \quad 1 - 6\nu = 11.4638
\]

\[
(1 - 2\lambda)^2 = 0.4894, \quad (1 - 2\mu)^2 = 0.1089, \quad (1 - 2\nu)^2 = 9.9515
\]

\[
k(1 - 6\lambda) = 0.0843, \quad k(1 - 6\mu) = 0.2619, \quad k(1 - 6\nu) = 9.7632
\]

\[
\ell(1 - 2\lambda)^2 = 0.8652, \quad \ell(1 - 2\mu)^2 = 0.1925, \quad \ell(1 - 2\nu)^2 = 17.3723
\]

\[
k(1 - 6\lambda) - \ell(1 - 2\lambda)^2 = -0.4809, \quad (1 - 2\lambda)(1 - 2\mu) = -1.0410
\]

\[
k(1 - 6\mu) - \ell(1 - 2\mu)^2 = -1.0544, \quad (1 - 2\mu)(1 - 2\nu) = -2.2070
\]

\[
k(1 - 6\nu) - \ell(1 - 2\nu)^2 = -2.3758, \quad (1 - 2\nu)(1 - 2\mu) = +0.3830
\]

\[
N_1 = 0.8129, \quad N_2 = 2.3271, \quad N_3 = -6.3202
\]

\[
\gamma - \mu = 1.7423, \quad \gamma - \lambda = 1.9241, \quad \mu - \lambda = 0.1848
\]

\[
(y - \lambda) (\mu - \lambda) = +0.3561, \quad D_1 = 1.2156
\]

\[
(\lambda - \mu) (\nu - \mu) = -0.3220, \quad D_2 = -1.09921
\]

\[
(\lambda - \nu) (\mu - \nu) = 3.3576, \quad D_3 = 11.4615
\]

\[
\bar{q}_1 = +0.6687, \quad \bar{q}_2 = -2.1171, \quad \bar{q}_3 = -0.6514
\]
The parabolas are then
\[ x^2 - K_2 y + T_2 x = 0 \quad (r = 1, 2, 3) \]
\[
\frac{x}{\frac{1}{J_2}} = \text{and } y = \frac{g_2}{J_2} T_2.
\]

The parabolas are
\[ y_1^2 - K_2 J_2 y_2 + 2 T_2 x = 0 \]
\[
\frac{1}{J_2 K_1} = + 0.3098, \quad \frac{1}{J_2 K_2} = + 0.1461, \quad \frac{1}{J_2 K_3} = -1.3970
\]

The envelope therefore consists of the parabolas

\[ A \quad y_2 = 0.3098 \left( y_1^2 + 1.337 y_1 \right) \]
\[ B \quad y_2 = 0.1461 \left( y_1^2 - 4.234 y_1 \right) \]
\[ C \quad y_2 = 1.3970 \left( 1.1028 - y_1^2 \right) \]

<table>
<thead>
<tr>
<th>( y_1 )</th>
<th>( y_1^2 )</th>
<th>( y_1^2 + 1.337 y_1 )</th>
<th>( y_2 (A) )</th>
<th>( y_2^2 - 4.234 y_1 )</th>
<th>( y_2 (B) )</th>
<th>( 1.1028 - y_2^2 )</th>
<th>( y_2 (C) )</th>
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</tbody>
</table>

The envelope is shown in red ink in figure 11. The close agreement
with the plotted orbit, offers an additional confirmation of the identity
of the series and elliptic function solutions.
2. Consider the case discussed by means of the elliptic function

solution on pages 29-41

\[ s = 0.0 \quad \theta = 0.02 \quad \lambda = 0.1 \]

\[ \lambda = 0.04921 \quad \mu = 0.32444 \quad \nu = 0.62635 \]

\[ s - 2\theta = -0.04 \quad \bar{M}_2 = 0.14142136 \quad k = -0.28284 \]

\[ \lambda = 0 \]

\[ 1 - 2\lambda = 0.90158 \quad 1 - 2\theta = 0.35112 \quad 1 - 2\nu = 0.25279 \]

\[ K_1 = -0.62744 \quad K_2 = -1.61107 \quad K_3 = 2.23854 \]

\[ 1 - 6\lambda = 0.104744 \quad 1 - 6\mu = -0.94564 \quad 1 - 6\nu = -2.75581 \]

\[ K_1 = -0.19933 \quad K_2 = 0.267475 \quad K_3 = 0.78010 \]

\[ r(1-6\lambda) = -0.08873 \quad r(1-6\mu) = -0.22783 \quad r(1-6\nu) = 0.31656 \]

\[ r_1 = +0.01969 \quad r_2 = -0.05100 \quad r_3 = +0.24695 \]

\[ r - \mu = 0.30191 \quad r - \lambda = 0.57414 \quad \mu - \lambda = 0.27523 \]

\[ (r - \lambda)(\mu - \lambda) = +0.15885 \]

\[ (r - \mu)(r - \nu) = -0.08309 \]

\[ A = -0.17992 \]

\[ B = +0.09400 \]

\[ D = -0.19713 \]

\[ \bar{P}_1 = -0.09843 \]

\[ \bar{P}_2 = -0.64894 \]

The parabolas forming the envelope are

\[ x^2 - K_n y + \bar{P}_n = 0 \quad (n = 1, 2, 3) \]

But \( x = \frac{y_1}{\bar{P}_2} \) \( + y = \frac{y_2}{\bar{P}_2} \), the parabolas are

\[ y_2 = \frac{1}{2\bar{P}_n} \left\{ y_1^2 + 2\bar{P}_n \right\} \quad (n = 1, 2, 3) \]

or \[ y_2 = 1.12699(0.1968 - y_1^2) \quad (A) \]

\[ y_2 = 0.43899(1.2978 - y_1^2) \quad (B) \]

\[ y_3 = 0.31588(y_1^2 - 2.50546) \quad (C) \]

We obtain therefore:
<table>
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<tr>
<th>$y_1$</th>
<th>$y_1^2$</th>
<th>$0.19656 - y_1^2$</th>
<th>$y_2$ (A)</th>
<th>$1.29768 - y_1^2$</th>
<th>$y_2$ (B)</th>
<th>$y_1^2 - 2.50546$</th>
<th>$y_2$ (C)</th>
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<td>2.38035</td>
<td>0.24060</td>
<td>0.08457</td>
<td>-0.64546</td>
<td>-0.17830</td>
<td></td>
</tr>
</tbody>
</table>

The parabolas are shown in red ink in figure 9. (p. 41)

Note. The system discussed by Beth (see, e.g., page 76) is slightly simpler than the one considered here, and reduces to this case if we take \( l = 0 \). Beth shows that the envelope breaks up into three parabolas which are the limiting forms of the parabolas \( y_1, y_2, y_3 \) when \( l = 0 \), i.e., \( s = 0 \).
It is at once evident that the relation (60) which is the only
simple relation which has been found to hold for points on the
boundary of the region of convergence, does not give any characteristic
form to the parabolae forming the envelope of the boundary.

Since, therefore, all analytical attempts to discover peculiarities
of these orbits have failed, we are led to the consideration of
particular numerical cases in which $s$ and $g$ satisfy the relation (61).
The simplest such case is where
\[ g = 0, \quad s = 2.52 x. \]

We take as before $\delta = 0.1, \quad s = 0.2828, 4.274, 2$

The cubic (26) then becomes
\[ 0.04 x^3 - (0.04 + 0.8) x^2 + 0.01 x = 0 \]
or \[ x \left( x^2 - 3x + 1 \right) = 0 \]

\[ x = 0; \quad \mu = \frac{3}{2} - \sqrt{2} = 0.0857, 5644; \quad \nu = \frac{3}{2} + \sqrt{2} = 2.9142, 1356 \]

Also \[ l + m = 2 \nu = 5.8384, 2712 + lm = 0.2\]

\[ m = 5.7052, 1350; \quad l = 0.0432, 1362; \quad m - l = 5.7419, 9988 \]

Also from (22) \[ k = \frac{m}{l} = 0.0074, 6966, 7 \]

\[ l - l = 0.0432, 1362; \quad \mu - l = 0.0428, 7282; \quad \nu - l = 2.8709, 9994 \]

\[ \frac{0.04 (l - l)(\mu - l)(\nu - l)}{k (m - l)} = 0.3388, 9994 \]
\[ \therefore \quad \mu = -0.3388, 9994 \]

\[ q_2 = \frac{0.0432, 1362 + 0.0432, 1362}{1 + 0.0074, 6966} \]
\[ q_2 = \frac{0.0432, 1362 + 0.0428, 9083}{1 + 0.0074, 6966} \]

\[ q_1 = 1 - 2q_2 \]
\[
k = \begin{pmatrix}
0.00 & 0.4 & 0.5 & 0.2 & 0.6 & 0.7 & 0.8 & 0.9 \\
0.00 & 0.00 & 0.5 & 0.4 & 0.2 & 0.1 & 0.0 & 0.0 \\
0.9 & 0.4 & 0.5 & 0.2 & 0.1 & 0.0 & 0.0 & 0.0 \\
0.9 & 0.4 & 0.5 & 0.2 & 0.1 & 0.0 & 0.0 & 0.0 \\
0.9 & 0.4 & 0.5 & 0.2 & 0.1 & 0.0 & 0.0 & 0.0 \\
0.9 & 0.4 & 0.5 & 0.2 & 0.1 & 0.0 & 0.0 & 0.0 \\
0.9 & 0.4 & 0.5 & 0.2 & 0.1 & 0.0 & 0.0 & 0.0 \\
0.9 & 0.4 & 0.5 & 0.2 & 0.1 & 0.0 & 0.0 & 0.0 \\
\end{pmatrix}
\]

\[
2m-1 = 10.57042700 \\
1-2l = 0.91357276 \\
\text{Swa} = 11.57042700 \\
\text{Swa}^2 = 13.87478285 \\
k^2\text{Swa}^2 = 0.0007469677 \\
k^2 = 13.87478285 \\
c = 11.57042700 \\
\text{du}^2a = 0.992530323 \\
\text{dua} = 0.996258161 \\
k_c = 0.0085103841 \\
\frac{\text{dua}}{k_c} = 11.57042647 \\
C = 0.04321362 \\
1-2l = 0.91357276 \\
s = 0.28542712 \\
l_s = 0.012222166 \\
1-2l + l_s = 0.925745842 \\
N = 0.3388589994 \\
N(1-2l) = 0.30960062 \\
\frac{1-2l+k_s}{N(1-2l)} = 2.99028930 \\
\frac{2\sqrt{n}}{k_n} = 1.00001822 \\
\]

\[
k^2a = 1.00000000 \\
\therefore \text{Swa} = \frac{1}{k^2} \\
\therefore a = k + \frac{k^2}{2}
\]
Writing \( \alpha = 2K \pi / \pi \) we get

\[
\frac{1}{2} \log \frac{\theta'(u-a)}{\theta'(u+a)} = \frac{1}{2} \log \frac{\theta'(u-v)}{\theta'(u+v)} = - \frac{1}{2} \left[ \frac{2g \sin \left( \frac{\pi}{2K} + \frac{\pi K'}{2K} \right)}{1 - q^2} + \frac{2g \sin \left( \frac{\pi}{2K} + \frac{\pi K'}{2K} \right)}{1 - q^2} \right]
\]

Also writing \( u = 2K \pi \nu \).

\[
\frac{1}{2} \log \frac{\theta'(u-a)}{\theta'(u+a)} = \frac{1}{2} \log \frac{\theta'(u-v)}{\theta'(u+v)} = - \frac{1}{2} \left[ \frac{2g \sin \left( \frac{\pi}{2K} + \frac{\pi K'}{2K} \right)}{1 - q^2} + \frac{2g \sin \left( \frac{\pi}{2K} + \frac{\pi K'}{2K} \right)}{1 - q^2} \right]
\]

\[
\pi = 0.1080705
\]

\[
2K = 6.28325085
\]

\[
\pi K' = 6.28318511
\]

\[
\frac{\pi}{2K} = 0.434294819
\]

\[
\gamma = \log_{10} e = 0.434294819
\]

\[
2g \sinh \left( \frac{\pi}{2K} + \frac{\pi K'}{2K} \right) = 0.00106399
\]

\[
\frac{1}{2} \log \theta'(u-a) = 0.00106399
\]

\[
\frac{1}{2} \log \theta'(u+a) = 0.00106399
\]

\[
\frac{1}{2} \log \theta'(u-v) = 0.00106399
\]

\[
\frac{1}{2} \log \theta'(u+v) = 0.00106399
\]

\[
\frac{1}{2} \log \theta'(u-a) = -0.00106399
\]

\[
\frac{1}{2} \log \theta'(u+a) = -0.00106399
\]

\[
\frac{1}{2} \log \theta'(u-v) = -0.00106399
\]

\[
\frac{1}{2} \log \theta'(u+v) = -0.00106399
\]

\[
\pi = \arctan \left\{ \frac{11.57042647}{2} \right\} + \frac{\pi}{2}
\]

\[
\beta_1 = 2.99215325\ u + \frac{1}{4} \arctan \left\{ \frac{11.57042647}{2} \right\} + \frac{\pi}{2}
\]

\[
\frac{1}{2} \log \theta'(u-a) = 0.0100001341\ u + \frac{\pi}{2}
\]

\[
\frac{1}{2} \log \theta'(u+a) = 0.0100001341\ u + \frac{\pi}{2}
\]

\[
\beta_2 = 0.0100001341\ u + \frac{\pi}{2}
\]
\[ g_1 = 0.04321362 + 0.04289083 \sin \left( \frac{1 + 0.00746967 \sin \varpi}{1 - 2g_1 \cos 0} \right) \]

\[ \sin \varpi = 1.000001622 \left[ \frac{\sin \varpi - g_1 \sin \varpi}{1 - 2g_1 \cos 0} \right] \]

\[ v = 0.99996005 \quad u = 1.00001395 \quad \theta = 0.00000346 \gamma \quad \eta = 0.00000000 \]

\[ \gamma_1 = 1 - 2g_1 \]

\[ p_1 = 2.992194999 - \frac{1}{2} \tan^2 \left[ 11.57004254 \varpi \right] \]

\[ d \varpi = 0.999998 \cdot \gamma \cdot \gamma \cdot \left( \frac{1 + 2g_1 \cos 2 \varpi}{\cos \varpi + g_1 \cos 3 \varpi} \right) \]

\[ s_1 = 1 \quad s_2 = 1.717157288 \]

\[ q_1 = \frac{1}{2g_1} \cos \theta_1 \quad q_2 = \frac{1}{2g_1} \cos \theta_2 \]

In order to satisfy the integral of energy \( g \) we must take \( \gamma_1 = 0 \quad \gamma_2 = 135^\circ 0.42 \)

The numerical results are given on pages 90-91 and the orbit is shown in Figure 5 (page 92).

The envelope of the orbit is given by the three parabolas

\[ p_2 = 0.190480939 \]

\[ q_2 = 0.15064912(\eta_2 - 1) \]

\[ q_2 = 1.84234370(1 - \eta_2) \]

These are indicated in red ink in the figure.
To obtain the envelope: — We have in this case.

\[ s = 2 \sqrt{2} \quad g = 0 \quad k = 2 \quad k = 2 \quad x = 0, \quad \mu = 2 - \sqrt{2}, \quad \nu = 2 + \sqrt{2} \]

The parabolas forming the envelope therefore reduce to

\[ x^2 - 4y = 0 \quad \text{(A)} \]
\[ x^2 - \frac{2}{\sqrt{2} - 1} y = 1 \quad \text{(B)} \]
\[ x^2 + \frac{2}{\sqrt{2} + 1} = 1 \quad \text{(C)} \]

Also \( x = \frac{y_1}{\sqrt{2}} \quad y = \frac{\sqrt{2}}{2} y_2 \quad \text{and} \quad s_x = 2(1 - d \sqrt{2}) \)

... these parabolas become

\[ y_1^2 - 4\sqrt{2} x_2 y_2 = 0 \quad \text{or} \quad y_1^2 = 5.2416141 \quad \text{(A)} \]
\[ y_2 = \frac{(y_1^2 - 2)}{2 \sqrt{2} (1 - \sqrt{2})} \quad \text{or} \quad y_2 = \left(\frac{y_1^2 - 2}{y_1^2 - 2}\right) 0.15804812 \quad \text{(B)} \]
\[ y_2 = \left(\frac{1}{y_2^2} - 1\right) \frac{1}{2 \sqrt{2} (1 - \sqrt{2})} \quad \text{or} \quad y_2 = \left(1 - \frac{y_2^2}{y_2^2} \right) 1.84234370 \quad \text{(C)} \]

<table>
<thead>
<tr>
<th>( y_1 )</th>
<th>( y_1^2 )</th>
<th>( y_2 ) (A)</th>
<th>( y_1^2 - 2 )</th>
<th>( y_2 ) (B)</th>
<th>( \frac{y_1^2 - 2}{y_1^2} )</th>
<th>( 1 - \frac{y_1^2}{\sqrt{2}} )</th>
<th>( y_2 ) (C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00</td>
<td>0.000000</td>
<td>-2.00</td>
<td>-0.31610</td>
<td>0.0000</td>
<td>1.0000</td>
<td>1.84234</td>
</tr>
<tr>
<td>0.1</td>
<td>0.01</td>
<td>0.010001</td>
<td>-1.99</td>
<td>-0.31452</td>
<td>0.0050</td>
<td>0.9950</td>
<td>1.83313</td>
</tr>
<tr>
<td>0.2</td>
<td>0.04</td>
<td>0.00763</td>
<td>-1.99</td>
<td>-0.30977</td>
<td>0.0200</td>
<td>0.9800</td>
<td>1.80550</td>
</tr>
<tr>
<td>0.3</td>
<td>0.09</td>
<td>0.01717</td>
<td>-1.91</td>
<td>-0.30187</td>
<td>0.0400</td>
<td>0.9600</td>
<td>1.75945</td>
</tr>
<tr>
<td>0.4</td>
<td>0.16</td>
<td>0.03562</td>
<td>-1.84</td>
<td>-0.29301</td>
<td>0.0600</td>
<td>0.9400</td>
<td>1.71496</td>
</tr>
<tr>
<td>0.5</td>
<td>0.25</td>
<td>0.04770</td>
<td>-1.78</td>
<td>-0.28416</td>
<td>0.0800</td>
<td>0.9200</td>
<td>1.67443</td>
</tr>
<tr>
<td>0.6</td>
<td>0.36</td>
<td>0.06086</td>
<td>-1.74</td>
<td>-0.27532</td>
<td>0.1000</td>
<td>0.9000</td>
<td>1.63072</td>
</tr>
<tr>
<td>0.7</td>
<td>0.49</td>
<td>0.07498</td>
<td>-1.69</td>
<td>-0.26651</td>
<td>0.1200</td>
<td>0.8800</td>
<td>1.58507</td>
</tr>
<tr>
<td>0.8</td>
<td>0.64</td>
<td>0.08953</td>
<td>-1.64</td>
<td>-0.25773</td>
<td>0.1400</td>
<td>0.8600</td>
<td>1.53747</td>
</tr>
<tr>
<td>0.9</td>
<td>0.81</td>
<td>0.10488</td>
<td>-1.60</td>
<td>-0.24899</td>
<td>0.1600</td>
<td>0.8400</td>
<td>1.48789</td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
<td>0.12098</td>
<td>-1.56</td>
<td>-0.24028</td>
<td>0.1800</td>
<td>0.8200</td>
<td>1.43634</td>
</tr>
<tr>
<td>1.1</td>
<td>1.21</td>
<td>0.13784</td>
<td>-1.52</td>
<td>-0.23159</td>
<td>0.2000</td>
<td>0.8000</td>
<td>1.38283</td>
</tr>
<tr>
<td>1.2</td>
<td>1.44</td>
<td>0.15542</td>
<td>-1.48</td>
<td>-0.22295</td>
<td>0.2200</td>
<td>0.7800</td>
<td>1.32740</td>
</tr>
<tr>
<td>1.3</td>
<td>1.69</td>
<td>0.17372</td>
<td>-1.44</td>
<td>-0.21435</td>
<td>0.2400</td>
<td>0.7600</td>
<td>1.27006</td>
</tr>
<tr>
<td>1.4</td>
<td>1.96</td>
<td>0.19273</td>
<td>-1.40</td>
<td>-0.20578</td>
<td>0.2600</td>
<td>0.7400</td>
<td>1.21185</td>
</tr>
<tr>
<td>1.5</td>
<td>2.25</td>
<td>0.21246</td>
<td>-1.36</td>
<td>-0.19724</td>
<td>0.2800</td>
<td>0.7200</td>
<td>1.15279</td>
</tr>
<tr>
<td>1.6</td>
<td>2.56</td>
<td>0.23304</td>
<td>-1.32</td>
<td>-0.18873</td>
<td>0.3000</td>
<td>0.7000</td>
<td>1.09298</td>
</tr>
</tbody>
</table>

The envelope is shown in red ink in figure 5 (page 92).
The characteristic features of this orbit are that for a certain value of the time \( t_e \) becomes zero, and that one of the parabolas forming the envelope of the orbit passes through the origin. These are evidently due to the fact that \( g = 0 \).

For \( g_e = 0 \) when \( \sin u = -1 \) provided that \( mk = l \). But if this condition is satisfied we must have \( \lambda = 0 \) and therefore \( g = 0 \).

The orbit therefore apparently presents no peculiarities which can be attributed to the fact that the values of \( s \) and \( g \) which have been chosen are on the boundary of the region of convergence.

As an additional case we will assume \( s = 0.2 \) and we have then to determine a value of \( g \) which will satisfy equation (61).

We take as before \( \delta x = 0.1 \)

the cubic (26) is now

\[
x^3 - 2x^2 + \left( \frac{1}{4} + \frac{g}{\delta x} \right) x - \frac{1}{4} \left( \frac{g}{\delta x} \right)^2 = 0
\]

\( g \) must be determined from the relation

\[
\frac{1}{E} = \frac{\delta x^2}{5 \pi^2} \left( 1 + \lambda - \frac{4g}{s} \right) + \frac{16\pi^2}{5 \pi^2} \left( 2\lambda - 3 \lambda^2 \right)^2 = 1
\]

This gives

\[
E = 2 + 4\lambda - 3\lambda^2 - \frac{4g}{s^2}
\]

We proceed to assume values for \( s^2 \) and calculate values for \( \lambda \) and \( E \) and find

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( E )</th>
<th>( 1 - E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.293</td>
<td>0.048</td>
<td>1.012</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0493, 372</td>
<td>0.9901, 838</td>
</tr>
<tr>
<td>0.2970</td>
<td>0.0487, 1531</td>
<td>1.0001, 1245</td>
</tr>
<tr>
<td>0.2971</td>
<td>0.0487, 4259</td>
<td>0.9999, 11358</td>
</tr>
<tr>
<td>0.2970, 3</td>
<td>0.0488, 2314, 36</td>
<td>1.0000, 2147, 63</td>
</tr>
<tr>
<td>0.2970, 4</td>
<td>0.0488, 2592, 14</td>
<td>0.9999, 9177, 38</td>
</tr>
<tr>
<td>0.2970, 37</td>
<td>0.0488, 2501, 0599</td>
<td>1.0000, 0039, 7399</td>
</tr>
<tr>
<td>0.2970, 372</td>
<td>0.0488, 2614, 3595</td>
<td>1.0000, 0009, 0460</td>
</tr>
<tr>
<td>0.2970, 373</td>
<td>0.0488, 2617, 1875</td>
<td>0.9999, 9999, 0414</td>
</tr>
<tr>
<td>0.2970, 3723</td>
<td>0.0488, 2617, 1928, 07</td>
<td>1.0000, 0000, 1329</td>
</tr>
</tbody>
</table>
We therefore take \( \varphi = 0.2970, 3728 \) \( \cdot \varphi = 0.0297, 0372, 3 \).

\( \lambda = 0.0488, 2515, 2 \); \( \mu = 0.2684, 8064, 3 \); \( \nu = 1.6826, 9420, 5 \).

\[ \nu - \mu = 1.4142, 1356, 2 = \sqrt{2} \text{ in agreement with theory (see Equation (63))} \]

\[ \nu - \lambda = 1.6338, 6905, 3 \]

\[ \frac{\nu - \mu}{\nu - \lambda} = 2.3106, 3977, 3 \]

\[ \frac{\nu - \mu}{\nu - \lambda} = 1.5200, 1388, 1 \]

\( m = \nu + \frac{1}{\nu - \mu} (\nu - \lambda) = 3.3224, 7307, 6 \)

\[ l = \nu - \frac{1}{\nu - \mu} (\nu - \lambda) = 0.1626, 1533, 4 \]

\[ k = 2 \nu - \mu - 2 \nu (\nu - \lambda) = 0.0360, 7864, 1 \]

\[ \rho = 0.1626, 1533, 4 + 0.1155, 5713, 9 \text{ sin}\theta \]

\[ = 0.1626, 1533, 4 + 0.1098, 0477, 5 \text{ sin}\theta \]

\[ = 1 - 0.29^2 \]

\[ \rho_1 = 1 - 2 \rho_2 \]
Functions of \( k \) etc.

\[
\begin{align*}
\frac{\pi}{2k'} &= 1.8707763268
\\
\log_{10} &= 2.300255450930
\\
N &= \frac{1}{k} \int \frac{0.01(\lambda-\mu)(\mu-\mu)}{(\nu-\nu)} d\theta
\\
\end{align*}
\]

\[
\begin{align*}
m &= 3.2024733046
\\
dm-1 &= 5.405546152
\\
dl &= 0.8225230668
\\
d-2l &= 0.6747969832
\\
\text{Sua} &= 8.010954108
\\
\text{Su}^2a &= 6.1753555720
\\
h^2\text{Sua}^2 &= 0.083535110
\\
C &= 63.174535720
\\
\text{C} &= 7.945294516
\\
du^2a &= 0.916464490
\\
du &= 0.957321428
\\
\text{Su}a \text{ dwa} &= 7.6659106043
\\
C - \text{Sua}dwa &= 0.2679324059
\\
KC &= 0.286763741
\\
kC\text{Sua} &= 2.297250782
\\
du + kC\text{Sua} &= 3.25487265
\\
k^2(du + kC\text{Sua}) &= 0.615817590
\\
\text{Su}^2b &= 0.451649928
\\
\end{align*}
\]

\[
\begin{align*}
\text{Sua} &= 3.33836429
\\
\text{KC} &= 3.64657164
\\
\ell &= a - k - \frac{1}{k'}
\\
\end{align*}
\]
Determination of \( \alpha \).

Put \( \beta = \frac{2K}{\pi} \) i.e. \( \alpha \beta = 2^{\frac{1}{2}} \left[ \frac{\sinh \beta - \frac{1}{2} \sinh 2\beta}{1 - 2\beta \cosh 2\beta} \right] \)

\[
\begin{align*}
\text{Sinh } \beta &= 0.45169928 \\
\cosh \beta &= 0.99983787 \\
\sinh 2\beta &= 0.45162605 \\
2 \beta &= 0.4585 \\
2 \beta_1 &= 0.8170 \\
\cosh 2\beta_1 &= 1.4075 \\
2 \cosh 2\beta_1 &= 0.9997492022 \\
1 - 2 \cosh 2\beta_1 &= 0.999772 \\
\sinh \beta_2 &= 0.451523 \\
\beta_2 &= 0.43443482 \\
2 \beta_2 &= 0.86886784 \\
\cosh 2 \beta_2 &= 1.40747 \\
2 \cosh 2 \beta_2 &= 0.99945873 \\
1 - 2 \cosh 2 \beta_2 &= 0.999499 \\
\text{Product} &= 0.45152264 \\
\beta_3 &= 1.3123.2 \\
\sinh 3 \beta_3 &= 0.722550 \\
\beta_3 &= 0.0000000066713 \\
\sinh \beta_3 &= 0.45152254 \\
\beta_3 &= 0.43443478 \\
2 \beta_3 &= 0.86886756 \\
\cosh 2 \beta_3 &= 1.81002222 \\
2 \cosh 2 \beta_3 &= 0.99979449 \\
1 - 2 \cosh 2 \beta_3 &= 0.999772 \\
\text{Product} &= 0.45152264 \\
\beta_3 &= 0.43443478 \\
2 \beta_3 &= 0.86886756 \\
\cosh 2 \beta_3 &= 1.81002222 \\
2 \cosh 2 \beta_3 &= 0.99979449 \\
1 - 2 \cosh 2 \beta_3 &= 0.999772 \\
\text{Product} &= 0.45152264 \\
\beta_3 &= 0.43443478 \\

\frac{b}{2} = \frac{2K}{\pi} \beta = 0.43755803 \\
\therefore \alpha = K + iK' + i \frac{b}{2} = 0.43755803
Coefficients of $p_i$

Using the formula on page 32 and writing $a = \frac{2K}{\pi} \left[ \frac{\pi}{2} + i \phi \right]$ we have

\[
\frac{\pi}{2} \Theta(a) = \frac{2K}{\pi} \left[ \frac{2a^2 \sinh^2 y - 2a^2 \sinh^4 y + 2a^3 \sinh 6y}{1 - y^2} \right]
\]

\[a = K + iK' + i \cdot 0.4375803\]

\[K' = 2.3547782\]

\[a = K + i \cdot 2.7923585\]

\[y = \frac{\pi}{2K} \cdot 2.7923585\]

\[\frac{\pi}{2K} = 0.999674451\]

\[y = 2.79144495\]

\[2y = 5.5828989\]

\[\exp(y) = 0.43429442819\]

\[\exp(-y) = 2.42463222\]

\[\exp(2y) = 2.6583412\]

\[\exp(-2y) = 0.00035\]

\[\exp \sinh 2y = 2.6583412\]

\[\exp \sinh 2y = 0.00000014072\]

\[2y \sinh 2y = 0.021641052\]

\[1 - y^2 = 0.99999999934\]

\[2y \sinh 2y = 0.021641052\]

\[2y = 11.1657978\]

\[4y = 4.84492444\]

\[-4y = -5.1507486\]

\[\exp 4y = 7.0671152\]

\[\exp \sinh 4y = 3.5335576\]

\[2y \sinh 4y = 0.00046835\]

\[\sinh = 16.746686\]

\[\sinh y = 7.2735666\]

\[\exp \sinh y = 1.84784400\]

\[2y \sinh y = 0.12450654\]

\[2y \sinh y = 0.000001014\]

\[\exp 2y = 2.23315966\]

\[\exp 2y = 9.69548848\]

\[\exp \sinh 2y = 10x + 1.527333\]

\[\exp \sinh 2y = 10^{-9} \times 4.478582\]

\[\exp 2y = 2.79144495\]

\[10y = 2.79144945\]

\[10y = 12.1231711\]

\[\exp 10y = 1.010 \times 1.327333\]

\[\exp 10y = 10^{-9} \times 4.478582\]

\[\frac{2K}{\pi} \Theta(a) = 0.02115265\]

\[\frac{2K}{\pi} \Theta(a) = 0.02117575\]
To evaluate \( \frac{1}{4} i \log \frac{\theta(u-a)}{\theta(u+a)} \n\nUsing the formulas of page 37, we write \( a = K + iK' - b, \) \( b = \frac{1}{2} \pi - c, \) \nwe have

\[
\frac{1}{4} i \log \frac{\theta(u-a)}{\theta(u+a)} = -\frac{1}{2} \pi + \frac{1}{2} \arctan \left\{ \frac{\text{tanh} \beta_1 \text{tanh} \beta_2}{\text{sech}^2 \beta_1 + \text{sech}^2 \beta_2} \right\}
\]

\[
+ \frac{1}{2} \left[ \frac{2q^6 \sinh 2\beta + \frac{1}{2} \sinh^2 \beta}{1-q^2} - \frac{2q^6 \sinh 4\beta + \frac{1}{2} \sinh^2 \beta}{2(1-q^2)} + \frac{2q^6 \sinh 6\beta + \frac{1}{2} \sinh^2 \beta}{3(1-q^2)} \right]
\]

Now \( a = K + iK' + i \cdot 0.4375 \pi \) and \( K' = 2.3547782 \)

\[
a = K + iK' = i \cdot 1.9171979 \quad b = i \cdot 1.9171979
\]

\[
\beta_1 = 1.9171979
\]

\[
\beta_2 = 1.9171979
\]

\[
\text{sech} \beta_1 = 0.3235748
\]

\[
\text{sech} \beta_2 = 0.3235748
\]

\[
\text{tanh} \beta_1 = 0.9576341
\]

\[
\text{tanh} \beta_2 = 0.9576341
\]

\[
\frac{1}{4} i \log \frac{\theta(u-a)}{\theta(u+a)} = -\frac{1}{2} \pi + \frac{1}{2} \arctan \left\{ \frac{0.9576341 \text{tanh} \beta_1}{0.000000015 \text{sech}^2 \beta_1} \right\}
\]

\[
\beta_1 = 4.06972325 \pi + \frac{1}{2} \arctan \left\{ 0.9576341 \text{tanh} \beta_2 \right\} + 0.0211755 \pi
\]

\[
- \left[ \frac{1}{2} \text{arc tan} \left\{ 0.9576341 \text{tanh} \beta_2 \right\} \right] + 0.00000015 \text{sech}^2 \beta_2 + \pi
\]

\[
= 4.0923122 \pi + \frac{1}{2} \arctan \left\{ 0.9576341 \text{tanh} \beta_2 \right\}
\]

\[
- \left[ \frac{1}{2} \text{arc tan} \left\{ 0.9576341 \text{tanh} \beta_2 \right\} \right] + 0.00000015 \text{sech}^2 \beta_2 + \pi
\]
\[
\begin{align*}
\text{coefficients of } a_n \quad & \\
\sin a' = 1.9695394016 & \\
\sin a = 8.8790854531 & \\
c = 3.86905854531 & \\
c' = 1.96999990984 & \\
R_{\sin a'} = 0.504927845 & \\
\sin a' = 0.49507155 & \\
\sin a' = 0.70361321 & \\
C'\sin a' = 13.740066549 & \\
1 - k = 0.963921353 & \\
(-k)\sin a' = 18.98481074 & \\
(-k)\sin a' - C\sin a' = 5.16474535 & \\
k\sin a' = 13.99521547 & \\
k\sin a' - 1 = 12.99521547 & \\
k'(k\sin a' - 1) = 2.46536061 & \\
\frac{\sin a'}{k'} = 2.05427623 & \\
\end{align*}
\]

Alternatively,
\[
\begin{align*}
\cos a' = 1.96999990985 & \\
\sin a' = 13.867783940 & \\
c = 8.867285188 & \\
C' = 0.709666666 & \\
k'\sin a' = 13.071166485 & \\
\sin a' = 0.708361321 & \\
\sin a' + k'\sin a' = 14.680747469 & \\
k'(\sin a' + k'\sin a') = 2.468852276 & \\
\frac{\sin a'}{k'} = 2.05427623 & \\
\end{align*}
\]
Calculation of $a'$.

$e_n b' = 2.08327623$

$k' = 0.9983783$

such $b' = 2.08398383$

\[
\begin{align*}
\beta_1' &= 1.465056 \\
2b' &= 2.93112 \\
\cos b_1' &= 0.843922 \\
\sin b_1' &= 0.566444 \\
\exp \beta_1' &= 4.83919199 \\
\exp(-\beta_1') &= 0.22483248 \\
2 \sinh \beta_1' &= 4.16135585 \\
\sinh \beta_1' &= 2.08067775 \\
\sinh b_1' - \sinh b_1' &= -1.724 \\
\cos \beta_1' &= 2.3085121 \\
\sinh b_1' &= 2.08067775 \\
\beta_1' - \beta_1' &= 1.47913797
\end{align*}
\]

\[
\begin{align*}
\beta_2' &= 1.444914 \\
2b_2' &= 2.975258 \\
\cos b_2' &= 0.820457 \\
\sin b_2' &= 0.566444 \\
\exp b_2' &= 4.83919199 \\
\exp(-b_2') &= 0.22483248 \\
2 \sinh b_2' &= 4.16135585 \\
\sinh b_2' &= 2.08067775 \\
\\sinh b_2' - \sinh b_2' &= -1.724 \\
\cos b_2' &= 2.3085121 \\
\sinh b_2' &= 2.08067775 \\
\beta_2' - \beta_2' &= 1.47913797
\end{align*}
\]

\[
\begin{align*}
\beta_3' &= 0.484294482 \\
2b_3' &= 0.642372344 \\
\cos b_3' &= 0.35761766 \\
\sin b_3' &= 0.80462874 \\
\exp b_3' &= 4.38971694 \\
\exp(-b_3') &= 0.22483248 \\
2 \sinh b_3' &= 4.16135585 \\
\sinh b_3' &= 2.08067775 \\
\sinh b_3' - \sinh b_3' &= -1.199 \\
\cos b_3' &= 2.3085121 \\
\sinh b_3' &= 2.08067775 \\
\beta_3' - \beta_3' &= 1.47913797
\end{align*}
\]

\[
\begin{align*}
\beta_4' &= 0.64235460 \\
-2b_4' &= 1.28476920 \\
-2 \cos b_4' &= 2.71538080 \\
\exp b_4' &= 4.38971694 \\
\exp(-b_4') &= 0.22483248 \\
2 \sinh b_4' &= 4.16135585 \\
\sinh b_4' &= 2.08067775 \\
\sinh b_4' - \sinh b_4' &= -1.199 \\
\cos b_4' &= 2.3085121 \\
\sinh b_4' &= 2.08067775 \\
\beta_4' - \beta_4' &= 1.47913797
\end{align*}
\]

\[
\begin{align*}
\beta_5' &= 1.47961900 \\
\bar{b}' &= 1.47961900 \\
\bar{a}' &= \frac{k + \bar{k} + \bar{b}'}{2}
\end{align*}
\]
Evaluation of \( \frac{1}{2} \cdot \frac{\Theta'(a')}{\Theta(a')} \)

\[
a' = k + k' = 1.147961900 + \frac{k'}{2} = 2.3547782
\]

\[
a' = k + ik' = 1.08749592
\]

Write \( a' = k + ik' - b', \quad b' = 1.08749592 \)

Also write \( \beta_i = \frac{\pi}{2k}, \quad b_i = \frac{\pi}{2k} \cdot 0.8749592 \)

and use the formula of page 35 viz

\[
\frac{1}{2} \cdot \frac{\Theta'(a')}{\Theta(a')} = \frac{\pi}{2k} \left[ 1 - \tan \beta_i - \frac{2\beta_i \sinh 2\beta_i}{1 + 2 \beta_i \cosh 2 \beta_i} \right]
\]

\[
\frac{\pi}{2k} = 0.999674451
\]

\[
\beta_1 = 0.8749592
\]

\[
\beta_{-1} = 0.3798668
\]

\[
\cosh(\beta) = 2.3980945
\]

\[
\sinh(\beta) = 0.4169678
\]

\[
2 \cdot \sinh \beta = 1.981091
\]

\[
2 \cdot \cosh \beta = 2.5159032
\]

\[
\tanh \beta = 0.7037418
\]

\[
1 - \tanh \beta = 0.2962582
\]

\[
1 + \tanh \beta = 0.1481293
\]

\[
\frac{1}{2} \cdot \frac{\Theta'(a')}{\Theta(a')} = 0.1480841
\]

We also have

\[
\frac{i \log \Theta(a-a')}{\Theta(a+a')} = -1.5 + 1 = \tan \left\{ 0.7037418 \right\}
\]
\[ \beta_2 = 0.33015416 \mu + \frac{1}{2} \arctan \left\{ 0.99147001 \text{deg} \cos \psi + 0.14808.11 \mu + \left[ \frac{Q}{2} - \frac{1}{2} \arctan \left\{ 0.7037413 \tan \psi \right\} \right] + \gamma_2 \right\} \]

\[ \beta_2 = 0.1844120 \, \varpi + \frac{1}{2} \arctan \left\{ 0.99147001 \text{deg} \cos \psi + \left[ \frac{Q}{2} - \frac{1}{2} \arctan \left\{ 0.7037413 \tan \psi \right\} \right] + \gamma_2 \right\} \]

In order to satisfy the integral of energy we must take

\[ \gamma_1 = 0 \quad \gamma_2 = 134.59.91 \]

We have therefore finally

\[ \varpi = 0.16261533 + 0.10968.478 \text{ sec} \quad 1 + 0.03607.865 \text{ sec} \]

\[ \gamma_1 = 1.2 \gamma_2 \]

\[ \sin \mu = 0.00016216 \text{ sec} \quad 1 - 0.00016281 \cos 2 \psi \]

\[ \beta_1 = 4.09228122.15 + \frac{1}{2} \arctan \left\{ 3.33836429 \text{ deg} \right\} \]

\[ \beta_2 = 0.1844120 \, \varpi + \frac{1}{2} \arctan \left\{ 0.99147001 \text{ deg} \cos \psi \right\} \]

\[ \text{deg} \cos \psi = 0.99983 \frac{1}{2} \left( 1 + 0.00016281 \cos 2 \psi \right) \]

\[ \gamma_2 = 134.59.91 \]

\[ \gamma_1 = \sqrt{2} \gamma_1 \cdot \cos \beta_1 \quad \gamma_2 = \sqrt{2} \gamma_2 \cdot \cos \beta_2 \]

The numerical results are given on pages 105 and 106 and the orbit is shown in Figure 6 (page 107).
Envelope of Orbit.

The equations of the three parabolas forming the envelope are given by equations (91) (92) (93) which we write in the form

\[ x^2 - T_1 y + N_1 \frac{D_1}{D} = 0; \quad x^2 - T_2 y + N_2 \frac{D_2}{D} = 0; \quad x^2 - T_3 y + N_3 \frac{D_3}{D} = 0 \]

where \( T_1 = \frac{2k}{1-2\lambda}; \quad T_2 = \frac{2k}{1-2\mu}; \quad T_3 = \frac{2k}{1-2\nu} \)

\( N_1 = (1-2\nu)(1-2\mu)k(1-6\lambda) - \ell(1-2\lambda)^2 \) + similar expressions for \( N_2 \) and \( N_3 \)

\( D_i = 4k(\nu-\lambda)(\mu-\lambda) \)

where \( k = \frac{S-2g}{32k} \) \quad and \quad \( l = \frac{S}{32k} \)

\[ g = 0.0264.7034.23 \]
\[ a = 0.0947.4074.46 \]
\[ b = 0.2 \]
\[ S-2g = 0.1405.9256.4 \]
\[ D = 0.1414.2136 \]
\[ k = 0.9941.3946 \]
\[ l = 1.4142.1356 \]
\[ 2k = 1.9002.7882 \]
\[ T_1 = 2.2013.44610 \]
\[ T_2 = 4.2939.7989 \]
\[ T_3 = -0.8405.1195 \]
\[ 1-6\lambda = +0.7070.4909 \]
\[ 1-6\mu = -0.6106.8368 \]
\[ 1-6\nu = -0.9096.1652 \]
\[ (1-2\lambda)^2 = 0.8174.2349.7 \]
\[ (1-2\mu)^2 = 0.2144.0484 \]
\[ (1-2\nu)^2 = 0.5595.0652.33 \]
\( k(1-6\lambda) = 0.7029.0540 \]
\( k(1-6\mu) = 0.6074.3075 \)
\( k(1-6\nu) = 0.3032.1425 \)

\[ k(1-2\lambda)^2 = 1.7151.5021.4 \]
\[ k(1-2\mu)^2 = 0.8104.1876 \]
\[ k(1-2\nu)^2 = -0.9126.1800 \]
\[ (1-2\lambda)^2 = 1.0142.856 \]
\[ (1-2\mu)^2 = 0.2856.897 \]
\[ (1-2\nu)^2 = -1.6955.4698.1 \]

\[ N_1 = +0.4913.3294 \]
\[ N_2 = +1.9446.1646 \]
\[ N_3 = -4.0843.8255 \]

\[ \lambda = 0.4555.2515.2 \]
\[ \mu = 0.2652.4064.3 \]
\[ \nu = 1.6526.0420.5 \]

\[ \lambda-\mu = 0.0976.0330.4 \]
\[ \lambda-\nu = 0.5834.5120.1 \]
\[ \lambda + \nu = 3.3653.8841.0 \]

\[ 1-2\lambda = 0.9023.4964 \]
\[ 1-2\mu = 0.7630.3874 \]
\[ 1-2\nu = 2.3653.8841.0 \]

\[ \lambda - \nu = 1.4142.1356 \]
\[ \lambda - \mu = 1.6338.6905 \]
\[ \lambda + \mu = 0.2196.5549 \]

\[ (\lambda-\nu)(\mu-\lambda) = +0.3580.8081 \]
\[ (\lambda-\mu)(\nu-\lambda) = -0.3106.3977 \]
\[ (\lambda+\nu)(\lambda-\mu) = +2.3106.3977 \]

\[ k = 3.9765.0784 \]
Now \( \phi_1 = \frac{2}{s_1} x \) \( \phi_2 = \frac{2}{s_2} y \)

\[ x^2 - \frac{q_1}{s_2} y - k_1 = 0 \]

\[ \phi_2 = \frac{2}{s_2} \left( \frac{\phi_1^2 + k_1}{2} \right) + \text{similar expressions for the other two parabolas} \]

\[ \frac{3}{s_2} = 1.34164079 \]
\[ \frac{3}{s_2} q_1 = 2.95623317 \]
\[ \frac{3}{s_2} q_2 = 3.76697856 \]
\[ \frac{3}{s_2} q_3 = 0.12774548 \]
\[ \frac{3}{s_2} q_1 = 0.67653662 \]
\[ \frac{3}{s_2} q_2 = 0.34716324 \]
\[ \frac{3}{s_2} q_3 = -1.77344900 \]

The three parabolas forming the envelope of the orbit are

\[ \phi_2 = 0.67653662 \left( \frac{\phi_1^2}{2} + 0.34716324 k_1 + 0.77101434 \phi_1 \right) \]
\[ \phi_2 = 0.34716324 \left( \frac{\phi_1^2}{2} - 1.77344900 k_1 - 0.77101434 \phi_1 \right) \]
\[ \phi_2 = 1.77344900 \left( 2 \phi_1 - \frac{\phi_1^2}{2} \right) \]

These are shown in red ink in figure 6 (page 104).
<table>
<thead>
<tr>
<th>Column 1</th>
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</thead>
<tbody>
<tr>
<td>Data 1</td>
<td>Data 2</td>
<td>Data 3</td>
<td>Data 4</td>
<td>Data 5</td>
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<tr>
<td>Data 7</td>
<td>Data 8</td>
<td>Data 9</td>
<td>Data 10</td>
<td>Data 11</td>
<td>Data 12</td>
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</tr>
</tbody>
</table>

Note: The table continues with similar columns and rows.
From the two particular cases of values of $s$ and $g$ on the boundary of the region of convergence which have been examined in §§18 and 19 it appears that the orbit for such a case exhibits no distinctive characteristics.

It has therefore been adequately demonstrated that there is no discontinuity in the system corresponding to a passage from values of $s$ and $g$ for which the series solution is convergent to values of which it is divergent. In other words the divergence of the series solution represents no discontinuity in the system but merely the failure of the series solution to represent the solution for such values of $s$ and $g$.

On the other hand it has been shown that the same remarks do not apply to that part of the boundary of the region of convergence which consists of the double line $s = 2g$; for values of $s$ and $g$ on this line we do have a discontinuity in the system.

To sum up the results obtained for this particular system: we have shown that real solutions of the problem exist for a range of values of $s$ and $g$ given from the discriminant of the cubic (26).

A solution has been obtained in terms of elliptic functions which is valid throughout the region of real solutions except on the double line $s = 2g$ where the solution degenerates into a particular asymptotic form.

The solution in trigonometric series however is only valid in a part of this region, corresponding to values of $s$ and $g$ for which the roots of the cubic (26) are admissible in the form of infinite series of positive powers of $t$, and the boundary of the region of convergence of the series solution has thus been defined.

It has further been shown that the divergence of the series solution represents no discontinuity in the system.

In the next chapter it is shown how these results obtained for the particular system considered may be extended to the general case.
CHAPTER 5

Extension to a general dynamical system
21. As has been shown in the preceding chapters the whole question of the convergence of the series solution has turned upon the cubic equation (26), the series solution being convergent and representing the orbit so long as the roots of the cubic may be expressed in the form of series of positive powers of \( z \).

Before proceeding to generalize the results I will show how the cubic equation (26) may be derived in a slightly different fashion.

The two integrals of the original system of differential equations were equations (5) and (6) namely

\[
\begin{align*}
8, q_1 + s_2 q_2 + x q_1 q_2^k \cos (2p_1 - p_2) &= h \\
q_1 + 2q_2 &= c
\end{align*}
\]

In these parts, in accordance with the preceding work,

\[
\begin{align*}
s_1 &= 1, \quad 2s_1 - s_2 = s, \quad \therefore s_2 = 2 - s, \quad h = 1 - q, \quad c = 1, \quad \text{giving}\\
q_1 + (2 - s) q_2 + x q_1 q_2^k \cos (2p_1 - p_2) &= 1 - q \\
q_1 + 2q_2 &= 1
\end{align*}
\]

Substituting from (95) in (94) we get

\[
\begin{align*}
1 - 2q_2 + (2 - s) q_2 + x (1 - 2q_2) q_2^k \cos (2p_1 - p_2) &= 1 - q \\
\text{or} \\
x (1 - 2q_2) q_2^k \cos (2p_1 - p_2) &= s q_2 - q \\
\end{align*}
\]

On squaring, this gives

\[
\begin{align*}
x^2 (1 - 2q_2)^2 q_2^2 \cos^2 (2p_1 - p_2) &= (s q_2 - q)^2 \\
\text{or} \\
x^2 \cos^2 (2p_1 - p_2) q_2^3 - \left\{ 4 x^2 \cos^2 (2p_1 - p_2) \right\} q_2 + \left\{ \frac{1}{4} x^2 \cos^2 (2p_1 - p_2) \right\} q_2^2 = 0
\end{align*}
\]

\[
\begin{align*}
q_2^3 - \left\{ 1 + \frac{5}{3} q_2^2 \right\} q_2^2 + \left\{ \frac{1}{4} + \frac{5}{4} q_2 \right\} q_2^2 - \frac{5}{2} x^2 \cos^2 (2p_1 - p_2) = 0
\end{align*}
\]

If in this we write \( \lambda' = x \cos (2p_1 - p_2) \) it becomes
This equation is seen to be identical with (26) except that we have, in (95), \( x' \) in place of \( x \).

For any particular value of \((x', 2p', -p')\) the roots of this equation (98) will represent the three possible values of \( q' \) corresponding to this value of \((x', 2p', -p')\). The roots of our original cubic equation (26) thus represent the three possible values of \( q' \) corresponding to values of \( p' \) and \( p \) for which
\[
\cos (2p, -p') = \pm 1.
\]

It is further apparent that if the roots of the cubic (26) can be expressed in any particular form e.g. in series of positive powers of \( x' \), then the roots of the cubic (95) can be expressed in an identical form if we replace \( x' \) by \( 2' \), i.e. in series of positive powers of \( x' \).

Now \( x' = x \cos (2p, -p') \) and we have inferred that when \( q' \) can be expressed in a series of positive powers of \( x' \), \( p' \), and \( p' \)\( -p' \), can be expressed in a similar form, and therefore \((2p, -p')\) and also \( \cos (2p, -p') \) can be expressed in a series of positive powers of \( x' \). A series of positive powers of \( x' \) is equivalent to a series of positive powers of \( x' \).

We therefore obtain another proof of the theorem already proved that the value of \( q' \) at any time can be expressed in the form of a series of positive powers of \( x' \) so long as the roots of the cubic (26) are expressible in series of positive powers of \( x' \). The terms of the series for \( q' \) will however obviously have factors which are trigonometric functions of the time.

In this form the results may be immediately extended to the general case.
It has been shown by Prof. Whittaker (Proc. Lond. Math. Soc. XXXI, (1902), p. 206. or Analytical Dynamics § 182-186) that in any dynamical system in which the motion is of a type not far removed from a steady motion or an equilibrium configuration, the equations of motion may be expressed in a general form applicable to all such cases. It has also been shown that the same general form is applicable to motion which is not of this character and in particular to motion such as that of the planets round the sun or the moon round the earth (see Delaunay, Théorie de la Lune et des Planètes, Annales de l'Obs. de Paris, Mémoires, XVII, 1885).

This general form may be stated as follows:

The equations of motion are

\[ \frac{d^2 q_r}{dt^2} = \frac{\partial H}{\partial q_r}, \quad \frac{dq_r}{dt} = -\frac{\partial H}{\partial q_r} \quad (r = 1, 2, \ldots, n) \quad (99) \]

where

\[ H = a_0, 0, \ldots, 0 + \sum a_{n_1 n_2 \ldots n_0} \cos (n_1 p_1 + n_2 p_2 + \ldots + n_0 p_0) \quad (100) \]

and the coefficients \( a_{n_1 n_2 \ldots n_0} \) are functions of \( q_1, q_2, \ldots, q_n \) only; moreover the periodic part of \( H \) is small compared with the non-periodic part \( a_0, 0, \ldots, 0 \); a term which has for argument \( (n_1 p_1 + n_2 p_2 + \ldots + n_0 p_0) \) has its coefficient, \( a_{n_1 n_2 \ldots n_0} \), at least of order \( \frac{1}{n_1 n_2 \ldots n_0} \) in the small quantities \( q_1, q_2, \ldots, q_n \); and the expansion of \( a_0, 0, \ldots, 0 \) begins with the terms \( (p_1 q_1, 1, p_2 q_2 + \ldots + 3 n_0 q_n) \).

Whittaker has also shown how these equations may be integrated, the coordinates \( p_r \) and \( q_r \) being expressed in the form of trigonometric series; the method consists in the repeated application of contact transformations, thereby removing periodic terms from \( H \) to ultimately reducing the problem to the equilibrium problem, and is essentially the method employed in Chapter 1 of this work.

In order to integrate the system we must be able to find in particular integrals, expressing relations between the \( q_r \)'s and the \( p_r \)'s.

One of these particular integrals will be the integral of energy, namely

\[ H = \text{const} = h \ (\text{say}) \quad (101) \]
Let us suppose that the remaining \((n-1)\) integrals are such that each of them involves some of the \(g_i\)'s i.e. they are not merely relations between the \(p_j\)'s alone, and further that the \(p_j\)'s only occur in the arguments of trigonometrical functions. These conditions will be satisfied in general in practical problems.

We may then use these \((n-1)\) equations to express all the \(g_i\)'s in terms of one of them, say \(g_1\), and certain trigonometrical functions of the \(p_j\)'s. When we substitute these values for the \(g_i\)'s in the integral of energy

\[ H = W \]

we shall obtain an equation in \(g_1\), involving also certain trigonometrical functions of the \(p_j\)'s. This equation may then be rationalised so that it becomes an equation involving positive integral powers of \(g_1\), and trigonometrical functions of the \(p_j\)'s. say

\[ F(g_1) = 0 \quad (102) \]

where \(F\) is a polynomial in \(g_1\), whose coefficients may involve trigonometrical functions of the \(p_j\)’s.

The degree of \(F\) will be at least that of the greatest of the expressions

\[ \frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k} \]

arising from the original \(H\) \((100)\)

For any particular values of the \(p_j\)'s the roots of the equation \((102)\) will give the corresponding possible values of \(g_1\).

If it is possible to find such values of the \(p_j\)'s that all the trigonometrical functions in \((102)\) have their maximum values we shall get an equation corresponding to the cubic equation \((26)\) of the particular case previously considered. The equation \((102)\) will correspond precisely to the generalised form of the cubic \((q^3)\).

If then the roots of this equation \((102)\) can be expressed in power series in any particular form, for any particular initial conditions then the coordinate \(g_1\) can also be expressed as a power series of the same form, whose coefficients will involve trigonometrical functions of the time, for the same initial conditions.

Conversely, if it is desired to ascertain when a solution for \(g_1\) (say), consisting of a power series, the coefficients of whose terms
involving trigonometric functions of a function of the time, is convergent, it is sufficient to ascertain when the roots of the equation (10.2) may be expressed in a similar form, the coefficients of the different terms in this case not involving trigonometric functions of a function of the time, but involving trigonometric functions of the p's.

From the discussion given in Chapter 4, it seems legitimate to infer that the divergence of the series solution in the general case does not necessarily imply any discontinuity in the system but merely denotes the failure of the series solution to represent the coordinates of the system throughout the whole range of real solutions.