THE DIELECTRIC CONSTANT

THESIS

Submitted by

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**INTRODUCTION**

It is known that if a piece of matter be subject to an external field, an electric moment will be induced within it.

The electric moment per unit volume, or the polarisation, at any point is related to the field at the same point by the relation

\[ P(\mathbf{r}) = \chi \mathbf{E}(\mathbf{r}) \]

where \( \chi \) is the susceptibility of the medium. The dielectric constant \( \varepsilon \) is defined by

\[ \varepsilon = 1 + 4\pi \chi \]

Passing from the macroscopic to the microscopic behaviour of the matter, the electric moment is actually the average of the separate electric moments of the molecules constituting the matter. The electric moment of a single molecule depends on two parameters, namely its polarisability \( \alpha \) and its permanent moment \( \rho \). Thus one should be able to calculate the dielectric constant \( \varepsilon \) in terms of the molecular constants \( \alpha \) and \( \rho \).

Several attempts have been made to relate the dielectric constant to the molecular constants \( \alpha \) and \( \rho \); the simplest of which is the Clausius-Mosotti relation,

\[ \frac{\varepsilon - 1}{\varepsilon + 2} = \frac{4\pi}{3} \frac{N\alpha}{\rho} \]

where \( N \) is the number of molecules per unit volume.
The relation is supposed to hold for non-polar molecules in the case of steady fields and for any molecules in the case of highly alternating fields. This relation which gives the same value $\alpha$ in the liquid state as that in the vapor state, is used for the calculation of the polarisability of molecules.

A simple relation, including the permanent moment $\rho$ as well, is that given by Debye

$$\frac{\alpha-1}{2\pi} = \frac{\alpha}{3N} \left( \alpha + \frac{\beta^2}{jK} \right),$$

where $T$ is the absolute temperature and $\alpha$ Boltzmann's constant. This relation when applied in the vapor state, is used to calculate the permanent moment $\rho$; but when applied in the liquid state, it does not give the same value of $\rho$ as given in the vapor state. For substances having highly polar molecules, the value of the permanent moment calculated in the liquid state is much lower than the actual value, namely that calculated for the substance in the vapor state.

This discrepancy was later explained by Debye as due to strong interactions between the dipoles.

Onsager\(^{(7)}\) has considered a molecule in a spherical cavity, replacing the other molecules by a homogeneous dielectric. In this way he replaced the effect of the interaction with the other molecules by the effect of the image field. His formula gives
values of the dielectric constant which are much lower than the observed ones in the case of highly polar substances.

Kirkwood (5) and then Fröhlich (11) constructed formulae which depend on the knowledge of the relative distribution of any two dipoles, or the probability that two dipoles have a certain configuration.

One object of the present paper is to construct a formula which gives the dielectric constant in terms of the molecular parameters  and . Since this formula will depend on the knowledge of the relative distribution of any two dipoles, it will be a very useful means of checking the accuracy of the distribution function of the dipoles, calculated by other methods.

A knowledge of the distribution function of the dipoles can also be used in calculating an additional average potential energy and an additional specific heat due to dipole interaction; the effect of dipole rotation on the X-ray scattering; the hydration of salts and many other things.
In the first chapter a formula is obtained by considering a big cavity containing many molecules, and thus allowing for the effect of the dipole interactions and the structure of the crystal concerned. This formula, which depends on the knowledge of the distribution of the dipoles reduces to that given by Kirkwood for cubic crystals and neglect of $\alpha$, but for other crystals it has a factor depending on the structure of the crystal.

In the second chapter a method for calculating the distribution function of two dipoles is suggested by considering the two molecules in question as in a cavity (or two cavities), finding the energy and then calculating the probability for such a distribution, using Boltzmann's law.

The calculation of such a distribution function will be carried out for ice in the third chapter, and it will be shown to give a satisfactory value for the dielectric constant.

The effect of thermal vibrations in crystals has not been taken into account. It is by no means clear however that the thermal motion will not materially affect the results so far obtained, and for this reason this effect will be considered in Chapter IV. A
formula applicable to liquids will also be considered in the same chapter.

It is observed experimentally that the dielectric constant decreases very rapidly, almost discontinuously, for polar substances at a lower temperature, usually coincident or a little lower than the melting point of the substance considered. This has been explained (21) as due to a transfer from a state of free rotations of the dipoles to a state where the dipoles are in ordered configurations, that is when a small number, have a certain configuration which is repeated throughout the whole crystal.

It was assumed in the formulae so far obtained that the dipoles are free to rotate, or in a state of disorder. In the last chapter the case of ordered dipoles will be considered, as well as the stability of such ordered configurations.
CHAPTER I.

A NEW FORMULA FOR THE DIELECTRIC CONSTANT.

The Effect of the Dipoles Interactions

Consider an assembly of \( N^{(o)} \) molecules confined within a spherical cavity of volume \( V \), equal to \( \frac{N'}{N} \), where \( N \) is the number of molecules per unit volume. The space outside the cavity will be treated macroscopically as a continuous medium of dielectric constant \( \varepsilon \). If the external field in the medium at a great distance from the cavity be \( E \), then the field inside the cavity will be \( E' \) given by (1.5).

Let the components of the field of the dipole moment of a molecule \( \mathbf{j} \) acting on a molecule \( \mathbf{i} \) be

\[
-\frac{1}{\alpha} \frac{\mathbf{z}^{(1)}}{\mathbf{z}^{(2)}} \mathbf{\lambda}^{(i \rightarrow j)} \mathbf{f}^{(i \rightarrow j)} \quad (s = 1, 2, 3)
\]

(1.26)

where \( \lambda^{(i \rightarrow j)} \) depends on the type of axes taken and will be given in the first appendix at the end of this chapter.

The field components \( f^{(i \rightarrow j)} \) acting on a molecule \( \mathbf{i} \) are
\[ F_s^{(ij)} E'_s - \frac{1}{\alpha} \sum_{j \neq i} \sum_{s'} \lambda_{ss'}^i \mu_{s'}^j \] (1.27)

In this expression the image field, created by the moments of the molecules confined within the cavity, has been omitted. In fact, since we are considering a large number of molecules in the cavity, we shall neglect the effect of the image field of the dipoles on the configuration of the confined molecules.

The components of the electric moment of a molecule \( i \) can be written, following equation (1.11) in the form

\[ \mu_{s}^{(ij)} = \rho_{s}^{(ij)} + \alpha F_{s}^{(ij)} \] (1.28)

Eliminating \( F_s^{(ij)} \) between equations (1.27) and (1.28) we obtain

\[ \mu_s^{(ij)} + \sum_{j \neq i} \sum_{s'} \lambda_{ss'}^{ij} \mu_{s'}^{j'} = \rho_s^{(ij)} + \alpha E'_s \]

or, if we define

\[ \lambda_{ss'}^{ij} = \delta_{ss'} \] (1.29)

then

\[ \sum_{j \neq i} \sum_{s'} \lambda_{ss'}^{ij} \mu_{s'}^{j'} = \rho_s^{(ij)} + \alpha E'_s \] (1.30)

Therefore

\[ \mu_s^{(ij)} = \frac{2}{\delta_{s's}} (\lambda^{-1})_{ss'}^{ij} (\rho_{s'}^{(ij)} + \alpha E'_s) \] (1.31)
where the matrix \( \mathbf{X}^{-1} \) is the reciprocal of the matrix \( \mathbf{X} \). The energy \( W \) of the dipoles in a given configuration will be defined by

\[
-\frac{2W}{\sum_{j=1}^{3} \mu_j} = \sum_{s=1}^{3} \xi_{s} \left[ \mathbf{F}_{s} \cdot \left( \mathbf{p}_{s}^{\prime} - \mathbf{p}_{s}^{\prime} \right) \right]
\]  

(1.32)

where

\[
\mathbf{q}_{s} = \xi_{s} \cdot \xi_{s} \]  

(1.33)

\( \xi_{s} \) (\( s = 1, 2, 3 \)) are unit vectors along the three axes of the crystal. From equations (1.27) and (1.32) one obtains

\[
-\mathbf{W} = \sum_{j=1}^{3} \mu_j q_{j} E_{j} - \frac{1}{2\alpha} \sum_{j=1}^{3} \sum_{k=1}^{3} (q_{j} - q_{k}) \mathbf{p}_{j} \cdot \mathbf{p}_{k} + \text{const.}
\]

(1.34)

or, extending the second summation over \( i = j \),

\[
-\mathbf{W} = \sum_{j=1}^{3} \mu_j q_{j} E_{j} - \frac{1}{2\alpha} \sum_{j=1}^{3} \sum_{k=1}^{3} (q_{j} - q_{k}) \mathbf{p}_{j} \cdot \mathbf{p}_{k} + \text{const.}
\]

(1.34)

Substituting for the value of \( \mu_{s}^{\prime} \) from equation (1.31) we obtain

\[
-\mathbf{W} = \sum_{j=1}^{3} \mu_j q_{j} X_{j}^{\prime} \mathbf{E}_{j} - \frac{1}{2\alpha} \sum_{j=1}^{3} \sum_{k=1}^{3} (q_{j} - q_{k}) \mathbf{p}_{j} \cdot \mathbf{p}_{k}
\]

(1.35)

to the first power in \( \mathbf{E} \).

The probability \( \mathbf{P} \) for the occurrence of a given configuration is given by (1.17). Putting

\[
\mathbf{W} = \mathbf{W}^{(0)} - \mathbf{W}^{(1)}
\]

(1.36)

where \( \mathbf{W}^{(0)} \) is the energy in the absence of external
field and $W^{(1)}$ is linear in $E'$. Then the probability $\rho$ is given by

$$\rho = \rho_0 \left(1 + \frac{W^{(1)}}{kT}\right)$$

(1.37)

to the first power in $E'$, $\rho_0$ being the probability in the absence of an external field.

Substituting for $W^{(1)}$ from equation (1.35) we obtain

$$\rho = \rho_0 \left[1 + \frac{1}{kT} \sum_{\gamma} \langle \rho_{\gamma} \overline{A}_{\gamma \gamma} \rangle E' \right]$$

(1.38)

Because

$$\sum_{\gamma} \langle \rho_{\gamma} \overline{A}_{\gamma \gamma} \rangle = \langle \rho \overline{A} \rangle$$

(1.39)

(independent of $\gamma$)

(See appendix II at the end of this Chapter).

Equation (1.38) becomes

$$\rho = \rho_0 \left[1 + \frac{1}{kT} \sum_{\gamma} \langle \rho_{\gamma} \overline{A}_{\gamma \gamma} \rangle E' \right]$$

(1.40)

So far we have neglected the effect of the image field which is not represented in the expression for the probability of a given configuration specified by equation (1.38). But when we pass on to find the average total electric moment of the confined molecules, namely the average of $\sum_{\gamma} \mu^{(\gamma)}_s$, we shall
not use the expression for $\mu_1^{(i)}$ given by equation (1.31), but instead, we shall use an expression containing an extra term resulting from the image field; this will be shown in the following.

If $\varepsilon^{(i)}_s$ are the components of the image field acting on a molecule (i) which is created by the presence of the molecules inside the cavity, then the total field acting on a molecule $i$ will be given, instead of (1.27), by

$$F_s^{(i)} = E_s^{(i)} + \varepsilon_s^{(i)} - \frac{1}{\alpha} \sum_{s'} \frac{\lambda^{(s')}_s}{s'} \mu_s^{(s')}, \quad (1.41)$$

Eliminating $F_s^{(i)}$ between (1.41) and (1.28) one obtains

$$\sum_{s'} \frac{\lambda^{(s')}_s}{s'} \mu_s^{(s')} = \alpha E_s^{(i)} + \alpha \varepsilon_s^{(i)} + \beta_s^{(i)}, \quad (1.42)$$

and therefore

$$\mu_s^{(i)} = \sum_{s'} \frac{(\lambda^{-1})^{(s')}_s}{s'} (\beta^{(s')}_s + \alpha E_s^{(s')} + \alpha \varepsilon_s^{(s')}) \quad (1.43)$$

This differs from equation (1.31) by the addition of an extra term due to the image field. Summing (1.43) over $i$, one finds

$$\sum_i \mu_i^{(i)} = \sum_{j^{(i)}} (X^{-1})^{(i)}_{s_s^{(i)}} (\beta^{(i)}_s + \alpha E_s^{(i)} + \alpha \varepsilon_s^{(i)}) \quad (1.44)$$

By a limiting procedure $\frac{1}{N^{(i)}} \sum_i \varepsilon_s^{(i)}$ can be shown to be equivalent to the space average of the
image field inside the cavity. This space average of the image field is shown in the third appendix to be equal to \( \frac{L}{N_a^2} \sum \frac{\mu}{d} \) and therefore

\[
\sum \frac{\mu}{d} = \lambda_0 \sum \frac{\mu}{d}.
\]

(1.45)

Using equations (1.45), equation (1.44) becomes

\[
\frac{\lambda_0}{\sum} \frac{\mu}{d} = \lambda_0 \sum \frac{\mu}{d} (\lambda'_{1}, \lambda'_{2}, \lambda'_{3}, \lambda'_{4}, \lambda'_{5}).
\]

that is

\[
\frac{\lambda_0}{\sum} \frac{\mu}{d} = \lambda_0 \sum \frac{\mu}{d} (\lambda'_{1}, \lambda'_{2}, \lambda'_{3}, \lambda'_{4}, \lambda'_{5}).
\]

(1.46)

For a cubic crystal one has the tensor equation

\( \lambda = 1 \)

and therefore equation (1.46) will be similar to equation (1.12) when only one molecule was supposed to be inside the cavity.

From equations (1.40), (1.46) one obtains for the product \( \frac{\lambda_0}{\sum} \frac{\mu}{d} \) the following expression

\[
\frac{\lambda_0}{\sum} \frac{\mu}{d} = \sum \frac{\lambda_0}{\sum} \frac{\mu}{d} \left( \sum \frac{\mu}{d} (\lambda'_{1}, \lambda'_{2}, \lambda'_{3}, \lambda'_{4}, \lambda'_{5}) \right) \frac{1}{k^1} \sum \frac{\mu}{d} \frac{1}{k^2} \sum \frac{\mu}{d} \frac{1}{k^3} \sum \frac{\mu}{d} \frac{1}{k^4} \sum \frac{\mu}{d} \frac{1}{k^5}
\]

(1.47)

to the first power in \( \lambda' \).

The average value \( \sum \frac{\mu}{d} \frac{\mu}{d} \) is the sum of \( \sum \frac{\mu}{d} \frac{\mu}{d} \) over all the orientations of the molecules.

Putting

\[
\frac{\lambda_0}{\sum} \frac{\mu}{d} = \sum \frac{\lambda_0}{\sum} \frac{\mu}{d} \frac{\mu}{d}
\]

(1.48)
the summation being taken over all the orientations of the molecules, and since \( \sum_{\ell', m'} e_{\ell} \sum_{s'} \vec{p}_{s} = 0 \) we obtain

\[
\frac{1}{N^{e}} \sum_{s'} \frac{\vec{p}_{s}^{2}}{kT} \sum_{l} \frac{\Delta_{X}^{l}}{\beta f X} \left\{ \alpha + \frac{f^{2}}{kT} (\Delta_{Y} X) \right\}_{s,s'} \frac{E_{s}^{l}}{E_{s}} \tag{1.49}
\]

where

\[
\beta^{2} Q_{s} = \frac{\sum_{s'} \vec{p}_{s}^{2}}{\sum_{s'} \vec{p}_{s}} \sum_{s'} \vec{p}_{s} \tag{1.50}
\]

Since

\[
\frac{1}{N^{e}} \sum_{s'} \frac{\vec{p}_{s}^{2}}{k^{2} T} \sum_{l} \frac{\Delta_{X}^{l}}{\beta f X} \left\{ \alpha + \frac{f^{2}}{kT} (\Delta_{Y} X) \right\}_{s,s'} \frac{E_{s}^{l}}{E_{s}} \tag{1.51}
\]

But by definition

\[
P_{s} = \frac{\sum_{s'} (\frac{\Delta_{X}^{s}}{\beta f X})_{s,s'} E_{s}^{l}}{E_{s}} \tag{1.52}
\]

then we obtain

\[
\frac{4}{3} = \frac{3 \Delta_{X}}{2kT} \left[ \beta_{\Delta_{Y}} X + \frac{f^{2}}{kT} \frac{X^{2}}{1 - \beta f X} \right] \tag{1.53}
\]
In this equation the symbol \( \varepsilon \) represents a tensor though the suffixes are not explicitly shown while \( \varepsilon_0 \) is a number taken to be \( \frac{1}{3} \frac{2}{4} (\varepsilon \beta)_{ss} \), also \( f \) is a number given by \( \frac{\varepsilon(3\varepsilon - 1)}{2\varepsilon + 1} \).

For a cubic crystal \( \lambda = 1 \) i.e. \( \lambda_{ss} = \delta_{ss} \) also \( \varepsilon_{ss} \beta_{ss} = \varepsilon_0 \delta_{ss} \) and therefore \( \varepsilon_{ss} = \varepsilon_0 \beta_{ss} \).

Hence equation (1.53) becomes for this special case

\[
\frac{\varepsilon_0^{-1}}{3} = \frac{1}{3} \varepsilon + \frac{3\varepsilon}{2\varepsilon + 1} \frac{\beta^2}{\alpha^0 k^2} Q_0 \tag{1.54}
\]

The evaluation of \( Q \) as will be seen in the next chapter, will again depend on an image field, and so on the dielectric constant. Therefore equation (1.54) can only be considered as a verification of the calculation of \( Q_0 \), or of the calculation of the configurational probability of the dipoles, and it cannot be used directly to calculate \( \varepsilon_0 \). Naturally there must be a value of \( \varepsilon_0 \) which satisfies (1.54) but to find such a value is a very lengthy and impracticable problem.
APPENDIX I

The Tensor $\lambda$

This tensor was defined by equations (1.26) and (1.29) for $i \neq j$ and $i = j$ respectively. It is required now to find an expression for it for any type of crystal.

Let $\phi^{(i)}$ be the value of the potential at the position occupied by the molecule $i$ due to the moment of the molecule $j$, then

$$-\frac{\partial \phi^{(i)}}{\partial x_j^i} = \sum_{s} \frac{a_s}{s} \frac{F_s^{(j)}}{s}$$

(1.55)

where the $a_s$ are defined by (1.33).

If $R_{ij}$ is the distance between the two molecules considered, then

$$\phi^{(i)} = -\sum_{s} \frac{2}{s} \frac{\partial \phi^{(j)}}{\partial x_j^s} (\frac{1}{R_{ij}}) \mu^j_s$$

(1.56)

Substituting for $F_s^{(j)}$ from equation (1.26) and for $\phi^{(j)}$ from (1.56), the equation (1.55) becomes

$$-\frac{1}{\alpha^2} \sum_{s} (\frac{\partial \phi^{(j)}}{\partial x_j^s} \mu^j_s) \cdot \sum_{s} \frac{2}{s} \frac{\partial}{\partial x_j^s} \frac{\partial \phi^{(j)}}{\partial x_j^s} (\frac{1}{R_{ij}}) \mu^j_s$$

then

$$\left(\frac{\partial \phi}{\partial x_j^s}\right)_{s' s} = -\frac{2}{\alpha^2} \frac{\partial}{\partial x_j^s} \frac{\partial \phi^{(j)}}{\partial x_j^s} \left(\frac{1}{R_{ij}} \right) \mu^j_s$$

(1.57)

Since

$$R_{ij}^2 = \sum_{s} \frac{a_s}{s} \frac{X_j^i X_j^j}{s}$$

and

$$\frac{\partial R_{ij}}{\partial x_j^s} = \sum_{s} \frac{a_s}{s} \frac{X_j^i X_j^j}{s} \quad \text{(1.58)}$$
then

\[
(g \lambda)_{ss}^j = \frac{\kappa}{R_j^3} g_{ss} - \frac{2 \kappa}{R_j^5} \gamma_j^s \gamma_s^j
\]  

(1.58)

or

\[
(g \lambda')_{ss}^j = \frac{\kappa}{R_j^3} g_{ss}^1 - \frac{2 \kappa}{R_j^5} \lambda_j^s \lambda_s^j
\]  

(1.59)
APPENDIX II

The Tensor $\mathbf{A}$

This tensor was defined by equation (1.39) namely

$$A_{ss'} = \sum_j A_{sj} \delta^j_{s'}$$

which is assumed to be independent on $i$.

It is required to prove that

$$\sum_j [f(\lambda)]_s^j = [f(\mathbf{A})]_{ss'}$$

(1.60)

where $f(\lambda)$ is any tensor function of $\lambda$,

$f(\mathbf{A})$ is the same function of $\mathbf{A}$.

To do this we start by considering $f(\lambda) = \lambda^2$,

$$\sum_j \lambda^2_j = \sum_j \lambda^i_j \lambda^{i'}_{j'} = \sum_i \lambda^i_{r'} \lambda^{i'}_{r} = \sum_r \mathbf{A}_{sr} \mathbf{A}_{sr'} = (\mathbf{A}^2)_{ss'}$$

using equation (1.39)

By successive pre-multiplication by $\lambda$ one obtains

$$\sum_j \lambda^2_j = (\mathbf{A}^2)_{ss'}$$

and hence the required formula (1.60) follows.
The Image Field inside the Cavity.

To find the image field due to the moment of a molecule situated at any point inside the spherical cavity, we start by considering a charge \( \epsilon \) situated at a point distant \( c_o \) from the centre of the spherical cavity whose radius is \( a_o \).

If the potential inside the cavity is \( \phi_1 \), and that in the dielectric is \( \phi_2 \) then

\[
\nabla^2 \phi_1 = 0 \quad \nabla^2 \phi_2 = 0 \tag{1.61}
\]

and at the boundary of the cavity

\[
\phi_1 = \phi_2 \quad \frac{\partial \phi_1}{\partial n} = \varepsilon \frac{\partial \phi_2}{\partial n} \tag{1.62}
\]

From (1.61) \( \phi_1 \), \( \phi_2 \) can be written in the form

\[
\phi_1 = \frac{\varepsilon}{n} \sum_{m=1}^{\infty} \left( \frac{\xi}{n} \right)^m P_m (\cos \theta) + \sum_{m=1}^{\infty} A_m \lambda^m P_m (\cos \theta)
\]

\[
\phi_2 = \sum_{m=1}^{\infty} \frac{B_m}{\lambda^m} P_m (\cos \theta)
\]

Applying the boundary conditions (1.62) one obtains

\[
A_n = - \frac{\varepsilon c^m}{\xi} \left( \frac{\xi - 1}{\xi + 1} \right)^{\frac{n+1}{m+\sigma}}
\]

where

\[
\sigma = \frac{\xi}{\xi + 1}
\]
Therefore the image potential inside the cavity is

\[ \phi = -\frac{2}{\sigma} \sum_{l=1}^{\infty} \frac{\alpha_0^n}{\alpha^n} \frac{P_l(\cos \theta)}{\cos \theta} \]

\[ = -\frac{e_0}{\sigma} \sum_{l=1}^{\infty} \frac{\alpha_0^n}{\alpha^n} \frac{\alpha_0^{n+1}}{\alpha^{n+1}} \frac{P_l(\cos \theta)}{\cos \theta} \]

(1.63)

Since

\[ \frac{1}{\sigma^n} = \int_{\xi_0^2}^{\infty} \frac{dx}{x^{n+1}} (\frac{\alpha_0^2}{\xi_0^2})^x \]

then the image field inside the cavity is due to

(i) a charge \(-\frac{e_0}{\xi_0} \frac{x}{x+1}\) at the inverse point \(x = \frac{\alpha_0^2}{\xi_0^2}\)

(ii) a line charge of density \(-\frac{e_0 x}{\xi_0} \frac{\alpha_0^{n+1}}{\alpha_0^{n+1}} (\frac{\alpha_0^2}{\xi_0^2})^x\) from \(x = \frac{\alpha_0^2}{\xi_0^2}\) to \(x = \infty\).

(1.64)

We now consider a dipole \(\vec{P}\) at \(x = \xi_0\). This is equivalent to a charge \(-e\) at \(\xi_0\) and a charge \(+e\) at \(\xi_0 = \xi_0 + \frac{1}{2} \vec{P}\).

In Fig. (2) \(\vec{OA}\) represents \(\xi_0\) and \(\vec{OA}'\) represents \(\xi_0'\). \(B\) \(B'\) are the inverse points of \(A\), \(A'\) respectively with respect to the hollow sphere. If \(\xi_0 = \xi_0 + \frac{1}{2} (\vec{B} \cdot \vec{k})\), then \(\xi_0' = \xi_0 + \frac{1}{2} (\vec{B} \cdot \vec{k})\).

The image system due to a charge \(+e\) at \(A'\) is

(i) a charge \(-\frac{e_0}{\xi_0} \frac{x}{x+1}\)

\[ = -\frac{e_0}{\xi_0} \frac{x}{x+1} + \frac{e_0}{\xi_0} \frac{x}{x+1} (\vec{A} \cdot \vec{k}) \] at \(B'\)

(ii) a line charge from \(B'\) to infinity. The charge between \(x'\) and \(x'+dx'\) is

\[ -\frac{e_0}{\alpha_0} \frac{x}{x+1} (\frac{\alpha_0^2}{\xi_0^2})^x dx' \]

where \(OX'\) in the figure represents \(x'\).
Since \( c' x' = c x \) then the latter is
\[
-\frac{e}{\alpha} \frac{i^{-1}}{2 + i} \left( \frac{\alpha^2}{\omega x} \right) \left( \frac{(\alpha^2)}{i + 1} \right) \left( \frac{(\alpha^2)}{\omega x} \right) \left( \mu \cdot \xi \right) d\xi
\]

Combining this image system of +c with the image system of -c at \( A \) we obtain for the image system due to the dipole, remembering that \( (BB') = \frac{\alpha^2}{\omega^2} (\xi \cdot \xi') \)
and \( (XX') = \frac{\alpha}{\xi} (\xi \cdot \xi') \).

(i) a charge \( \frac{\alpha}{\omega^2} \left( \frac{i^{-1}}{2 + i} \right) (\mu \cdot \xi) \) at \( x = \frac{\alpha^2}{\omega} \).

(ii) a line charge of density \( \frac{1}{\alpha} \frac{i^{-1}}{2 + i} \left( \frac{\alpha^2}{\omega x} \right) \left( \mu \cdot \xi \right) \)
from \( x = \frac{\alpha^2}{\omega} \) to \( x = \infty \).

(iii) a dipole \( -\frac{\alpha}{\omega^2} \left( \frac{i^{-1}}{2 + i} \right) [\mu - \frac{1}{2}(\mu \cdot \xi) \xi] \) at \( x = \frac{\alpha^2}{\omega} \).

(iv) a line dipole of density \( \frac{\alpha}{\omega^2} \left( \frac{i^{-1}}{2 + i} \right) [\mu - \frac{1}{2}(\mu \cdot \xi) \xi] \)
from \( x = \frac{\alpha^2}{\omega} \) to \( x = \infty \) (1.65)

The image field of the dipole inside the cavity is thus given by the field of the image system (1.65) which lies outside the cavity.

The space average value of the image field created by the dipole \( \mu \) (at distance \( \omega < a \) from the centre) inside the cavity is equal to the field of the image system (1.65) at the centre of the cavity.

The field, at the centre of the cavity, of the system given by (1.65) can easily be found to be
\[
\mathcal{J} = \frac{1}{2} \frac{2^{(2i-1)}}{2i+1} \mu \] (1.66)
Now if there are $N^0$ molecules inside the cavity, then the average image field of all the molecules inside the cavity is

$$\frac{f}{\mathcal{A}^2} \sum_i \mu^{(i)} = \frac{f}{N \mathcal{A}^2} \sum_i \mu^{(i)} \quad (1.67)$$
CHAPTER II

THE CONFIGURATIONAL PROBABILITY

In the last chapter the dielectric constant was shown to depend on a factor $Q$ given by equations (1.48) and (1.50). In this chapter a direct evaluation of $Q$ from those equations will be carried out.

Since the configurational probability $\zeta$ depends on the energy $W'$ of the dipoles in the given configuration, the evaluation of $Q$ will depend on a knowledge of $W'$.

The energy in a given configuration $W'$ is, according to equations (1.35), (1.36) expressed by

$$W' = \frac{1}{\alpha} \sum_{ss'} \mathcal{U}_{ss'}^{ij} \rho_{ij} \rho_{ij}^{*}$$

(2.1)

where

$$\mathcal{U}_{ss'}^{ij} = \frac{1}{\zeta} (g - g' \gamma')_{ss'}^{ij}$$

(2.2)

The sum over states $Z$ is defined by

$$Z = \frac{1}{\mathcal{H}} \exp \left\{ - \frac{1}{2kT} \sum_{ss'} \mathcal{U}_{ss'}^{ij} \rho_{ij} \rho_{ij}^{*} \right\}$$

(2.3)

where the summation is over all orientations of the dipoles. The tensor $\mathcal{U}$ as defined by (2.2) is clearly $\frac{1}{\alpha^3}$ multiplied by a function of $\beta$ (See (1.59)(1.13))

Then $Z$ as defined by (2.3) is a function of $\frac{\rho^2}{\alpha^3 kT}$ and $\beta$. The tensor

$$\frac{\partial Z}{\partial \alpha}$$

is given by

$$\frac{\partial Z}{\partial \mathcal{U}_{ss'}^{ij}} = - \frac{1}{2kT} \sum_{ij} \rho_{ij} \rho_{ij}^{*} \exp \left\{ - W' / kT \right\}$$

(2.4)
By using equation (1.48) one obtains

$$\frac{\partial^2 Z}{\partial u_s \partial u_s'} = \frac{Z}{Z} \frac{2}{\rho_s \rho_s'} \langle \log Z \rangle$$

and therefore by use of (1.50)

$$Q_{ss} = -\frac{2kT}{\rho_s \rho_s'} \frac{2}{\rho_s \rho_s'} \langle \log Z \rangle$$

which is clearly a function of \(\frac{\beta}{kT}\) and \(\beta^2\).

There is no way of finding \(\frac{\rho_s}{\rho_s'}\) from equation (2.5) or (1.48) by evaluating the sums given. These sums, however, can be evaluated in the case of thermal vibrations, with which we shall deal in the fourth chapter.

We therefore have to use some other approximate method to find \(\frac{\rho_s}{\rho_s'}\).

The expression \(\frac{\rho_s}{\rho_s'}\) does not, however, require a knowledge of the probability of a configuration of the whole set of dipoles, but can be evaluated if the probability of only the two molecules concerned is known for any configuration. This is clear since

$$\sum_{\rho_s \rho_s'} \rho_s \rho_s' C(\rho_s \rho_s') = \sum_{\rho_s \rho_s'} \rho_s \rho_s' C(\rho_s \rho_s')$$

To evaluate the probability \(C_{ss}\) of the two molecules \(i, j\), we adopt the approximate method of replacing the rest of the molecules macroscopically by a homogeneous dielectric. The most convenient shape of the internal boundary of this dielectric will be considered later. The steps required for the evaluation of \(C_{ss}\) are as follows.
We first calculate the field acting on the two molecules in a given configuration. This field will depend on the dielectric constant, since the two molecules concerned lie in the neighbourhood of a homogeneous dielectric. We then formulate the mutual potential energy in the same configuration, which will depend on the induced moment as well as on the permanent moment. The probability \( \Theta \) is then found by Boltzman's law. In this method, as well as in the exact method shown at the beginning of this chapter, \( Q \) will depend on \( p \) and \( \alpha \) through the quantities \( \frac{p^2}{2kT} \) and \( \beta \). The factor \( Q \) in this method will also depend on the dielectric constant.

The field acting on the two molecules.

The two events when the two molecules concerned are not first neighbours, and when they are first neighbours, will be treated separately. When the two molecules are not first neighbours, the internal boundary of the dielectric will be chosen to be two separate spheres with the two molecules concerned at their centres. The volume of each sphere is taken to be equal \( \frac{1}{N} = \frac{4\pi}{3} a^3 \). A method of successive images will be explained in the first appendix to find the field at the centres of the two
spheres of a dipole $\mathbf{\mu}$ at the centre of one of them. The field inside the sphere in which the dipole lies is found to be equal to

$$\frac{f}{a^2} \mathbf{\mu},$$

and the field at the centre of the other sphere is the same as the field of a dipole,

$$\frac{9\varepsilon}{(2n+1)^2} \mathbf{\mu}$$

at the centre of the first sphere.

There are other terms due to further successive image fields and containing the parameter $\frac{R}{c^2}$ where $c$ is the distance between the two molecules. These terms were found small, even in the case when the two molecules are second neighbours, and therefore are neglected. Now if the molecule at the centre of one of the two spherical cavities has the moment $\mathbf{\mu}$ and the molecule at the centre of the other spherical cavity has the moment $\mathbf{\mu}$ then the field acting on the first molecule, following equations (2.8) and (2.9) is given by

$$F = \frac{f}{a^2} \mathbf{\mu} - \frac{9\varepsilon}{(2n+1)^2} \mathbf{\mu} + \mathbf{\mu} \left( \mathbf{\mu} \cdot \mathbf{k} \right) \mathbf{k}$$

(2.10)

If we write the field in the form

$$-\mathbf{F} = (\mathbf{\mu} \cdot \mathbf{k}) \mathbf{k} + \mathbf{\mu} \left( \mathbf{\mu} \cdot \mathbf{k} \right) \mathbf{k}$$

(2.11)
then from (2.10)

\[ A_0 = B_0 = 1 - \alpha f \]  

(2.12)

\[ A = \frac{9 \xi}{(2 \xi + 1)^2} \frac{\kappa}{c^2} \quad B = -2 A \]  

(2.13)

When the molecules are first neighbours, the two spherical cavities surrounding the two molecules will overlap; and thus form a single cavity of volume less than the required volume, i.e. \( \frac{2}{N} \). We might therefore consider the radius of each of the overlapping spheres to be \( \chi c \) where \( c \) is the distance between first neighbours, as shown in figure (3). In this case the volume of the cavity would be

\[ \frac{\pi c^3}{12} (2 \xi + 1)^2 (4 \xi - 1) \]

Equating this to \( \frac{2 \pi}{3} a^3 \) we obtain

\[ \frac{L}{X^3} - \frac{12}{(1 + \frac{32 a^3}{c^3})} \frac{1}{X} - \frac{16}{(1 + \frac{32 a^3}{c^3})} = 0 \]  

(2.14)

which is a cubic equation to determine \( \frac{L}{X} \) and hence the radius \( \chi c \) of each of the overlapping spheres.

The evaluation of the field inside such a cavity is, however, found to be very difficult and therefore a cavity in the form of an ellipsoid of revolution, with its axis of revolution passing through the two molecules, will actually be considered. The
equation (2.14) will be used to fix the dimensions of the cavity. If the distance between the foci is \( 2b \) and the major axis is \( 2a \omega h t \), then the volume of the cavity is

\[
\frac{4\pi}{3} b^3 [c \omega h t]^3 - (c \omega h t)
\]

Equating this to \( \frac{8\pi}{3} a^3 \), one obtains

\[
(c \omega h t)^3 - (c \omega h t) - \frac{2a^3}{b^3} = 0
\]

(2.15)

This equation, although it has fixed the volume of the cavity, has not fixed the form of the ellipsoid of revolution, since it connects two parameters \( b \) and \( c \omega h t \).

We choose the form of the ellipsoid to coincide as much as possible with the cavity of the two over-lapping spheres whose dimensions are given by equation (2.14).

This method of choosing the form of the ellipsoid may be criticized as not precise; but bearing in mind that the image field inside the cavity does not change appreciably in changing the shape of the ellipsoid, this method will be accurate enough.

The additional or the image field inside the cavity created by the presence of a dipole \( \mu \), on the line between the foci and at a distance \( \frac{1}{2} c < b \) from the centre, will be given in the second appendix.
The field at the position of the dipole (due to the image field of this dipole alone) is found to be

\[ C_0 \mu + \frac{D_0}{(\mu - \xi)} \xi \]  \hspace{1cm} (2.16)

and the image field at the position of the other dipole (which was not taken into account) is

\[ C \mu + \frac{D}{(\mu - \xi)} \xi \]  \hspace{1cm} (2.17)

where

\[
C_0 = -\frac{1}{b^3} \sum_{m=0}^{\infty} \left[ \frac{2(m+1)}{\pi} \frac{Q_m'(i\alpha b)}{P'(i\alpha b)} \frac{1}{1 - X_m} \right] \]

\[
C = -\frac{1}{b^3} \sum_{m=0}^{\infty} \left[ (-1)^m \frac{2(m+1)}{\pi} \frac{Q_m'(i\alpha b)}{P'(i\alpha b)} \frac{1}{1 - X_m} \right] \]

\[
C + D = -\frac{2}{b^3} \sum_{m=0}^{\infty} \left[ (-1)^m \frac{2(m+1)}{\pi} \frac{Q_m'(i\alpha b)}{P'(i\alpha b)} \frac{1}{1 - X_m} \right] \]

where \( X_m \) and \( X_m^{(1)} \) are given by the equations

\[
(\alpha - 1) \frac{X_m}{X_m^{(1)}} = \frac{Q_m'(i\alpha b) P'(i\alpha b) - Q_m''(i\alpha b) P'(i\alpha b)}{P'(i\alpha b) Q_m'(i\alpha b)} \]

\[
(\alpha - 1) \frac{X_m^{(1)}}{X_m} = \frac{\sin^2 b - Q_m'(i\alpha b) P''(i\alpha b) - Q_m''(i\alpha b) P'(i\alpha b)}{P''(i\alpha b) \sin b Q_m'(i\alpha b) + \sin^2 b Q_m''(i\alpha b)} \]

\( 2b \) is the distance between the foci and \( 2b \cot \alpha \) is the major axis, both being determined by equation (2.15) and the choice of the form of the ellipsoid with the help of (2.14).
\( P_n \), \( Q_n \) are Legendre functions and Legendre associated functions respectively. The evaluation of the field thus depends on the computation of infinite series. The numerical calculation, as will be found in the next chapter, is not very difficult, since only a few terms in the series given by (2.18) and (2.19) will be needed.

Now let the two molecules, on the axis of the ellipsoidal cavity and at distances \( \frac{1}{2} c \) on both sides from the centre, have moments \( \mu^{(u)} \) and \( \mu^{(v)} \). Writing the field acting on the dipole \( \mu^{(v)} \) in the form (2.11), then the constants \( A_0, B_0, A, \) and \( B \) in that equation will be given (using equations (2.16), (2.17), (2.18) and (2.19) and taking into account the field of \( \mu^{(u)} \) at \( \mu^{(v)} \) ) by

\[
A_0 = -\frac{\pi}{b^2} \sum_{m=0}^{\infty} \left( \frac{n+1}{2} \right) \frac{P_n'(\xi_0)}{P_n(\cos \theta)} \cdot \frac{1}{1 - X_m^{(v)}}
\]

\[
B_0 = -\frac{2\pi}{b^2} \sum_{m=0}^{\infty} \left( \frac{n+1}{2} \right) \frac{P_n'(\xi_0)}{P_n(\cos \theta)} \cdot \frac{1}{1 - X_m^{(v)}}
\]

\[
A = \frac{\pi}{\alpha^2} \sum_{m=0}^{\infty} (-1)^m \left( \frac{n+1}{2} \right) \frac{P_n'(\xi_0)}{P_n(\cos \theta)} \cdot \frac{1}{1 - X_m^{(v)}}
\]

\[
B = -\frac{2\pi}{\alpha^2} \sum_{m=0}^{\infty} (-1)^m \left( \frac{n+1}{2} \right) \frac{P_n'(\xi_0)}{P_n(\cos \theta)} \cdot \frac{1}{1 - X_m^{(v)}}
\]
The Mutual Potential Energy.

The field acting on one of the two molecules given by equation (2.11) in vector form, can be re-written in the tensor form

\[-\alpha F_i^j = \sum_{\delta}s_s (\lambda^{-1})_{\delta_s^i} \mu_{\delta_s^j} \tag{2.23}\]

(not to be confused with \( \lambda \) defined by (1.59))

If the axes considered are orthogonal then \( \lambda \) will be given by

\[\lambda_{ss}'' = \lambda_{ss'}'' = \lambda_{ss''}'' = A_s s_s \delta_{ss'} + (B_s - A_s) k_s k_s = \delta_{ss'} \]

\[\lambda_{ss}''' = \lambda_{ss'}''' = A_s s_s \delta_{ss'} + (B_s - A_s) k_s k_s = \chi_{ss'} \]

where \( k \) is a unit vector along the line joining the two molecules. Omitting the suffixes \( s, s' \) then

\[\lambda = \begin{pmatrix} \chi \delta \\ \delta \chi \end{pmatrix} \tag{2.24}\]

where

\[\chi = A_s + (B_s - A_s) (kk) \]

\[\delta = A_s + (B_s - A_s) (kk) \tag{2.25}\]

It was already explained in the first chapter that if the field acting on a molecule \( i \) is given by the expression (2.23) then the mutual potential energy is given by

\[W = \frac{1}{2\alpha} \sum_i \sum_j (1 - \lambda^{-1})_{\delta_s^i} \mu_{\delta_s^j} \]

In this case \( i, j \) have only the two values one and two and so (2.26) can be written in the form...
It is proved in the third appendix that if \( \lambda \) is given by (2.24) and \( f(\lambda) \) is any tensor function of \( \lambda \) then

\[
[f(\lambda)]^n = \frac{1}{2} f(\lambda + \delta) + \frac{1}{2} f(\lambda - \delta)
\]

\[
[f(\lambda)]^p = \frac{1}{2} f(\lambda + \delta) - \frac{1}{2} f(\lambda - \delta)
\]

(2.28)

In particular taking \( f(\lambda) = 1 - \lambda^{-1} \) then omitting suffixes \( s, s' \)

\[
(1 - \lambda^{-1})^n = \frac{x_0}{(1 - \lambda^{-1})} + F(\delta, \delta) - 1
\]

\[
(1 - \lambda^{-1})^p = \frac{\delta}{(1 - \lambda^{-1})} + F(\delta, \delta)
\]

(2.29)

The expression (2.27) can then be written, using (2.29), in the form

\[
W = \frac{1}{2\lambda} \sum_{ss'} \left[ F(\delta, \delta) \right]_{ss'ss'} \left[ \lambda^{\alpha} P_{s}^{(\alpha)} + \lambda^{\beta} P_{s'}^{(\beta)} \right] + \frac{1}{2} \sum_{ss'} \left[ F(\delta, \delta) \right]_{ss'ss'} P_{s}^{(\alpha)} P_{s'}^{(\beta)}
\]

(2.30)

where \( \delta_0, \delta' \) are given by equation (2.25). It is proved in the fourth appendix that if \( F(\delta, \delta) \) is a tensor function of the two tensors \( \delta_0 \) and \( \delta' \) given by (2.25), then

\[
F(\delta_0, \delta') = F(A, A) + \left[ F(\delta, \delta') - F(A, A) \right]^{(k)}(k)
\]

(2.31)
A similar equation holds for \( F(\gamma, \delta) \).

Substituting in (2.30) the values of \( F(\gamma, \delta) \) and \( F(\gamma', \delta') \) given by (2.31) and remembering that
\[
\sum_{s_0} \bar{p}_{s_0}^{(u)} \bar{p}_{s_0}^{(v)} \bar{p}_{s_0}^{(w)} \bar{p}_{s_0}^{(s)} = (\bar{p}_{s_0}^{(u')} \bar{p}_{s_0}^{(v')} \bar{p}_{s_0}^{(w')} \bar{p}_{s_0}^{(s')})
\]
and
\[
\sum_{s_0} \bar{p}_{s_0}^{(u'')} \bar{p}_{s_0}^{(v'')} \bar{p}_{s_0}^{(w'')} \bar{p}_{s_0}^{(s'')} = (\bar{p}_{s_0}^{(u''')} \bar{p}_{s_0}^{(v'')} \bar{p}_{s_0}^{(w'')} \bar{p}_{s_0}^{(s''')})
\]
we obtain for the final formula for \( W \)
\[
W = \frac{1}{2 \alpha} \left\{ F(A, A)(\bar{p}_{s_0}^{(u')} \bar{p}_{s_0}^{(v')} \bar{p}_{s_0}^{(w')} \bar{p}_{s_0}^{(s')}) + \left( F(A, A) - F(B, B) \right) \left( \bar{p}_{s_0}^{(u''')} \bar{p}_{s_0}^{(v'')} \bar{p}_{s_0}^{(w'')} \bar{p}_{s_0}^{(s''')} \right) \right\}
\]
\[
+ \frac{1}{2 \alpha} \left\{ F(A, A)(\bar{p}_{s_0}^{(u''')} \bar{p}_{s_0}^{(v'')} \bar{p}_{s_0}^{(w'')} \bar{p}_{s_0}^{(s''')}) + \left( F(B, B) - F(A, A) \right) \left( \bar{p}_{s_0}^{(u''')} \bar{p}_{s_0}^{(v'')} \bar{p}_{s_0}^{(w'')} \bar{p}_{s_0}^{(s''')} \right) \right\}
\]
(2.32)

The functions \( F_0 \) and \( F_0' \) are given by (2.29) on replacing \( \gamma, \delta \) by the constants \( A_0, A \) or \( B_0, B \). For first neighbours \( A_0, A, B \) and \( B_0, B \) are given by equations (2.21) and (2.22); and for other neighbours they are given by equations (2.12) and (2.13).

The Tensor \( Q \).

The tensor \( Q \), given by equation (1.50), depends on the average \( \bar{p}_{s_0}^{(u')} \bar{p}_{s_0}^{(v')} \bar{p}_{s_0}^{(w')} \bar{p}_{s_0}^{(s')} \) for any two molecules \( i, j \). Let \( \mathbf{e} \left( \mathbf{r} \right) \) be a unit vector, having components \( \bar{k}_s \left( \mathbf{r} \right) \), along the line joining the molecule \( i \) to the molecule \( j \).

To evaluate the tensor \( \bar{p}_{s_0}^{(u')} \bar{p}_{s_0}^{(v')} \bar{p}_{s_0}^{(w')} \bar{p}_{s_0}^{(s')} \) we may first determine it for three orthogonal axes, one along the line joining the two molecules and the other two perpendicular to it. We can then find it with respect to the crystal axes by a tensor transformation.
Let \( \ell_s^\sigma \ (s = 1, 2, 3) \) be the components of a unit vector along the axis \( \sigma \) of the orthogonal set referred to the crystal axes. If we choose the axes \( \sigma = 1, 2 \) to be perpendicular to the line joining the two molecules and the axis \( \sigma = 3 \) along it, then \( \ell_s^3 = k_s \). The components of a unit vector along a crystal axis \( s \) referred to the orthogonal set of axes will then be

\[
\ell_s^\sigma = \sum_{s'} T_{ss'} \ell_s^{s'} \ (s = 1, 2, 3)
\]  

(2.33)

\( T_{ss'} \) was given by equation (1.33).

The angle between two crystal axes \( s, s' \) is then \( T_{ss'} h_s^\sigma h_s'^\sigma \) and therefore

\[
T_{ss'} = \sum_\sigma h_s^\sigma h_s'^\sigma
\]  

(2.34)

Using equation (2.33) one then obtains

\[
T_{ss'}^{-1} = \sum_\sigma \ell_s^\sigma \ell_s'^\sigma
\]  

(2.35)

where \( T_{ss'}^{-1} \) is the reciprocal of \( T_{ss'} \).

If \( \beta_s^{(ij)} \) are the components of \( \beta_s^{(ij)} \) along the orthogonal axes and \( \beta_s^{(ij)} \) are those along the crystal axes then

\[
\beta_s^{(ij)} = \sum_{\sigma} \ell_s^\sigma \beta_s^{(ij)}
\]  

(2.36)

and therefore

\[
\beta_{s's'}^{(ij)} = \sum_{\sigma} \ell_s^\sigma \ell_{s'}^{s'} \beta_{s's'}^{(ij)}
\]  

(2.37)
This equation gives the tensor \( \mathbf{T}^{\alpha \beta} \) referred to the crystal axes in terms of the tensor \( \mathbf{T}^{\alpha \beta} \) referred to the orthogonal axes, namely the one along the line joining the two molecules and the other two perpendicular to it. In the calculation of the tensor \( \mathbf{T}^{\alpha \beta} \) we use polar co-ordinates.

\[
\mathbf{T}^{\alpha \beta} = \mathbf{T}^{\alpha \beta} \left( \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta \right) \quad (\alpha, \beta, \gamma)
\]

(2.38)

The expression for \( W \) given by (2.32) can then be written in the form

\[
- \frac{W}{kT} = \mathcal{G} (\cos \theta + \cos \phi') + \mathcal{L} \cos \theta + \mathcal{M} \sin \phi \cos (\phi - \phi')
\]

(2.39)

where

\[
\mathcal{G} = \frac{1}{2 \alpha} \left[ F(\theta, \phi) \right] - F(\lambda, \lambda) \frac{k^2}{kT}
\]

\[
\mathcal{L} = -\frac{1}{\alpha} F(\lambda, \lambda) \frac{k^2}{kT}
\]

\[
\mathcal{M} = -\frac{1}{\alpha} F(\lambda, \lambda) \frac{k^2}{kT}
\]

(2.40)

Now \( \mathbf{T}^{\alpha \beta} \) is given by

\[
\mathbf{T}^{\alpha \beta} = \frac{1}{4 \pi} \int \mathbf{T}^{\alpha \beta} \frac{k}{kT} \sin \theta \sin \phi \sin \phi' \sin \psi \, d\theta d\phi d\phi' d\psi
\]

(2.41)

where

\[
\kappa_0 = \frac{1}{(4\pi)^2} \int \mathbf{T}^{\alpha \beta} (-\frac{k}{kT}) \sin \theta \sin \phi \sin \phi' \sin \psi \, d\theta d\phi d\phi' d\psi
\]

(2.42)

\( W \) is given by (2.39) and \( \mathbf{T}^{\alpha \beta} \) by (2.38).
Putting $\phi' = \phi$ and integrating over $\phi'$ one obtains

$$\overline{p^{(\phi')} p^{(\phi')}} = \sum_{n=0}^{\infty} \frac{\overline{p^{n} p^{n}}}{\rho_{n}}$$  \hspace{1cm} (2.43)

$$\overline{p^{(\phi')} p^{(\phi')}} = 2 \rho^{2} \overline{\sin^{2} \theta \overline{\sin^{2} \theta}} = 2 \rho^{2} \overline{\sin^{2} \theta}$$  \hspace{1cm} (say) \hspace{1cm} (2.44)

$$\overline{p^{(\phi')} p^{(\phi')}} = \rho^{2} \overline{\sin \theta \overline{\sin \theta}} = \rho^{2} \overline{\sin \theta}$$  \hspace{1cm} (say) \hspace{1cm} (2.45)

where

$$K_{2} = \frac{2\pi}{(2\pi)^2} \int_{0}^{2\pi} \int_{0}^{2\pi} \sin \theta \sin' \theta \exp \{ 2 \rho \sin^{2} \theta \overline{\sin^{2} \theta} \} d\theta d\theta'$$  \hspace{1cm} (2.46)

$$K_{2} = \frac{2\pi}{(2\pi)^2} \int_{0}^{2\pi} \int_{0}^{2\pi} \cos \theta \cos' \theta \exp \{ 2 \rho \sin^{2} \theta \overline{\sin^{2} \theta} \} d\theta d\theta'$$  \hspace{1cm} (2.47)

Also from (2.42)

$$K_{0} = \frac{2\pi}{(2\pi)^2} \int_{0}^{2\pi} \int_{0}^{2\pi} \sin \theta \sin' \theta \exp \{ 2 \rho \sin^{2} \theta \overline{\sin^{2} \theta} \} d\theta d\theta'$$  \hspace{1cm} (2.48)

Performing the integration over $\phi$ by use of the formula

$$\int_{0}^{2\pi} \cos \phi \sin^{n} \phi \sin^{n} \phi = 2\pi i_{n} F_{n}(\lambda) = 2\pi i_{n} \lambda_{n}(\lambda)$$

where $i_{n}(\lambda)$ is a Bessel function of order $n$.

Afterwards for the integrals over $\theta$, $\theta'$, we use the formula

$$\int F(\theta, \theta') d\theta d\theta' = \int_{0}^{\pi} \left[ F(\theta, \theta') + F(\theta, \pi - \theta') + F(\pi - \theta, \theta') + F(\pi - \theta, \pi - \theta') \right] d\theta d\theta'$$

one then obtains the following expression for
\[ K_0 = \frac{1}{2} \int_0^1 \int_0^1 \sinh(L \cos \phi) \mu(L \phi) J_0(H \sin \phi) \sin \phi d\phi dL \]  

(2.49)

\[ K_1 = \int_0^1 \int_0^1 \sinh(L \cos \phi) \mu(L \phi) J_1(H \sin \phi) \sin \phi d\phi dL \]  

(2.50)

\[ K_2 = \int_0^1 \int_0^1 \sinh(L \cos \phi) \mu(L \phi) J_2(H \sin \phi) \sin \phi d\phi dL \]  

(2.51)

Returning to the object of finding \( \frac{\beta^{(i)} \beta^{(j)}}{s_i s_j} \)
referred to the crystal axes, equation (2.37) becomes, 
by use of equations (2.43), (2.44) and (2.45)

\[ \frac{\beta^{(i)} \beta^{(j)}}{s_i s_j} = \left( \frac{\ell^{(i)} \ell^{(j)}}{s_i s_j} \right) \frac{1}{2} \rho^2 \frac{K_i}{K_0} + \frac{\ell^{(3)} \ell^{(1)}}{s_i s_j} \rho^2 \frac{K_1}{K_0} \]  

(2.52)

Finally using equation (2.35) one obtains the formula

\[ \frac{\beta^{(i)} \beta^{(j)}}{s_i s_j} = \frac{1}{2} \rho^2 \frac{K_i}{K_0} \delta^{ij} + \rho^2 \frac{K_i - \frac{1}{2} K_2}{K_0} \delta^{ij} \frac{K_1}{K_0} \]  

(2.53)

In this expression \( K_0, K_1, K_2 \) depend on the 
distance between the two molecules but not on the 
direction, while \( K_2 \delta^{ij} \) depends on the direction 
of the line joining the two molecules concerned.

If \( i = j \) then \( \frac{\beta^{(i)} \beta^{(i)}}{s_i s_i} \) referred to any 
orthogonal axes is given

\[ \frac{\beta^{(i)} \beta^{(i)}}{s_i s_i} = \frac{1}{2} \rho^2 \delta^{ii} \]  

(2.54)

Then by use of (2.37) and (2.35), \( \frac{\beta^{(i)} \beta^{(j)}}{s_i s_j} \) referred 
to the crystal axes is given by

\[ \frac{\beta^{(i)} \beta^{(j)}}{s_i s_j} = \frac{1}{2} \rho^2 \delta^{ij} \]  

(2.55)
From (2.53), (2.55) one can then easily evaluate $Q$ by use of (1.50).
The internal boundary of the dielectric is two separate spheres each of radius $a$, and the distance between their centres is $c$. (See fig. (4)).

It is required to find the field at the centres of the two spheres created by the presence of a dipole $\mu$ at the centre of one of them.

We may first find the field of a dipole $\mu'$ at distance $c$ from the centre, inside and outside the hollow sphere of radius $a$ in the following way.

If there is a charge $e$ at distance $c > a$ in the dielectric, let $\phi_1$ be the potential inside the cavity and $\phi_2$ the potential outside. Following the procedure in the third appendix of the first chapter one obtains

$$\phi_1 = \sum_{m=0}^{\infty} A_m \frac{e}{r^m} \quad \phi_2 = \sum_{n=0}^{\infty} B_n \frac{e}{r^{n+1}} + \frac{e}{c} \sum_{m=0}^{\infty} \left( \frac{c}{r} \right)^m P_m$$

where

$$A_m = \frac{e c}{c^{m+1}} \left[ 2 - \frac{1}{m+1} \right] \quad B_m = \frac{e^2}{c^{m+1}} \left( \frac{1}{m+1} \right) \left[ 1 - \frac{c}{r} \right]$$

showing that the field inside the cavity is due to

(i) a charge $2e \sigma$ at $x = c$

(ii) a line charge of density $-\frac{e c}{c^{m+1}} \left( \frac{c}{r} \right)^m$ from $x = c$ to $x = \infty$ and the field outside the cavity is due to (besides the field of the charge) (4.16)

(i) a charge $\frac{e a}{c} \left( \frac{c}{r} \right)^m$ at $x = \frac{a}{c}$
(ii) a line charge \(-\frac{ce}{a^{3}} \frac{e-1}{z+1} \left(\frac{x}{R}\right)^{n-1}\)
from \(x = 0\) to \(x = \frac{a^{2}}{z}\)

The field at the centre of the sphere can be easily shown to be the same as the field of a charge

\[-\frac{3e}{2n+1} e \quad \text{at } x = c\]  

(2.58)

If there is a dipole \(\mathbf{\mu}'\) at distance \(c\) from the centre, one finds by the same method in the third appendix of Chapter I that the field outside the cavity is due to

(i) a dipole \(\frac{\mathbf{\mu}}{a} \left[ \mathbf{\mu}' - 2\mathbf{\mu} \cdot \mathbf{k} \right] \mathbf{k} \quad \text{at } x = \frac{a^{2}}{z}\)

(ii) a charge \(-\frac{\mathbf{\mu}}{a} \frac{e-1}{z+1} \left(\mathbf{\mu}' \cdot \mathbf{k}\right) \quad \text{at } x = \frac{a^{2}}{z}\)

(iii) a line dipole of density \(\frac{\mathbf{\mu}}{a} \left[ (\mathbf{\mu}' \cdot \mathbf{k}) \mathbf{k} - 2\mathbf{\mu} \cdot \mathbf{k} \mathbf{k} \right] \quad \text{from } x = 0 \text{ to } x = \frac{a^{2}}{z}\)

(iv) a line charge of density \(\frac{\mathbf{\mu}}{a} \frac{e-1}{z+1} \left(\mathbf{\mu}' \cdot \mathbf{k} \right) \left(\frac{x}{a^{2}}\right)^{n-1} \quad \text{from } x = 0 \text{ to } x = \frac{a^{2}}{z}\)

(2.59)

The field at the centre of the sphere, following (2.58) is the same as the field of a dipole

\[-\frac{3e}{2n+1} \mathbf{\mu}' \quad \text{at } x = c\]  

(2.60)

\(\mathbf{\mu}'\) is a unit vector from the centre of the sphere towards the dipole.

If one now considers only the sphere in which the dipole lies, then the field inside is

\[-\frac{\mathbf{f}}{a^{3}} \mathbf{\mu} \quad \text{from } x = 0 \text{ to } x = \frac{a^{2}}{z}\]  

(2.61)

(See (1.6) )
and the field outside is the same as the field of a dipole

\[
\frac{3}{2z+1} \mu \quad (2.62)
\]

at the position of \( \mu \) \( (x = c) \)

Taking into account the second hollow sphere, then the field at its centre, according to \( 2.60 \), \( (2.62) \), will be the same as the field of a dipole

\[
\frac{3}{2z+1} \cdot \frac{3}{2z+1} \mu = \frac{9z^2}{(2z+1)^2} \mu \quad (2.63)
\]

at \( x = c \)

and there will be an additional field outside, due to the following system. \( (\text{Using } 2.59, 2.62) \)

(i) a dipole \( \frac{\alpha^3}{c} \frac{1}{z+1} \left( \frac{3}{2z+1} \right) (\mu - \frac{1}{\mu}) \kappa \) at \( x = \frac{a^2}{c} \)

(ii) a charge \( -\frac{\alpha^3}{c} \frac{1}{z+1} \left( \frac{1}{\mu} \kappa \right) \) at \( x = \frac{a^2}{c} \)

(iii) a line dipole of density \( \frac{\alpha^3}{c} \frac{1}{z+1} \left( \frac{1}{\mu} \kappa \right) \) from \( x = 0 \) to \( x = \frac{a^2}{c} \)

(iv) a line charge of density \( \frac{\alpha^3}{c} \frac{1}{z+1} \left( \frac{1}{\mu} \kappa \right) \) from \( x = 0 \) to \( x = \frac{a^2}{c} \)

Taking into account the first sphere again (in which the dipole lies), then owing to the image system given by \( 2.64 \), there will be an additional field at the centre of the first sphere given by \( (\text{Using } 2.60, 2.62) \)

\[
- \frac{9z^2(1-\eta)}{(2z+1)^2(3z+1)^2} \cdot \frac{\alpha^3}{c^6} \left[ \frac{1}{(\frac{a^2}{c})^3} \right] - \frac{1}{c^6} \left[ \frac{\frac{3}{2} \frac{d \xi}{dx}}{\xi} \left( \frac{1}{\frac{a^2}{c^3}} \right)^2 \right] \mu = \frac{9z^2(1-\eta)}{(2z+1)^2(3z+1)^2} \frac{\alpha}{c^4}
\]

\[
\left[ \frac{1}{(\frac{a^2}{c^3})^2} \right] - \frac{1}{c^6} \left[ \frac{\frac{3}{2} \frac{d \xi}{dx}}{\xi} \left( \frac{1}{\frac{a^2}{c^3}} \right)^2 \right] (\kappa \kappa) \kappa
\quad (2.65)
\]
We can keep on in this way indefinitely to find the fields inside the two spheres. For the Ice crystal at \( T = 273 \), \( \varepsilon = 74.6 \), the expression (2.65) was found to be, when the two molecules are second neighbours

\[-\frac{1}{a^3} \left( 0.0024 \right) \mathbf{A} - \frac{1}{a^3} \left( 0.0032 \right) (\mathbf{A} \cdot \mathbf{k}) \mathbf{k}\]  

(2.66)

which is very small compared with the expression (2.61). For further neighbours it is clear that it will be much smaller. We therefore neglect the fields of further successive image systems and take the expression (2.61) for the field at the centre of the first cavity, and the field at the centre of the other cavity to be the same as the field of the dipole given by (2.63) at the centre of the first sphere.
APPENDIX II.

The internal boundary of the dielectric is an ellipsoid of revolution of major axis (along the axis of revolution) \(2b\,\coth t\) and the distance between the foci is \(2b\). A dipole \(\mu\) lies at distance \(\xi < b\) from the centre of the ellipsoid and on the line joining the two foci. It is required to find the field at the position of the dipole and at a point on the line joining the two foci at distance \(c\) from the dipole \(\mu\). (See Fig. (5))

We take then orthogonal axes \(x, x_2, x_3\) the first along the axis of revolution, the second in the plane of the dipole \(\mu\) and the first axis. The co-ordinates of the dipole are then \((\xi, 0, 0)\) and the co-ordinates of the other point at which we want to find the field is \((-\xi, 0, 0)\).

Using orthogonal co-ordinates \((t, \phi, \phi')\) given by

\[
x_1 = b \coth \theta \cos \phi \quad x_2 = b \sinh \theta \sin \phi \sin \phi \quad x_3 = b \sinh \theta \sin \phi \sin \phi
\]

then the general potential \(V\) satisfying \(\nabla^2 V = 0\) can be written in terms of these new co-ordinates

\[
V(t, \phi, \phi') = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [a_m^m P^m(\cos \phi) + b_m^m Q^m(\cos \phi)] T_m^m(t, \phi) \cos m\phi \tag{2.67}
\]

where

\[
T_m^m(x) = (x^2 - 1)^{m/2} \frac{d^m}{dx^m} P_m^m, \quad P_m^m(x) = (x^2 - 1)^{m/2} \frac{d^m}{dx^m} P_m
\]

\[
Q_m^m(x) = (x^2 - 1)^{m/2} \frac{d^m}{dx^m} Q_m
\]
The co-ordinates of the dipole $\mu$ in the new system are $(\phi, \theta, \phi)$ and those of the other point $(-\frac{\pi}{2}, 0, 0)$ are $(\theta, 0, 0)$ where $\cos \theta = \frac{\vec{R}}{R}$.

If $R$ is the distance between $(\theta, \phi, \phi')$ and $(-\frac{\pi}{2}, 0, 0)$ then

$$
\frac{1}{R^2} = \frac{1}{b^2} \sum_{m=0}^{\infty} \left[ \sum_{l=0}^{m} P^m_0 (\cos \theta) \cos \phi' \left( \frac{\Delta}{\theta} \right) \right]
$$

Taking a charge $-e$ at $t=0 \theta=\theta_o$ and a charge $+e$ at $t=t' \theta=\theta'$ then if the dipole makes an angle $\alpha$ with the direction of the axis $\hat{x}$, then

$$
\frac{\mu}{\epsilon_0} \sin \alpha = \sin \theta' \sin \theta
$$

and to first power in $\frac{\mu}{\epsilon_0}$ we have

$$
\sin \theta' \sin \theta = \frac{\mu}{\epsilon_0} \sin \alpha
$$

$$
\cos \theta' = \cos \theta + \frac{\mu}{\epsilon_0} \cos \alpha
$$

The potential of the dipole $\mu$ at $(\phi, \theta, \phi)$ is then, by use of equations (2.68) (2.69) given by

$$
V = \frac{\mu}{\epsilon_0} \sum_{m=0}^{\infty} \left( \frac{\Delta}{\theta} \right) \sum_{l=0}^{m} \left[ \cos \phi' \left( \frac{\Delta}{\theta} \right) \frac{1}{l} \right] \frac{\sin \phi}{\sin \phi'}
$$

To find the image field inside the cavity, let the potentials outside and inside the cavity be (following (2.67))

$$
V = \sum_{m=0}^{\infty} \sum_{l=0}^{m} \frac{C^m_{l} C^{\prime m}_{l} (\cos \phi')}{l} \frac{1}{l} \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{\infty} \left( \frac{\Delta}{\theta} \right) \cos \phi \right] \frac{1}{l} \sum_{m=0}^{\infty} \left( \frac{\Delta}{\theta} \right) (\cos \phi')
$$

$$
V_i = \sum_{m=0}^{\infty} \sum_{l=0}^{m} \left[ \frac{C^m_{l} C^{\prime m}_{l} (\cos \phi')}{l} \frac{1}{l} \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{\infty} \left( \frac{\Delta}{\theta} \right) (\cos \phi') \right] \frac{1}{l} \sum_{m=0}^{\infty} \left( \frac{\Delta}{\theta} \right) (\cos \phi') \right]
$$

(2.71)
The second part in the expression for $V_i$ is due to the dipole $\mu_0$ at $(0, \theta_0, 0)$ and is given by (2.70); the first part is the additional or image potential inside the cavity. Applying the boundary conditions

$$V_0 = V_i \quad \text{and} \quad \frac{2V_0}{\partial t} = \frac{2V_i}{\partial t} \quad \text{at the boundary} \quad t = t_i,$$

one obtains

$$A_m^* = - \frac{Q_m'(\cos h t_i)}{P_m'(\cos h t_i)} \frac{1}{(1 - X_m^1)} B_m^*$$

(2.72)

where

$$(\xi - 1) X_m^1 = \left[ \frac{Q_m'^* \frac{\partial P_m}{\partial t} - P_m'^* \frac{\partial Q_m}{\partial t}}{P_m'' \frac{\partial Q_m'^*}{\partial t}} \right]_{t = t_i}$$

(2.73)

Using the expression for $B_m^*$ from (2.70) we obtain

$$A_m^* = - \frac{Q_m'(\cos h t_i)}{P_m'(\cos h t_i)} \frac{1}{(1 - X_m^1)} \frac{\mu \cos \alpha (\xi m + 1)}{b^2 m(\xi m + 1)} P_m'(\cos \theta)$$

$$A_m^{(1)} = - \frac{Q_m'(\cos h t_i)}{P_m'(\cos h t_i)} \frac{1}{(1 - X_m^1)} \frac{\mu \sin \alpha (\xi m + 1)}{b^2 m(\xi m + 1)} P_m'(\cos \theta)$$

(2.74)

where

$$(\xi - 1) X_m^1 = \left( \frac{Q_m P_m^* - P_m Q_m^*}{P_m Q_m'^*} \right)_{t = t_i}$$

$$(\xi - 1) X_m^{(1)} = \frac{\sin h t_i}{b \sqrt{\cos h t_i - \cos \phi}} \left( \frac{Q_m P_m^* - P_m Q_m^*}{P_m Q_m'^*} \right)_{t = t_i}$$

(2.75)

The field components are given by

$$- \frac{1}{b \sqrt{\cos h t_i - \cos \theta}} \frac{\partial V}{\partial t}, \quad - \frac{1}{b \sqrt{\cos h t_i - \cos \theta}} \frac{\partial V}{\partial \theta}$$

and at $\theta = \pi - \theta_0$, $t = 0$ the image field inside is given by

$$F_\theta = - \frac{1}{b} \sum_{n=0}^{\infty} \left( \frac{c_\theta}{P_m'(\cos \phi)} \right)^m m(\xi m + 1) \cos \phi$$

$$F_\phi = - \frac{1}{b} \sum_{n=0}^{\infty} \left( \frac{c_\phi}{P_m'(\cos \phi)} \right)^m m(\xi m + 1)$$

(2.76)

(See Fig. 5)

and at $\theta = \theta_0$, $t = 0$
\[
\begin{align*}
F_t &= \frac{1}{b} \sum_{n=0}^{\infty} A_n (\omega t) \frac{n+1}{2} \cos \phi \\
F^\alpha_t &= \frac{1}{b} \sum_{n=0}^{\infty} A_n (\omega t) R_n (\omega \theta)
\end{align*}
\]  \( (2.77) \)

Putting the fields at \( x_1 = \frac{\xi}{2} \) and \( x_1 = -\frac{\xi}{2} \) as

\[
F^\alpha_t = C \alpha + D_0 (\alpha \cdot k) k \quad \text{and} \quad F = C \alpha + D (\alpha \cdot k) k
\]  \( (2.78) \)

respectively,

\[
F_t = C \alpha \cos \phi \\
F^\alpha_t = - (C + D) \alpha \cos \phi
\]  \( (2.79) \)

\[
F_t = C \alpha \sin \phi \\
F^\alpha_t = - (C + D) \alpha \sin \phi
\]  \( (2.80) \)

Using equations \( (2.74), (2.76), (2.77), (2.79) \) and \( (2.80) \) we finally obtain

\[
C_0 = \frac{1}{b^3} \sum_{n=0}^{\infty} \frac{Q_n (\omega t \phi)}{Q_n (\omega t \phi)} \frac{1}{(n+\frac{1}{2})} \left\{ \frac{P_n^\prime (\omega \theta)}{P_n (\omega \theta)} \right\}^2
\]

\[
C = \frac{1}{b^3} \sum_{n=0}^{\infty} \frac{Q_n^\prime (\omega t \phi) \ell}{Q_n (\omega t \phi)} \frac{1}{(n+\frac{1}{2})} \left\{ \frac{P_n^\prime (\omega \theta)}{P_n (\omega \theta)} \right\}^2
\]

\[
C + D_0 = \frac{2}{b^3} \sum_{n=0}^{\infty} \frac{Q_n (\omega t \phi) \ell}{Q_n (\omega t \phi)} \frac{1}{(n+\frac{1}{2})} \left\{ \frac{P_n^\prime (\omega \theta)}{P_n (\omega \theta)} \right\}^2
\]

\[
C + D = \frac{2}{b^3} \sum_{n=0}^{\infty} \frac{Q_n (\omega t \phi) \ell}{Q_n (\omega t \phi)} \frac{1}{(n+\frac{1}{2})} \left\{ \frac{P_n^\prime (\omega \theta)}{P_n (\omega \theta)} \right\}^2
\]  \( (2.81) \)
APPENDIX III.

If \( \lambda \) is a tensor given by

\[
\lambda = \begin{pmatrix} \partial_x & \partial_y \\ \partial_y & \partial_x \end{pmatrix},
\]

(2.82)
to prove that if \( f(\lambda) \) is any tensor function of \( \lambda \), then

\[
[f(\lambda)]'' = [f(\lambda)]' = \frac{1}{2} f(\partial_x + \partial_y) + \frac{1}{2} f(\partial_x - \partial_y)
\]

\[
[f(\lambda)]' = [f(\lambda)]' = \frac{1}{2} f(\partial_x + \partial_y) - \frac{1}{2} f(\partial_x - \partial_y)
\]

Suppose that

\[
\lambda'' = \begin{pmatrix} \delta_x(n) & \delta_y(n) \\ \delta_y(n) & \delta_x(n) \end{pmatrix}
\]

(2.83)

Multiplying the tensor \( \lambda \) given by (2.82) by the tensor \( \lambda\) given by (2.83) we obtain

\[
\begin{align*}
\delta_x(n+1) &= \delta_x \delta_x(n) + \delta_y \delta_y(n) \\
\delta_y(n+1) &= \delta_y \delta_x(n) + \delta_y \delta_y(n)
\end{align*}
\]

(2.84)

Eliminating \( \delta_y(n) \) from equations (2.84) one obtains

\[
\delta_x(n+2) - 2 \delta_x \delta_x(n+1) + (\delta_x^2 - \delta_y^2) \delta_x(n) = 0
\]

(2.85)

By eliminating \( \delta_y(n) \) we obtain a similar equation for \( \delta_y(n) \).

Putting \( \delta_x(n) \propto x^n \) we obtain from (2.85)

\[
x^2 - 2 \delta_x x + \delta_x^2 - \delta_x^2 = 0
\]

(2.86)
giving

\[
x = \delta_x \pm \delta
\]

The general expression for \( \delta_x(n) \) is therefore
\[ \chi_0(n) = A_0 (\chi_0 + \xi)^n + B_0 (\chi_0 - \xi)^n \], similarly
\[ \chi(n) = A (\chi + \xi)^n + B (\chi - \xi)^n \] (2.87)

where \( A_0, B_0, A, B \) are arbitrary constants.

Since the determinant of \( \lambda \) is \( \chi^2 - \xi^2 \) and that of \( \lambda^n \) is \( [\chi(n)]^2 - [\xi(n)]^2 \) then

\[ [\chi(n)]^2 - [\xi(n)]^2 = (\chi^2 - \xi^2)^n \] (2.88)

Also
\[ \chi_0(n) + \xi(n) = (\chi_0 + \xi)^n \] (2.89)

then from (2.88) and (2.89)

\[ \chi_0(n) - \xi(n) = (\chi_0 - \xi)^n \] (2.90)

By use of (2.89) (2.90) equation (2.97) becomes

\[ \chi_0(n) = \frac{1}{2} (\chi_0 + \xi)^n + \frac{1}{2} (\chi_0 - \xi)^n = (\chi^n)^{n-1} = (\lambda^n)^{n-1} \]
\[ \chi(n) = \frac{1}{2} (\chi_0 + \xi)^n - \frac{1}{2} (\chi_0 - \xi)^n = (\chi^n)^{n-1} = (\lambda^n)^{n-1} \] (2.91)

Hence the required formula follows.

**APPENDIX IV.**

The tensors \( \chi_0, \chi \) are given by

\[ \chi_0 = A_0 s_{ss} + (B_0 - A_0) k_s k_s' \]
\[ \chi = A s_{ss} + (B - A) k_s k_s' \] (2.92)

where \( A_0, B_0, A, B \) are constants and \( k \) is a unit vector. It is required to prove that if \( F(\chi^n, \chi) \) is any tensor function of \( \chi^{n-1}, \chi^0 \), then

\[ \{F(\chi^n, \chi) \}_{ss'} = F(A, A) s_{ss'} + \{F(B, B) - F(A, A) \} k_s k_s' \] (2.93)
We first consider the tensor \((\gamma^n)_{ss'}\); omitting the suffixes \(s, s'\) we have

\[
\gamma = A + (B - A)(k k)
\]

\[
\gamma^n = \sum_{\lambda=0}^{n} A^{n-\lambda} (B - A)^\lambda \binom{n}{\lambda} (k k)^\lambda
\]  
(2.94)

We also have

\[
[(k k)^1]_{ss'} = \sum_{s''} k_s k_{s''} k_{s''} k_{s'} = k_s^2 k_{s''} = (k k)_{ss'}
\]

by successive premultiplication by \((k k)\) we obtain

\[
(k k)^n = (k k)_{ss'}^n \quad n \geq 1
\]  
(2.95)

Substituting from (2.95) into (2.94) we obtain

\[
\gamma^n = A^n + (k k) \sum_{\lambda=0}^{n} A^{n-\lambda} (B - A)^\lambda \binom{n}{\lambda} (k k)
\]

\[
= A^n + (B^n - A^n)(k k)
\]  
(2.96)

Similarly

\[
\delta_0^n = A_0^n + (B_0^n - A_0^n)(k k)
\]  
(2.97)

From (2.95), (2.96) and (2.97)

\[
\delta_0^m \gamma^n = (B^n B^n - A^n A^n)(k k)
\]  
(2.98)

Hence the required formula (2.93) follows.
CHAPTER III

APPLICATION TO ICE.

The Structure of the Crystal.

The positions of the oxygen atoms in the crystal are known from X-ray diffraction experiments. They form a hexagonal crystal with four oxygen atoms in a unit cell. The oxygen atoms lie in planes normal to the axis of the crystal; the distances between successive planes are \( c \) and \( \frac{2}{3} \) alternately as shown in Fig. 6; \( c \) being the distance between first neighbours.

The positions of the atoms in the planes drawn in black in Fig. 6 are shown in black in Fig. 7, while the positions of the atoms in the planes drawn in red in Fig. 6 are shown in red in Fig. 7.

If \( \mathbf{k}_s \) \((s=1,2,3,4)\) are unit vectors along the lines joining an atom and its four first neighbours then

\[
\mathbf{k}_s \cdot \mathbf{k}_s' = -\frac{1}{3} \quad (s \neq s')
\]

which means that each oxygen atom is symmetrically surrounded by four oxygen atoms.

If \( (a_1, a_2, a_3) \) are the fundamental translations of the unit cell then it is clear from Figs. 6 and 7 that

\[
a_1 = a_2 = \sqrt{\frac{2}{3}} c \quad a_3 = \frac{8}{3} c
\]
giving the dimensions of a unit cell in terms of the distance between first neighbours.

Also the tensor $g$ defined by (1.33) is given, for the Ice crystal, by

$$
g = \begin{pmatrix}
1 & -\frac{1}{2} & 0 \\
-\frac{1}{2} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad (3.3)
$$

If $v$ is the volume of a unit cell then $v^2$ is given by the determinant

$$
v^2 = \begin{vmatrix}
\frac{2}{3}c^2 & -\frac{8}{3}c^2 & 0 \\
-\frac{8}{3}c^2 & \frac{8}{3}c^2 & 0 \\
0 & 0 & \frac{64}{9}c^2
\end{vmatrix} = 3\left(\frac{32}{9}\right)^2c^6,
$$

or

$$
v = \frac{12\sqrt{3}}{\pi}c^3 \quad (3.4)
$$

Since there are four molecules in each unit cell then

$$
\frac{1}{N} = \frac{8\sqrt{3}}{9}c^3 = \frac{4\pi}{3}a^3 \quad (3.5)
$$

where $N, a$ are defined by (1.4).

The distance $O-H$ in the steam molecule is $0.95 \text{ A}^\text{w}$ while the angle $H-O-H$ is about 109°. It has been assumed that the hydrogen atoms will retain the same relative positions in Ice.

Since the angle between the two lines joining an oxygen atom and any two of the four first neighbours is $105°(\frac{-1}{2}) = 109° \text{ A}^\text{w}$, i.e., nearly the same as the angle $H-O-H$ in the steam molecule, then it is possible geometrically for the molecule to have a
configuration similar to that in the gas with the two hydrogen atoms lying on two lines joining the oxygen atom of the same molecule and two of its first neighbours.

There are \( \binom{6}{2} = 6 \) such orientations for the molecule. The question now is whether the molecule rotates freely in the solid or assumes such definite restricted orientations. It will be shown that the high dielectric constant of ice near the melting point can only be explained if the dipolar molecules are free, or nearly free to rotate. At lower temperatures when the dielectric constant drops to a small value, it may happen that the dipoles restrict their rotations to between the six orientations mentioned before. The distribution of the dipoles at any instant is then very probably regular, i.e., the dipoles are divided into groups, or unit cells, which are similar in their configurations.

**Experimental values of the Dielectric Constant.**

Curves for the dielectric constant of ice, for the lower frequencies, are shown on the same scale in Figs. (8), (9), (10) and (11), showing values of: due to Errera (1924), the International Critical Tables (1929), Winteh (1932), and Smyth and Hitchcock (1932), respectively.
Murphy (1934) has recorded, for 1 = 1 cycle/sec, a value of \( \varepsilon = 150 \) at \( T = -90^\circ \), this value decreasing gradually with rising temperatures and rapidly with falling temperatures. His values \( \varepsilon = 95 \) at \( T = -7.5^\circ \) for \( 1 = 30 \) and \( \varepsilon = 77 \) at \( T = 0 \) for \( 1 = 1000 \) are much higher than the corresponding values due to Smyth and Hitchcock. Drawing curves relating the dielectric constant and the frequency (at zero temperature) for different authors, one would expect by extrapolating to zero frequency, the following values for the dielectric constant at zero temperature and for a steady field:

Errera \( \varepsilon = 81 \), Wintoch \( \varepsilon = 86 \), Smyth and Hitchcock \( \varepsilon = 74 \), Granier (1924) has given the high value \( \varepsilon = 153 \), and Weymann (1931) has given the value \( \varepsilon = 88 \), the same as that of water at zero degree.

It is noticed that the value given by Smyth and Hitchcock is the lowest of all. It was explained by them that the low values of the dielectric constant they recorded, were due to the purity of the sample they experimented upon. It is shown in their paper that the existence of small quantities of KCl in water will raise the dielectric constant of ice near the melting point to a much higher value. The higher values of the dielectric constant recorded by others may then be due to existence of small quantities of such impurities.
Darsey (1940) has given an empirical equation which fitted the curves given by Smyth and Hitchcock for all frequencies. For \( t = 0 \) and a steady field it gives

\[
\begin{align*}
\varepsilon &= 74.6 \\
\end{align*}
\]  

(3.6)

The Tensor \( \mathbf{A} \).

In order to calculate the dielectric constant of ice from equation (1.53), we start by calculating the tensor \( \mathbf{A} \) appearing in that equation and defined by equations (1.39) and (1.59).

Putting

\[
\mathbf{A}_{ss'} = \alpha \sum_{j} \frac{x_j x_{s} y_j y_{s'}}{R_{j}^{2}}
\]

(3.7)

then

\[
\alpha \sum_{j} \frac{1}{R_{j}^{2}} = \sum_{ss'} \mathbf{A}_{ss'} = 2 \mathbf{A}_{b} + \mathbf{A}_{12} + \mathbf{A}_{33}
\]

(3.8)

Substituting from (3.7), (3.8) in (1.39) one obtains

\[
(\mathbf{A} - 1)_{ss'} = 2 \mathbf{A}_{b} + \mathbf{A}_{12} + \mathbf{A}_{33} \delta_{ss'} - 3(\theta \cdot \mathbf{A})_{ss'}
\]

(3.9)

It is noticed, referring to Fig. (7), that in any plane there are always three molecules with co-ordinates.

\[
(x_{1}, x_{2}) (-x_{1} + x_{2}, -x_{1}) (-x_{2}, x_{1} - x_{2})
\]

These co-ordinates are equidistant from the origin and
at an equal angular separation \( \frac{2\pi}{3} \). For three such molecules

\[
\sum_{i=1}^{3} \chi^{(i)}_s = 0, \quad \sum_{i=1}^{3} x^{(i)}_s x^{(i)}_s = \left( \frac{2\lambda}{3}, \lambda^2 \right)
\]  
(3.10)

where \( \lambda \) is the distance of any of them from the origin in the plane.

If there are \( m \) molecules in a plane at an equal distance \( \lambda \) from the origin, then

\[
\sum_{i=1}^{m} \chi^{(i)}_s = 0, \quad \sum_{i=1}^{m} x^{(i)}_s x^{(i)}_s = \left( \frac{2m\lambda}{3}, \frac{m\lambda^2}{3} \right)
\]  
(3.11)

The tensor \( A \) defined by (3.7) can then be written

\[
\begin{align*}
A_{11} &= A_{22} = \frac{2}{3} \alpha \sum_{i=1}^{m} \frac{m_i \delta^2}{R^2_i} \\
A_{33} &= \alpha \sum_{i=1}^{m} \frac{m_i \lambda^2}{R^2_i} \\
A_{12} &= A_{13} = \frac{1}{2} A_{11} \\
A_{23} &= A_{32} = 0
\end{align*}
\]
(3.12)

where \( m_j \) is the number of molecules in the plane \( \ell_j \) at equal distances \( \lambda_j \) from the origin in that plane; \( R_j \) is the distance of the set of molecules from the origin of co-ordinates, i.e.,

\[
R_j^2 = \ell_j^2 + \lambda_j^2
\]

Since, using (3.12) and (3.3)

\[
(g \Lambda)_{AB} = \begin{pmatrix}
\frac{2}{3} A_{11} & 0 & 0 \\
0 & \frac{1}{3} A_{11} & 0 \\
0 & 0 & A_{33}
\end{pmatrix}
\]
(3.13)

then equation (3.9) becomes

\[
\begin{align*}
A_{11} - 1 &= A_{33} - \frac{2}{3} A_{11} = B \quad \text{(say)} \\
A_{33} - 1 &= -2 A_{33} + \frac{3}{2} A_{11} = -2 B
\end{align*}
\]
(3.14)

and the non-diagonal elements vanish.
\[ B = \frac{\alpha}{c^3} \sum_{j} \frac{c^3 m_j}{R_j^2} (l_j^2 - \frac{1}{3} k_j^2) \]  \quad (3.15)

In tables (2) and (3) the number \( m_j \) of molecules at the same distance from the origin and \( R_j^2 \), the square of their distances from the origin in their plane are tabulated for planes marked in black and red (See Fig. (7)) respectively.

In table (4) \( m_j \), \( R_j^2 \), \( l_j^2 \) and \( R_j^2 \) are tabulated for molecules in all planes up to the 100th nearest neighbours. It is found that

\[ \frac{10^4}{27} \sum_j \frac{c^3 m_j}{R_j^2} (l_j^2 - \frac{1}{3} k_j^2) \approx 14 \] \quad (3.16)

The quantity \( \frac{\alpha}{c^3} \) appearing in (3.15) is given, by use of equation (3.5), by

\[ \frac{\alpha}{c^3} = \frac{8\sqrt{3}}{9} N \alpha = \frac{8\sqrt{3}}{9} \frac{\alpha d}{M} \times \frac{10^{23}}{(166)} \]

where \( \alpha \) is the density and \( M \) the molecular weight. For \( T = 273 \) we have \( \alpha = 9.168 \)

also \( \frac{M}{\alpha} = 18.0156 \)

\[ \alpha = 0.16 \times 10^{-23} \] \quad (3.17)

then

\[ \frac{\alpha}{c^3} = 0.7575 \] \quad (3.18)

Substituting from equations (3.18) and (3.16) into (3.15) one finally obtains

\[ B = 0.003 \] \quad (3.19)

to three places of decimals
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Extrapolating for \(\sum 9 \frac{n^2}{c^2} (-\frac{c^3}{2}) = 0\),

we obtain

\[
\sum 10^4 \frac{nc^3}{R^5} (-\frac{c^3}{2}) = -14.4326
\]
The Mutual Potential Energy.

The field acting on one of the two first neighbours, included inside a cavity of the form of an ellipsoid of revolution, is given by equation (2.11) where $A_0$, $B_0$, $\lambda$ and $\beta$ appearing in that equation are given by (2.21) and (2.22). The dimensions of the ellipsoid of revolution are determined by the distance between the foci $2b$ and the major axis $2b \cosh \lambda$; these are related by equation (2.15).

It is required that the ellipsoid should coincide as nearly as possible with a cavity of two overlapping spheres whose dimensions are given by (2.14).

Substituting in equation (2.14) $\frac{a^3}{c^3} = \frac{2}{\pi \sqrt{3}}$ given by (3.5), one finds that

$$x = \frac{7}{3}$$

(3.20)

which determines the radius $xc$ of each of the overlapping spheres.

Putting now $\cosh \lambda = \ldots 1.2, 1.3, 1.4, \ldots$ one obtains from (2.15) $2b \cosh \lambda = \ldots 2.9032, 2.4331, 2.1261$, respectively. It is found that for $\cosh \lambda = 1.3$ the ellipsoid will coincide as nearly as possible with the cavity of the two overlapping spheres. This is clearly shown in Fig. (12).

For $\cosh \lambda = 1.3$

$$\frac{c^3}{b^3} = 1.2202335$$

$$\frac{c}{2b} = 0.34299$$

(3.21)
In Table (6) values of \( P_n(x, \omega t), P'_n(x, \omega t), P''_n(x, \omega t) \)
\( Q_n(x, \omega t), Q'_n(x, \omega t), Q''_n(x, \omega t) \)
for \( \omega t = 1.3 \) are tabulated for \( n = 1 \) to \( n = 10 \).
In Table (6) those for \( P'_n(x, \omega t) \) are given.
In Table (7) the values of \( (\varepsilon - 1) X_0', (\varepsilon - 1) X_0^0 \)
are tabulated for \( \omega t = 1.3 \).

Finally in Table (8) values of

\[
\frac{\varepsilon^3}{b_1^3} \left( n + \frac{1}{2} \right) \left[ \frac{P'_n(x, \omega t)}{P_n(x, \omega t)} \right]^2 \frac{Q'_n}{P'_n} \frac{1}{1 - X'_n}
\]
and

\[
\frac{\varepsilon^3}{b_1^3} \left( n + \frac{1}{2} \right) \left[ \frac{P'_n(x, \omega t)}{P_n(x, \omega t)} \right]^2 \frac{Q''_n}{P''_n} \frac{1}{1 - X''_n}
\]
are tabulated, using for \( \varepsilon \) the value given by (3.6).

The values of \( A_0, B_0, A, B \) are then

\[
\begin{align*}
A_0 &= 1 - \frac{\varepsilon}{c_3} (2.6114) = 0.8022 \\
B_0 &= 1 - \frac{2\varepsilon}{c_3} (0.99545) = 0.8092 \\
A &= \frac{\varepsilon}{c_3} - \frac{\varepsilon}{c_3} (0.6405) = 0.0272 \\
B &= -2\frac{\varepsilon}{c_3} - 2\frac{\varepsilon}{c_3} (0.0045) = -0.1522
\end{align*}
\tag{3.22}
\]
using the value of \( \frac{\varepsilon}{c_3} \) given by (2.18).

The mutual potential energy is expressed by (2.39) where \( G, L, M \) are given by (2.40) and (2.29). Using the values provided by (3.22) one obtains

\[
\begin{align*}
F(0, B) - F(A_0, A) &= 0.3135 \\
F(A_0, A) &= 0.4232 \\
F(B, B) &= -0.21806
\end{align*}
\tag{3.23}
\]
Taking \( \rho = 1.845 \times 10^{-18} \)

\[
\tag{3.24}
\]

then for \( \mathcal{T} = 273 \)
Table (5)

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Table (6)

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\[ \frac{\beta^2}{\sigma^2} = 56.27518 \]  

Therefore from (3.23), (3.25) and (2.40)

\[ G = 3.824 \quad L = 12.2714 \quad M = 2.3816 \]  

(3.26)

The Tensor \( Q \).

The average \( \beta_0^0 \beta_0^0 \) is given by (2.53), (2.55) for \( i \neq j \), \( i = j \) respectively; \( \kappa_0, \kappa_1 \) and \( \kappa_2 \) appearing in equation (2.53) are expressed by (2.49) (2.50) and (2.51). The evaluation of the double integrals over \( \theta, \theta' \) between the limits \( \theta = 0 \rightarrow 90^\circ, \theta' = 0 \rightarrow 90^\circ \) will be effected numerically by dividing the range between the two limits into \( 9 \times 9 = 81 \) intervals; the difference between the upper and lower limit of both \( \theta, \theta' \) for each interval is 10 degrees.

The values of the integrands at the middle of each interval are tabulated in Table (9). There are 9 diagonal intervals \( (\theta = \theta') \) and \( 2 \times 36 = 72 \) non-diagonal elements \( (\theta \neq \theta') \). The values of \( \kappa_0, \kappa_1, \kappa_2 \) are found to be

\[
\begin{align*}
\kappa_0 &= (\frac{\pi}{\beta})^2 \cdot 55.26 \cdot 89 \\
\kappa_1 &= (\frac{\pi}{\beta})^2 \cdot 514.7 \cdot 15 \\
\kappa_2 &= (\frac{\pi}{\beta})^2 \cdot 324.27
\end{align*}
\]  

(3.27)

then

\[
\frac{\kappa_1 + \kappa_2}{\kappa_0} = 0.73678
\]  

(3.28)

In Table (4) it is noticed that \( \sum \frac{\lambda^y_{15}}{\beta} \) over
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first neighbours, vanishes, i.e.
\[
\sum_{\delta=1}^{\infty} (\delta_{\delta}^{-1} - 3 \ k_5^{\delta} \ k_{s'}^{\delta}) = 0
\]  
(3.29)
where \( k_{s'}^{\delta} \) are the components of a unit vector along the line joining the molecules \( \omega', \omega' \). Then
\[
\sum_{\delta=1}^{\infty} k_{s'}^{\delta} \ k_{s'}^{\delta} = \frac{4}{3} \ \delta_{s's'}
\]
(3.30)
Summing \( \delta_{s's'} \) over first neighbours using equation (3.30) one obtains
\[
\frac{1}{\mu^2} \sum_{\delta=1}^{\infty} p_{s'}^{\delta} p_{s'}^{\delta} = \frac{4}{3} \ \frac{K_o}{K_o} \ \delta_{s's'}^{\delta} + \frac{4}{3} \ (\frac{K_1}{K_o} - \frac{1}{2} \ \frac{K_2}{K_o}) \ \delta_{s's'}^{\delta}
\]
\[
= \frac{4}{3} \ (\frac{K_1 + K_2}{K_o}) \ \delta_{s's'}
\]
(3.31)
For further neighbours \( A_0, B_0, A, B \) are given by (2.12) and (2.13). Since, for ice, \( A \) is very small we may neglect its square and higher powers, and so
\[
F (A_0 A) = A \quad F (B_0 B) = B = -2 A \quad \text{then}
\]
\[
G = 0 \quad M = \frac{9 \ \varepsilon}{(2 \alpha + 1)^2} \ \frac{\mu^2}{K_o} \quad L = -2 M
\]
(3.32)
where \( R \) is the distance of the molecule considered from the molecule taken as origin.

Neglecting second and higher powers of \( M \) in the double integrals for \( K_0, K_1, K_2 \) we obtain
\[
K_0 = 1 \quad K_1 = \frac{L}{4} \quad K_2 = \frac{2M}{9}
\]
(3.33)
Therefore \( \frac{h^{(i)}_{p_j}}{h^{(i)}_{p_j}} \) for second and higher neighbours will be given by (from (2.53))

\[
\frac{1}{p_j} \frac{\partial}{\partial y_j} \phi_j = -\frac{\varepsilon}{(\ell + 1)^2} \frac{p_i^2}{K_j^2} (9_{ss}^{(i)} - 3 \alpha \kappa \phi_j)
\]

(3.34)

Remembering, according to (3.29) that the sum on the right hand side of (3.34) vanishes for first neighbours, it is seen that the sum of the left hand side for second and further neighbours is

\[
\frac{1}{p_j} \sum_{j > 4} \frac{\partial}{\partial y_j} \phi_j = -\frac{\varepsilon}{(2 \ell + 1)^2} \frac{p_i^2}{\alpha \kappa T} \left[ \frac{1}{2}(A - 1) \right]_{ss}
\]

(3.35)

using the definition of \( \bar{A} \) given by (1.39).

Combining (3.31), (3.35) one finally obtains

\[
Q_{ss} = \frac{3}{2} \phi_{ss}^{(i)} + \frac{5}{p_j} \sum_{j = 1}^{5} \frac{\partial}{\partial y_j} \phi_j^{(i)} + \frac{1}{p_j} \sum_{j > 4} \frac{\partial}{\partial y_j} \phi_j^{(i)}
\]

\[
= \frac{3}{2} \phi_{ss}^{(i)} \left[ 1 + 4 \frac{\eta_i + \eta_2}{\Omega_0} \right] - \frac{\varepsilon}{(2 \ell + 1)^2} \frac{p_i^2}{\alpha \kappa T} \left[ \frac{1}{2}(A - 1) \right]_{ss}.
\]

(3.36)

Expressing

\[
Q_o = \frac{1}{3} \left[ 1 + 4 \frac{\eta_1 + \eta_2}{\Omega_0} \right]
\]

(3.37)

\[
S_o = \frac{\varepsilon}{(2 \ell + 1)^2} \frac{p_i^2}{\alpha \kappa T}
\]

(3.38)

then

\[
\left( \phi \phi \right)_{ss} = Q_o \delta_{ss} - S_o (A - 1)_{ss}
\]

(3.39)
By use of (3.6), (3.25) and (3.28) one obtains

\[ Q_0 = 1.31570 \]
\[ S_0 = 1.186087 \]  \hspace{1cm} (3.40)

To calculate the dielectric constant from equation (1.53) we express it in the form

\[ \frac{\varepsilon_{33} - \varepsilon_{33}}{3} = \frac{3 \varepsilon_0}{2 \varepsilon_0 + 1} \left[ F(A) \right]_{15} \]  \hspace{1cm} (3.41)

where \( F(A) \) is a function of the tensor \( A \), then

\[ \frac{\varepsilon_{33} - 1}{3} = \frac{3 \varepsilon_0}{2 \varepsilon_0 + 1} F(A_{11}) \] \hspace{1cm} (3.42)

\[ \frac{\varepsilon_{33} - 1}{3} = \frac{3 \varepsilon_0}{2 \varepsilon_0 + 1} F(A_{33}) \]

The non-diagonal elements of the tensor \( A \) and therefore of \( \varepsilon \) vanish.

Since \( A_{11} = 1 + B \quad A_{33} = 1 - 2B \) \hspace{1cm} (See (3.14)

then we can express

\[ \varepsilon_{33} = \varepsilon_0 + A \quad \varepsilon_{33} = \varepsilon_0 - 2A \] \hspace{1cm} (3.43)

where

\[ \frac{\varepsilon_{33} - 1}{3} = \frac{3 \varepsilon_0}{2 \varepsilon_0 + 1} \left[ F(A) \right]_{A=1} \]

\[ A = \frac{3 \varepsilon_0}{2 \varepsilon_0 + 1} \left( \frac{\partial F(A)}{\partial A} \right)_{A=1} B \] \hspace{1cm} (3.44)

Equations (3.44) can be expressed in the form

\[ \frac{\varepsilon_{33} - 1}{3} = \frac{\varepsilon_0 + A}{3} + \left( \frac{3 \varepsilon_0}{2 \varepsilon_0 + 1} \right) \frac{B}{a^3 kT} Q_0 \] \hspace{1cm} (3.45)
\[ A = - \frac{9 \varepsilon_0}{2 \varepsilon_{0+1}} \left[ \frac{\beta}{(1-\beta^2)^2} + \frac{2-\beta^2}{(1-\beta^2)^2} \frac{\beta^2}{\alpha^2 k^T} \frac{1}{(1-\beta^2)\alpha^2 k^T} S_0 \right] B \]  

Substituting from (3.45) into (3.46) one obtains

\[ - \frac{A}{B} = \frac{1}{(1-\beta^2)} \left[ 2(\varepsilon_{0+1}) - \beta (\varepsilon_{0+2}) \right] + \frac{1}{(1-\beta^2)} \frac{\beta^2}{\alpha^2 k^T} \frac{9 \varepsilon_0}{2 \varepsilon_{0+1}} S_0 \]  

Putting

\[ \beta = 0.2061 \quad \frac{\beta^2}{\alpha^2 k^T} = 11.5984 \]  

and substituting for \( \varepsilon_0 \) from (3.40), equation (3.45) becomes

\[ 1.5878 \varepsilon_0^2 - 139.3706 \varepsilon_0 - 1.4122 = 0 \]

giving

\[ \varepsilon_0 = 87.84 \]  

Also from (3.40) (3.47) (3.48) and (3.19)

\[ - \Delta = 0.62 \]  

By use of equations (3.43) (3.49) and (3.50) we finally obtain

\[ \varepsilon_u = \varepsilon_2 = 87.22 \quad \varepsilon_3 = 89.08 \]  

and the non-diagonal elements vanish.

One notices from this result obtained the following:

1. The dielectric constant along the axis is somewhat bigger than that perpendicular to the axis; in spite of the fact that the results of all the experiments available do not distinguish between the values along
the axis and those perpendicular to the axis.

(ii) The value of the dielectric constant calculated is higher than that given by the most accurate experiments, and which we used to calculate the tensor \( Q \). This may be explained by our assumption that the molecules in ice are free to rotate. There may be some restriction on the rotations of the molecules which will lower the dielectric constant.

**Restricted Rotations.**

In this section we shall consider any molecule in the ice crystal to have only 6 definite orientations. The rotation of the molecules between these orientations can either happen in an orderly way, that is, at any instant the molecules are divided into groups with similar configurations, or in a disorderly way. The former case is the more probable and will be dealt with in the last chapter. The latter case will be treated in this chapter, just to show that it will not give the expected high dielectric constant near the melting point.

If \( \mathbf{f}_i \), \( (s = 1, 2, 3, 4) \) are unit vectors along the four lines joining a molecule to its four first neighbours; then a unit vector along the dipole has one of the 6 values

\[
\frac{\sqrt{3}}{2} (\mathbf{f}_s + \mathbf{f}_{s'}) \quad (s \neq s')
\]  

(3.52)
The cosine of the angle between such a dipole and either \( \mathbf{E}_1 \) or \( \mathbf{E}_2 \) is \( \frac{1}{\sqrt{3}} \); and with the other two lines it makes an angle whose cosine is \( \frac{1}{\sqrt{3}} \).

Three of the six vectors given by equation (3.52) by use of (3.1), are mutually perpendicular and opposite to the other three.

The six states of the molecule will be labelled

\[
\sigma = (1, 2, 3, \overline{1}, \overline{2}, \overline{3})
\]

so that the direction \( \sigma \) is opposite to the direction \( \sigma' \). These states are shown in Fig. (13) for the middle molecule and one of the upper three molecules. (By upper three we mean those in red plane in Fig. 6). In Fig. (14) they are shown for the middle molecule and the lower molecule.

We consider the mutual potential energy \( W_{\sigma \sigma'} \) between the middle molecule in state \( \sigma \) and one of the first neighbours in state \( \sigma' \). It is clear that

\[
W_{\sigma \sigma'} = - W_{\overline{\sigma} \overline{\sigma}'}
\]

so that we need only write down the energy tensor for \( \sigma, \sigma' = 1, 2, 3 \). Using the expression for \( W \) given by equation (2.39), we may write

\[
- \frac{W_{\sigma \sigma'}}{kT} = L \left( \cos \theta \cos \theta' \right)_{\sigma \sigma'} + M \left( \sin \theta \sin \theta' \cos \phi \right)_{\sigma \sigma'}.
\]

The term containing \( Y \) has been omitted since

\[
\cos^2 \theta = \frac{1}{3}
\]

for all orientations of the molecule.

For the middle molecule and one of the upper three

\[
\cos \theta \cos \theta' = \begin{pmatrix}
\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
-\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{pmatrix}
\]

(3.54)
and for the middle and the lower molecules

\[
\cos \theta \cos \theta' \equiv \begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}
\]  
\tag{3.55}

The tensors \( W, \ W' \) for the upper, and the lower molecules respectively can then be written, following equations (3.53), (3.54) and (3.55)

\[
-\frac{W}{kT} \equiv \begin{bmatrix} \frac{1}{3}L+\frac{1}{3}M, \frac{1}{3}L+\frac{1}{3}M, \frac{1}{3}L+\frac{1}{3}M \\ \frac{1}{3}L+\frac{1}{3}M, \frac{1}{3}L+\frac{1}{3}M, \frac{1}{3}L+\frac{1}{3}M \\ \frac{1}{3}L+\frac{1}{3}M, \frac{1}{3}L+\frac{1}{3}M, \frac{1}{3}L+\frac{1}{3}M \end{bmatrix}
\]  
\tag{3.56}

\[
-\frac{W'}{kT} \equiv \begin{bmatrix} \frac{1}{3}L+\frac{1}{3}M, \frac{1}{3}L+\frac{1}{3}M, \frac{1}{3}L+\frac{1}{3}M \\ \frac{1}{3}L+\frac{1}{3}M, \frac{1}{3}L+\frac{1}{3}M, \frac{1}{3}L+\frac{1}{3}M \\ \frac{1}{3}L+\frac{1}{3}M, \frac{1}{3}L+\frac{1}{3}M, \frac{1}{3}L+\frac{1}{3}M \end{bmatrix}
\]  
\tag{3.57}

Using for the probability Boltzmann's expression (1.17) we may write the probability tensors \( \rho, \rho' \) corresponding to the energy tensors \( W, W' \) in the form.

\[
\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} \\ \rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55} \end{pmatrix}
\]  
\tag{3.58}
where, using for $L, M$ the values given by $(3.26)$

\[
\begin{align*}
36 \epsilon_n &= 0.26488 \\
36 \epsilon_n &= 0.0017 \\
36 \epsilon_{iz} &= 0.0178 \\
36 \epsilon_{iz} &= 2.86650 \\
36 \epsilon_{iz} &= 5.06298 \\
36 \epsilon_{iz} &= 0.46784 \\
\end{align*}
\]  

We now consider the average $\overline{p^0_p^L_p^M}$, where $p$ is the component along the axis of the ice crystal, for the middle molecule and one of the upper three molecules

\[
\overline{p^0_p^L_p^M} = \frac{1}{2} p^2 \left[ 6 \epsilon_n + 8 \epsilon_{iz} + 4 \epsilon_{iz} - 6 \epsilon_{iz} - 8 \epsilon_{iz} + 4 \epsilon_{iz} \right] \\
= \frac{1}{2} p^2 \left[ 3 (\epsilon_n - \epsilon_{iz}) + 2 (\epsilon_{iz} - \epsilon_{iz}) \right]  
\]  

and for the middle and lower molecules

\[
\overline{p^0_p^L_p^M} = \frac{1}{2} p^2 \left[ -6 \epsilon_n + 12 \epsilon_{iz} + 6 \epsilon_{iz} + 12 \epsilon_{iz} \right] \\
= \frac{1}{2} p^2 \left[ (\epsilon_n - \epsilon_{iz}) + 2 (\epsilon_{iz} - \epsilon_{iz}) \right]  
\]  

Summing $\overline{p^0_p^L_p^M}$ over the four first neighbours then

\[
\frac{1}{2} \sum_{p_l} \overline{p^0_p^L_p^M} = 6 (\epsilon_n - \epsilon_{iz}) + 4 (\epsilon_{iz} - \epsilon_{iz}) + 2 (\epsilon_{iz} - \epsilon_{iz}) + 4 (\epsilon_{iz} - \epsilon_{iz})  
\]  

Neglecting $\overline{p^0_p^L_p^M}$ for further neighbours, we obtain,
by use of equations (3.60) (3.61) and (3.64)

\[ Q_0 = \frac{1}{3} + \mu \frac{2}{3} \rho \frac{\rho_z}{\rho_1} = 0.191887 \quad (3.65) \]

Substituting for this value of \( Q_0 \) in (3.45)

we obtain

\[ 1.5878 \varepsilon_0^2 - 2.0305 \varepsilon_0 - 1.4122 = 40.9074 \varepsilon_0 \]

giving

\[ \varepsilon_0 = 27.09 \quad (3.66) \]

which is much lower than the value we used for the
calculation of \( L, M \), namely \( \varepsilon_0 = 74.6 \)
CHAPTER IV.

AN EXPRESSION FOR THE CORRELATION COEFFICIENT $\rho$ IN LIQUIDS AND THE EFFECT OF THERMAL VIBRATIONS IN SOLIDS.

An Expression of $\rho$ for Liquids.

The positions of the molecules, in the liquid state, are neither fixed nor confined to certain volumes. A single molecule will be assumed to have the same probability of occupying any position throughout the liquid.

The tensor $\overline{P^{00}_{\rho}}$ (\(p_{\rho}\)), defined by equation (2.53) will then depend on the simultaneous positions $X^{00}$ of the two molecules $\psi_{\rho}$ in question.

The tensor $\rho$ will then be defined by

$$N^0\rho_{ss'} = \frac{1}{3} N^0 \delta_{ss'} + \sum_{n} \int \frac{q_{ss'}^{n}}{V} \rho(X, X') \, dx \, dx'$$  \hspace{1cm} (4.1)

(using orthogonal axes)

where

$$N^{-1} \rho_{ss'}^{n} = \overline{P^{00}_{\rho}} P^{00}_{\rho}$$  \hspace{1cm} (4.2)

$N^0$ is the total number of molecules and $\rho(X, X')$ is the probability that two molecules $\psi_{\rho}$ occupy positions $X, X'$ simultaneously. Since both $\rho(X, X')$ and $q_{ss'}^{n}$ depend on the difference $X - X'$, then, integrating over $X'$ in (4.1), remembering that the volume occupied by $N^0$ molecules is $\frac{N^0}{N}$, we obtain

$$\rho_{ss'} = \frac{1}{3} \delta_{ss'} + \frac{1}{N} \sum_{n} \int \frac{q_{ss'}^{n}}{V} \rho(X, X') \, dx$$  \hspace{1cm} (4.3)
Substituting for \( \varphi'(x) \) from (2.53), the integral in (4.3) becomes

\[
\int \left[\frac{K_1}{K_0} \delta(x - \frac{K_1}{K_0}) - \frac{1}{\pi} \right] K_1 e^{i\varphi(x)} x \, dx \, d\mathbf{r} = \frac{\pi}{\delta} \int_0^\infty \left(\frac{K_1 + K_2}{K_0} \right) e^{i\varphi(x)} x \, dx (4.4)\]

Writing

\[
e(x) = \frac{Z}{\delta} e^{i\varphi(x)} (4.5)
\]

for the probability that any two molecules be found simultaneously at distance \( x \) apart, then by use of (4.5) and (4.4) equation (4.3) becomes

\[
Q = \frac{1}{3} \left[1 + \frac{1}{N} \int_0^\infty \left(\frac{K_1 + K_2}{K_0} \right) 4\pi e(x) x^2 \, dx \right] (4.6)
\]

Since the liquid is isotropic we put \( A = 1 \) and the dielectric constant will be given by equation (1.54) where \( Q_0 \) is given by (4.6).

\( K_0, K_1, \) and \( K_2 \) are functions of \( x \) and are given for 1st neighbours by equations (2.49) (2.50) and (2.51).

For further neighbours we shall neglect \( \frac{K_1 + K_2}{K_0} \) and therefore the integral in (4.6) will have the upper limit \( n_1 \), instead of \( \infty \) (See Fig. (15)).

The expression \( \frac{1}{N} + \pi e(x) x^2 \, dx \) is the number of molecules which lie between spheres \( x \) and \( x + dx \) relative to the molecule at the centre of the sphere; calling this expression \( m(x) \, dx \) then

\[
Q = \frac{1}{3} \left[1 + \int_0^{n_1} \left(\frac{K_1 + K_2}{K_0} \right) m(x) \, dx \right] (4.7)
\]
Let
\[ z = \int_{0}^{1} n(h) \, dh \] 
be the co-ordination number, that is the number of first neighbours, then \( Q_0 \) may be given approximately by
\[ Q_0 = \frac{1}{J} \left[ 1 + z \left( \frac{h_1 + h_2}{h_0} \right) \right] \]
where \( h_0 \) is the value for which \( n(h) \) is greatest or more correctly given by
\[ h_0 = \frac{1}{J} \int_{0}^{1} h \, n(h) \, dh \]

**Thermal Vibrations in Solids**

If \( \mathbf{x}^{(0)} \) is the undisturbed position of a molecule \( i \) in a crystal and \( \mathbf{u}^{(0)} \) its thermal displacement, then its position vector will be given by
\[ \mathbf{x}'_{(0)} = \mathbf{x}^{(0)} + \mathbf{u}_{(0)} \]

We assume a certain thermal configuration of the whole set of molecules and proceed in the same way as we have done in the second section of Chapter I. Instead of equations (1.40) and (1.43), we have
\[ \mathbf{e} = \mathbf{e}^0 \left[ 1 + \frac{kT}{J} \sum_{j} \sum_{s} p^{(0)}_{j} (\mathbf{X}^{(0)})_{s}^{j} \mathbf{E}_{s}^{(0)} \right] \]
\[ \sum_{c} \mu_{c}^{(0)} = \sum_{j} \sum_{s} (\mathbf{X}^{(0)})_{s}^{j} \left[ p^{j}_{s} + \alpha \mathbf{E}_{s}^{(0)} + \alpha \mathbf{Y}_{s}^{j} \right] \]
The tensor \( X \) appearing in these equations depends on the new positions \( \mathbf{x}' \) given by (4.11).
The image field $\mathbf{E}_i$ will depend on the given thermal configuration, as well as on the configuration of the dipoles. Forming the product $\mathbf{E}_i \cdot \mathbf{E}_j$, and averaging over all the orientations of the dipoles leaving the thermal configuration as it is, we obtain

$$
\sum_{\mathbf{E}_j} \mathbf{E}_j \cdot \left( \mathbf{E}_i \cdot \mathbf{E}_j \right) = \frac{1}{2} \sum_{\mathbf{E}_j} \mathbf{E}_j \cdot \mathbf{E}_j - \frac{1}{2} \sum_{\mathbf{E}_j} \mathbf{E}_j \cdot \mathbf{E}_j.
$$

(4.14)

Other terms which will vanish after thermal averaging are omitted.

The tensor $q_i'$, defined by (4.2), will also depend on the new positions $\mathbf{x}'$ given by (4.11).

We may expand the tensors $\mathbf{x}'$, $q_i'$ in powers of the thermal displacements as follows

$$(\mathbf{x}')_{ij} = \mathbf{x}_{ij} + \sum \frac{2 \mathbf{x}_{ij} \cdot \mathbf{q}_i q_j}{\mathbf{x}_{ij} \cdot \mathbf{x}_{ij}} + \sum \frac{\mathbf{q}_i q_j}{\mathbf{x}_{ij} \cdot \mathbf{x}_{ij}} \mathbf{x}_{ij} \cdot \mathbf{x}_{ij},$$

(4.15)

$$(q_i')_{ij} = q_{ij} + \sum \frac{2 q_{ij} \cdot q_{ij}}{\mathbf{x}_{ij} \cdot \mathbf{x}_{ij}} q_{ij} + \sum \frac{q_{ij} q_{ij}}{\mathbf{x}_{ij} \cdot \mathbf{x}_{ij}} \mathbf{x}_{ij} \cdot \mathbf{x}_{ij},$$

(4.16)

where

$$\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$$

(4.17)

and the tensors $\mathbf{x}_{ij}'$, $q_{ij}'$ are defined by equations (1.29), (1.59), (2.53) (2.55) and (4.2).

The tensors $(\mathbf{x}')_{ij}$, $(q_i)'_{ij}$ appearing in equation (4.14) can be expanded, with the help of (4.15) and (4.16) in first and second powers of the thermal displacements.

If $\phi(\mathbf{x}')$ is the mutual potential energy
between molecules \( i \) and \( j \) then the mutual potential energy of the whole crystal is

\[
\mathcal{E}_{\text{mutual}} = \sum_i \sum_{j \neq i} \frac{x_i \cdot \nabla_j \cdot \mathcal{F}_{ij}}{2} \cdot n_i \cdot n_j.
\]  
(4.18)

where

\[
\mathcal{F}_{ij} = \sum \frac{\partial \mathcal{F}}{\partial x_i} \frac{\partial \mathcal{F}}{\partial x_j},
\]  
(4.19)

The probability of the given thermal displacement is

\[
\rho = \frac{1}{Z} \exp \left\{ -\frac{\mathcal{F}}{kT} \right\}
\]  
(4.20)

where

\[
Z = \int \exp \left\{ -\frac{1}{2} \sum_{i} \sum_{j} \frac{1}{kT} \mathcal{F}_{ij} \cdot n_i \cdot n_j \right\} \, \text{d}n_i \cdot \text{d}n_j
\]

\[
= \left( \frac{2\pi kT}{\alpha_{ij}} \right)^{i \times j}
\]
(4.21)

\( \mathcal{F}_{ij} \) is the determinant of \( \mathcal{F}_{ij} \).

Following equations (2.3) and (2.5) we have

\[
\overline{\alpha_i \cdot \alpha_i} = -2kT \frac{2}{\mathcal{F}_{ij}} \log Z
\]
(4.22)

By use of equation (4.21) we obtain

\[
\overline{\alpha_i \cdot \alpha_i} = \frac{kT}{\alpha_{ij}} \frac{\partial \mathcal{F}_{ij}}{\partial \alpha_{ij}} \frac{2}{\mathcal{F}_{ij}} = kT \left( \frac{\mathcal{F}^{-1}}{\mathcal{F}_{ij}} \right)
\]
(4.23)

where \( \mathcal{F}^{-1} \) is the reciprocal of the matrix \( \mathcal{F} \).
Since the average of terms linear in the thermal displacements vanish and the averages of quadratic terms, according to (4.23) are proportional to the temperature, we may write

\[ \sum_{\delta} (x'_{\delta})_{11} = (x_{11}^{\text{tr}})_{11} + \xi_{11}^T \]

(4.24)

\[ \sum_{\delta} (x'_{\delta} a_{\delta}^{-1} b_{\delta})_{11} = (x^{\text{tr}}Q a_{\delta})_{11} + \gamma_{11}^T \]

(4.25)

The tensor \( \xi_{11} \) depends on the structure of the crystal as well as on the form of the potential field of a molecule. The tensor \( \gamma_{11} \) depends in addition on the temperature \( T \).

Proceeding from equation (4.14) in the same way as we have done in the second section of Chapter I, we obtain, instead of (1.53),

\[ \frac{1}{3} = \frac{2}{2\xi + 1} \left[ \beta + \frac{\mu^2}{\alpha^2} \right] \frac{2\xi + 1}{2\xi + 1} \left[ \frac{\mu^2}{\alpha^2} \right] \frac{1}{1 - \beta f(\lambda^2 + \gamma^2)T} \]

(4.26)

For an isotropic crystal, putting \( \lambda = 1 \),

\[ \frac{2\xi + 1}{2\xi + 1} \left[ \beta + \frac{\mu^2}{\alpha^2} \right] \frac{2\xi + 1}{2\xi + 1} \left[ \frac{\mu^2}{\alpha^2} \right] \frac{1}{1 - \beta f(\lambda^2 + \gamma^2)T} \]

(4.27)

Neglecting second and higher powers of \( \xi \), \( \gamma \), one obtains

\[ \left[ \frac{2\xi + 1}{3} - \beta \frac{\mu^2}{\alpha^2} \right] = \frac{2\xi + 1}{2\xi + 1} \left[ \frac{\mu^2}{\alpha^2} (\beta + \frac{\mu^2}{\alpha^2}) + \frac{\xi}{\alpha^2} \right] \]

(4.28)

Putting

\[ \xi = \xi^{(0)} + \xi^{(1)} \]

(4.29)
where $\xi^{(0)}$ is the excess due to the thermal motion, then

$$\frac{\xi^{(1)}}{3} = \beta \frac{\xi^{(0)} + \xi}{3} - \frac{3 \xi^{(0)}}{2^2 + 1} \cdot \frac{\beta^2 T}{a^2 k T} = 0$$

$$\frac{\xi^{(1)}}{3 \xi^{(0)}(2 \xi^{(0)} + 1)} \left[ 2 \xi^{(0)} (1-\beta T) + (\beta T) \right] = \frac{1}{T} \left[ \frac{2}{3} \beta \frac{\xi^{(0)}}{3} + \frac{\beta^2}{a^2 k T} \cdot \frac{3 \xi^{(0)}}{2^2 + 1} \right]$$

(4.30)

In the particular case when $\beta = 0$

$$\xi^{(1)} - \beta \left( \xi^{(0)} + \xi \right) = 0$$

(4.31)

$$\xi^{(0)} = \frac{3 \beta}{(\xi^{(0)} - \xi)} \xi^{(0)} T$$

Since $\xi^{(0)}$ is independent of $T$, the last equation is in agreement with the experimental results of the slightly linear increase of $\xi$ with the temperature.
CHAPTER V.

THE TRANSITION FROM FREE ROTATIONS TO FIXED ORIENTATIONS OF POLAR MOLECULES IN THE SOLID STATE.

Experimental Evidence.

The dependence of the dielectric constant on temperature for nitrobenzene, as shown in Fig. (16) shows a discontinuity at the melting point. The difference between the dielectric constant of the liquid and the solid at the melting point is so high that it cannot be explained by the slight change of volume on the transition from the liquid to the solid state. Substances which show similar behaviour are ammonia, acetone and nitromethane.

The discontinuity in the dielectric constant may occur in the solid state as for hydrochloric acid (14), (26) (Fig. (18)) and methyl alcohol.

There is more than one discontinuity in the case of hydrogen sulphide (Fig. (19)), and finally in the case of water (Fig. (17)), there is no discontinuity in the dielectric constant but it decreases rapidly to a much lower value.

The specific heats of HCl and H2S show corresponding transitions at a transition temperature the same as that of the dielectric constant. For phosphine
there are three transitions in the solid state.

This abrupt decrease of the dielectric constant, either at the melting point or in the solid state, was explained as due to a transition from free rotations of the molecules to fixed orientations; small rotational oscillations round these fixed orientations occur similar to the thermal vibrations of the molecules. A thorough treatment of the subject with regard to the experiments and explanations was given by Smythe (21).

Frenkel considered \( \sqrt{\theta^2} \) where \( \theta \) is the angle of vibration of a dipole, as a measure of the degree of disorder; when this attains a certain value (\( J \) say) it passes to a state of complete disorder (free rotations). For a system of dipoles he uses the mutual energy between the dipoles instead of the average field adopted by others. A system of linear dipoles vibrating in one plane was treated in his book.

We shall adopt the same idea as his, treating each dipole as vibrating in two dimensions instead of one.

The Dielectric Constant Below the Transition Point.

We shall consider the dipoles below the transition point as vibrating round certain orientations, regularly
arranged throughout the crystal. As pointed out by Frenkel, there may be more than one of such arrangements and they may pass from one arrangement to another. We shall only calculate the electric moment for one regular arrangement. The electric moment for several arrangements can then be found easily if we know the probability of each arrangement.

Let \( x^{(i)} (r = 1, 2, 3) \) be the co-ordinates of a unit vector along the dipole referred to orthogonal axes, one (\( r = 3 \)) along the stationary orientation of the dipole, then

\[
\chi_{1}^{(i)} + \chi_{2}^{(i)} + \chi_{3}^{(i)} = 1 \quad (5.1)
\]

If we regard \( x^{(i)} , \chi_{j}^{(i)} \) as small then \( \chi_{j}^{(i)} \) can be written

\[
\chi_{j}^{(i)} = 1 - \frac{1}{2} x_{j}^{(i)} - \frac{1}{2} \chi_{j}^{(i)^{2}} \quad (5.2)
\]

up to the second power in \( x_{j}^{(i)}, \chi_{j}^{(i)} \).

Let \( \ell_{s}^{(r)} (s = 1, 2, 3) \) be the components of a unit vector along the axis \( r \) of the orthogonal set of a molecule \( i \), referred to the crystal axes.

The components of a unit vector, along the dipole, referred to the crystal axes, will then be

\[
\chi_{s}^{(r)} = \frac{3}{2} \phi_{s}^{(r)} x^{(i)} \chi_{r}^{(i)}
\]

\[
= \phi_{s}^{(r)} x^{(i)} + \sum_{r=1}^{3} \phi_{s}^{(r)} x_{r}^{(i)} \chi_{s}^{(i)} + \frac{3}{2} \phi_{s}^{(r)} x_{r}^{(i)} \chi_{s}^{(i)^{2}} \quad (5.3)
\]

also

\[
\chi_{s}^{(r)} x_{s}^{(r)} = \phi_{s}^{(r)} x_{s}^{(r)} + \sum_{r=1}^{3} \phi_{s}^{(r)} x_{s}^{(r)} \chi_{s}^{(i)} + \frac{3}{2} \phi_{s}^{(r)} x_{s}^{(r)} \chi_{s}^{(i)^{2}} \]

\[
+ \sum_{r=1}^{3} \phi_{s}^{(r)} x_{s}^{(r)} \chi_{s}^{(i)^{2}} - \frac{1}{2} \phi_{s}^{(r)} x_{s}^{(r)} \chi_{s}^{(i)^{2}} \phi_{s}^{(r)} x_{s}^{(r)^{2}} \quad (5.4)
\]
Since the resultant moment in the absence of an external field is zero, then

\[ \sum_{f} f_s^{(1)}(d) = 0 \]  

(5.5)

Substituting for \( \kappa'_s(i) \), \( \kappa'_s(i) \kappa'_s(d) \) from (5.3) (5.4) into the expression for \( W \) given by (1.35), then equating the linear terms in \( W \) to zero, we obtain

\[ \sum_{f} (g - g^{-1}) y^{i}_{ss'} f_s^{(1)}(d) = 0 \]  

(5.6)

By use of equations (5.5) (5.6) we obtain for \( W \) (Considering only dipolar interactions) the expression

\[ W = -\frac{\beta^2}{2\alpha} \sum_{f} (g - g^{-1}) y^{i}_{ss'} f_s^{(1)}(i) f_s^{(1)}(d') \]

\[ + \frac{\beta}{2\pi} \sum_{f} (g - g^{-1}) y^{i}_{ss'} f_s^{(1)}(i) f_s^{(1)}(d') \kappa'_s(k) \kappa'_s(k') \]

\[ + \frac{\beta}{\rho} \sum_{f} (g - g^{-1}) y^{i}_{ss'} f_s^{(1)}(i) f_s^{(1)}(d') \kappa'_s(k) \kappa'_s(k') \]

(5.7)

Putting

\[ \kappa'_s(i) = \frac{\beta^2}{\alpha kT} \sum_{f} (g - g^{-1}) y^{i}_{ss'} f_s^{(1)}(i) f_s^{(1)}(d') \]

\[ + \frac{\beta}{kT} \sum_{f} (g - g^{-1}) y^{i}_{ss'} f_s^{(1)}(i) f_s^{(1)}(d') \]

(5.8)

\[ \alpha'_s = -\frac{\beta}{kT} \sum_{f} (g - g^{-1}) y^{i}_{ss'} f_s^{(1)}(i) f_s^{(1)}(d') \]

(5.9)

then

\[ \frac{W}{kT} = \frac{\sum_{f} \sum_{d} y^{i}_{ss'} \alpha'_s(i) \alpha'_s(d') + \sum_{f} \alpha'_s(i) \kappa'_s(k)}{\sum_{f} \sum_{d} y^{i}_{ss'} \alpha'_s(i) \alpha'_s(d') + \sum_{f} \alpha'_s(i) \kappa'_s(k)} \]  

(5.10)
The probability for a given configuration is given by

\[ Z = \exp \left\{ -\frac{1}{2} \sum_{j} \sum_{\sigma} \left[ \chi_{\sigma j}^{\dagger} \chi_{\sigma j} - \frac{\lambda_{\sigma j}^{\dagger} \lambda_{\sigma j}}{2} \right] \right\} \]  

(5.11)

where

\[ Z = \left\{ \frac{(\alpha^\dagger)^{2N}}{4} \right\}^{\frac{1}{2}} \exp \left\{ \frac{1}{2} \sum_{\sigma} \left[ \sum_{j} \chi_{\sigma j}^{\dagger} \chi_{\sigma j} - \frac{\lambda_{\sigma j}^{\dagger} \lambda_{\sigma j}}{2} \right] \right\} \]  

(5.12)

\[ \left| \alpha \right| \text{ is the } (2N \times 2N) \text{ determinant of } \alpha_{\sigma \sigma'}^{\dagger}, \]

and \( (\alpha^\dagger) \) is the reciprocal of the matrix \( \alpha \).

From (5.12), the averages of \( \chi_{\sigma}^{\dagger} \) and \( \chi_{\sigma}^{\dagger} \chi_{\sigma}^{\dagger} \) are given by

\[ \frac{\partial Z}{\partial \chi_{\sigma}^{\dagger}} \]

\[ \frac{\partial Z}{\partial \chi_{\sigma}^{\dagger} \chi_{\sigma}^{\dagger}} = -z \frac{\partial Z}{\partial \alpha_{\sigma \sigma'}^{\dagger}} \]

(5.13)

also, by use of (5.3),

\[ \bar{p}_{(\sigma)}^{(i)} = \bar{p}_{(\sigma)}^{(i)} - \frac{1}{2} \sum_{\sigma} \bar{p}_{(\sigma)}^{(i)} \bar{x}_{\sigma}^{(i)} - \frac{1}{2} \sum_{\sigma} \bar{p}_{(\sigma)}^{(i)} \bar{x}_{\sigma}^{(i)} \]

(5.14)

By use of (5.12) and (5.13), and retaining only terms linear in \( E' \) we obtain

\[ \bar{p}_{(\sigma)}^{(i)} = -\bar{p}_{(\sigma)}^{(i)} \sum_{\sigma} \bar{p}_{(\sigma)}^{(i)} \bar{x}_{\sigma}^{(i)} \]

(5.15)

where

\[ \alpha_{\sigma \sigma'}^{\dagger} \]

is given instead of (5.3), by

\[ \alpha_{\sigma \sigma'}^{\dagger} = \frac{\bar{p}_{(\sigma)}^{(i)}}{\alpha_{\sigma \sigma'}^{(i)}} \bar{x}_{\sigma}^{(i)} \bar{x}_{\sigma'}^{(i)} \]

(5.16)
Let
\[ C_{ss} = \frac{1}{N_0} \sum_{\gamma} \left( \frac{Z}{r_\gamma} \right)^{\frac{2}{3}} (\hat{A}^{-1})_{\alpha \gamma} \hat{A}_{\alpha \beta} \hat{f}_{i}^{j} (\alpha) \]  
(5.17)

then
\[ \sum_{\gamma} \vec{P}_s^{(i)} = N_0 \sum_{\gamma} (C \vec{f} A)_{ss} \hat{f}_{i}^{j} \]  
(5.18)

By use of equations (1.46) and (5.18) one obtains
\[ \frac{1}{\beta} \frac{3}{\beta} \mu_s = \frac{3}{\beta} (A)_{ss} \left( \beta + \frac{A_{ss}}{1 - \beta \frac{2}{3} A} \right) \]  
(5.19)

Using the definition (1.52) for \( \hat{F}_s \) we obtain the tensor equation
\[ \frac{\xi_s}{3} = \frac{\xi_s}{2s+1} \left[ \frac{A_{ss}}{1 - \beta \frac{2}{3} A} \beta + \frac{A_{ss}}{1 - \beta \frac{2}{3} A} \right] \]  
(5.20)

This equation is the same as (1.53) with the tensor
\[ C \]  
replacing the tensor \( \frac{A_{ss}}{3 \beta \frac{2}{3} A} \). From (5.17)

\[ C \]  
is independent of \( \hat{F}_s \) or \( T \).

For isotropic solids
\[ \frac{\xi_s}{3} = \frac{3 \beta}{3} \beta + \frac{3 \xi_s}{2s+1} \]  
(5.21)

Using the fact that the stationary orientations of the dipoles are regularly distributed we may refer to a molecule \( i \) by means of two letters \( h, k \); the first defining the unit cell and the second a particular molecule inside the unit cell. Then \( \xi_s^{\alpha}(h) \) is the same for all \( h \) and this will be written
\[ \xi_s^{\alpha}(k) \]
also we may express
\[ \gamma_{\alpha \beta}^{kk'} = \sum_{h} \lambda_{\alpha \beta}^{hk} \]  
(5.22)
The conditions (5.5) and (5.6) can then be written

\[ \sum_k \beta_s^{s'}(k) = 0 \]

\[ \sum_k \left( \frac{r - r'}{ \beta } \right)^2 (k) = 0 \]  \hspace{1cm} (5.23)

If \( m \) is the number of dipoles in a unit cell then, from (5.17)

\[ C_{ss'} = \frac{1}{m} \sum_{kk'} \sum_{\alpha \alpha'} \left( A^{-1}_{\alpha} \right)_{kk'} \beta_s^{s'}(k) \beta_s^{s'}(k') \]  \hspace{1cm} (5.24)

where, according to (5.16) and (5.22)

\[ A_{kk'}^{\alpha \alpha'} = \sum_{ss'} \left( \frac{r - r'}{ \beta } \right)^2 (k) \beta_s^{s'}(k) \beta_s^{s'}(k') \]  \hspace{1cm} (5.25)

\textbf{Stability Below the Transition Point.}

In the absence of an external field, the configuration probability, instead of (5.11) and (5.12), is given by

\[ e^Z = \exp \left\{ -\frac{1}{2} \sum_{\alpha \alpha'} \sum_{\nu \nu'} \chi_{\alpha}^{\nu} \chi_{\alpha}^{\nu} \chi_{\alpha'}^{\nu} \chi_{\alpha'}^{\nu} \right\} \]  \hspace{1cm} (5.26)

\[ Z = \frac{\left( \pi m \right)^{\frac{N}{2}}}{\left( \pi \right)^{\frac{N}{2}}} \]  \hspace{1cm} (5.27)

Integrating equations (5.26) over all \( \chi \) except \( \chi^{(i)} \), one obtains

\[ e^{(i)Z^{(i)}} = \exp \left\{ -\frac{1}{2} \sum_{\alpha \alpha'} \sum_{\nu \nu'} \beta_{\alpha \alpha'}^{(i)} \chi_{\alpha}^{(i)} \chi_{\alpha'}^{(i)} \chi_{\alpha}^{(i)} \chi_{\alpha'}^{(i)} \right\} \]  \hspace{1cm} (5.28)

\[ Z^{(i)} = \frac{\left( \pi m \right)^{\frac{N}{2}}}{\left| \beta^{(i)}_{\alpha \alpha'} \right|^2} \]  \hspace{1cm} (5.29)

where \( |\beta^{(i)}_{\alpha \alpha'}| \) is a 2 x 2 determinant of \( \beta^{(i)}_{\alpha \alpha'} \).
By use of (5.13) and (5.27) one obtains

$$\overline{X^a X^a} = (\alpha^1)_{\alpha a}$$

(5.30)

and from (5.29)

$$\overline{X^a X^a} = (\beta^{ii})_{\alpha a}$$

(5.31)

One sees from (5.30), (5.31) that the tensor $\beta^{ii}$ appearing in (5.28) is the reciprocal of the 2 x 2 tensor $(\alpha^1)^{ii}$. According to equation (5.28) the projection of a unit vector along the dipole on a plane normal to its stationary orientation has the same probability everywhere on the curve

$$\sum_{\alpha a} \beta^{ii}_{\alpha a} X^a X^a = \eta_i$$

(5.32)

This is an ellipse or hyperbola according as

$$|\beta^{ii}| \geq 0$$

(5.33)

or, since $|\beta^{ii}| = \frac{1}{|(\alpha^1)^{ii}|}$, the condition (5.33) is equivalent to

$$|(\alpha^1)^{ii}| \geq 0$$

(5.34)

Therefore a first condition for harmonic oscillations round stationary orientations is

$$|(\alpha^1)^{ii}| > 0$$

for all $i$.

If $a^{ii}_1$, $a^{ii}_2$ are the semi-major and -minor axes respectively of the ellipse (5.32) $[(\beta^{ii}) > 0]$. 
then
\[
\overline{r_i^2 (\mathbf{u}^{(i)})} = a_1^{(i)} a_2^{(i)} = \frac{\mathbf{u}^{(i)}}{|\mathbf{u}^{(i)}|^2} (5.36)
\]
where \(r_i (\mathbf{u}^{(i)})\) is a radius vector from the centre of the ellipse (5.32) to a point on it.

Since the elementary volume between the ellipses \(\mathbf{u}^{(i)}\) and \(\mathbf{u}^{(i)} + d\mathbf{u}^{(i)}\) is \(\frac{\pi}{8} d\mathbf{u}^{(i)}\), then
\[
Z_i^{(i)} \overline{r_i^2} = \int_0^\infty \overline{r_i^2 (\mathbf{u}^{(i)})} \frac{d\mathbf{u}}{|\mathbf{u}^{(i)}|^2} (5.37)
\]

\[
\overline{\mathbf{r}_i^2} = \frac{1}{8} < 1
\]

for all \(i\), then
\[
2\left| (\mathbf{\alpha}^i)^{\alpha^i} \right|^{1/2} < 1
\]

for all \(i\). This is a second condition for harmonic oscillations round the stationary orientation.

By use of (5.16) one can rewrite the two conditions for harmonic oscillations
\[
\frac{|(\mathbf{A}^{-1})^{\alpha^i}|}{\frac{\hbar^2}{\Delta^2}} > 0
\]
\[
\frac{|(\mathbf{A}^{-1})^{\alpha^i}|^{1/2}}{a^2 k^2} < \frac{\hbar^2}{\Delta^2} (5.40)
\]

for all \(i\).

A transition from rotational oscillations to free rotations will happen at the temperature \(T_0\) given by
for the greatest \( \left| \left( A^{-1} \right)^{\alpha} \right| \) in the unit cell.

It may be noticed from (5.40) that the higher the value of \( \rho \) and the lower the temperature, the more stable is the ordered configuration of the dipoles. The experimental confirmation of this result has been noticed by Smythe (21).

\[
2 \left| \left( A^{-1} \right)^{\alpha} \right| = \frac{J^2}{a^3 k T_0}
\]

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REFERENCES.


(4) Clusius, Z. Elektroehen, 39, 598 (1933).


(22) Smythe, C.P. and Hitchcock, J., Am. Chem. Soc. 54, 4651 (1932).
REFERENCES (Contd.)


