Numerical Approximations of Stochastic Optimal Stopping and Control Problems

David Šiška

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Abstract

We study numerical approximations for the payoff function of the stochastic optimal stopping and control problem. It is known that the payoff function of the optimal stopping and control problem corresponds to the solution of a normalized Bellman PDE.

The principal aim of this thesis is to study the rate at which finite difference approximations, derived from the normalized Bellman PDE, converge to the payoff function of the optimal stopping and control problem. We do this by extending results of N.V. Krylov from the Bellman equation to the normalized Bellman equation. To our best knowledge, unitl recently, no results about the rate of convergence of finite difference approximations to Bellman equations have been known. A major breakthrough has been made by N. V. Krylov. He proved rate of convergence of $\tau^{1/4} + h^{1/2}$ where τ and h are the step sizes in time and space respectively.

We will use the known idea of randomized stopping to give a direct proof showing that optimal stopping and control problems can be rewritten as pure optimal control problems by introducing a new control parameter and by allowing the reward and discounting functions to be unbounded in the control parameter.

We extend important results of N. V. Krylov on the numerical solutions to the Bellman equations to the normalized Bellman equations associated with the optimal stopping of controlled diffusion processes. We obtain the same rate of convergence of $\tau^{1/4}+h^{1/2}$. This rate of convergence holds for finite difference schemes defined on a grid on the whole space $[0,T]\times\mathbb{R}^d$ i.e. on a grid with infinitely many elements. This leads to the study of localization error, which arises when restricting the finite difference approximations to a cylindrical domain.

As an application of our results, we consider an optimal stopping problem from mathematical finance: the pricing of American put option on multiple assets. We prove the rate of convergence of $\tau^{1/4} + h^{1/2}$ for the finite difference approximations.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*David Šiška*)

To my Grandmother.

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Contents

Notation

$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, where $\mathcal{F}$ is a σ-algebra of subsets of Ω and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. If S is a topological space then $\mathfrak{B}(S)$ denotes the Borel σ-algebra of subsets of S. The mathematical expectation of a random variable X is denoted $\mathbb{E}X$. If $(x_t^{\alpha,s,x})_{t\geq 0}$ is a stochastic process depending on some parameters α, s and x and h is some functional then $\mathbb{E}^{\alpha}_{s,x} h((x_t)_{t\geq 0}) := \mathbb{E}h((x_t^{\alpha,s,x})_{t\geq 0})$.

We will use $:=$ to mean "equal by definition". $\mathbb{N}$ denotes the natural numbers, $\mathbb{R}$ denotes the real numbers, $\mathbb{R}^d$ denotes a d dimensional real vector space with the inner product (scalar product) $(.,.)$, x^i the ith coordinate of the point $x \in \mathbb{R}^d$ ($i = 1, 2, \dots, d$), $xy = (x, y)$ the scalar product of vectors $x, y \in \mathbb{R}^d$.

A matrix with elements σ^{ij} is denoted $\sigma = (\sigma^{ij})$. The transpose of σ is denoted σ^T. Trace of a matrix is denoted by

$$\operatorname{tr} \sigma := \prod_i \sigma^{ii}.$$

Product of the matrix σ and vector x is σx, while $x\sigma y = (x, \sigma y)$. If x is a vector then $|x| := \sqrt{(x,x)}$ is the norm of x, while if σ is a matrix then

$$|\sigma| := (\operatorname{tr} \sigma\sigma^T)^{1/2} = \left(\sum_{i,j} |\sigma^{ij}|^2 \right)^{1/2}$$

is the norm of σ.

$$t \wedge s := \min(t, s), \quad t \vee s := \max(t, s),$$
$$t^+ := t_+ := (1/2)(|t| + t), \quad t^- := t_- := (1/2)(|t| - t).$$

For $u(t, x)$ let

$$u_t := \frac{\partial}{\partial t} u, \quad u_{x^i} := \frac{\partial u}{\partial x^i}, \quad u_{x^i x^j} := \frac{\partial^2 u}{\partial x^i \partial x^j}.$$

If ℓ is a vector in $\mathbb{R}^d$ then the first and second directional derivatives along ℓ are

$$\mathrm{D}_\ell\, u = \sum_i u_{x^i} \ell^i \quad \text{and} \quad \mathrm{D}_\ell^2\, u = \sum_{i,j} u_{x^i x^j} \ell^i \ell^j$$

respectively. The space of smooth (infinitely differentiable) functions defined on some domain D is denoted $C^\infty(D)$ and the space of smooth functions with bounded support defined on some domain D is $C_0^\infty(D)$. Let $\mathrm{D}_x^n\, u$ denote the collection of all nth order derivatives of u with respect to x. Finally, for a fixed $T > 0$, $h > 0$, $\tau \in (0, T]$, let $\tau_T(t) := \tau$ for $t \leq T - \tau$ and $\tau_T(t) := T - t$ for $t > T - \tau$. Then

$$\begin{aligned}
\delta_\tau\, u(t,x) &:= \frac{u(t + \tau_T(t), x) - u(t,x)}{\tau_T(t)}, \\
\delta_{h_k,\ell_k}\, u(t,x) &:= \frac{u(t, x + h\ell) - u(t,x)}{h}, \\
\Delta_{h_k,\ell_k}\, u &:= -\,\delta_{h_k,\ell_k}\, \delta_{h,-\ell}\, u = \frac{1}{h}(\delta_{h_k,\ell_k}\, u + \delta_{h,-\ell}\, u).
\end{aligned}$$

Chapter 1

Introduction

Optimal stopping and control problems are part of stochastic control theory. In the broadest sense stochastic control theory deals with "optimal" decision making about some noisy system. One considers a given system which develops in time. One may be able to influence, or "control" the system. One is also given some performance criterion to evaluate the various behaviors of the system. Then two questions arise naturally: What is the, in some sense, best (or "optimal") behavior we can expect to see from the system? What controls should we choose at each time to achieve this behavior? A related problem is that of optimal stopping. Sat that on top of controlling the system, we are allowed to stop the system at a time of our choice (e.g. stop playing a game, sell or buy an asset, harvest the fields etc.). Again the natural question to ask is: How well can we do now that we can stop at any time? And what is the "optimal" time when to stop?

Example 1.0.1. A very simple example of an optimal stopping problem is the following: given a fair dice we are told that we're allowed to roll the dice for up to three times. At each time we can either choose to stop the game and take the "winnings" which are equal to the number we rolled on the dice, or to carry on. Unless, of course this is the 3rd time we rolled the dice in which case we have to accept whichever number it is we got. In this case solving the problem is a matter of simple calculation, working backward in time. If we're in the third round then we stop, because we have no choice. In the second round we stop only if we rolled $4, 5$ or 6, while in the first round it is optimal to stop if we got 5 or 6. The optimal expected "payoff" for this optimal stopping problem is $4 + \frac{2}{3}$.

In the previous example the system developed in discrete time and naturally the decision making was in discrete time also. These discrete time problems are treated in the theory of discrete stochastic programming initially in the 1950s by Howard [14], Bellman [3] and then furthermore for example by Derman [6], Mine and Osaki [30] and Dynkin and Yushkevich [9]. In this thesis, however, we will be concerned with systems modeled in continuous time. Our model for the system will be a differential equation with the noise added as a continuous random process. Consequently, the control process will be allowed to change in continuous time, not just at discrete time points. The time continuous case has been studied extensively, see [19] and also [11], [5] and references therein.

We will now present briefly the terminology trying not to go into technical details which we will discuss later. Our model for the systems is a stochastic differential equation. As we said, one is able to control the system by choosing parameters α from some set A.

$$x_t = x + \int_0^t \sigma^{\alpha_u}(s+u, x_u)dw_u + \int_0^t \beta^{\alpha_u}(s+u, x_u)du.$$

For $t \in [0,T]$, $x_t \in \mathbb{R}^d$, w_t is a d' dimensional Wiener process and $\sigma^\alpha(t,x)$ is a $d \times d'$ matrix (the "diffusion" coefficient) and b is a d dimensional vector (the "drift" coefficient). The control process $(\alpha_t)_{t\in[0,T]}$ is, for each t, a mapping from $(x_s)_{s\in[0,t]}$ to some $\alpha \in A$. This α is then another parameter in the drift and diffusion coefficients. We see that the control control at time t can only depend on the behavior of the system up to time t. We will assume that for some choice of the control process $(\alpha_t)_{t\in[0,T]}$ the SDE has a unique solution $(x_t)_{t\in[0,T]}$, depending on the control process. We will only consider such processes where the SDE has a unique solution.

The solution to this stochastic differential equation, which we denote $(x_t)_{t\in[0,T]}$ depends on the initial time s, initial position x and the control process $(\alpha)_{t\in[0,T]}$ used. So we should write $(x_t^{\alpha,s,x})_{t\in[0,T]}$ instead of $(x_t)_{t\in[0,T]}$.

Furthermore we need to somehow distinguish which trajectories of x_t are more desirable and which ones less. To that end we're given an "instantaneous reward" function $f^\alpha(t,x)$ a "terminal reward" function $g(t,x)$ and a "discounting factor" $c^\alpha(t,x)$. Then for a particular trajectory of the controlled process x_t the total reward is

$$\int_0^{T-s} f^{\alpha_t}(s+t, x_t^{\alpha,s,x}) e^{-\int_0^t c^{\alpha_u}(s+u, x_u^{\alpha,s,x})du} dt.$$

If we're only considering optimal control (without optimal stopping) then our aim is to find

$$v(s,x) = \sup_{\alpha_s} \mathbb{E}_{s,x}^{\alpha} \left(\int_0^T f^{\alpha_t}(s+t, x_t) e^{-\varphi_t} dt + g(x_T) e^{-\varphi_T} \right), \tag{1.0.1}$$

where

$$\varphi_t := \exp\Big(\int_0^t c^{\alpha_u}(s+u, x_u)du\Big)$$

and where the supremum is taken over all control processes that we're considering. Furthermore we will use $\mathbb{E}^{\alpha}_{s,x}$ to denote the mathematical expectation of the expression behind it, with $x_t^{\alpha,s,x}$ in place of x_t everywhere.

If we're interested in the optimal stopping and control problem then we wish to find

$$w(s,x) = \sup_{\tau}\sup_{\alpha_s}\mathbb{E}^{\alpha}_{s,x}\left(\int_0^{\tau} f^{\alpha_t}(s+t, x_t)e^{-\varphi_t}dt + g(x_\tau)e^{-\varphi_\tau}\right), \tag{1.0.2}$$

where the first supremum is taken over all stopping times less than T.

Example 1.0.2. Let $r \geq 0$, $\sigma > 0$ be given constants, w_t a one dimensional Wiener process and

$$dS_t = S_t(rdt + \sigma dw_t).$$

Consider the optimal stopping problem

$$v(t,x) = \sup_{\tau\in\mathfrak{T}[0,T]}\mathbb{E}_{t,x}(e^{-r\tau}[S_\tau - K]_+),$$

where the supremum is taken over all stopping times not greater than T. To illustrate the terminology introduced $[S-K]_+$ is the terminal reward function, the instantaneous reward function is 0, the discounting factor is constant r. The diffusion coefficient is $\sigma(t,x) = \sigma x$ and the drift coefficient is $\beta(t,x) = rx$. We see that there is no control involved. This is a very simple optimal stopping problem because it can be solved directly. First observe that the process $e^{-rt}S_t$ is a martingale and hence for $s < t$

$$\mathbb{E}(e^{-rt}(S_t - K)|\mathcal{F}_s) \geq \mathbb{E}(e^{-rt}S_t|\mathcal{F}_s) - Ke^{-rs} = e^{-rs}(S_s - K).$$

Since $[S-K]_+$ is convex in S we can use Jensen's inequality

$$\mathbb{E}(e^{-rt}[S_t - K]_+|\mathcal{F}_s) \geq \big[\mathbb{E}(e^{-rt}(S_t-K)|\mathcal{F}_s)\big]_+ \geq \big[e^{-rs}(S_s - K)\big]_+$$

So $e^{-rt}[S_t - K]_+$ is a submartinale. If we apply Doob's optimal sampling theorem we get that for $\tau < T$,

$$v(t,x) = \sup_{\tau \in \mathfrak{T}[0,T]} \mathbb{E}_{t,x}(e^{-r\tau}[S_\tau - K]_+) \geq \mathbb{E}_{t,x}(e^{-r\tau}[S_T - K]_+).$$

On the other hand, due to the fact that T is also a stopping time

$$\mathbb{E}_{t,x}(e^{-r\tau}[S_T - K]_+) \leq \sup_{\tau \in \mathfrak{T}[0,T]} \mathbb{E}_{t,x}(e^{-r\tau}[S_\tau - K]_+) = v(t,x).$$

Hence it always optimal to stop at time T. In fact there is an analytical expression for $v(t,x)$ and it's just the European option price (see for example [28]).

In a sense the example just given is misleading. It gives the impression that problems in stochastic control theory can be easily solved just by considering the expressions (1.0.1) or (1.0.2) directly. If one tries then one will quickly convince himself that this is not the case. For most problems evaluating v and w directly is not a feasible proposition. However, it turns out, that it is possible to prove that v satisfies a nonlinear partial differential equation, the Bellman PDE (sometimes referred to as the Hamilton-Jacobi-Bellman equation, or HJB equation). The Bellman PDE is a nonlinear parabolic PDE:

$$\begin{aligned} v_t + \sup_{\alpha \in A}(\mathrm{L}^\alpha v + f^\alpha) = 0 \quad &\text{on} \quad [0,T) \times \mathbb{R}^d \\ v(T,x) = g(x) \quad &\text{for} \quad x \in \mathbb{R}^d, \end{aligned} \tag{1.0.3}$$

where v_t is the partial derivative of v with respect to t and

$$\mathrm{L}^\alpha v = \sum_{i,j} a^\alpha_{i,j} v_{x^i x^j} + \sum_i \beta^\alpha_i v_{x^i} - c^\alpha v$$

where u_{x^i} is the partial derivative of v with respect to the ith coordinate of x while $u_{x^i x^j}$ is the second partial derivative of v with respect to the ith and jth coordinates of x. Proving the Bellman equation rests on Bellman principle which itself is a consequence of the Markov property of solutions to stochastic differential equations. A rigorous proof is beyond the scope of this thesis, but a very well presented and rigorous exposition is to be found in [19].

It is possible to show that the optimal stopping problem can in fact be regarded as an optimal control problem (this is the well known method of randomized stopping see for example [19] again). The resulting control problem involves discounting and instantaneous reward functions unbounded in the control parameter. It turns out that in such a case, or when the drift or diffusion coefficients are not bounded in the control parameter, (1.0.3) is no longer valid for w. Instead one should consider the normalized Bellman equation

$$\begin{aligned} \sup_{\alpha\in A} m^\alpha (w_t + \mathrm{L}^\alpha w + f^\alpha) = 0 \quad \text{on} \quad [0,T)\times\mathbb{R}^d \\ w(T,x) = g(x) \quad \text{for} \quad x\in\mathbb{R}^d, \end{aligned} \tag{1.0.4}$$

where $m^\alpha(t,x)$ is a positive function called the normalizing factor. It has the crucial property that when the drift coefficient, the diffusion coefficient, the instantaneous reward function and the discounting function are multiplied by the normalizing factor, then the product is bounded in the control parameter.

While there are some stochastic control problems which can be solved with the aid of the Bellman equation, one has to resort to solving the problems numerically in most cases. There are two approaches to this. One is to approximate the original controlled diffusion process by an appropriate controlled Markov chain on a finite state space. An thorough account of this method is available in [27]. Apart from presenting various techniques for constructing the approximations, they also prove convergence of the approximations. However they do not have any rate of convergence results. The other approach is to use a finite difference approximation to (1.0.3) when solving control problems with bounded coefficients, or to (1.0.4) in the case when one either solves optimal stopping and control problem or in the case when the control problems itself involves unbounded coefficients.

The main aim of this thesis is to consider the rate of convergence numerical approximations to the payoff function of the optimal stopping and control problem (1.0.2). The numerical approximations we will use will be the finite difference approximations derived from (1.0.4). First, we will look at stochastic control in more detail in Chapter 2. Then we will use the method of randomized stopping to translate the optimal stopping and control problem into a control problem. This is done in Chapter 3 where we present a new direct proof of for randomized stopping. In Chapter 4 we prove the rate of convergence of $\tau^{1/4}+h^{1/2}$ (where τ and h are the steps in time and space respectively) for the finite difference approximations to (1.0.4). We do this by following the methods presented in [24]. There has been a lot of work done recently on rates of convergence of finite difference approximations to Bellman PDE recently. We discuss this at the beginning of Chapter 4. The rate of convergence is proved on a grid over the whole of $[0,T]\times\mathbb{R}^d$ and hence this grid contains infinitely many elements. In order to solve this problem numerically one has to consider localization to a finite grid and the error arising from this. We do this in Chapter 5, where we also apply our results to the problem of valuing American put option. If one is solving an optimal control problem numerically then one also faces the issue of finding a discrete representation for the space of control parameters. In Chapter 5 we consider one case when this can be done easily.

We have avoided the question of how to find the optimal stopping time (optimal stopping rule) that would achieve the optimal payoff. Indeed the numerical method only gives an approximation for the payoff functions. Finding the optimal stopping rule, given the payoff function of optimal stopping problems is often done by finding the stopping boundary in the domain of the problem. That is, once the process x_t enters the set given by the stopping boundary then one can prove that it is optimal to stop. See [33] (or the original Russian version [32]) and [31].

Chapter 2

Stochastic control theory

In this chapter we try to briefly mention some basic concepts in stochastic control theory. After fixing some terminology in section 2.1 we define what we mean by a solution to a stochastic differential equation and what are the conditions for its existence. We will also state a result on the estimate of moments for the solutions of stochastic differential equations. That will be done in section 2.2. In section 2.3 we state the minimal assumption about the optimal stopping and control problem that we will need throughout the thesis. In section 2.4 we state the Bellman principle. We will provide some justification for it but no proof. We will show how it can be used to, heuristically, derive the Bellman PDE. Finally we derive the normalized Bellman PDE in section 2.5.

2.1 Some definitions

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a probability space with a right-continuous filtration, such that $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets.

Definition 2.1.1. Given a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a Wiener process $(w_t)_{t\geq 0}$ we say that w_t is $\mathcal{F}_t$ Wiener martingale if $(w_t)_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and if for all $t, h \geq 0$, $w_{t+h} - w_t$ is independent of $\mathcal{F}_t$.

Definition 2.1.2. Let X be a metric space and $\mathfrak{B}(X)$ be the Borel σ algebra on $\mathfrak{B}$. We call the X valued process $(x_t)_{t\geq 0}$ progressively measurable with respect to $\mathcal{F}_t$ if for all $t \geq 0$ and $A \in \mathfrak{B}(X)$ the set

$$\{(s, \omega) \in [0, t] \times \Omega : x_s(\omega) \in A\}$$

belongs to the product sigma algebra $\mathfrak{B}([0,t]) \times \mathcal{F}_t$. In other words the mapping

$$(s,\omega) \to x_s(\omega) : ([0,t] \times \Omega, \mathfrak{B}([0,t]) \times \mathcal{F}_t) \to (X, \mathfrak{B}(X))$$

is measurable for each $t \geq 0$.

Please note that any continuous adapted stochastic process is progressively measurable.

2.2 Stochastic differential equations

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a probability space with a right-continuous filtration, such that $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets. Let $(w_t, \mathcal{F}_t)$ be a d' dimensional Wiener martingale. Fix $T > 0$. Let $(\xi_t)_{t\in[0,T]}$ be a progressively measurable, $\mathbb{R}^d$ valued process such that

$$\mathbb{E} \int_0^T |\xi_t|^2 dt < \infty.$$

For every $x \in \mathbb{R}^d$ let $(\sigma(t,x))_{t\in[0,T]}$ and $(\beta(t,x))_{t\in[0,T]}$ be $(\mathcal{F}_t)_{t\in[0,T]}$ progressively measurable processes that are Borel measurable in (t,x) such that $\sigma(t,x)$ is $d \times d'$ dimensional matrix and $\beta(t,x)$ is a d dimensional vector. We will refer to $\sigma(t,x)$ as the diffusion coefficient and to $\beta(t,x)$ as the drift coefficient.

Assumption 2.2.1. Let $K, K_1 > 0$ be fixed constants independent of $t \in [0,T]$,$x \in \mathbb{R}^d$ and $\omega \in \Omega$. For any x, y in $\mathbb{R}^d$, $t \in [0,T]$ and $\omega \in \Omega$,

$$|\sigma(t,x) - \sigma(t,y)| \leq K|x-y|, \quad |\beta(t,x) - \beta(t,y)| \leq K|x-y|,$$

$$|\sigma(t,x)| \leq K(1+|x|), \quad |\beta(t,x)| \leq K(1+|x|).$$

Under the term "solution" of the stochastic differential equation

$$x_t = \xi_t + \int_0^t \sigma(s,x_s)dw_s + \int_0^t \beta(s,x_s)ds, \quad t \in [0,T], \tag{2.2.1}$$

we understand a process $(x_t)_{t\in[0,T]}$ that is progressively measurable with respect to the filtration $\mathcal{F}_t$ and such that the right hand side of (2.2.1) is well defined and coincides with the left hand side for all $t \in [0,T]$, $\omega \in \Omega'$ for some Ω' of $\mathbb{P}$-measure one.

Theorem 2.2.2. *If Assumption 2.2.1 is satisfied then* (2.2.1) *has a solution* $(x_t)_{t\in[0,T]}$ *such that*

$$\mathbb{E}\int_0^T |x_t|^2 < \infty.$$

Furthermore if $(x_t)_{t\in[0,T]}$, $(y_t)_{t\in[0,T]}$ *are two solutions of* (2.2.1)*, then*

$$\mathbb{P}\{\sup_{t\in[0,T]} |x_t - y_t| \neq 0\} = 0.$$

This is proved as Theorem 2.5.7 in [19]. The following theorem is a moments estimate for solutions of SDEs.

Theorem 2.2.3. *Let* x_t *be the solution of* (2.2.1)*. Let* x'_t *be a solution of*

$$x'_t = x' + \int_0^t \sigma'(s,x'_s)dw_s + \int_0^t \beta'(s,x'_s)ds.$$

Let Assumption 2.2.1 be satisfied by σ, β *and by* σ', β'*. Then for all* $q \geq 1$*,* $t \in [0,T]$

$$\begin{aligned}\mathbb{E}\sup_{s\leq t}|x_s - x'_s|^{2q} \leq & Ne^{Nt}|x-y|^{2q} \\ & + Nt^{q-1}e^{Nt}\mathbb{E}\int_0^t |\beta(s,x'_s) - \beta'(s,x'_s)|^{2q} + |\sigma(s,x'_s) - \sigma'(s,x_s)|^{2q}ds,\end{aligned}$$

where N *depends only on the Lipschitz constant* K *in Assumption 2.2.1 and on* q*.*

This theorem is proved, in a slightly more general form, as Theorem 2.5.9 in [19]. Finally we state a simpler form of Corollary 2.5.12 from [19].

Corollary 2.2.4. *Let* x_t *be the solution of* (2.2.1) *with* σ, β *satisfying Assumption 2.2.1. Let* K_1 *be a constant such that for all* $t \in [0,T]$*,* $x \in \mathbb{R}^d$

$$|\sigma(t,x)| \leq K_1(1+|x|) \quad \textit{and} \quad |\beta(t,x)| \leq K_1(1+|x|).$$

Then there exists a constant N *depending on* K_1 *and* q *such that for all* $q \geq 0$*,* $t \in [0,T]$

$$\mathbb{E}\sup_{s\leq t}|x_s|^q \leq Ne^{Nt}(1+|x|)^q.$$

2.3 The optimal control problem

Fix $T \in [0,\infty)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\mathcal{F}_t)_{t\geq 0}$ a right continuous filtration, such that $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets. Let $(w_t, \mathcal{F}_t)$ be a d' dimensional Wiener martingale. Let

A be a separable metric space. For every $t \in [0,T]$, $x \in \mathbb{R}^d$ and $\alpha \in A$ we are given a $d \times d'$ dimensional matrix $\sigma^\alpha(t,x)$, a d dimensional vector $\beta^\alpha(t,x)$ and real numbers $c^\alpha(t,x)$, $f^\alpha(t,x)$ and $g(x)$.

Assumption 2.3.1. σ, β, c, f are Borel functions of (α, t, x). The function g is continuous in x. There exist an increasing sequence of subsets A_n of A, and positive real constants K, K_n, and m, m_n, such that $\bigcup_{n\in\mathbb{N}} A_n = A$ and for each $n \in \mathbb{N}$, $\alpha \in A_n$,

$$\begin{aligned} |\sigma^\alpha(t,x) - \sigma^\alpha(t,y)| + |\beta^\alpha(t,x) - \beta^\alpha(t,y)| &\leq K_n|x-y|, \\ |\sigma^\alpha(t,x)| + |\beta^\alpha(t,x)| &\leq K_n(1+|x|), \\ |c^\alpha(t,x)| + |f^\alpha(t,x)| \leq K_n(1+|x|)^{m_n}, \quad |g(x)| &\leq K(1+|x|)^m \end{aligned} \tag{2.3.1}$$

for all $x \in \mathbb{R}^d$ and $t \in [0,T]$.

We say that $\alpha \in \mathfrak{A}_n$ if $\alpha = (\alpha_t)_{t\geq 0}$ is a progressively measurable process with values in A_n. Let $\mathfrak{A} = \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$. Then, due to Assumption 2.3.1 and Theorem 2.2.2, for each $s \in [0,T]$, $x \in \mathbb{R}^d$ and $\alpha \in \mathfrak{A}$ there is a unique solution $\{x_t : t \in [0, T-s]\}$ of

$$x_t = x + \int_0^t \sigma^{\alpha_u}(s+u, x_u)dw_u + \int_0^t \beta^{\alpha_u}(s+u, x_u)du, \tag{2.3.2}$$

denoted by $x_t^{\alpha,s,x}$. For $s \in [0,T]$ we use the notation $\mathfrak{T}(T-s)$ for the set of stopping times $\tau \leq T-s$. Define the payoff function to the optimal stopping and control problem as

$$w(s,x) = \sup_{\alpha\in\mathfrak{A}} \sup_{\tau\in\mathfrak{T}(T-s)} v^{\alpha,\tau}(s,x), \tag{2.3.3}$$

where

$$v^{\alpha,\tau}(s,x) = \mathbb{E}^\alpha_{s,x}\left[\int_0^\tau f^{\alpha_t}(s+t, x_t)e^{-\varphi_t}dt + g(x_\tau)e^{-\varphi_\tau}\right],$$

$$\varphi_t = \int_0^t c^{\alpha_r}(s+r, x_r)dr,$$

and $\mathbb{E}^\alpha_{s,x}$ means expectation of the expression behind it, with $x_t^{\alpha,s,x}$ in place of x_t everywhere. It is worth noticing that for

$$w_n(s,x) := \sup_{\alpha\in\mathfrak{A}_n} \sup_{\tau\in\mathfrak{T}(T-s)} v^{\alpha,\tau}(s,x)$$

we have $w_n(s,x) \uparrow w(s,x)$ as $n \to \infty$.

Theorem 2.3.2. *There exists a constant N_n depending on n, m, K, T such that for all $s \in$*

$[0,T]$, $x \in \mathbb{R}^d$

$$|w_n(s,x)| \le N_n(1+|x|)^m.$$

Proof.

$$\begin{aligned}
|v^{\alpha,\tau}(s,x)| &\le \mathbb{E}^{\alpha}_{s,x}\left[\int_0^{\tau} |f^{\alpha_t}(s+t,x_t)|dt + |g(x_\tau)|\right] \\
&\le K_n \mathbb{E}^{\alpha}_{s,x}\left[\int_0^{\tau} (1+|x_t|)^m dt + (1+|x_\tau|)^m\right] \\
&\le K_n(1+T-s)\mathbb{E}^{\alpha}_{s,x} \sup_{t\in[0,T-s]} (1+|x_t|)^m \le N_n(1+|x|)^m,
\end{aligned}$$

where we used Theorem 2.2.4 to get the last estimate. □

Thus $w_n(s,x)$ is bounded from above and from below and so $w(s,x)$ is bounded from below. Notice however that it can be equal to $+\infty$. The payoff function for the optimal control problem without stopping is defined for any $s \in [0,T]$ and $x \in \mathbb{R}^d$ by

$$v(s,x) = \sup_{\alpha\in\mathfrak{A}} v^{\alpha}(s,x), \tag{2.3.4}$$

$$v^{\alpha}(s,x) = \mathbb{E}^{\alpha}_{s,x}\left[\int_0^{T-s} f^{\alpha_t}(s+t,x_t)e^{-\varphi_t}dt + g(x_{T-s})e^{-\varphi_{T-s}}\right], \tag{2.3.5}$$

where x_t is the solution to (2.3.2).

2.4 Bellman principle and Bellman PDE

A very important concept in stochastic control is the Bellman principle. In particular it allows one to derive the Bellman PDE. For us this is very important, because we will use the Bellman PDE to derive the finite difference scheme to approximate the payoff functions. However we will prove convergence of the finite difference approximations to v as given by 2.3.4 not as a solution to the normalized Bellman equation. This is why we need not insist on a rigorous derivation of the Bellman PDE. A rigorous proof of the Bellman Principle is in Theorem 3.1.6 and 3.1.9 of [19], for control problems with σ^α, b^α, f^α and c^α bounded in the control parameter. Furthermore Theorem 6.1.5 of [19] proves Bellman principle for control problems where σ^α, b^α, f^α and c^α may be unbounded in the control parameter. Chapter 4 and Chapter 6 in [19] are devoted to rigorous derivations of the Bellman PDE and the normalized Bellman PDE.

We will now try to formally justify the Bellman principle. Our aim here is to provide a convincing heuristic argument, not a proof. First of all we have to accept that instead of using control strategies α_t which at time t depend on the whole $(x_s)_{s\in[0,t]}$ we can just use Markov controls i.e. α_t which at time t depend only on x_t (the current position of the controlled process in $\mathbb{R}^d$). This is not at all obvious, but let us assume that it is the case (see [19] for a rigorous explanation of the relationship between admissible and Markov strategies and the effect of considering only Markov ones). Now it seems possible to convince oneself of the following: Imagine that we have been controlling our diffusion process "optimally" from time t until time t'. Then our payoff so far, for the "optimal" strategy $(\alpha_s)_{s\in[t,t']}$, is given by

$$\mathbb{E}^{\alpha}_{t,x}\int_0^{t'-t} f^{\alpha_s}(t+s,x_s)e^{-\varphi_s}ds.$$

Of course the trajectory of the process x_s depends on it's initial value x_t and on the control process. However, if we're only using Markov strategies then the diffusion is a Markov Process and one could argue that if at any point on the trajectory of x_t a certain control is the optimal one, then the control is optimal no matter what the trajectory of the process has been in the past. So for any time after t' the trajectory of the process depends only on $x_{t'-t}$. The maximum payoff from time t' until time T is given, by definition, by (2.3.4) i.e.

$$v(t',x_{t'-t}) = \sup_{\alpha\in\mathfrak{A}}\mathbb{E}^{\alpha}_{t',x}\left(\int_0^{T-t'} f^{\alpha_s}(t+s,x_s)e^{-\varphi_s}ds + g(x_{T-t'})e^{-\varphi_{T-t'}}\right).$$

Hence our total payoff on the interval $[t,T]$, where we use the strategy $(\alpha_s)_{s\in[t,t']}$ on the time interval $[t,t']$ and then add the optimal payoff $v(t',x_{t'-t})$, is

$$\mathbb{E}^{\alpha}_{t,x}\int_0^{t'-t} f^{\alpha_s}(t+s,x_s)e^{-\varphi_s}ds + v(t',x_{t'-t})e^{-\varphi_{t'-t}}.$$

Since we're arguing the we have used the optimal control on the interval $[t,t']$ and since we believe that the optimal control at time s depends only depends on x_s, so that the payoff between $[t',T]$ depends only on $x_{t'}$, we get

$$v(t,x) = \sup_{\alpha_s\in\mathfrak{A}}\mathbb{E}^{\alpha}_{t,x}\left(\int_0^{t'-t} f^{\alpha_s}(t+s,x_s)e^{-\varphi_s}ds + v(t',x_{t'-t})e^{-\varphi_{t'-t}}\right). \tag{2.4.1}$$

We will now state the assumptions under which the Bellman Principle holds. For the proof we refer the reader to the proof of Theorem 6.1.5 in [19].

Assumption 2.4.1. The functions σ, b, c and f are continuous with respect to (α,x) and also that for each n and t they are continuous with respect to x uniformly with respect to $\alpha\in A_n$.

Theorem 2.4.2. *Let Assumptions 2.3.1 and 2.4.1 hold. Then* (2.4.1) *holds.*

If we are convinced of the validity of Bellman principle then we may proceed to the formal derivation of the Bellman PDE. Assume that all the functions σ^α, β^α, c^α and f^α are sufficiently smooth in t and x. Let us assume that the function v is sufficiently smooth so that we can apply Ito's formula to $v(t+s, x_s)e^{-\varphi_s}$ on $[t, t+t']$, for some $t' > t$. Let

$$a = \frac{1}{2}\sigma\sigma^*,$$

$$\mathrm{L}^\alpha u = \sum_{i,j=1}^{d} (a^\alpha)^{ij} u_{x^i x^j} + \sum_{i=d}^{d} (b^\alpha)^i u_{x^i} - c^\alpha.$$

Then

$$v(t+t', x_{t'})e^{-\varphi_{t'}} - v(t,x) = \int_0^{t'} \left(\frac{\partial}{\partial t} v(t+s, x_s) + \mathrm{L}^{\alpha_s} v(t+s, x_s) \right) e^{-\varphi_s} ds$$
$$+ \int_0^{t'} \sum_{i=1}^{d} \sigma(t+s, x_s) v_{x^i}(t+s, x_s) e^{-\varphi_s} dw_s.$$

Assume that

$$\mathbb{E} \int_0^{t'} [\sigma(t+s, x_s) v_{x^i}(t+s, x_s) e^{-\varphi_s}]^2 ds < \infty.$$

Then

$$v(t,x) = \mathbb{E} v(t+t', x_{t'})e^{-\varphi_{t'}} - \mathbb{E} \int_0^{t'} \left(\frac{\partial}{\partial t} v(t+s, x_s) + \mathrm{L}^{\alpha_s} v(t+s, x_s) \right) e^{-\varphi_s} ds.$$

Using this and the Bellman principle gives

$$0 = \sup_{\alpha_s \in \mathfrak{A}} \left(\mathbb{E} \int_0^{t'} \left(f^{\alpha_s}(t+s, x_s) + \frac{\partial}{\partial t} v(t+s, x_s) + \mathrm{L}^{\alpha_s} v(t+s, x_s) \right) e^{-\varphi_s} ds \right).$$

Divide the expression by t'. Assume that we can formally take the limit $t' \to 0$. Then

$$0 = \sup_{\alpha \in A} \left(v_t + \mathrm{L}^\alpha v + f^\alpha \right), \quad \text{on} \quad H_T. \tag{2.4.2}$$

The boundary condition

$$v(T, x) = g(x), \quad \text{for} \quad x \in \mathbb{R}^d,$$

follows directly from the definition of v.

We present an example that shows that if one of σ^α, β^α, c^α or f^α are unbounded as functions of the parameter α then the payoff function does not satisfy the Bellman PDE.

Example 2.4.3. Let $T > 0$, $\sigma^r(t,x) = 0$, $b^r(t,x) = -1$, $f^r(t,x) = rg(x)$, and $c^r(t,x) = r$ and $g(x) = x$. Then for $s \in [0, T-t]$

$$x_s = x - s$$

and

$$v(t,x) = \sup_{r_s \in \mathfrak{R}} \mathbb{E}_{t,x} \left(\int_0^{T-t} r_s g(x_s) e^{-\int_0^s r_u du} ds + g(x_{T-t}) e^{\int_0^{T-t} r_u du} \right).$$

This corresponds to the optimal stopping of the process x_t (see chapter 3). Hence $v(t,x) \geq g(x)$ and in fact $v(t,x) = x$. This is because x_t is a decreasing function of t and as g is increasing in x it is always optimal to stop immediately. On the other hand if we write down the Bellman PDE we get

$$u_t + u_x + \sup_{r \geq 0}(-ru + rg) = 0, \quad \text{for} \quad t \in [0,T)$$

$$u(T,x) = x, \quad \text{for} \quad x \in \mathbb{R}.$$

We can see that $\sup_{r \geq 0} r(g-u) = 0$, because if the solution u corresponds to the payoff function of the control problem then $g \leq u$. Hence

$$u_t + u_x = 0, \quad \text{for} \quad (t,x) \in [0,T) \times \mathbb{R}.$$

But $v_t = 0$ and $v_x = 1$ so v does not satisfy the Bellman PDE.

2.5 Normalized Bellman PDE

Notice that if σ^α, b^α, c^α or f^α are unbounded as functions of α, then one can't take the limit in (2.4.2) uniformly with respect to the strategies α_s.

One can use the method of random time change (see Chapter 1 of [19]) to overcome this problem. We need to assume that there is a uniformly bounded function $m^\alpha(t,x)$ such that $\sqrt{m^\alpha}\sigma^\alpha$, $m^\alpha b^\alpha$, $m^\alpha c^\alpha$ and $m^\alpha f^\alpha$ are bounded as functions of the control parameter. For $t \in [0, T-s]$ let

$$\psi(t) := \int_0^t \frac{1}{m^{\alpha_r}(s+r, x_r)} dr.$$

Let τ be the inverse of ψ. Consider a new process $z_t := x_{\tau(t)}$. Let $\beta_t = \alpha_{\tau(t)}$. Then

$$\begin{aligned} z_t = x_{\tau(t)} &= x + \int_0^{\tau(t)} \sigma^{\alpha_r}(s+r, x_r)dw_r + \int_0^{\tau(t)} b^{\alpha_r}(s+r, x_r)dr \\ &= x + \int_0^t \sigma^{\beta_t}(s+\tau(r), z_r)\sqrt{m^{\beta_r}(s+\tau(r), z_r)}d\xi_r \\ &+ \int_0^t b^{\beta_t}(s+\tau(r), z_r)m^{\beta_r}(s+\tau(r), z_r)dr, \end{aligned}$$

where $\xi_t = \int_0^{\tau(t)} \sqrt{m^{\alpha_r}(s+r, r)}dw_r$. One can argue that ξ_t is a Wiener process adapted to the filtration $\mathcal{F}_{\tau(t)}$. Let

$$\varphi_1(\tau(t)) = \int_0^{\tau(t)} c^{\alpha_u}(s+u, x_u)du = \int_0^t c^{\beta_u}(s+\tau(u), z_u)m^{\beta_u}(s+\tau(u), z_u)du.$$

Let $\sigma_1 = \sqrt{m}\sigma$, $b_1 = mb$, $c_1 = mc$ and $f_1 = mf$. Then (with v given by (2.3.4))

$$\begin{aligned} v(s,x) = \sup_{\beta_t} \big(\mathbb{E}^{\alpha}_{s,x} \int_0^{\psi(T-s)} & f_1^{\beta_t}(s+\tau(t), z_t)e^{-\varphi_1(\tau(t))}dt \\ & + g(z_{\psi(T-s)})e^{-\varphi_1(\psi(T-s))}\big). \end{aligned}$$

We formally apply the Bellman principle to the process z_t. Then for some $t' \le \psi(T-s)$:

$$v(s,x) = \sup_{\beta_t} \mathbb{E}^{\alpha}_{s,x}\left(\int_0^{t'} f_1^{\beta_t}(s+\tau(t), z_t)e^{-\varphi_1(\tau(t))}dt + v(s+\tau(t'), z'_t)e^{-\varphi_1(t')}\right).$$

As before we formally apply the Itô formula to $ve^{-\varphi}$, but this time on the interval $[s, s+\tau(t')]$.

$$\begin{aligned} &\mathbb{E}v(s+\tau(t'), x_{\tau(t')})e^{-\varphi_{\tau(t')}} \\ &= v(s,x) + \mathbb{E}\int_0^{\tau(t')} \left(\frac{\partial}{\partial t}v(s+r, x_r) + \mathrm{L}^{\alpha_r}\, v(s+r, x_r)\right) e^{-\varphi_1(r)}dr, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \mathbb{E}v(s+\tau(t'), z_t)e^{\varphi_1(t')} &= v(s,x) \\ &+ \mathbb{E}\int_0^{t'} m^{\beta_r}(s+\tau(r), z_r)\left(\frac{\partial}{\partial t} + \mathrm{L}^{\beta_r}\right) v(s+\tau(r), z_r)e^{-\varphi_1(\tau(r))}dr. \end{aligned}$$

Hence

$$0 = \sup_{\beta_r} \mathbb{E} \int_0^{t'} m^{\beta_r}(s+\tau(r), z_r) \left(\frac{\partial}{\partial t} + \mathrm{L}^{\beta_r} \right) v(s+\tau(r), z_r) e^{-\varphi(\tau(r))}$$
$$+ f_1^{\beta_r}(s+\tau(r), z_r) e^{-\varphi_1(\tau(r))} dr.$$

Divide by this t'. Since $\sqrt{m^\alpha}\sigma^\alpha$, $m^\alpha b^\alpha$, $m^\alpha c^\alpha$ and $m^\alpha f^\alpha$ are bounded as functions of the control parameter, we can argue that it should be possible to formally take the limit as $t' \to 0$. Hence we obtain the normalized Bellman equation

$$\begin{aligned} \sup_{\alpha \in A} m^\alpha \left(u_t + \mathrm{L}^\alpha u + f^\alpha \right) = 0 \quad \text{on} \quad [0,T) \times \mathbb{R}^d \\ u(T,x) = g(x) \quad \text{for} \quad x \in \mathbb{R}^d. \end{aligned} \tag{2.5.1}$$

A reasonable question is to consider what would happen if we applied the previous argument in the case when σ^α, b^α, c^α and f^α are bounded functions of the control parameter, with some bounded $m^\alpha > 0$. The result of the above formal argument will again be the normalized Bellman equation (2.5.1) which is different than (2.4.2). One can check that in the case that for some $\delta > 0$, $m^\alpha \geq \delta > 0$, then (2.5.1) and (2.4.2) are equivalent. If $\inf_\alpha m^\alpha = 0$ then this is not the case. We will therefore impose the condition $m^\alpha(1 + c^\alpha) \geq \delta > 0$.

Chapter 3

Randomized stopping

We will use the well known method of randomized stopping to embed the optimal stopping (and optimal stopping and control problem) into the class of optimal control problems with unbounded coefficients.

It is known that optimal stopping problems for controlled diffusion processes can be transformed into optimal control problems by using the method of randomized stopping (see [19] and [10]). Since only a few optimal stopping problems can be solved analytically (see [32]), one has to resort to numerical approximations of the solution. In such case one would like to know the rate of convergence of these approximations. Embedding optimal stopping problems into the class of stochastic control problems allows one to apply numerical methods developed for stochastic control as we will show later. The price one pays for it is the unboundedness of the reward function, as a function of the control parameter.

Our main result, Theorem 3.1.1, formulates the method of randomized stopping in a general setting. Applying it to optimal stopping problems of controlled diffusion processes we easily get, see Theorem 3.2.1, that under general conditions the payoff function of optimal stopping problem of controlled diffusions equals the payoff function of the control problem obtained by randomized stopping. This result is known from [19] in the case when the coefficients of the controlled diffusions are bounded in the control parameter (see section 4 of Chapter 3 in [19]). In Theorem 3.2.1 the coefficients of the diffusions and the functions, defining the payoff may be unbounded functions of the control parameter. Also they need not satisfy those conditions on continuity which are needed in [19]. Theorem 3.1.1 can also be applied to optimal stopping of stochastic processes from a larger classes than that of diffusion processes.

3.1 A general result

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$, such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\mathfrak{T}$ denote the set of finite stopping times. Let $\mathfrak{F}$ denote the set of all processes $F = (F_t)_{t\geq 0}$ which are $\{\mathcal{F}_t\}_{t\geq 0}$ adapted, right continuous and increasing, such that

$$\forall \omega \in \Omega, \quad F_0(\omega) = 0 \quad \text{and} \quad \lim_{t\to\infty} F_t(\omega) = 1.$$

Let $\bar{\mathfrak{R}}$ be a class of nonnegative adapted locally integrable stochastic processes $r = (r_t)_{t\geq 0}$ such that

$$\int_0^\infty r_t dt = \infty.$$

Let $\mathfrak{R}_n$ denote those stochastic processes from $\bar{\mathfrak{R}}$ which take values in $[0, n]$. Let $\mathfrak{R} = \bigcup_{n\in\mathbb{N}} \mathfrak{R}_n$.

Theorem 3.1.1. *Let $(h_t)_{t\geq 0}$ be a progressively measurable process with sample paths continuous at 0. Assume that for all $t \geq 0$, $|h_t| \leq \xi$ for some random variable ξ satisfying $\mathbb{E}\xi < \infty$. Then*

$$\sup_{\tau\in\mathfrak{T}} \mathbb{E}h_\tau = \sup_{F\in\mathfrak{F}} \mathbb{E} \int_0^\infty h_t dF_t < \infty. \tag{3.1.1}$$

Theorem 3.1.1 is the main theorem of this section and will be proved later. For now we look at one consequence of this theorem.

Theorem 3.1.2. *Let $(h_t)_{t\geq 0}$ be an adapted cadlag process such that*

$$\mathbb{E} \sup_{t\geq 0} |h_t| < \infty.$$

Then

$$\sup_{r\in\bar{\mathfrak{R}}} \mathbb{E} \int_0^\infty h_t r_t e^{-\int_0^t r_u du} dt = \sup_{r\in\mathfrak{R}} \mathbb{E} \int_0^\infty h_t r_t e^{-\int_0^t r_u du} dt = \sup_{\tau\in\mathfrak{T}} \mathbb{E}h_\tau < \infty. \tag{3.1.2}$$

Proof. Let $r \in \bar{\mathfrak{R}}$. If $d\varphi_t = -de^{-\int_0^t r_u du}$, $\varphi_0 = 1$, then $1 - \varphi \in \mathfrak{F}$. Hence by Theorem 3.1.1

$$\mathbb{E} \int_0^\infty h_t r_t e^{-\int_0^t r_u du} dt \leq \sup_{F\in\mathfrak{F}} \mathbb{E} \int_0^\infty h_t dF_t = \sup_{\tau\in\mathfrak{T}} \mathbb{E}h_\tau < \infty.$$

On the other hand, for $\tau \in \mathfrak{T}$, let $r_t^n = 0$ for $t < \tau$, and $r_t^n = n$ for $t \geq \tau$. Set

$$F_t^n := \begin{cases} 0, & \text{for} \quad t < \tau \\ 1 - e^{-n(t-\tau)}, & \text{for} \quad t \geq \tau. \end{cases}$$

Then for any $\omega \in \Omega$ and for any $\delta > 0$,

$$\int_0^\infty h_t r_t^n e^{-\int_0^t r_u^n du} dt = \int_0^\infty h_t dF_t^n = \int_\tau^\infty h_t dF_t^n = I_n + J_n + K_n,$$

where

$$I_n := \int_\tau^{\tau+\delta} h_\tau \, dF_t^n, \quad J_n := \int_\tau^{\tau+\delta} (h_t - h_\tau) \, dF_t^n, \quad K_n := \int_{\tau+\delta}^\infty h_t \, dF_t^n.$$

Notice that as $n \to \infty$,

$$I_n = h_\tau (1 - e^{-n\delta}) \to h_\tau \quad \text{and} \quad |K_n| \leq \sup_{t \geq 0} |h_t| e^{-(\tau+\delta)n} \to 0.$$

Furthermore

$$|J_n| \leq \sup_{t \in [\tau, \tau+\delta]} |h_t - h_\tau| \to 0 \quad \text{as} \quad \delta \to 0.$$

Hence for any $\omega \in \Omega$, taking first the limit as $n \to \infty$ and then the limit as $\delta \to 0$,

$$\int_0^\infty h_t r_t^n e^{-\int_0^t r_u^n du} dt \to h_\tau.$$

Clearly,

$$|\int_0^\infty h_t r_t^n e^{-\int_0^t r_u^n du} dt| \leq \sup_{t \geq 0} |h_t|.$$

Hence by Lebesgue's Theorem

$$\sup_{r \in \mathfrak{R}} \mathbb{E} \int_0^\infty h_t r_t e^{-\int_0^t r_u du} dt \geq \lim_{n \to \infty} \mathbb{E} \int_0^\infty h_t r_t^n e^{-\int_0^t r_u^n du} dt$$

$$= \mathbb{E} \lim_{n \to \infty} \int_0^\infty h_t r_t^n e^{-\int_0^t r_u^n du} dt = \mathbb{E} h_\tau.$$

$\square$

The proof of Theorem 3.1.1 is based on the following lemmas. We use the notation $\mathbf{1}_S$ for the indicator function of a set S.

Lemma 3.1.3. *Let $(h_i)_{i=1}^n$, $(p_i)_{i=1}^n$ be sequences of random variables adapted to a filtration $(\mathcal{F}_i)_{i=1}^n$, where $\mathcal{F}_1$ contains all $\mathbb{P}$-null sets, such that for all i, $p_i \geq 0$, $\mathbb{E}|h_i| < \infty$ and*

$$\sum_{i=1}^n p_i = 1 \quad (a.s.).$$

Then there exist disjoint sets $(A_i)_{i=1}^n$, $A_i \in \mathcal{F}_i$, $A_1 \cup A_2 \cup \ldots \cup A_n = \Omega$ such that almost surely

$$\mathbb{E}(p_1 h_1 + p_2 h_2 + \ldots + p_n h_n | \mathcal{F}_1) \leq \mathbb{E}(h_1 \mathbf{1}_{A_1} + h_2 \mathbf{1}_{A_2} + \ldots + h_n \mathbf{1}_{A_n} | \mathcal{F}_1).$$

Proof. For $n = 1$ the statement of the lemma is obvious. Assume $n = 2$ (this will illustrate the general case better). Then, since $p_1 h_1$ is $\mathcal{F}_1$ measurable, $p_2 = 1 - p_1$ is also $\mathcal{F}_1$ measurable, and

$$I := p_1 h_1 + \mathbb{E}(p_2 h_2 | \mathcal{F}_1) = p_1 h_1 + p_2 \mathbb{E}(h_2 | \mathcal{F}_1).$$

Let $A_1 = \{h_1 \geq \mathbb{E}(h_2 | \mathcal{F}_1)\}$. This is an $\mathcal{F}_1$ set and

$$I \leq h_1 \mathbf{1}_{A_1} + \mathbf{1}_{\Omega \setminus A_1} \mathbb{E}(h_2 | \mathcal{F}_1) = \mathbb{E}(h_1 \mathbf{1}_{A_1} + h_2 \mathbf{1}_{\Omega \setminus A_1} | \mathcal{F}_1).$$

Assume that the lemma holds for $n - 1 \geq 1$. Let us prove that it remains true for n. Let $B = \{p_1 < 1\}$.

$$I := \mathbb{E}(h_1 p_1 + h_2 p_2 + \ldots + h_n p_n | \mathcal{F}_1)$$

$$= \mathbb{E}(\mathbf{1}_B (h_1 p_1 + h_2 p_2 + \ldots + h_n p_n) | \mathcal{F}_1) + \mathbb{E}(\mathbf{1}_{B^c} h_1 | \mathcal{F}_1) =: I_1 + I_2.$$

Then

$$I_1 = \mathbb{E}\left(\mathbf{1}_B \left(h_1 p_1 + \mathbb{E}(h_2 p_s + \ldots + h_n p_n | \mathcal{F}_1)\right) | \mathcal{F}_1\right)$$
$$= \mathbb{E}\left(\mathbf{1}_B \left(h_1 p_1 + (p_2 + \ldots + p_n) \mathbb{E}\left(\frac{h_2 p_2 + \ldots + h_n p_n}{p_2 + \ldots + p_n} | \mathcal{F}_1\right)\right) | \mathcal{F}_1\right),$$

since $p_2 + \ldots + p_n > 0$ is $\mathcal{F}_1$ measurable. Let

$$A_1 = \left\{h_1 \geq \mathbb{E}\left(\frac{h_2 p_2 + \ldots + h_n p_n}{p_2 + \ldots + p_n} | \mathcal{F}_1\right)\right\} \cap B.$$

Since A_1 is an $\mathcal{F}_1$ set,

$$I_1 \leq \mathbb{E}\left(h_1 \mathbf{1}_{A_1} + \mathbf{1}_B \mathbb{E}\left(\mathbf{1}_{\Omega \setminus A_1} \frac{h_2 p_2 + \ldots + h_n p_n}{p_2 + \ldots + p_n} | \mathcal{F}_1\right) | \mathcal{F}_1\right)$$

$$= \mathbb{E}\left(h_1 \mathbf{1}_{A_1} | \mathcal{F}_1\right) + \mathbb{E}\left(\mathbb{E}(h'_2 p'_2 + \ldots + h'_n p'_n | \mathcal{F}_1) | \mathcal{F}_1\right),$$

where $h'_i = h_i \mathbf{1}_B \mathbf{1}_{\Omega \setminus A_1}$, $p'_i = \frac{p_i}{p_2 + \ldots + p_n}$ on B and $p'_i = (n-1)^{-1}$ on B^c, for $2 \leq i \leq n$. Then $p'_2 + \ldots + p'_n = 1$. Apply the inductive hypothesis to $(h'_i)_{i=2}^n$, $(p'_i)_{i=2}^n$, $(\mathcal{F}_i)_{i=2}^n$. Then there are disjoint $A'_2 \cup A'_3 \cup \ldots \cup A'_n = \Omega$ such that $A'_i \in \mathcal{F}_i$ for $i = 2, \ldots n$, and

$$\mathbb{E}(h'_2 p'_2 + \ldots + h'_n p'_n | \mathcal{F}_1) \leq \mathbb{E}(h'_2 \mathbf{1}_{A'_2} + \ldots + h'_n \mathbf{1}_{A'_n} | \mathcal{F}_1).$$

Hence

$$I_1 \leq \mathbb{E}\left(h_1 \mathbf{1}_{A_1} | \mathcal{F}_1\right) + \mathbb{E}\left(\mathbf{1}_B \mathbb{E}\left(\mathbf{1}_{\Omega \setminus A_1}(h_2 \mathbf{1}_{A'_2} + \ldots h_n \mathbf{1}_{A'_n}) | \mathcal{F}_1\right) | \mathcal{F}_1\right).$$

We see that $(\Omega \setminus A_1) \cap (A'_2 \cup \ldots \cup A'_n) = \Omega \setminus A_1$. For $1 < i \leq n$, define $A_i = B \cap A'_i \cap (\Omega \setminus A_1)$. Such A_i are disjoint, $\mathcal{F}_i$ measurable and

$$B^c \cup A_1 \cup A_2 \cup \ldots \cup A_n = \Omega.$$

Thus

$$I_1 \leq \mathbb{E}\left(h_1 \mathbf{1}_{A_1} + h_2 \mathbf{1}_{A_2} + \ldots + h_n \mathbf{1}_{A_n} | \mathcal{F}_1\right).$$

Finally

$$I = I_1 + I_2 \leq \mathbb{E}\left(h_1(\mathbf{1}_{A_1} + \mathbf{1}_{B^c}) + h_2 \mathbf{1}_{A_2} + \ldots + h_n \mathbf{1}_{A_n} | \mathcal{F}_1\right).$$

□

Lemma 3.1.4. *Let $(h_t)_{t \geq 0}$ be a $\mathcal{B}([0,\infty)) \times \mathcal{F}$- measurable process and let $F \in \mathfrak{F}$ such that*

$$\mathbb{E} \int_0^\infty |h_t| dF_t < \infty.$$

Then for each $n \in \mathbb{N}$ there exists a finite sequence of stopping times $\tau_i^{(n)}$ such that

$$0 = \tau_0^{(n)} < \tau_1^{(n)} \leq \ldots \leq \tau_{M(n)}^{(n)} < \infty,$$

and for

$$\phi_t^{(n)} = \sum_{i=1}^{M(n)} h_{\tau_i^{(n)}} \mathbf{1}_{(\tau_{i-1}^{(n)}, \tau_i^{(n)}]}, \tag{3.1.3}$$

one has

$$\mathbb{E} \int_0^\infty |h_t - \phi_t^{(n)}| dF_t \to 0 \quad as \quad n \to \infty.$$

This lemma is a particular case of Proposition 1, part ii, in [12]. We present a proof for the convenience of the reader.

Proof. It is known (see [8] or [29]) that if an integrable stochastic process $\{g(r) : r \in [0,1]\}$ satisfies

$$\mathbb{E}\int_0^1 |g(r)|dr < \infty,$$

then for every integer $n \geq 1$ there exists a partition

$$0 = t_0^n < \ldots < t_M^n < t_{M+1}^n = 1$$

with some $M = M(n)$, such that

$$\lim_{n\to\infty} \mathbb{E}\int_0^1 |g(\kappa_n^j(r)) - g(r)|dr = 0, \tag{3.1.4}$$

where $j = 1, 2$,

$$\kappa_n^1(r) := t_{i-1}^n \quad \text{for} \quad r \in [t_{i-1}^n, t_i^n), \quad i = 1, \ldots, M.$$

$$\kappa_n^2(r) := t_i^n \quad \text{for} \quad r \in (t_{i-1}^n, t_i^n], \quad i = 1, \ldots, M.$$

Let

$$\beta(r) := \inf\{t \geq 0 : F_t \geq r\}.$$

Then $\beta(r)$ is a stopping time for every $r \geq 0$, and

$$\mathbb{E}\int_0^\infty h_t dF_t = \mathbb{E}\int_0^1 h_{\beta(r)} dr.$$

Consider $g(r) := h_{\beta(r)}$, $\tau_i^n := \beta(t_i^n)$ for $i = 0, \ldots, M$, and notice that $\tau_M^n < \infty$. If $r \in (t_{i-1}^n, t_i^n]$ then

$$\phi_{\beta(r)}^{(n)} = \begin{cases} h_{\beta(t_i^n)} = h_{\beta(\kappa_n^2(r))} & \text{if} \quad \beta(t_{i-1}^n) < \beta(r), \\ h_{\beta(t_{i-1}^n)} = h_{\beta(\kappa_n^1(r))} & \text{if} \quad \beta(t_{i-1}^n) = \beta(r). \end{cases}$$

So there exist $S_n \in \mathcal{F} \times \mathfrak{B}(\mathbb{R})$ such that

$$\phi_{\beta(r)}^{(n)} = \mathbf{1}_{S_n} h_{\beta(\kappa_n^2(r))} + (1 - \mathbf{1}_{S_n}) h_{\beta(\kappa_n^1(r))}.$$

Hence by (3.1.4)

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \mathbb{E} \int_0^1 \left| \phi_{\beta(r)}^{(n)} - h_{\beta(r)} \right| dr \\
&= \lim_{n\to\infty} \mathbb{E} \int_0^\infty \left| \sum_{i=1}^M h_{\tau_i^n} \mathbf{1}_{(\tau_{i-1}^n, \tau_i^n]}(s) - h_s \right| dF_s.
\end{aligned}
$$

□

Proof of Theorem 3.1.1.

$$
\sup_{F\in\mathfrak{F}} \mathbb{E} \int_0^\infty h_t dF_t \le \sup_{F\in\mathfrak{F}} \mathbb{E} \sup_{t\ge 0} |h_t| \int_0^\infty dF_t = \mathbb{E} \sup_{t\ge 0} |h_t| \le \mathbb{E}\xi < \infty.
$$

Let $\tau \in \mathfrak{T}$. Define

$$
G_t^n := \mathbf{1}_{\{\tau>0\}} \mathbf{1}_{\{t\ge\tau\}} + \mathbf{1}_{\{\tau=0\}} \left(1 - e^{-nt}\right).
$$

clearly $G_t^n \in \mathfrak{F}$. Using the argument of the proof of Theorem 3.1.2 it is easy to see that

$$
\mathbb{E} h_\tau = \lim_{n\to\infty} \mathbb{E} \int_0^\infty h_t dG_t^n \le \sup_{F\in\mathfrak{F}} \mathbb{E} \int_0^\infty h_t dF_t.
$$

This is bounded from above independently of $\tau \in \mathfrak{T}$ and so we can take the supremum over $\mathfrak{T}$. To complete the proof one needs to show that

$$
\sup_{\tau\in\mathfrak{T}} \mathbb{E} h_\tau \ge \sup_{F\in\mathfrak{F}} \mathbb{E} \int_0^\infty h_t dF_t
$$

holds. First, for some sequence $0 = \tau_0 < \tau_1 \le \ldots \le \tau_M < \infty$ of stopping times, consider

$$
\phi_t = \sum_{i=1}^M h_{\tau_i} \mathbf{1}_{(\tau_{i-1}, \tau_i]}, \quad \text{for } t \ge 0,
$$

and find a stopping time τ such that

$$
I := \mathbb{E} \int_0^\infty \phi_t dF_t \le \mathbb{E} h_\tau.
$$

To this end notice that

$$
I = \mathbb{E} \int_0^\infty \sum_{i=1}^M h_{\tau_i} \mathbf{1}_{(\tau_{i-1}, \tau_i]} dF_t = \mathbb{E} \sum_{i=1}^M h_{\tau_i} \int_{\tau_{i-1}}^{\tau_i} dF_t := \mathbb{E} \sum_{i=1}^M h_{\tau_i} p^{(i)}.
$$

Since $h = (h_t)_{t\ge 0}$ is progressively measurable, h_{τ_i} is $\mathcal{F}_{\tau_i}$ measurable. Also, $p^{(i)}$ is $\mathcal{F}_{\tau_i}$ measurable. By Lemma 3.1.3 there exist disjoint sets

$$A_1 \in \mathcal{F}_{\tau_1}, A_2 \in \mathcal{F}_{\tau_2}, \dots, A_M \in \mathcal{F}_{\tau_M}, \text{ such that } \bigcup_{i=1}^{M} A_i = \Omega$$

and

$$\mathbb{E}\left(\int_0^\infty \phi_t dF_t\right) \le \mathbb{E}\left(\sum_{i=1}^{M} h_{\tau_i} \mathbf{1}_{A_i}\right).$$

Let $\tau = \tau_i < \infty$ on A_i for each i. By definition of $\mathcal{F}_{\tau_i}$, $\{\tau_i \le s\} \cap A_i \in \mathcal{F}_s$. Thus

$$\{\tau \le s\} = \bigcup_{i=1}^{M} \{\tau_i \le s\} \cap A_i \in \mathcal{F}_s, \quad \text{for all } s \ge 0,$$

which shows that τ is a (finite) stopping time. Using Lemma 3.1.4 we obtain a sequence $\phi^{(n)}$ of the form (3.1.3) such that, for h satisfying the hypothesis of the theorem,

$$\mathbb{E}\int_0^\infty |h_t - \phi_t^{(n)}| dF_t \to 0 \quad \text{as} \quad n \to \infty.$$

For each $\phi^{(n)}$ there exists a stopping time $\tau^{(n)}$ such that

$$\mathbb{E}\int_0^\infty \phi_t^{(n)} dF_t \le \mathbb{E}\phi^{(n)}_{\tau^{(n)}} = \mathbb{E}h_{\tau^{(n)}}.$$

Take $\liminf$ as $n \to \infty$ on both sides of the above inequality. Then

$$\mathbb{E}\int_0^\infty h_t dF_t \le \liminf_{n\to\infty} \mathbb{E}\phi^{(n)}_{\tau^{(n)}} = \liminf_{n\to\infty} \mathbb{E}h_{\tau^{(n)}} \le \sup_{\tau \in \mathfrak{T}} \mathbb{E}h_\tau.$$

□

Remark 3.1.5. Using the method of the proof of Theorem 3.1.1 we can see that the following generalization holds: let $F \in \mathfrak{F}$ and $(h_t)_{t\ge 0}$ be a progressively measurable process such that $\mathbb{E}|h_\tau| < \infty$ for all finite stopping times τ, and

$$\mathbb{E}\int_0^\infty |h_t| dF_t < \infty.$$

Then for any $\varepsilon > 0$ there is a finite stopping time τ such that

$$\mathbb{E}\int_0^\infty |h_t| dF_t \le \mathbb{E}h_\tau + \varepsilon.$$

3.2 Application to diffusion processes

We will recall the notation from Chapter 2. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a probability space with a right-continuous filtration, such that $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets. Let $(w_t, \mathcal{F}_t)$ be a d' dimensional Wiener martingale. Let A be a separable metric space. Let $T \in [0, \infty)$. For every $t \in [0, T]$, $x \in \mathbb{R}^d$ and $\alpha \in A$ we are given a $d \times d'$ dimensional matrix $\sigma^\alpha(t, x)$, a d dimensional vector $\beta^\alpha(t, x)$ and real numbers $c^\alpha(t, x)$, $f^\alpha(t, x)$ and $g(t, x)$. We say that $\alpha \in \mathfrak{A}_n$ if $\alpha = (\alpha_t)_{t\geq 0}$ is a progressively measurable process with values in A_n. Let $\mathfrak{A} = \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$. Then under Assumption 2.3.1 it is well known that for each $s \in [0, T]$, $x \in \mathbb{R}^d$ and $\alpha \in \mathfrak{A}$ there is a unique solution $\{x_t : t \in [0, T-s]\}$ of

$$x_t = x + \int_0^t \sigma^{\alpha_u}(s+u, x_u)dw_u + \int_0^t \beta^{\alpha_u}(s+u, x_u)du,$$

denoted by $x_t^{\alpha,s,x}$. For $s \in [0, T]$ we use the notation $\mathfrak{T}(T-s)$ for the set of stopping times $\tau \leq T - s$. Recall that we've defined

$$w(s, x) = \sup_{\alpha\in\mathfrak{A}} \sup_{\tau\in\mathfrak{T}(T-s)} v^{\alpha,\tau}(s, x),$$

where

$$v^{\alpha,\tau}(s, x) = \mathbb{E}^\alpha_{s,x} \left[\int_0^\tau f^{\alpha_t}(s+t, x_t)e^{-\varphi_t}dt + g(s+\tau, x_\tau)e^{-\varphi_\tau} \right],$$

$$\varphi_t = \int_0^t c^{\alpha_r}(s+r, x_r)dr.$$

Let $\mathfrak{R}_n$ contain all progressively measurable, locally integrable processes $r = (r_t)_{(t\geq 0)}$ taking values in $[0, n]$ such that $\int_0^\infty r_t dt = \infty$. Let $\mathfrak{R} = \bigcup_{n\in\mathbb{N}} \mathfrak{R}_n$.

Next we prove a theorem which is known from [19] in the special case when $A = A_n$, $K = K_n$, $m = m_n$ for $n \geq 1$ (see Exercise 3.4.12, via Lemma 3.4.3(b) and Lemma 3.4.5(c)). Our proof is a straightforward application of Theorem 3.1.2. Since Theorem 3.1.2 is separated from the theory of controlled diffusion processes developed in [19], we do not require that σ, b, c, f and g be continuous in (α, x) and be continuous in x, uniformly in α for each t, which conditions are needed in [19].

Theorem 3.2.1. *Let Assumption 2.3.1 hold. Then for all* $(s, x) \in [0, T] \times \mathbb{R}^d$*, either both*

$w(s,x)$ and

$$\sup_{\alpha\in\mathfrak{A}}\sup_{r\in\mathfrak{R}}\mathbb{E}^{\alpha}_{s,x}\bigg\{\int_0^{T-s}\big[f^{\alpha_t}(s+t,x_t)e^{-\varphi_t} + r_t g(s+t,x_t)e^{-\varphi_t}\big]e^{-\int_0^t r_u du}dt + g(T,x_{T-s})e^{-\varphi_{T-s}-\int_0^{T-s}r_u du}\bigg\} \tag{3.2.1}$$

are finite and equal, or they are both infinite.

Proof. Without loss of generality we may assume that $s=0$. Let $r\in\mathfrak{R}$. For $t>T$ let $f^{\alpha}(t,x)=0$, $c^{\alpha}(t,x)=0$. For fixed $(\alpha_t)\in\mathfrak{A}$ set

$$f_t = f^{\alpha_t}(t,x_t)e^{-\varphi_t}, \quad \text{for } t\geq 0,$$

$$g_t = \begin{cases} g(t,x_t)e^{-\varphi_t} & \text{for} \quad t\leq T \\ g(T,x_T)e^{-\varphi_T} & \text{for} \quad t>T. \end{cases}$$

Recall that if $r\in\mathfrak{R}$ then $\int_0^\infty r_t dt=\infty$. Clearly, $f_t=0$ for $t>T$, and

$$\int_T^\infty g_t r_t e^{-\int_0^t r_u du}dt = g_T e^{-\int_0^T r_u du}.$$

Thus

$$\int_0^T (f_t+g_t r_t)e^{-\int_0^t r_u du}dt + g_T e^{-\int_0^T r_u} du = \int_0^\infty (f_t+g_t r_t)e^{-\int_0^t r_u du}dt$$
$$= \int_0^\infty\left(\int_0^t f_s ds + g_t\right) r_t e^{-\int_0^t r_u du}dt,$$

where the last equality comes from integrating by parts. Check that for

$$h_t := \int_0^t f_s ds + g_t,$$

for each $\alpha\in\mathfrak{A}$, $\mathbb{E}\sup_{t\geq 0}|h_t|<\infty$ holds. Indeed, if $\alpha\in\mathfrak{A}_n$, then

$$\mathbb{E}\sup_{t\geq 0}|h_t| \leq \mathbb{E}\sup_{t\in[0,T]}\left(\int_0^t |f^{\alpha_s}(s,x_s)|e^{-\varphi_s}ds + |g(t,x_t)|e^{-\varphi_t}\right)$$

$$\leq TK_n\mathbb{E}\sup_{t\in[0,T]}(1+|x_t|)^{m_n} + TK\mathbb{E}\sup_{t\in[0,T]}(1+|x_t|)^m < \infty,$$

due to estimates of moments of solutions to SDEs (Theorem 2.2.3). Since $g=g(t,x)$ is continuous, h_t is continuous. Hence by Theorem 3.1.2,

$$\sup_{r\in\mathfrak{R}} \mathbb{E} \int_0^\infty \left(\int_0^t f_s ds + g_t \right) r_t e^{-\int_0^t r_u du} dt = \sup_{\tau\in\mathfrak{T}} \mathbb{E} \int_0^\tau f_s ds + g_\tau$$

$$= \sup_{\tau\in\mathfrak{T}(T)} \mathbb{E} \int_0^\tau f_s ds + g_\tau,$$

because $f_t = 0$ and $g_t = g_T$ for $t > T$, so nothing can be gained or lost by stopping later. Hence for any $\alpha \in \mathfrak{A}$:

$$\sup_{\tau\in\mathfrak{T}(T)} \mathbb{E}\bigg\{ \int_0^\tau f^{\alpha_t}(t, x_t) e^{-\varphi_t} dt + g(\tau, x_\tau) e^{-\varphi_\tau} \bigg\}$$

$$= \sup_{r\in\mathfrak{R}} \mathbb{E}\bigg\{ \int_0^T \left[f^{\alpha_t}(t, x_t) e^{-\varphi_t} + g(t, x_t) r_t \right] e^{-\int_0^t r_u du} dt$$

$$+ g(T, x_T) e^{-\varphi_T - \int_0^T r_u du} \bigg\},$$

which proves the theorem. □

Chapter 4

Rate of convergence of finite difference approximations

In this chapter we use the normalized Bellman PDE to derive a finite difference approximation scheme for approximating optimal stopping and control problems. We prove that the rate of convergence is $\tau^{1/4} + h^{1/2}$, where τ and h are the mesh sizes in time and space respectively.

We are interested in the rate of convergence of finite difference approximations to the payoff function of optimal control problems when the reward and discounting functions may be unbounded in the control parameter. This allows us to treat numerically the optimal stopping of controlled diffusion processes by randomized stopping, i.e. by transforming the optimal stopping into a control problem. This leads us to approximating a normalized degenerate Bellman equation.

Until quite recently, there were no results on the rate of convergence of finite difference schemes for degenerate Bellman equations. A major breakthrough was achieved in Krylov [21] for Bellman equations with constant coefficients, followed by rate of convergence estimates for Bellman equations with variable coefficients in [22] and [23]. The estimate from [23] is improved in [2] and [1]. Finally, Krylov [24] (published in [25]) establishes the rate of convergence $\tau^{1/4} + h^{1/2}$ of finite difference schemes to degenerate Bellman equations with Lipschitz coefficients, where τ and h are the mesh sizes in time and space respectively.

We extend this estimate to a class of normalized degenerate Bellman equations arising in optimal stopping of controlled diffusion processes with variable coefficients. Adapting ideas and techniques of [24] we obtain the rate of convergence $\tau^{1/4} + h^{1/2}$, as in [24]. The main ingredient of the proof is a gradient estimate for the solution to the discrete normalized Bellman PDE. This is an extension of the gradient estimate from [24] to our case. After writing our account, we read [26] where an essentially more general gradient estimate is proved. This opens the way to proving the same rate of convergence result for normalized Bellman PDEs in more general setting then we present below.

Rate of converge results for optimal stopping are proved for general consistent approximation schemes in [16]. However, the rate $\tau^{1/4} + h^{1/2}$ is obtained only when the diffusion coefficients are independent of the time and space variables. For further results on numerical approximations for Bellman equations we refer to [17], [18] and [4].

This chapter is organized as follows. The main result is formulated in section 4.1. In section 4.2 the existence and uniqueness of the solution to finite difference schemes is proved together with a comparison result. The main technical result, the gradient estimate of solutions to finite difference schemes, is stated in section 4.3. Some useful analytic properties of payoff functions are presented in section 4.4. Finally, the main result is proved in section 4.7.

We recall some notation from section 2.3. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a probability space with a right-continuous filtration, such that $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets. Let $(w_t, \mathcal{F}_t)$ be a d' dimensional Wiener martingale. Let A be a separable metric space. Let $T \in [0, \infty)$. For every $t \in [0, T]$, $x \in \mathbb{R}^d$ and $\alpha \in A$ we are given a $d \times d'$ dimensional matrix $\sigma^\alpha(t, x)$, a d dimensional vector $\beta^\alpha(t, x)$ and real numbers $c^\alpha(t, x)$, $f^\alpha(t, x)$ and $g(x)$.

Recall from Chapter 3 that we can express the payoff function for the optimal stopping and control problem as an optimal control problem. For $s \in [0, T]$ and $x \in \mathbb{R}^d$ this is

$$v(s, x) = \sup_{\alpha \in \mathfrak{A}, r \in \mathfrak{R}} v^{\alpha, r}(s, x), \tag{4.0.1}$$

$$v^{\alpha, r}(s, x) = \mathbb{E}^\alpha_{s,x} \left[\int_0^{T-s} \left(f^{\alpha_t}(s + t, x_t) + r_t g(x_t)\right) e^{-\varphi_t} dt + g(x_{T-s}) e^{-\varphi_{T-s}} \right], \tag{4.0.2}$$

where

$$\varphi_t = \int_0^t r_u + c^{\alpha_u}(s + u, x_u) du,$$

and $x_t = x_t^{\alpha, s, x}$ is the solution of (2.3.2). It is this payoff function which we wish to approximate.

4.1 The main result

From [19] we know that under some assumptions (more restrictive than Assumptions 2.3.1) the payoff function $v(t,x)$ satisfies the normalized Bellman PDE

$$\sup_{\alpha\in A, r\geq 0}\left(\frac{1}{1+r}(v_t+\mathrm{L}^\alpha v+f^\alpha)+\frac{r}{1+r}(g-v)\right)=0 \quad \text{on} \quad [0,T)\times\mathbb{R}^d \qquad (4.1.1)$$
$$v(T,x)=g(x) \quad \text{for} \quad x\in\mathbb{R}^d,$$

where

$$\mathrm{L}^\alpha u=\sum_{i,j} a^\alpha_{i,j}u_{x^ix^j}+\sum_i \beta^\alpha_i u_{x^i}-c^\alpha u \qquad (4.1.2)$$

Therefore it is natural to derive the finite difference approximating scheme from this. It is worth noticing that to obtain the rate of convergence of the solutions of the finite difference scheme to the payoff function of the optimal control problem, we don't use any result about the solvability of the normalized Bellman PDE.

Assumption 4.1.1. There exist a natural number d_1, vectors $\ell_k\in\mathbb{R}^d$ and functions

$$\sigma^\alpha_k:[0,T)\times\mathbb{R}^d\to\mathbb{R},\quad b^\alpha_k:[0,T)\times\mathbb{R}^d\to\mathbb{R},\quad \forall k=\pm1,\dots,d_1$$

such that $\ell_k=-\ell_{-k}$, $|\ell_k|\leq K$, $\sigma^\alpha_k=\sigma^\alpha_{-k}, b^\alpha_k\geq 0$ for $k=\pm1,\dots,d_1$ for any $\alpha\in A$ and $(t,x)\in[0,T)\times\mathbb{R}^d$ and such that L^α given by (4.1.2) satisfies

$$\mathrm{L}^\alpha u=\sum_k \sigma^\alpha_k \mathrm{D}^2_{\ell_k} u+b^\alpha_k \mathrm{D}_{\ell_k} u-c^\alpha u, \qquad (4.1.3)$$

for any smooth u.

Fix $\tau>0$, $h>0$ and define the grid

$$\bar{\mathcal{M}}_T:=\{(t,x)\in[0,T]\times\mathbb{R}^d:(t,x)=((j\tau)\wedge T,h(i_1\ell_1+\cdots+i_{d_1}\ell_{d_1})),$$
$$j\in\{0\}\cup\mathbb{N},i_k\in\mathbb{Z},k=\pm1,\dots,\pm d_1\}.$$

Let $\tau_T(t):=\tau$ for $t\leq T-\tau$ and $\tau_T(t):=T-t$ for $t>T-\tau$. So $t+\tau_T(t)=(t+\tau)\wedge T$. Let Q be a non-empty subset of

$$\mathcal{M}_T:=\bar{\mathcal{M}}_T\cap\big([0,T)\times\mathbb{R}^d\big).$$

Let $\mathrm{T}_\tau u(t,x) = u(t+\tau_T(t),x)$, $\mathrm{T}_{h,\ell_k} u(t,x) = u(t,x+h_k\ell_k)$,

$$\begin{aligned} \delta_\tau u(t,x) &:= \frac{u(t+\tau_T(t),x)-u(t,x)}{\tau_T(t)}, \\ \delta_{h_k,\ell_k} u(t,x) &:= \frac{u(t,x+h_k\ell_k)-u(t,x)}{h_k}, \\ \Delta_{h_k,\ell_k} u &:= -\,\delta_{h_k,\ell_k}\,\delta_{h_k,-\ell_k}\,u = \frac{1}{h_k}(\delta_{h_k,\ell_k}\,u + \delta_{h_k,-\ell_k}\,u). \end{aligned} \tag{4.1.4}$$

Let $\delta_{0,\ell} := 0$. Let $a_k^\alpha := (1/2)(\sigma_k^\alpha)^2$. Consider the following finite difference problem:

$$\begin{aligned} \sup_{\alpha\in A, r\geq 0}\left(\frac{1}{1+r}(\delta_\tau u + \mathrm{L}_h^\alpha u + f^\alpha) + \frac{r}{1+r}(g-u)\right) = 0 \quad \text{on} \quad Q, \\ u = g \quad \text{on} \quad \bar{\mathcal{M}}_T \setminus Q, \end{aligned} \tag{4.1.5}$$

where

$$\mathrm{L}_h^\alpha u = \sum_k a_k^\alpha\,\Delta_{h_k,\ell_k} u + \sum_k b_k^\alpha\,\delta_{h_k,\ell_k} u - c^\alpha u.$$

Assumption 4.1.2. Let $0 \leq \lambda < \infty$. Functions σ_k^α, b_k^α, f^α, $c^\alpha \geq \lambda$, g are Borel in t and continuous in $\alpha \in A$ for each $k = \pm 1, \ldots, d_1$. For any $t \in [0,T]$, $x, y \in \mathbb{R}^d$ and $\alpha \in A$:

$$\begin{aligned} |\sigma_k^\alpha(t,x) - \sigma_k^\alpha(t,y)| + |b_k^\alpha(t,x) - b_k^\alpha(t,y)| + |c^\alpha(t,x) - c^\alpha(t,y)| \\ + |f^\alpha(t,x) - f^\alpha(t,y)| + |g(x) - g(y)| \leq K|x-y|, \\ |\sigma_k^\alpha| + b_k^\alpha + |f^\alpha| + c^\alpha + |g| \leq K. \end{aligned} \tag{4.1.6}$$

We also need the time Hölder continuity of σ^α, b^α, c^α and f^α.

Assumption 4.1.3. For all $\alpha \in A$, $x \in \mathbb{R}^d$ and $t, s \in [0,T]$

$$\begin{aligned} &|\sigma^\alpha(s,x) - \sigma^\alpha(t,x)| + |b^\alpha(s,x) - b^\alpha(t,x)| \\ &+ |c^\alpha(s,x) - c^\alpha(t,x)| + |f^\alpha(s,x) - f^\alpha(t,x)| \leq K|t-s|^{1/2}. \end{aligned}$$

The following theorem is the main result of this chapter. When reading further, the reader will notice that the results in sections 4.2 and 4.3 are for normalized Bellman PDEs slightly more general than just the one corresponding to the payoff of the optimal stopping and control problem. To obtain the rate of convergence we need to prove Hölder continuity in time for v and $v_{\tau,h}$. Example 4.5.1 demonstrates that we don't get continuity in time under the generality of sections 4.2 and 4.3. Since the case of optimal stopping and control is the most interesting from a practical point of view, we didn't attempt to find the exact conditions needed to state the result in the greatest possible generality.

Theorem 4.1.4. *Let v be the function given by* (4.0.1)-(4.0.2)*. Let Assumptions 2.3.1, 4.1.1, 4.1.2, and 4.1.3 be satisfied. Then* (4.1.5) *with $Q = \mathcal{M}_T$ has a unique bounded solution $v_{\tau,h}$ and*

$$|v - v_{\tau,h}| \le N_T(\tau^{1/4} + h^{1/2}) \quad on \quad \mathcal{M}_T, \tag{4.1.7}$$

where N_T is a constant depending on K, d, d_1, T, λ.

We briefly outline how this is proved in Section 4.7. The proof follows closely that of Theorem 2.2 in [24]. We "shake" the finite difference scheme (4.1.5), we smooth the corresponding solution to get a supersolution of the normalized Bellman PDE (4.1.1). Hence we obtain the estimate $v \le v_{\tau,h} + N(\tau^{1/4} + h^{1/2})$ by using a comparison of v with supersolutions to (4.1.1). To get $v_{\tau,h} \le v + N(\tau^{1/4} + h^{1/2})$ we use the same approach with the roles of v and $v_{\tau,h}$ interchanged. We "shake" the optimal control problem (4.0.1)-(4.0.2), smooth the resulting payoff function to obtain a supersolution of (4.1.5) and use a comparison theorem established for (4.1.5).

The reader will notice that the estimate given by Theorem 4.1.4 is also true for w given by (2.3.3). Indeed we only need to use Theorem 3.2.1 to see that $w(t, x)$ given by (2.3.3) is equal to (4.0.1)-(4.0.2).

Corollary 4.1.5. *The system* (4.1.5) *is equivalent to:*

$$\begin{aligned} &\delta_\tau u + \sup_{\alpha \in A} [\mathrm{L}_h^\alpha u + f^\alpha] \le 0, \quad g - u \le 0 \quad on \quad Q, \\ &\delta_\tau u + \sup_{\alpha \in A} [\mathrm{L}_h^\alpha u + f^\alpha] = 0, \quad g - u < 0 \quad on \quad Q \\ &\qquad and \quad u = g \quad on \quad \bar{\mathcal{M}}_T \setminus Q. \end{aligned} \tag{4.1.8}$$

Proof. Let $\varepsilon = \frac{1}{1+r}$ in (4.1.5) and take the supremum over $\varepsilon \in [0, 1]$. Hence (4.1.5) can be rewritten as

$$\sup_{\varepsilon \in [0,1]} \left[\varepsilon \sup_{\alpha \in A} (\delta_\tau u + \mathrm{L}_h^\alpha u + f^\alpha) + (1 - \varepsilon)(g - u) \right] = 0 \quad \text{on} \quad Q,$$

$$u = g \quad \text{on} \quad \bar{\mathcal{M}}_T \setminus Q.$$

Notice that the supremum over ε is achieved by either taking $\varepsilon = 0$ or $\varepsilon = 1$. Hence it can be seen that this is equivalent to (4.1.8).

□

4.2 On the finite difference scheme

In this section we will consider the following finite difference problem:

$$\sup_{\alpha\in A} m^\alpha \left(\delta_\tau u + \mathrm{L}^\alpha_h u + f^\alpha\right) = 0 \quad \text{on} \quad Q, \tag{4.2.1}$$

$$u = g \quad \text{on} \quad \bar{\mathcal{M}}_T \setminus Q, \tag{4.2.2}$$

where m^α is a positive function of $\alpha \in A$ taking values in $(0,1]$, such that the following conditions hold.

Assumption 4.2.1. Let $0 \le \lambda < \infty$. Functions σ^α_k, b^α_k, f^α, $c^\alpha \ge \lambda$, g are Borel in t and continuous in $\alpha \in A$ for each $k = \pm 1, \ldots, d_1$. For any $t \in [0,T]$, $x, y \in \mathbb{R}^d$ and $\alpha \in A$:

$$\begin{aligned} |\sigma^\alpha_k(t,x) - \sigma^\alpha_k(t,y)| + |b^\alpha_k(t,x) - b^\alpha_k(t,y)| + |c^\alpha(t,x) - c^\alpha(t,y)| \\ + m^\alpha |f^\alpha(t,x) - f^\alpha(t,y)| + |g(x) - g(y)| \le K|x-y|, \\ |\sigma^\alpha_k| + b^\alpha_k + m^\alpha|f^\alpha| + m^\alpha c^\alpha + |g| \le K \end{aligned} \tag{4.2.3}$$

and

$$\frac{1}{K} \le m^\alpha \left(1 + c^\alpha(t,x) - \lambda\right). \tag{4.2.4}$$

The reader will notice that (4.1.5) is a special case of (4.2.1)-(4.2.2). This is obvious if we take $A \times [0,\infty)$ instead of A, (α, r) instead of α and $1/(1+r)$, $f^\alpha + rg$ and $c^\alpha + r$ instead of m^α, f^α and c^α respectively. Now we present two simple examples which justify the condition (4.2.4).

Example 4.2.2. Consider $A = [0,\infty)$, $m^\alpha = (1+\alpha)^{-1}$ and the equation

$$\sup_{\alpha\in A} m^\alpha \left(\delta_\tau u\right) = 0 \quad \text{on} \quad \mathcal{M}_T$$

with the terminal condition $u = 1$ on $\bar{\mathcal{M}}_T \setminus \mathcal{M}_T$. If $u : \mathcal{M}_T \to \mathbb{R}$ is any non-increasing function in t, then $m^\alpha\, \delta_\tau u \le 0$. Hence, letting $\alpha \to \infty$, we see that u satisfies the equation. Consequently the solution to the above problem is not unique.

Let $f^\alpha = 1 + \alpha$. Consider now the equation

$$\sup_{\alpha\in A} m^\alpha(\delta_\tau u + f^\alpha) = \sup_{\alpha\in A} m^\alpha\, \delta_\tau u + 1 = 0 \quad \text{on} \quad \mathcal{M}_T.$$

If u is a solution then we have $m^\alpha \delta_\tau u \le 0$. Hence $\sup_{\alpha\in A} m^\alpha \delta_\tau u = 0$, which contradicts the equation. Thus the above equation has no solution.

Let Assumption 4.2.1 be satisfied throughout the remainder of this section.

Lemma 4.2.3. *There is a unique bounded solution of the finite difference problem* (4.2.1)-(4.2.2) *on* Q.

Proof. Let $\gamma = (0,1)$ and define ξ recursively as follows: $\xi(T) = 1$, $\xi(t) = \gamma^{-1}\xi(t+\tau_T(t))$ for $t < T$. Then for any function v

$$\delta_\tau(\xi v) = \gamma\xi\,\delta_\tau v - \nu\xi v, \quad \text{where} \quad \nu = \frac{1-\gamma}{\tau}.$$

To solve (4.2.1)-(4.2.2) for u, we could equivalently solve the following for v, with $u = \xi v$:

$$v = H[v] := H[(f^\alpha), g, v] := \mathbf{1}_{\bar{\mathcal{M}}_T\setminus Q}\frac{1}{\xi}g + \mathbf{1}_Q G[v], \tag{4.2.5}$$

where for any $\varepsilon > 0$,

$$G[v] := v + \varepsilon\xi^{-1}\sup_\alpha m^\alpha\left(\delta_\tau u + \mathrm{L}_h^\alpha u + f^\alpha\right). \tag{4.2.6}$$

Then, using the convention that repeated indices indicate summation and always summing up before taking the supremum,

$$G[v] = \sup_\alpha\left[p_\tau^\alpha\,\mathrm{T}_\tau v + p_k^\alpha\,\mathrm{T}_{h,l_k} v + p^\alpha v + \varepsilon m^\alpha(\mathrm{T}_\tau\xi^{-1})f^\alpha\right],$$

with

$$p_\tau^\alpha = \varepsilon\gamma\tau^{-1}m^\alpha \ge 0, \quad p_k^\alpha = \varepsilon(2h^{-2}a_k^\alpha + h^{-1}b_k^\alpha)m^\alpha \ge 0,$$

$$p^\alpha = 1 - p_\tau^\alpha - \sum_k p_k^\alpha - \varepsilon\nu m^\alpha - \varepsilon m^\alpha c^\alpha.$$

Notice that

$$p_k^\alpha \le K\varepsilon\left(\frac{2K}{h^2} + \frac{K}{h}\right)$$

and

$$\varepsilon\nu m^\alpha + \varepsilon m^\alpha c^\alpha \le \varepsilon\tau^{-1}K + \varepsilon K, \quad p_\tau \le \varepsilon\tau^{-1}K,$$

so for all ε smaller than some ε_0, $p^\alpha \geq 0$. Also

$$0 \leq \sum_k p_k^\alpha + p^\alpha + p_\tau = 1 - \varepsilon m^\alpha(\nu + c^\alpha) \leq 1 - (1 \vee \nu)\varepsilon m^\alpha(1 + c^\alpha)$$

$$\leq 1 - (1 \vee \nu)\varepsilon K^{-1} =: \delta < 1,$$

for some sufficiently small $\varepsilon > 0$. Since the difference of supremums is less than the supremum of a difference

$$|H[v](t,x) - H[w](t,x)| \leq \delta \sup_{\bar{\mathcal{M}}_T} |v - w|.$$

Thus the operator H is a contraction on the space of bounded functions on $\bar{\mathcal{M}}_T$. By Banach's fixed point theorem (4.2.5) has a unique bounded solution. □

Remark 4.2.4. Let v be a function defined on $\bar{\mathcal{M}}_T$. The operator H defined by (4.2.5) has the following property: if there exists $R > 0$ such that $v(t,x) = f^\alpha(t,x) = g(x) = 0$ for all $\alpha \in A$, $t \in [0,T]$ and $|x| > R$ then there exists R' such that if $|x| > R'$ then for all $t \in [0,T]$,

$$H[(f^\alpha), g, v](t,x) = 0.$$

Corollary 4.2.5. *Let u be the solution of* (4.2.1)-(4.2.2) *on $Q = \mathcal{M}_T$. Assume there exists R such that for all $\alpha \in A$ and $t \in [0,T]$,*

$$f^\alpha(t,x) = g(x) = 0 \quad \textit{for} \quad |x| \geq R.$$

Then

$$\lim_{|x|\to\infty} \sup_{[0,T]} |u(t,x)| = 0.$$

Proof. Let ξ be defined as in the proof of Lemma 4.2.3 and let $v = \xi u$. From the proof of Lemma 4.2.3 we see that H is a contraction on the space of bounded functions on $\bar{\mathcal{M}}_T$. Hence for any $\varepsilon > 0$ there is n_0 such that

$$\sup_{\bar{\mathcal{M}}_T} |H^n[0] - v| < \varepsilon, \quad \text{for} \quad n > n_0.$$

By Remark 4.2.4 there exist R_ε such that $H^{n_0}[0](t,x) = 0$ for all $t \in [0,T]$ and $|x| > R_\varepsilon$. Hence

$$\sup_{(t,x)\in\bar{\mathcal{M}}_T, |x|>R_\varepsilon} |v(t,x)| < \varepsilon.$$

□

Lemma 4.2.6. *Consider f_1^α, f_2^α defined on $A \times \mathcal{M}_T$ and assume that in Q*

$$\sup_\alpha m^\alpha f_2^\alpha < \infty, \quad f_1^\alpha \le f_2^\alpha.$$

Let there be functions u_1, u_2 defined on $\bar{\mathcal{M}}_T$ and a constant $\mu \ge 0$ such that $u_1 e^{-\mu|x|}$, $u_2 e^{-\mu|x|}$ are bounded and for some $C \ge 0$

$$\begin{aligned} &\sup_\alpha m^\alpha \left(\delta_\tau u_1 + \mathrm{L}_h^\alpha u_1 + f_1^\alpha + C\right) \\ &\qquad \ge \sup_\alpha m^\alpha \left(\delta_\tau u_2 + \mathrm{L}_h^\alpha u_2 + f_2^\alpha\right) \quad on \quad Q \end{aligned} \tag{4.2.7}$$

and $u_1 \le u_2$ on $\bar{\mathcal{M}}_T \setminus Q$. Then there exists a constant τ^* depending only on K, d_1, μ such for $\tau \in (0, \tau^*)$

$$u_1 \le u_2 + TC \quad on \quad \bar{\mathcal{M}}_T. \tag{4.2.8}$$

If u_1, u_2 are bounded on $\mathcal{M}_T$ then (4.2.8) *holds for any τ.*

To prove this lemma we use the following observation from [24].

Remark 4.2.7. Let $\mathrm{D}_\ell u := u_{x^i}\ell^i$, $\mathrm{D}_\ell^2 u := u_{x^i x^j}\ell^i \ell^j$, $a_k^\alpha = \frac{1}{2}(\sigma_k^\alpha)^2$. Let D_x^n denote the collection of all n-th order derivatives in x. Consider

$$\mathrm{L}^\alpha(t,x)u(t,x) = \sum_k a_k^\alpha(t,x)\,\mathrm{D}_{\ell_k}^2 u(t,x) + b_k^\alpha(t,x)\,\mathrm{D}_{\ell_k} u(t,x).$$

For any sufficiently smooth function $\eta(x)$, by Taylor's theorem,

$$\eta(x+h\ell) - \eta(x) = \mathrm{D}_\ell\,\eta(x)h + \mathrm{D}_\ell^2\,\eta(x)\frac{h^2}{2!} + \ldots + \mathrm{D}_\ell^n\,\eta(x)\frac{h^n}{n!} + R_n(x),$$

with

$$R_n(x) = \int_0^h \mathrm{D}_\ell^{n+1}\,\eta(x+s\ell)\frac{(h-s)^n}{n!}ds.$$

Thus

$$\delta_{h,\ell}\,\eta(x) = \mathrm{D}_\ell\,\eta(x) + \frac{1}{2h}\int_0^h \mathrm{D}_\ell^2\,\eta(x+s\ell)(h-s)ds$$

also, since $\mathrm{D}_{-\ell} = \frac{\partial}{\partial x^i}(-\ell^i) = -\mathrm{D}_\ell$ and $\mathrm{D}_{-\ell}^2 = \mathrm{D}_\ell^2$,

$$\Delta_{h,\ell}\,\eta(x) = \mathrm{D}_\ell^2\,\eta(x) + \frac{1}{6h^2}\int_{-h}^h \mathrm{D}_\ell^4(x+s\ell)(h-|s|)^3 ds.$$

Now notice that

$$|\,\mathrm{D}^4_{\ell_k}\,\eta| \le N(d,K)|\,\mathrm{D}^4_x\,\eta| \quad \text{and} \quad |\,\mathrm{D}^2_{\ell_k}\,\eta| \le N(d,K)|\,\mathrm{D}^2_x\,\eta|,$$

$$\int_0^h \mathrm{D}^2_{\ell_k}\,\eta(x+s\ell_k)(h-s)ds \le \sup_{B_K(x)} |\,\mathrm{D}^2_x\,\eta|\frac{h^2}{2}$$

and

$$\int_{-h}^h \mathrm{D}^4_{\ell_k}\,\eta(x+s\ell_k)(h-|s|)^3 ds \le \sup_{B_K(x)} |\,\mathrm{D}^4_x\,\eta|\frac{h^4}{2}.$$

Hence

$$\begin{aligned}
&|\,\mathrm{L}^\alpha\,\eta(x) - \mathrm{L}^\alpha_h\,\eta(x)| \\
&= \left| a^\alpha_k \frac{1}{6h^2}\int_{-h}^h \mathrm{D}^4_{\ell_k}(x+s\ell)(h-|s|)^3 ds + b^\alpha_k \frac{1}{2h}\int_0^h \mathrm{D}^2_{\ell_k}\,\eta(x+s\ell)(h-s)ds \right| \\
&\le N(h^2 \sup_{B_K(x)} |D^4_x \eta| + h \sup_{B_K(x)} |D^2_x \eta|).
\end{aligned}$$

Proof of Lemma 4.2.6. Let $w = u_1 - u_2 - C(T-t)$. From (4.2.7)

$$\sup_\alpha m^\alpha\,(\delta_\tau\,w + \mathrm{L}^\alpha_h\,w) \ge 0$$

hence for any $\varepsilon > 0$

$$w + \varepsilon \sup_\alpha m^\alpha\,(\delta_\tau\,w + \mathrm{L}^\alpha_h\,w) \ge w.$$

In the proof of Lemma 4.2.3, choose ε such that p^α, p^α_k are nonnegative. So G, defined in (4.2.6), is a monotone operator. Hence for any $\psi \ge w$,

$$\psi + \varepsilon \sup_\alpha m^\alpha\,(\delta_\tau\,\psi + \mathrm{L}^\alpha_h\,\psi) \ge w. \tag{4.2.9}$$

Let $\gamma \in (0,1)$. Use ξ from the proof of Lemma 4.2.3. Fix $\gamma \in (0,1)$ later. Then

$$\delta_\tau\,\xi = \xi\frac{1}{\tau}(\gamma - 1).$$

Let

$$N_0 = \sup_{\bar{\mathcal{M}}_T} \frac{w_+}{\zeta} < \infty \quad \text{and} \quad \psi := N_0\zeta \ge \zeta\frac{w_+}{\zeta} \ge w,$$

where $\eta(x) = \cosh(\mu|x|)$ and $\zeta = \eta\xi$. Then by Remark 4.2.7 and since for $r \ge 0$

$$\frac{\cosh(\mu r)}{r^2} - \frac{\sinh(\mu r)}{r^3} \le \cosh(\mu r)$$

we get

$$m^\alpha \mathrm{L}_h^\alpha \eta(x) \le m^\alpha L^\alpha \eta(x) + N_1(h^2+h)\cosh(\mu|x|+\mu K) \le N_2 \cosh(\mu|x|+\mu K),$$

with N_1, N_2 depending only on μ, d_1, K. Hence

$$m^\alpha \left(\delta_\tau \zeta + \mathrm{L}_h^\alpha \zeta\right) \le \zeta[\tau^{-1}(\gamma-1)+N_3], \quad N_3 := N_2 \frac{\cosh(\mu|x|+\mu K)}{\cosh(\mu|x|)} < \infty.$$

Then

$$\begin{aligned} w &\le \psi + \varepsilon \sup_\alpha m^\alpha(\delta_\tau \psi + \mathrm{L}_h^\alpha \psi) \\ &\le N_0 \zeta \left[1+\varepsilon(\tau^{-1}(\gamma-1)+N_3)\right]. \end{aligned}$$

Define $\kappa(\gamma) = \tau^{-1}(\gamma-1)+N_3$. Then for $\tau < \tau^* := N_3^{-1}$ one has $\kappa(0) < 0$ and $\kappa(1) > 0$. So one can have γ such that $\kappa < 0$ and $1+\varepsilon\kappa > 0$. By (4.2.9)

$$w \le \psi + \varepsilon \sup_\alpha m^\alpha \left(\delta_\tau \psi + \mathrm{L}_h^\alpha \psi\right) \le N_0 \xi \left(1+\varepsilon\kappa\right) \quad \text{on} \quad Q. \tag{4.2.10}$$

If $w \ge 0$ then $w_+ = w$. Hence

$$N_0 = \sup_{\bar{\mathcal{M}}_T} \frac{w_+}{\xi} \le \sup_{\bar{\mathcal{M}}_T} \frac{N_0 \xi(1+\varepsilon\kappa)}{\xi}. \tag{4.2.11}$$

By the hypothesis $w \le 0$ on $\bar{\mathcal{M}}_T \setminus Q$. But $\kappa < 0$, hence N_0 must be 0 and so $w \le 0$ on $\bar{\mathcal{M}}_T$. Finally observe that if $\mu = 0$, $N_2 = N_3 = 0$. □

Corollary 4.2.8. *If v is the solution of* (4.2.1)-(4.2.2) *then*

$$v \le N_T + \sup_x |g(x)|.$$

Proof. Apply Lemma 4.2.6 to v and

$$\xi(t) = K^2\left[(T-t)+1\right] + N_1,$$

where $N_1 := \sup_x |g(x)|$. Then on Q

$$\begin{aligned} &\sup_\alpha m^\alpha(\delta_\tau \xi + \mathrm{L}_h^\alpha \xi + f^\alpha) \\ = &\sup_\alpha \left[-m^\alpha(K^2 + c^\alpha(K^2\left[(T-t)+1\right]+N_1)\right] \quad \le \quad 0, \end{aligned}$$

while on $\bar{\mathcal{M}}_T \setminus Q$

$$\xi(T) \geq \sup_x |g(x)| \geq g(x).$$

□

Corollary 4.2.9. *Assume there exists R such that*

$$f^\alpha(t,x) = g(x) = 0 \quad \textit{for} \quad |x| \geq R.$$

Then for some $\gamma \in (0,1)$ and N_T depending only on K, d, T

$$|v_{\tau,h}(t,x)| \leq N_T e^{\gamma R} \exp(-\gamma |x|)$$

and hence

$$\lim_{|x|\to\infty} \sup_{[0,T]} |v_{\tau,h}(t,x)| = 0.$$

Proof. Let $\gamma \in (0,1)$ be some constant to be chosen later. Let $\xi(t) = \xi(t,x)$ be defined recursively as follows:

$$\xi(T) = 1; \quad \xi(t) = \gamma^{-1}\xi(t + \tau(t)) \quad \text{for} \quad t < T.$$

Take an arbitrary unit $l \in \mathbb{R}^d$. Define $\zeta = \xi\eta$, where

$$\eta(x) = e^{\gamma(x,l)}.$$

We are going to use Lemma 4.2.6. Observe that by Taylor's Theorem

$$\begin{aligned}
\Delta_{h_k,\ell_k} \eta(y) &= \mathrm{D}^2_{\ell_k} \eta(y) + \frac{1}{6h^2} \int_{-h}^{h} \mathrm{D}^4_{\ell_k} \eta(y + s l_k)(h - |s|)^3 ds \\
&\leq \mathrm{D}^2_{\ell_k} \eta(y) + \frac{1}{6h^2} \sup_{s \in (-h,h)} |\mathrm{D}^4_{\ell_k} \eta(y + s l_k)| \int_{-h}^{h} (h - |s|^3) ds \\
&= \mathrm{D}^2_{\ell_k} \eta(y) + \frac{h^2}{12} \sup_{s \in (-h,h)} |\mathrm{D}^4_{\ell_k} \eta(y + s l_k)|.
\end{aligned}$$

Also, recalling that $|l_k| < K$, $\gamma \in (0,1)$, for some N depending only on K, d

$$\mathrm{D}^2_{\ell_k} \eta = \eta\gamma^2 (l, l_k)^2 \leq N\gamma\eta.$$

Similarly

$$\mathrm{D}^4_{\ell_k}\,\eta \le N\gamma\eta.$$

Hence

$$\sup_{s\in(-h,h)} |\,\mathrm{D}^4_{\ell_k}\,\eta(y+sl_k)| \le N\gamma \sup_{s\in(-h,h)} |\exp(y,l)\exp(sl_k,l_k)| \le N\gamma\eta,$$

provided $h < K$. Thus

$$\Delta_{h_k,\ell_k}\,\eta \le N\gamma\eta$$

and due to similar considerations

$$\delta_{h_k,\ell_k}\,\eta \le N\gamma\eta.$$

Using

$$\delta_\tau(\xi\eta) = \tau^{-1}(\gamma-1)\xi\eta$$

we see that for sufficiently small γ

$$m^\alpha(\delta_\tau\,\zeta + \mathrm{L}^\alpha_h\,\zeta) \le (\tau^{-1}(\gamma-1)+N\gamma)\zeta \le 0 \quad \text{on} \quad H_T.$$

Let $Q = \{(t,x) \in \mathcal{M}_T : (x,l) \le -R\}$. Since $-R \ge (x,l) = |x|\cos\theta \ge -|x|$, $Q \subset \{(t,x) \in \mathcal{M}_T : |x| \ge R\}$. Recall that $f^\alpha(t,x) = 0$ for $|x| \ge R$. Hence for any constant $N > 0$,

$$m^\alpha(\delta_\tau\,N\zeta + \mathrm{L}^\alpha_h\,N\zeta + f^\alpha) \le 0 \quad \text{on} \quad Q.$$

For $(t,x) \in \bar{\mathcal{M}}_T \setminus Q$, either $t < T$ and $(x,l) > -R$ or $t = T$ and $(x,l) \ge -|x|$. Hence either

$$\zeta(t,x) \ge e^{-\gamma R} \quad \text{or} \quad \zeta(t,x) \ge e^{-\gamma|x|}.$$

In the first case, we know from Corollary 4.2.8 that $v_{\tau,h}$ is bounded by a constant depending on K, T, d, d_1 only. So for large N, $e^{\gamma R}N\zeta \ge v_{\tau,h}$. In the second case, $t = T$ and so $v_{\tau,h} = g$. As $g(x) = 0$ for $x \ge R$ we only need to consider $|x| < R$ and so for large N, $e^{\gamma R}N\zeta \ge g = v_{\tau,h}$. Either way, for large N (depending on K, T, d, d_1), $Ne^{\gamma R}\zeta \ge v_{\tau,h}$ on $\bar{\mathcal{M}}_T \setminus Q$. By Lemma 4.2.6 $v_{\tau,h} \le Ne^{\gamma R}\zeta$ in $\mathcal{M}_T$. Since the choice of the unit vector l was arbitrary we can see that in $\mathcal{M}_T$

$$v_{\tau,h} \le N_T e^{\gamma R}\exp(-\gamma|x|).$$

Analogous application of Lemma 4.2.6 would yield that in $\mathcal{M}_T$

$$v_{\tau,h} \geq -N_T e^{\gamma R} \exp(-\gamma|x|).$$

□

Finally we can state a lemma identical (with an identical proof) to Lemma 3.8 of [24].

Lemma 4.2.10. *Let v be the solution of* (4.2.1)-(4.2.2)*, functions f_n^α and g_n, $n = 1, 2, \dots$ satisfy the same conditions as f^α, g with the same constants and let v_n be the solutions of* (4.2.1)-(4.2.2)*, where f_n^α, g_n take the place of f^α, g respectively. Assume that on $[0,T] \times \mathbb{R}^d$*

$$\lim_{n\to\infty} \sup_\alpha (|f^\alpha - f_n^\alpha| + |g - g_n|) = 0.$$

Then, as $n \to \infty$, $v_n \to v$ on $\bar{\mathcal{M}}_T$.

4.3 Gradient estimate for the finite difference problem

We present a simple example of a gradient estimate for a solution to a simple parabolic PDE. We have learned this from [20]. This could help one to understand the method which is used to obtain the gradient estimate for solutions to (4.2.1)-(4.2.2).

Example 4.3.1. Consider $D \subset \mathbb{R}$ open and bounded. Let $\bar{D}$ denote the closure of D and $\partial D = \bar{D} \setminus D$. Let $G = [0,T) \times D$ and let a, f and u be a functions defined on $\bar{G} := [0,T] \times \bar{D}$ such that

$$u_t + a u_{xx} + f = 0, \quad \text{on} \quad G.$$

We wish to find an upper bound for $|u_x|$ on $\mathcal{G}$. We assume $0 < \delta \leq a$, $|a| \leq K$, $|a_x| \leq K$, $|f| \leq K$ and $|f_x| \leq K$ on g. Let $\mu \in \mathbb{R}$ and $v(t,x) = u(t,x)e^{-\mu t}$. Then

$$0 = v_t + a v_{xx} + \mu v + e^{-\mu t} f, \quad \text{on} \quad G. \tag{4.3.1}$$

Let $V = v^2 + (v_x)^2$. Let $(t_0, x_0) \in \bar{G}$ be the point where V attains it's maximum. If $(t_0, x_0) \in \partial G := \bar{G} \setminus G$, then

$$|v_x| \leq \sup_{\partial G} |v| + \sup_{\partial G} |v_x|.$$

Assume that $(t_0, x_0) \in G$. Assume that the maximum principle is satisfied i.e. at (t_0, x_0)

$$0 \geq V_t + aV_{xx} = 2v_t v + 2v_x \frac{\partial}{\partial t} v_x + 2a((v_x)^2 + vv_{xx} + (v_{xx})^2 + v_x v_{xxx}). \tag{4.3.2}$$

Differentiating (4.3.1) with respect to x and multiplying by v_x gives

$$v_x \frac{\partial}{\partial t} v_x + v_x(av_{xxx} + v_{xx}a_x) + v_x \mu v_x + v_x e^{-\mu t} f_x = 0.$$

Hence (4.3.2) becomes

$$0 \geq 2v_t v - 2(v_x v_{xx} a_x + (v_x)^2 \mu + v_x e^{-\mu t} f_x) + 2a(v_x)^2 + 2avv_{xx} + 2a(v_{xx})^2,$$

at (t_0, x_0). Using (4.3.1) we obtain

$$(v_x)^2(2a - 2\mu) \leq 2v(\mu v + e^{-\mu t} f) - 2a(v_{xx})^2 + 2v_x v_{xx} a_x + 2v_x e^{-\mu t} f_x.$$

Let $M := \sup_{\bar{G}} |v|$. Notice $v_x v_{xx} a_x - a(v_x)^2 \leq (v_x)^2 (4\delta)^{-1} K^2$ and $v_x f_x \leq |v_x| K$. Hence

$$(v_x)^2 \left(2a - 2\mu - \frac{K^2}{4\delta} \right) \leq 2\mu M^2 + e^{-\mu t} MK + 2|v_x| e^{-\mu t} K.$$

Assume $|v_x| \geq 1$ and divide the above inequality by $|v_x|$. Then choose $\mu \leq -(1/2)(1 + K^2(4\delta)^{-1})$ to obtain

$$|v_x| \leq KMe^{-\mu t} + 2Ke^{-\mu t}.$$

Hence we can conclude that on $\bar{G}$

$$|u_x| \leq \sup_{\partial G} |u_x| + K(2 + e^{-\mu T} \sup_{G} |u|),$$

where $\mu < 0$ depends on K and δ.

The following lemma states some properties of the operators $\delta_{h,l}$ and $\Delta_{h,l}$, which will be used in this section.

Lemma 4.3.2. *Let u and v be functions on $\mathbb{R}^d$, l and x_0 vectors in $\mathbb{R}^d$ and $h > 0$. Then*

$$\delta_{h,l}(uv) = (\delta_{h,l}\, u)v + (\mathrm{T}_{h,l}\, u)\, \delta_{h,l}\, v = v\, \delta_{h,l}\, u + u\, \delta_{h,l}\, v + h(\delta_{h,l}\, u)(\delta_{h,l}\, v), \tag{4.3.3}$$

which can be thought of as the discrete Leibnitz rule. Also

$$\begin{aligned}\Delta_{h,l}(uv) &= v\,\Delta_{h,l}\,u + (\delta_{h,l}\,u)\,\delta_{h,l}\,v + (\delta_{h,-l}\,u)\,\delta_{h,-l}\,v + u\,\Delta_{h,l}\,v,\\ \Delta_{h,l}(u^2) &= 2u\,\Delta_{h,l}\,u + (\delta_{h,l}\,u)^2 + (\delta_{h,-l}\,u)^2.\end{aligned} \tag{4.3.4}$$

Furthermore if $v(x_0) \leq 0$ then at x_0 it holds that

$$-\,\delta_{h,l}\,v \leq \delta_{h,l}(v_-), \quad -\,\Delta_{h,l}\,v \leq \Delta_{h,l}(v_-), \tag{4.3.5}$$

$$-\,\delta_{h,l}(u_-) \leq [\delta_{h,\ell}((u+v)_-)]_- + [\delta_{h,\ell}(v_-)]_+\,. \tag{4.3.6}$$

$$\begin{aligned}(\Delta_{h,\ell}\,u)_- \leq [\delta_{h,-\ell}((\delta_{h,\ell}\,u + v)_-)]_- &+ [\delta_{h,\ell}((\delta_{h,-\ell}\,u + w)_-)]_-\\ &+ [\delta_{h,-\ell}(v_-)]_+ + [\delta_{h,\ell}(w_-)]_+.\end{aligned} \tag{4.3.7}$$

$$|\,\Delta_{h,l}\,u| \leq |\,\delta_{h,-l}((\delta_{h,l}\,u)_-)| + |\,\delta_{h,l}((\delta_{h,-l}\,u)_-)|, \tag{4.3.8}$$

$$|\,\Delta_{h,l}\,u| \leq |\,\delta_{h,-l}((\delta_{h,l}\,u)_+)| + |\,\delta_{h,l}((\delta_{h,-l}\,u)_+)|, \tag{4.3.9}$$

Proof. We get (4.3.3) with a simple calculation:

$$\begin{aligned}\delta_{h,l}(uv) &= h^{-1}\left((\mathrm{T}_{h,l}\,u)(\mathrm{T}_{h,l}\,v) - uv\right)\\ &= h^{-1}\left((\mathrm{T}_{h,l}\,u)(\mathrm{T}_{h,l}\,v) - v(\mathrm{T}_{h,l}\,u) + v(\mathrm{T}_{h,l}\,u) - uv\right)\\ &= (\mathrm{T}_{h,l}\,u)\,\delta_{h,l}\,v + v\,\delta_{h,l}\,u\\ &= v\,\delta_{h,l}\,u + u\,\delta_{h,l}\,v + (\mathrm{T}_{h,l}\,u)\,\delta_{h,l}\,v - u\,\delta_{h,l}\,v\\ &= v\,\delta_{h,l}\,u + u\,\delta_{h,l}\,v + h(\delta_{h,l}\,u)(\delta_{h,l}\,v).\end{aligned}$$

Another simple calculation yields 4.3.4:

$$\begin{aligned}\Delta_{h,l}(uv) &= -v\,\delta_{h,l}\,\delta_{h,-l}\,u - (\mathrm{T}_{h,l}\,\delta_{h,-l}\,u)\,\delta_{h,l}\,v - (\delta_{h,-l}\,v)(\delta_{h,l}\,\mathrm{T}_{h,-l}\,u)\\ &\quad - (\mathrm{T}_{h,l}\,\mathrm{T}_{h,-l}\,u)\,\delta_{h,l}\,\delta_{h,-l}\,v\\ &= v\,\Delta_{h,l}\,u + (\delta_{h,l}\,u)(\delta_{h,l}\,v) + (\delta_{h,-l}\,u)(\delta_{h,-l}\,v) + u\,\Delta_{h,l}\,v.\end{aligned}$$

Note that for any α, $-\alpha \leq \alpha_-$ and if $v \leq 0$ then $v = -v_-$. Then at x_0

$$-(\mathrm{T}_{h,l}\, v - v) \leq (\mathrm{T}_{h,}\, v)_- + v- = (\mathrm{T}_{h,}\, v)_- - v_-$$

and the first inequality of (4.3.5) follows and we can use it to obtain the second inequality in (4.3.5) by observing that

$$-\Delta_{h,l}\, v = -h^{-1}(\delta_{h,l}\, v + \delta_{h,-l}\, v) \leq h^{-1}(\delta_{h,l}\, v_- + \delta_{h,-l}\, v_-).$$

To prove (4.3.6) we first notice that for any real numbers α and β, $(\alpha+\beta)_- \leq \alpha_- + \beta_-$ and hence

$$-u_- \leq -(u+v)_- + v_-, \quad -\mathrm{T}_{h,l}\, u_- \leq -\mathrm{T}_{h,\ell}(u+v)_- + \mathrm{T}_{h,\ell}\, v_-.$$

If $u(x_0) \geq 0$ then the left hand side of (4.3.6) is less than or equal to zero. Hence we can assume that $u(x_0) < 0$. Then (recall that we're assuming that $v(x_0) \leq 0$), at x_0

$$u_- = (u+v)_- - v_-.$$

Hence

$$-\mathrm{T}_{h,\ell}\, u_- + u_- \leq -\mathrm{T}_{h,\ell}(u+v)_- + -(u+v)_- + \mathrm{T}_{h,\ell}\, v_- - v_-.$$

Dividing this by h the inequality (4.3.6) follows. To prove (4.3.7) we may assume that

$$\Delta_{h,\ell}\, u(x_0) \leq 0.$$

Then at x_0,

$$(\Delta_{h,\ell}\, u)_- = \delta_{h,-\ell}\, \delta_{h,\ell}\, u = \delta_{h,-\ell}((\delta_{h,\ell}\, u)_+) - \delta_{h,-\ell}((\delta_{h,\ell}\, u)_-).$$

Noticing that for any function $h(x)$, $[\mathrm{T}_{h,\ell}\, h(x)]_+ = \mathrm{T}_{h,\ell}[h(x)]_+$, we get

$$\begin{aligned}\delta_{h,-\ell}((\delta_{h,\ell}\, u)_+) &= \delta_{h,-\ell}((-\mathrm{T}_{h,\ell}\, \delta_{h,-\ell}\, u)_+) = \delta_{h,-\ell}(\mathrm{T}_{h,\ell}(-\delta_{h,-\ell}\, u)_+) \\ &= \mathrm{T}_{h,\ell}\, \delta_{h,-\ell}((-\delta_{h,-\ell}\, u)_+) = -\delta_{h,\ell}((-\delta_{h,-\ell}\, u)_+) \\ &= -\delta_{h,\ell}((\delta_{h,-\ell}\, u)_-).\end{aligned}$$

Thus

$$(\Delta_{h,\ell}\, u)_- = \delta_{h,\ell}(-(\delta_{h,-\ell}\, u)_-) + \delta_{h,-\ell}(-(\delta_{h,\ell}\, u)_-). \tag{4.3.10}$$

We now first apply (4.3.6) with $-(\delta_{h,-\ell}\, u)_-$ playing the role of u in (4.3.6) and then with $-(\delta_{h,\ell}\, u)_-$ playing the role of u in (4.3.6). Substituting this in (4.3.10), (4.3.7) follows.

To prove (4.3.8), notice that if $0 \geq \Delta_{h,\ell}\, u$ then it follows immediately from (4.3.7) with $u = w = 0$, while if $0 \leq \Delta_{h,\ell}\, u = (\delta_{h,\ell} - u)_-$ then it follows again from (4.3.7) applied to $-u$ again with $v = w = 0$. Finally, (4.3.9) follows from (4.3.8) when applied to $-u$. □

Let T' be the smallest integer multiple of τ which is greater than or equal to T. Choose an arbitrary $\varepsilon \in (0, Kh]$ and $l \in R^d$. Let $h_r = h$ for $r = \pm 1, \ldots, \pm d_1$ and $h_r = \varepsilon$ for $r = \pm(d_1 + 1)$. Let

$$\bar{\mathcal{M}}_T(\varepsilon) := \{(t, x + i\varepsilon l) : (t, x) \in \bar{\mathcal{M}}_T, i = 0, \pm 1, \ldots\}.$$

For a finite $Q \subset \bar{\mathcal{M}}_T(\varepsilon)$ let

$$Q^0_\varepsilon := \{(t, x) : (t + \tau_T(t), x) \in Q, (t, x + h_r l_r) \in Q, \forall r = \pm 1, \ldots, \pm(d_1 + 1)\}$$

and $\partial_\varepsilon Q := Q \setminus Q^0_\varepsilon$. Recall that $a^\alpha_k = (1/2)(\sigma^\alpha_k)^2$.

Assumption 4.3.3. For any $(t, x) \in Q^0_\varepsilon$ and $r = \pm 1, \ldots, \pm(d_1 + 1)$,

$$|\, \delta_{h_r,l_r}\, b^\alpha_k| \leq K, \quad m^\alpha |\, \delta_{h_r,l_r}\, f^\alpha| \leq K, \quad |\, \delta_{h_r,l_r}\, c^\alpha| \leq K, \tag{4.3.11}$$

$$|\, \delta_{h_r,l_r}\, a^\alpha_k| \leq K\sqrt{a^\alpha_k} + Kh. \tag{4.3.12}$$

Theorem 4.3.4. *Let Assumption 4.2.1 hold with some $\lambda \geq 0$. Let u be a function defined on $\bar{\mathcal{M}}_T(\varepsilon)$ satisfying (4.2.1) on Q and Assumption 4.3.3 be satisfied. Then there is a constant $N_0 > 1$, such that, for any $c_0 \geq 0$ satisfying*

$$\lambda + \frac{1}{\tau}(1 - e^{-c_0\tau}) > N_0, \tag{4.3.13}$$

then on Q,

$$|\delta_{\varepsilon,\pm l} u| \leq K N_0 e^{c_o T'}\big(1 + |u|_{0,Q} + \sup_{k,\partial_\varepsilon Q}\, (|\, \delta_{h_k,\ell_k}\, u| + |\delta_{\varepsilon,\pm l} u|)\big). \tag{4.3.14}$$

Proof. Let

$$v_r = \delta_{h_r,l_r} v, \quad v = \xi u, \quad \xi(t) = \begin{cases} e^{c_0 t} & t < T, \\ e^{c_0 T'} & t = T. \end{cases}$$

Let $(t_0, x_0) \in Q$ be the point where

$$V = \sum_r (v_r^-)^2$$

is maximized. By definition, for any $(t,x) \in Q^o_\varepsilon$ we know that

$$(t, x + h_r l_r) \in Q.$$

Then either

$$v_r(t,x) \le 0 \quad \text{or} \quad -v_r(t,x) = v_{-r}(t, x + h_r l_r) \le 0.$$

In either case, for any $(t,x) \in Q^0_\varepsilon$

$$|v_r(t,x)| \le V^{1/2}.$$

Hence

$$M_1 := \sup_{Q,r} |v_r| \le \sup_{\partial_\varepsilon Q, r} |v_r| + V^{1/2}, \tag{4.3.15}$$

$$|\delta_{\varepsilon,\pm l} u| \le e^{c_0 T'} \sup_{\partial_\varepsilon Q, r} |\delta_{h_r,l_r} u| + V^{1/2} \tag{4.3.16}$$

on Q. So we only have to estimate V on Q. If (t_0, x_0) belongs to $\partial_\varepsilon Q$, then the conclusion of the theorem is trivially true. Thus, we may assume that $(t_0, x_0) \in Q^0_\varepsilon$. Then for any $\bar\varepsilon_0 > 0$ there exists $\alpha_0 \in A$ such that at (t_0, x_0),

$$m^{\alpha_0} \left(\delta_\tau u + a_k^{\alpha_0} \Delta_{h_k,\ell_k} u + b_k^{\alpha_0} \delta_{h_k,\ell_k} u - c^{\alpha_0} u + f^{\alpha_0} \right) + \bar\varepsilon_0 \ge 0.$$

and so for some $\bar\varepsilon \in [0, \bar\varepsilon_0]$

$$m^{\alpha_0} \left(\delta_\tau u + a_k^{\alpha_0} \Delta_{h_k,\ell_k} u + b_k^{\alpha_0} \delta_{h_k,\ell_k} u - c^{\alpha_0} u + f^{\alpha_0} \right) + \bar\varepsilon = 0. \tag{4.3.17}$$

Furthermore (thanks to the fact that $(t_0, x_0) \in Q^0_\varepsilon$)

$$\mathrm{T}_{h_r,\ell_r} \left[m^{\alpha_0} \left(\delta_\tau u + a_k^{\alpha_0} \Delta_{h_k,\ell_k} u + b_k^{\alpha_0} \delta_{h_k,\ell_k} u - c^{\alpha_0} u + f^{\alpha_0} \right) \right] \le 0. \tag{4.3.18}$$

We subtract (4.3.17) from (4.3.18) and divide by h_r to obtain that for each r

$$m^{\alpha_0}\,\delta_{h_r,l_r}\left(\delta_\tau\,u + a_k^{\alpha_0}\,\Delta_{h_k,\ell_k}\,u + b_k^{\alpha_0}\,\delta_{h_k,\ell_k}\,u + f^{\alpha_0} - c^{\alpha_0}u\right) - \frac{\bar\varepsilon}{h_r} \le 0.$$

For each r, by the discrete Leibnitz rule (4.3.3)

$$\begin{aligned} m^{\alpha_0}\left(\delta_\tau(\xi^{-1}v_r) + \xi^{-1}\left[a_k^{\alpha_0}\,\Delta_{h_k,\ell_k}\,v_r + I_{1r} + I_{2r} + I_{3r}\right] + \delta_{h_r,l_r}\,f^{\alpha_0}\right) \\ -\xi^{-1}\,\delta_{h_r,l_r}(m^{\alpha_0}c^{\alpha_0}v) - \frac{\bar\varepsilon}{h_r} \le 0, \end{aligned} \tag{4.3.19}$$

where

$$\begin{aligned} I_{1r} &= (\delta_{h_r,l_r}\,a_k^{\alpha_0})\,\Delta_{h_k,\ell_k}\,v, \\ I_{2r} &= h_r(\delta_{h_r,l_r}\,a_k^{\alpha_0})\,\Delta_{h_k,\ell_k}\,v_r, \\ I_{3r} &= (\mathrm{T}_{h_r,\ell_r}\,b_k^{\alpha_0})\,\delta_{h_k,\ell_k}\,v_r + (\delta_{h_r,l_r}\,b_k^{\alpha_0})\,\delta_{h_k,\ell_k}\,v. \end{aligned}$$

Notice (first equality by (4.3.4), second inequality by (4.3.5), that

$$\begin{aligned} 0 &\ge \Delta_{h_k,\ell_k}\sum_r (v_r^-)^2 = 2v_r^-\,\Delta_{h_k,\ell_k}\,v_r^- + \sum_r\left[\delta_{h_k,\ell_k}(v_r^-)^2 + \delta_{h_k,l_{-k}}(v_r^-)^2\right] \\ &\ge -2v_r^-\,\Delta_{h_k,\ell_k}\,v_r + \sum_r(\delta_{h_k,\ell_k}\,v_r^-)^2 + \sum_r(\delta_{h_k,-\ell_k}\,v_r^-)^2, \end{aligned}$$

which gives $0 \le v_r^-\,\Delta_{h_k,\ell_k}\,v_r$ and

$$I := \sum_r a_k^{\alpha_0}(\delta_{h_k,\ell_k}\,v_r^-)^2 + \sum_r a_k^{\alpha_0}(\delta_{h_k,-\ell_k}\,v_r^-)^2 \le v_r^-\,a_k^{\alpha_0}\,\Delta_{h_k,\ell_k}\,v_r.$$

Go back to (4.3.19), multiply by ξv_r^- and sum up in r to get

$$\begin{aligned} m^{\alpha_0}\Big(\xi v_r^-\,\delta_\tau(\xi^{-1}v_r) + \frac{1}{2}a_k^{\alpha_0}v_r^-\,\Delta_{h_k,\ell_k}\,v_r + \frac{1}{2}I + v_r^-[I_{1r} + I_{2r} \\ +I_{3r} + \xi\,\delta_{h_r,l_r}\,f^{\alpha_0}]\Big) - v_r^-\,\delta_{h_r,l_r}(m^{\alpha_0}c^{\alpha_0}v) - \sum_r \xi(v_r^-)\frac{\bar\varepsilon}{h_r} \le 0. \end{aligned} \tag{4.3.20}$$

Since $-v_r^- v_r = (v_r^-)^2$ and due to (4.3.11), $m^\alpha \, \delta_{h_r,l_r} \, f_k^\alpha \geq -K$,

$$
\begin{aligned}
& m^{\alpha_0} v_r^- \xi \, \delta_{h_r,l_r} \, f^{\alpha_0} - v_r^- \, \delta_{h_r,l_r} (m^{\alpha_0} c^{\alpha_0} v) \\
& \qquad = m^{\alpha_0} v_r^- \xi \, \delta_{h_r,l_r} \, f^{\alpha_0} - m^{\alpha_0} v_r^- (\delta_{h_r,l_r} \, c^{\alpha_0}) v - m^{\alpha_0} v_r^- (\mathrm{T}_{h_r,\ell_r} \, c^{\alpha_0}) v_r \\
& \qquad \geq -e^{c_o T'} K v_r^- - m^{\alpha_0} v_r^- | \, \delta_{h_r,l_r} \, c^{\alpha_0} || v | + m^{\alpha_0} (\mathrm{T}_{h_r,\ell_r} \, c^{\alpha_0}) (v_r^-)^2 \\
& \qquad \qquad \geq -e^{c_o T'} K M_1 - m^{\alpha_0} K M_1 M_o + (\mathrm{T}_{h_r,\ell_r} \, m^{\alpha_0} c^{\alpha_0}) V.
\end{aligned}
$$

Use the fact that V attains its maximum at (t_o, x_o), discrete Leibnitz rule and (4.3.5) in order to get a lower bound for the term containing I_{3r}. First,

$$
\begin{aligned}
0 \quad & \geq \quad \sum_r \delta_{h_k,\ell_k} (v_r^-)^2 = 2 v_r^- \, \delta_{h_k,\ell_k} \, v_r^- + \sum_r h_k (\delta_{h_k,\ell_k} \, v_r^-)^2 \\
& \geq \quad 2 v_r^- \, \delta_{h_k,\ell_k} \, v_r^- \geq -2 v_r^- \, \delta_{h_k,\ell_k} \, v_r .
\end{aligned}
$$

Next recall that $b_k^\alpha \geq 0$ and $| \, \delta_{h_r,l_r} \, b_k^\alpha | \leq K$, and so

$$
-v_r^- (\mathrm{T}_{h_r,\ell_r} \, b_k^{\alpha_0}) \, \delta_{h_k,\ell_k} \, v_r \leq 0,
$$

hence

$$
v_r^- I_{3r} \geq -v_r^- | \, \delta_{h_r,l_r} \, b_k^{\alpha_0} || \, \delta_{h_k,\ell_k} \, v | \geq -K M_1^2 .
$$

Apply discrete Leibnitz rule to the very first term of (4.3.20). Then

$$
\begin{aligned}
\xi v_r^- \, \delta_\tau (\xi^{-1} v_r) \quad & = \quad \xi v_r^- \left[\xi^{-1} (t_o + \tau) \, \delta_\tau \, v_r + v_r \, \delta_\tau \, \xi^{-1} \right] \\
& = \quad e^{-c_o \tau} v_r^- \, \delta_\tau \, v_r - V \xi \, \delta_\tau \, \xi^{-1} \geq -V \xi \, \delta_\tau \, \xi^{-1} \\
& = \quad V \frac{1}{\tau} (1 - e^{-c_o \tau}).
\end{aligned}
$$

Using the above estimates we see

$$
\begin{aligned}
& m^{\alpha_0} V \frac{1}{\tau} (1 - e^{-c_o \tau}) + (\mathrm{T}_{h_r,\ell_r} \, m^{\alpha_0} c^{\alpha_0}) V + \frac{1}{2} m^{\alpha_0} a_k^{\alpha_0} v_r^- \, \Delta_{h_k,\ell_k} \, v_r + \frac{1}{2} m^{\alpha_0} I \\
& \qquad + v_r^- m^{\alpha_0} [I_{1r} + I_{2r}] - \sum_r \xi v_r^- \frac{\bar{\varepsilon}}{h_r} \leq K M_1 (e^{c_o T'} + M_0 + m^{\alpha_0} M_1).
\end{aligned}
$$

Hence

$$m^{\alpha_0} V \frac{1}{\tau}(1-e^{-c_o\tau}) + (\mathrm{T}_{h_r,\ell_r}\, m^{\alpha_0} c^{\alpha_0})V \le KM_1(e^{c_oT'} + M_0 + m^{\alpha_0} M_1)$$
$$+m^{\alpha_0} v_r^- |I_{1r}| + m^{\alpha_0} v_r^- |I_{2r}| - \frac{1}{2} m^{\alpha_0} a_k^{\alpha_0} v_r^- \, \Delta_{h_k,\ell_k} v_r - \frac{1}{2} m^{\alpha_0} I + \sum_r \xi v_r^- \frac{\bar{\varepsilon}}{h_r}.$$

Define

$$J_1 := v_r^- |(\delta_{h_r,l_r}\, a_k^{\alpha_o})\, \Delta_{h_k,\ell_k} v| - \frac{1}{4}\sum_r a_k^{\alpha_0} (\delta_{h_k,\ell_k} v_r^-)^2,$$
$$J_2 := J_3 - \frac{1}{2} a_k^{\alpha_0} v_r^- \, \Delta_{h_k,\ell_k} v_r - \frac{1}{4}\sum_r a_k^{\alpha_0} (\delta_{h_k,\ell_k} v_r^-)^2$$
$$J_3 := h_r v_r^- |(\delta_{h_r,l_r}\, a_k^{\alpha_0})\, \Delta_{h_k,\ell_k} v_r|.$$

One can rewrite the above inequality as

$$m^{\alpha_0} V \frac{1}{\tau}(1-e^{-c_o\tau}) + (\mathrm{T}_{h_r,\ell_r}\, m^{\alpha_0} c^{\alpha_0})V$$
$$\le KM_1(e^{c_oT'} + M_0 + m^{\alpha_0} M_1) + m^{\alpha_0}(J_1 + J_2) + \sum_r \xi v_r^- \frac{\bar{\varepsilon}}{h_r}.$$

So we need to estimate J_1, J_2. By (4.3.8)

$$h|\,\Delta_{h_k,\ell_k} v| \le h\sum_r |\,\delta_{h_k,\ell_k} v_r^-| + h\sum_r |\,\delta_{h_k,l_{-k}} v_r^-| \le NM_1.$$

We turn our attention to J_1. Notice that

$$\begin{aligned} v_r^- |(\delta_{h_r,l_r}\, a_k^{\alpha_0})\, \Delta_{h_k,\ell_k} v| &\le M_1 K |(\sqrt{a_k^{\alpha_0}} + h)\, \Delta_{h_k,\ell_k} v| \\ &\le KM_1 \sqrt{a_k^{\alpha_0}} |\,\Delta_{h_k,\ell_k} v| + NM_1^2 \\ &\le NM_1^2 + 2KM_1\sqrt{a_k^{\alpha_0}} \sum_r \delta_{h_k,\ell_k} v_r^- \\ &\le NM_1^2 + \sum_r 8KM_1^2 + \frac{1}{4}\sum_r a_k^{\alpha_0}(\delta_{h_k,\ell_k} v_r^-)^2, \end{aligned}$$

by Young's inequality. So $J_1 \le NM_1^2$. Since $h_r = h$ or $\varepsilon \in (0, Kh]$,

$$J_3 \le Khv_r^- \sqrt{a_k^{\alpha_0}} |\,\Delta_{h_k,\ell_k} v_r| + Kh^2 v_r^- |\,\Delta_{h_k,\ell_k} v_r|.$$

Also $h^2|\,\Delta_{h_k,\ell_k} v_r| \le 4M_1$ and in general $|a| = 2a^- + a$ and so

$$J_3 \le 2Khv_r^- \sqrt{a_k^{\alpha_0}}(\Delta_{h_k,\ell_k} v_r)_- + Khv_r^- \sqrt{\alpha_k^{\alpha_0}}\,\Delta_{h_k,\ell_k} v_r + NM_1^2.$$

Due to (4.1.4),

$$\begin{aligned}
&h(\Delta_{h_k,\ell_k} v_r)_- \le h|\,\Delta_{h_k,\ell_k}(v_r^-)| \le |\,\delta_{h_k,\ell_k}(v_r^-)| + |\,\delta_{h_k,l_{-k}}(v_r^-)|,\\
&J_3 \le 2Kv_r^- h\sqrt{a_k^{\alpha_0}}\,\Delta_{h_k,\ell_k} v_r + 4KM_1\left(\sum_r a_k^{\alpha_0}(\delta_{h_k,\ell_k} v_r^-)^2\right)^{1/2} + NM_1^2\\
&\quad \le NM_1^2 + 2Kv_r^- h\sqrt{a_k^{\alpha_0}}\,\Delta_{h_k,\ell_k} v_r + \frac{1}{4}\sum_r a_k^{\alpha_0}(\delta_{h_k,\ell_k} v_r^-)^2,\\
&J_2 \le NM_1^2 - \frac{1}{2}(a_k^{\alpha_0} - 4Kh\sqrt{a_k^{\alpha_0}})v_r^-\,\Delta_{h_k,\ell_k} v_r.
\end{aligned}$$

Consider $\mathcal{K} := \left\{k : a_k^{\alpha_0} - 4Kh\sqrt{a_k^{\alpha_0}} \ge 0\right\}$. For all k not in $\mathcal{K}$ we have $a_k^{\alpha_0} < Nh^2$. Also

$$h^2|\,\Delta_{h_k,\ell_k} v_r| \le 4M_1.$$

Hence $J_2 \le NM_1^2$ and

$$\begin{aligned}
m^{\alpha_0}V\frac{1}{\tau}(1 - e^{-c_o\tau}) &+ (\mathrm{T}_{h_r,\ell_r}\, m^{\alpha_0}c^{\alpha_0})V\\
&\le NM_1\left(e^{c_oT'} + M_0 + m^{\alpha_0}M_1\right) + \sum_r \xi v_r^- \frac{\bar{\varepsilon}}{h_r}.
\end{aligned}$$

We know from (4.3.15), that $M_1 \le \mu + V^{1/2}$, where

$$\mu := \sup_{\partial_\varepsilon Q, r} |v_r| \le e^{c_oT'} \sup_{\partial_\varepsilon Q, r} |\,\delta_{h_r,l_r} u| =: \bar{\mu}.$$

One also defines

$$\bar{M_o} = |u|_{0,Q} \ge e^{-c_0T'}M_0, \quad \bar{V} = e^{-2c_0T'}V$$

and so, using Young's inequality

$$\begin{aligned}
&m^{\alpha_0}V\frac{1}{\tau}(1 - e^{-c_o\tau}) + (\mathrm{T}_{h_r,\ell_r}\, m^{\alpha_0}c^{\alpha_0})V\\
&\qquad \le N(\bar{\mu} + \bar{V}^{1/2})\left(1 + \bar{M_0} + \bar{\mu} + m^{\alpha_0}\bar{V}^{1/2}\right) + \sum_r \xi v_r^- \frac{\bar{\varepsilon}}{h_r}\\
&\qquad\qquad \le N^*\left(1 + \bar{M_0}^2 + \bar{\mu}^2 + m^{\alpha_0}\bar{V}\right) + \sum_r \xi v_r^- \frac{\bar{\varepsilon}}{h_r}. \qquad (4.3.21)
\end{aligned}$$

Assume that for a constant c_0

$$\lambda + \frac{1}{\tau}(1 - e^{-c_o\tau}) > 1 + N^* =: N_0.$$

Due to $\bar{V} \le V$, (4.3.21) becomes

$$m^{\alpha_0} V \left(1 + \mathrm{T}_{h_r,\ell_r}\, c^{\alpha_0} - \lambda\right) \le N^* \left(1 + \bar{M_0}^2 + \bar{\mu}^2\right) + \sum_r \xi v_r^- \frac{\bar{\varepsilon}}{h_r}.$$

By Assumptions (4.2.4), $m^{\alpha_0}(1 + \mathrm{T}_{h_r,\ell_r}\, c^{\alpha_0} - \lambda) \ge K^{-1}$ and so

$$V \le KN^* \left(1 + \bar{M_0}^2 + \bar{\mu}^2\right) + K \sum_r \xi v_r^- \frac{\bar{\varepsilon}}{h_r}.$$

Noticing that $\xi v_r^- \le e^{c_o T'} 2M_0 h_r^{-1}$

$$V \le N^* \left(1 + \bar{M_0}^2 + \bar{\mu}^2\right) + e^{c_o T'} N M_0 \frac{\bar{\varepsilon}}{h_r^2}.$$

Neither N, nor M_0 depend on choice of α_0 and so we can let $\bar{\varepsilon} \to 0$. Then

$$V \le N^* \left(1 + \bar{M_0}^2 + \bar{\mu}^2\right).$$

□

Corollary 4.3.5. *Let u be the solution to (4.2.1)-*(4.2.2) *on $\bar{\mathcal{M}}_T(\varepsilon)$ and let Assumption 4.2.1 and 4.3.3 be satisfied. If 4.3.13 holds, then on $\bar{\mathcal{M}}_T(\varepsilon)$,*

$$|\delta_{\varepsilon,\pm l} u| \le N e^{c_0 T'}, \tag{4.3.22}$$

where N depends on K, d and d_1 only.

Proof. First consider u to be solution to (4.2.1)-(4.2.2) on $\bar{\mathcal{M}}_T(\varepsilon)$ with $g(x)$ and $f^\alpha(t,x)$ zero for all x outside B_R, where B_R denotes the open ball centered at origin with radius R. Then by Theorem 4.3.4 applied to $Q_n := \bar{\mathcal{M}}_T(\varepsilon) \cap ([0,T] \times B_n)$,

$$|\delta_{\varepsilon,\pm l} u| \le N e^{c_o} T' \big(1 + |u|_{0,Q_n} + \sup_{k,\partial_\varepsilon Q_n} \left(|\, \delta_{h_k,\ell_k}\, u| + |\delta_{\varepsilon,\pm l} u|\right)\big),$$

on Q_n, with the same constant N for each n. Let $\partial\bar{\mathcal{M}}_T(\varepsilon) = \{(T,x) \in \bar{\mathcal{M}}_T(\varepsilon)\}$. Due to

Lemma 4.2.9

$$\lim_{n\to\infty} \sup_{k,\partial_\varepsilon Q_n \setminus \partial \bar{\mathcal{M}}_T(\varepsilon)} (|\,\delta_{h_k,\ell_k}\, u| + |\delta_{\varepsilon,\pm l} u|) = 0$$

and hence on $\bar{\mathcal{M}}_T(\varepsilon)$

$$|\delta_{\varepsilon,\pm l} u| \le N e^{c_o} T' \big(1 + |u|_{0,\bar{\mathcal{M}}_T(\varepsilon)} + \sup_{k,\partial \bar{\mathcal{M}}_T(\varepsilon)} (|\,\delta_{h_k,\ell_k}\, u| + |\delta_{\varepsilon,\pm l} u|)\big). \tag{4.3.23}$$

Now consider any f^α and g that satisfy Assumptions 4.2.1 and 4.3.3. For such f^α and g we can find functions f_n^α and g_n that vanish outside B_n,

$$\lim_{n\to\infty} \sup_\alpha (|f^\alpha - f_n^\alpha| + |g - g_n|) = 0$$

and furthermore f_n^α and g_n satisfy Assumptions 4.2.1 and 4.3.3 with f_n^α, g_n and $2K$ in place of f^α, g and K respectively. Let u_n be the solution to (4.2.1)-(4.2.2) on $\bar{\mathcal{M}}_T(\varepsilon)$. Then from (4.3.23) we see that for each $n \in \mathbb{N}$,

$$|\delta_{\varepsilon,\pm l} u_n| \le N e^{c_o} T' \big(1 + |u_n|_{0,\bar{\mathcal{M}}_T(\varepsilon)} + \sup_{k,x\in\mathbb{R}^d} (|\,\delta_{h_k,\ell_k}\, g_n| + |\delta_{\varepsilon,\pm l} g_n|)\big),$$

with N independent of n. We can estimate $|u_n|_{0,\bar{\mathcal{M}}_T(\varepsilon)}$ using Corollary 4.2.8 and $|\,\delta_{h_k,\ell_k}\, g_n| + |\delta_{\varepsilon,\pm l} g_n|$ using the Lipschitz continuity of g_n. Finally we use Lemma 4.2.10 when we take the limit as $n \to \infty$. □

Corollary 4.3.6. *Let Assumption 4.2.1 hold. Then there is a constant N such that for any $t \in [0,T]$ and $x, y \in \mathbb{R}^d$,*

$$|v_{\tau,h}(t,x) - v_{\tau,h}(t,y)| \le N T e^{NT} |x - y|. \tag{4.3.24}$$

Lemma 4.3.7. *Let u and $\hat{u}$ be functions satisfying (4.2.1) on $\mathcal{M}_T$ with the coefficients σ, b, c, f and $\hat{\sigma}, \hat{b}, \hat{c}, \hat{f}$ respectively. Assume that $\varepsilon \in (0,h]$ and for all $\alpha \in A$*

$$|b_k^\alpha - \hat{b}_k^\alpha| + m^\alpha |f^\alpha - \hat{f}^\alpha| + |c^\alpha - \hat{c}^\alpha| + |g - \hat{g}| \le K\varepsilon,$$
$$|a_k^\alpha - \hat{a}_k^\alpha| \le K\varepsilon \sqrt{a_k^\alpha \wedge \hat{a}_k^\alpha} + K\varepsilon h,$$

on $\mathcal{M}_T$. Then on $\mathcal{M}_T$,

$$|u - \hat{u}| \le \varepsilon N e^{c_0 T'}.$$

Proof. We follow the idea of [24] to deduce this lemma from the gradient estimate (4.3.22).

Let $(t,x) = (t,x',x^{d+1}) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$. Let $l = (0,\dots,0,1) \in \mathbb{R}^{d+1}$,

$$\tilde{Q} := \mathcal{M}_T \times \{0, \pm\varepsilon, \pm 2\varepsilon, \dots\},$$
$$\partial\tilde{Q} := \{(t,x',x^{d+1}) \in \bar{\mathcal{M}}_T \times \{0, \pm\varepsilon, \pm 2\varepsilon, \dots\} : t = T\}.$$

Let

$$\tilde{a}_k^\alpha(t,x',x^{d+1}) = \begin{cases} a_k^\alpha(t,x') & \text{if} \quad x^{d+1} > 0, \\ \hat{a}_k^\alpha(t,x') & \text{if} \quad x^{d+1} \le 0 \end{cases}$$

and define $\tilde{b}_k^\alpha, \tilde{c}_k^\alpha$, $\tilde{f}^\alpha$ and $\tilde{u}$ in a similar way. Then $\tilde{u}$ satisfies (4.2.1) with $\tilde{a}_k^\alpha, \tilde{b}_k^\alpha, \tilde{c}^\alpha$ and $\tilde{f}^\alpha$ in place of a_k^α, b_k^α, c^α and f^α respectively and with the normalizing factor m^α, on the domain $\tilde{Q}$. To apply Corollary 4.3.5 to $\tilde{u}$ one needs to check Assumption 4.3.3. For $r = \pm 1, \dots, \pm d_1$ (4.3.11) follows from Assumption 4.2.1. Furthermore, for $r = \pm(d_1+1)$ notice that

$$\delta_{\varepsilon,\ell}\, \tilde{a}_k^\alpha(t,x) = \begin{cases} 0 & \text{if} \quad x^{d+1} \neq 0, \\ \varepsilon^{-1}(a_k^\alpha(t,x') - \hat{a}_k^\alpha(t,x')) & \text{if} \quad x^{d+1} = 0 \end{cases}$$

and similarly for $\tilde{b}_k^\alpha, \tilde{c}^\alpha$ and $\tilde{f}^\alpha$. So for $\tilde{b}_k^\alpha$, $\tilde{c^\alpha}$ and $\tilde{f^\alpha}$ (4.3.11) holds, while (4.3.12) follows from:

$$\varepsilon^{-1}|a_k^\alpha(t,x') - \hat{a}_k^\alpha(t,x')| \le K\sqrt{\tilde{a}_k^\alpha(t,x',0)} + Kh.$$

By Corollary 4.3.5, for all $(t,x') \in \mathcal{M}_T$

$$\varepsilon^{-1}|u(t,x') - \hat{u}(t,x')| = |\,\delta_{\varepsilon,\ell}\, \tilde{u}(t,x',0)| \le NTe^{c_0(T+\tau)}.$$

□

Theorem 4.3.8. *Let*

$$\varepsilon = \sup_{\mathcal{M}_T, A, k} \left(|\sigma_k^\alpha - \hat{\sigma}_k^\alpha| + |b_k^\alpha - \hat{b}_k^\alpha| + |c^\alpha - \hat{c}^\alpha| + m^\alpha |f^\alpha - \hat{f}^\alpha| + |g - \hat{g}| \right).$$

Assume that the functions $u, \hat{u}$ satisfy (4.2.1)-(4.2.2) *on $\mathcal{M}_T$, with σ, b, c, f, g and $\hat{\sigma}, \hat{b}, \hat{c}, \hat{f}, \hat{g}$ respectively. Then*

$$|u - \hat{u}| \le Ne^{c_0 T'} T' \varepsilon. \tag{4.3.25}$$

Proof. From our definition of ε we clearly have

$$|b_k^\alpha - \hat{b}_k^\alpha| + m^\alpha |f^\alpha - \hat{f}^\alpha| + |c^\alpha - \hat{c}^\alpha| + |g - \hat{g}| \le \varepsilon.$$

Assume initially that $\varepsilon \in (0,h]$. Then, using that for any $a,b \geq 0$,

$$|a^2 - b^2| = (a+b)|a-b| = 2(a \wedge b)|a-b| + |a-b|^2,$$

we get

$$|a_k^\alpha - \hat{a}_k^\alpha| \leq (|\sigma_k^\alpha| \wedge |\hat{\sigma}_k^\alpha|)|\sigma_k^\alpha - \hat{\sigma}_k^\alpha| + |\sigma_k^\alpha - \hat{\sigma}_k^\alpha|^2 \leq 2\varepsilon\sqrt{a_k^\alpha \wedge \hat{a}_k^\alpha} + \varepsilon h.$$

Hence by Lemma 4.3.7, $|u - \hat{u}| \leq \varepsilon N e^{c_0 T'}$ on $\bar{\mathcal{M}}_T$. Now consider the case $\varepsilon > h$. For $\theta \in [0,1]$, let u^θ be the solution of

$$\sup_\alpha m^\alpha(\delta_\tau u^\theta + a_k^{\theta\alpha} \Delta_{h_k,\ell_k} u^\theta + b_k^{\theta\alpha} \delta_{h_k,\ell_k} u^\theta - c^{\theta\alpha} u^\theta + f^{\theta\alpha}) = 0 \quad \text{on} \quad \mathcal{M}_T$$
$$g^\theta = u^\theta \quad \text{on} \quad \{(T,x) \in \bar{\mathcal{M}}_T\},$$

where

$$(\sigma_k^{\theta\alpha}, b_k^{\theta\alpha}, c^{\theta\alpha}, f^{\theta\alpha}, g^\theta) = (1-\theta)(\sigma_k^\alpha, b_k^\alpha, c^\alpha, f^\alpha, g) + \theta(\hat{\sigma}_k^\alpha, \hat{b}_k^\alpha, \hat{c}^\alpha, \hat{f}^\alpha, \hat{g})$$

and $a_k^{\theta\alpha} = (1/2)|\sigma_k^{\theta\alpha}|^2$. For any $\theta_1, \theta_2 \in [0,1]$,

$$|\sigma_k^{\theta_1\alpha} - \sigma_k^{\theta_2\alpha}| + |b_k^{\theta_1\alpha} - b_k^{\theta_2\alpha}| + |c^{\theta_1\alpha} - c^{\theta_2\alpha}| + m^\alpha|f^{\theta_1\alpha} - f^{\theta_2\alpha}| + |g^{\theta_1} - g^{\theta_2}| \leq |\theta_1 - \theta_2|\varepsilon.$$

Hence if θ_1, θ_2 satisfy $|\theta_1 - \theta_2|\varepsilon \leq h$, then, thanks to the first part of the proof (where u^{θ_1} plays the part of u while u^{θ_2} plays the part of $\hat{u}$),

$$|u^{\theta_1} - u^{\theta_2}| \leq N|\theta_1 - \theta_2|\varepsilon T' e^{c_0(T+\tau)}.$$

Split the interval $[0,1]$ into intervals of appropriate length to complete the proof for any $\varepsilon > 0$. □

4.4 Some properties of payoff functions

We assume in the whole section that Assumption 2.3.1 and the following assumption hold and that v is the function given by (2.3.4).

Assumption 4.4.1. There is a function $m : A \to (0,1]$, such that for all $\alpha \in A$, $t \in [0,T)$,

$x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$

$$m^\alpha c^\alpha(t,x) + m^\alpha \geq K^{-1}, \quad |m^\alpha f^\alpha(t,x)| \leq K, \tag{4.4.1}$$

$$m^\alpha |f^\alpha(t,x) - f^\alpha(t,y)| \leq K|x-y|, \tag{4.4.2}$$

$$|\sigma^\alpha(t,x) - \sigma^\alpha(t,y)| \leq K|x-y| \quad \text{and} \quad |\beta^\alpha(t,x) - \beta^\alpha(t,y)| \leq K|x-y|. \tag{4.4.3}$$

Lemma 4.4.2. *Let Assumption 4.4.1 be satisfied. Then there exists a constant N_T such that:*

1. *For any $(s,x) \in [0,T] \times \mathbb{R}^d$, $v(s,x) \leq N_T$.*
2. *For any $(s,x), (s,y) \in [0,T] \times \mathbb{R}^d$,*

$$|v(s,x) - v(s,y)| \leq N_T|x-y|.$$

Proof. Condition (4.4.1) yields $|f^\alpha(s,x)| \leq K^2(1 + c^\alpha(s,x))$. This and the assumption on the polynomial growth of g together with the moments estimates for solutions to SDEs give

$$\begin{aligned}
|v^\alpha(s,x)| \leq & \mathbb{E}^\alpha_{s,x} \int_0^{T-s} |f^{\alpha_t}(s+t,x_t)| e^{-\varphi_t} dt + \mathbb{E}^\alpha_{s,x} |g(x_{T-s})| e^{-\varphi_{T-s}} \\
\leq & K^2 \mathbb{E}^\alpha_{s,x} \int_0^{T-s} (1 + c^{\alpha_t}(s+t,x_t)) e^{-\int_0^t c^{\alpha_u}(s+u,x_u)du} dt + N \\
\leq & N(1 + (T-s)) + K^2 \mathbb{E} \int_0^{T-s} c^{\alpha_t}(s+t,x_t) e^{-\int_0^t c^{\alpha_u}(s+u,x_u)du} dt \\
\leq & N(1 + (T-s)),
\end{aligned}$$

which proves (1). To prove (2) we first get an estimate for a fixed $\alpha \in \mathfrak{A}$.

$$\begin{aligned}
|v^\alpha(s,x) - v^\alpha(s,y)| \leq \mathbb{E} & \int_0^{T-s} |f^{\alpha_t}(s+t, x_t^{s,x})| |e^{-\varphi_t^{\alpha,s,x}} - e^{-\varphi_t^{\alpha,s,y}}| dt \\
& + \mathbb{E} \int_0^{T-s} |f^{\alpha_t}(s+t, x_t^{s,x}) - f^{\alpha_t}(s+t, x_t^{s,y})| e^{-\varphi_t^{\alpha,s,y}} dt \\
& + \mathbb{E} |g(x_{T-s}^{s,x}) - g(x_{T-s}^{s,y})| =: I_1 + I_2 + I_3.
\end{aligned}$$

Estimating the above integrals separately:

$$I_1 \leq \mathbb{E} \int_0^{T-s} |f^{\alpha_t}(s+t, x_t^{s,x})| |\varphi_t^{\alpha,s,x} - \varphi_t^{\alpha,s,y}| e^{-\min(\varphi_t^{\alpha,s,x}, \varphi_t^{\alpha,s,y})} dt.$$

Since (4.4.1) holds,

$$|f^\alpha(s,x)| \leq \frac{K}{m^\alpha} \quad \text{and} \quad c^\alpha(s,x) \geq \frac{1}{Km^\alpha} - 1.$$

Therefore, using Lipschitz continuity of c when estimating $|\varphi_t^{\alpha,s,x} - \varphi_t^{\alpha,s,y}|$,

$$
\begin{aligned}
I_1 &\leq (T-s)e^{T-s}K^3\mathbb{E}\int_0^{T-s} \frac{1}{Km^{\alpha_t}} \sup_{r\leq t} |x_r^{\alpha,s,x} - x_r^{\alpha,s,y}| e^{-\int_0^t \frac{1}{Km^{\alpha_u}}du} dt \\
&\leq (T-s)e^{T-s}K^3\mathbb{E} \sup_{t\leq T-s} |x_t^{\alpha,s,x} - x_t^{\alpha,s,y}|.
\end{aligned}
$$

Now

$$
I_2 \leq \int_0^{T-s} (m^{\alpha_t})^{-1} |x_t^{\alpha,s,x} - x_t^{\alpha,s,y}| e^{-\varphi^{\alpha,s,y}} dt
$$

$$
\leq \mathbb{E}\Big(\sup_{t\leq T-s} |x_t^{\alpha,s,x} - x_t^{\alpha,s,y}| \int_0^{T-s} K(1 + c^{\alpha_t}(s+t, x_t^{\alpha,s,y}) e^{-\varphi^{\alpha,s,y}} dt \Big)
$$

$$
\leq [K(t-s) + K]\, \mathbb{E} \sup_{t\leq T-s} |x_t^{\alpha,s,x} - x_t^{\alpha,s,y}|
$$

and

$$
I_3 \leq K(T-s)\mathbb{E} \sup_{t\leq T-s} |x_t^{\alpha,s,x} - x_t^{\alpha,s,y}|.
$$

Consequently

$$
|v^\alpha(s,x) - v^\alpha(s,y)| \leq K^3(T-s)^2 e^{(T-s)} \mathbb{E} \sup_{t\leq T-s} |x_t^{\alpha,s,x} - x_t^{\alpha,s,y}|.
$$

Notice that

$$
\mathbb{E} \sup_{t\leq T-s} |x_t^{\alpha,s,x} - x_t^{\alpha,s,y}| \leq Ne^{NT}|x-y|,
$$

with constant N depending only on the Lipschitz constant of σ^α and β^α (Theorem 2.2.3). □

Let $H_T := [0,T) \times R^d$ and $\partial H_T := \{T\} \times \mathbb{R}^d$.

Lemma 4.4.3. *Let $\psi \in C^\infty(H_T)$ such that its first order partial derivatives in x grow at most polynomially and for all $\alpha \in A$ it satisfies*

$$
\frac{\partial}{\partial t}\psi + L^\alpha \psi + f^\alpha \leq 0 \quad on \quad H_T.
$$

Let v be the payoff function of the stochastic control problem (2.3.4). Then

$$
v \leq \psi + \sup_{\partial H_T} [v - \psi]_+ \quad on \quad H_T. \tag{4.4.4}
$$

Hence, clearly, if $v \leq \psi$ on ∂H_T, then $v \leq \psi$ on H_T.

Proof. For any $\varepsilon > 0$ there is a control process $\alpha_t \in \mathfrak{A}$ such that

$$
\begin{aligned}
v(s,x) &\leq \varepsilon + \mathbb{E}_{s,x}\left(\int_0^{T-s} f^{\alpha_t}(s+t,x_t)e^{-\varphi_t}dt + g(x_{T-s})e^{-\varphi_{T-s}}\right) \\
&\leq \varepsilon - \mathbb{E}_{s,x}\int_0^{T-s} e^{-\varphi_t}\left[\frac{\partial}{\partial t} + L^{\alpha_t}\right]\psi(s+t,x_t)dt + \mathbb{E}_{s,x}g(x_{T-s})e^{-\varphi_{T-s}}.
\end{aligned}
$$

Applying Itô's formula to $\psi(s+r,x_r)e^{-\varphi_r}$ on the interval $[0,T-s]$ and taking expectation we get

$$
\psi(s,x) = \mathbb{E}_{s,x}\Big(\psi(T,x_{T-s})e^{-\varphi_{T-s}} - \int_0^{T-s} e^{-\varphi_t}\left[\frac{\partial}{\partial t} + L^{\alpha}\right]\psi(s+t,x_t)dt\Big),
$$

by noting that the Itô integral has zero expectation, due to moment estimates for x_r, since ψ has polynomially growing first order partial derivatives in x. Hence

$$
v(s,x) \leq \varepsilon + \psi(t,x) + \sup_{x\in\mathbb{R}^d}\left[v(T,x) - \psi(T,x)\right]_+,
$$

for any $\varepsilon > 0$, which yields (4.4.4). □

4.5 Hölder continuity in time of v and $v_{\tau,h}$

We present an example which illustrates that Assumption 2.3.1 and 4.4.1 are not sufficient to ensure continuity of $v(t,x)$ in time.

Example 4.5.1. Let

$$
v(t) = \sup_{r\in\mathfrak{R}}\mathbb{E}\int_0^{T-t} r_s e^{-\int_0^s r_u du}ds.
$$

Then for $t \in [0,T]$

$$
v(t) = \sup_{r\in\mathfrak{R}}\mathbb{E}\left(1 - e^{\int_0^{T-t} r_u du}\right) = \mathbf{1}_{t<T},
$$

which is not continuous at T.

The following Lemma states a known fact. See [19].

Lemma 4.5.2. *Let v be the payoff function for the optimal stopping and control problem* (4.0.1)-(4.0.2). *Let Assumption 4.1.2 be satisfied. Then there exists a constant N depending on K and d only, such that for (s,x), $(t,x) \in H_T$,*

$$
|v(s,x) - w(t,x)| \leq N(\nu+1)|s-t|^{1/2}.
$$

We were unable to prove the following lemma for solutions to (4.2.1)-(4.2.2) under the Assumption 4.2.1. Unlike the results in sections 4.2 and 4.3, we only consider solutions to (4.1.5) under the Assumption 4.1.2. The proof then follows that of a similar result in [24] with only minor changes.

Lemma 4.5.3. *Fix* $(s_0, x_0) \in \bar{\mathcal{M}}_T$. *Let Assumptions 4.1.1, 4.1.2 be satisfied. Let* v *be a solution of* (4.1.5) *and*

$$\nu := \sup_{(s_0,x)\in\bar{\mathcal{M}}_T} \frac{|v(s_0,x) - v(s_0,x_0)|}{|x - x_0|}.$$

Then for all $(t_0, x_0) \in \bar{\mathcal{M}}_T$ *such that* $t_0 \in [s_0 - 1, s_0]$,

$$|v(s_0, x_0) - v(t_0, x_0)| \le N(\nu + 1)|s_0 - t_0|^{1/2}.$$

Proof. Assume, without loss of generality, that $\nu > 1$. Indeed if $\nu < 1$, then in the proof of the theorem use $\bar{\nu} = \nu + 1$. If the result holds for $\bar{\nu}$, then

$$N(\bar{\nu} + 1) \le N(\nu + 1 + 1) \le 2N(\nu + 1).$$

Assume that $s_0 > 0$. Shift the time axis so that $t_0 = 0$. Then $s_0 \le 1$. Let

$$\gamma = s_0^{-1/2} \quad \text{and} \quad \xi(t) = \begin{cases} e^{s_0 - t} & \text{for} \quad t < s_0, \\ 1 & \text{for} \quad t \ge s_0. \end{cases}$$

Notice that $\gamma \ge 1$. Define

$$\psi = \gamma\nu\,[\zeta + \kappa(s_0 - t)] + K(s_0 - t) + \gamma^{-1}\nu + v(s_0, x_0),$$

where $\zeta = \eta\xi$, $\eta(x) = |x - x_0|^2$ and κ is a (large) constant to be chosen later, depending on K and d_1. The aim now is to apply Lemma 4.2.6 to v and ψ on $\mathcal{M}_{s_0}$. In order to do that we have to check that

$$\sup_{r\ge 0}\left[\frac{1}{1+r}\left(\delta_\tau\,\psi + \sup_{\alpha\in A}\left(\mathrm{L}_h^\alpha\,\psi + f^\alpha\right)\right) + \frac{r}{1+r}(g - \psi)\right] \le 0 \quad \text{on} \quad \mathcal{M}_{s_0}. \qquad (4.5.1)$$

and that for all x

$$\psi(s_0, x) \ge v(s_0, x). \qquad (4.5.2)$$

Inequality (4.5.1) will be satisfied if

$$\delta_\tau \psi + \sup_{\alpha \in A} (\mathrm{L}_h^\alpha \psi + f^\alpha) \le 0 \tag{4.5.3}$$

and

$$g - \psi \le 0 \tag{4.5.4}$$

are satisfied on $\mathcal{M}_{s_0}$. Observe that

$$\delta_\tau \xi(t) = -\theta \xi, \quad \text{where} \quad \theta = \tau^{-1}(1 - e^{-\tau}) \ge K^{-1}(1 - e^{-K})$$

and, with $(.,.)$ denoting the inner product in $\mathbb{R}^d$,

$$\mathrm{L}_h^\alpha \eta = 2a_k^\alpha |l_k|^2 + b_k^\alpha (l_k, 2(x - x_0) + hl_k) - c^\alpha \eta \le N_1(1 + |x - x_0|).$$

Hence

$$\begin{aligned} \delta_\tau \zeta + \mathrm{L}_h^\alpha \zeta &\le \xi(t) \left[N_1(1 + |x - x_0|) - \theta |x - x_0|^2 \right] \\ &\le N_2(1 + |x - x_0|) - \theta |x - x_0|^2, \end{aligned}$$

where constants N_1, N_2 depend only on K and d_1. Thus in $\mathcal{M}_{s_0}$

$$\delta_\tau \psi + \mathrm{L}_h^\alpha \psi + f^\alpha \le \gamma\nu[N_2(1 + |x - x_0|) - \theta |x - x_0|^2 - \kappa].$$

Hence one may choose $\kappa > 0$ depending only on K and d_1 such that the inequality (4.5.3) holds. Furthermore, for all x,

$$\begin{aligned} \psi(s_0, x) = \gamma\nu |x - x_0|^2 + \nu\gamma^{-1} + v(s_0, x) &\ge \gamma\nu |x - x_0|^2 + \nu\gamma^{-1} - \nu |x - x_0| \\ &\ge (3/4)\nu\gamma^{-1} + v(s_0, x_0). \end{aligned}$$

Hence the inequality (4.5.2) also holds. Observe that if v is the solution to (4.1.5) then $v(t, x) \ge g(x)$ for all $(t, x) \in \mathcal{M}_T$. Since (4.5.2) holds,

$$g(x) \le v(s_0, x) \le \psi(s_0, x) \le \psi(t, x)$$

for all $(t, x) \in \mathcal{M}_{s_0}$. Hence (4.5.4) holds. By Lemma 4.2.6

$$v(t_0, x_0) \le \kappa(1 + \nu)|t_0 - s_0|^{1/2} + v(s_0, x_0).$$

To get the estimate from the other side, consider

$$\psi = -\gamma\nu\left[\zeta + \kappa(s_0 - t)\right] - \gamma^{-1}\nu - K(s_0 - t) + v(s_0, x_0).$$

We need to show that

$$\sup_{r\geq 0}\left[\frac{1}{1+r}\left(\delta_\tau\,\psi + \sup_{\alpha\in A}\left(\mathrm{L}_h^\alpha\,\psi + f^\alpha\right)\right) + \frac{r}{1+r}(g-\psi)\right] \geq 0 \quad \text{on} \quad \mathcal{M}_{s_0} \tag{4.5.5}$$

and that for all x

$$\psi(s_0, x) \leq v(s_0, x). \tag{4.5.6}$$

Using similar estimates as before, we obtain that for all $(t, x) \in \mathcal{M}_{s_0}$

$$\delta_\tau\,\psi + \sup_{\alpha\in A}\left(\mathrm{L}_h^\alpha\,\psi + f^\alpha\right) \geq 0, \tag{4.5.7}$$

for large κ depending on K and d_1. Notice that now, if $\psi \leq g$ on $\mathcal{M}_{s_0}$ then (4.5.5) is already satisfied. On the other hand, if $\psi > g$ on $\mathcal{M}_{s_0}$, then the supremum on the left hand side of (4.5.5) is equal to $\delta_\tau\,\psi + \sup_{\alpha\in A}\left(\mathrm{L}_h^\alpha\,\psi + f^\alpha\right)$, which we already know is not less than 0. Hence (4.5.5) holds. We can use similar estimate as we used to obtain (4.5.2) to show that (4.5.7) holds. By Lemma 4.2.6

$$v(t_0, x_0) \geq \kappa(1+\nu)|t_0 - s_0|^{1/2} + v(s_0, x_0).$$

□

4.6 Smoothing

As we mentioned at the beginning of this chapter, obtaining the error estimates hinges on Krylov's idea of "shaking" the coefficients. This allows the application of smoothing convolutions to the relevant functions (while controlling the error). That in turn allows us to use estimates arising from Taylor's Theorem. We now introduce the smoothing convolutions. For any two functions u, v for which it makes sense define their convolution as

$$u * v(t, x) = \int_{\mathbb{R}^{d+1}} u(s, y)v(t-s, x-y)dsdy.$$

We will use $\zeta(x), \zeta_1(t)$ which are infinitely differentiable functions of x and respectively, with support in B_1 and $(-1,0)$ respectively and such that

$$\int_{\mathbb{R}^d} \zeta(x)dx = 1, \quad \int_{-\infty}^{\infty}\int_{\mathbb{R}^d} \zeta_1(t)\zeta(x)dxdt = 1.$$

Let $\zeta(t,x) = \zeta_1(t)\zeta(x)$. For any $\varepsilon > 0$ and locally integrable functions $u(x)$, $u(t,x)$ define the smoothing with respect to x:

$$u^{(\varepsilon)}(x) = \varepsilon^{-d}\zeta\left(\frac{x}{\varepsilon}\right) * u(x),$$
$$u^{(0,\varepsilon)}(t,x) = \varepsilon^{-d}\zeta\left(\frac{x}{\varepsilon}\right) * u(t,x)$$

and the smoothing with respect to (t,x):

$$u^{(\varepsilon)}(t,x) = \varepsilon^{-d-2}\zeta\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) * u(t,x).$$

We will make use of the following standard property of convolutions.

Lemma 4.6.1. *Let B be a measurable subset of a unit ball. Let $\zeta \in C_0^\infty(\mathbb{R}^d)$, $\zeta \geq 0$, $\zeta = 0$ on B^c such that $1 = \int_{\mathbb{R}^d} \zeta(x)dx$. Assume that for all $x, y \in \mathbb{R}^d$ a function w satisfies*

$$|w(x) - w(y)| \leq K|x-y|^\alpha,$$

for some $K \geq 0$, $\alpha > 0$. Define $\zeta_\varepsilon(x) := \varepsilon^{-kd}\zeta(\varepsilon^{-k}x)$ and

$$w^{(\varepsilon)} := \int_{\mathbb{R}^d} w(y)\zeta_\varepsilon(x-y)dy.$$

Then

$$\left|\operatorname{D}_x^n w^{(\varepsilon)}(x)\right| \leq N\varepsilon^{k(\alpha-n)},$$

where the constant N depends on K and ζ only.

Proof.

$$w^{(\varepsilon)}(x) = \varepsilon^{-kd}\int_{\mathbb{R}^d} w(y)\zeta\left(\frac{x-y}{\varepsilon^k}\right)dy.$$

Then,

$$\begin{aligned}\operatorname{D}_x^n w^{(\varepsilon)}(x) &= \varepsilon^{-kd}\varepsilon^{-nk}\int_{\mathbb{R}^d} w(y)\operatorname{D}_x^n \zeta\left(\frac{x-y}{\varepsilon^k}\right)dy \\ &= -\varepsilon^{-nk}\int_{\mathbb{R}^d} w(x-\varepsilon^k z)\operatorname{D}_x^n \zeta(z)dz,\end{aligned}$$

where first equality follows by Lebesgue's Dominated Convergence Theorem and the second one from substitution. Notice that $0 = w(x)\int_{\mathbb{R}^d} D_x^n \zeta(z)dz$ and thus

$$D_x^n w^{(\varepsilon)}(x) = -\varepsilon^{-nk}\int_{\mathbb{R}^d}[w(x-\varepsilon^k z) - w(x)]\, D_x^n \zeta(z)dz.$$

Hence

$$\begin{aligned}\left| D_x^n w^{(\varepsilon)}(x)\right| &\le \varepsilon^{-nk}\int_B |w(x-\varepsilon^k z) - w(x)||\, D_x^n \zeta(z)|dz \\ &\le K\varepsilon^{\alpha k}\varepsilon^{-nk}\int_B |\, D_x^n \zeta(z)|dz \le N\varepsilon^{k(\alpha-n)}.\end{aligned}$$

□

4.7 Shaking the coefficients

In this section we are going to need Hölder continuity in time of v and $v_{\tau,h}$. In section 4.5 we proved the Hölder continuity in time of v and $v_{\tau,h}$ given by (4.0.1)-(4.0.2) and (4.1.5) respectively. Thus we are going to assume that v corresponds to the payoff function of the optimal stopping and control problem. Let $A = \tilde{A} \times [0,\infty)$, $(\tilde{\alpha}, r) \in A$, $\sigma_k^\alpha = \sigma_k^{\tilde{\alpha}}$, $b_k^\alpha = b_k^{\tilde{\alpha}}$,

$$\begin{aligned} f^\alpha(t,x) &= f^{(\tilde{\alpha},r)}(t,x) = \tilde{f}^{\tilde{\alpha}}(t,x) + rg(x),\\ c^\alpha(t,x) &= c^{(\tilde{\alpha},r)}(t,x) = \tilde{c}^\alpha(t,x) + r \quad \text{and}\\ m^\alpha &= m^{(\tilde{\alpha},r)} = \frac{1}{1+r},\end{aligned}$$

where here and throughout this section we assume that $\sigma_k^{\tilde{\alpha}}$, $b_k^{\tilde{\alpha}}$, $\tilde{c}^{\tilde{\alpha}}$ and $\tilde{f}^{\tilde{\alpha}}$ satisfy Assumptions 4.1.1, 4.1.2 and 4.1.3. We take v to be the payoff function given by

$$v(s,x) = \sup_{\alpha\in\mathfrak{A}} v^\alpha(s,x), \tag{4.7.1}$$

$$v^\alpha(s,x) = \mathbb{E}_{s,x}^\alpha\left[\int_0^{T-s} f^{\alpha_t}(s+t,x_t)e^{-\varphi_t}dt + g(x_{T-s})e^{-\varphi_{T-s}}\right] \tag{4.7.2}$$

$$= \mathbb{E}_{s,x}^\alpha\left[\int_0^{T-s}\left(\tilde{f}^{\tilde{\alpha}_t}(s+t,x_t) + r_t g(x_t)\right)e^{-\tilde{\varphi}_t}dt + g(x_{T-s})e^{-\tilde{\varphi}_{T-s}}\right], \tag{4.7.3}$$

where

$$\varphi_t = \int_0^t c^{\alpha_u}(s+u,x_u)du = \int_0^t r_u + \tilde{c}^{\tilde{\alpha}}(s+u,x_u)du = \tilde{\varphi}_t.$$

By $v_{\tau,h}$ we denote the solution to

$$\sup_{\alpha\in A} m^{\alpha}\left(\delta_{\tau}\, u + \mathrm{L}_h^{\alpha}\, u + f^{\alpha}\right) = 0 \quad \text{on} \quad Q, \tag{4.7.4}$$

with the boundary condition

$$u = g \quad \text{on} \quad \bar{\mathcal{M}}_T \setminus Q, \tag{4.7.5}$$

which is easily seen to be equivalent to

$$\sup_{\alpha\in A, r\geq 0}\left(\frac{1}{1+r}(\delta_{\tau}\, u + \mathrm{L}_h^{\tilde{\alpha}}\, u + \tilde{f}^{\tilde{\alpha}}) + \frac{r}{1+r}(g-u)\right) = 0 \quad \text{on} \quad Q,$$
$$u = g \quad \text{on} \quad \bar{\mathcal{M}}_T \setminus Q,$$

where

$$\mathrm{L}_h^{\tilde{\alpha}}\, u = \sum_k \sigma_k^{\tilde{\alpha}}\, \Delta_{h_k,\ell_k}\, u + \sum_k b_k^{\tilde{\alpha}}\, \delta_{h_k,\ell_k}\, u - \tilde{c}^{\tilde{\alpha}} u.$$

The method of shaking the coefficients first introduced in [22] and [23] will be used. Recall that $H_T = [0,T)\times\mathbb{R}^d$. Notice that (4.7.4)-(4.7.5) can be considered on H_T instead of $\mathcal{M}_T$.

Let S be nonempty a subset of B_1, the unit ball centered at the origin, in R^d. Let Λ be a nonempty subset of $(-1,0)$. Fix $\varepsilon > 0$. Let $v_{\tau,h}^{\varepsilon}$ be the unique solution of

$$\sup_{\alpha\in A, y\in S, r\in\Lambda} m^{\alpha}\Big[\,\delta_{\tau}\, u(t,x) + \mathrm{L}_h^{\alpha}(t+\varepsilon^2 r, x+\varepsilon y)u(t,x)$$
$$+ f^{\alpha}(t+\varepsilon^2 r, x+\varepsilon y)\Big] = 0 \quad \text{on} \quad H_T, \tag{4.7.6}$$
$$u(T,x) = \sup_{y\in S} g(x+\varepsilon y), \quad \text{for} \quad x\in\mathbb{R}^d.$$

Assumption 4.2.1 is satisfied by (4.7.6) and so the solution exists, is unique and has all the other properties proved in section 4.2. Assume, from now on that (4.3.13) holds.

Lemma 4.7.1. *Let Assumption 4.2.1 and 4.1.3 hold. Then there is a constant N, such that*

$$|v_{\tau,h}^{\varepsilon} - v_{\tau,h}| \leq N e^{NT} T\varepsilon.$$

Proof. Let the space of controls be $A\times S\times\Lambda$. Let $\hat{g}(x) = \sup_{y\in S} g(x+\varepsilon y)$,

$$(\hat{\sigma}_k^{\alpha}, \hat{b}_k^{\alpha}, \hat{c}^{\alpha}, \hat{f}^{\alpha})(t,x) := (\sigma_k^{\alpha}, b_k^{\alpha}, c^{\alpha}, f^{\alpha})(t+\varepsilon^2 r, x+\varepsilon y).$$

Then

$$\sup_{\mathcal{M}_T, A\times S\times\lambda, k} \left(|\sigma_k^\alpha - \hat{\sigma}_k^\alpha| + |b_k^\alpha - \hat{b}_k^\alpha| + |c^\alpha - \hat{c}^\alpha| + m^\alpha|f^\alpha - \hat{f}^\alpha| + |\hat{g} - g|\right) \le 10K\varepsilon.$$

Apply Theorem 4.3.8 to get the conclusion. □

Corollary 4.7.2. *Let Assumptions 4.1.1, 4.2.1 and 4.1.3 be satisfied by $\sigma_k^{\tilde\alpha}$, $b_k^{\tilde\alpha}$, $\tilde c^{\tilde\alpha}$ and $\tilde f^{\tilde\alpha}$. Then there is a constant N such that for any $s, t \in [0, T]$ and $x \in \mathbb{R}^d$,*

$$|v_{\tau,h}(t,x) - v_{\tau,h}(s,x)| \le NTe^{NT}|t-s|^{1/2}. \tag{4.7.7}$$

Proof. Say $s - t = n\tau + \gamma$ for some $n = 0, 1, \ldots$ and $\gamma \in [0, \tau)$. Then by Lemma 4.5.3 and Corollary 4.3.6

$$|v_{\tau,h}(t,x) - v_{\tau,h}(s,x)| \le N(n\tau)^{1/2} + |v_{\tau,h}(n\tau,x) - v_{\tau,h}(s,x)|.$$

Hence we only need to show that for $\gamma \in [0, \tau)$,

$$|v_{\tau,h}(t,x) - v_{\tau,h}(t+\gamma,x)| \le N\gamma^{1/2}.$$

Assume for now that $\tau \le T$. Let

$$[\hat\sigma_k^\alpha, \hat b_k^\alpha, \hat c^\alpha, \hat f^\alpha](t,x) := [\sigma_k^\alpha, b_k^\alpha, c^\alpha, f^\alpha](t+\gamma, x).$$

Then if we solve (4.1.5). with $\hat\sigma_k^\alpha, \hat b_k^\alpha, \hat c^\alpha$ and $\hat f^\alpha$ in place of $\sigma_k^\alpha, b_k^\alpha, c^\alpha$ and f^α respectively, denoting the solution $\hat v$, then we see that by uniqueness $v_{\tau,h}(t,x) = \hat v(t-\gamma, x)$. Furthermore using Assumption 4.1.3 and Theorem 4.3.8 we get

$$|v_{\tau,h}(t,x) - v_{\tau,h}(t+\gamma,x)| = |v_{\tau,h}(t,x) - \hat v(t,x)| \le K\gamma^{1/2} \quad \text{on} \quad \mathcal{M}_T.$$

Finally if $\tau > T$ then we note that $\mathcal{M}_T = \bar{\mathcal{M}}_T \cap \{t = 0\}$. Furthermore $\hat v(0,x) = v_{\tau,h}(\gamma, x)$ and $\hat v(T,x) = g(x)$ satisfies (4.2.1)-(4.2.2) corresponding to $\hat\sigma_k^\alpha, \hat b_k^\alpha, \hat c^\alpha$ and $\hat f^\alpha$ in place of $\sigma_k^\alpha, b_k^\alpha, c^\alpha$ and f^α. By Theorem 4.3.8,

$$|v_{\tau,h}(0,x) - v_{\tau,h}(\gamma,x)| = |v_{\tau,h}(0,x) - \hat v(0,x)| \le N\gamma^{1/2}.$$

□

We will also introduce shaking to the payoff function of the optimal control problem (2.3.4). Let Assumption 2.3.1 and 4.1.1 hold. Consider the separable metric space

$$C = A \times \{(\tau, \xi) \in (-1, 0) \times B_1\},$$

with the metric which comes from taking the sum of the metric for A and the metrics which are induced by natural norms on $(-1, 0)$ and B_1. Extend all the functions σ, β, f, c for negative t by $\sigma^\gamma(t, x) = \sigma^\gamma(0, x)$ etc. For a fixed $\varepsilon \in (0, 1)$, for $\gamma = (\alpha, \tau, \xi)$ let

$$\sigma^\gamma(t, x) = \sigma^\alpha(t + \varepsilon^2\tau, x + \varepsilon\xi) \tag{4.7.8}$$

and similarly for β, c and f. Let $x_t^{\gamma,s,x}$ be the solution of

$$x_t^{s,x} = x + \int_0^t \sigma^\gamma(s + u, x_u) dw_u + \int_0^t \beta^\gamma(s + u, x_u) du. \tag{4.7.9}$$

Let $C_n := A_n \times \{(\tau, \xi) \in (-1, 0) \times B_1\}$, $C = \bigcup_{n\in\mathbb{N}} C_n$. Let $\mathfrak{C}_n$ be the spaces of admissible control processes defined analogously to $\mathfrak{A}_n$ and $\mathfrak{C}$ defined analogously to $\mathfrak{A}$. Let

$$\begin{aligned} w^\gamma(s, x) &= \mathbb{E}^\gamma_{s,x}\left[\int_0^{T-s} f^{\gamma_t}(t, x_t) e^{-\varphi_t} dt + g(x_{T-s}) e^{-\varphi_{T-s}}\right], &(4.7.10)\\ w_n(s, x) &= \sup_{\gamma\in\mathfrak{C}_n} w^\gamma(s, x), &(4.7.11)\\ w(s, x) &= \sup_{\gamma\in\mathfrak{C}} w^\gamma(s, x). &(4.7.12)\end{aligned}$$

Lemma 4.7.3. *If w^γ and w are defined by (4.7.10) and (4.7.12), respectively, then*

$$|w^\gamma - v^\alpha| \le N e^{NT} \varepsilon \quad \textit{and} \quad |w - v| \le N e^{NT}\varepsilon \quad \textit{on} \quad H_T. \tag{4.7.13}$$

Proof. Let $x^{\gamma,s,x}$ and $x^{\alpha,s,x}$ denote the solutions to (4.7.9) and (2.3.2) respectively. By Theorem 2.2.3

$$\begin{aligned}\mathbb{E} \sup_{t\le T-s} |x_t^{\gamma,s,x} - x_t^{\alpha,s,x}|^2 \le & N e^{NT} \sup\big[|\sigma^\alpha(t + \varepsilon^2\tau, x + \varepsilon\xi) - \sigma^\alpha(t, x)|^2 \\ & + |\beta^\alpha(t + \varepsilon^2\tau, x + \varepsilon\xi) - \beta^\alpha(t, x)|^2\big].\end{aligned} \tag{4.7.14}$$

Hence

$$\mathbb{E} \sup_{t\le T-s} |x_t^{\gamma,s,x} - x_t^{\alpha,s,x}| \le N e^{NT}\varepsilon. \tag{4.7.15}$$

With this in mind the reader could use the same technique used in proving the second part of

Lemma 4.4.2 in order to get (4.7.13). □

The following Lemma is the same as Theorem 2.1 in [23], inequality (2.1). The statement and proof are included for the convenience of the reader.

Lemma 4.7.4. *Let $\zeta \in C_0^\infty((-1,0) \times B_1)$ be non-negative with unit integral. Let $\varepsilon > 0$. Let $\zeta_\varepsilon(t,x) := \varepsilon^{-d-2}\zeta(t/\varepsilon^2, x/\varepsilon)$. Let w_n be defined by* (4.7.11)*. Then for any $n \in \mathbb{N}$ the function $u_n = w_n * \zeta_\varepsilon$ satisfies*

$$\frac{\partial}{\partial t}u_n + \mathrm{L}^\alpha u_n + f^\alpha \leq 0 \quad on \quad H_T, \tag{4.7.16}$$

for all $\alpha \in A_n$.

Proof. First, we need to assume that σ and β are twice continuously differentiable in x for each $t \in [0,T]$, with the derivatives bounded. Then Theorem 4.1.5 of [19] applies and hence for any smooth function η

$$\int_{H_T} \Big(- w_n \frac{\partial}{\partial t}\eta + w_n \mathrm{L}^{\gamma *} \eta + f^\gamma \eta \Big) dxdt \leq 0,$$

where

$$\mathrm{L}^{\gamma *} \eta = \sum_{i,j=1}^{d} [a^{ij}(\gamma,t,x)\eta]_{x^i x^j} - \sum_{i=1}^{d} [b^i(\gamma,t,x)\eta]_{x^i} - c^\gamma(t,x)\eta.$$

That is

$$\frac{\partial}{\partial t}w_n(t,x) + \mathrm{L}^\alpha(t+\varepsilon\tau, x+\varepsilon^2\xi)w_n(t,x) + f^\alpha(t+\varepsilon\tau, x+\varepsilon^2\xi) \leq 0,$$

holds in the sense of generalized functions on $(0,T)\times\mathbb{R}^d$ for each $(\alpha,\tau,\xi) \in A\times(-1,0)\times B_1$. We can argue that the same inequality holds for generalized functions on $(0,T)\times\mathbb{R}^d\times(-1,0)\times B_1$. Then

$$\frac{\partial}{\partial t}w_n(t-\varepsilon\tau, x-\varepsilon^2\xi) + \mathrm{L}^\alpha(t,x)w_n(t-\varepsilon\tau, x-\varepsilon^2\xi) + f^\alpha(t,x) \leq 0,$$

holds in the sense of generalized functions on $(-1,0) \times B_1$ for each $\alpha \in A$ and any $(t,x) \in [0,T] \times \mathbb{R}^d$. Hence in particular using ζ as the test function we obtain (4.7.16) in the particular case of smooth σ and β.

We can now approximate general σ and β using smooth functions σ_m, β_m. Then due to Theorem 3.1.13 of [19], $w_{n,m}$ converges to w_n uniformly on any bounded subset of $[0,T]\times\mathbb{R}^d$, as $m \to \infty$. □

Theorem 4.7.5. *For any $\varepsilon \in (0,1]$ there exists u in $C^\infty([0,T]\times\mathbb{R}^d)$ such that*

$$\sup_{\alpha\in A}(u_t + \mathrm{L}^\alpha u + f^\alpha) \le 0 \quad on \quad H_T \tag{4.7.17}$$

$$|u-v| \le Ne^{NT}\varepsilon \quad on \quad H_T, \tag{4.7.18}$$

$$\begin{aligned} |\,\mathrm{D}_t^2 u|_{0,[0,T]\times\mathbb{R}^d} + |\,\mathrm{D}_x^4 u|_{0,[0,T]\times\mathbb{R}^d} &\le Ne^{NT}\varepsilon^{-3} \quad on \quad H_T, \\ |\,\mathrm{D}_x^2 u|_{0,[0,T]\times\mathbb{R}^d} &\le Ne^{NT}\varepsilon^{-1} \quad on \quad H_T. \end{aligned} \tag{4.7.19}$$

Proof. Let $\zeta \in C_0^\infty((-1,0)\times B_1)$ be non-negative with unit integral. Let

$$\zeta_\varepsilon(t,x) := \varepsilon^{-d-2}\zeta(t/\varepsilon^2, x/\varepsilon).$$

Due to Lemma 4.7.4, $u_n = w_n * \zeta_\varepsilon$ satisfies

$$\frac{\partial}{\partial t}u_n + \mathrm{L}^\alpha u_n + f^\alpha \le 0 \quad \text{on} \quad H_T,$$

for all $\alpha \in A_n$. Let w be defined by (4.7.12) and let $u := w * \zeta_\varepsilon$. By Lemma (4.4.2) the functions w, w_n are bounded in absolute value by a constant independent of n. Then by Lebesgue Theorem

$$\begin{aligned} \lim_{n\to\infty}\frac{\partial}{\partial t}u_n(t,x) &= \lim_{n\to\infty}\frac{\partial}{\partial t}\int_{(-1,0)\times B_1}\zeta_\varepsilon(t-s,x-y)w_n(s,y)dsdy \\ &= \int_{(-1,0)\times B_1}\lim_{n\to\infty} w_n(s,y)\frac{\partial}{\partial t}\zeta_\varepsilon(t-s,x-y)dsdy = \frac{\partial}{\partial t}u(t,x). \end{aligned}$$

Hence $\lim_{n\to\infty}\frac{\partial}{\partial t}u_n = \frac{\partial}{\partial t}u$. Similarly $\lim_{n\to\infty}\mathrm{L}^\alpha u_n = \mathrm{L}^\alpha u$ on H_T, for any $\alpha \in A$. For each fixed $\alpha \in A$, let $n\to\infty$ in (4.7.16). Then for any $\alpha \in A$

$$\frac{\partial}{\partial t}u + \mathrm{L}^\alpha u + f^\alpha \le 0 \quad \text{on} \quad H_T,$$

which shows (4.7.17). By Lemma 4.4.2 and Corollary 4.5.2

$$|w(t,x) - w(s,y)| \le Ne^{NT}(|t-s|^{1/2} + |x-y|).$$

Then (4.7.19) follows from known properties of convolutions. Now we want to show (4.7.18).

Since $u = w * \zeta_\varepsilon$, $|u(s,x) - w(s,x)|$ is estimated by the right hand side of (4.7.18). So we need only estimate $|w(s,x) - v(s,x)|$, for $(s,x) \in H_T$. To this end we use the fact that difference of supremums is less than the supremum of a difference to see that we need only estimate

$$\begin{aligned}\mathbb{E}_{s,x} \int_0^{T-s} &|f^{\gamma_t}(s+t, x_t^\gamma) e^{-\varphi_t^\gamma} - f^{\alpha_t}(s+t, x_t^\alpha) e^{-\varphi_t^\alpha}| dt,\\ &+ \mathbb{E}_{s,x} |g(x_{T-s}^\gamma) e^{-\varphi_{T-s}^\gamma} - g(x_{T-s}^\alpha) e^{-\varphi_{T-s}^\alpha}| =: I + J,\end{aligned}$$

where $\gamma_t = (\alpha_t, \tau_t, \xi_t)$ for τ_t, ξ_t progressively measurable processes taking values in $(-1,0)$ and B_1 respectively. Clearly

$$\begin{aligned}J &\le K\Big[(2T+1)\varepsilon \mathbb{E}_{s,x} \sup_{t \le T-s} |x_t^\gamma - x_t^\alpha|\Big],\\ I &\le \mathbb{E}_{s,x} \int_0^{T-s} |f^{\gamma_t}(s+t, x_t^\gamma)| |e^{-\varphi_t^\gamma} - e^{-\varphi_t^\alpha}| dt\\ &+ \mathbb{E}_{s,x} \int_0^{T-s} |f^{\gamma_t}(s+t, x_t^\gamma) - f^{\alpha_t}(s+t, x_t^\alpha)| e^{-\varphi_t^\alpha} dt =: I_1 + I_2.\end{aligned}$$

Like in the proof of Lemma 4.4.2 (using Assumptions 4.2.1, 4.1.3 and the definition of c^γ), we get

$$\begin{aligned}I_1 &\le \mathbb{E}_{s,x} \int_0^{T-s} K^2 (m^{\alpha_t})^{-1} t \left[2\varepsilon + \sup_{u \le t} |x_u^\gamma - x_u^\alpha|\right] e^{-\min(\varphi_t^\gamma, \varphi_t^\alpha)} dt\\ &\le K^3 e^T \Big[2\varepsilon + \mathbb{E}_{s,x} \sup_{t \le T-s} |x_t^\gamma - x_t^\alpha|\Big].\end{aligned}$$

By Assumption 4.2.1 and the definition of f^γ

$$\begin{aligned}I_2 &= \mathbb{E}_{s,x} \int_0^{T-s} |f^{\alpha_t}(s+t+\varepsilon^2\tau, x_t^\gamma + \varepsilon\xi) - f^{\alpha_t}(s+t, x_t^{\alpha_t})| e^{-\varphi_t^\alpha} dt\\ &\le \mathbb{E}_{s,x} \int_0^{T-s} |K(m^{\alpha_t})^{-1}| \, [2\varepsilon + |x_t^\gamma - x_t^\alpha|] \, e^{-\varphi_t^\alpha} dt \le N\Big[\varepsilon + \mathbb{E}_{s,x} \sup_{t \le T-s} |x_t^\gamma - x_t^\alpha|\Big].\end{aligned}$$

By Theorem 2.2.3

$$\mathbb{E}_{s,x} \sup_{u \le T-s} |x_u^\gamma - x_u^\alpha| \le N e^{NT} \varepsilon,$$

where x^α is a solution of (2.3.2), while x^γ is a solution of (4.7.9). Hence, noting that the estimate for $I + J$ is independent of α, τ and ξ, we get (4.7.18). □

To prove Theorem 4.1.4 we follow closely the proof of Theorem 2.2 of [24].

Lemma 4.7.6. *Let Assumptions 2.3.1, 4.1.1, 4.2.1 and 4.1.3 hold. Then*

$$v \le v_{\tau,h} + N_T(\tau^{1/4} + h^{1/2}). \tag{4.7.20}$$

Proof. Recall that for $\varepsilon > 0$, $v^{\varepsilon}_{\tau,h}$ is defined as the unique bounded solution to (4.7.6) and by Lemma 4.7.1

$$v^{\varepsilon}_{\tau,h} \le v_{\tau,h} + N\varepsilon.$$

Assume, without loss of generality, that $h \le 1$. Let $\xi \ge 0$ be a $C^{\infty}_0([0,T] \times \mathbb{R}^d)$ function with a unit integral and support in $(-1,0) \times B_1$. For any function w defined on $(-\infty, T) \times \mathbb{R}^d$, for which it makes sense, let

$$w^{(\varepsilon)}(t,x) = \int_{H_T} w(t - \varepsilon^2 r, x - \varepsilon y)\xi(r,y)drdy.$$

If the function $w(t,x)$ is not defined for negative t then extend it for $t < 0$ by defining $w(t,x) = w(0,x)$. For any $\alpha \in A$ and for all $r \in \Lambda$, $y \in S$

$$\delta_\tau\, v^{\varepsilon}_{\tau,h}(t - \varepsilon^2 r, x - \varepsilon y) + \mathrm{L}^{\alpha}_h(t,x) v^{\varepsilon}_{\tau,h}(t - \varepsilon^2 r, x - \varepsilon y) + f^{\alpha}(t,x) \le 0 \quad \text{on} \quad H_T.$$

Multiply this by $\xi(r,y)$ and integrate over H_T with respect to r and y. Then

$$\delta_\tau\, v^{\varepsilon(\varepsilon)}_{\tau,h} + \mathrm{L}^{\alpha}_h\, v^{\varepsilon(\varepsilon)}_{\tau,h} + f^{\alpha} \le 0 \quad \text{on} \quad H_T.$$

Let $\varepsilon = (\tau + h^2)^{1/4}$. Use Taylor's Theorem, to get that on $H_{T-2\varepsilon^2}$:

$$\begin{aligned} |\,\delta_\tau\, v^{\varepsilon(\varepsilon)}_{\tau,h} - \mathrm{D}_t\, v^{\varepsilon(\varepsilon)}_{\tau,h}| &\le N\tau |\,\mathrm{D}^2_t\, v^{\varepsilon(\varepsilon)}_{\tau,h}|_{0,H_{T-2\varepsilon^2}} =: M_1 \\ |\,\mathrm{L}^{\alpha}_h\, v^{\varepsilon(\varepsilon)}_{\tau,h} - \mathrm{L}^{\alpha}\, v^{\varepsilon(\varepsilon)}_{\tau,h}| &\le N h^2 |\,\mathrm{D}^4_x\, v^{\varepsilon(\varepsilon)}_{\tau,h}|_{0,H_{T-2\varepsilon^2}} \\ &\quad + h |\,\mathrm{D}^2_x\, v^{\varepsilon(\varepsilon)}_{\tau,h}|_{0,H_{T-2\varepsilon^2}} =: M_2. \end{aligned}$$

Hence with $M := M_1 + M_2$, for any $\alpha \in A$ on $H_{T-2\varepsilon^2}$

$$[\mathrm{D}_t + \mathrm{L}^{\alpha}]\,(v^{\varepsilon(\varepsilon)}_{\tau,h} + (T-t)M) + f^{\alpha} \le 0.$$

By Lemma 4.4.3

$$v \le v^{\varepsilon(\varepsilon)}_{\tau,h} + 2(T-t)M + \sup_{\partial H_{T-2\varepsilon^2}} (v - v^{\varepsilon(\varepsilon)}_{\tau,h})_+.$$

By Hölder continuity in time of v and also $v^{\epsilon(\epsilon)}_{\tau,h}$:

$$\sup_{\partial H_{T-2\varepsilon^2}} (v - v^{\varepsilon(\varepsilon)}_{\tau,h})_+ \le \sup_{\partial H_{T-2\varepsilon^2}} (|v-g| + |v^{\varepsilon(\varepsilon)}_{\tau,h} - g|) \le N\varepsilon.$$

By standard properties of convolutions:

$$\begin{aligned}|\,\mathrm{D}^2_t\, v^{\varepsilon(\varepsilon)}_{\tau,h}|_{0,H_{T-2\varepsilon^2}} + |\,\mathrm{D}^4_x\, v^{\varepsilon(\varepsilon)}_{\tau,h}|_{0,H_{T-2\varepsilon^2}} + |\,\mathrm{D}^2_x\, v^{\varepsilon(\varepsilon)}_{\tau,h}|_{0,H_{T-2\varepsilon^2}}& \\ \le N\varepsilon^{-3} + N\varepsilon^{-3} + N\varepsilon^{-1}.&\end{aligned}$$

Hence

$$v \le v^{\varepsilon(\varepsilon)}_{\tau,h} + 4N\varepsilon + N\tau\varepsilon^{-3} + Nh\varepsilon^{-1} + Nh^2\varepsilon^{-3} \le v^{\varepsilon(\varepsilon)}_{\tau,h} + N(\tau + h^2)^{1/4}.$$

Recall that $|v^{\varepsilon}_{\tau,h} - v_{\tau,h}| \le N\varepsilon$ and clearly $|v^{\varepsilon(\varepsilon)}_{\tau,h} - v^{\varepsilon}_{\tau,h}| \le N\varepsilon$. Hence

$$v \le v_{\tau,h} + N(\tau^{1/4} + h^2) \quad \text{on} \quad H_T.$$

□

Lemma 4.7.7. *Let Assumptions 2.3.1, 4.1.1, 4.2.1 and 4.1.3 hold. Then*

$$v_{\tau,h} \le v + N_T(\tau^{1/4} + h^{1/2}) \quad \textit{on} \quad H_T. \tag{4.7.21}$$

Proof. Let $\varepsilon = (\tau + h^2)^{1/4}$. On $(T-\varepsilon^2, T]$ the estimate is trivial consequence of the Hölder continuity in time of both v and $v_{\tau,h}$. Let $S := T - \varepsilon^2$. It remains to prove the estimate on H_S. By Theorem 4.7.5, there is a smooth function u defined on $[0,T] \times \mathbb{R}^d$ satisfying

$$\sup_{\alpha \in A} (u_t + \mathrm{L}^\alpha u + f^\alpha) \le 0 \quad \text{on} \quad H_S. \tag{4.7.22}$$

Apply Taylor's Theorem to see that in $H_{T-\tau}$

$$\begin{aligned} |\,\delta_\tau\, u - \mathrm{D}_t\, u| &\le N\tau |\,\mathrm{D}^2_t\, u|_{0,H_T} =: M_1 \\ |\,\mathrm{L}^\alpha_h\, u - \mathrm{L}^\alpha\, u| &\le Nh^2 |\,\mathrm{D}^4_x\, u|_{0,H_T} + h |\,\mathrm{D}^2_x\, u|_{0,H_T} =: M_2 \end{aligned} \tag{4.7.23}$$

Let $M = M_1 + M_2$. By Theorem 4.7.5:

$$M_1 + M_2 \le N_T \tau\varepsilon^{-3} + N_T h^2 \varepsilon^{-3} + N_T h\varepsilon^{-1} \le N_T(\tau + h^2)^{1/4}.$$

Since $\tau < 1$, $\varepsilon^2 > \tau$, we have $H_S \subset H_{T-\tau}$. By (4.7.22) and (4.7.23)

$$\sup_\alpha m^\alpha(\delta_\tau v_{\tau,h} + \mathrm{L}_h^\alpha v_{\tau,h} + f^\alpha) = 0 \geq \sup_\alpha(\delta_\tau u + \mathrm{L}_h^\alpha u + f^\alpha - M) \quad \text{on} \quad H_S.$$

Let $u' = \sup_{H_T \setminus H_S}(v_{\tau,h} - u)_+ + u$. The aim now is to apply Lemma 4.2.6 to $v_{\tau,h}$ and u'. On $H_T \setminus H_S$, $u' \geq v_{\tau,h}$. By Lemma 4.2.6

$$v_{\tau,h} \leq u + TM + \sup_{H_T \setminus H_S}(v_{\tau,h} - u)_+. \tag{4.7.24}$$

By Hölder continuity in time of $v_{\tau,h}$ and v and by Theorem 4.7.5

$$\sup_{H_T \setminus H_S}(v_{\tau,h} - u)_+ \leq \sup_{H_T \setminus H_S}|v_{\tau,h} - g| + \sup_{H_T \setminus H_S}|g - v| + \sup_{H_T \setminus H_S}|v - u| \leq N_T\varepsilon.$$

By Theorem 4.7.5, $|v - u| \leq N_T\varepsilon$. Hence

$$\begin{aligned} v_{\tau,h} &\leq v + N_T(\varepsilon + \tau^{1/2} + (\tau + h^2)^{1/4}) \leq v + N_T(\tau + h^2)^{1/4} \\ &\leq v + N_T(\tau^{1/4} + h^{1/2}) \quad \text{on} \quad H_S. \end{aligned}$$

□

Chapter 5

Practical considerations

The aim of this chapter is to try to complement results from Chapter 4 from the point of view of applicability. The reader noticed that the rate of convergence result proved in Chapter 4 applies to finite difference schemes on grids over the whole space $[0, T] \times \mathbb{R}^d$. In section 5.1 we consider what happens if we wish to restrict the problem to a domain Q with finitely many elements. This is done in two steps. First we estimate the error arising from replacing f and g in (2.3.5) by $\bar{f}$ and $\bar{g}$ which are equal to f and g for x in some ball centered at the origin, but which have support in some $[0, T] \times B_{R_1}$. The second step is to estimate the error arising from solving (4.2.1)-(4.2.2) on $Q = ([0, T] \times B_R) \cap \mathcal{M}_T$ instead of the whole $\mathcal{M}_T$ and with f^α and g replaced by $\bar{f}$ and $\bar{g}$ respectively. In section 5.2 we will prove a lemma telling us when the elliptic operator satisfies the Assumption 4.1.1. We also give specific examples. In section 5.3 we show how all the previous pieces of our work fit together when applied to calculating prices of American put options. Finally in section 5.4 we give an example when we're able to discretize the space of control parameters easily.

5.1 Approximations in cylindrical domains

Recall from Section 4.1 that

$$\bar{\mathcal{M}}_T := \{(t, x) \in [0, T] \times \mathbb{R}^d : (t, x) = ((j\tau) \wedge T, h(i_1 \ell_1 + \cdots + i_{d_1} \ell_{d_1})),$$
$$j \in \{0\} \cup \mathbb{N}, i_k \in \mathbb{Z}, k = \pm 1, \ldots, \pm d_1\}.$$

and

$$\mathcal{M}_T := \bar{\mathcal{M}}_T \cap \big([0, T) \times \mathbb{R}^d\big).$$

The following lemma tells us that in the particular case when f^α and g are zero outside a ball of radius R_1 then the difference between $v_{\tau,h}$ which is the solution of

$$\sup_{\alpha\in A} m^\alpha\left(\delta_\tau u + \mathrm{L}_h^\alpha u + f^\alpha\right) = 0 \quad \text{on} \quad Q, \tag{5.1.1}$$

$$u = g \quad \text{on} \quad \bar{\mathcal{M}}_T \setminus Q, \tag{5.1.2}$$

on $Q = \mathcal{M}_T$ and u which is the solution of the same finite difference problem but with $Q = ([0,T) \times B_R) \cap \mathcal{M}_T$ decreases exponentially as we increase R.

Lemma 5.1.1. *Let Assumption 4.1.1 and 4.2.1 be satisfied. Let there be R_1 such that*

$$f^\alpha(t,x) = g(x) = 0 \quad \textit{for} \quad |x| \geq R_1.$$

Let $v_{\tau,h}$ be a function satisfying (5.1.1) *on $\mathcal{M}_T$ and such that $v_{\tau,h} = g$ on $\bar{\mathcal{M}}_T \setminus \mathcal{M}_T$. For $R \geq R_1$ consider*

$$Q = ([0,T) \times B_R) \cap \mathcal{M}_T$$

and a function u satisfying (5.1.1) *on Q and $u = g$ on $\bar{\mathcal{M}}_T \setminus Q$. Then for some constants $\gamma \in (0,1)$ and N depending only on K, d, d_1 and T,*

$$|v_{\tau,h} - u| \leq N e^{\gamma(R_1 - R)} \quad \textit{on} \quad \bar{\mathcal{M}}_T.$$

Proof. From Lemma 4.2.9 we know that for $|x| \geq R$, $|v_{\tau,h}| \leq N e^{\gamma(R_1-R)}$. Since $v_{\tau,h}$ satisfies (4.2.1) on $\mathcal{M}_T$ and u satisfies (4.2.1) on Q,

$$\begin{aligned}
&\sup_{\alpha\in A} m^\alpha(\delta_\tau u + L_h^\alpha u + f^\alpha) = 0 \\
&= \sup_{\alpha\in A} m^\alpha(\delta_\tau v_{\tau,h} + L_h^\alpha v_{\tau,h} + f^\alpha) \\
&\leq \sup_{\alpha\in A} m^\alpha(\delta_\tau v_{\tau,h} + L_h^\alpha v_{\tau,h} + c^\alpha N e^{\gamma(R_1-R)} + f^\alpha) \\
&= \sup_{\alpha\in A} m^\alpha\left(\delta_\tau(v_{\tau,h} - N e^{\gamma(R_1-R)}) + L_h^\alpha(v_{\tau,h} - N e^{\gamma(R_1-R)}) + f^\alpha\right) \quad \text{on} \quad Q.
\end{aligned}$$

For $(t,x) \in \bar{\mathcal{M}}_T \setminus Q$ either $(t,x) \in [0,T) \times B_R^c \cap \mathcal{M}_T$ and so

$$v_{\tau,h} - N e^{\gamma(R_1-R)} \leq 0 = g = u$$

or $t = T$ and so

$$v_{\tau,h} - Ne^{\gamma(R_1 - R)} = g - Ne^{\gamma(R_1-R)} \leq g = u.$$

Hence by Lemma 4.2.6 applied to $v_{\tau,h} - Ne^{\gamma(R_1-R)}$ and u on Q

$$v_{\tau,h} \leq u + Ne^{\gamma(R_1-R)} \quad \text{on} \quad \bar{\mathcal{M}}_T.$$

Similar argument for $v_{\tau,h} + Ne^{\gamma(R_1-R)}$ gives

$$v_{\tau,h} + Ne^{\gamma(R_1-R)} \geq u \quad \text{on} \quad \bar{\mathcal{M}}_T.$$

□

Now recall that, as always, we let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a probability space with a right-continuous filtration, such that $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets. Let $(w_t, \mathcal{F}_t)$ be a d' dimensional Wiener martingale. Let A be a separable metric space. For every $t \in [0,T]$, $x \in \mathbb{R}^d$ and $\alpha \in A$ we are given a $d \times d'$ dimensional matrix $\sigma^\alpha(t,x)$, a d dimensional vector $\beta^\alpha(t,x)$ and real numbers $c^\alpha(t,x)$, $f^\alpha(t,x)$ and $g(x)$. Furthermore recall that the payoff function for the optimal control problem is, by definition (see (2.3.4)-(2.3.5)),

$$\begin{aligned} v^\alpha(t,x) &:= \mathbb{E}^\alpha_{t,x}\left(\int_0^{T-t} f^{\alpha_s}(s+t, x_s)e^{-\varphi_s}ds + g(x_{T-t})e^{-\varphi_{T-t}}\right), \\ v^\alpha(t,x) &:= \sup_{n\in\mathbb{N}}\sup_{\alpha\in\mathfrak{A}_n} v^\alpha(t,x). \end{aligned} \tag{5.1.3}$$

Lemma 5.1.2. *Assume that 2.3.1 and on* $[0,T]\times\mathbb{R}^d$

$$|f^\alpha(t,x)| \leq \frac{K}{m^\alpha}(1+|x|)^m, \quad |g(x)| \leq K(1+|x|)^m, \quad m^\alpha(1+c^\alpha) \geq \frac{1}{K} > 0. \tag{5.1.4}$$

Let $R_1 > 0$ *be given. For* $r \geq 0$ *let*

$$\xi_{R_1}(r) = \mathbf{1}_{[0,R_1]}(r) + (R_1 + 1 - r)\mathbf{1}_{(R_1,R_1+1]}(r).$$

Define $\bar{g}(x) := \xi_{R_1}(|x|)g(x)$, $\bar{f}^\alpha(t,x) := \xi_{R_1}(|x|)f^\alpha(t,x)$. *Then functions* $\bar{f}$ *and* $\bar{g}$ *are Lips-*

chitz continuous in x and zero outside the ball B_{R_1+1} and furthermore for

$$\bar{v}^{\alpha}(t,x) := \mathbb{E}^{\alpha}_{t,x}\left(\int_0^{T-t} \bar{f}^{\alpha_s}(s+t,x_s)e^{-\varphi_s}ds + \bar{g}(x_{T-t})e^{-\varphi_{T-t}}\right),$$
$$\bar{v}(t,x) := \sup_{n\in\mathbb{N}}\sup_{\alpha\in\mathfrak{A}_n}\bar{v}^{\alpha}(t,x),$$

we have

$$|v^{\alpha}(t,x) - \bar{v}^{\alpha}(t,x)| \le |Nt^q e^{NT}\mathbb{P}^{\alpha}_{t,x}\{\tau_{R_1} < T-t\},$$

where N depends on K, d, q and m and

$$\tau^{\alpha,t,x}_{R_1} := \inf_s(|x^{\alpha,t,x}_u| \ge R_1).$$

Proof. Notice that $\bar{g}$ and $\bar{f}$ are both Lipschitz continuous in x with the constant $2K$, provided that f and g are Lipschitz continuous in x with a constant K. Clearly

$$\begin{aligned}
&|v^{\alpha}(t,x) - \bar{v}^{\alpha}(t,x)| \\
&\le \mathbb{E}^{\alpha}_{t,x}\big[|\int_{\tau_{R_1}}^{T-t}|f^{\alpha_s}(s+t,x_s)|e^{-\varphi_s}ds| + \mathbf{1}_{\{\tau_{R_1}<T-t\}}|g(x_{T-t})|\big] \\
&\le \mathbb{E}^{\alpha}_{t,x}\mathbf{1}_{\{\tau_{R_1}<T-t\}}\left[\int_0^{T-t}|f^{\alpha_s}(s+t,x_s)|e^{-\varphi_s}ds + |g(x_{T-t})|\right].
\end{aligned}$$

Then, due to (5.1.4)

$$\begin{aligned}
|v^{\alpha}(t,x) - \bar{v}^{\alpha}(t,x)| \le & \mathbb{E}^{\alpha}_{t,x}\mathbf{1}_{\{\tau_{R_1}<T-t\}}K\sup_{0\le s\le T-t}(1+|x_s|)^m \\
&\times\left[\int_0^{T-t}\frac{1}{m^{\alpha_s}}e^{-\int_0^s\frac{\delta}{m^{\alpha_u}}-1du}ds + 1\right] \\
\le & e^T 2K^2\mathbb{E}^{\alpha}_{t,x}\mathbf{1}_{\{\tau_{R_1}<T-t\}}\sup_{0\le s\le T-t}(1+|x_s|)^m \\
\le & e^T 2K^2\mathbb{P}^{\alpha}_{t,x}\{\tau_{R_1}<T-t\}\mathbb{E}^{\alpha}_{t,x}\sup_{0\le s\le T-t}(1+|x_s|)^{2m},
\end{aligned}$$

where we have used Cauchy's inequality. By Assumption 2.3.1 all $t\in[0,T]$ and $x\in\mathbb{R}^d$,

$$|\beta(t,x)| \le K(1+|x|) \quad \text{and} \quad |\sigma(t,x)| \le K(1+|x|),$$

and so we can apply Corollary 2.2.4 to conclude that,

$$\mathbb{E}^{\alpha}_{t,x}\sup_{0\le s\le T-t}(1+|x_s|)^{2m} \le Nt^q e^{NT}.$$

Hence

$$|v^\alpha(t,x) - \bar{v}^\alpha(t,x)| \le N t^q e^{NT} \mathbb{P}^\alpha_{t,x}\{\tau_{R_1} < T - t\},$$

where N depends on K, d, q and m. □

The next step is to estimate, for any $R > 0$, $\mathbb{P}^\alpha_{t,x}\{\tau_R < T - t\}$ by some constant. In order to do that we will use the following lemma.

Lemma 5.1.3. *Let Assumption 2.2.1 be satisfied and let $(x_t)_{t\in[0,T]}$ be the solution of*

$$x_t = \xi + \int_0^t \beta(s, x_s)ds + \int_0^t \sigma(s, x_s)dw_s.$$

Let $a(t,x) = \frac{1}{2}\sigma\sigma^*$. If for all $t \in [0,T]$

$$2x\beta(t,x) + |\sigma(t,x)|^2 + (a(t,x)x, x) \le K(1 + |x|^2) \tag{5.1.5}$$

and

$$\mathbb{E}e^{\xi^2} < \infty$$

then there exists $\mu > 0$ such that

$$\mathbb{E} \sup_{t\in[0,T]} e^{\mu x_t^2} \le C(\mu)(1 + \mathbb{E}e^{\xi^2}), \tag{5.1.6}$$

where $C(\mu)$ is a constant.

Proof. We begin by applying Itô formula to the process x_t.

$$dx_t^2 = 2x_t\sigma(t,x_t)dw_t + (2x_t\beta(t,x_t) + |\sigma(t,x_t)|^2)dt.$$

$$de^{x_t^2} = e^{x_t^2}\left[2x_t\sigma(t,x_t)dw_t + (2x_t\beta(t,x_t) + |\sigma(t,x_t)|^2)dt + 2(a(t,x_t)x_t, x_t)dt\right].$$

Let $\psi_t := \exp(y_t)$, where $y_t := e^{-\lambda t}x_t^2$ and $\lambda > 0$ is a constant to be chosen later. Then

$$\begin{aligned} d\psi_t = e^{-\lambda t}\psi_t\big[&2x_t\sigma(t,x_t)dw_t + (2x_t\beta(t,x_t) + |\sigma(t,x_t)|^2 \\ &+ e^{-\lambda t}(a(t,x_t)x_t, x_t))dt - \lambda x_t^2 dt\big]. \end{aligned}$$

By our hypothesis (5.1.5)

$$\exp(e^{-\lambda t}x_t^2) \le \exp(\xi^2) + \int_0^t 2e^{-\lambda s}\exp(e^{-\lambda s}x_s^2)x_s\sigma(s,x_s)dw_s$$
$$+ \int_0^t e^{-\lambda s}\exp(e^{-\lambda s}x_s^2)\left(K(1+|x_s|^2) - \lambda x_s^2\right)ds.$$

Thus, for any stopping time $\tau \le T$ and any $A \in \mathcal{F}_0$

$$\mathbb{E}\mathbf{1}_A \exp(e^{-\lambda\tau}x_\tau^2) \le \mathbb{E}\mathbf{1}_A\exp(\xi^2) + \mathbb{E}\int_0^\tau e^{-\lambda s}\exp(e^{-\lambda s}x_s^2)\left(K(1+|x_s|^2) - \lambda x_s^2\right)ds$$
$$=: \mathbb{E}\mathbf{1}_A\exp(\xi^2) + I.$$

Now

$$I \le \mathbb{E}\mathbf{1}_A\left[\int_0^\tau \mathbf{1}_{\{x_s^2<1\}}e^{-\lambda s}\exp(e^{-\lambda s}x_s^2)\left(K(1+|x_s|^2) - \lambda x_s^2\right)ds\right.$$
$$\left.+\int_0^\tau \mathbf{1}_{\{x_s^2\ge 1\}}e^{-\lambda s}\exp(e^{-\lambda s}x_s^2)\left(K(1+|x_s|^2) - \lambda x_s^2\right)ds\right].$$

Choose $\lambda > 0$ large such that, $2K \le \lambda$. Then

$$I \le \mathbb{E}\mathbf{1}_A\int_0^\tau \mathbf{1}_{\{x_s^2<1\}}e^{-\lambda s}\exp(e^{-\lambda s}x_s^2)\left(K(1+|x_s|^2) - \lambda x_s^2\right)ds$$
$$\le 2K\mathbb{E}\mathbf{1}_A\int_0^\tau e^{-\lambda s}\exp(e^{-\lambda s})ds \le \frac{2K}{\lambda} \le \mathbb{E}\mathbf{1}_A.$$

Hence for any stopping time $\tau \le T$

$$\mathbb{E}\mathbf{1}_A\exp(e^{-\lambda\tau}x_\tau^2) \le \mathbb{E}\mathbf{1}_A(1+\exp(\xi^2)).$$

Then for any $\delta \in (0,1)$

$$\mathbb{E}\sup_{t\in[0,T]}\exp(\delta e^{-\lambda t}x_t^2) \le \frac{2-\delta}{1-\delta}(1+\mathbb{E}\exp(\delta\xi^2)),$$

by Lemma 3.2 from [13]. □

Corollary 5.1.4. *Under the assumption of Lemma 5.1.3 there exists* $\mu > 0$ *such that*

$$\mathbb{P}(\tau_R \le T) \le e^{-\mu R^2}C(\mu)(1+\mathbb{E}e^{\xi^2}),$$

where

$$\tau_R := \inf_s \{|x_t| \geq R\}.$$

Proof. We will use Lemma 5.1.3 and Markov inequality.

$$\mathbb{P}(\tau_R < T) = \mathbb{P}\left(\sup_{t\in[0,T]} |x_t| \geq R\right) = \mathbb{P}\left(\sup_{t\in[0,T]} \exp(\mu x_t^2) \geq \exp(\mu R^2)\right)$$
$$\leq e^{-\mu R^2}\mathbb{E}\sup_{t\in[0,T]} e^{x_t^2} \leq e^{-\mu R^2} C(\mu)(1+\mathbb{E}e^{\xi^2}).$$

□

Finally we are ready to state the main result of this section. Recall that $H_T = [0,T) \times \mathbb{R}^d$.

Theorem 5.1.5. *Let $R > R_1 > 0$ be given. Let Assumptions 2.3.1, 4.1.1, 4.1.2 and 4.1.3 be satisfied. Let*

$$\xi_{R_1}(r) := \mathbf{1}_{[0,R_1]}(r) + (R_1 + 1 - r)\mathbf{1}_{(R_1,R_1+1]}(r).$$

and $g_{R_1}(x) := \xi_{R_1}(|x|)g(x)$, $f^\alpha_{R_1}(t,x) := \xi_{R_1}(|x|)f^\alpha(t,x)$. Let $Q_R = ([0,T) \times B_R) \cap \mathcal{M}_T$. Let $u^{R,R_1}_{\tau,h}$ be the unique solution to

$$\sup_{\alpha\in A, r\geq 0}\left(\tfrac{1}{1+r}\left(\delta_\tau\, u^{R,R_1}_{\tau,h} + \mathrm{L}^\alpha_h\, u^{R,R_1}_{\tau,h} + f^\alpha_{R_1}\right) + \tfrac{r}{1+r}\left(g - u^{R,R_1}_{\tau,h}\right)\right) = 0 \text{ on } Q_R, \quad (5.1.7)$$

$$u^{R,R_1}_{\tau,h} = g_{R_1} \quad \text{on} \quad \bar{\mathcal{M}}_T \setminus Q_R. \quad (5.1.8)$$

Let v be the payoff function for the optimal stopping and control problem (4.0.1)-(4.0.2). *Then on* $[0,T) \times B_{R_1}$

$$|v - u^{R,R_1}_{\tau,h}| \leq N(e^{-\mu {R_1}^2} + \tau^{1/4} + h^{1/2} + e^{\gamma(R_1 - R)})$$

for some $\mu > 0$ and $\gamma \in (0,1)$, where N is a constant independent of τ, h, R and R_1.

Proof. Due to Lemma 5.1.2 and Corollary 5.1.4 there is $\mu > 0$ such that

$$|v - \bar{v}| \leq Ne^{-\mu R_1^2}, \quad \text{on} \quad [0,T) \times B_{R_1}, \quad (5.1.9)$$

with N independent of R_1, where $\bar{v}$ is the payoff function to the optimal control problem (given in Lemma 5.1.2) with f^α and g replaced by $f^\alpha_{R_1}$ and g_{R_1} respectively. Let $v^{R_1}_{\tau,h}$ denote the solution to (5.1.7)-(5.1.8) with Q_R replaced by $\mathcal{M}_T$. This means that $v^{R_1}_{\tau,h}$ is a solution to the discretized problem on the whole space, but with f^α and g vanishing outside ball of radius

R_1. Due to Theorem 4.1.4

$$|\bar{v} - v_{\tau,h}^{R_1}| \leq N(\tau^{1/4} + h^{1/2}) \quad \text{on} \quad H_T, \tag{5.1.10}$$

with N independent of τ and h. Due to Lemma 5.1.1 there exists $\gamma \in (0,1)$ such that

$$|v_{\tau,h}^{R_1} - u_{\tau,h}^{R,R_1}| \leq Ne^{R_1 - R} \quad \text{on} \quad [0,T) \times B_R, \tag{5.1.11}$$

where N is a constant independent of R and R_1. Then, since

$$|v - u_{\tau,h}^{R,R_1}| \leq |v - \bar{v}| + |\bar{v} - v_{\tau,h}^{R_1}| + |v_{\tau,h}^{R_1} - u_{\tau,h}^{R,R_1}|,$$

we use inequalities (5.1.9), (5.1.10) and (5.1.11) to obtain the conclusion of the theorem. □

5.2 Constructing the approximating schemes

Recall that to obtain the rate of convergence result we have proved in Chapter 4 we need the elliptic operator (4.1.2), associated with the controlled diffusion process to look like (4.1.3). To be precise recall that we need to assume that there exist a natural number d_1, vectors $\ell_k \in \mathbb{R}^d$ and functions

$$\sigma_k^\alpha : [0,T) \times \mathbb{R}^d \to \mathbb{R}, \quad b_k^\alpha : [0,T) \times \mathbb{R}^d \to \mathbb{R}, \quad \forall k = \pm 1, \ldots, d_1$$

such that $\ell_k = -\ell_{-k}$, $|\ell_k| \leq K$, $\sigma_k^\alpha = \sigma_{-k}^\alpha, b_k^\alpha \geq 0$ for $k = \pm 1, \ldots, d_1$ and L^α given by (4.1.2) can in fact be written as

$$\mathrm{L}^\alpha u = \sum_k \sigma_k^\alpha \mathrm{D}_{\ell_k}^2 u + b_k^\alpha \mathrm{D}_{\ell_k} u - c^\alpha u. \tag{5.2.1}$$

Example 5.2.1. First we look at the one dimensional case, i.e. $x \in \mathbb{R}$. Then if

$$\mathrm{L}^\alpha u = \frac{1}{2}(\sigma^\alpha)^2 u_{xx} + \beta^\alpha u_x - c^\alpha u,$$

then with $\ell_1 = 1$, $\ell_{-1} = -1$, $\sigma_1^\alpha := \sigma_{-1}^\alpha := \frac{1}{4}(\sigma^\alpha)^2$ and $b_1^\alpha = (\beta^\alpha)^+$, $b_{-1}^\alpha = (\beta^\alpha)^-$, it is easy to verify that

$$\mathrm{L}^\alpha u = \sum_{k=\pm 1} \sigma_k^\alpha \mathrm{D}_{\ell_k}^2 u + \sum_{k=\pm 1} b_k^\alpha \mathrm{D}_{\ell_k} u - c^\alpha u,$$

since $u_{xx} = \mathrm{D}_1^2 u = \mathrm{D}_{-1}^2 u$ and $u_x = \mathrm{D}_1 u = -\mathrm{D}_{-1} u$.

We have just seen that in the one dimensional case, finding the required form of the operator L^α is straightforward. In fact the same approach works when even for $d > 1$, as long as the coefficients $a_{ij}^\alpha = 0$ for $i \neq j$.

Example 5.2.2. Let $x \in \mathbb{R}^d$ and $(a^\alpha) = (1/2)\sigma^\alpha(\sigma^\alpha)^T$ be such that $a_{ij}^\alpha = 0$ when $i \neq j$. Then

$$\mathrm{L}^\alpha = \sum_i a_{i,i}^\alpha u_{x^i x^i} + \sum_i \beta_i^\alpha u_{x^i} - c^\alpha u.$$

Let $\ell_k = e_k$ where e_k is the k-th vector in the standard basis for $\mathbb{R}^d$. Let $\ell_{-k} = -e_k$. Let $\sigma_k^\alpha := \sigma_{-k}^\alpha := a_{kk}^\alpha$ and let $b_k^\alpha = (\beta_k^\alpha)^+$, $b_{-k}^\alpha = (\beta_k^\alpha)^-$. It is easy to see that

$$\mathrm{L}^\alpha u = \sum_{k=\pm 1,\ldots,\pm d} \sigma_k^\alpha \mathrm{D}_{\ell_k}^2 u + \sum_{k=\pm 1,\ldots,\pm d} b_k^\alpha \mathrm{D}_{\ell_k} u - c^\alpha u,$$

Since $u_{x^i x^i} = \mathrm{D}_{\ell_i}^2 u = \mathrm{D}_{-\ell_i}^2 u$ and $u_{x^i} = \mathrm{D}_{\ell_i} u = -\mathrm{D}_{-\ell_i} u$.

The following lemma and its corollary (from [7]) gives a sufficient condition for this to be the case.

Lemma 5.2.3. *If* $\bar{a}^\alpha := \operatorname{tr} a^\alpha > 0$ *and if there is a finite difference operator*

$$\mathrm{L}_h^\alpha u(x) = \sum_{y \in B} p_h^\alpha(y) u(x + hy), \tag{5.2.2}$$

for some $B \subset \mathbb{R}^d$ *finite, such that:*

1. *For all* $t \in [0, T]$, $x \in \mathbb{R}^d$, *for all* $\alpha \in A$ *and any smooth function* u
$$\sum_{i,j} a_{ij}^\alpha(t,x) u_{x^i x^j} u(x) = \lim_{h \downarrow 0} \mathrm{L}_h^\alpha u(x).$$
2. *The set* B *contains* 0 *and* $p_h(0) < 0$.
3. *For* $y \in B$, $y \neq 0$, $p_h(y) \geq 0$.

Then there exist $\ell_k \in \mathbb{R}^d$ *and* $\sigma_k^\alpha(t,x) \in \mathbb{R}$, $k = \pm 1, \ldots, \pm m$, *for some* $m \in \mathbb{N}$, *such that*

$$\mathrm{L}^\alpha u = \sum_k \sigma_k^\alpha \mathrm{D}_{\ell_k}^2 u.$$

Proof. We will omit α from the notation as it is fixed throughout the proof and hence it does not play a role. Let $B = \{0, l_1, \dots, l_m\}$. Let k be an index always running through $1, \dots, m$. Consider a function

$$v(x) := u(x) + u(-x)$$

and notice that $v_{x^i x^j} = 2u_{x^i x^j}$. Furthermore

$$\mathrm{L}_h\, v(0) = 2p_h(0)u(0) + \sum_k p_h(l_k)(u(hl_k) + u(-hl_k)).$$

and

$$0 = \sum_{i,j} a^{ij} 1_{x^i x^j} = \lim_{h \downarrow 0} \left(p_h(0) + \sum_k p_h(l_k) \right)$$

and so

$$\begin{aligned} \mathrm{L}\, v(0) &= \lim_{h\downarrow 0} L_h v(0) = \lim_{h \downarrow 0} \left(\sum_k p_h(l_k) \left(u(hl_k) - 2u(0) + u(-hl_k) \right) \right) \\ &= \lim_{h \downarrow 0} \sum_k p_h(l_k) h^2\, \Delta_{h,l_k}\, u(0). \end{aligned}$$

We will now show that

$$\bar{a} = \lim_{h \downarrow 0} \sum_k p_h(l_k) h^2 |l_k|^2.$$

Indeed,

$$\bar{a} = \sum_i a^{ii} = \frac{1}{2} \sum_{i,j} a^{ij} |x|^2_{x^i x^j}.$$

Then with $v(x) = \frac{1}{2}(|x|^2 + |-x|^2) = |x|^2$ and $x = 0$,

$$\bar{a} = \frac{1}{2} \lim_{h \downarrow 0} \sum_k p_h(l_k) h^2\, \Delta_{h,l_k}\, |x|^2 = \lim_{h \downarrow 0} \sum_k p_h(l_k) h^2 |l_k|^2.$$

Let

$$\bar{p}_h := \left(\sum_k 2p_h(l_k) h^2 |l_k|^2 \right)^{-1}$$

and notice that $\lim_{h\downarrow 0} \frac{1}{\bar{p}_h} = \bar{a}$. Now

$$h^2 p_h(l_k) \bar{p}_h = \frac{p_h(l_k)}{\sum_k p_h(l_k)|l_k|^2} \leq |l_k|^{-2},$$

is a sequence bounded independent of h and so it must have a convergent subsequence along some h_n. Let

$$\bar{a}_k := \lim_{n\to\infty} h_n^2 p_{h_n}(l_k)\bar{p}_{h_n}.$$

Then

$$\sum_{i,j} a^{ij} u_{x^i x^j}(0) = \lim_{n\to\infty} \sum_k h_n^2 p_{h_n}(l_k) \frac{\bar{p}_{h_n}}{\bar{p}_{h_n}} \Delta_{h_n, l_k} u(0) = \bar{a} \sum_k \bar{a}_k \, \mathrm{D}^2_{l_k} u(0).$$

Notice that $\mathrm{D}_{l_k} = \mathrm{D}_{-l_k}$. Let $\ell_k = l_k$, $\ell_{-k} = -l_k$, $\sigma_k = \frac{\bar{a}}{2} a_k$ and $\sigma_{-k} = \frac{\bar{a}}{2} a_k$. Then

$$\bar{a} \sum_{k=1,\dots,m} \bar{a}_k \, \mathrm{D}^2_{l_k} u(0) = \sum_{k=\pm 1,\dots,\pm m} \sigma_k \, \mathrm{D}^2_{\ell_k} u(0).$$

□

Corollary 5.2.4. *Let*

$$\mathrm{L}^\alpha u = \sum_{i,j} a^\alpha_{i,j} u_{x^i x^j} + \sum_i \beta^\alpha_i u_{x^i} - c^\alpha u.$$

If the conditions of Lemma 5.2.3 are satisfied then there exist ℓ_k, σ^α_k and b^α_k such that

$$\mathrm{L}^\alpha u = \sum_k \sigma^\alpha_k \, \mathrm{D}^2_{\ell_k} u + b^\alpha_k \, \mathrm{D}_{\ell_k} u - c^\alpha u.$$

Proof. Clearly the term $c^\alpha u$ does not really play a role. Again, $\alpha \in A$ will be fixed and we will omit it from the notation. To find a representation for

$$\sum_{i=1}^d b^i u_{x^i} \quad \text{as} \quad \sum_k b_k \, \mathrm{D}_{\ell_k} u$$

we can start be choosing $\ell_k = e_k$, $\ell_{-k} = -e_k$ (where e_i is the i-th vector of the standard basis of $\mathbb{R}^d$) for $k = 1, \dots, d$. Then let $b_k = (b^i)^+$ and $b_{-k} = (b^i)^-$ to see that

$$\sum_k b_k \, \mathrm{D}_{\ell_k} u = \sum_k b_k \sum_{i=1}^d u_{x^i} \ell^i_k = \sum_{i=1}^d u_{x^i} \sum_k b_k \ell^i_k = \sum_{i=1}^d u_{x^i} (b^i)_+ - (b^i)_-$$
$$= \sum_{i=1}^d b^i u_{x^i}.$$

So to express L in the form (5.2.1) we only need to really consider

$$\sum_{i,j} a_{ij} u_{x^i x^j}. \tag{5.2.3}$$

We now apply Lemma 5.2.3 to complete the proof. $\square$

Corollary 5.2.4 provides a general condition under which the differential operator

$$\mathrm{L}^\alpha u = \sum_{i,j} a^\alpha_{i,j} u_{x^i x^j} + \sum_i \beta^\alpha_i u_{x^i} - c^\alpha u.$$

can be written in the form (5.2.1). That is, the Corollary provides conditions under which there there exist vectors $\ell_k = -\ell_{-k}$, functions $\sigma^\alpha_k(t,x) = \sigma^\alpha_k(t,x) \geq 0$ and $b^\alpha_k(t,x) \geq 0$, such that

$$\mathrm{L}^\alpha u = \sum_k \sigma^\alpha_k \mathrm{D}^2_{\ell_k} u + b^\alpha_k \mathrm{D}_{\ell_k} u - c^\alpha u.$$

We will now give an example that uses Corollary 5.2.4 to find the desired form of L^α.

Example 5.2.5. Consider

$$\mathrm{L}^\alpha u(t,x) = \sum_{i,j} a^\alpha_{i,j} u_{x^i x^j} + \sum_i \beta^\alpha_i u_{x^i}.$$

We would like to find find an example of approximating finite difference operator

$$\mathrm{L}^\alpha_h u(x) = \sum_{y \in B} p^\alpha_h(y) u(x + hy),$$

satisfying the conditions of Lemma 5.2.3. We are going to use ideas from Chapter 5 of [27]. Let e_i denote the i-th element in the standard basis for $\mathbb{R}^d$. The following standard finite difference approximations for $u_{x^i x^j}(t,x)$ will be used: if $i = j$ then

$$\Delta_{ii} u(t,x) := \frac{1}{h^2}\big(u(t, x + he_i) - 2u(t,x) + u(t, x - he_i)\big).$$

If $i \neq j$ and $a^\alpha_{ij}(t,x) \geq 0$ then

$$\begin{aligned}\Delta^+_{ij} u(t,x) :=& \frac{1}{2h^2}\Big(2u(t,x) + u(t, x + h(e_i + e_j)) + u(t, x - h(e_i + e_j)\Big) \\ &- \frac{1}{2h^2}\Big(u(t, x + he_i) + u(t, x - he_i) + u(t, x + he_j) + u(t, x - he_j)\Big).\end{aligned}$$

If $a^\alpha_{ij}(t,x) < 0$ then

$$\begin{aligned}\Delta^-_{ij} u(t,x) :=& -\frac{1}{2h^2}\Big(2u(t,x) + u(t, x + he_i - he_j) + u(t, x - he_i + he_j)\Big) \\ &+ \frac{1}{2h^2}\Big((u(t, x + he_i) + u(t, x - he_i) + u(t, x + he_j) + u(t, x - he_j)\Big).\end{aligned}$$

For $u_{x^i}(t,x)$ we will use

$$\delta_i^+ u(t,x) := h^{-1}\big(u(t,x+he_i) - u(t,x)\big), \quad \text{if} \quad \beta_i^\alpha(t,x) \geq 0$$

and

$$\delta_i^- u(t,x) := h^{-1}\big(u(t,x) - u(t,x-he_i)\big), \quad \text{if} \quad \beta_i^\alpha(t,x) < 0.$$

For any $(t,x) \in H_T$, let

$$\mathrm{L}_h^\alpha u = \sum_i \Delta_{ii} a_{ii}^\alpha u + \sum_{i,j:j\neq i} (a_{ij}^\alpha)^+ \Delta_{ij}^+ u - (a_{ij}^\alpha)^- \Delta_{ij}^- u + \sum_i (\beta_i^\alpha)^+ \delta_i^+ u - (\beta_i^\alpha)^- \delta_i^- u.$$

Let $B = \{0, e_i, e_i + e_j, e_i - e_j : i,j = 1,\ldots,d, i \neq j\}$ and let

$$p_h(0) = \frac{1}{h^2} \sum_{i,j:j\neq i} |a_{ij}^\alpha(t,x)| + \sum_i -\frac{a_{ii}^\alpha(t,x)}{h^2} - \frac{(\beta_i^\alpha(t,x))^+}{h} - \frac{(\beta_i^\alpha(t,x))^-}{h},$$

$$p_h(e_i) = \frac{1}{h^2}\Big(a_{ii}^\alpha(t,x) - \sum_{j:j\neq i} |a_{ij}^\alpha(t,x)| + h(\beta_i^\alpha(t,x))^\pm\Big),$$

$$p_h(e_i + e_j) = p_h(-e_i - e_j) = 2h^{-2}(a_{ij}^\alpha(t,x))^+, \quad \text{for} \quad i \neq j,$$

$$p_h(e_i - e_j) = p_h(-e_i + e_j) = 2h^{-2}(a_{ij}^\alpha(t,x))^-, \quad \text{for} \quad i \neq j.$$

Then

$$\mathrm{L}_h^\alpha u(t,x) = \sum_{y\in B} p_h(y) u(t, x+hy),$$

notice that $p_h(y)$ all depend on $\alpha \in A$ and $(t,x) \in H_T$. Furthermore it is easy to check that for any u infinitely differentiable in x, $\mathrm{L}_h^\alpha u(t,x)$ converges to $\mathrm{L}^\alpha u(t,x)$ for any $\alpha \in A$ and $(t,x) \in H_T$. In other words (5.2.2) holds. Finally if we assume that

$$a_{ii}^\alpha(t,x) - \sum_{j:j\neq i} |a_{ij}^\alpha(t,x)| > 0$$

then $p_h(0) < 0$ and for $0 \neq y \in B$, $p_h(y) \geq 0$. Thus the conditions of Lemma 5.2.3 are satisfied.

5.3 Application to American put option price

We present an example application which puts together the randomized stopping result (Chapter 3) and the rate of convergence result (Chapter 4) together with the note about restricting the approximations to domains with finitely many elements (section 5.1). Our example of choice is the price of American put option in a Black-Scholes model. For description of what American options are from a finance point of view see for example Chapter 1 of [15]. For the mathematical theory of option pricing using the Black-Scholes model see for example Chapter 4 in [28]. It is well known (see again e.g. [28] section 5.3), that the value of the American put options can be computed using the finite difference method. However, to our best knowledge, there are no results giving the rate of convergence of such approximations.

Our model will be a slight generalization of the basic Black-Scholes model in the sense that we consider d risky assets and that we allow the volatility and interest rate to change (deterministically) in time. In our model we will have one risk-less security

$$dB_t = \rho(t)B_t dt$$

and d different risky assets $S_t = (S_t^1, \dots S_t^d)$:

$$dS_t^i = S_t^i\Big(\sum_{j=1}^{d} \sigma_{ij}(t)dw_t^j + \mu_i(t)dt\Big), \quad i = 1, \dots, d, \tag{5.3.1}$$

where w_t is a d-dimensional Wiener process. It is well known that if there is a unique measure $\mathbb{Q}$ such that the process $e^{-\int_0^t \rho(u)du} S_t$ is a $\mathcal{F}_t$ martingale, then the American put option price corresponds to the following optimal stopping problem:

$$w(t,x) = \sup_{\tau \in \mathfrak{T}[0,T-t]} \mathbb{E}_{t,x}^{\mathbb{Q}}\left(e^{-\int_0^\tau \rho(u)du}\Big[K - \sum_{i=1}^{d} \lambda_i S_\tau^i\Big]_+\right), \tag{5.3.2}$$

where K and λ_i for $(i = 1, \dots, d)$ are positive real numbers. Recall that for any real number z we use $[z]_+$ to denote the positive part of z. The option strike is K. The numbers λ_i determine the "weights" of the different stocks in the portfolio.

We can now state assumptions under which the measure $\mathbb{Q}$ exists and is unique and under which we can approximate the error arising in finite difference approximation calculation of w.

Assumption 5.3.1. Let $a = (1/2)\sigma\sigma^T$. For each i, j, for all $t, s \in [0,T]$,

$$|a_{ij}(t) - a_{ij}(s)| \leq K|t-s|^{1/2}, \quad |\mu_i(t) - \mu_i(s)| \leq K|t-s|^{1/2},$$

$$|\rho(t) - \rho(s)| \leq K|t-s|^{1/2}, \quad |\rho(t)| \leq K, \quad |a_{ij}(t)| \leq K.$$

For every $t \in [0,T]$, it is assumed that

$$a_{ii}(t) - \sum_{j:j\neq i} |a_{ij}(t)| > 0. \tag{5.3.3}$$

Furthermore assume that there exists a unique vector γ_t such that for all t

$$\sum_{j=1}^{d} \sigma_{ij}(t)\gamma_t^j = \mu_t^i - \rho(t), \quad \text{for} \quad i = 1,\dots,d. \tag{5.3.4}$$

and

$$\int_0^T |\gamma_t|^2 dt < \infty,$$

Remark 5.3.2. Notice that assumption on unique existence of γ allows finding the measure $\mathbb{Q}$ uniquely with the use of Girsanov Theorem. Furthermore, this assumption is satisfied for example if σ is invertible with a bounded inverse. The boundedness and continuity assumptions on a_{ij}, μ_i and ρ are sufficient for (5.3.1) to have a solution, for (5.3.2) to be well defined and so the continuity and boundedness the assumptions in section 4.1 hold for $\ln S_t^i$. The assumption (5.3.3) on a_{ij} is sufficient to see that L^α can be written in form (5.2.1). See Example 5.2.5.

Lemma 5.3.3. *If Assumption 5.3.1 is satisfied then there is a unique measure $\mathbb{Q}$ such that*

$$e^{-\int_0^t \rho(u)du} S_t$$

is a $\mathcal{F}_t$ martingale under $\mathbb{Q}$ and $(\tilde{w}_t)_{t\in[0,T]}$ given by

$$\tilde{w}_t^i := w_t^i + \int_0^t \gamma_s^i ds \tag{5.3.5}$$

is a $\mathbb{Q}$ Wiener martingale. Furthermore.

$$dS_t^i = S_t^i\Big(\sum_{j=1}^{d} \sigma_{ij}(t) d\tilde{w}_t^j + \rho(t)dt\Big).$$

Proof. This is a straightforward application of Girsanov Theorem. Let $\mathbb{Q}$ be given by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\Big(\sum_{i=1}^{d} \int_0^T \gamma_t^i dw_t^i - \frac{1}{2}\int_0^T |\gamma_t|^2 dt\Big).$$

then $\tilde{w}$ is a $\mathbb{Q}$ Wiener process. Furthermore the discounted stock price process

$$e^{-\int_0^t \rho(u)du} S_t$$

satisfies

$$d\tilde{S}_t^i = \tilde{S}_t^i \Big(\sum_{j=1}^d \sigma_{ij}(t) d\tilde{w}_t^j + \big(\mu_t^i - \rho(t) - \sum_{j=1}^d \sigma_{ij}(t)\gamma_t^j\big) dt \Big).$$

Hence due to (5.3.4)

$$d\tilde{S}_t^i = \tilde{S}_t^i \Big(\sum_{j=1}^d \sigma_{ij}(t) d\tilde{w}_t^j \Big).$$

Due to Assumption 5.3.1 $\sigma(t)$ is bounded (deterministic), so it's clearly square integrable and so $\tilde{S}$ is a $\mathbb{Q}$ Wiener martingale. □

We now wish to use results from Section 4.1 and 5.1 to estimate the error arising in finite difference approximation of w. But the reader will notice that S_t has got the drift and diffusion coefficients growing linearly in the spatial variable, which is a case that Theorem 4.1.4 does not cover. However, if one applies Itô formula to $x_t^i = \ln S_t^i$ then

$$dx_t^i = \Big(\rho(t) - \frac{1}{2}\sum_{j=1}^d \sigma_{ij}(t)^2\Big) dt + \sum_{j=1}^d \sigma_{ij}(t) d\tilde{w}_t^j.$$

Let

$$g(x) := \Big[K - \sum_{i=1}^d \lambda_i e^{x_i}\Big]_+ \tag{5.3.6}$$

and notice that $|g(x)| \leq K$. Furthermore

$$\begin{aligned} w(t,x) &= \sup_{\tau \in \mathfrak{T}[0,T-t]} \mathbb{E}_{t,x}^{\mathbb{Q}} (e^{-\int_0^\tau \rho(u)du} g(x_\tau)) \\ &= \sup_{\tau \in \mathfrak{T}[0,T-t]} \mathbb{E}_{t,x}^{\mathbb{Q}} \Bigg(e^{-\int_0^\tau \rho(u)du} \Big[K - \sum_{i=1}^d \lambda_i e^{x_\tau^i}\Big]_+ \Bigg). \end{aligned}$$

So we see that we can in fact consider the American option price w in terms of the process x_t which has drift and diffusion coefficients bounded in space. We now recall some notation from Section 4.1:

$$\begin{aligned} \bar{\mathcal{M}}_T := \{(t,x) \in [0,T] \times \mathbb{R}^d : (t,x) = ((j\tau) \wedge T, h(i_1\ell_1 + \cdots + i_{d_1}\ell_{d_1})), \\ j \in \{0\} \cup \mathbb{N}, i_k \in \mathbb{Z}, k = \pm 1, \ldots, \pm d_1\}. \end{aligned}$$

Further recall that $\tau_T(t) := \tau$ for $t \le T-\tau$ and $\tau_T(t) := T-t$ for $t > T-\tau$ and

$$\begin{aligned} \delta_\tau\, u(t,x) &:= \frac{u(t+\tau_T(t),x)-u(t,x)}{\tau_T(t)}, \\ \delta_{h_k,\ell_k}\, u(t,x) &:= \frac{u(t,x+h_k\ell_k)-u(t,x)}{h_k}, \\ \Delta_{h_k,\ell_k}\, u &:= -\,\delta_{h_k,\ell_k}\,\delta_{h_k,-\ell_k}\, u = \frac{1}{h_k}(\delta_{h_k,\ell_k}\, u + \delta_{h_k,-\ell_k}\, u). \end{aligned}$$

Theorem 5.3.4. *Let w be given by* (5.3.2). *Let $R > R_1 > 0$ be given. Let $Q := ([0,T)\times B_R)\cap \mathcal{M}_T$. Let*

$$\xi_{R_1}(r) = \mathbf{1}_{[0,R_1]}(r) + (R_1+1-r)\mathbf{1}_{(R_1,R_1+1]}(r)$$

and $g_{R_1}(x) := \xi_{R_1}(|x|)g(x)$. We can find ℓ_k, $\sigma_k(t)\ge 0$ and $b_k(t)\ge 0$, $k=\pm1,\dots,\pm d_1$ such that if L^α_h is taken as

$$\mathrm{L}_h\, u := \sum_k \sigma_k\,\Delta_{h_k,\ell_k}\, u + \sum_k b_k\,\delta_{h_k,\ell_k}\, u - \rho u$$

and $u^{R,R_1}_{\tau,h}$ is the solution to

$$\begin{aligned} \delta_\tau\, u^{R,R_1}_{\tau,h} + \mathrm{L}_h\, u^{R,R_1}_{\tau,h} - (u^{R,R_1}_{\tau,h} - g_{R_1}) \le 0, &\quad g_{R_1} - u^{R,R_1}_{\tau,h} \le 0 \quad \textit{on} \quad Q_R, \\ \delta_\tau\, u^{R,R_1}_{\tau,h} + \mathrm{L}_h\, u^{R,R_1}_{\tau,h} - (u^{R,R_1}_{\tau,h} - g_{R_1}) = 0, &\quad g_{R_1} - u^{R,R_1}_{\tau,h} < 0 \quad \textit{on} \quad Q_R, \\ \textit{and} \quad u^{R,R_1}_{\tau,h} = g_{R_1} &\quad \textit{on} \quad \bar{\mathcal{M}}_T \setminus Q_R, \end{aligned} \tag{5.3.7}$$

then

$$|w - u^{R,R_1}_{\tau,h}| \le N(e^{-\mu R_1^2} + \tau^{1/4} + h^{1/2} + e^{\gamma(R_1-R)}) \quad \textit{on} \quad [0,T]\times B_{R_1}.$$

Proof. First, we use the method of randomized stopping to consider w as a payoff function to an optimal control problem (apply Theorem 3.2.1 to w)

$$w(t,x) = \sup_{r_s\in\mathfrak{R}} \mathbb{E}_{t,x}\Big(\int_0^{T-t} r_s g(x_s) e^{-\int_0^s \rho(u)+r_u du} ds + g(x_{T-t}) e^{-\int_0^{T-t}\rho(u)+r_u du}\Big).$$

Since we're assuming that (5.3.3) holds and due to the work we've done in Example 5.2.5 and due to Corollary 5.2.4 we know that there exist the appropriate ℓ_k, $\sigma_k(t)\ge 0$ and $b_k(t)\ge 0$, $k=\pm1,\dots,\pm d_1$ such that for smooth u

$$\mathrm{L}\, u = \sum_k \sigma_k\,\mathrm{D}^2_{\ell_k}\, u + \sum_k b_k\,\mathrm{D}_{\ell_k}\, u - \rho u.$$

Hence Assumption 4.1.1 is satisfied. We will use Corollary 4.1.5 rewrite (5.3.7) as

$$\sup_{r\geq 0}\frac{1}{1+r}\left[\delta_\tau\, u^{R,R_1}_{\tau,h} + \mathrm{L}_h\, u^{R,R_1}_{\tau,h} - r(u^{R,R_1}_{\tau,h} - g_{R_1})\right] = 0 \quad \text{on} \quad Q,$$
$$u^{R,R_1}_{\tau,h} = g_{R_1} \quad \text{on} \quad \bar{\mathcal{M}}_T \setminus Q. \tag{5.3.8}$$

Finally we can check that due to Assumption 5.3.1, all the assumptions needed to apply Theorem 5.1.5 (that is Assumptions 2.3.1, 4.1.1, 4.2.1 and 4.1.3) are satisfied. Hence

$$|w - u^{R,R_1}_{\tau,h}| \leq N(e^{-\mu R_1^2} + \tau^{1/4} + h^{1/2} + e^{\gamma(R_1-R)}) \quad \text{on} \quad [0,T]\times B_{R_1}.$$

□

Remark 5.3.5. It also worth noting that an algorithm computing the exact value of $u^{R,R_1}_{\tau,h}$ given by 5.3.7 is described for example in [28].

5.4 Approximations in the policy space

We briefly look at discretizing the space A. We are not going to present any very interesting result. We simple aim to point out that this is something one still has to consider if one wishes to solve optimal control problems numerically. We consider $A = [0,1]$ but the argument would be the same on any A which is a bounded subset of R^n. All we will do is choose a uniform grid on A and then use Theorem 4.3.8 to estimate the error arising in this discretization.

Example 5.4.1. Let $A = [0,1]$. Let $\alpha \in A$, $t \in [0,T]$ and $x \in \mathbb{R}^d$. Assume that we're given functions $\sigma^\alpha_k(t,x)$, $b^\alpha_k(t,x)$, $c^\alpha(t,x)$, $f^\alpha(t,x)$ and $g(x)$ that satisfy Assumption 4.2.1. Further assume that for $\alpha, \alpha' \in A$

$$|\sigma^\alpha_k - \sigma^{\alpha'}_k| + |b^\alpha_k - b^{\alpha'}_k| + |c^\alpha - c^{\alpha'}| + |f^\alpha| - f^{\alpha'}| \leq K|\alpha - \alpha'| \quad \text{on} \quad [0,T]\times\mathbb{R}^d.$$

The solution to

$$\sup_{\alpha\in A} m^\alpha\left(\delta_\tau v + L^\alpha_h v - c^\alpha v + f^\alpha\right) = 0 \quad \text{on} \quad \mathcal{M}_T,$$
$$v = g \quad \text{on} \quad \bar{\mathcal{M}}_T \setminus \mathcal{M}_T$$

exists and is unique. Let $\tilde{A} = \{x : x = i/n; i = 0,\ldots,n\}$, $C = \tilde{A}\times[0,1/n]$ and $\gamma = (\alpha,\varepsilon) \in C$. Define $m^\gamma = m^{\alpha,\varepsilon} = m^\alpha$, $\sigma^\gamma_k = \sigma^{\alpha,\varepsilon}_k = \sigma^\alpha_k$, $b^\gamma_k = b^{\alpha,\varepsilon}_k = b^\alpha_k$, $c^\gamma = c^{\alpha,\varepsilon} = c^\alpha$ and

$f^\gamma = f^{\alpha,\varepsilon} = f^\alpha$. Let u denote the solution to

$$\sup_{\gamma\in C} m^\gamma \left(\delta_\tau\, u + L_h^\gamma u - c^\gamma u + f^\gamma\right) = 0 \quad \text{on} \quad \mathcal{M}_T,$$

$$u = g \quad \text{on} \quad \bar{\mathcal{M}}_T \setminus \mathcal{M}_T.$$

Define $\hat{m}^\gamma = m^{\alpha+\varepsilon}$, $\hat{\sigma}_k^\gamma = \sigma_k^{\alpha+\varepsilon}$, $\hat{b}_k^\gamma = b_k^{\alpha+\varepsilon}$, $\hat{c}^\gamma = c^{\alpha+\varepsilon}$ and $\hat{f}^\gamma = f^{\alpha+\varepsilon}$. Let $\hat{u}$ denote the solution to

$$\sup_{\gamma\in C} m^\gamma \left(\delta_\tau\, \hat{u} + L_h^\gamma \hat{u} - c^\gamma \hat{u} + f^\gamma\right) = 0 \quad \text{on} \quad Q,$$

$$\hat{u} = g \quad \text{on} \quad \bar{\mathcal{M}}_T \setminus Q.$$

Then, if (4.3.13) holds then by Theorem 4.3.8,

$$|u - \hat{u}| \le N e^{NT} \varepsilon.$$

Due to uniqueness of the solutions for the discrete Bellman PDE, v and $\hat{u}$ coincide. Hence

$$|u - v| \le N e^{NT} \varepsilon.$$

By definition of σ_k^γ, b_k^γ, c^γ and f^γ and by uniqueness again, u coincides with the solution to

$$\sup_{\alpha\in\tilde{A}} m^\alpha \left(\delta_\tau\, u + L_h^\alpha u - c^\alpha u + f^\alpha\right) = 0 \quad \text{on} \quad Q,$$

$$u = g \quad \text{on} \quad \bar{\mathcal{M}}_T \setminus Q.$$

Hence in this particular case, we know what the error, arising from discretization of the space of controls, is.

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