THE CONSTRUCTION OF ORTHOGONAL REPRESENTATIONS
OF THE SYMMETRIC GROUP.

By

Hamish G. Anderson,
M.A.

Thesis for Degree of Ph.D. at Edinburgh University, 1949.
SUMMARY.

Section 1, The Symmetric Group, defines the notation used for permutations, and reviews some properties of the group of permutations.

Section 2, Group Representations, introduces the representation of a group by matrices and defines the regular representation. Group characters, and the permutation matrices are also introduced.

Section 3, Young's Tableaux, describes the tableaux dealt with by Young, and gives a summary of Young's method, as revised by Rutherford, for defining the semi-normal and orthogonal units, and the representations associated with them. The orthogonal representations of certain transpositions, which generate the symmetric group, are deduced. As an appendix, the generators for three representations of the group of degree 6 are given.

Section 4, Central Cores of Invariant Matrices, gives a method, described in a series of lectures by Prof. Aitken, of finding the group matrix for the irreducible representations from the invariant matrices. With the aid of the representations of the transpositions as found in section 3, the form of the matrix, which reduces the central core of an invariant matrix to /
to the irreducible orthogonal representation, is found.

Section 5, Calculation of Orthogonal Representations, gives a summary of the methods available for obtaining the representations, and introduces a method due to Aitken for obtaining in a simple manner the representations associated with unicursal partitions. As an appendix, the representations of the groups of degree 3, 4 and 5 are calculated.

The principal references are:


and the series of Papers by A. Young "On Quantitative Substitutional Analysis", in Proc. London Math. Soc. 33 (1900); 34 (1902); (2) 28 (1927); (2) 31 (1929);
(2) 34 (1921); (2) 36 (1932) and (2) 37 (1933).
1. **THE SYMMETRIC GROUP.**

The operation which re-arranges a set of n elements \((z_1, z_2, \ldots, z_n)\) or briefly \((1, 2, \ldots, n)\) is called a permutation \(P\). If \(i_1\) is an element which is changed, suppose that \(i_1\) replaces \(i_2\), \(i_2\) replaces \(i_3\) and so on, till eventually \(i_r\) replaces \(i_1\). This set of replacements, forming part of \(P\), is called a cycle of order \(r\), and will be written

\[(i_1, i_2, \ldots, i_r)\]

Then if any elements of the set are not included in this cycle, another cycle may be constructed, and so on till the elements \((1, 2, \ldots, n)\) are all included. Any element which is unchanged will be considered as a cycle of order 1. So \(P\) consists of a set of cycles \((i_1, i_2 \ldots, i_r)(j_1, j_2 \ldots, j_s)(k_1 \ldots)\) where the cycles involve different elements. The set of orders of the cycles of \(P\) is thus a partition of \(n\), and is called the cycle type of \(P\). In specifying a cycle, the element placed first is immaterial.

As an example, the permutation which replaces \((1234567)\) by \((4635127)\) consists of the cycles \((1, 5, 4), (2, 6), (3)\) and \((7)\) and is written \(P = (1, 5, 4)(2, 6)(3)(7)\) or \((1, 5, 4)(2, 6)\) since a cycle of order 1 leaves its element unchanged.

If \(P_1\) and \(P_2\) are any two permutations, the result of performing first \(P_1\) and then \(P_2\) on the re-arranged set is called the product \(P_3 = P_2P_1\). If \(P_2\) be performed first, followed by \(P_1\), the product \(P_4 = P_1P_2\) is not in general equal to \(P_3\). As an example,
let \( P_1 = (1, 2, 3) (4) \) \( P_2 = (13) (24) \)

Then \( P_3 = (13) (24) (123) (4) = (1) (243) \)

\[ P_4 = (123) (4) (13) (24) = (142) (3) \]

If however, \( P_1 \) and \( P_2 \) change distinct sets of elements, they will commute. So the cycles into which a permutation is separable, and which involve distinct sets of elements, commute.

The permutation which leaves every element unchanged is the identity permutation, written \( \varepsilon = (1) \). Every permutation \( P \) has an inverse \( Q \) such that \( PQ = \varepsilon \), since when \( P \) is written in cycles, \( Q \) is the permutation with every cycle written in the reverse order. It is easily seen that if \( PQ = \varepsilon \) \( QP = \varepsilon \), so that the inverse of a permutation is unique, commutes with the permutation, and is of the same cycle type.

It is easily verified that the associative law holds for permutations. So the \( n! \) permutations of \( n \) elements obey all the group postulates, and the group so formed is called the symmetric group of degree \( n \).

A cycle involving only two elements is called a transposition. Since

\[ (i_1, i_2 \cdots i_r) = (i_r, i_{r-1})(i_{r-1}, i_{r-2}) \cdots (i_2, i_1) \]

any cycle can be resolved into a product of transpositions. Again, if \( i_r < i_s \)

\[ (i_r, i_s) = (i_s, i_{s-1})(i_{s-1}, i_{s-2}) \cdots (i_{r+2}, i_{r+1}) \]

\[ (i_r, i_{r+1})(i_{r+1}, i_{r+2}) \cdots (i_{s-1}, i_s) \]

so that any transposition is expressible as a product of transpositions each of which involves two adjacent elements.

So /
So any permutation may be expressed as a product of transpositions, each of the form \((k-1, k)\).

If \(P_1\) and \(P_2\) are any two permutations, the product \(P_2P_1P_2^{-1}\) is that permutation obtained by applying \(P_2\) to \(P_1\) regarding \(P_1\) as the operand on which \(P_2\) operates. For, if \(P_1\) changes \(r\) to \(s\), and \(P_2\) changes \(r\) to \(r_1\), \(s\) to \(s_1\), then \(P_2^{-1}\) changes \(r_1\) to \(r\), \(s_1\) to \(s\). Thus \(P_2P_1P_2^{-1}\) consists of the changes \(r_1\) to \(r\), \(r\) to \(s\), \(s\) to \(s_1\) i.e. \(r_1\) to \(s_1\). So the cycle type of \(P_2P_1P_2^{-1}\) is the same as that of \(P_1\). Also, any other permutation of the same cycle type as \(P_1\) can be obtained by means of some \(P_2\). The set of all permutations which are related by \(P_1 = QP_2Q^{-1}\) for some \(Q\) is called a class, and in the case of the symmetric group, the class and a cycle-type consist of the same elements.

The effect of any permutation \(P\) on the alternant

\[
\Delta = \prod_{1 \leq j < k} (Z_j - Z_k)
\]

is to leave it unaltered apart possibly from the sign. If \(PA = \Delta\), \(P\) is called even, if \(PA = -\Delta\), \(P\) is called odd. Any transposition clearly is odd, since in \(\Delta\) it alters the sign of an odd number of factors.

The set of all permutations which permute \(i_1, i_2 \ldots i_r\) among themselves is a group of order \(r!\), and will be called the positive symmetric group on \(i_1 \ldots i_r\). The sum of all its elements will be denoted by \(\{i_1, i_2 \ldots i_r\}\).

The set of all expressions \(\pm P\) where \(P\) is an element of the positive symmetric group on \(i_1 \ldots i_r\), and the sign is
is positive if $P$ is even, negative if $P$ is odd also constitutes a group. This is called the negative symmetric group on $i_0, \ldots, i_r$, and the sum of its elements is denoted by
\[
\{i_0, i_1, \ldots, i_r\}'.
\]
Thus
\[
\{1, 2, 3\} = \varepsilon + (1,2) + (1,3) + (2,3) + (1,2,3) + (1,3,2).
\]
\[
\{1, 2, 3\}' = \varepsilon - (1,2) - (1,3) - (2,3) + (1,2,3) + (1,3,2).
\]
2. GROUP REPRESENTATIONS.

If there is a correspondence such that to every element $P_i$ of a group there corresponds a matrix $R_i$, and $R_i R_j = R_k$ whenever $P_i P_j = P_k$, the matrices $R_i$ are said to form a matrix representation of the group. It may happen that the $R_i$ are not all different. Indeed, if every $R_i$ is a unit matrix, this forms a representation of the group. Obviously, if $R_i$ forms a representation of the group, so does $HR_iH^{-1}$, where $H$ is any non-singular matrix. Two such representations are said to be equivalent.

The representation is said to be reducible if a non-singular matrix $Q$ can be found such that, for every $R_i$

$$QR_iQ^{-1} = \begin{bmatrix} S_i & \cdot \\ U_i & T_i \end{bmatrix}$$

where $S_i$ and $T_i$ are square sub-matrices. The sets of matrices $S_i$ and $T_i$ then form representations of the group.

In the case of a finite group, and so of the symmetric group, it has been shown that if $R_i$ is reducible, $Q$ can be determined so that in addition to the zero submatrix above the diagonal, $U_i$ can be made zero. The representation is then said to be completely reducible, and in finite groups, reducibility implies complete reducibility. The matrices $S_i$ and $T_i$ may in turn be reducible, and eventually the representation may be reduced to the equivalent form

$$R_{i1}$$
where $R_{i_1}$, $R_{i_2}$ --- are irreducible.

The matrix $R = \sum x_i R_i$ is called the group matrix of the representation, and has the same properties of reducibility and equivalence as the representation $R_i$.

Schur has shown that, if $R_1$ and $R_2$ are irreducible group matrices of order $r_1 \times r_1, r_2 \times r_2$ respectively and $P$, of order $r_1 \times r_2$, is such that $R_1 P = P R_2$, then either $P = 0$ or $r_1 = r_2$ and $P$ is non-singular. So if a non-zero $P$ can be found to satisfy $R_1 P = P R_2$ the representations $R_1$ and $R_2$ are equivalent. From this it follows that if $R_1$ is irreducible the only matrices which commute with $R_1$ are scalar matrices. For, if $P$ with a latent root $\rho$, satisfies $P R_1 = R_1 P$, then $(P - \rho I)R_1 = R_1 (P - \rho I)$ and $P - \rho I$, being singular, is zero.

For any group, a multiplication table can be constructed as follows: Associate the $r$th row with the element $a_r^{-1}$, the $s$th column with $a_s$, and let the entry in the $r$th row, $s$th column be $a_r^{-1} a_s$. As an example, for the symmetric group of degree 3, the table is

\[
\begin{array}{ccccccc}
\varepsilon & (12) & (13) & (23) & (123) & (132) \\
(12)^{-1} & (12) & \varepsilon & (23) & (123) & (132) \\
(13)^{-1} & (13) & (123) & \varepsilon & (132) & (12) \\
(23)^{-1} & (23) & (132) & (13) & \varepsilon & (123) \\
(123)^{-1} & (132) & (15) & (23) & (12) & \varepsilon \\
(132)^{-1} & (123) & (23) & (12) & (13) & \varepsilon \\
\end{array}
\]
This table is such that no element appears twice in any row or in any column, and the identity element appears in the diagonal.

If, in this multiplication table, the elements $a_i$ are replaced by indeterminates $x_i$, the matrix $X$ obtained is Frobenius' regular group matrix. In this case, with $x_0 = e, x_1 = (12), x_2 = (13), x_3 = (23), x_4 = (123), x_5 = (132)$

$$X = \begin{bmatrix}
x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\
x_1 & x_0 & x_4 & x_5 & x_2 & x_3 \\
x_2 & x_5 & x_0 & x_4 & x_3 & x_1 \\
x_3 & x_4 & x_5 & x_0 & x_1 & x_2 \\
x_5 & x_2 & x_3 & x_1 & x_0 & x_4 \\
x_4 & x_3 & x_1 & x_2 & x_5 & x_0
\end{bmatrix}$$

Let $X = \sum x_i A_i$ and so each $A_i$ will be a permutation matrix, with one unit in each row and in each column.

Now there is a unit element at $(r, s)$ in $A_i$ if $a_i = a_{r}^{-1} a_{s}$ and at $(s, t)$ in $A_j$ if $a_j = a_{s}^{-1} a_{t}$.

So in the product $A_i A_j$, which is a permutation matrix, there is a unit element at $(r, t)$ and

$$a_i a_j = a_{r}^{-1} a_{s} a_{s}^{-1} a_{t} = a_{r}^{-1} a_{t}$$

If $a_i a_j = a_k$, then $A_k$ will have a unit at $(r, t)$ since $a_k = a_{r}^{-1} a_{t}$.

So the matrices $A_i$ represent the group elements $a_i$, and $X$ is the group matrix of the representation.

Let a group be represented by the matrices $A_i$ ($i = 0, 1, 2 \ldots h-1$). The traces of these matrices form a vector /
vector of \( h \) elements, called the character of the representation. Since the trace of \( H A H^{-1} \) is the same as the trace of \( A \), the characters of equivalent representations are the same. For the same reason, all the elements in the vector which correspond to the elements of a conjugate class in the group are equal. The character can be specified by quoting one element for each class of the group.

If the representation is irreducible, the character is called a simple character. For simple characters, a number of important orthogonal relations hold. Let

\[
\begin{align*}
\chi^{(\mu)} &= \sum x_i A_i \\
\chi^{(\nu)} &= \sum x_i B_i
\end{align*}
\]

be two non-equivalent irreducible representations of the same symmetric group, and let

\[
\begin{align*}
\chi^{(\mu)}(A_i) &= \text{trace of } A_i \\
\chi^{(\nu)}(B_i) &= \text{trace of } B_i
\end{align*}
\]

Then the orthogonal relations are

\[
\begin{align*}
\sum_1^{\mu} \chi^{(\mu)}(A_i) \chi^{(\nu)}(B_i) &= 0 \quad (\mu \neq \nu) \\
\sum_1^{\mu} \left( \chi^{(\mu)}(A_i) \right)^2 &= h
\end{align*}
\]

These relations can be put in matrix form. Let \( G \) be the matrix of group characters, \( D \) be the diagonal matrix with elements \( h_\mu \), the number of elements in the class \( \mu \), and \( h \) the order of the group.

Then /
Then \( GDG^1 = hI \)
and \( G^1G = hD^{-1} \)
are the orthogonal relations of the group characters.

Now suppose the character vector of any representation \( R \) is \( \chi^1 \), a column vector. Then, if \( \chi^{(\mu)} \) is the row vector of characters for the \( \mu \) representation, and \( \chi^{(\mu)}D\chi^1 = m\,h \), then the representation \( R \) contains \( \mu \) \( m \) times in direct sum, in the canonical form of \( R \). This gives a method of analysing an unknown representation to obtain the irreducible components.

As an example, consider the regular representation. The only matrix which has non-zero elements in the diagonal is that representing the identity permutation, and it is the unit \( h \times h \) matrix whose trace is \( h \). The character is \( \{ h, o, o \ldots \ldots o \} \). Then if any irreducible representation is of order \( f \times f \), the first element in the character vector is \( f \).

So using the orthogonal relations,
\[
hf = mh \quad m = f
\]
and the representation is contained as often as the order of the matrix.

Let the permutation
\[
P = (i_1,i_2 \ldots \ldots i_r)(j_1,j_2 \ldots \ldots j_s) \quad ---
\]
be associated with the permutation matrix which has units at the positions \( (i_1,i_2), (i_2,i_3) \ldots \ldots (i_r,i_1),(j_1,j_2) \ldots \ldots (j_s,j_1) \quad ---. \) Then this forms a representation of the symmetric group.

To see that this is the case, observe first that the product of two permutation matrices is a permutation matrix, and so /
so is associated with a permutation. Let $P_1$ contain $r$ replaces $s$ and $P_2$ contain $s$ replaces $t$. Then $P_3 = P_1 P_2$ contains $r$ replaces $t$. The matrix associated with $P_1$ has 1 at $(r,s)$ and that associated with $P_2$ has 1 at $(s,t)$. So the product $P_1 P_2$ has a unit at $(r,t)$ which means that the permutation contains the replacement of $t$ by $r$. 
3. YOUNG'S TABLEAUX.

Let \( \lambda = (\lambda_1, \lambda_2, \lambda_3 \ldots) \), \( \lambda_1 > \lambda_2 > \lambda_3 \) \ldots; \( \sum \lambda_r = n \) be a partition of \( n \). Construct a shape with \( \lambda_1 \) spaces in the first row, \( \lambda_2 \) spaces in the second row and so on, and arrange the symbols 1, 2 \ldots n in any order in the shape. This arrangement is called a tableau of shape \( \lambda \), and was studied by A. Young. The tableaux

\[
\begin{align*}
1 & \ 2 \ 3 & 1 & \ 3 \ 6 & 6 & \ 1 \ 3 \\
4 & \ 5 & 2 & \ 5 & 4 & \ 2 \\
6 & & 4 & & 5
\end{align*}
\]

are of the shape \([3, 2, 1]\) a partition of 6. Any tableau may be transformed to any other tableau of the same shape by some permutation \( \sigma \) of the \( n \) symbols. If \( S^\lambda_s \) and \( S^\lambda_r \) are two tableaux of shape \( \lambda \), there is a permutation \( \sigma^\lambda_{rs} \) such that

\[
\sigma^\lambda_{rs} \cdot S^\lambda_s = S^\lambda_r \quad \text{and if} \quad (\sigma^\lambda_{rs})^{-1} = \sigma^\lambda_{sr}
\]

There are in all \( n! \) tableaux of any shape, \( \lambda \), involving 1, 2, \ldots n. There is a certain number, \( f^\lambda \), of these which are such that the symbols in each row and in each column are in natural order. Such tableaux are called standard, and they may be arranged in a sequence called the last-letter ordering, according to the following rule. The letter \( n \) appears at the right of a row and the foot of a column. Let all tableaux in which \( n \) is in the last row precede those in which \( n \) is in the second last row, these in turn preceding those in which \( n \) is in the second last row. In the case of two tableaux /
tableaux in which \( n \) appears in the same row, the order is decided by the order of the tableaux \( S^\pi \), which are derived from \( S \) by deleting \( n \). These are, of course, standard, and the order depends on the position of the last letter, \( n-1 \). If these derived tableaux have \( n-1 \) in the same row, a repetition of the process will eventually lead to a symbol which is not in the same row for both tableaux, and this symbol determines the order. For example, the tableaux for the partition \([4, 2]\) of 6 are arranged

\[
\begin{array}{cccccc}
1234 & 1235 & 1245 & 1345 & 1236 \\
56 & 46 & 36 & 26 & 45 \\
1246 & 1346 & 1256 & 1356 \\
35 & 25 & 34 & 24 \\
\end{array}
\]

A tableau may be such that, while the elements in the rows are in natural order, those in columns are not, or it may have the columns in standard order but not the rows. Such tableaux will be called row-standard and column-standard respectively. Every standard tableau is both row- and column-standard.

A transposition of two adjacent symbols, \( \sigma = (k, k+1) \) may have either of two effects on a standard tableau \( T \). If the two symbols are in the same row or in the same column of \( T \), \( \sigma T \) is non-standard, while otherwise \( \sigma T \) is a standard tableau distinct from \( T \). In the case of a row-standard tableau, the symbols must be in the same row if \( \sigma T \) is not to be row-standard, and similarly for column-standard.

In the last-letter ordering, any standard tableau \( T \) (except the first) may be transformed to an earlier tableau by a /
a suitably chosen transposition of the type \((k, k+1)\). For, if the symbol in the "last" position, at the end of the last row, is not \(n\), let it be \(k\). Then \(\sigma = (k, k+1)\) will bring \((k+1)\) to a later row, and, since \(k+1\) cannot be in the same row or column as \(k\), \(T\) will be standard. Thus \(T\) will be earlier in the sequence than \(T\). If \(n\) occupies the "last" position, consider the tableau \(T^x\) in which either \((n-1)\) is in the "last" position, or the above method of making \(T^x\) earlier in the sequence applies. The only tableau with which a stage is not reached at which such a transposition may be found is that which has every symbol in the latest position. This is the tableau \(T_1\) which is first in the sequence.

Since the set of standard tableaux for the conjugate partition is simply the set of transposed tableaux, and the order is exactly reversed, this proof shows that a tableau can also be made later in the sequence by the choice of a suitable transposition.

For example the tableau 1 3 5 contains 4 in the 2 6
4 "last" position, and the interchange \((4, 5)\) gives the tableau 1 3 4
2 6
5
which is earlier in the sequence. To obtain a later tableau, the position to be considered is that which would be last were the tableau transposed, namely the position occupied by 5 in the original tableau. The interchange is now \((5, 6)\) and the new tableau 1 3 6 /
which is seen to be later in the sequence. Again 1 3 4
contains 6 in the "last" position, and the tableau derived
by deleting 6 contains 5 in its "last" position. The tableau
derived by deleting both 5 and 6 contains 2 in its "last"
position, and the interchange (2, 3) gives the tableau 1 2 4
which precedes the original tableau. To obtain a later tableau
in this case, 4 occupies the position at the foot of the last
column, and the interchange (4, 5) gives 1 3 5
which is later than the original tableau.
A theorem concerning row- and column-standard tableaux,
which will be required later, is proved here.

Every column-standard tableau $S$ may be transformed
by a series of transpositions $\sigma = (k, k+1)$, each involving
adjacent symbols which appear in the same row in $S$ and in $T_1$,
the first standard tableau, into either (1) $T_1$, the first
standard tableau or (2) a tableau $S^1$ which
has two adjacent symbols $r, r+1$ in the same column, $r$ and $r+1$
being in the same row of $T_1$.

Proof: /
Proof: Suppose the earliest row of $S$ which does not consist of the same symbols as are in the corresponding row of $T_1$, contains $r, r+1, \ldots, r+s$ but not $r+s+1$. It must contain at least one element $r$ in common with $T_1$ since the columns of $S$ are in standard order. Then $(r+s+1)$ must be in the row next below this, and must have one of $r, (r+1) \ldots (r+s)$ immediately above it. Now by a series of transpositions of adjacent symbols of the set $r, (r+1) \ldots (r+s)$, all of which are in the same row in $S$ and in $T_1$, the element above $(r+s+1)$ can be made $(r+s)$. This satisfies alternative (2). If no row of $S$ contains a set of symbols different from those of the corresponding row of $T_1$, then interchanges within each row in turn transform $S$ into $T_1$.

When the tableaux concerned here are all transposed, the corresponding theorem for row-standard tableaux is obtained. The first standard tableau $T_1$ is replaced by the last, $T_\ell$, in the statement of the theorem.

As an example to clarify the theorem, consider the tableaux $S_1$: 2 1 6 3  
5 4 8 7  
7 8  

$S_2$: 2 1 4 3  
7 5 6  
8  

$S_3$: 2 4 1 3  
5 8 6 7  
7 8  

and $T_1$: 1 2 3 4  
5 6 7  
8  

In the case of $S_1$, the first row differs from the first row of
of \( T_1 \), but contains 1, 2 and 3, while 4 is in the second row of \( S_1 \). The interchanges \((1, 2)\) and \((2, 3)\) bring 3 immediately above 4, and the tableau \( S_1' \) contains 3 and 4 in the same column, while they are in the same row of \( T_1 \). In the case of \( S_2 \) the rows are simply re-arrangements of the corresponding rows of \( T_1 \), and the transpositions \((1, 2)\), \((3, 4)\), \((6, 7)\) and \((5, 6)\) applied to \( S_2 \) give \( T_1 \). In \( S_3 \) the first row is a re-arrangement of the first row of \( T_1 \), but the second contains only 5 and 6 in common with the second row of \( T_1 \). 7 appears directly below 5, and the interchange \((5, 6)\) gives a tableau \( S_3' \) which contains 6 and 7 in the same column, while they are in the second row of \( T_1 \).

The number of column-standard tableaux which in this theorem transform to \( T_1 \) is

\[(\lambda_1 !) (\lambda_2 !) (\lambda_3 !) \] where \((\lambda_1, \lambda_2, \lambda_3 \ldots)\) is the partition defining the shape of the tableaux \( S \) and \( T \). This is easily proved, as every possible permutation of each row of \( T_1 \) may be combined to form a column-standard tableau which satisfies this alternative. In the case of row-standard tableaux, the number is obtained similarly from the partition related to columns, the conjugate of \((\lambda_1, \lambda_2, \lambda_3, \ldots)\).

Let \( P \) denote the sum of the elements in the product of the positive symmetric groups of the rows of a tableau \( S \) and \( N \) the sum of the elements in the product of the negative symmetric groups of the columns of \( S \). Thus for
P = \{1, 2, 3\} \{4, 5\} \{6\}; \quad N = \{1, 4, 6\} \{2, 5\} \{3\}

If \(S_r\) and \(S_s\) are any two tableaux of the same shape, it can be shown that either \(P_rN_s = N_sP_r = 0\) or there is a tableau \(S_t\) such that \(P_r = P_t\), \(N_s = N_t\) and \(P_rN_s = P_tN_t \neq 0\).

The expressions \(E_{rs}^\alpha\) are defined by the relation

\[ E_{rs}^\alpha = \sigma_{rs}^\alpha P_s^\alpha N_s^\alpha = P_r^\alpha \sigma_{rs}^\alpha N_s^\alpha = P_r^\alpha N_r^\alpha \sigma_{rs}^\alpha \]

In particular, when \(r = s\), the expression \(E_{rr}^\alpha\) is written

\[ E_{rr}^\alpha = P_r^\alpha N_r^\alpha = E_r^\alpha \neq 0 \]

These expressions have the properties

\[ E_r^\alpha E_r^\beta = \Theta_r^\alpha E_r^\beta \]

where \(\Theta_r^\alpha\) is a numerical constant, independent of \(r\), and non-zero, and if \(X\) is any substitutional expression,

\[ E_{r\mu}^\alpha X E_{\nu s}^\beta = \Theta_r^{\nu \mu} E_{rs}^\alpha \]

where \(\Theta_r^{\nu \mu}\) is the coefficient of \(\epsilon\) in \(E_{\nu \mu}^\alpha X\).

When tableaux of different shapes \(\alpha\) and \(\beta\) are considered, it is easily shown that, if \(\alpha > \beta\),

\[ N_s^\beta P_r^\alpha = P_r^\alpha N_s^\beta = 0 \]

and deduced that, if \(X\) is any substitutional expression,

\[ E_r^\alpha X E_s^\beta = 0 \] for any \(\alpha \neq \beta\).

It follows that \(E_{r\mu}^\alpha X E_{r\nu s}^\beta = 0\).

The results given here can be used, by putting \(X = \epsilon\), to obtain

\[ E / \]
\[ E^\alpha_{\mu \nu} E^\beta_{\nu \sigma} = 0 \quad \text{if} \quad \alpha \neq \beta \]
\[ E^\alpha_{\mu \nu} E^\beta_{\nu \sigma} = 0 \quad \text{if} \quad N^\mu_{\mu} P^\nu_{\nu} = 0 \]
\[ E^\alpha_{\mu \nu} E^\beta_{\nu \sigma} = \mp \theta^\alpha \theta^\beta E^\alpha_{\nu \sigma} \quad \text{if} \quad N^\mu_{\mu} P^\nu_{\nu} \neq 0, \]

the sign being determined by finding \( S_t \), where \( N^\mu_{\mu} = N^t_{\mu} P^\nu_{\nu} = P^t \) and if \( \sigma \) is even, sign is positive; if \( \alpha \) is odd, sign is negative.

Let \( S^\alpha_{\alpha} \) be a standard tableau of shape \( \alpha \), \( S^\alpha_{\alpha} \) be the tableau derived from \( S^\alpha_{\alpha} \) by deleting \( n \), and so on. The sequence of expressions \( e^\alpha_{\alpha}, \ e^\alpha_{\alpha}, \ e^\alpha_{\alpha}, \ldots \) is defined by
\[
e^\alpha_{\alpha} = \frac{1}{\theta} e^{*\alpha} E^\alpha_{\alpha} e^{*\alpha}
\ne^{*\alpha} = \frac{1}{\theta} e^{*\alpha} E^\alpha_{\alpha} e^{*\alpha}
\]
\[
\therefore \ e^\alpha_{\alpha} = \theta, \ \text{the identity permutation.}
\]

In this definition \( E^\alpha = PN, \ e^\alpha = P^\alpha N^\alpha \) and so on.

The expressions so defined satisfy the relations
\[
e^\alpha_{\alpha} e^\beta_{\beta} = \delta^\alpha_{\alpha} \delta^\beta_{\beta} e^\alpha_{\alpha} e^\beta_{\beta}
\]
\[
\sum_{\alpha \in \tau} e^\alpha_{\alpha} = \theta
\]

These expressions are particular cases of the expressions \( e^\alpha_{\alpha} \) defined by
\[
e^\alpha_{\alpha} e^\alpha_{\alpha} = \frac{1}{\theta} e^{*\alpha} E^\alpha_{\alpha} e^{*\alpha}
\]
so that \( e^\alpha_{\alpha} = e^\alpha_{\alpha} \) as previously defined. These expressions are called the semi-normal units, and satisfy the relations
\[
e^\alpha_{\alpha} e^\beta_{\beta} = \delta^\alpha_{\alpha} \delta^\beta_{\beta} e^\alpha_{\alpha} e^\beta_{\beta}
\]
\[
E^\alpha_{\alpha} e^\alpha_{\alpha} E^\alpha_{\alpha} = \theta E^\alpha_{\alpha}
\]
\[
e^\alpha_{\alpha} E^\alpha_{\alpha} e^\alpha_{\alpha} = \theta e^\alpha_{\alpha}
\]
The semi-normal units may be expressed in terms of the permutations \( \tau_i \) of the group, say
\[
e_{rs} = \sum_i \nu_{rs \tau_i} \tau_i
\]
with numerical coefficients \( \nu_{rs \tau_i} \).

The units can be shown to be linearly independent and to be equinumerous with the permutations. The reciprocal transformation
\[
\tau_i = \sum_{r,s} u_{rs} \tau_{rs \tau_i}
\]
therefore exists and expresses the permutations in terms of the semi-normal units.

Now
\[
\sum_{r,s} u_{rs} (\tau_i \tau_j) e_{rs} = \tau_i \tau_j
\]

The semi-normal units are linearly independent, and so
\[
u_{rs} (\tau_i \tau_j) = \sum_{r,t} u_{rt \tau_i} u_{ts \tau_j}
\]

Let the matrix \( U_{\tau_i} \) have elements \( u_{rs \tau_i} \). The relation proved shows that
\[
U_{(\tau_i \tau_j)} = U_{\tau_i} U_{\tau_j}
\]
so that the matrices \( U_{\tau_i} \) form a representation of the symmetric group, associated with the partition \( \alpha \) of \( n \). If any element \( u_{rs \tau_i} \) were zero for every \( \tau \), the \( n! \) permutations would be expressible in terms of fewer than \( n! \) expressions, and so would be linearly dependent. This is impossible, and it is easily deduced that no transformation \( H \) can be found such that
\[
H (U_{\tau_i}) H^{-1}
\]
contains a zero element in the same position for every /
every \( \tau \). Hence the representation \( U_{\tau}^{\alpha} \) is irreducible.

If \( X \) is any substitutional expression, given by
\[
X = \sum \lambda_i \tau_i,
\]
its representation is
\[
U_X = \sum \lambda_i U_{\tau_i},
\]
and it has been shown by Schur that if \( U_X^\alpha C = C U_X^\beta \) for every \( X \), then either \( \alpha = \beta \), and \( C \) is a scalar matrix, or else \( C = 0 \). As a corollary to Schur's theorem, it is seen that the representations associated with different partitions \( \alpha \), and \( \beta \), of \( n \) are not equivalent.

This method thus obtains an irreducible representation of the symmetric group of degree \( n \) associated with each partition of \( n \), no two representations being equivalent. In all but the simplest cases, the calculation of the semi-normal units is very lengthy, but the matrices representing the transpositions \( (k, k+1) \) take a simple form, and from these matrices the representation of any element of the group may be found.

The representation \( U^\alpha \) consists of matrices of \( f^\alpha \) rows and columns, each row and each column being associated with a standard tableau of shape \( \alpha \). The matrices may be divided into submatrices according to the position of the last letter in the tableau. The ordering of the tableaux places first all those tableaux with \( n \) in the last row, then those with \( n \) in the second last row and so on. By considering the expression of a permutation, which does not affect the last letter, in terms of the semi-normal units of the group of degree \( n \) and also those of the group of degree \( (n-1) \), it is seen /
seen that any submatrix which is not on the leading diagonal is zero, while the submatrices on the diagonal are the representations, of the same permutation, in the group of degree (n-1), the partition being that given by the tableau $\mathbf{S}^\pi$.

For example, if $\gamma$ is a permutation not affecting 10, but permuting 1, 2 --- 9, the matrix $\bigcup_{\gamma}^{[5,3,2]}$ takes the form

$$\bigcup_{\gamma}^{[5,3,2]} = \begin{bmatrix}
\bigcup_{\gamma}^{[5,3,1]} & \cdot & \cdot \\
\cdot & \bigcup_{\gamma}^{[5,2,2]} & \cdot \\
\cdot & \cdot & \bigcup_{\gamma}^{[4,3,2]}
\end{bmatrix} = \bigcup_{\gamma}^{[5,3,1]} + \bigcup_{\gamma}^{[5,2,2]} + \bigcup_{\gamma}^{[4,3,2]}$$

where $+$ denotes direct summation.

The process may be repeated, so that, if $\gamma$ does not affect 9 or 10, the form of $\bigcup_{\gamma}^{[5,3,2]}$ is

$$\bigcup_{\gamma}^{[5,3,2]} = \bigcup_{\gamma}^{[5,3]} + \bigcup_{\gamma}^{[5,2,1]} + \bigcup_{\gamma}^{[4,3,1]} + \bigcup_{\gamma}^{[5,2,1]} + \bigcup_{\gamma}^{[4,2,2]} + \bigcup_{\gamma}^{[4,3,1]} + \bigcup_{\gamma}^{[4,2,2]} + \bigcup_{\gamma}^{[3,3,2]}$$

The conception of the axial distance between two elements of a tableau is required in the construction of the matrices for transpositions. If in a tableau $\mathbf{S}$ the element $k$ is in row $i_k$, column $j_k$, while $l$ is in row $i_l$ column $j_l$, the axial distance from $k$ to $l$ is defined as

$$d_{k,l} = (i_l - j_l) - (i_k - j_k).$$

This is obtained graphically by following any rectangular route /
route from \( k \) to \( l \), and counting +1 for every step down or to the left, -1 for every step up or to the right. The resultant total is \( d_{k,l} \).

Consideration of the form of the matrix representing \( E_{1}^{x} \), and the matrix \( \bigcup (n-1,n) \) representing the transposition \( (n-1,n) \) which commutes with every permutation affecting only the letters 1, 2, ..., \( n-2 \), and is such that \( (n-1,n)^{2} = e \), leads to the theorem defining the matrix for any transposition of the form \( (k-1, k) \). The theorem is here stated without proof.

The matrix \( \bigcup (k-1,k) \) has elements

(i) +1 at \((r,r)\) if \( S_{r}^{x} \) has \( k-1 \) and \( k \) in the same row

(ii) -1 at \((r,r)\) if \( S_{r}^{x} \) has \( k-1 \) and \( k \) in the same column

(iii) -\( \frac{1}{\rho} \) \( 1-\rho^{2} \) at \((r,r)\) \((r,s)\)

\[ l + \frac{1}{\rho} \] \((s,r)\) \((s,s)\) respectively, if \( r < s \), and \( S_{s}^{x} \) is obtained from \( S_{r}^{x} \) by interchange of \( k-1 \) and \( k \), where \( \frac{1}{\rho} \) is the axial distance in \( S_{r}^{x} \) from \( k-1 \) to \( k \).

(iv) 0 in every other position.

Two particular representations may be noticed here.

For \( x = [n] \) \( f^{x} = 1 \) and every element is in the same row. So every transposition \( (k-1,k) \) is represented by 1. It is easily seen that in this case every permutation is represented by unity. Again, for \( x = [1^{n}] \) \( f^{x} = 1 \) and every transposition \( (k-1,k) \) is represented by -1. The representation in this case is such that every even permutation is represented by +1, and every odd permutation by -1.

Since it has been shown that every permutation may be written /
written down as a product of transpositions of the form \((k-1, k)\) it is clear that the representation of any permutation may be obtained by means of this theorem. The representation is not orthogonal, but may easily be made so, as will be shown.

The function \(\phi_s(n)\) for a tableau \(S\) of \(n\) letters is defined as \(\phi_s(n) = 1\) if \(n\) appears in the first row of \(S\) and otherwise as \(\phi_s(n) = \frac{1}{n^s}(1 + c_{n\sigma})\) where there is one factor for every row \(\sigma\) of \(S\) above that in which \(n\) lies, and \(\frac{1}{c_{n\sigma}}\) is the axial distance from the last letter of row \(\sigma\) to \(n\).

\[
\phi_s(n-1) = \phi_s(n-1) \\
\phi_s(n-2) = \phi_s(n-2)
\]
define the corresponding functions of other elements. The tableau function \(\psi_r\) of a tableau \(S_r\) is defined by

\[
\psi_r = \phi_r(n) \phi_r(n-1) \ldots \phi_r(1)
\]
where \(\phi_r\) is written for \(\phi_{sr}\).

Consideration of \(\psi_r\) and \(\psi_s\) in the case where \(S_r\) and \(S_s\) differ only by the interchange of \(k-1\) and \(k\) leads to the equation

\[
\psi_r = (1 - \rho^2) \psi_s \quad (r < s)
\]
where \(\rho\) is the axial distance from \(k-1\) to \(k\) in \(S_r\).

Confining attention to one partition \(\lambda\), let the matrix \(H\) be a diagonal matrix with \(r\) th element \(\sqrt{\psi_r}\), the tableau being \(S_r^{\lambda}\). Consider the representation of the symmetric group given by \(\psi_\lambda = H^{-1} \psi_\lambda^\lambda H\)

which is equivalent to the representation already discussed. In the case \(\gamma = (k-1, k)\), \(V_\gamma\) is the same as \(U_\gamma\) except that

\[
-\rho^2 \quad 1 - \rho^2 \quad \text{is replaced by} \quad -\rho \quad \frac{1 - \rho^2}{\sqrt{1 - \rho^2}} \quad \text{and} \quad 1 + \rho \quad \frac{\sqrt{1 - \rho^2}}{\rho}
\]
Thus the matrix $V_\tau$ for a transposition $(k-1, k)$ is orthogonal. So for any $\tau$, when $V_\tau$ is a product of orthogonal matrices, the matrix is orthogonal.

The representation derived here is the orthogonal representation of the symmetric group associated with the partition $\lambda$.

3 (a) Representations of Group of Degree 6.

To illustrate the use of the theorem giving the form of the generators $(k, k+1)$, the orthogonal representations of the elements $(1 2)$, $(2 3)$, $(3 4)$, $(4 5)$ and $(5 6)$ of the group of degree 6 are obtained here, for three partitions.

(a) $[5 \, 1]$ representation.

The standard tableaux are

\[
S_1 : \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6
\end{array} \\
S_2 : \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6
\end{array} \\
S_3 : \begin{array}{cccc}
1 & 2 & 3 & 5 \\
4 & 6
\end{array} \\
S_4 : \begin{array}{cccc}
1 & 2 & 4 & 5 \\
6 & 3
\end{array} \\
S_5 : \begin{array}{cccc}
1 & 3 & 4 & 5 \\
6
\end{array}
\]

and the representations are

\[
(1 2) = \begin{bmatrix}
1 & \ldots & \ldots & \\
\cdot & 1 & \ldots & \\
\ldots & \cdot & 1 & \\
\ldots & \ldots & \ldots & -1
\end{bmatrix}
\]

\[
(2 3) = \begin{bmatrix}
1 & \ldots & \ldots & \\
\cdot & 1 & \ldots & \\
\ldots & \cdot & 1 & \\
\ldots & \ldots & \cdot & -\frac{1}{2} \frac{\sqrt{2}}{2}
\end{bmatrix}
\]

\[
(3 4) = \begin{bmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\]
(b) $[4\ 2]$ representation.

The standard tableaux are

<table>
<thead>
<tr>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3 4 5 6</td>
<td>1 2 3 5 6</td>
<td>1 2 4 5 6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$S_4$</th>
<th>$S_5$</th>
<th>$S_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 3 4 5 2 6</td>
<td>1 2 3 6 4 5</td>
<td>1 2 4 6 3 5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$S_7$</th>
<th>$S_8$</th>
<th>$S_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 3 4 6 2 5</td>
<td>1 2 5 6 3 4</td>
<td>1 3 5 6 2 4</td>
</tr>
</tbody>
</table>
and the representations are

\[
\begin{array}{cccccccccc}
1 & & & & & & & & & \\
& 1 & & & & & & & & \\
& & 1 & & & & & & & \\
& & & -1 & & & & & & \\
& & & & 1 & & & & & \\
& & & & & 1 & & & & \\
& & & & & & -1 & & & \\
& & & & & & & -1 & & \\
& & & & & & & & 1 & \\
\end{array}
\]

\[
(1 \ 2)
\]

\[
\begin{array}{cccccccccc}
1 & & & & & & & & & \\
& 1 & & & & & & & & \\
& & & \frac{1}{2} & & & & \frac{\sqrt{3}}{2} & & \\
& & & \frac{\sqrt{3}}{2} & & & & & & \\
& & & & 1 & & & & & \\
& & & & & \frac{1}{2} & & \frac{\sqrt{3}}{2} & & \\
& & & & & & & \frac{1}{2} & & \\
& & & & & & & & \frac{1}{2} & \\
\end{array}
\]

\[
(2 \ 3)
\]

\[
\begin{array}{cccccccccc}
1 & & & & & & & & & \\
& 1 & & & & & & & & \\
& & & \frac{1}{2} & & & & \frac{\sqrt{3}}{2} & & \\
& & & \frac{\sqrt{3}}{2} & & & & & & \\
& & & & 1 & & & & & \\
& & & & & \frac{1}{2} & & \frac{\sqrt{3}}{2} & & \\
& & & & & & & \frac{1}{2} & & \\
& & & & & & & & \frac{1}{2} & \\
\end{array}
\]

\[
(3 \ 4)
\]
\[
\begin{pmatrix}
1 & & & & & & \\
& \frac{1}{3} & \frac{2\sqrt{3}}{3} & & & & \\
& \frac{2\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{2} & & & \\
& & -\frac{1}{3} & \frac{2\sqrt{3}}{3} & \frac{2\sqrt{2}}{2} & & \\
& & \frac{2\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\
& & & & & & \mathbf{1}
\end{pmatrix}
\]
(c) $[3^2]$ representation.

The standard tableaux are

$S_1 : \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & \end{array}$

$S_2 : \begin{array}{cccc} 1 & 2 & 4 & 3 \\ & 3 & 5 & 6 \end{array}$

$S_3 : \begin{array}{cccc} 1 & 3 & 4 & 2 \\ 3 & 4 & 6 & 5 \end{array}$

$S_4 : \begin{array}{cccc} 1 & 2 & 5 & 3 \\ 3 & 4 & 6 & 2 \end{array}$

$S_5 : \begin{array}{cccc} 1 & 3 & 5 & 2 \\ 2 & 4 & 6 & 3 \end{array}$

and the representations are

\begin{align*}
(1 \ 2) & \left[ \begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \\ & & & -1 \end{array} \right] \\
(2 \ 3) & \left[ \begin{array}{cccc} 1 & & & \\ & -\frac{1}{2} & & \\ & & 1 & \\ & & & -\frac{1}{2} \\ & & & \frac{\sqrt{2}}{2} \end{array} \right] \\
(3 \ 4) & \left[ \begin{array}{cccc} -\frac{1}{3} & \frac{\sqrt{3}}{3} & & \\ & \frac{1}{3} & & \\ & & 1 & \\ & & & 1 \\ & & & -1 \end{array} \right] \\
(4 \ 5) & \left[ \begin{array}{cccc} 1 & & & \\ & -\frac{1}{2} & & \\ & & 1 & \\ & & & -\frac{1}{2} \\ & & & \frac{\sqrt{2}}{2} \end{array} \right] \\
(5 \ 6) & \left[ \begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \\ & & & -1 \end{array} \right]
\end{align*}
4. **CENTRAL CORES OF IN Variant MATRICES.**

An invariant matrix \( T(A) \) is a matrix which has elements polynomial and homogeneous in the elements of a matrix \( A \), and which satisfies

\[
T(AB) = T(A)T(B)
\]

It is assumed that \( T(A) \) is non-singular in the sense that for an arbitrary non-singular \( A \), \( T(A) \) is non-singular.

\( T(A) \) may be reducible, if there is a constant matrix \( H \) such that for any \( A \),

\[
H^{-1}T(A)H = \begin{bmatrix} T_1(A) & U(A) \\ 0 & T_2(A) \end{bmatrix}
\]

where \( T_1(A) \) and \( T_2(A) \) are square sub-matrices, easily seen to be invariant matrices. If also \( U(A) = 0 \), \( T(A) \) is completely reducible.

Certain properties of invariant matrices are easily proved e.g.

\[
T(I) = I \\
T(A^{-1}) = \{T(A)\}^{-1}
\]

Corresponding to Schur's theorem for representations of groups, there is a theorem for irreducible invariant matrices, that if

\[
T_1(A) P = P T_2(A)
\]

either \( P = 0 \) or \( P \) is non-singular and \( T_1(A) \) is equivalent to \( T_2(A) \). It follows that all matrices which commute with an irreducible \( T(A) \) are scalar matrices.

It is well-known that the compound matrix \( A^{(m)} \) satisfies /
satisfies the Binet-Cauchy relation
\[ A^{(m)} B^{(m)} = (AB)^{(m)} \]
and so the compound matrix is an invariant matrix. In the same way, the induced matrix \( A[m] \) satisfies the relation
\[ A[m] B[m] = (AB)[m] \]
and is also an invariant matrix.

If \( T_1(A) \) and \( T_2(A) \) are invariant matrices, the direct product \( T_1(A) \times T_2(A) \) obeys the product law
\[
[T_1(A) \times T_2(A)][T_1(B) \times T_2(B)] = [T_1(A) T_1(B)] \times [T_2(A) T_2(B)]
\]
and so is also an invariant matrix. It follows that the direct product of any number of compound or induced matrices is an invariant matrix.

It is easily shown that the latent roots of an invariant matrix are of the form
\[ \lambda_1, \lambda_2, \ldots, \lambda_N \]
where \( \lambda_1, \lambda_2, \ldots, \lambda_N \) are the latent roots of \( A \), and that the trace of \( T(A) \) is a symmetric polynomial in the roots of \( A \).

In the case of \( A^{(m)} \) the trace is the \( m \)th symmetric function \( \xi_m \), while the trace of \( A[m] \) is the complete symmetric function \( h_m \). The trace of the direct product is the product of the traces of the matrices in the product.

The invariant matrices constructed from direct products of compound and induced matrices are generally reducible. Let a column of \( T(A) \) be denoted by \( t(A) \). The existence of some non-zero column \( t(A) \) with linearly dependent elements /
elements leads to the conclusion that $T(A)$ is reducible.

For, there is a non-singular constant matrix $H$ such that

$$H \; t(A) = \begin{bmatrix} t_1(A) & 0 \end{bmatrix}$$

where $t_1(A)$ has elements that are linearly independent.

Therefore

$$H \; T(A) \; H^{-1} \; t(X) = H \; T(A) \; t(X) = H \; t(AX)$$

Partitioning this:

$$\begin{bmatrix} T_1(A) & U(A) \\ V(A) & T_2(A) \end{bmatrix} \begin{bmatrix} t_1(X) \\ 0 \end{bmatrix} = \begin{bmatrix} t_1(AX) \\ 0 \end{bmatrix}$$

and so

$$V(A) \; t_1(X) = 0$$

But $t_1(X)$ has linearly independent elements, functions of the arbitrary $X$, and hence $V(A) = 0$, so that $T(A)$ is reducible.

For example $T(A) = A \; x \; A$ has columns $t(X)$, the first of which is

$$\begin{bmatrix} x_{11}^2 \\ x_{11} \; x_{21} \\ x_{21} \; x_{11} \\ x_{21}^2 \end{bmatrix}$$

This has two equal elements, and so the matrix $T(A) = A \; x \; A$ is reducible. The matrix $H$ which performs the reduction is

$$H = \begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & -1 & 1 & . \end{bmatrix}$$

and $H^{-1} = \begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{bmatrix}$

The reduced form of $T(A)$ is

$$all^2 /$$
\[
\begin{bmatrix}
  a_{11}^2 & 2a_{11}a_{12} & a_{12}^2 & a_{12}a_{11} \\
  a_{11}a_{21} & a_{11}a_{22} & a_{12}a_{21} & a_{12}a_{22} \\
  a_{21}^2 & 2a_{21}a_{22} & a_{22}^2 & a_{22}a_{21} \\
  \vdots & \vdots & \vdots & \vdots \\
  a_{22} & a_{22} & a_{22} & a_{22} - a_{12}a_{21}
\end{bmatrix}
\]

In the case if \( T(A) = A \times A^{(2)} \) there is a vector
\[
\begin{bmatrix} x_{11} & x_{21} & x_{31} \end{bmatrix} \times \begin{bmatrix} |x_{11} x_{22}| & |x_{11} x_{32}| & |x_{21} x_{32}| \end{bmatrix}
\]
containing nine elements. These are linearly dependent, the relation
\[
x_{11} \cdot |x_{21} x_{32}| - x_{21} \cdot |x_{11} x_{32}| + x_{31} \cdot |x_{11} x_{22}| = 0
\]
being the expansion of a determinant with equal columns. The matrix \( H \) reducing \( T(A) \) is that which gives the operation row 3 - row 5 + row 7, and \( H^{-1} \) consists of col. 5 + col. 3 : col. 7 - col. 3. When \( HT(A)H^{-1} \) is calculated, \( |a_{11} a_{22} a_{33}| \) appears semi-isolated at position \((3,3)\). The remaining submatrix is of 8 rows and 8 columns,
\[
\begin{bmatrix}
  a_{11} \cdot |a_{11} a_{22}| & a_{11} \cdot |a_{11} a_{23}| & \vdots & a_{13} \cdot |a_{12} a_{23}| \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{31} \cdot |a_{21} a_{32}| & a_{31} \cdot |a_{21} a_{33}| & \vdots & a_{33} \cdot |a_{22} a_{33}|
\end{bmatrix}
\]
the fourth and sixth rows containing elements of a more complicated nature.

The direct product of compound or induced matrices is characterised by the fact that the elements in each column are associated with a fixed selection of columns of \( A \), and the elements /
elements in each row with a fixed selection of rows of A. The nature of the reduction by the matrix $H$ is such that only those columns or rows which are derived from the same selection of columns or rows of $A$ are combined with each other. Thus in the example $T(A) = A \times A^{(2)}$ the columns combined have one element selected from each column of A, and the rows one element from each row of A.

If the elements of $T(A)$ have degree equal to $n$, the number of rows and columns in A, certain rows and certain columns of $T(A)$ are associated with the selection consisting of each row or each column taken once. These elements are sums of products containing one factor from each row or each column. In the example, the fourth and sixth rows and columns are those with this property. The elements common to these rows and columns form a square submatrix which is called the central core of the invariant matrix. In the example this central core is

$$
\begin{bmatrix}
    a_{11} & a_{22} & a_{33} & a_{21} & a_{12} & a_{33} & -a_{31} & a_{22} & a_{13} & -a_{21} & a_{32} & a_{13} \\
    a_{11} & a_{32} & a_{23} & +a_{31} & a_{12} & a_{23} & -a_{21} & a_{32} & a_{13} & -a_{31} & a_{22} & a_{13} \\
    a_{11} & a_{22} & a_{23} & +a_{21} & a_{32} & a_{13} & -a_{21} & a_{12} & a_{33} & -a_{31} & a_{12} & a_{23} \\
    a_{11} & a_{32} & a_{33} & +a_{31} & a_{22} & a_{13} & -a_{21} & a_{12} & a_{33} & -a_{31} & a_{12} & a_{23}
\end{bmatrix}
$$

The central core so constructed is a group matrix for the appropriate symmetric group. The group matrix is of the form $\sum_i \gamma_i (a_{11} \quad a_{22} \quad a_{33}) P_i$ where $P_i$ is the representation of the permutation $\gamma_i$, and $\gamma_i (a_{11} \quad a_{22} \quad a_{33} \ldots)$ is the result /
result of the operation $T_i'$ on the column suffixes.

The proof of this theorem is as follows.

Consider the product of permutations $T_i' T_j' = T_k'$ and the permutation matrix product $A_i A_j = A_k$. In $A_i$, every element $a_{rs}$ which occurs in the product $T_i' (a_{11} a_{22} a_{33} \ldots)$ is unity, while every other element is zero.

In any column of $T(A)$ the elements are associated with a constant selection of column suffixes, but varying selections of row suffixes. For example $a_{11} a_{22} a_{33} a_{44}$ in the core will have, in the same column $a_{11} a_{12} a_{23} a_{24}$. So in $T(A)$ all the terms which occur in the same columns as the central core, but in rows other than those occupied by the central core, will have at least two factors from one row but different columns, of $A$. When $A$ is a permutation matrix $A_i$, one factor at least will be zero, so that the sub-matrices in the same columns as the central core, but different rows, are zero. In the same way, the sub-matrices in the same rows as the central core, but different columns, are zero. So $T(A_i)$ is of the form

$$
\begin{bmatrix}
A_{11} & 0 & A_{13} \\
0 & A_{22} & 0 \\
A_{31} & 0 & A_{33}
\end{bmatrix}
$$

and matrices of this shape conserve their form in multiplication. So the central core $P_k$ of $T(A_k) = T(A_i A_j) = T(A_i) T(A_j)$ is $P_i P_j$. This shows that the $P_i$ obey the same multiplication rules as $T_i'$; in fact, $P_i$ is a matrix representation /
representation of the group, and the central core of $T(A)$ is the group matrix for this representation.

A case of particular interest is the central core of the $n$th direct power $A^n$. The $i$th row of the central core contains a product with one factor chosen from each row of $A$, the rows being taken in the order $\eta_i (1,2,3--n)$ and the $j$th column similarly from columns taken in the order $\eta_j (1,2,3--n)$. Thus the product at position $(i, j)$ is $A_{i1}A_{i2}A_{i2}---A_{in}j_n$ where $\eta_i (1,2,3--n) = (i_1, i_2--i_n)$ and $\eta_j (1,2,3--n) = (j_1, j_2--j_n)$. To re-arrange the row suffixes in the natural order, the permutation $\eta_i^{-1}$ must be applied to both sets of suffixes, so that the product appears as $a_{k1}a_{k2}a_{k3}---a_{kn}$ where $(k_1, k_2--k_n) = \eta_i^{-1}(j_1, j_2--j_n) = \eta_i^{-1}\eta_j (1,2,3--n)$. This means that the group matrix contains a term $\eta_k(a_{11}a_{22}---)$ at position $(i, j)$ when

$$\eta_i^{-1}\eta_j = \eta_k.$$  

This is exactly the condition previously found for the element in the regular group matrix of Frobenius at position $(i, j)$ to be $x_k$. It follows that the central core of the $n$th direct power $A^n$ is the regular group matrix.

As an example, consider $A^3$ where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$
The central core $C$ of $A^{[3]}$ is

\[
\begin{bmatrix}
    a_1 b_2 c_3, a_1 c_2 b_3, b_1 a_2 c_3, b_1 c_2 a_3, c_1 a_2 b_3, c_1 b_2 a_3 \\
    a_1 b_3 c_2, a_1 c_3 b_2, b_1 a_3 c_2, b_1 c_3 a_2, c_1 a_3 b_2, c_1 b_3 a_2 \\
    a_2 b_1 c_3, a_2 c_1 b_3, b_2 a_1 c_3, b_2 c_1 a_3, c_2 a_1 b_3, c_2 b_1 a_3 \\
    a_2 b_3 c_1, a_2 c_3 b_1, b_2 a_3 c_1, b_2 c_3 a_1, c_2 a_3 b_1, c_2 b_3 a_1 \\
    a_3 b_1 c_2, a_3 c_1 b_2, b_3 a_1 c_2, b_3 c_1 a_2, c_3 a_1 b_2, c_3 b_1 a_2 \\
    a_3 b_2 c_1, a_3 c_2 b_1, b_3 a_2 c_1, b_3 c_2 a_1, c_3 a_2 b_1, c_3 b_2 a_1 
\end{bmatrix}
\]

and if the products are replaced in turn

\[
\begin{align*}
    a_1 b_2 c_3 &= x_0 & a_2 b_3 c_1 &= x_5 \\
    a_1 b_3 c_2 &= x_3 & a_3 b_1 c_2 &= x_4 \\
    a_2 b_1 c_3 &= x_1 & a_3 b_2 c_1 &= x_2
\end{align*}
\]

the central core becomes the group matrix given on page $1^*$, with a re-arrangement of rows and the congruent re-arrangement of columns.

The representations so obtained from the central cores of invariant matrices are not orthogonal, but by a modification of the method by which they are derived, they may be made orthogonal.

It is easily seen that compound matrices satisfy

\[(A^{(m)})' = (A')^{(m)}\]

and hence, when $A$ is orthogonal

\[(A^{(m)})' (A^{(m)}) = (A'A)^{(m)} = I\]

so that the compound is also orthogonal.

In the case of induced matrices, it is necessary first to "prepare" or "normalise" $A^{[m]}$. 
For example, if
\[
A = \begin{bmatrix}
  a_1 & b_1 \\
  a_2 & b_2
\end{bmatrix}
\]
then
\[
A^2 = \begin{bmatrix}
  a_1^2 + 2a_1b_1 & b_1^2 \\
  a_1a_2 + a_1b_2 + a_2b_1 & b_1b_2 \\
  a_2^2 + 2a_2b_2 & b_2^2
\end{bmatrix}
\]

To normalise this, the second column is multiplied by \(1/\sqrt{2}\) while the second row is multiplied by \(\sqrt{2}\). A procedure of this type always ensures that the prepared induced matrix is orthogonal when \(A\) is orthogonal. It will be assumed henceforward that all induced matrices have been normalised.

The direct product of orthogonal matrices is always orthogonal.

To illustrate the method by which the orthogonality of the central core may be preserved, the case \(A^2 \times A\) will be considered. The central core is
\[
C = \begin{bmatrix}
  a_1b_2 & c_3 & a_1c_2 & b_3 & b_1c_2 & a_3 \\
  a_1b_3 & c_2 & a_1c_3 & b_2 & b_1c_3 & a_2 \\
  a_2b_3 & c_1 & a_2c_3 & b_1 & b_2c_3 & a_1
\end{bmatrix}
\]

In the first column of \(A^2 \times A\), in the rows to which the central core belongs, \(\sqrt{2} a_1a_2a_3\) appears in each position. This is a linearly dependent vector, and the matrix \(H\) which removes the dependence is
\[
H = \begin{bmatrix}
  1 & & \\
  -1 & 1 & \\
  -1 & & 1
\end{bmatrix}
\]
The last row of \( H \) may be normalised, giving

\[
\begin{bmatrix}
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\end{bmatrix}
\]

Next, the preceding row is made orthogonal to the last row, and the new row normalised, giving

\[
\begin{bmatrix}
\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\end{bmatrix}
\]

The row above that is now made orthogonal to both the new rows, and then normalised, giving

\[
\begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\end{bmatrix}
\]

The same procedure could be continued until every row had been dealt with, and the complete orthogonal matrix so derived is

\[
K = \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\end{bmatrix}
\]

The calculation of \( K \ C \ K' \) shows that \( a_1 b_2 c_3 \) is isolated completely in the first row and column, and the group matrix for another representation appears in the 2nd and 3rd rows and columns.

The complete isolation is not accidental, but appears in /
in every case. The premultiplication of \( C K' \) by \( K \) produces semi-isolation while the post-multiplication of \( K C \) by \( K' \) produces a transposed semi-isolation. The combination of the two shows that complete isolation of one group matrix, here \( |a_1 b_2 c_3| \), must occur.

It has now been shown that when the central core of a direct product of normalised induced matrices or of compound matrices is orthogonally reduced to a direct sum of irreducible central cores, each of these is a group matrix for an irreducible orthogonal representation.

The central cores so obtained are the central cores of orthogonal invariant matrices. An outline of the method by which the invariant matrices may be obtained from the direct product of compound or induced matrices will be given next.

In a direct product of compound matrices the first column contains elements such as

\[
|a_1 b_2 c_3|, |a_4 b_5 c_6| \quad \text{and} \quad |a_1 b_3 c_5|, |a_2 b_4 c_6|.
\]

These elements are known as Clebsch products. There is a set of such elements, for each partition, which are linearly independent, and all other elements are linearly dependent on this set. The elements are associated with tableaux, those for the examples given being

\[
\begin{align*}
T_1 &: \quad 1 \ 4 \ \ 1 \ 2 \\
T_2 &: \quad 2 \ 5 \\
T_3 &: \quad 3 \ 4 \\
T_4 &: \quad 5 \ 6 \\
T_5 &: \quad \ 6 \ 5
\end{align*}
\]

and it will be shown that the set which forms a basis for all elements consists of those which are associated with the standard /
standard tableaux.

The order given to the tableaux will here be determined by the rule: Read down the columns of the tableaux till the first pair of differing indices is found. The earlier tableau is the one with the earlier index of this pair. In the case given $T_1$ contains 2 which precedes the 3 of $T_j$.

It will be shown that the leading term in the Clebsch product does not occur among the terms of the expansion of any earlier Clebsch product. Compare $T_1$ and $T_j$, with reference to the earliest differing index, here 2 of $T_1$ and 3 of $T_j$. The only way to produce $a_1 b_2 c_5$ in a term of $|a_1 b_2 c_5|$ is to consider the expansion as $|a_1 b_3 c_2|, |a_4 b_5 c_6|$ or $|a_1 b_3 c_2|, |a_6 b_4 c_5|$. The term is then $a_1 b_3 c_5 \cdot a_4 b_6 c_2$ or $a_1 b_3 c_5 \cdot a_6 b_4 c_2$, both of which are non-standard. To obtain the first product from the tableau 1 4 2 5 3 6 the procedure has been: Interchange 2 and 3, then 5 and 6, then 5 and 2. Both the interchange of 2 and 3, and of 5 and 2 have made the new tableau non-standard.

This holds in the general case, and proves the assertion that the leading term of the Clebsch product does not occur in any earlier product. So the set of Clebsch products associated with standard tableaux is linearly independent. This is similar to the linear independence of Young's tableau operators.

The next stage of the proof consists in showing that
any product with a non-standard tableau can be expressed in terms of the standard products. It may be possible to transform to a standard tableau by the interchange of columns of equal length. If this is so, the Clebsch product is equal to that for a standard tableau. If this cannot be done, the non-standardness may be removed by stages.

Thus, in a tableau such as

\[
\begin{array}{cccc}
1 & 2 & 5 & 3 \\
4 & 6 & 7 & 8 \\
\end{array}
\]

the earliest index out of order is 3. The tableau is associated with the partitioned determinant

\[
\begin{vmatrix}
A_1 & A_2 & A_3 & A_5 & A_6 & A_7 & A_8 \\
A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 \\
\end{vmatrix}
\]

where \( A_i = \{ a_i, b_i, c_i, d_i \} \)

The overlap of the two rows is determined by following the second column to the out of order index 3, then reverting to the first column, giving 2 3 5 6 7 as the overlap. The remainder of the first column is used in row 1 of the determinant, the second column in row 2. Expand the determinant by a Laplacian expansion, noting that the determinant vanishes.

\[
\begin{vmatrix}
a_1 b_2 c_3 d_5 \\
a_4 b_6 c_7 d_8 \\
\end{vmatrix} + \cdots + \begin{vmatrix}
a_1 b_5 c_6 d_7 \\
\end{vmatrix} = 0
\]

and the last term, which is the latest in order, is expressible in terms of other Clebsch products. If these are non-standard they may be expressed in terms of still earlier products, until a stage is reached at which all the products are standard.

It is easily seen that the above result applies equally.
equally to tableaux with repeated indices, the basis being the set of standard tableaux with the same selection of indices. If two products correspond to tableaux with different sets of indices, they have no terms in common, and so are linearly independent.

The result shows that there is a vector of Clebsch products corresponding to all standard tableaux of shape \([\lambda]\), the elements of the vector being linearly independent, and forming a basis for the set of all Clebsch products of shape \([\lambda]\). It is associated with the conjugate partition \([\lambda]^t\), since the number of elements is then equal to the number of terms in the bialternant \(h_{[\lambda]}\).

The construction of an irreducible orthogonal \(T(A)\) of type \([\lambda]\) is therefore as follows. Construct the direct product of compound matrices \(A^{(\lambda'_1)}\) where \(\lambda'_1\) takes in turn the value of each part of \([\lambda]^t\). In the first column bring the Clebsch vector to the leading position, by a rearrangement of rows and the congruent change of columns. The rest of the first column is linearly dependent on this vector and so there is a matrix \(H\) which semi-isolates the leading sub-matrix. The rows of \(H\) which produce the zeros below this sub-matrix may be orthogonalised without affecting the result, and a continuation of the orthogonalising process through the whole of \(H\) produces the orthogonal \(K\) which completely isolates the matrix \(A^{[\lambda]}\) in leading position. This is the irreducible \(T(A)\) corresponding to partition \([\lambda]\).

The central core of the matrix \(A^{[\lambda]}\) is the orthogonal /
orthogonal representation of the symmetric group, associated with the partition \( [N] \). It may be obtained directly by the reduction of the central core of the direct product of compound matrices. Let C be the central core, R the representation corresponding to \( [N] \), and it is known that an orthogonal matrix M exists such that R is the leading submatrix of \( MCC' \), say

\[
MCC' = \begin{bmatrix} R & \cdot \\ \cdot & X \end{bmatrix}
\]

and

\[
MC = \begin{bmatrix} R & \cdot \\ \cdot & X \end{bmatrix} M
\]

Partition M by rows conformably with this, so that

\[
\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} C = \begin{bmatrix} R & \cdot \\ \cdot & X \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}
\]

and

\[
M_1 C = RM_1
\]

Now the form of R for the transpositions \( \sigma = (k,k+1) \) has been determined using the Young's tableaux method. With \( \rho \) as defined in section 3,

\[
\tau_{ii} = +1 \text{ if both symbols of } \sigma \text{ are in the same row of } T_i
\]

\[
\tau_{ii} = -1 \text{ if both symbols of } \sigma \text{ are in the same column of } T_i
\]

\[
\tau_{ij} = -\rho \text{ if } \sigma T_i = T_j \quad (i < j)
\]

\[
\tau_{ij} = +\rho \text{ if } \sigma T_i = T_j \quad (i > j)
\]

\[
\tau_{ij} = (1-\rho^2) \text{ if } \sigma T_i = T_j \quad (i \neq j)
\]

and in all other cases \( \tau_{ij} = 0 \).

The form of C is readily found. There is associated with
with the s-th row or column of $C$, a column-standard tableau, in the same way as the Clebsch product is associated with a tableau. If an interchange of two elements in a column of such a tableau is regarded as leaving the same tableau, the elements of $C$ are

$$C_{ss} = -1 \text{ if } \sigma S_s = S_s$$
$$C_{st} = +1 \text{ if } \sigma S_s = S_t$$
$$C_{st} = 0 \text{ in all other cases.}$$

The elements of $M_1C$ and $RM_1$ can now be written down. Consider the element at position $(i, s)$.

In $M_1C$: If $\sigma S_s = S_s$, $\sum_{mir}^c = -m_{is}$

If $\sigma S_s = S_t$, $\sum_{mir}^c = m_{it}$

In $RM_1$: If both symbols of $\sigma$ are in the same row of $T_i$

$$\sum_{rir}^m = m_{is}$$

If both symbols of $\sigma$ are in the same column of $T_i$

$$\sum_{rir}^m = -m_{is}$$

If $\sigma T_i = T_j$ and $i < j$

$$\sum_{rir}^m = -p m_{is} + \sqrt{1 - p^2} m_{js}$$

If $\sigma T_i = T_j$ and $i > j$

$$\sum_{rir}^m = p m_{is} + \sqrt{1 - p^2} m_{js}$$

When the equality $M_1C = RM_1$ is considered, the following cases arise:

1. $\sigma S_s = S_s$

   (a) $\sigma$ in a row of $T_i$ $\quad -m_{is} = m_{is} = 0$

   (b) $\sigma$ in a column of $T_i$ $\quad -m_{is} = -m_{is}$

   (c) $\sigma T_i = T_j$ $(i < j)$ $\{ -m_{is} = -p m_{is} + \sqrt{1 - p^2} m_{js} \}$

   $\sigma T_j = T_i$ $\{ -m_{js} = p m_{js} + \sqrt{1 - p^2} m_{is} \}$

2. /
2. $\sigma S_s = S_t$

(a) $\sigma$ in a row of $T_i$
$$m_{it} = m_{is}$$

(b) $\sigma$ in a column of $T_i$
$$m_{it} = -m_{is}$$

(c) $\sigma T_i = T_j \quad (i \neq j)$

1. $m_{it} = -\rho m_{is} + (1 - \rho^2) m_{js}$
2. $m_{is} = -\rho m_{it} + (1 - \rho^2) m_{jt}$
3. $m_{jt} = \rho m_{js} + (1 - \rho^2) m_{is}$
4. $m_{js} = \rho m_{jt} + (1 - \rho^2) m_{it}$

Two deductions may be made directly from these. If two adjacent symbols are in the same row of $T_i$ and in the same column of $S_s$, the $(i, s)$ element of $M$ is zero. If two symbols are in the same row of $T_i$ and $\sigma S_s = S_t$, the elements $m_{is}$ and $m_{it}$ are equal, while if the symbols are in the same column of $T_i$ and $\sigma S_s = S_t$, the elements $m_{is}$ and $m_{it}$ differ only in sign.

Consider the row of $M_1$ related to the first standard tableau $T_1$. It has been shown that a series of transpositions of the form $(k, k+1)$ where $k$ and $k+1$ are in the same row of $T_1$ and of $S$, can be found such that, for any $S$, either

(a) $\sigma_1 \sigma_2 \ldots \sigma_r S = T_1$

or (b) $\sigma_1 \sigma_2 \ldots \sigma_r S = S'$

where $S'$ contains two adjacent symbols in the same column, the symbols being in the same row of $T_1$. In both cases, the elements of $M_1$ in the columns associated with $S$, $\sigma_r S$, $\ldots$, $\sigma_1 \sigma_2 \ldots \sigma_r S$ are, by case 2 (a), all equal. By case 1 (a) the /
the element for $S'$ is zero, and so all elements in the set corresponding to (b) are zero. The complete row of $M_1$ corresponding to $T_1$ thus consists of $m_{11}$ wherever alternative (a) holds and zero elsewhere. The number of column-standard tableaux for which the first alternative holds is, as already shown

$$\left( \lambda_1' \right) \left( \lambda_2' \right)$$

Since the matrix $M$ is orthogonal, the rows of $M_1$ are normalised, and so

$$(m_{11})^2 + (m_{11})^2 = 1$$

i.e. $$\left\{ \left( \lambda_1' \right) \left( \lambda_2' \right) \right\} (m_{11})^2 = 1$$

or $$m_{11} = \frac{1}{\sqrt{\left( \lambda_1' \right) \left( \lambda_2' \right)}}$$

This determines completely the first row of $M_1$.

The tableau function $\psi_{\ell}$, defined in section 3, for the last standard tableau $T_1$ has the value

$$\psi_{\ell} = \prod_{t=1}^{n} \phi_{\ell}(t)$$

and

$$\phi_{\ell}(n) = n \left( 1 + \frac{1}{\lambda_{n-1}} \right)$$

$$= \left( 1 + \frac{1}{\lambda_{n-1}} \right) \left( 1 + \frac{1}{\lambda_{n-2}} \right)$$

$$= \lambda'$$

$$\phi_{\ell}(n-1) = \lambda' - 1$$

until finally

$$\psi_{\ell} = \left( \lambda_1' \right) \left( \lambda_2' \right)$$

If the function $\psi'$ is defined as the tableau function of the transposed tableau,

$$\psi' = \psi_{\ell} = \left( \lambda_1' \right) \left( \lambda_2' \right)$$

and
and so

$$m_{11} = \frac{1}{\sqrt{\psi_1}}$$

Corresponding to the result

$$\psi_r = (1 - \rho^2) \psi_s \quad (r < s)$$

for the tableau function $\psi$, there is the result

$$\psi_s' = (1 - \rho^2) \psi_r' \quad (r < s)$$

for the conjugate tableau function $\psi'$.

Now consider the row of $M_1$ related to any other standard tableau $T_j$. As has already been shown, there is a transposition $\sigma_1$ of the form $(k, k+1)$ such that $\sigma_1 T_j = T_i$, where $j > i$. Thus any row of $M_1$ can be related to an earlier row.

For any column standard tableau $S_s$, related to a column of $M_1$, either $\sigma_1 S_s = S_s$ or $\sigma_1 S_s = S_t$, where $\sigma_1$ is the transposition just defined.

1. If $\sigma_1 S_s = S_s$.

By case 1: (c) $m_{is} = -\rho m_{is} + \sqrt{1 - \rho^2} m_{js}$

i.e. $m_{js} = (\rho - 1) m_{is} / \sqrt{1 - \rho^2}$

and the element in row $j$ can be obtained from the elements in an earlier row.

2. If $\sigma_1 S_s = S_t$, then $\sigma_1 S_t = S_s$.

By case 2: (c)(1) $m_{it} = -\rho m_{is} + \sqrt{1 - \rho^2} m_{js}$

so that $m_{js} = (m_{it} + \rho m_{is}) / \sqrt{1 - \rho^2}$

and similarly $m_{jt} = (m_{is} + \rho m_{it}) / \sqrt{1 - \rho^2}$

Again every element in row $j$ can be obtained from the elements in an earlier row.

Thus /
Thus the whole matrix $M_1$ may be built up row by row, based on the first row of which the elements are already determined.

If the elements $\sqrt{\psi_i} m_{is}$ are calculated instead of $m_{is}$, it is found that all the elements are rational. This provides an arithmetical simplification of the work. For

$$\sqrt{\psi_i} m_{11} = 1$$

and so all elements in the modified first row are unity or zero.

Also, if (1) holds

$$m_{js} = (\rho - 1)^{m_{is} / \sqrt{1 - \rho^2}}$$

so that

$$\sqrt{\psi_j} m_{js} = (\rho - 1) \sqrt{\psi_i} m_{is} \quad (A)$$

since

$$\psi_j = (1 - \rho^2) \psi_i \quad (i < j)$$

while if (2) holds, a similar substitution gives

$$\sqrt{\psi_j} m_{js} = \sqrt{\psi_i} m_{jt} + \rho \sqrt{\psi_i} m_{is} \quad (B)$$

$$\sqrt{\psi_j} m_{jt} = \sqrt{\psi_i} m_{is} + \rho \sqrt{\psi_i} m_{it} \quad (C)$$

The equations (A), (B) and (C) are sufficient to determine row $j$. Thus all elements are obtained as rational combinations of elements of earlier rows.

As an example to illustrate the construction of $M_1$, consider the $[2^2, 1^2]$ representation of the symmetric group of degree 6.

The standard tableaux in order, the transpositions by which they are obtained from an earlier tableau, the values of /
The rational elements $\sqrt{v_i} m_{1s}$ for the column standard tableaux $S_s$ are

\[
\begin{array}{cccccccccccc}
15 & 14 & 14 & 13 & 13 & 12 & 12 & 12 & 21 & 21 & 21 & 21 & 31 \\
26 & 26 & 25 & 26 & 25 & 24 & 36 & 35 & 34 & 36 & 35 & 34 & 43 & 42 \\
3 & 3 & 3 & 4 & 4 & 5 & 4 & 4 & 5 & 5 & 4 & 4 & 5 & 5 \\
4 & 5 & 6 & 5 & 6 & 6 & 5 & 6 & 6 & 5 & 6 & 6 & 6 & 6 \\
\end{array}
\]

$2\ m_{1s}$ 

$\sqrt{5} m_{2s}$ 

$\sqrt{3} m_{3s}$ 

$\frac{3}{2} m_{4s}$ 

$\frac{1}{2} m_{5s}$ 

$\frac{2}{\sqrt{3}} m_{6s}$ 

$\frac{1}{\sqrt{3}} m_{7s}$ 

$\frac{4}{3} m_{8s}$ 

$\frac{5}{3} m_{9s}$

The matrix $M_1$ is easily found from this, and the representation is derived from

$$R = M_1 C M_1'$$

where $C$ is the central core of $A(4) \times A(2)$.

An exactly similar process may be applied to the central core of $A^\frac{4}{3} \times A^\frac{4}{3}$ to obtain the representation associated with the partition $[\lambda_1 \lambda_2 \ldots]$. There are certain modifications which have to be applied to the calculation of the /
the matrix $N_1$ which is such that

$$ R = N_1 D N_1' $$

where $D$ is the central core.

The central core $D$ has elements

$$ d_{ss} = 1 \quad \text{if } \sigma R_s = R_s $$
$$ d_{st} = 1 \quad \text{if } \sigma R_s = R_t $$

and $d_{st} = 0$ otherwise.

Here $R$ is the row-standard tableau which will be associated with the rows and columns of the central core. Its rows represent the permanents which appear in $D$. As with $S$, $R$ is considered identical with a tableau formed by a rearrangement within any of its rows.

The equating of elements in $R N_1 = N_1 D$ leads to the cases

1. $\sigma R_s = R_s$

   (a) $\sigma$ in a row of $T_i$
   
   $$ n_{is} = n_{is} $$

   (b) $\sigma$ in a column of $T_i$
   
   $$ n_{is} = -n_{is} = 0 $$

   (c) $\sigma T_i = T_j$ \quad (i < j)

   $$ n_{is} = -\rho n_{is} + \sqrt{1 - \rho^2} n_{js} $$
   $$ n_{js} = \rho n_{js} + \sqrt{1 - \rho^2} n_{is} $$

2. $\sigma R_s = R_t$

   (a) $\sigma$ in a row of $T_i$
   
   $$ n_{it} = n_{is} $$

   (b) $\sigma$ in a column of $T_i$
   
   $$ n_{it} = -n_{is} $$

   (c) $\sigma T_i = T_j$ \quad (i < j)

   (1) $n_{it} = -\rho n_{is} + \sqrt{1 - \rho^2} n_{js}$
   
   (2) $n_{is} = -\rho n_{it} + \sqrt{1 - \rho^2} n_{jt}$
   
   (3) $n_{jt} = \rho n_{js} + \sqrt{1 - \rho^2} n_{is}$
   
   (4) $n_{js} = \rho n_{jt} + \sqrt{1 - \rho^2} n_{it}$
In this case, if two adjacent symbols are in the same column of $T_i$ and the same row of $R_s$, $n_{is}$ is zero. If two symbols are in the same row of $T_i$ and $\sigma S_s = S_t$, then $n_{is}$ and $n_{it}$ are equal while if the symbols are in the same column of $T_i$, and $\sigma S_s = S_t$, $n_{is}$ and $n_{it}$ differ only in sign.

Here it is the last tableau $T_2$ which is associated with the row which has elements either $\pm n_{\ell 1}$ or zero. The value of $n_{\ell 1}$ is $\sqrt[\ell]{\psi}_{\ell}$, where $\psi_{\ell}$ is the original tableau function. The other elements in the row of $N_1$ may have a negative sign, every transposition introducing a factor $-1$, as is seen by case 2 (b).

The earlier rows are obtained from the equations

$$\sqrt[\ell]{\psi}_{\ell} n_{is} = (1 - \rho) \sqrt[\ell]{\psi}_{\ell} n_{js}$$

$$\sqrt[\ell]{\psi}_{\ell} n_{is} = -\rho \sqrt[\ell]{\psi}_{\ell} n_{js} + \sqrt[\ell]{\psi}_{\ell} n_{jt}$$

according as $\sigma R_s = R_s$ or $\sigma R_s = R_t$.

The matrix $N_1$ for the $[4, 2]$ representation of the group of degree 6 will be given, to illustrate the method.

The standard tableaux in order, the transpositions by which they are obtained from a later tableau, the values of $\rho$ and the values of the tableau function are as follows:

Transposition /
53.

Transposition from $\Psi_1$

<table>
<thead>
<tr>
<th></th>
<th>Transposition</th>
<th>$\Psi_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$</td>
<td>1 2 3 4</td>
<td>(4, 5)</td>
</tr>
<tr>
<td>$T_2$</td>
<td>1 2 3 5</td>
<td>(5, 6)</td>
</tr>
<tr>
<td>$T_3$</td>
<td>1 2 4 5</td>
<td>(5, 6)</td>
</tr>
<tr>
<td>$T_4$</td>
<td>1 3 4 5</td>
<td>(5, 6)</td>
</tr>
<tr>
<td>$T_5$</td>
<td>1 2 3 6</td>
<td>(3, 4)</td>
</tr>
<tr>
<td>$T_6$</td>
<td>1 2 4 6</td>
<td>(4, 5)</td>
</tr>
<tr>
<td>$T_7$</td>
<td>1 3 4 6</td>
<td>(4, 5)</td>
</tr>
<tr>
<td>$T_8$</td>
<td>1 2 5 6</td>
<td>(2, 3)</td>
</tr>
<tr>
<td>$T_9$</td>
<td>1 3 5 6</td>
<td>-</td>
</tr>
</tbody>
</table>

The rational elements $\Psi_i$ are for the row-standard tableaux.
A further illustration of the method is provided by the matrix which reduces the permutation matrices to irreducible representations.

For example, if \( n = 3 \), \( A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \), the group matrix associated with the representation by permutation matrices is

\[
\begin{bmatrix}
  a_1 b_2 c_3 + a_1 b_3 c_2 & a_2 b_1 c_3 + a_3 b_1 c_2 & a_2 b_3 c_1 + a_3 b_2 c_1 \\
  a_2 b_1 c_3 + a_2 b_3 c_1 & a_1 b_2 c_3 + a_5 b_2 c_1 & a_1 b_3 c_2 + a_3 b_1 c_2 \\
  a_3 b_1 c_2 + a_3 b_2 c_1 & a_1 b_3 c_2 + a_2 b_3 c_1 & a_1 b_2 c_3 + a_2 b_1 c_3 \\
\end{bmatrix}
\]

which is equal to

\[
\begin{bmatrix}
  a_1 \begin{array}{c} b_2 \\ c_3 \end{array} & b_1 \begin{array}{c} a_2 \\ c_3 \end{array} & c_1 \begin{array}{c} a_2 \\ b_3 \end{array} \\
  a_2 \begin{array}{c} b_1 \\ c_3 \end{array} & b_2 \begin{array}{c} a_1 \\ c_3 \end{array} & c_2 \begin{array}{c} a_1 \\ b_3 \end{array} \\
  a_3 \begin{array}{c} b_1 \\ c_2 \end{array} & b_3 \begin{array}{c} a_1 \\ c_2 \end{array} & c_3 \begin{array}{c} a_1 \\ b_2 \end{array} \\
\end{bmatrix}
\]

In general, the leading element is \( a_1 \begin{array}{c} b_2 \\ c_3 \end{array} \) and the other elements are of similar nature. This is the central core of the direct product \( A \times A^{[n-1]} \) and the representation contains the \( [n-1, 1] \) representation, in direct sum with the \( [n] \), unit scalar, representation. The orthogonal reduction is performed by a matrix \( N_1 \) constructed by the method just given.

The case \( n = 5 \) will serve as an illustration of the general form of \( N_1 \). The standard tableaux concerned are:
56.

\[
\begin{array}{cccc}
T_1 & 1 & 2 & 3 & 4 \\
5 & 4 & 3 & 2 & 1 \\
T_2 & 1 & 2 & 3 & 5 \\
4 & 5 & 3 & 2 & 1 \\
T_3 & 1 & 2 & 4 & 5 \\
3 & 5 & 1 & 2 & 4 \\
T_4 & 1 & 3 & 4 & 5 \\
2 & - & - & - & - \\
\end{array}
\]

Transposition from \( (4, 5) \) to \( (3, 4) \) to \( (2, 3) \) to \( (2, 3) \) to \( (4, 5) \).

The rational elements \( t \) for the row-standard tableaux are

\[
\begin{array}{cccccc}
R & 2345 & 1345 & 1245 & 1235 & 1234 \\
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\sqrt{5} & n_{1s} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1 \\
\sqrt{3} & n_{2s} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & 1 \\
\sqrt{2} & n_{3s} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 \\
\sqrt{2} & n_{4s} & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

the tableaux \( R \) being arranged according to the arrangement in the group matrix.

The general form of the matrix \( N_1 \) is clearly seen from this example. For the group of degree \( n \),

\[
N_1 = \begin{bmatrix}
\frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\
\frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{(n-1)(n-2)}} \\
\frac{1}{6} & \frac{1}{6} & \cdots & \cdots \\
\frac{1}{2} & \frac{1}{2} & \cdots & \cdots \\
\end{bmatrix}
\]
The complete orthogonal matrix $N$ has a row $N_2$ in which every element is $1/\sqrt{n}$. This row produces the unit scalar representation in the reduction of the permutation matrices.
5. **Calculation of Orthogonal Representations.**

Three main methods may be used to calculate the orthogonal representations of a symmetric group. These are

1. Use of the representations of the transpositions \((k, k+1)\);
2. Reduction of the central core of the direct product of compound or induced matrices;
3. For some representations only, the compounds of the permutation matrices.

The first method consists of expressing the group element of which the representation is required as a product of transpositions of the form \((k, k+1)\). In section 1, it is shown that this is possible. The matrices representing these transpositions may be written down, by the method given in section 3, and the representation of the group element is obtained as a product of these matrices.

The second method consists of calculating the matrix \(M_1\) or \(N_1\), obtained as in section 4, and applying to the central core the transformation \(M_1 C M_1'\) or \(N_1 D N_1'\). This produces the group matrix of the representation. An alternative to calculating the group matrix is to use the central core derived from a particular permutation matrix, and obtain by transformation the representation of that permutation.

The third method, which is due to Aitken, is useful for obtaining the representations associated with partitions of /
of rank 1, or unicursal partitions. The rank of a partition is determined by the number of asterisks in the leading diagonal of the Ferrers-Sylvester diagram of the partition. Thus \[ \begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \\
\ast & \ast & \\
\end{array} \] has a diagram \( x \ast \ast \ast \) and is of rank 3.

A unicursal partition, of rank 1, has a diagram of the form \( x \ast \ast \ast \) and is expressible as \( [n - k, 1^k] \).

It has already been observed that the \( [n - 1, 1] \) representation is easily derived from the permutation matrices, and it will be shown that the successive compounds of this representation are the \( [n - 2, 1^2], [n - 3, 1^3] \ldots [1^n] \) representations.

If a permutation matrix \( A \) represents a cycle of order \( r \), \( A^r = I \) and so the characteristic polynomial of \( A \) is \( 1 - t^r \). Applying this to all the cycles of a permutation matrix, the characteristic polynomials have the form \( \frac{1}{n!} (1 - t^n) \) where \( r \) takes the values of all cycles.

For example, if \( n = 4 \), the cycle types and their characteristic polynomials are

\[
\begin{array}{c}
[1^4] \\
[2, 1^2] \\
[2^2] \\
[3, 1] \\
[4]
\end{array}
\]

\( (1 - t)^4, (1 - t)^2(1 - t^2), (1 - t^2)^2, (1 - t^3)(1 - t), 1 - t^4 \)

There is in every case a factor \( 1 - t \) due to the \( [4] \) representation,
representation, so for the $[3, 1]$ representation
\[
\begin{align*}
(1 - t)^3 & \quad (1 - t)(1 - t^2) & \quad (1 - t^2)(1 - t) & \quad l + t + t^2 + t^3
\end{align*}
\]

The table of coefficients of $1, -t, t^2, \text{ and } -t^3$ is

<table>
<thead>
<tr>
<th>Class</th>
<th>$[1^4]$</th>
<th>$[2, 1^2]$</th>
<th>$[2^2]$</th>
<th>$[3, 1]$</th>
<th>$[4]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-t</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$t^2$</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$-t^3$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

and this is observed to be the table of group characters, with the $[2^2]$ character omitted.

The trace of $A(k)$ is the coefficient of $(-t)^k$ in
the characteristic polynomial of $A$, since it is the $k$ th
 elementary symmetric function of the latent roots of $A$. Define a set of symmetric functions by the relation

$$s_r = 1 - tr^r \quad r = 1, 2, \ldots$$

The generating function of the complete symmetric functions $h_r$ can be shown to be

$$h_r = (1 - tx)(1 - x)^{-1}$$

and so $h_r$, the coefficient of $x^r$ is

$$h_r = 1 - t \quad \text{ (r \neq 0)}$$

$$h_0 = 1$$

The bialternants become, for example

$$h_{[2, 2]} /$$
\[ h_{[2,2]} = \begin{vmatrix} 1-t & 1-t \\ 1-t & 1-t \end{vmatrix} = 0 \]
\[ h_{[2,1^2]} = \begin{vmatrix} 1-t & 1-t & 1-t \\ 1 & 1-t & 1-t \\ 0 & 1 & 1-t \end{vmatrix} = t^2(1-t) \]

and in general, the operations row 1 - row 2; row 2 - row 3; -- show that the bialternant for a partition \([n-k, 1^k]\) is \((-t)^{n-k}(1-t)\) while for any partition of greater rank, the bialternant vanishes.

Frobenius' relations between products of power sums and bialternants give

\[ \left\{ S_{[\ell]} \right\} = G \left\{ h_{[\ell]} \right\} \]

where \(\{S_{[\ell]}\}\) is the column vector of products of power sums, \(\{h_{[\ell]}\}\) is the column vector of bialternants, and \(G\) is the group character matrix. Inserting for \(s_{[\ell]}\) and \(h_{[\ell]}\) the values found for the set of functions considered, and comparing \(G\)' with the table of coefficients, it is seen that the rows of the table are the characters for the representation: \([n], [n-1, 1], [n-2, 1^2]\) ------.

Thus the compounds \(A^{(0)} = 1, A^{(1)} = A, A^{(2)}, \ldots\) of the \([n-1, 1]\) representation are respectively the \([n], [n-1, 1], [n-2, 1^2]\) representations. If \(A\) is orthogonal, so are all its compounds.
62.

5 (a). REPRESENTATIONS OF GROUPS OF DEGREE 3, 4 AND 5.

The orthogonal representations of the symmetric groups of degree 3, 4 and 5 have been calculated by the methods set out here. The details of the method found most convenient are as follows.

1. Group of degree 3.

\([2, 1]\) representation. Order 2.

Reduction of permutation matrices by

\[
N_1 = \begin{bmatrix}
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{\sqrt{3}}{3} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\end{bmatrix}
\]

2. Group of degree 4.

(a) \([3, 1]\) representation. Order 3.

Reduction of permutation matrices by

\[
N_1 = \begin{bmatrix}
\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{\sqrt{3}}{2} \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{3}}{\sqrt{3}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\
\end{bmatrix}
\]

(b) \([2^2]\) representation. Order 2.

Use of the generators \((1, 2), (2, 3), (3, 4)\) as obtained using Young's tableaux method.

(c) \([2, 1^2]\) representation. Order 3.

This is the second compound of the \([3, 1]\) representation.
3. Group of degree 5.

(a) \([4, 1]\) representation. Order 4.

Reduction of permutation matrices by

\[
N_1 = \begin{pmatrix}
-\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\
-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{3} & . \\
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} & . & . \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & . & . & .
\end{pmatrix}
\]

(b) \([3, 1^2]\) representation. Order 6.

This is the second compound of the \([4, 1]\) representation.

(c) \([3, 2]\) representation. Order 5.


\[
N_1 = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{\sqrt{3}} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} \\
\frac{1}{\sqrt{3}} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]

(d) \([2^2, 1]\) representation. Order 5.

This is not quoted explicitly. It may be obtained /
obtained by forming the direct product of the $[3, 2]$ and alternating scalar representations. This gives a representation which is equivalent to that which would be obtained by the other methods. To make it identical with the representation given by Young's tableaux method, it should be transformed by $K( )K'$ where

$$K = \begin{bmatrix}
1 & & & -1 \\
& 1 & & \\
& & 1 & \\
-1 & & & \\
1 & & & \\
\end{bmatrix}$$


This is not quoted explicitly, but may be obtained from the $[4, 1]$ representation as described for the $[2^2, 1]$ representation. The transforming matrix is

$$K = \begin{bmatrix}
\cdots & -1 \\
\cdots & 1 \\
\cdots & -1 \\
1 & \cdots \\
\end{bmatrix}$$
1. Group of Degree 3

\[
\begin{bmatrix}
1 & \cdot \\
\cdot & 1
\end{bmatrix}
\]

(12)

\[
\begin{bmatrix}
1 & \cdot \\
\cdot & -1
\end{bmatrix}
\]

\[
(13)
\begin{bmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{-\sqrt{3}}{2} & -\frac{1}{2}
\end{bmatrix}
\]

(23)

\[
\begin{bmatrix}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
\frac{-\sqrt{3}}{2} & -\frac{1}{2}
\end{bmatrix}
\]
2. Group of Degree 4.

\[ \begin{array}{c}
\text{(a)} & \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \\
\text{(b)} & \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \\
\text{(c)} & \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \\
\end{array} \]

\[ \begin{array}{c}
\text{(12)} & \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \\
\text{(13)} & \begin{bmatrix}
1 & -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\
-\frac{\sqrt{3}}{2} & 1 & -\frac{1}{2} & 0 \\
0 & -\frac{1}{2} & 1 & -\frac{\sqrt{3}}{2} \\
-\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 1 \\
\end{bmatrix} \\
\text{(14)} & \begin{bmatrix}
\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\
0 & \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\
0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\
\end{bmatrix} \\
\text{(23)} & \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \\
\text{(24)} & \begin{bmatrix}
\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\
0 & \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\
0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\
\end{bmatrix} \\
\end{array} \]
\[(34) \quad \begin{bmatrix} -\frac{1}{3} & \frac{2\sqrt{2}}{3} \\ \frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2\sqrt{2}}{3} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{1}{2} \end{bmatrix} \]

\[(123) \quad \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \]

\[(132) \quad \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \]

\[(124) \quad \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \]

\[(142) \quad \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \]

\[(134) \quad \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \]

\[
E
\]

\[
(1\ 2)
\]
\[
\begin{bmatrix}
\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\
-\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\frac{1}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{12}}{3} \\
-\frac{\sqrt{3}}{3} & \frac{5}{3} & -\frac{1}{3} \\
\frac{\sqrt{12}}{3} & \frac{1}{3} & \frac{2}{3}
\end{bmatrix}
\]

\[
(24)
\]

\[
\begin{bmatrix}
-\frac{1}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{12}}{3} \\
-\frac{\sqrt{3}}{3} & \frac{5}{3} & -\frac{1}{3} \\
\frac{\sqrt{12}}{3} & \frac{1}{3} & \frac{2}{3}
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\frac{1}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{12}}{3} \\
-\frac{\sqrt{3}}{3} & \frac{5}{3} & -\frac{1}{3} \\
\frac{\sqrt{12}}{3} & \frac{1}{3} & \frac{2}{3}
\end{bmatrix}
\]

\[
(25)
\]

\[
\begin{bmatrix}
-\frac{1}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{12}}{3} \\
-\frac{\sqrt{3}}{3} & \frac{5}{3} & -\frac{1}{3} \\
\frac{\sqrt{12}}{3} & \frac{1}{3} & \frac{2}{3}
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\frac{1}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{12}}{3} \\
-\frac{\sqrt{3}}{3} & \frac{5}{3} & -\frac{1}{3} \\
\frac{\sqrt{12}}{3} & \frac{1}{3} & \frac{2}{3}
\end{bmatrix}
\]
\[
(132)
\]

\[
\begin{pmatrix}
1 & -1 & 1 \\
-\frac{1}{2} & 1 & -\frac{1}{2} \\
-\frac{1}{2} & -1 & -\frac{1}{2}
\end{pmatrix}
\]

\[
(124)
\]

\[
\begin{pmatrix}
1 & -1 & 1 & -1 \\
-\frac{1}{2} & 1 & -\frac{1}{2} & 1 \\
-\frac{1}{2} & -1 & -\frac{1}{2} & 1 \\
-\frac{1}{2} & -1 & -\frac{1}{2} & 1 \\
\end{pmatrix}
\]
\[
\begin{bmatrix}
-\frac{1}{3} & -\frac{1}{6\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{\sqrt{5}}{2\sqrt{2}} & \frac{\sqrt{5}}{2\sqrt{2}} \\
-\frac{1}{6\sqrt{3}} & -\frac{5}{12} & \frac{1}{2\sqrt{3}} & \frac{\sqrt{5}}{4\sqrt{3}} & -\frac{\sqrt{5}}{2\sqrt{2}} \\
-\frac{1}{2\sqrt{3}} & -\frac{1}{6\sqrt{3}} & 3 & 4 & \frac{\sqrt{5}}{4\sqrt{3}} & \frac{\sqrt{5}}{2\sqrt{2}} \\
\frac{1}{6\sqrt{3}} & -\frac{1}{6\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{\sqrt{5}}{12} & -\frac{5}{6\sqrt{3}} & \frac{1}{3} \\
\frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{2\sqrt{3}} & \frac{\sqrt{5}}{2\sqrt{2}} & 0 & \frac{1}{2} & \frac{\sqrt{5}}{2} & \frac{1}{2} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\frac{1}{3} & -\frac{1}{3\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\
-\frac{1}{3\sqrt{3}} & -\frac{1}{3} & -\frac{1}{2} & \frac{2\sqrt{3}}{3} & -\frac{2\sqrt{3}}{3} \\
-\frac{1}{\sqrt{3}} & -\frac{1}{2} & 1 & 2 & -1 \\
\frac{\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} & -\frac{2\sqrt{3}}{3} & 1 & 2 \\
\frac{1}{3} & -\frac{2\sqrt{3}}{3} & -\frac{2\sqrt{3}}{3} & 1 & 2 \\
\end{bmatrix}
\]
\[
(1+5)
\]
\[(154)\]

\[
\begin{bmatrix}
\frac{1}{3} & \frac{1}{6} & \frac{1}{3} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\
\frac{\sqrt{2}}{3} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\
\frac{1}{3} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\
\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
-\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8}
\end{bmatrix}
\]

\[(234)\]

\[
\begin{bmatrix}
\frac{1}{3} & \frac{1}{6} & \frac{1}{3} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\
\frac{\sqrt{2}}{3} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\
\frac{1}{3} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\
\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
-\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8}
\end{bmatrix}
\]
\[
\begin{vmatrix}
\frac{1}{3} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{3} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{3} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{3} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{3} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{3} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{3} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{3} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\end{vmatrix}
\]
\[(354)\]

\[
\begin{bmatrix}
1/3 & -1/352 & 1/4 & 1/5 \\
-2\sqrt{3} & -1/12 & 5/3 & 2\sqrt{3} \\
\sqrt{5} & 1/4 & -1/2 & 2\sqrt{3}/3 \\
1/4 & -\sqrt{3}/2 & \sqrt{5}/2 & 1/3 \\
\end{bmatrix}
\]

\[(12)(34)\]

\[
\begin{bmatrix}
1 & -1/3 & 2\sqrt{2}/3 \\
-1/3 & 2\sqrt{2}/3 & 1/3 \\
2\sqrt{2}/3 & 1/3 & 1 \\
\end{bmatrix}
\]
(12)(35)

\[
\begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\]

(12)(45)

\[
\begin{bmatrix}
\sqrt{\frac{5}{4}} & 1 & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{2}
\end{bmatrix}
\]
(13)(24)

\[
\begin{pmatrix}
1 & -\frac{1}{3} & -\frac{\sqrt{5}}{3} \\
-\frac{1}{3} & 1 & -\frac{\sqrt{5}}{3} \\
-\frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{3} & 1
\end{pmatrix}
\]

(13)(25)

\[
\begin{pmatrix}
1 & -\frac{1}{3} & -\frac{\sqrt{5}}{3} \\
-\frac{1}{3} & 1 & -\frac{\sqrt{5}}{3} \\
-\frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{3} & 1
\end{pmatrix}
\]
<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[(14)(25)\]

\[
\begin{array}{c|cccc}
\frac{1}{2} & \frac{5}{8} & -\frac{5}{8} & \frac{5}{8} & -\frac{5}{8} \\
\frac{5}{8} & \frac{115}{8} & -\frac{115}{8} & \frac{115}{8} & -\frac{115}{8} \\
\frac{5}{3} & \frac{15}{8} & -\frac{15}{8} & \frac{15}{8} & -\frac{15}{8} \\
\frac{15}{3} & -\frac{15}{3} & \frac{15}{3} & -\frac{15}{3} & \frac{15}{3} \\
\end{array}
\]

\[(14)(35)\]

\[
\begin{array}{c|cccc}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array}
\]
(15) (23)
\[(15)(34)\]

\[
\begin{bmatrix}
-\frac{1}{4} & -\frac{\sqrt{15}}{4} & -\frac{5}{6} & -\frac{5}{2} & -\frac{\sqrt{15}}{2} \\
\frac{\sqrt{15}}{4} & \frac{5}{12} & \frac{7}{6} & \frac{1}{2} & \frac{1}{2} \\
-\frac{5}{6} & \frac{1}{6} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
-\frac{\sqrt{15}}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

\[(23)(45)\]

\[
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\]
We have two sets of equations:

1. \((24) (35)\)

\[\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}\]

2. \((25) (34)\)

\[\begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}\]
\[
\begin{pmatrix}
1 & -\frac{1}{3} & -\frac{5\sqrt{2}}{3} & -\frac{\sqrt{3}}{3} \\
-\frac{1}{3} & 2 & -\frac{2\sqrt{2}}{3} & -\frac{2\sqrt{2}}{3} \\
-\frac{5\sqrt{2}}{3} & -\frac{2\sqrt{2}}{3} & 2 & -\frac{\sqrt{3}}{3} \\
-\frac{\sqrt{3}}{3} & -\frac{2\sqrt{2}}{3} & -\frac{\sqrt{3}}{3} & 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -\frac{1}{3} & -\frac{5\sqrt{2}}{3} & -\frac{\sqrt{3}}{3} \\
-\frac{1}{3} & 2 & -\frac{2\sqrt{2}}{3} & -\frac{2\sqrt{2}}{3} \\
-\frac{5\sqrt{2}}{3} & -\frac{2\sqrt{2}}{3} & 2 & -\frac{\sqrt{3}}{3} \\
-\frac{\sqrt{3}}{3} & -\frac{2\sqrt{2}}{3} & -\frac{\sqrt{3}}{3} & 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -\frac{1}{3} & -\frac{5\sqrt{2}}{3} & -\frac{\sqrt{3}}{3} \\
-\frac{1}{3} & 2 & -\frac{2\sqrt{2}}{3} & -\frac{2\sqrt{2}}{3} \\
-\frac{5\sqrt{2}}{3} & -\frac{2\sqrt{2}}{3} & 2 & -\frac{\sqrt{3}}{3} \\
-\frac{\sqrt{3}}{3} & -\frac{2\sqrt{2}}{3} & -\frac{\sqrt{3}}{3} & 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -\frac{1}{3} & -\frac{5\sqrt{2}}{3} & -\frac{\sqrt{3}}{3} \\
-\frac{1}{3} & 2 & -\frac{2\sqrt{2}}{3} & -\frac{2\sqrt{2}}{3} \\
-\frac{5\sqrt{2}}{3} & -\frac{2\sqrt{2}}{3} & 2 & -\frac{\sqrt{3}}{3} \\
-\frac{\sqrt{3}}{3} & -\frac{2\sqrt{2}}{3} & -\frac{\sqrt{3}}{3} & 2
\end{pmatrix}
\]
\[
\theta(25) = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\theta(52) = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
-\frac{1}{3} & -\frac{1}{6} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
-\frac{1}{6} & -\frac{1}{2} & \frac{7}{8} & \frac{3\sqrt{3}}{8} \\
-\frac{1}{2} & \frac{7}{8} & \frac{5}{8} & \frac{3\sqrt{3}}{8} \\
\frac{\sqrt{3}}{2} & \frac{3\sqrt{3}}{8} & \frac{5}{8} & \frac{1}{8} \\
\frac{3\sqrt{3}}{2} & \frac{5}{8} & \frac{1}{8} & \frac{1}{8} \\
\frac{\sqrt{3}}{2} & \frac{3\sqrt{3}}{8} & \frac{5}{8} & \frac{1}{8} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\
-\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\
-\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\
-\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\
\end{bmatrix}
\]
(1 2 4 5)
\[
\begin{bmatrix}
-1 & -\frac{\sqrt{5}}{4} & -\frac{\sqrt{5}}{4}
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\frac{\sqrt{5}}{2} & -\frac{5}{4} & -\frac{\sqrt{3}}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\frac{\sqrt{5}}{2} & -\frac{5}{4} & -\frac{\sqrt{3}}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\frac{\sqrt{5}}{2} & -\frac{5}{4} & -\frac{\sqrt{3}}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\frac{5}{8} & \frac{5}{8} & -\frac{5}{8}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{5}{8} & \frac{5}{8} & -\frac{5}{8}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{5}{8} & \frac{5}{8} & -\frac{5}{8}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{5}{8} & \frac{5}{8} & -\frac{5}{8}
\end{bmatrix}
\]
\[
(15 \ 34)
\]

\[
\begin{array}{cccccc}
1 & -\frac{1}{3} & \frac{5}{12} & \frac{5}{3} & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{2} & \frac{3}{8} & \frac{1}{8} & \frac{5}{8} & \frac{5}{8} & \frac{5}{8} \\
-\frac{1}{2} & \frac{1}{8} & \frac{3}{2} & -\frac{1}{2} & \frac{5}{2} & \frac{5}{2} \\
-\frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{2} & \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{2} \\
-\frac{\sqrt{15}}{2} & -\frac{\sqrt{15}}{2} & \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{15}}{2} & \frac{\sqrt{15}}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array}
\]

\[
(15 \ 43)
\]

\[
\begin{array}{cccc}
-\frac{1}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{5}}{4} & \frac{1}{2} \\
-\frac{\sqrt{15}}{4} & -\frac{\sqrt{15}}{4} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
-\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\end{array}
\]
$$
\begin{bmatrix}
\frac{1}{3} & \frac{12}{5} & -\frac{1}{5} & 1 \\
-\frac{1}{3}\frac{5}{2} & 2 & -\frac{3}{5} & 8 \\
\frac{5}{6} & \frac{8}{5} & \frac{8}{5} & 8 \\
\frac{8}{5} & \frac{8}{5} & 8 & 8
\end{bmatrix}
$$

$$
\begin{bmatrix}
\frac{1}{3} & -\frac{12}{5} & \frac{1}{5} & -1 \\
\frac{1}{3}\frac{5}{2} & -2 & \frac{3}{5} & -8 \\
\frac{5}{6} & -\frac{8}{5} & -\frac{8}{5} & -8 \\
\frac{8}{5} & -\frac{8}{5} & -8 & -8
\end{bmatrix}
$$

$$
\begin{bmatrix}
\frac{1}{3} & \frac{1}{8} & \frac{1}{2} & -\frac{1}{4} \\
\frac{1}{3}\frac{5}{2} & -\frac{1}{8} & -\frac{3}{8} & \frac{1}{8} \\
\frac{5}{6} & \frac{8}{5} & -\frac{8}{5} & \frac{1}{8} \\
\frac{8}{5} & \frac{8}{5} & \frac{8}{5} & -\frac{1}{8}
\end{bmatrix}
$$

$$
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{8} & \frac{1}{2} & \frac{1}{4} \\
\frac{1}{3}\frac{5}{2} & \frac{1}{8} & \frac{3}{8} & -\frac{1}{8} \\
\frac{5}{6} & -\frac{8}{5} & \frac{8}{5} & -\frac{1}{8} \\
\frac{8}{5} & -\frac{8}{5} & -\frac{8}{5} & \frac{1}{8}
\end{bmatrix}
$$
\[
\begin{align*}
& (2435) \\
& \begin{bmatrix}
\frac{1}{216} & 216 & \frac{15}{6} & \frac{15}{6} & \frac{15}{2} \\
\frac{1}{216} & \frac{1}{216} & \frac{15}{6} & \frac{15}{6} & \frac{15}{2} \\
\frac{1}{216} & \frac{1}{216} & \frac{15}{6} & \frac{15}{6} & \frac{15}{2} \\
\frac{1}{216} & \frac{1}{216} & \frac{15}{6} & \frac{15}{6} & \frac{15}{2} \\
\frac{1}{216} & \frac{1}{216} & \frac{15}{6} & \frac{15}{6} & \frac{15}{2}
\end{bmatrix}
\end{align*}
\]

\[
(2453) \\
\begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\]

\[
\begin{align*}
& \begin{bmatrix}
\frac{1}{3} & \frac{216}{3} & \frac{1}{216} & \frac{1}{216} & \frac{1}{216} \\
\frac{1}{3} & \frac{216}{3} & \frac{1}{216} & \frac{1}{216} & \frac{1}{216} \\
\frac{1}{3} & \frac{216}{3} & \frac{1}{216} & \frac{1}{216} & \frac{1}{216} \\
\frac{1}{3} & \frac{216}{3} & \frac{1}{216} & \frac{1}{216} & \frac{1}{216} \\
\frac{1}{3} & \frac{216}{3} & \frac{1}{216} & \frac{1}{216} & \frac{1}{216}
\end{bmatrix}
\end{align*}
\]
\[
\begin{bmatrix}
\frac{1}{25} & -\frac{1}{24} & \frac{\sqrt{5}}{20} & -\frac{\sqrt{5}}{15} & \frac{\sqrt{5}}{15} \\
3 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
-\frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{2} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\end{bmatrix}
\]
\[(13)(245)\]

\[
\begin{bmatrix}
\frac{1}{3} & \frac{12}{3} & \frac{-12}{5} & \frac{15}{4} & \frac{3}{2} \\
-\frac{1}{3} & \frac{1}{5} & \frac{-15}{2} & \frac{5}{2} & \frac{-15}{12} \\
-\frac{1}{2} & \frac{6}{12} & \frac{-4}{3} & \frac{-4}{5} & \frac{4}{2} \\
-\frac{1}{2} & \frac{12}{24} & \frac{-15}{3} & \frac{25}{12} & \frac{-15}{2} \\
-\frac{1}{2} & \frac{15}{25} & \frac{-4}{5} & \frac{-4}{3} & \frac{1}{2} \\
\end{bmatrix}
\]

\[(13)(254)\]

\[
\begin{bmatrix}
-\frac{1}{3} & -\frac{\sqrt{15}}{2} & -\frac{\sqrt{15}}{4} & \frac{\sqrt{15}}{2} & \frac{\sqrt{15}}{4} \\
-\frac{1}{3} & -\frac{\sqrt{15}}{4} & \frac{\sqrt{15}}{12} & \frac{\sqrt{15}}{2} & \frac{\sqrt{15}}{4} \\
-\frac{1}{3} & -\frac{\sqrt{15}}{2} & \frac{\sqrt{15}}{4} & \frac{\sqrt{15}}{2} & \frac{\sqrt{15}}{4} \\
-\frac{1}{3} & -\frac{\sqrt{15}}{4} & \frac{\sqrt{15}}{12} & \frac{\sqrt{15}}{2} & \frac{\sqrt{15}}{4} \\
-\frac{1}{3} & -\frac{\sqrt{15}}{2} & \frac{\sqrt{15}}{4} & \frac{\sqrt{15}}{2} & \frac{\sqrt{15}}{4} \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
\]

\[
(15) (234)
\]

\[
\begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}
\end{bmatrix}
\]

\[
(15) (243)
\]

\[
\begin{bmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
-\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\
\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}
\end{bmatrix}
\]

\[
(15) (256)
\]
\[
(2\ 3) (1\ 4\ 5)
\]

\[
\begin{bmatrix}
-\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\
\frac{1}{3} & 1 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & 1 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & 0 \\
\end{bmatrix}
\]

\[
(2\ 3) (1\ 5\ 4)
\]

\[
\begin{bmatrix}
-\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
1 & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & 1 \\
\frac{1}{3} & \frac{1}{3} & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
1 & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & 1 \\
\frac{1}{3} & \frac{1}{3} & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} \\
\end{bmatrix}
\]
(24)(135)

\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

(24)(153)

\[
\begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\]
$$\begin{array}{cccc}
(25)(134) & -1 & \frac{1}{4} & \frac{\sqrt{5}}{3} \\
(25)(143) & -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{5}}{2} \\
\end{array}$$

$$\begin{array}{cccc}
-1 & \frac{1}{4} & -\frac{\sqrt{5}}{4} & \frac{\sqrt{5}}{2} \\
-\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{2} \\
\end{array}$$

$$\begin{array}{cccc}
\frac{1}{3} & \frac{1}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\
-\frac{1}{3} & \frac{1}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\
\end{array}$$

$$\begin{array}{cccc}
\frac{1}{4} & -\frac{\sqrt{5}}{4} & -\frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{2} \\
\frac{1}{4} & -\frac{\sqrt{5}}{4} & -\frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{2} \\
\end{array}$$
\[(35)(124)\]

\[
\begin{array}{cccc}
\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{3} \\
-\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \\
\end{array}
\]

\[(35)(142)\]

\[
\begin{array}{cccc}
\frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{3} \\
-\frac{1}{2} & \frac{1}{2} & 1 & 0 \\
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \\
\end{array}
\]
\[
(4 \ 5) (1 \ 2 \ 3)
\]

\[
\begin{bmatrix}
-1 & 1 & \frac{\sqrt{15}}{4} \\
\frac{\sqrt{15}}{4} & 1 & -1 \\
-1 & -1 & \frac{\sqrt{15}}{2} \\
\end{bmatrix}
\]

\[
(4 \ 5) (1 \ 3 \ 2)
\]

\[
\begin{bmatrix}
-1 & 1 & \frac{\sqrt{15}}{4} \\
\frac{\sqrt{15}}{4} & 1 & -1 \\
-1 & -1 & \frac{\sqrt{15}}{2} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 1 & \frac{\sqrt{15}}{4} \\
\frac{\sqrt{15}}{4} & 1 & -1 \\
-1 & -1 & \frac{\sqrt{15}}{2} \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
\]
\[
\begin{bmatrix}
\frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{3}}{3} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{bmatrix}
\]

\[
\frac{1}{3} - \frac{\sqrt{2}}{3} - \frac{\sqrt{3}}{3}
\]

\[
\begin{bmatrix}
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{3}}{3} \\
-\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\
\frac{1}{3\sqrt{3}} & -\frac{1}{3\sqrt{3}} & -\frac{1}{3\sqrt{3}}
\end{bmatrix}
\]

\[
\frac{1}{3} - \frac{\sqrt{2}}{3} - \frac{\sqrt{3}}{3}
\]

\[
\begin{bmatrix}
\frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{3}}{3} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{bmatrix}
\]

\[
\frac{1}{3} - \frac{\sqrt{2}}{3} - \frac{\sqrt{3}}{3}
\]

\[
\begin{bmatrix}
\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{3}}{3} \\
\frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\
\frac{1}{3\sqrt{3}} & \frac{1}{3\sqrt{3}} & \frac{1}{3\sqrt{3}}
\end{bmatrix}
\]

\[
\frac{1}{3} - \frac{\sqrt{2}}{3} - \frac{\sqrt{3}}{3}
\]
\[
\begin{bmatrix}
1 & -\frac{1}{6} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\
-\frac{1}{\sqrt{2}} & 1 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
\]

\[
(13524)
\]

\[
\begin{bmatrix}
-\frac{1}{3} & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\
\frac{1}{3} & 1 & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \\
\frac{1}{3} & 1 & 1 & -\frac{1}{3}
\end{bmatrix}
\]

\[
(13542)
\]

\[
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{6\sqrt{2}} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
-\frac{1}{6\sqrt{2}} & \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
\frac{1}{6\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{6\sqrt{2}} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
\]
\[
\begin{align*}
(1 & 4 2 3 5) \\
\begin{bmatrix}
\frac{1}{2} & -\frac{1}{6} & -\frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{3} \\
\frac{1}{2} & -\frac{1}{6} & \frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{3} \\
\frac{1}{2} & -\frac{1}{6} & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{3} \\
\frac{1}{2} & -\frac{1}{6} & \frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{3} \\
\frac{1}{2} & -\frac{1}{6} & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{3}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
(1 & 4 2 5 3) \\
\begin{bmatrix}
\frac{1}{3} & -\frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{3} \\
\frac{1}{3} & -\frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{3} \\
\frac{1}{3} & -\frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3} \\
\frac{1}{3} & -\frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{3} \\
\frac{1}{3} & -\frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3}
\end{bmatrix}
\end{align*}
\]
\[(14325)\]

\[
\begin{bmatrix}
-\frac{1}{2} & \frac{1}{4} & \frac{\sqrt{5}}{8} & \frac{\sqrt{5}}{8} & \frac{\sqrt{5}}{8} \\
-\frac{1}{8} & \frac{7}{8} & -\frac{\sqrt{5}}{8} & -\frac{\sqrt{5}}{8} & -\frac{\sqrt{5}}{8} \\
-\frac{1}{8} & \frac{5}{8} & \frac{\sqrt{5}}{8} & -\frac{\sqrt{5}}{8} & -\frac{\sqrt{5}}{8} \\
\frac{\sqrt{15}}{8} & \frac{\sqrt{15}}{8} & \frac{7}{8} & -\frac{1}{8} & -\frac{1}{8} \\
\frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\end{bmatrix}
\]

\[(14352)\]

\[
\begin{bmatrix}
-\frac{1}{2} & \frac{1}{4} & -\frac{\sqrt{5}}{8} & -\frac{\sqrt{5}}{8} & -\frac{\sqrt{5}}{8} \\
\frac{1}{8} & -\frac{7}{8} & \frac{\sqrt{5}}{8} & \frac{\sqrt{5}}{8} & \frac{\sqrt{5}}{8} \\
\frac{1}{8} & \frac{5}{8} & -\frac{\sqrt{5}}{8} & \frac{\sqrt{5}}{8} & \frac{\sqrt{5}}{8} \\
\frac{\sqrt{15}}{8} & \frac{\sqrt{15}}{8} & \frac{7}{8} & \frac{1}{8} & \frac{1}{8} \\
\frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\end{bmatrix}
\]
\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}}
\end{pmatrix}
\]
\[
\begin{pmatrix}
\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}
\end{pmatrix}
\]