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Essays in Game Theory and Bankruptcy

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Doctor of Philosophy
University of Edinburgh
October 2015
Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Ercan Aslan)
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I am deeply grateful to my thesis supervisor, Prof. Jonathan Thomas, for his invaluable guidance throughout the present thesis. My work would not have been possible without his motivation, brilliant ideas and his patience. He acted not only my supervisor but almost as a "father" during my PhD.

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Abstract

In Chapter 1 I study the iterative strategy elimination mechanisms for normal form games. The literature is mostly clustered around the order of elimination. The conventional elimination also requires more strict knowledge assumptions if the elimination is iterative. I define an elimination process which requires weaker rationality. I establish some preliminary results suggesting that my mechanism is order independent whenever iterative elimination of weakly dominated strategies (IEWDS) is so. I also specify conditions under which the “undercutting problem” occurs. Comparison of other elimination mechanisms in the literature (Iterated Weak Strategy Elimination, Iterated Strict Strategy Elimination, Generalized Strategy Eliminability Criterion, RBEU, Dekel-Fudenberg Procedure, Asheim-Dufwenberg Procedure) and mine is also studied to some extent. In Chapter 2 I study the axiomatic characterization of a well-known bankruptcy rule: Proportional Division (PROP). The rule allocates shares proportional to agents’ claims and hence, is intuitive according to many authors. I give supporting evidence to this opinion by first defining a new type of consistency requirement, i.e., union—consistency and showing that PROP is the only rule that satisfies anonymity, continuity and union—consistency. Note that anonymity and continuity are very general requirements and satisfied by almost all the rules that have been studied in this literature. Thus, I prove that we can choose a unique rule among them by only requiring union—consistency. Then, I define a bankruptcy operator and give some intuition on it. A bankruptcy operator is a mapping from
the set of bankruptcy operators to itself. I prove that any rule will converge to
PROP under this operator as the claims increase. I show nice characteristics of
the operator some of which are related to PROP. I also give a definition for con-
tinuity of an operator. In Chapter 3 investigate risk-averse investors’ behaviour
towards a risky firm. In order to find Pareto Optimal allocations regarding a joint
venture, I employ a 2-stage game, first stage of which involves a social-planner
committing to an ex-post bankruptcy rule. A bankruptcy rule is a set of sug-
gestions for solving each possible bankruptcy problem. A bankruptcy problem
occurs when there is not enough endowment to allocate to the agents each of
whom has a claim on it. I devise the game-theoretic approach posed in Kıbrıs
and Kıbrıs (2013) and extend it further. In fact, that paper considers a compar-
ison among 4 renowned bankruptcy rules whereas mine do not restrict attention
to any particular rule but rather aim to find a Pareto Optimal(PO) one. I start
with 2 agent case in order to give some insight to the reader and then, generalise
the results to an arbitrary number of investors. I find socially desirable (PO)
allocations and show that the same can be achieved through financial markets by
the help of some well-known results.
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Chapter 1

Iterative Elimination with Recall

1.1 Motivation and Literature Review

As is well known, the most widely studied and probably the most important issue in game theory is that of making predictions about outcomes of games or at least making predictions about payoffs that might be obtained, to the extent that the predictions about the outcomes allow. Broadly speaking, one can consider two main approaches used in the literature regarding this endeavor. The first approach, which is also widely used in other disciplines such as evolutionary biology, computer science and political science, involves the well-known concept of "equilibrium", which prescribes strategy profiles that might emerge as outcomes. The second approach are iterative methods in which unanticipated strategies are removed from consideration. Such methods focus on which strategies cannot be played rather than which can be played. In both approaches, the state of knowledge the players are in and their ability to use reason and deduce from others’ reasonings, play an important role. There is, however, a significant difference between the two approaches in terms of how they use internally consistent belief systems. Nash equilibrium assumes certain restrictions on agents’ expectations and argues that agents will expect others to play equilibrium strategies in order to justify the outcomes it suggests. By contrast, iterative methods are concerned
with making predictions using rationality alone. Bernheim (1984) says that

"...the notion of an equilibrium has little intrinsic appeal within a strategic context. When an agent reaches a decision in ignorance of the strategies adopted by other players, rationality consists of making a choice which is justifiable by an internally consistent system of beliefs, rather than one which is optimal, post hoc. This point of view is not original; indeed, most serious justifications of the Nash hypothesis embrace such an approach, arguing that agents will expect the game to yield a Nash outcome, and consequently will choose their equilibrium strategies. Nevertheless, when we think in terms of maximizing utility subject to expectations rather than realizations, it becomes clear that the Nash hypothesis, far from being a consequence of rationality, arises from certain restrictions on agents’ expectations which may or may not be plausible, depending upon the game being played. We are then quite naturally led to ask: are there any restrictions of individuals’ expectations (and hence choices) which are required by rationality alone, rather than by (subjective) plausibility?"

The most common assumption is that the players use a common criterion when throwing strategies out, and that this criterion is common knowledge. Strict dominance and weak dominance are at the core of such possible criteria. Although the term ”dominance solvability” is coined by Moulin (1979), the tradition of using such criteria dates back to Luce and Raiffa (1957), and is also used as early as Farquharson (1969) to study voting schemes.

Using such criteria for eliminating strategies and assuming common knowledge may lead to further eliminations as each player will also take into account which strategies her opponents will eliminate, and as a consequence which strategies of her own will become ”eligible” for elimination. However, the players may iterate different sequences of reasoning and draw conclusions that do not agree with each other. Nevertheless, it is commonly known that iterated elimination of strictly dominated strategies (IESDS) results in a unique set of strategies in a finite normal form game. Moreover, Moulin (1984) shows that the same applies to Cournot duopoly, i.e., only the Nash equilibrium remains in a Cournot game after sequential elimination of different levels of output. In addition, Dufwenberg and Stegeman (2002) prove that IESDS may be an order-dependent procedure
when strategy spaces larger than finite sets are considered, and that it may gen-
erate spurious Nash equilibria. They also establish that if the strategy spaces are
compact in Hausdorff spaces and the payoff functions are upper semi-continuous,
IEWDS succeeds in yielding a prediction, whereas it is not the case in most of
the larger classes of games. They prove an order-independence result for IESDS
under such a class of games. Additionally, they establish that when the players
have well-defined best-response correspondences, IESDS preserves Nash equilib-
ria. Gilboa et al. (1990) establish sufficient conditions for order-independence
for various types of eliminations and show that IESDS satisfy them in finite
normal-form games.

On the other hand, the conditions in Gilboa et al. (1990) are not satisfied by
iterative elimination of weakly dominated strategies (IEWDS). This is why things
are not as straightforward when it comes to IEWDS as the order-independence
problem persists even if we restrict attention to finite normal-form games. Be-
sides, there has been a long discussion on whether the knowledge of not playing
weakly dominated strategies automatically leads to IEWDS. Samuelson (1992)
argues that the answer to this question is “no”. He proves that common knowl-
edge does not guarantee order-independence, nor does it guarantee a solution to
the players. Hillas and Samet (2014b) write:

"Despite the awareness of the problem, no suggestion has been made how
to fix the process of iterated elimination of weakly dominated strategies in
order to capture common knowledge of weak dominance rationality, due to
the lack of formalization of weak dominance rationality”.

They show that when it is common knowledge that the players do not play
weakly dominated strategies, they must play profiles that survive flaws of the
weakly dominated strategies process, which is described by Stalnaker (1994).
Hillas and Samet (2014a) establish weak/strong non-probabilistic correlated equi-
librium which suggests typically a collection of profiles as it can be perceived in the
ploys common knowledge of Bayesian rationality and assumes that beliefs are derived from a common prior. Unlike his paper, Hillas and Samet (2014a) use weak/strong rationality. A player is weakly rational if she does not play strictly dominated strategies, and strongly rational if she does not play weakly dominated strategies.

Besides defining the concept of dominance solvability, Moulin (1979) also uses it to show that some important classes of voting games are dominance solvable. A game is said to be dominance solvable if all outcomes obtained by applying IEWDS yield the same payoff profile. Mariotti (2000) defines "maximum games" and shows that they are dominance solvable. He also establishes that an important subclass of such games is dominance solvable on the unique Pareto dominant outcome. Ewerhart (2002) proves that any 2-person strictly competitive game with \( n \) outcomes is solvable in \( (n - 1) \) stages of IEWDS. Kukushkin (2012) studies dominance solvability and best-response dominance solvability in finite games. Börgers and Janssen (1995) establish a condition which is necessary and sufficient for a Cournot game to be dominance solvable.

The reason why common knowledge of players’ rationality does not directly justify IEWDS, is that the principle of rationality is not fully applied. In other words, the strategies that were weakly dominated in some stage of elimination, may become weakly undominated later on. There are numerous solutions offered in order to deal with this issue. One such solution is the reasoning based expected utility procedure (RBEU), suggested by Cubitt and Sugden (2011). RBEU comes to a halt, producing a trinary partition of strategies, and this provides a partial answer to the question of order-independence and full employment of common rationality. Unlike many other procedures, RBEU generates a trinary partition as there remains a category of strategies about which the mechanism does not make any definite assertions. Nevertheless, it provides a reasoning procedure which removes mutually inconsistent conclusions that may be held by different
An alternative approach is chosen by Börgers (1994) by replacing common knowledge of rationality with “approximate common knowledge” which is discussed by Monderer and Samet (1989) and Stinchcombe (1988). Using this approach, he justifies the procedure introduced by Dekel and Fudenberg (1990). The Dekel and Fudenberg procedure utilises maximal elimination of weakly dominated strategies at the first stage, and then continues with IESDS. Cheng and Wellman (2007) study a modified version of IEWDS. They weaken the weak dominance condition and allow a more aggressive pruning of strategies, and also show some important implications of this technique regarding the equilibria that survive iterated elimination. As with many other papers in the literature, they consider elimination by mixed strategies but impose a condition which does not permit a strategy $s_i$ to be eliminated by a mixture that also includes $s_i$. They define $\delta$—dominance such that a mixed strategy $\sigma_i$ can eliminate a pure strategy $s_i$ although it yields payoffs lower by $\delta$ against some opponent profiles. In some sense, $\sigma_i$ can be said to approximately weakly dominate $s_i$. They also establish that the equilibria of a game obtained by iteratively eliminating $\delta$—dominated strategies will be approximate equilibria of the original game. Unfortunately, their procedure is order-dependent, and the approximate equilibria that survive depend on the order of elimination as well.

The closest procedure to ours in the literature is Asheim and Dufwenberg (2003). They define fully permissible strategy sets using an algorithm that eliminates subsets of the entire strategy set. In the procedure they establish, players treat sets of strategies as choice sets. Their work relies on the assumption that players hold a common belief that each player prefers $s_i$ to $t_i$ if and only if $s_i$ weakly dominates $t_i$ on the set of the opponent’s strategies, or on the union of choice sets that are deemed possible for the opponent. This assumption constructs a link between the strategies that survive up to some stage of elimination.
and the entire set. Notice that in a given stage, a strategy which is not weakly
dominated on the set of the opponent’s strategies that also survive until that
stage, may be eliminated due to being weakly dominated when the entire set is
considered. In the present paper, we show that such an ad hoc assumption is not
necessary in an important class of games. That is, once this condition on choice
sets is removed, the Asheim and Dufwenberg procedure produces the same results
as ours. However, while they prove their findings for the 2-player case only, we
allow for an arbitrary number of players. In other words, we show that EWR is
well defined in TDI games and produces the same admissible set of strategies as
IEWDS.

It is noteworthy that although many of the papers above consider elimination
by mixed strategies, Marx and Swinkels (1997), which is the results-wise closest
paper to ours, also considers elimination by pure strategies alone. The following
example demonstrates that allowing elimination by mixtures may have an effect
on which strategies should be eliminated.

**Example 1.**

<table>
<thead>
<tr>
<th></th>
<th>U</th>
<th>M</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>.,10</td>
<td>.,0</td>
<td>.,0</td>
</tr>
<tr>
<td>R</td>
<td>.,0</td>
<td>.,4</td>
<td>.,10</td>
</tr>
</tbody>
</table>

In the example above, although M is not weakly dominated by either U or
D, it is so by the mixture \((U, 0.5; D, 0.5)\) where both U and D are played with
probability 0.5. We also know that if a pure strategy \(s_i\) yields a higher payoff than
a mixed strategy \(\sigma_i\) against some mixed opponent profile \(\sigma_{-i}\), then \(\sigma_{-i}\) assigns
positive probability to a pure strategy \(s_j\) against which \(s_i\) generates a higher
payoff than \(\sigma_i\). That is why weak domination against mixtures does not increase
the chance of a strategy being recalled, an operation we define in the following
section. However, as the effect of deletion of an opponent’s strategy on one’s
own strategies is uncertain, we are far from making any statements on whether
allowing mixtures will make our procedure stronger or not.
1.2 Model

Let $I = \{1, 2, ..., n\}$ be the set of players. A typical strategy of player $i$ is denoted by $s_i$. Let $S_i$ be the finite set of strategies that are available to player $i$. A strategy profile is a vector which is an ordered collection of strategies denoted by $s = (s_1, s_2, ..., s_n)$ whose $i$th component shows $s_i$ of $i$. $S = \times_{i \in I} S_i$ is the set of strategy profiles. $\pi_i : S \rightarrow \mathbb{R}$ is the payoff function of player $i$. The structure of the game is common knowledge and each player is assumed to be rational. Furthermore, we are interested in games where rationality of agents is also common knowledge (so that iterative reasoning applies). $\Pi = (\pi_1, \pi_2, ..., \pi_n)$ is the payoff function. A finite normal-form game is an ordered tuple $\Gamma = (I, S, \Pi)$.

By $s_{-i}$, $\pi_{-i}$ we will denote the strategy profile and payoff functions of the opponents’ of player $i$, respectively. $S_{-i} = \times_{j \neq i} S_j$ and $\Pi_{-i} = (\pi_1, \pi_2, ..., \pi_{i-1}, \pi_{i+1}, ..., \pi_n)$ will denote the set of strategy profiles and payoff functions of $i$’s opponents, respectively. Let $\Phi = \bigcup_{i \in I} S_i$ and $W \subseteq \Phi$. $W$ is said to be a restriction of $\Phi$ if it includes at least one strategy of each player, i.e., $W \cap S_i \neq \emptyset$ for each $i \in I$. The strategies in a given restriction $W$ of $\Phi$ that belong to player $i$ are denoted by $W_i = W \cap S_i$. The set of strategy profiles that can be constructed by using strategies in a restriction $W$ are given by $S^w = \times_{i \in I} W_i$, with a typical element represented by $s^w \in S^w$. We will denote by $s^w_i \in W_i$ player $i$’s strategy in $s^w$.

We need to make the distinction between $W$ and $s$, as $W$ might include more than one strategy of any agent and it does not pair up a player’s strategies with the strategies of her opponents.

**Definition 1.**  

i) For any $t_i, s_i \in S_i$, $t_i$ strictly dominates $s_i$ on $W_{-i}$, if we have $\pi_i(t_i, s^w_{-i}) > \pi_i(s_i, s^w_{-i})$ for all $s^w_{-i} \in S^w_{-i}$.

ii) For any $t_i, s_i \in S_i$, $t_i$ weakly dominates $s_i$ on $W_{-i}$, if we have $\pi_i(t_i, s^w_{-i}) \geq \pi_i(s_i, s^w_{-i})$ for all $s^w_{-i} \in S^w_{-i}$ and $\pi_i(t_i, z_{-i}) > \pi_i(s_i, z_{-i})$ for some $z_{-i} \in S^w_{-i}$.

The exercise of eliminating weakly dominated strategies is performed at the
thought level and does not involve any commitment. It is merely a process of reasoning. Therefore, the players are not bound by the remaining strategies but rather benefit from not playing them as a consequence of their rationality. Once the players eliminate weakly dominated strategies, they face a similar situation: A new game possibly comprising strategies which became weakly dominated after the first stage of elimination. It is natural to expect the other players to follow a similar approach. Thus, the same reasoning will apply iteratively until no weakly dominated strategy remains. As there is more than one path of elimination, in order for the players to reach a final set of surviving strategies, some coherency among the reasoning of the players is required. One may argue that in order to further proceed with the iterative elimination, a player needs to be certain about which strategies are deleted in the previous stages by his opponents so that he can adequately choose strategies to delete at that stage. Although there might be more than one path of elimination and the player cannot know which one to follow, she can work out the resulting "reduced game" of each path and still act if all paths lead to the survival of the same strategies. If she knows that it is the unique reduced game regardless of the elimination path followed and that other players work it out too, and others also know that each player works out the same outcome and so on, the deletion occurs just as it does when there is a single path. Hence, what matters is the reduced game. With regard to this point, Gretlein (1983) proves order independence of elimination paths for games where players have strict preferences over the outcomes. In such games, given her opponents’ strategy vector, a player can be indifferent between two strategies only if both result in the same outcome. Rochet (1980) identifies a class of games which also satisfy order independence, namely, any finite normal form game with a payoff matrix that satisfies the following condition:
\[ \pi_i(s) = \pi_i(t) \implies \pi_j(s) = \pi_j(t) \text{ for all } i, j \in I \text{ and for all } s, t \in S. \]

Marx and Swinkels (1997) prove order independence for a wider class of games, games that satisfy transference of decision maker indifference (TDI).

**Definition 2.** A normal form game \( \Gamma = (I, S, \Pi) \) satisfies TDI if we have for all \( i, r_i, t_i \in S_i \) and \( s_{-i} \in S_{-i} \):

\[ \pi_i(r_i, s_{-i}) = \pi_i(t_i, s_{-i}) \implies \pi_j(r_i, s_{-i}) = \pi_j(t_i, s_{-i}) \]

Note that Marx and Swinkels (1997) extend their results to mixed strategies as well. TDI is very similar to non-bossiness condition in social choice. Basically, it states that, given a strategy profile, no player should be able to change some other player’s payoff without changing his own payoff by playing a different strategy. Marx and Swinkels (1997) provide many examples that satisfy TDI including patent races, oligopoly with an endogenous number of firms, first price auctions, public good provision games etc. We take the following definition from Marx and Swinkels (1997).

**Definition 3.** Let \( V \) be a restriction of \( \Phi \) and let \( W \) be a restriction of \( V \). Then, \( W \) is a reduction of \( V \) by weak dominance if \( W = V \setminus X^1, X^2, \ldots, X^m \) where \( \forall k, X^k \subset \Phi \) and \( \forall x_i \in X^k, \exists z_i \in V \setminus X^1_i, \ldots, X^k_i \) such that \( z_i \) weakly dominates \( x_i \) on \( V \setminus X^1_{-i}, \ldots, X^k_{-i} \). \( W \) is a full reduction of \( V \) by weak dominance if \( W \) is a reduction of \( V \) by weak dominance and no strategies in \( W_i \) are weakly dominated on \( W_{-i} \) for all \( i \in I \).

The above definition is saying that a set is a reduction of its superset only if the difference consists of strategies that were weakly dominated on the superset. In other words, in order to obtain a reduction of a set, either some of the weakly
dominated strategies should be removed or the set should remain the same. As opposed to $EWR$ new strategies cannot be added. Nor can strategies that are not weakly dominated be eliminated. Note that according to the definition above, a reduction of $\Phi$ is also a reduction of itself. Below is the definition from Hillas and Samet (2014b) which is equivalent.

**Definition 4.** A process of iterated elimination of weakly dominated strategies consists of sequences of strategy profile sets $(S^0, S^1, \ldots, S^m)$, where $S^0 = S$, and for $k \geq 1$, $S^k = \times_i S^k_i$ where $S^k_i$ is obtained from $S^{k-1}_i$ by eliminating some strategies in the latter set which are weakly dominated relative to $S^{k-1}_i$. In the sets $S^m_i$ there are no weakly dominated strategies relative to $S^m_i$.

Next we define iterated elimination with recall (hereafter $EWR$).

**Definition 5.** Let $\Psi$ be a restriction of $\Phi$. A process of $EWR$ is a sequence of restrictions $(\Psi^0, \Psi^1, \ldots, \Psi^m)$ such that $\Psi^0 = \Psi$ and for $k \in \{1, \ldots, m\}$ $\forall z_i \in \Psi_i^{k-1}\setminus \Psi_i^k$, $\exists x_i \in \Psi_i^k$ where $x_i$ weakly dominates $z_i$ on $\Psi_i^{k-1}$, $\forall z_i \in \Psi_i^k/\Psi_i^{k-1}$, $\exists x_i \in \Psi_i^k \cap \Psi_i^{k-1}$ where $x_i$ weakly dominates $z_i$ on $\Psi_i^k \cap \Psi_i^{k-1}$ and $\forall z_i \in \Psi \setminus \Psi_i^m$, $\exists x_i \in \Psi_i^m$ such that $x_i$ weakly dominates $z_i$ on $\Psi_i^m$. In the sets $\Psi_i^m$ there are no weakly dominated strategies relative to $\Psi_i^m$.

Unlike $IEWDS$, the size of the set does not necessarily shrink at each step. At each $EWR$ stage, first some of the weakly dominated strategies are eliminated. Then, before proceeding to the next stage, some of the previously removed strategies are recalled back if they are not weakly dominated with respect to the new set. Notice that a strategy which became weakly undominated after some strategies are deleted, does not have to be recalled right away. On the other hand, a strategy which is recalled has to be weakly undominated with respect to the new set obtained after the deletion. This process goes on until there is no weakly dominated strategy left to eliminate and no strategy to recall. Regarding the extent to which rationality and common knowledge of rationality assumptions are
employed, we can consider EWR as a criticism of iterated elimination of weakly
dominated strategies (hereafter IEWDS). Since a player’s opponents’ strategies
are also going through the process of elimination in IEWDS, some of his strate-
gies which were eliminated at previous stages might become undominated. As
applying rationality does not exclude such strategies from being played, his op-
ponents need to consider them valid while further iterating the elimination. Also
when deleting strategies at following stages, the player knows that his opponents
consider such strategies of his admissible and then, the same kind of iterated logic
follows. Consider the following example:

Example 2.

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a_1$</td>
<td>$a_2$</td>
<td>$a_3$</td>
<td>$a_4$</td>
<td>$a_5$</td>
</tr>
<tr>
<td>$b_1$</td>
<td>5, 0</td>
<td>5, 1</td>
<td>3, 6</td>
<td>4, 1</td>
<td>2, 6</td>
</tr>
<tr>
<td>$b_2$</td>
<td>3, 1</td>
<td>2, 2</td>
<td>3, 7</td>
<td>3, 0</td>
<td>2, 0</td>
</tr>
</tbody>
</table>

The game has 3 Nash Equilibria (NE) and these are $(a_3, b_1), (a_3, b_2), (a_5, b_1)$. A possible path of IEWDS is eliminating all weakly dominated strategies of a
player at once. Notice that $b_1$ and $a_3$ are weakly dominating strategies for player
1 and player 2, respectively. Therefore, the remaining strategies would be $a_3$
and $b_1$. Another path of elimination might be $(b_2, a_1, a_2, a_4)$ with the remaining
strategies $\{a_3, a_5, b_1\}$. The latter path retains the NE $(a_3, b_1)$ but also $(a_5, b_1)$.

On the other hand, all paths of EWR eliminate $(a_1, a_2, a_4, a_5)$ and the strategies
$\{a_3, b_1, b_2\}$ survive. For instance, we have

$$
\begin{align*}
\Psi^0 &= \{\{b_1, b_2\} \cup \{a_1, a_2, a_3, a_4, a_5\}\} \\
\Psi^1 &= \{\{b_1\} \cup \{a_3\}\} \\
\Psi^2 &= \{\{b_1\} \cup \{a_3, a_5\}\} \\
\Psi^3 &= \{\{b_1, b_2\} \cup \{a_3, a_5\}\} \\
\Psi^4 &= \{\{b_1, b_2\} \cup \{a_3\}\}
\end{align*}
$$
\[ \Psi^0 = \{\{b_1, b_2\} \cup \{a_1, a_2, a_3, a_4, a_5\}\} \]
\[ \Psi^1 = \{\{b_1, b_2\} \cup \{a_3\}\} \]

which is more straightforward.

The intuition is that each player has a weakly dominating strategy and against player 1’s weakly dominating \(b_1\), player 2 wishes to play either \(a_3\) or \(a_5\) against player 2’s weakly dominating \(a_3\), player 1 is indifferent between playing \(b_1\) or \(b_2\). However, player 1 is still as well-off by playing either of the 2 strategies given that player 2 chooses \(a_5\) (She can play \(a_5\) if she anticipates that player 1 will play \(b_1\)). Yet, recalling \(b_2\) will make \(a_5\) weakly dominated again and player 2 will adhere to \(a_3\).

1.2.1 The Undercutting Problem

EWR gets stuck in an infinite cycle when applied to some games. \textit{i.e.} players cannot certainly predict which strategies are admissible, which strategies their opponents think are admissible, which strategies their opponents think they think are admissible and so on ad infinitum. In such games, no matter which successive stages of reasoning are iterated by the players, it is impossible to reach a conclusion about which strategies are permitted as a result of perfectly rational calculation and common knowledge of perfectly rational calculation. Unfortunately, this argument would still be valid even if the players could correlate their reasonings or commit to the same steps of iterated elimination and recall. The epistemic foundations of problems which may arise due to a tension between “common knowledge that the players don’t play weakly dominated strategies” and admissibility were laid by Samuelson (1992) without asserting a new proce-
dure of elimination. Cubitt and Sugden (2011) also addresses the same problem and suggests RBEU. Asheim and Dufwenberg (2003) resolves this issue by imposing that no strategy which is weakly dominated in the entire set of strategies can be permitted in the choice sets that survive iterated steps of elimination. Our paper shows that without such an additional requirement, EWR is sufficient to yield the desired result. More formally, we say that an EWR process $\Psi$ ends up in an infinite cycle if there does not exist a sequence of restrictions $(\Psi^0, \Psi^1, ..., \Psi^n)$ where a strategy is weakly dominated if and only if it does not belong to $\Psi^n$.

Consider the example below:

$$
\begin{array}{c|cc}
\hline
 & a_1 & a_2 \\
\hline
b_1 & 1,1 & 1,0 \\
b_2 & 1,-1 & -1,1 \\
\hline
\end{array}
$$

There is a unique IEWDS path which is:

\[
\Delta^0 = \{\{b_1, b_2\} \cup \{a_1, a_2\}\}
\]
\[
\Delta^1 = \{\{b_1\} \cup \{a_1, a_2\}\}
\]
\[
\Delta^2 = \{\{b_1\} \cup \{a_1\}\}
\]

There is also a unique EWR path which is an infinite sequence of restrictions:

For $k = 0, 1, 2, ...$

\[
\begin{array}{l}
\Psi^{4k} = \{\{b_1, b_2\} \cup \{a_1, a_2\}\} \\
\Psi^{4k+1} = \{\{b_1\} \cup \{a_1, a_2\}\} \\
\Psi^{4k+2} = \{\{b_1\} \cup \{a_1\}\} \\
\Psi^{4k+3} = \{\{b_1, b_2\} \cup \{a_1\}\}
\end{array}
\]

A presumably fruitful way of attempting to characterize the existence of EWR and its uniqueness is to tackle these two problems by considering sets of strategies and relations among those sets rather than dealing with the elimination and recall processes themselves. This is the same approach utilised in Asheim and
Dufwenberg (2003) with some minor changes. It is also along the lines of admissible sets approach and related common knowledge of rationality assumptions which are widely discussed in Samuelson (1992), Brandenburger and Friedenberg (2010), Brandenburger et al. (2008) and Börgers (1994). On that account, the following discussion of "fixed restrictions’ and some conjectures we formulise might be helpful and provide some insight on what actually changes in the structure of surviving strategies and the deleted ones when we switch from IEWDS to EWR.

We will postulate the following conjecture:

**Claim 1.** Let $\Theta = \bigcup_{W \subseteq \Phi \text{ and } \forall i \in I W \cap S_i \neq \emptyset} W$ i.e. $\Theta$ is the set of all restrictions. Let $F : \Theta \to 2^\Theta$ be a mapping such that $F(W) = \bigcup_{1 \leq j \leq n} F_j(W)$ where $F_j(W) = \{s_i \in S_i : s_i \text{ is not weakly dominated by some } t_i \in S_i \text{ on } W \setminus i\}$. Then, EWR is order-independent if and only if for any $W, W' \in \Theta$ such that $F(W) = W$ and $F(W') = W'$, we have either $W \subseteq W'$ or $W' \subseteq W$.

If $F(W) = W$, then we call $W$ a fixed restriction of $\Phi$. Notice that $F$ maps each restriction $W$ of $\Phi$ to a restriction of $\Phi$. The condition required by the claim above is ruling out cases where we have a fixed restriction which contains all other fixed restrictions but also there exists two fixed restrictions each of which contains at least one strategy that is not included in the other. i.e. whenever we have $W, W' \subseteq W^{\ast}$ and $W/W' \neq \emptyset \wedge W'/W \neq \emptyset$ where $F(W) = W$, $F(W') = W'$and $F(W^{\ast}) = W^{\ast}$, EWR is order-dependent. In other words, it requires a sequence of restrictions $(W^1, W^2, ..., W^k)$ where $W^1 \subseteq W^2 \subseteq ... \subseteq W^k$. In a 2-person game, however, there might be two strategies constituting a Nash-Equilibrium by yielding extremely large payoffs against each other. Say $s_1 = \max_{s \in S_1} \pi_1(s, s_2)$ and $s_2 = \max_{s \in S_2} \pi_2(s_1, s)$ where $s_1$ and $s_2$ are player 1’s and player 2’s strategies, respectively. i.e. there is no other strategy of player 1 which gives a payoff higher than or equal to what $s_1$ gives against $s_2$, and vice versa. In this case, there is no EWR stage where one of these two strategies can be eliminated. Also, we have $F(\{s_1, s_2\}) = \{s_1, s_2\}$. Still, these two strategies might generate lower payoffs.
than other strategies that are available. Assume we have \( F(W) = W \) where \( W = S/\{s_1, s_2\} \). Then, no strategy is eliminated and the only EWR reduction is the set of all strategies.

Instead, we can possibly revise the condition in four ways: First, we can allow for fixed restrictions whose intersection is empty, yet with an exception of a fixed restriction which is superset of other fixed restrictions. Second, we can allow for the first revision above but without such a superset. Third, we can allow for fixed restrictions \( W \) and \( W' \) where \( W/W' \neq \emptyset \land W'/W \neq \emptyset \) and \( W \cap W' \neq \emptyset \). Fourth, we can also require a superset as in the first revision and use the third revision along with this requirement. As we also want to account for redundant strategies, arguably the best way to construct the conjecture is the fourth. Hence, we will incorporate the following condition, instead:

"For a given set of fixed restrictions \( W = \{W_1, W_2, ..., W_k\} \), exists \( W^h \in W \) with \( W^h = \bigcup_{1 \leq j \leq k} W_j \) if and only if EWR is order-independent".

In addition, an alternative and convenient approach may be taken into consideration by defining a fixed restriction in a different way:

**Fixed Restriction.** For each \( i \in I \), let

\[
F_i(W) = \left\{ s_j \in S_j : j \neq i \text{ and } s_j \text{ is a component of some profile } s_{-i} \text{ in a restriction on which the set of weakly undominated strategies that belong to player } i \text{ is } W_i \right\}
\]

Then, \( W \) is a fixed restriction of \( F \) if for each \( i \in I \) we have \( W_{-i} \in F_i(W) \).

Note that with this definition of a fixed restriction, we are actually considering fixed points of \( n \) different correspondences.

One might easily recognise that according to the former definition of a fixed restriction, any strategy of player \( i \) which is not weakly dominated on \( W_{-i} \) is included in \( F_i(W) \) whereas we have only opponents’ strategies in the set \( F_i(W) \) with respect to the latter definition. In fact, there might exist more than one set
of opponents’ profiles on which the set of weakly undominated strategies that belong to player $i$ is the same. Therefore, each $F_i(W)$ involves a correspondence which links player $i$’s strategy subsets to subsets of opponents’ strategies. Notice that although the two definitions are equivalent, they may require different technicalities. With the first sort of fixed restriction used in such assertions as ours, we are more likely to encounter fixed point theorems in the conventional sense. On the other hand, the second definition promises a rather intricate but possibly dynamic and cyclical structure which requires different tools to describe and to deal with.

**Conjecture 1.** Undercutting problem occurs if and only if $\not\exists$ a restriction $W$ of $\Phi$ such that $F_i(W{-}i) = W_i$ for all $i \in I$.

**Conjecture 2.** For any $\Gamma$, $\exists$ a path of IEWDS which gives EWR($\Gamma$) if $\exists$ a non-empty restriction $W$ of $\Phi$ such that $F_i(W{-}i) = W_i$ for all $i \in I$.

**Theorem 1.** Let $\Gamma$ be a finite normal form game which satisfies TDI. Then, EWR($\Gamma$) is well-defined. Furthermore, there exists an EWR path that gives the same reduction as IEWDS.

*Proof.* Let $\Delta = (\Delta^0, \Delta^1, ..., \Delta^m)$ be a process of IEWDS with maximal elimination. i.e. Each player removes all weakly dominated strategies at once at a given stage. The EWR path $\Psi$ we are going to define involves no recall for the first $m$ stages. i.e. $\Psi^h = \Delta^h$ for $0 \leq h \leq m$. Consider the strategies in $\Delta^{m-1}/\Delta^m$. Let $s_j \in \Delta^{m-1}/\Delta^m$ be such that it can be recalled at stage $(m + 1)$. As $s_j$ is weakly dominated by some other strategy on $\Delta^{m-1}$ and $\Delta^m \subset \Delta^{m-1}$, $\exists t_j \in \Delta^m$ such that $\not\exists s_{-j} \in \Delta^m_j$ with $\pi_j(s_j, s_{-j}) > \pi_j(t_j, s_{-j})$. Then, $\forall s_{-j} \in \Delta^m_j$, we have $\pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j})$. Define $\Psi^{m+1} = \Delta^m \cup \{s_j\}$. Namely, we recall only a single strategy, if there is any, and it belongs to the set of strategies eliminated at the final stage of IEWDS. There is no weakly dominated strategy in $\Psi^{m+1}$ and we recall another strategy $s_i \in \Delta^{m-1}/\Delta^m$, if there is any where $i \in I$. i.e. $i$ is
not necessarily a different player, we can have \( i = j \). Since \( s_i \) is also eliminated at stage \( m \), the argument we used for \( s_j \) applies to \( s_i \) as well. i.e. \( \exists t_i \in \Delta^m \) such that either \( \forall s_{-i} \in \Delta^m_{-i} \cup \{ s_j \} \), we have \( \pi_i(s_i, s_{-i}) = \pi_i(t_i, s_{-i}) \) or \( \forall s_{-i} \in \Delta^m_{-i}, \pi_i(t_i, s_{-i}) \geq \pi_i(s_i, s_{-i}) \) and for some \( s_{-i} \in \Delta^m_{-i} \), \( \pi_i(t_i, s_{-i}) > \pi_i(s_i, s_{-i}) \) and for some \( s_{-ij} \in \Delta^m_{-ij} \) we have \( \pi_i(s_i, s_j, s_{-ij}) > \pi_i(t_i, s_j, s_{-ij}) \) (One of these two conditions should apply to any strategy \( t_j \in \Delta^m \) with which \( s_j \) generates equal payoffs on \( \Delta^m_j \). Hence, it should apply to the strategy which eliminated \( s_j \) at stage \( m \) as \( s_j \) is recalled back to a restriction with same payoff structure as the IEWDS reduction) Suppose the latter holds. Then, since \( s_{-j}, s_{-ij} \in \Delta^m \implies s_{-j}, s_{-ij} \in \Delta^{m-1} \), \( s_i \) wouldn’t be weakly dominated by \( t_i \) on \( \Delta^{m-1}_{-i} \). Thus, \( \forall s_{-i} \in \Delta^{m-1}_{-i} \cup \{ s_j \} \), \( \pi_i(s_i, s_{-i}) = \pi_i(t_i, s_{-i}) \). As a consequence, \( \forall s_{-ij} \in \Delta^{m-1}_{-ij} \), by TDI we have

\[
\pi_j(t_j, t_i, s_{-ij}) = \pi_j(s_j, t_i, s_{-ij}) \implies \pi_I(t_j, t_i, s_{-ij}) = \pi_I(s_j, t_i, s_{-ij}) \quad (1.1)
\]

\[
\pi_i(s_j, t_i, s_{-ij}) = \pi_i(s_j, s_i, s_{-ij}) \implies \pi_I(s_j, t_i, s_{-ij}) = \pi_I(s_j, s_i, s_{-ij}) \quad (1.2)
\]

\[
\pi_i(t_j, t_i, s_{-ij}) = \pi_i(t_j, s_i, s_{-ij}) \implies \pi_I(t_j, t_i, s_{-ij}) = \pi_I(t_j, s_i, s_{-ij}) \quad (1.3)
\]

Hence, by (1.1), (2) and (3),

\[
\pi_I(t_j, t_i, s_{-ij}) = \pi_I(s_j, t_i, s_{-ij}) = \pi_I(t_j, s_i, s_{-ij}) = \pi_I(s_j, s_i, s_{-ij}).
\]

i.e. the payoff structure of \( \Delta^m \) is preserved by \( \Delta^m \cup \{ s_i, s_j \} \) as any payoff profile that can be constructed by the latter set can also be constructed by the former. In other words, \( s_i \) and \( s_j \) are redundant to \( \Delta^m \). Furthermore, there is no weakly dominated strategy in the set \( \Delta^m \cup \{ s_i, s_j \} \). One can easily show that the same argument applies if we continue to recall strategies one by one from the set \( \Delta^{m-1} / \Delta^m \). If there isn’t any strategy to recall in the set \( \Delta^{m-1} / \Delta^m \) to begin with, then we apply the same procedure to \( \Delta^{m-2} / \Delta^{m-1} \). If there is no strategy to recall in \( \Delta^{m-2} / \Delta^{m-1} \), then we recall a single strategy from \( \Delta^{m-3} / \Delta^{m-2} \) and
For $1 < k < m$, assume there exists a strategy $s_j \in \Delta^{m-p}/\Delta^{m-p+1}$ for some $p > k$ which is not weakly dominated on $\Psi^k$. If for each $t_j \in \Psi^k$, $\exists t_{-j} \in \Psi_{-j}^k$ such that $\pi_j(s_j, s_{-j}) > \pi_j(t_j, s_{-j})$, then since $\Psi^k \subset \Delta^{m-p}$, $\exists t_j \in \Delta^{m-p}$ such that $t_j$ weakly dominates $s_j$ on $\Delta^{m-p}$ and $s_j \in \Delta^{m-p+1}$. Contradiction. Thus, $\exists t_j \in \Psi^k$ such that $\forall s_{-j} \in \Psi_{-j}^k$ we have $\pi_j(t_j, s_{-j}) \geq \pi_j(s_j, s_{-j})$ and as $s_j$ is not weakly dominated on $\Psi^k$, $\forall s_{-j} \in \Psi_{-j}^k$, $\pi_j(t_j, s_{-j}) = \pi_j(s_j, s_{-j})$. For each step $k$ of $\Psi$ with $k > 1$ and after each strategy $s_j$ we recall at that stage, we are going to recall all strategies that are eliminated after $s_j$ in $\Delta$ process and became weakly undominated once $s_j$ is recalled. Let $s_i \in S_i$ be a such strategy. Then, $\exists t_i \in \Psi^k$ such that $t_i$ weakly dominates $s_i$ on $\Psi_{-i}^k$ but $\exists s_{-i} \in \Psi_{-i}^{k+1}$ such that $\pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i})$. Then, $s_j$ is a component of $s_{-i}$ as $\Psi^{k+1}/\Psi^k = \{s_j\}$, i.e. $\pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i})$, for some $s_{-ij} \in \Psi_{-ij}^k$. Since $\pi_j(t_i, s_j, s_{-ij}) = \pi_j(t_i, t_j, s_{-ij})$, by TDI we have $\pi_i(t_i, s_j, s_{-ij}) = \pi_i(t_i, t_j, s_{-ij})$. Again by TDI, if $\pi_j(s_i, s_{-ij}) = \pi_j(t_i, t_j, s_{-ij})$, then $\pi_i(s_i, s_j, s_{-ij}) = \pi_i(t_i, t_j, s_{-ij})$. Thus, $\pi_i(s_i, s_{-ij}) > \pi_i(t_i, s_{-ij}) \implies \pi_i(s_i, t_j, s_{-ij}) > \pi_i(t_i, t_j, s_{-ij})$. As $t_j$ and $s_{-ij}$ are not weakly dominated at the stage where $s_i$ is weakly dominated by $t_i$, we have a contradiction. Since $s_i$ is an arbitrarily chosen among the strategies eliminated after $s_j$, no strategy eliminated after $s_j$ will be recalled after $s_j$. Finally we run induction on strategies which are eliminated at a given stage and then, on stages until we get to the first stage of $\Psi$. Hence, proof is complete.

Next, we evoke the issue of order-independence and show that a result similar to the one which holds for IEWDS is also true for EWR. In the following theorem, we abuse the notation for convenience and write that a profile is an element of a set of strategies whenever it is constructed solely by strategies from that set. A similar notation is used for opponent strategies etc. All the superscripts are for stages and all the subscripts are for players.

For the first two parts of the following theorem, we employ the same approach.
We take two sets one of which is a subset of the other, $A \subset B$ where $A$ is the set obtained by $IEWDS(EWR)$ and $B$ is the set obtained by applying a correspondence (that will be defined in the proof) on the reduction set obtained by $IEWDS$ and on the reduction set obtained by $EWR$ for the first and the second parts’ respectively. We consider an arbitrarily chosen $s_i \in B/A$ and show that $s_i$ cannot be eliminated by a strategy from $A, B/A$ or $S/B$ when $IEWDS$ is applied for the first case and $EWR$ for the second.

**Theorem 2.** For a finite normal form game $\Gamma$ that satisfies TDI, we have $EWR(\Gamma) = IEWDS(\Gamma)$ where $IEWDS(\Gamma)$ is the unique reduction obtained by applying TDI. i.e. the reduction obtained by $EWR$ and the reduction obtained by $IEWDS$ are equivalent up to redundant strategies. Therefore, for such games $EWR$ is also order-independent.

*Proof.* Let $\Delta = (\Delta^0, \Delta^1, ..., \Delta^m)$ be a sequence of $IEWDS$ restrictions where $\Delta^0 = \Phi$ and $\Delta^m$ is the $IEWDS$ reduction of $\Phi$. Let $\Psi = (\Psi^0, \Psi^1, ..., \Psi^n)$ be a sequence of $EWR$ restrictions with $\Psi^0 = \Phi$ and $\Psi^n$ being an $EWR$ reduction of $\Phi$. We want to show that for any $i \in I$ and $s_i \in S_i$, $s_i \in EWR(\Gamma) \iff s_i \in IEWDS(\Gamma)$. In order to do so, we are going to show that neither $\Psi^n$ nor $\Delta^m$ is a proper subset of the other and it is not the case that both of them include a strategy which is not an element of the other. i.e. $\neg[\Psi^n \subset \Delta^m]$ and $\neg[\Delta^m \subset \Psi^n]$ and $\neg[\Delta^m/\Psi^n \neq \emptyset$ and $\Delta^m/\Psi^n \neq \emptyset$]. Assume $\Delta^m \subset \Psi^n$ with $s_i \in \Psi^n/\Delta^m$. Then, there exists $\Delta^k$ and $\Delta^{k-1}$ such that $s_i \in \Delta^{k-1}/\Delta^k$ and $t_i \in S_i$ such that $t_i$ weakly dominates $s_i$ on $\Delta^{k-1}$. We also know that if a strategy $x_i$ weakly dominates $y_i$ on a superset of $\Delta^m$, then $\nexists$ any opponent profile $s_{-i} \in \Delta^m$ such that $\pi_i(y_i, s_{-i}) > \pi_i(x_i, s_{-i})$ (otherwise, since strategies that construct $s_{-i}$ are also elements of the superset of $\Delta^m$, $x_i$ wouldn’t weakly dominate $y_i$). Since $\Delta^m \subset \Delta^{k-1}$ and $t_i$ weakly dominates $s_i$ on $\Delta^{k-1}$, $\nexists$ any opponent profile $s_{-i} \in \Delta^m$ such that $\pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i})$. If $t_i$ is eliminated in some interim stage $r$ such that $k - 1 < r < m$, then there exists $p_i \in S_i$ such that $p_i$ weakly dominates
\[ t_i \text{ on } \Delta_{-i} \text{ and } \forall s_{-i} \in \Delta_{m-i} \text{ such that } \pi_i(t_i, s_{-i}) > \pi_i(p_i, s_{-i}). \] Hence, \( \forall s_{-i} \in \Delta_{m-i} \) such that \( \pi_i(s_i, s_{-i}) > \pi_i(p_i, s_{-i}). \) If \( p_i \) is also eliminated at stage \( s \) with \( r < s < m \), then there exists \( v_i \) such that \( v_i \) weakly dominates \( p_i \) on \( \Delta_{-i} \) and for all \( s_{-i} \in \Delta_{m-i} \) we have \( \pi_i(v_i, s_{-i}) \geq \pi_i(p_i, s_{-i}) \) and so on. Since we have a finite number of stages and strategies, there exists \( z_i \in \Delta^m \) for which \( \forall s_{-i} \in \Delta_{m-i} \) such that \( \pi_i(s_i, s_{-i}) > \pi_i(z_i, s_{-i}). \) On the other hand, \( z_i, s_i \in \Psi^n \), i.e., \( z_i \) does not weakly dominate \( s_i \) on \( \Psi_{n-i}. \) If \( \forall s_i \in \Psi^n / \Delta^m \) and such \( z_i \in \Delta^m, \) then \( \forall s_{-i} \in \Psi_{n-i} \) we have \( \pi_i(s_i, s_{-i}) = \pi_i(z_i, s_{-i}) \) and the claim is true. Assume not. Then, there exists \( s_{-i} \in \Psi_{n-i} \) with at least one strategy \( s_j \in \Psi^n / \Delta^m \) such that \( \pi_i(s_i, s_{-i}) > \pi_i(z_i, s_{-i}). \) By the same token, \( s_j \in \Psi^n / \Delta^m \implies \exists t_j \in \Delta^m \) such that \( \forall s_{-j} \in \Delta_{m-j} \), we have \( \pi_j(t_j, s_{-j}) \geq \pi_j(s_j, s_{-j}). \) Then, either

\[ \exists s_{-j} \in \Psi^n \setminus \Delta^m \quad \text{s.t.} \quad \pi_j(s_j, s_{-j}) > \pi_j(t_j, s_{-j}) \quad \text{and} \]

\[ \exists s_{-j} \in \Delta_{m-j} \quad \text{s.t.} \quad \pi(t_j, s_{-j}) > \pi(s_j, s_{-j}) \quad \text{(1.4)} \]

or

\[ \forall s_{-j} \in \Psi^n, \pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j}). \quad \text{(1.5)} \]

If (1.4) holds, then \( s_j \notin \Psi^n \) as it would be weakly dominated by \( t_j \) on \( \Psi_{n-j}. \) A contradiction. If (1.5) holds, by \textit{TDI} \( \pi_i(s_j, s_{-j}) > \pi_i(t_j, s_{-j}), \) i.e., for all \( s_{-ij} \in \Psi_{n-ij}, \)

\[ \pi_j(s_i, s_j, s_{-ij}) = \pi_j(s_i, t_j, s_{-ij}) \implies \pi_i(s_i, s_j, s_{-ij}) = \pi_i(s_i, t_j, s_{-ij}) \quad \text{(1.6)} \]

and

\[ \pi_j(t_i, s_j, s_{-ij}) = \pi_j(t_i, t_j, s_{-ij}) \implies \pi_i(t_i, s_j, s_{-ij}) = \pi_i(t_i, t_j, s_{-ij}). \quad \text{(1.7)} \]
Since \( \pi_i(s_i, s_j, s_{-ij}) > \pi_i(t_i, s_j, s_{-ij}) \) for some \( s_{-ij} \in \Psi^n_{-ij} \), by (1.6) and (1.7) we have
\[
\pi_i(s_i, t_j, s_{-ij}) > \pi_i(t_i, t_j, s_{-ij})
\]
(1.8) for such \( s_{-ij} \in \Psi^n_{-ij} \).

Therefore, if for each component \( s_k \in \Psi^n/\Delta^n \) of \( s_{-j} \) (2) holds, then following from (1.8), we have \( \pi_i(s_i, t_{-i}) > \pi_i(t_i, t_{-i}) \) where \( t_{-i} \in \Delta^n_{-i} \). A contradiction. Hence, there exists an \( s_j \in \Psi^n/\Delta^n \), an \( s_{-j} \) with at least one component from \( \Psi^n/\Delta^n \) and a \( t_j \in \Delta^n \) such that \( t_j \) weakly dominates \( s_j \) on \( \Delta^n_{-j} \) and \( \pi_j(s_j, s_{-j}) > \pi_j(t_j, s_{-j}) \) with \( s_j \) being a component of \( s_{-i} \) where \( \pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i}) \). Therefore, for each \( s_i \in \Psi^n/\Delta^n \) for which \( \neq t_i \in \Delta^n \) such that for all \( s_{-i} \in \Psi^n_{-i} \), \( \pi_i(s_i, s_{-i}) = \pi_i(t_i, s_{-i}) \) there exists a such \( s_j \). Denote the set of such \( s_j \) by \( \alpha(s_i, t_i) \). Let \( \alpha(s_i) = \bigcup_{t_i \in \Delta^n} \alpha(s_i, t_i) \). Let \( \alpha(\alpha(s_i)) = \alpha^2 \) be the union of set of strategies obtained by applying \( \alpha \) to each strategy in \( \alpha(s_i) \). Consider the sequence \( (\alpha(s_i), \alpha^2(s_i), \ldots) \). Since we have a finite number of strategies, we have \( \alpha^k(s_i) \subseteq \bigcup_{1 \leq j \leq k-1} \alpha^j(s_i) \) for some finite \( k \). Set \( \alpha^k(s_i) = \alpha \). Consider the first strategy eliminated from \( \alpha \) by \( IEWDS \). For each \( t_i \in \Delta^n \) and \( s_i \in \alpha \), there exists either
\( (a) \) \( t_{-i} \in \Delta^n_{-i} \) such that \( \pi_i(s_i, t_{-i}) > \pi_i(t_i, t_{-i}) \) or
\( (b) \) \( s_{-i} \in \Delta^n_{-i} \cup \alpha \) such that \( \pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i}) \)

Therefore, the first strategy cannot be eliminated by a strategy from \( \Delta^n \).

the first strategy is eliminated by a strategy \( g_i \in S_i/\Psi^n_i \), then \( \pi_i(g_i, s_{-i}) \geq \pi_i(s_i, s_{-i}) \ \forall s_{-i} \in \Delta^n \cup \alpha \). Note that we can construct an opponent profile \( s_{-i} \in \Delta^n \cup \alpha \) since each player has at least one strategy in \( \Delta^n \) (because it is the reduction obtained by \( IEWDS \)). We also know that here exists a strategy \( e_i \in \Psi^n_i \) such that \( e_i \) weakly dominates \( g_i \) on \( \Psi_{-i} \) and for each \( s_i \in \Psi^n_i \) there exists a profile \( s_{-i} \in \Psi^n_{-i} \) such that \( \pi_i(s_i, s_{-i}) > \pi_i(g_i, s_{-i}) \) since otherwise \( g_i \) would be recalled back to \( \Psi^n \). By (a) and (b), there exists \( s_{-i} \in \Delta^n \cup \alpha \) for each \( s_i \in \alpha \) and each
\[t_i \in \Delta^m\] such that \(\pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i})\). Therefore, for each \(t_i \in \Delta^m\) there exists \(s_{-i} \in \Delta^m \cup \alpha\) such that \(\pi_i(g_i, s_{-i}) > \pi_i(t_i, s_{-i})\). Moreover, for each \(y_i \in \Psi^n\) such that \(\pi_i(y_i, s_{-i}) = \pi_i(t_i, s_{-i})\) where \(t_i \in \Delta^m\) and \(s_{-i} \in \Psi^n_i\) (i.e. \(y_i \in \Psi^n / [\Delta^m \cup \alpha]\)), we have \(\pi_i(g_i, s_{-i}) > \pi_i(y_i, s_{-i})\) for some \(s_{-i} \in \Delta^m \cup \alpha\). Hence, a strategy which weakly dominates \(g_i\) on \(\Psi^n_i\) cannot be an element of \(\Psi^n / [\Delta^m \cup \alpha]\) or \(\Delta^m\) and has to be an element of \(\alpha\). Since \(\pi_i(g_i, s_{-i}) \geq \pi_i(s_i, s_{-i})\) \(\forall s_{-i} \in \Delta^m \cup \alpha\) and \(\pi_i(s_i, s_{-i}) \geq \pi_i(g_i, s_{-i})\) \(\forall s_{-i} \in \Psi^n_i\), we have \(\pi_i(g_i, s_{-i}) = \pi_i(s_i, s_{-i})\) \(\forall s_{-i} \in \Delta^m \cup \alpha\). By \(TDI\), \(\pi_i(g_i, s_{-i}) = \pi_i(s_i, s_{-i}) \implies \pi_i(g_i, s_{-i}) = \pi_i(s_i, s_{-i})\). i.e.

For each \(s_j \in \alpha\) and \(t_i \in \Delta^m\) and \(s_{-ij} \in \Delta^m \cup \alpha\), we have \(\pi_j(s_i, s_{-ij}) > \pi_j(g_i, s_{-ij})\). Therefore, \(\alpha(s_i) = \alpha(g_i)\) and \(\alpha' = \alpha(s_1) \cup \alpha(s_2) \cup \ldots \alpha(s_{-i}) \cup \ldots \cup \alpha(g_i) \cup \alpha(g_{-i}) \cup \ldots \cup \alpha(s_j)\) where \(I\) also represents the number of players in the set \(I\). (\(g_i\) replaces \(s_i\) in \(\Delta^m \cup \alpha\) and the payoff structures are preserved, so are the weak dominance relations).

Then we consider the first strategy eliminated from \(\Delta^m \cup \alpha'\) by \(IEWDS\). By the same arguments we used for the first strategy eliminated from \(\Delta^m \cup \alpha\), it cannot be eliminated by a strategy from \(\Delta^m\). If it’s eliminated by a strategy from \(S/\Psi^n\), then we have a new collection \(\alpha''\) of sets of strategies such that the first strategy eliminated from \(\Delta^m \cup \alpha''\) by \(IEWDS\) cannot be eliminated by a strategy from \(\Delta^m\) and so on. Since we have a finite number of strategies, the first strategy eliminated from one of those collections must be eliminated by a strategy which is also an element of the same collection. Without loss of generality, say \(s_i \in \alpha\) is eliminated by \(h_i \in \alpha\). Then, for each \(s_{-i} \in \Delta^m \cup \alpha\), \(\pi_i(h_i, s_{-i}) \geq \pi_i(s_i, s_{-i})\).

If there exists \(s_{-i} \in \Delta^m \cup \alpha\) such that \(\pi_i(h_i, s_{-i}) > \pi_i(s_i, s_{-i})\), then there exists \(s_{-i} \in \Psi^n\) with at least one component from \(\Psi^n / [\Delta^m \cup \alpha]\) such that \(\pi_i(s_i, s_{-i}) > \pi_i(h_i, s_{-i})\). For each such component \(s_j \in \Psi^n / [\Delta^m \cup \alpha]\), since \(\pi_j(s_i, s_j, s_{-ij}) = \pi_j(s_i, t_j, s_{-ij})\) for some \(t_j \in \Delta^m\), by \(TDI\) we have \(\pi_i(s_i, s_j, s_{-ij}) = \pi_i(s_i, t_j, s_{-ij})\) and \(\pi_i(h_i, s_j, s_{-ij}) = \pi_i(h_i, t_j, s_{-ij})\) for all \(s_{-ij} \in \Psi^n_{-ij}\). Therefore, \(\pi_i(h_i, s_j, s_{-ij}) > \pi_i(s_i, s_j, s_{-ij}) \implies \pi_i(h_i, t_j, s_{-ij}) = \pi_i(s_i, t_j, s_{-ij})\). By replacing each such com-

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ponent \( s_j \in \Psi^n / [\Delta^m \cup \alpha] \) of \( s_i \), where \( \pi_i(s_i, s_{-i}) > \pi_i(h_i, s_{-i}) \) with \( t_j \in \Delta^m \) such that \( \pi_j(t_j, s_{-j}) = \pi_j(s_j, s_{-j}) \forall s_{-j} \in \Psi^n \). We construct \( s_{-i} \in \Delta^m \cup \alpha \) such that \( \pi_i(s_i, s_{-i}) > \pi_i(h_i, s_{-i}) \). A contradiction. Hence, \( \pi_i(s_i, s_{-i}) = \pi_i(h_i, s_{-i}) \) for all \( s_{-i} \in \Delta^m \cup \alpha \). By TDI, \( \forall i \in \alpha \) such that \( \pi_i(t_i, s_{-i}) \leq \pi_i(s_i, s_{-i}) \) we have \( \forall i \in \alpha \) such that \( \pi_j(s_j, s_{-j}) \leq \pi_j(h_j, s_{-j}) \). Therefore, \( \alpha(s_i) = \alpha(h_i) \) and \( \alpha' = \alpha(s_1) \cup \alpha(s_2) \cup \ldots \cup \alpha(s_{-i}) \cup \ldots \cup \alpha(h_i) \cup \alpha(s_{i+1}) \cup \ldots \cup \alpha(s_I) \) where \( I \) also represents the number of players in the set \( I \). \( h_i \) replaces \( s_i \) in \( \Delta^m \cup \alpha \) and the payoff structures are preserved, so are the weak dominance relations.

Then, we consider the first strategy eliminated from \( \Delta^m \cup \alpha' \) by IEWDS. By the same arguments we used for the first strategy eliminated from \( \Delta^m \cup \alpha \), it cannot be eliminated by a strategy from \( \Delta^m \). If it’s eliminated by a strategy from \( S / \Psi^n \), then we have a new collection \( \alpha'' \) of sets of strategies such that the first strategy eliminated from \( \Delta^m \cup \alpha'' \) by IEWDS cannot be eliminated by a strategy from \( \Delta^m \) and so on. Since we have a finite number of strategies, the first strategy eliminated from one of those collections must be eliminated by a strategy which is also an element of the same collection and so on.

On the contrary, assume \( \Psi^n \subset \Delta^m \). For some \( k \in \mathbb{N} \), we have \( s_j \in \Psi^{k-1} / \Psi^k \implies s_j \notin \Delta^m / \Psi^n \) where \( 1 \leq h \leq k \). Let \( s_i \in \Delta^m / \Psi^n \) with \( s_i \in \Psi^k / \Psi^{k+1} \) such that for \( 1 \leq h \leq k \) we have \( s_j \in \Delta^m / \Psi^n \implies s_j \notin \Psi^{h-1} / \Psi^h \). Namely, \( s_i \) is the first strategy eliminated from \( \Delta^m / \Psi^n \) by EWR. Assume that for \( k \) such that \( s_j \in \Psi^k \implies s_j \in \Psi^{k+1} \) where \( s_j \in \Delta^m / \Psi^n \), we have \( s_i \in \Psi^k \implies s_t \in \Psi^{k+1} \) for all \( t \in I \). That is, prior to the elimination of the first strategy from \( \Delta^m / \Psi^n \), no strategy is eliminated from \( \Psi^n \), either. Consequently, before the stage where \( s_i \) is weakly dominated, all the other strategies in \( \Delta^m \) have survived the previous stages, too. \( s_i \) is weakly dominated for the first time at some stage by \( t_i \in \Delta^m \implies [s_j \in \Delta^m \implies s_j \in \Psi^k] \). Since \( s_i \in \Delta^m / \Psi^n \), for each \( t_i \in \Psi^n \) such that \( t_i \) weakly dominates \( s_i \) on \( \Psi^n \), \( \exists s_{-i} \in \Delta^m \) with at least one strategy
from $\Delta^m/\Psi^n$ such that $\pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i})$. As the components of $s_{-i}$ are also elements of $\Psi^k$, $s_i$ cannot be eliminated by $t_i \in \Psi^n_i$.

Conversely, assume that $\exists t_j \in \Psi^n$ and $k \in \mathbb{N}$ such that $t_j \notin \Psi^k$ and $s_i \in \Psi^k$. Indeed, let $t_j$ be the first strategy eliminated from $\Psi^n$ (hence recalled at a later stage) before any strategy is eliminated from $\Delta^m/\Psi^n$. (Here, $j$ and $i$ aren’t necessarily distinct players as they are throughout the rest of the proof). Since $t_j \in \Psi^n$, we have either

For each $t_j \in \Psi^n_j$ and $s_j \in S$, $\exists t_{-j} \in \Psi^n_{-j}$ s.t. $\pi_j(t_j, t_{-j}) > \pi_j(s_j, t_{-j})$ (1.9)

or

$\exists e_j \in \Psi^n_j$ such that $\pi_j(t_j, t_{-j}) = \pi_j(e_j, t_{-j}) \forall t_{-j} \in \Psi^n_{-j}$ and $\exists s_{-j}$ with

- at least one component from $S/\Psi^n$ such that $\pi_j(e_j, s_{-j}) > \pi_j(t_j, s_{-j})$
- and there does not exist any $s_{-j} \in \Psi^n_{-j}$ such that $\pi_j(t_j, s_{-j}) > \pi_j(e_j, s_{-j})$

where $t_j$ is weakly dominated by $e_j$ at stage $k$. (1.10)

If (1.9) holds, then $t_j$ cannot be the first strategy eliminated from $\Psi^n$. Assume (1.10) holds. Then, since $e_j, k_j \in \Psi^n \subset \Delta^m$, for all $s_{-j} \in \Delta^m_{-j}$, we have $\pi_j(t_j, s_{-j}) = \pi_j(e_j, s_{-j})$. By TDI, $\pi_i(t_j, s_{-j}) = \pi_i(e_j, s_{-j})$ for all $s_{-j} \in \Delta^m_{-j}$.

In particular, $\pi_i(s_i, t_j, s_{-ij}) = \pi_i(s_i, e_j, s_{-ij})$ and $\pi_i(t_i, t_j, s_{-ij}) = \pi_i(t_i, e_j, s_{-ij})$ where $s_{-ij} \in \Delta^m_{-ij}$ and $s_i$ is the first strategy eliminated from $\Delta^m/\Psi^n$ with $t_i$ being the strategy that weakly dominates it. Since $\pi_i(s_i, t_j, s_{-ij}) > \pi_i(t_i, t_j, s_{-ij}) \implies \pi_i(s_i, e_j, s_{-ij}) > \pi_i(t_i, e_j, s_{-ij})$, the elimination of $t_j$ does not make $s_i$ weakly dominated by $t_i$.

Assume that before the first strategy from $\Delta^m/\Psi^n$ is eliminated and after the first strategy from $\Psi^n$ is eliminated, some strategy $g_k \in \Psi^n_k$ is also eliminated.
(Notice that all the statements are true regardless of whether only one strategy is eliminated from $\Psi^n$ at stage $k$ where $1 \leq h \leq k - 1$, $s_j \in \Psi^n_j \implies s_j \in \Psi^h_j$ or many.) Since for each $s_{-j} \in \Delta^m_{-j} \exists e_j$ such that $\pi_j(t_j, s_{-j}) = \pi_j(e_j, s_{-j})$, for each $s_k \in \Delta^m_k$, and each $s_{-kj} \in \Delta^m_{-kj}$ by TDI we have:

$$\pi_j(g_k, t_j, s_{-kj}) = \pi_j(g_k, e_j, s_{-kj}) \implies \pi_k(g_k, t_j, s_{-kj}) = \pi_k(g_k, e_j, s_{-kj})$$ (1.11)$$

$$\pi_j(s_k, t_j, s_{-kj}) = \pi_j(s_k, e_j, s_{-kj}) \implies \pi_k(s_k, t_j, s_{-kj}) = \pi_k(s_k, e_j, s_{-kj})$$ (1.12)

From (1.11) and (1.12) we have

$$\pi_k(g_k, t_j, s_{-kj}) > \pi_k(s_k, t_j, s_{-kj}) \implies \pi_k(g_k, e_j, s_{-kj}) > \pi_k(s_k, e_j, s_{-kj}).$$

Hence, $g_k \in \Psi^n_k$ cannot be weakly dominated by some strategy $s_k \in \Delta^m_k$. For each $s_k \in \Delta^m/\Psi^n$, $\exists s_{-k} \in \Psi^n_{-k}$ such that $\pi_k(g_k, s_{-k}) > \pi_k(s_k, s_{-k})$(Otherwise $s_k$ would be recalled). One component of $s_{-k} \in \Psi^n_{-k}$ may be eliminated in one of the earlier stages but (8) and (9) apply again. Therefore, $g_k$ can be eliminated only by a strategy also from $\Psi^n_k$. Thus, the first strategy eliminated from $\Delta^m/\Psi^n$ cannot be eliminated by a strategy from $\Psi^n$.

Assume that the first strategy $s_i$ eliminated from $\Delta^m/\Psi^n$ is eliminated by a strategy $t_i$ which is also from $\Delta^m/\Psi^n$. Since $s_i, t_i \in \Delta^m$, we have either

(c) $\forall s_{-i} \in \Delta^m_{-i}$, $\pi_i(s_i, s_{-i}) = \pi_i(t_i, s_{-i})$ or

(d) $\exists s_{-i}, t_{-i} \in \Delta^m_{-i}$ such that $\pi_i(s_i, s_{-i}) = \pi_i(t_i, s_{-i})$ and $\pi_i(t_i, t_{-i}) > \pi_i(s_i, t_{-i})$

If (d) holds, since $s_i$ is the first strategy to be eliminated from $\Delta^m/\Psi^n$ and all such $s_{-i}, t_{-i} \in \Delta^m_{-i}$ profiles have survived, then $s_i$ is not weakly dominated at this stage. Assume (c) holds. Since $s_i \in \Delta^m/\Psi^n$, $\exists k_i$ such that $k_i$ weakly dominates $s_i$ on $\Psi^n$ and $\exists s_{-i} \in \Delta^m_{-i}$ with at least one component from $\Delta^m/\Psi^n$ such that $\pi_i(s_i, s_{-i}) > \pi_i(k_i, s_{-i})$ for such $s_i$ and $k_i$. Then, we de-
fine correspondence $\beta$ which is similar to $\alpha$ we had defined earlier: $\beta(s_i, k_i) = \{s_j : j \neq i$ and $s_j$ is a component of some $s_{-i}$ such that $\pi_i(s_i, s_{-i}) > \pi_i(k_i, s_{-i})\}$ Since $\pi_i(s_i, s_{-i}) > \pi_i(k_i, s_{-i}) \implies \pi_i(t_i, s_{-i}) > \pi_i(k_i, s_{-i})$ for all $s_{-i} \in \Delta^m_{-i}$, $\beta(s_i, k_i) = \beta(t_i, k_i)$. By the same reasoning we employed for the case with $\alpha$, we conclude that the first strategy eliminated from $\Delta^m/\Psi^n$ is not eliminated by a strategy from $\Delta^m/\Psi^n$. Therefore, we have $\neg[\Psi^n \subset \Delta^m]$.

(Notice that the first strategy eliminated from $\Delta^m/\Psi^n$ may be recalled at a later stage. Our result still holds, though)

Finally, assume we have both $\Delta^m/\Psi^n \neq \emptyset$ and $\Psi^n/\Delta^m \neq \emptyset$. Then, there are two possible cases, either $\Delta^m \cap \Psi^n \neq \emptyset$ or $\Delta^m \cap \Psi^n = \emptyset$. Assume $\Delta^m \cap \Psi^n \neq \emptyset$. \exists a $k \in \mathbb{N}$ and a restriction $\Delta^k$ from the sequence $(\Delta^0, \Delta^1, ..., \Delta^m)$ such that $s_i \in \Psi^n/\Delta^m \implies s_i \in \Delta^h$ for $1 \leq h \leq k$ and $\exists s_i \in \Psi^n/\Delta^m$ such that $s_i \in \Delta^{k-1}/\Delta^k$.

i.e. $s_i$ is the first strategy eliminated from $\Psi^n/\Delta^m$ in a given IEWDS process. Therefore, $\exists t_i \in \Delta^k$ such that for all $s_{-i} \in \Delta^{k-1}$, $\pi_i(t_i, s_{-i}) \geq \pi_i(s_i, s_{-i})$. Since no strategy is yet eliminated from $\Psi^n/\Delta^m$ in stage $(k-1)$ and all the strategies in $\Delta^m$ (and hence $\Delta^m \cap \Psi^n$) survive IEWDS, we have $\Psi^n \subset \Delta^{k-1}$. Then, for each $s_{-i} \in \Psi^n_{-i}$, $\pi_i(t_i, s_{-i}) \geq \pi_i(s_i, s_{-i})$ and $t_i \in \Psi^n$. Since $s_i$ is also an element of $\Psi^n$, we have $\pi_i(t_i, s_{-i}) = \pi_i(s_i, s_{-i})$ for all $s_{-i} \in \Psi^n_{-i}$. Let $G^k$ be the set of strategies which are elements of $\Psi^n/\Delta^m$ and which are eliminated at stage $k$. Let the strategies $G^k = \{s_i, g_j, ..., x_p\}$ be eliminated by $\{t_i, h_j, ..., y_p\}$, respectively. Then, $\{t_i, h_j, ..., y_p\} \subset \Psi^n$. By TDI, $\forall s_{-ijkp} \in \Psi^n_{-ijkp}$, $\forall c_k, d_k \in \Psi^n_k$,

$$\pi_i(s_i, g_j, x_p, c_k, s_{-ijkp}) = \pi_i(t_i, g_j, x_p, c_k, s_{-ijkp}) \implies$$

$$\pi_k(s_i, g_j, x_p, c_k, s_{-ijkp}) = \pi_k(t_i, g_j, x_p, c_k, s_{-ijkp})$$

and

$$\pi_i(s_i, g_j, x_p, d_k, s_{-ijkp}) = \pi_i(t_i, g_j, x_p, d_k, s_{-ijkp}) \implies$$

$$\pi_k(s_i, g_j, x_p, d_k, s_{-ijkp}) = \pi_k(t_i, g_j, x_p, d_k, s_{-ijkp}).$$
By (1.13) and (1.14),
\[
\pi_k(s_i, g_j, x_p, c_k, s_{-ijkp}) \geq \pi_k(s_i, g_j, x_p, d_k, s_{-ijkp}) \quad \Rightarrow \quad (1.15)
\]
\[
\pi_k(t_i, g_j, x_p, c_k, s_{-ijkp}) \geq \pi_k(t_i, g_j, x_p, d_k, s_{-ijkp}),
\]
without loss of generality. By the same token,
\[
\pi_j(t_i, g_j, x_p, c_k, s_{-ijkp}) = \pi_j(t_i, h_j, x_p, c_k, s_{-ijkp}) \quad \Rightarrow \quad (1.16)
\]
\[
\pi_i(t_i, g_j, x_p, c_k, s_{-ijkp}) = \pi_i(t_i, h_j, x_p, c_k, s_{-ijkp}) \quad \Rightarrow \quad (1.17)
\]
\[
\pi_k(t_i, g_j, x_p, c_k, s_{-ijkp}) = \pi_k(t_i, h_j, x_p, c_k, s_{-ijkp}),
\]
and
\[
\pi_j(t_i, g_j, x_p, d_k, s_{-ijkp}) = \pi_j(t_i, h_j, x_p, d_k, s_{-ijkp}) \quad \Rightarrow \quad (1.17)
\]
\[
\pi_i(t_i, g_j, x_p, d_k, s_{-ijkp}) = \pi_i(t_i, h_j, x_p, d_k, s_{-ijkp}) \quad \Rightarrow \quad (1.17)
\]
\[
\pi_k(t_i, g_j, x_p, d_k, s_{-ijkp}) = \pi_k(t_i, h_j, x_p, d_k, s_{-ijkp}).
\]
By (1.15), (1.16), (1.17), we have
\[
\pi_k(t_i, h_j, x_p, c_k, s_{-ijkp}) \geq \pi_k(t_i, h_j, x_p, d_k, s_{-ijkp}),
\]
and iterating further
\[
\pi_k(t_i, h_j, y_p, c_k, s_{-ijkp}) \geq \pi_k(t_i, h_j, y_p, d_k, s_{-ijkp}),
\]
and so on. Notice that for any combination of strategies from $G^k$, we have a corresponding combination of strategies from $\Delta_m \cap \Psi^m$ which preserves the weak dominance relation between $c_k$ and $d_k$. Hence, for any $s_{-k} \in \Psi^m_{-k}$, $\exists \ t_{-k} \in \Psi^m / G^k$ such that for all $c_k, d_k \in \Psi^n$, we have $\pi_k(c_k, s_{-k}) \geq \pi_k(d_k, s_{-k}) \quad \Rightarrow \quad \pi_k(c_k, t_{-k}) \geq \pi_k(d_k, t_{-k})$.
Consider the strategies eliminated at stage \((k + 1)\) in \(\Psi^n/\Delta^m\), if any. For each \(s_i \in G^{k+1}\), \(\exists t_i \in \Delta^{k+1}\) such that \(t_i\) weakly dominates \(s_i\) on \(\Delta^k\). If \(t_i \notin \Psi^n\), then \(\exists h_i \in \Psi^n\) such that \(h_i\) weakly dominates \(t_i\) on \(\Psi^n/G_i^k\). Since \(s_{-i} \in \Psi^n/G^k \implies s_{-i} \in \Delta^k\), we have \(s_{-i} \in \Psi^n/G^k \implies \pi_i(t_i, s_{-i}) \geq \pi_i(s_i, s_{-i})\). Then, as \(s_{-i} \in \Psi^n/G^k\), we have \(\pi_i(h_i, s_{-i}) \geq \pi_i(t_i, s_{-i})\) and therefore, \(\pi_i(h_i, s_{-i}) \geq \pi_i(s_i, s_{-i})\)

Since for each \(s_{-i} \in \Psi^n/G^k\), \(\exists t_{-i} \in \Psi^n_{-i}\) such that \(\pi_i(s_i, s_{-i}) = \pi_i(s_i, t_{-i})\) and \(\pi_i(h_i, s_{-i}) = \pi_i(h_i, t_{-i})\) as shown above, we conclude that \(\pi_i(h_i, t_{-i}) \geq \pi_i(s_i, t_{-i})\) for all \(t_{-i} \in \Psi^n_{-i}\). Since both \(s_i\) and \(h_i\) are elements of \(\Psi^n\), then \(\pi_i(h_i, t_{-i}) = \pi_i(s_i, t_{-i})\) for all \(t_{-i} \in \Psi^n_{-i}\). Thus, by iterating the argument in the equations above and running induction on \(G^k, G^{k+1}, \ldots, G^m\), for each \(s_i \in G^k \cup G^{k+1} \cup \ldots \cup G^m\), \(\exists h_i \in \Delta^m \cap \Psi^n\) such that \(\pi_i(h_i, s_{-i}) = \pi_i(s_i, s_{-i})\) for all \(s_{-i} \in \Psi^n_{-i}\). Hence, \(\Psi^n\) and \(\Delta^m \cap \Psi^n\) are equivalent up to redundant strategies. Consequently, it is enough to show that \(\Delta^m\) is also strategically equivalent to \(\Delta^m \cap \Psi^n\) (up to redundant strategies).

Consider the first strategy \(s_j\) eliminated from \(\Delta^m/\Psi^n\) in the EWR process \(\Psi\), i.e. \(\exists k \in \mathbb{N}\) and a restriction \(\Psi^k\) from the sequence \((\Psi^0, \Psi^1, \ldots, \Psi^n)\) such that \(s_j \in \Delta^m/\Psi^n \implies s_j \in \Psi^h\) for \(1 \leq h < k - 1\) and \(\exists s_j \in \Delta^m/\Psi^n\) with \(s_j \in \Psi^{k-1}/\Psi^k\). Suppose that \(\exists t_j \in \Delta^m \cap \Psi^n\) such that \(t_j\) weakly dominates \(s_j\) on \(\Psi^{k-1}\). As \(t_j, s_j \in \Delta^m\), we have either

\[\forall s_{-j} \in \Delta_{-j}, \pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j})\]  

\((e)\)

or

\[\exists s_{-j}, t_{-j} \in \Delta_{-j}^m \text{ s.t. } \pi_j(s_j, s_{-j}) > \pi_j(t_j, s_{-j}) \text{ and } \pi_j(t_j, t_{-j}) > \pi_j(s_j, t_{-j}).\]  

\((f)\)

If \((f)\) holds, then \(s_j\) cannot be weakly dominated by \(t_j\) as \(\Delta^m \subseteq \Delta^{k-1}\). Assume \((e)\) holds. Then, \(s_j\) is redundant. Then, it follows from the same idea in (1.15),
that either all strategies in $\Delta^m/\Psi^n$ are redundant or the first strategy which is not redundant and eliminated cannot be eliminated by some $t_j \in \Delta^m \cap \Psi^n$.

If the first strategy $s_j$ eliminated from $\Delta^m/\Psi^n$ is eliminated by a strategy $t_j \in \Delta^m/\Psi^n$, since $\Delta^m \subseteq \Delta^{k-1}$ and $s_j, t_j \in \Delta^m$, then $\pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j})$ for all $s_{-j} \in \Delta_{-j}^m$. Then, it follows from the same idea in (1.15), (1.16), (1.17) that $s_j$ is not weakly dominated by some $t_j \in \Delta^m/\Psi^n$.

Assume $s_j$ is eliminated by some $t_j \notin \Delta^m \cup \Psi^n$, then. Since $t_j \notin \Delta^m$, $\exists g_j \in \Delta^m$ such that $\pi_j(g_j, g_{-j}) \geq \pi_j(t_j, g_{-j})$ for all $g_{-j} \in \Delta_{-j}^m$. If $t_j$ weakly dominates $s_j$ on $\Delta^m$, then $\pi_j(g_j, g_{-j}) \geq \pi_j(t_j, g_{-j}) \geq \pi_j(s_j, g_{-j})$ for all $g_{-j} \in \Delta_{-j}^m$ and $\pi_j(g_j, g_{-j}) \geq \pi_j(t_j, g_{-j}) > \pi_j(s_j, g_{-j})$ for some $g_{-j} \in \Delta_{-j}^m$. i.e. $g_j$ also weakly dominates $s_j$ on $\Delta^m$ but as $s_j, g_j \in \Delta^m$, we have a contradiction. Thus, $\pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j})$ for all $s_{-j} \in \Delta_{-j}^m$. If $\nexists g_j \in \Delta^m$ such that $\pi_j(s_j, s_{-j}) = \pi_j(g_j, s_{-j})$ for all $s_{-j} \in \Delta_{-j}^m$, then, for each $g_j \in \Delta^m \exists s_{-j} \in \Delta_{-j}^m$ such that $\pi_j(s_j, s_{-j}) > \pi_j(g_j, s_{-j})$ As $\forall s_{-j} \in \Delta_{-j}^m$ we have $\pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j})$, then for each $g_j \in \Delta^m \exists s_{-j} \in \Delta_{-j}^m$ such that $\pi_j(t_j, s_{-j}) > \pi_j(g_j, s_{-j})$. Hence, $t_j \in \Delta^m$. Contradiction. Thus, $s_j$ is not eliminated by some $t_j \notin \Delta^m \cup \Psi^n$ and we cannot have $\Delta^m/\Psi^n \neq \emptyset$ and $\Psi^n/\Delta^m \neq \emptyset$ and $\Delta^m \cap \Psi^n \neq \emptyset$.

On the contrary, suppose $\Delta^m \cap \Psi^n = \emptyset$. Consider the first strategy $s_j$ eliminated from $\Psi^n$ in a given IEWDS process. Let $s_j \in \Delta^{k-1}/\Delta^k$. Since $\Psi^n \subseteq \Delta^{k-1}$, for each $t_j \notin \Psi^n \exists s_{-j} \in \Psi_{-j}^n$ such that $\pi_j(s_j, s_{-j}) > \pi_j(t_j, s_{-j})$. Therefore, no $t_j \notin \Psi^n$ can weakly dominate $s_j$ on $\Delta^{k-1}$. Suppose $s_j$ is weakly dominated by some $t_j \in \Psi^n$ on $\Delta^{k-1}$. As $s_j, t_j \in \Psi^n$ and $\Psi^n \subseteq \Delta^{k-1}$, we have $\pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j})$ for all $s_{-j} \in \Psi_{-j}^n$. It follows from the same idea in (1.15), (1.16), (1.17) that elimination of $s_j$ does not change the payoff structure. Thus, the proof is complete.

While EWR and IEWDS suggest payoff equivalent sets of solutions in TDI games, one may wonder if one concept prevails over the other when we turn
attention to games that don’t satisfy TDI. We have already seen that IEWDS has an advantage over EWR when it comes to making predictions about games with 'undercutting problem’. In order to demonstrate that it is not always the case, let’s compare EWR and IEWDS by using our first example in the light of TDI. There are two different reductions one can obtain by devising IEWDS which are \{a_3, b_1\} and \{a_3, a_5, b_1\}. Notice that these reductions are not payoff equivalent as we have \(\pi(b_1, a_3) = (6, 2) \neq (6, 3) = \pi(b_1, a_3)\). This is due to the fact that the game does not satisfy TDI. We have \(\pi_2(b_1, a_3) = \pi_2(b_1, a_5) = 6\), although \(\pi_1(b_1, a_3) = 3\) and \(\pi_1(b_1, a_5) = 2\). i.e. When player 2 changes her strategy from \{a_3\} to \{a_5\} while player 1 is playing \{b_1\}, she does not change her own payoff but her opponent’s. Nevertheless, we have a unique EWR reduction which includes the strategy profile \(b_2, a_3\) and the associated payoff profile \(3, 7\), a payoff profile that cannot be obtained by strategy profiles that survive at least one of the two IEWDS paths. This counterexample proves two things: First, EWR and IEWDS are not equivalent solution concepts if we don’t restrict attention to TDI games but consider the entire set of finite normal-form games. Second, there are games in which EWR is order-independent but IEWDS is not. An interesting question which might arouse the reader’s curiosity as EWR and IEWDS predict different Nash-Equilibria (NE) in the example mentioned above: Is there any logical relation between Nash-Equilibria deemed possible to arise by EWR and IEWDS? Under which conditions can one expect a NE to become more likely to be played if the strategies involved survive both concepts of iterative elimination?
Chapter 2

A Characterization of The Proportional Rule

2.1 Motivation and Literature Review

A bankruptcy arises when there is a scarce resource and conflicting claims over it. Since the available resource is in sufficient to honour all the claims, many different suggestions on how to divide may arise. The common aspect of these suggestions is that no claimant gets more than his claim and nobody gets a negative share. A very common example of a bankruptcy problem is the process of liquidation of an insolvent firm among its creditors.

The study of bankruptcy problems has a historical tradition and dates back to the Babylonian Talmud. The Talmud involves two examples regarding bankruptcy situations one of which is about two men conflicting over how to share a garment. The second one involves a man who leaves mutually inconsistent bequests to his three wives. Nevertheless, it provides only numerical examples and no generalization. The most popular generalization of those numbers is probably suggested by Aumann and Maschler (1985). Although the problem is very intuitive and old, the formal study of it started as late as O’Neill (1982). Traditionally,
there has been two main branches in the literature, axiomatic bankruptcy which
aims to characterize bankruptcy rules by some normative but highly regarded
properties and game theoretical approach which aims to design games solutions
of which coincide with bankruptcy rules. In the present paper, we take the first
approach and define union—consistency in order to characterize the proportional
rule. Consistency and its variations have been widely used for characterizing
rules. Young (1987) uses consistency along with equal treatment of equals and
continuity to characterize parametric rules. Kaminski (2006) generalizes Young’s
result. Thomson (2007) develops a technique which determines whether a rule
which is defined for 2-agents can be generalized to an arbitrary number of agents
Dagan et al. (1997) shows that there may be particular solutions without a con-
sistent extension. Moulin (2000) uses consistency in order to obtain a joint char-
acterization of the proportional rule, the constrained equal awards rule and the
constrained equal losses rule. Chun (1999) proves that a consistent rule also
satisfies converse consistency Herrero and Villar (2001) considers the same class
of rules as in Moulin (2000) and obtain separate characterization results for the
constrained equal awards rule, the constrained equal losses rule and the Talmud
rule. Since our union—Consistency is a group property, it is noexaggeration to
say that it is loosely related to Chambers and Thomson (2002) which study group
order preservation. They also use a similar approach to ours in their proof. That
is why they require claims continuity as well. They characterise PROP. as our
paper do but they use group order preservation, claims continuity and consist-
tency. Since claims continuity and consistency are satisfied by a large class of
rules, the key assumption in their result is group order preservation which is de-
defined as ”given two groups of claimants, suppose that the sum of the claims of
the members of the first group is greater than or equal to the sum of the claims
of the members of the second group. Then, similar inequalities should hold for
the sums of the awards to the members of the two groups, and for the sums of
the losses incurred by the members of two groups”, as it appears in their paper. However, the assertion of union consistency does not stick to the original claims problem, as the union of groups chosen for comparison does not have to be equal to that of the original problem’s, but rather decides what each group should get by evaluating the conflicting claims of the groups in a new problem created by considering the sum of the claims of the members of each group as an individual claim. In addition, union consistency requires an equality between the awards received by the groups whereas group order preservation requires a "greater than or equal to”. Dagan and Volij (1997) also prove results on extensions of bilateral rules using consistency and average consistency. They analyze how to extend a given bilateral principle to a unique consistent rule and relate it to a family of binary relations.

It is worth mentioning that Thomson (2003) provides a detailed survey on bankruptcy rules.

2.1.1 union—Consistency

Definition 6. (Claims Problem) A claims problem is an ordered pair \((E; d) \in \mathbb{R}_+ \times \mathbb{R}_+^n\) where \(d = (d_1, \ldots, d_n)\) and \(\sum_{i=1}^{n} d_i > E\).

We denote by \(N = \{1, 2, \ldots, n\}\) a set of claimants each of whom has some claim on an endowment \(E\). The class of claims problems involving \(n\) agents and the class of all claims problems are denoted by \(D^n\) and \(D\), respectively. i.e \(D = \bigcup_{N \in \eta} D^N\) where \(\eta\) is the set of all non-empty subsets of \(\mathbb{N}\) where it denotes natural numbers.

Definition 7. An \(n\)-tuple vector \(x = (x_1, x_2, \ldots, x_n)\) is said to be a solution to or an allocation for the claims problem \((E; d)\) if

\(i)\ 0 \leq x_i \leq d_i\) for all \(i \in N\),

\(ii)\ \sum_{i \in N} x_i = E.\)
Here, $x_i$ is interpreted as claimant $i$’s share from $E$. Let’s denote the family of allocations for a given claims problem $(E; d)$ by $A(E; d)$.

**Definition 8. (Bankruptcy Rule)** A bankruptcy rule $\phi : D \to \bigcup_{N \subseteq \eta} R_+^N$ is a function that maps each each claims problem to an allocation.

**Definition 9. (Proportional Rule)** For each $(E; d) \in D$, $PROP(E; d) \equiv (E, \frac{d_i}{\sum_{j \in N} d_j})_{i=1}^n$

**Definition 10. (Anonymity)** A rule $\phi$ satisfies anonymity if for each $(E; d) \in D$, each $\pi \in \Pi^N$ and each $i \in N$, $\phi_{\pi(i)}((E; d_{\pi(i)})) = \phi_i(E; d)$ where $\Pi^N$ denotes the class of bijections from $N$ into itself.

**Definition 11. (Claims Continuity)** A rule $\phi$ satisfies claims continuity if for each sequence $\{(E^v; d^v)\}_{v=1}^\infty$ of elements of $D^n$ and each $(E; d) \in D^n$, if $(E^v; d^v) \to (E; d)$ and for each $v \in \mathbb{N}$ $E^v = E$, then $\phi(E^v; d^v) \to \phi(E; d)$.

**Definition 12. (Union-consistency)** A rule $\phi$ is said to be union-consistent iff for each $(E; d)$ and for any 2 non-empty, disjoint subsets $S_1$ and $S_2$ of $N$ we have:

$$\sum_{i \in S_1} \phi_i(E; d) = \phi_{S_1} \left( \sum_{j \in S_1 \cup S_2} \phi_j(E; d); \left( \sum_{i \in S_1} d_i, \sum_{j \in S_2} d_j \right) \right)$$

In a sense, union consistency requires fairness among all subgroups. The intuition is that some group of claimants may appeal to court claiming that some other group is favoured against them. From a different perspective, addition of new claimants and extra endowment for the new claimants should not favour one group over the other.

**Proposition 1.** $PROP$ is the only rule that satisfies both anonymity, union-consistency and continuity.

**Proof. (Sufficiency)** $PROP$ is trivially anonymous. For union-consistency, consider an arbitrary $(E; d) \in D$. For any $S_1, S_2$ with $S_1 \cap S_2 = \emptyset$, $S_1, S_2 \neq \emptyset$ and
\[ S_1, S_2 \subset N, \text{ we have} \]

\[
\sum_{i \in S_1} \text{PROP}_1(E; d) = \sum_{i \in S_1 \cup S_2} E. \frac{d_i}{\sum_{j \in N} d_j} \text{PROP}_1 \left( E. \sum_{i \in S_1 \cup S_2} \frac{d_i}{\sum_{j \in N} d_j} (\sum_{i \in S_1} d_i, \sum_{j \in S_2} d_j) \right)
\]

\[ = E. \sum_{i \in S_1 \cup S_2} \frac{d_i}{\sum_{j \in N} d_j} \sum_{i \in S_1} d_i = \frac{E. \sum_{i \in S_1 \cup S_2} d_i}{\sum_{j \in N} d_j}. \]

\text{(Necessity) On the contrary, let R satisfy continuity and union-consistency. Assume that } \sum_{i \in N} d_i \text{ is rational and there exists a rational } d_j \text{ for some } j \in N. \]

For an arbitrary \((E; d)\), consider the following 3 problems;

i) \((E; (d_1, d_2, \ldots, d_n))\),

ii) \((E; (d_1, \sum_{j=2}^N d_j))\),

iii) \((E; (d_1', d_2', \ldots, d_{k+M}'))\)

where \(d_i' = d_j'\) for all \(i, j \in \{1, 2, \ldots, k+M\}\) and \(\sum_{j=1}^k d_j' = d_1', \sum_{j=k+1}^{k+M} d_j' = \sum_{j=2}^N d_j\).

Notice that iii) is well defined due to our assumption.(By anonymity, we can assume \(d_1\) is rational. Set \(d_1 = \frac{a}{b}\) and \(\sum_{j=2}^N d_j = \frac{c}{d}\) where \(a, b, c, d \in \mathbb{Z}_+\). Then, \(\sum_{i \in N} d_i = \frac{a+b+c}{b}\).

Consider \((d_1', d_2', \ldots, d_k', \ldots, d_{k+m}')\) where \(d_i' = d_j' = \frac{1}{b} \text{ for all } i, j \in \{1, 2, \ldots, k+M\}\) and \(\sum_{j=1}^k d_j' = \frac{a}{b}, \sum_{j=k+1}^{k+M} d_j' = \sum_{j=2}^N \frac{c}{d} \text{ i.e. } \frac{k}{b} = \frac{a}{b} \Rightarrow k = \frac{a}{l/b} = a.d \in \mathbb{Z}_+ \text{ and } M = \frac{c}{l/b} = c.b \in \mathbb{Z}_+.\) i.e. we have a positive integer number of claimants. However, if either \(d_i\) is irrational for all \(i \in N\) or \(\sum_{i \in N} d_i\) is irrational, then this is not necessarily the case.)

Since by anonymity, \(R_i(E; (d_1', d_2', \ldots, d_{k+M}')) = R_j(E; (d_1', d_2', \ldots, d_{k+M}'))\) \(= E. \frac{1}{k+M} = E. \frac{1}{a.d+b.c},\) we have by union-consistency,

\[
R_i(E; (d_1, d_2, \ldots, d_n)) = R_i(E; (d_1, \sum_{j=2}^N d_j)) = \sum_{j=1}^k R_j(E; (d_1', d_2', \ldots, d_{k+M}'))
\]

\[= \sum_{j=1}^k \frac{E}{a.d+b.c} = \frac{a.d}{a.d+b.c}.E = E. \frac{d_i}{\sum_{j \in N} d_j} = \text{PROP}_1(E; (d_1, d_2, \ldots, d_N)) \]

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as desired.

Let either \( \sum_{i \in \mathbb{N}} d_i \) or each \( d_i \) be irrational. Either case, we have at least one irrational \( d_i \). Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), we can choose \( (d^v_j)_{j \in \mathbb{N}} \) such that \( d^v_j \) is rational and \( d^v_j \to d_j \) where \( d_j \) is irrational. We also have \( d \to d^* \iff d_i \to d^*_i \) for each \( i \in \mathbb{N} \). Let \( (E; d^v) \) be sequence of claims problems with \( d^v_i \in \mathbb{Q} \) for \( i = 1, 2, ..., N \) and \( v = 1, 2, ... \) and \( (E; d^v) \to (E; d) \). Since \( R \) is union-consistent and continuous, we have \( R(E; d^v) = \text{PROP}(E; d^v) \) and \( \lim_{d^v \to d} \text{PROP}(E; d^v) = \lim_{d^v \to d} R(E; d^v) = R(E; d) = \text{PROP}(E; d) \) as \( \text{PROP} \) is continuous.

\( \square \)
Chapter 3

Risk Averse Investors Behavior towards a Risky Firm

We extend the investment game which was first suggested in Kıbrıs and Kıbrıs (2013). In the model, there is an arbitrary number of agents each of which is endowed with a Constant-Absolute Risk Aversion (CARA) utility function. The agents are presented the opportunity to invest in a risky project whose outcome will either be success or be failure based on a Bernoulli Distribution function. In case of a failure, total value of the investment is allocated among the agents according to a bankruptcy rule. Kıbrıs and Kıbrıs (2013) compare 4 most common rules in the literature whereas we relax the assumption that no agent can receive more than his investment, in case of a failure. By doing so, we are allowing the agents to receive amounts which are not possible under standard bankruptcy rules. Hence, we are looking for optimality in a larger set of possible payoffs.

In real world situations, there are different types of agents some of which can receive gains on investment even in the case of a failure. For instance, as opposed to shareholders, some creditors may receive more than what they put in. In that sense, our model can be considered as a better approximation to reality. We also drop the assumption that each agent will receive a constant interest rate in case of a success. Our aim is to account for all possible bankruptcy rules, and
even other allocations which do not adhere to properties that are imposed by
bankruptcy rules, and to determine the optimal allocation a social planner would
choose without violating the individual rationality constraints. We first show
that only the total amount of investment matters, i.e. as long as the level of total
investment is the same, how much each investor contributes has no influence on
welfare and proceed to show that the optimal allocation can be obtained via a
free-market mechanism in which agents can trade payments in different states of
the world. We also show that no matter which rule is announced by the social
planner, the optimal amount of total investment is equal to the amount which
also emerges from the competitive game that utilises the proportional rule.

Following the main idea of Kıbrıs and Kıbrıs (2013) we investigate the Pareto
Optimal allocations regarding a joint venture(risky firm).

3.1 The Setting of Kıbrıs and Kıbrıs (2013)

Denote the set of investors/agents by \( N = \{1, \ldots, n\} \). Each agent \( i \)'s preferences
are represented by a constant-absolute risk aversion (CARA) utility function
\( u_i : \mathbb{R}_+ \rightarrow \mathbb{R} \) where \( u_i(x) = -e^{-a_i x} \). Here, \( a_i \) is agent \( i \)'s risk-aversion constant
and their analysis relies on the assumption that \( a_i > 0 \ \forall i \in N \), that is, the
agents are risk averse. Also, \( a_1 \leq ... \leq a_n \) without loss of generality.

There is a risky firm whose ex-ante value is determined by the total value of
investments made by the agents, i.e., \( \sum_i s_i \) where \( s_i \) is the investment of agent
\( i \). Furthermore, agents choose their investments \( s_i \in \mathbb{R}_+ \) simultaneously. They
borrow from an outside market and the interest rate is normalized to 0. There
are 2 states of the world: with probability \( p \in (0,1) \) the firm succeeds and the
total value of the firm becomes \( (1 + r) \sum_i s_i \) and with \( (1 - p) \) probability it fails
and the total value shrinks to \( \beta \sum_i s_i \). It is assumed that \( r > 0 \) and \( 0 < \beta < 1 \).
After the realization of the state of the world, the total value is to be distributed
among the agents.
The division method used at this point might capture some desirable properties. For example:

1. Given the method, this competitive game might lead to a Pareto Optimal allocation in either the utilitarian sense or the egalitarian sense.

2. It might maximize the volume of investments.

 Kıbrıs and Kıbrıs (2013) do not check for Pareto Optimality but rather compare a limited subset of possible division methods evaluated according to utilitarian and egalitarian welfare. While employing the same notions in order to measure welfare, we consider the set of all possible allocations of the ex-post total value and, therefore, characterize the Pareto optimal ones.

Moreover, the welfare levels in Kıbrıs and Kıbrıs (2013) concern only 2 agents with equal Pareto weights (the latter is valid for utilitarian welfare).

Their definitions as they appear in their paper are as follows:

**Definition 13** (Utilitarian Social Welfare). *The utilitarian social welfare at the Nash Equilibrium (NE, henceforth) of the 2 agent game induced by an allocation rule F is given by:*

\[ UT^F(p, r, \beta, a_1, a_2) = U^F_1(\varepsilon(G^F)) + U^F_2(\varepsilon(G^F)) \]

where the game \( G^F \) is defined by the parameters \((p, r, \beta, a_1, a_2)\), \( U^F_i(\cdot) \) is agent i’s utility and \( \varepsilon(G^F) \) is an equilibrium of the game.

**Definition 14** (Egalitarian Social Welfare). *The egalitarian social welfare at the N.E. of the 2 agent-game induced by an allocation rule F is given by:

\[ EG^F(p, r, \beta, a_1, a_2) = \min \{ U^F_1(\varepsilon(G^F)), U^F_2(\varepsilon(G^F)) \} \]

where again the game \( G^F \) is defined by the parameters \((p, r, \beta, a_1, a_2)\) and \( \varepsilon(G^F) \) is
an equilibrium; it is the minimum utility an agent gets at an equilibrium induced by \( F \).

### 3.1.1 Common Allocation (Bankruptcy Rules)

Bankruptcy occurs in the state of the world where the value of output is \( \beta \) of the initial investment, but Kıbrıs and Kıbrıs (2013) assume that in the other state each agent gets the full return \((1 + r)\) times their investment.

**Definition 15 (Proportional Rule).** For each \( i \in N \), \( \text{pro}_i(s) = \beta s_i \) where \( s_i \) is the investment choice of agent \( i \in N \) and \( s \) is the investment vector i.e. each agent gets a proportion \( \beta \) of his claim where we equate “claim” with the value of the initial investment.

**Definition 16 (Constrained Equal Losses).** For each \( i \in N \), \( \text{CEL}_i(s) = \max \{ s_i - \varphi, 0 \} \) where \( \varphi \in \mathbb{R}_+ \) satisfies \( \sum_{i \in N} \max \{ s_i - \varphi, 0 \} = \beta \sum_{i \in N} s_i \).

Note that in order to find equilibrium under CEA and CEL, Kıbrıs and Kıbrıs first prove that not all combinations of parameters lead to a NE under CEL/CEA. They then show that for the 2 agents case if there exists a NE it must be identical to those of EL/EA respectively where EL/EA are allocations such that agents lose/receive the same amount.

**Definition 17.** For each \( i \in N \), \( \text{CEA}_i(s) = \min \{ \varphi, s_i \} \) where \( \varphi \in \mathbb{R}_+ \) satisfies \( \sum_{i \in N} \min \{ \varphi, s_i \} = \beta \sum_{i \in N} s_i \).

**Proposition 2 (Kıbrıs and Kıbrıs (2013)).** If \( \ln \left( \frac{rp}{(1-p)(1-\beta)} \right) \leq 0 \), then Game\(^{\text{PROP}}\) has a unique dominant strategy equilibrium \((0, \ldots, 0)\). Otherwise, the game has a unique dominant strategy equilibrium \( s^* \) in which each agent \( i \) chooses a positive investment level \( s^*_i \) given by

\[
 s^*_i = \frac{1}{a_i(r + 1 - \beta)} \ln \left( \frac{rp}{(1 - p)(1 - \beta)} \right)
\]
3.2 Our Setting

We relax the assumption that no agent can receive more than his investment, in case of a failure. We also drop the assumption that each agent will get an interest rate of $r$, in case of success. Our aim is to determine the optimal allocation social planner (S.P. hereafter) chooses without violating the individual rationality constraints.

Therefore, we first present the social planner problem for the 2 agent case without imposing the constraints $F^s_i \geq 0$ and $F^f_i \geq 0$ where $F^s_i$ and $F^f_i$ represent player $i$’s shares in the case of success and failure respectively. We shall see that they are not binding for a fairly large set of parameters.

We solve the problem in 2 steps for simplicity: First, we assume that agents’ investment levels ($x_1, x_2$) are given and find the allocations in the case of success and failure as a function of them. Then, we see that what matters to the agents is not the individual investments but rather the total investment. Then, we solve the total utility maximization problem with respect to the total investment.

3.2.1 Social Planner’s Problem for 2 Agents

\[
\max \lambda_1 \left[ p u_1 (F^s_1 - x_1) + (1 - p) u_1 (F^f_1 - x_1) \right] + \lambda_2 \left[ p u_2 (F^s_2 - x_2) + (1 - p) u_2 (F^f_2 - x_2) \right]
\]

such that

\[
F^s_1 + F^s_2 = (1 + r)(x_1 + x_2) \tag{1}
\]

and

\[
F^f_1 + F^f_2 = \beta(x_1 + x_2). \tag{2}
\]

Alternatively, we can plug (1) and (2) in and solve the unconstrained problem and if $0 \leq F^s_i \leq (1 + r)(x_1 + x_2)$ and $0 \leq F^f_i \leq \beta(x_1 + x_2)$ then the constraints
\( F^s_i \geq 0 \) and \( F^f_i \geq 0 \) for \( i = 1, 2 \) are not binding. Then, the Lagrangian is:

\[
L = \lambda_1 \left[ -p.e^{-a_1(F^s_1 - x_1)} + (1 - p)(-e^{-a_1(F^f_1 - x_1)}) \right] + \\
\lambda_2 \left[ -p.e^{-a_2((1+r)(x_1 + x_2) - F^s_1 - x_2)} + (1 - p)(-e^{-a_2(\beta(x_1 + x_2) - F^f_1 - x_2)}) \right].
\]

Then solving for the 1st order conditions we find:

\[
F^s_1 = \frac{\ln\left(\frac{a_1\lambda_1}{a_2\lambda_2}\right) + a_2(1 + r)(x_1 + x_2) - a_2 x_2 + a_1 x_1}{a_1 + a_2},
\]

\[
F^s_2 = (1 + r)(x_1 + x_2) - \frac{\ln\left(\frac{a_1\lambda_1}{a_2\lambda_2}\right) + a_2(1 + r)(x_1 + x_2) - a_2 x_2 + a_1 x_1}{a_1 + a_2},
\]

\[
F^f_1 = \frac{\ln\left(\frac{a_1\lambda_1}{a_2\lambda_2}\right) + a_2 \beta(x_1 + x_2) - a_2 x_2 + a_1 x_1}{a_1 + a_2},
\]

and

\[
F^f_2 = \beta(x_1 + x_2) - F^f_1 = \beta(x_1 + x_2) - \frac{\ln\left(\frac{a_1\lambda_1}{a_2\lambda_2}\right) + a_2 \beta(x_1 + x_2) - a_2 x_2 + a_1 x_1}{a_1 + a_2}.
\]

This is the solution to the planner’s problem given \( x_1 \) and \( x_2 \). Now we consider the prior choice of investments. Set \( \Psi = (x_1 + x_2) \).

We then have

\[
u_1(F^s_1 - x_1) = -e^{-a_1[\ln\left(\frac{a_1\lambda_1}{a_2\lambda_2}\right)]\frac{1}{1+\alpha_2}} e^{-\frac{a_1\lambda_1}{a_1+\alpha_2}}
\]

\[
= -e^{-\frac{a_1\lambda_1}{a_1+\alpha_2}} \left( \frac{a_1\lambda_1}{a_2\lambda_2} \right)^{\frac{a_1}{1+\alpha_2}}
\]

\[
u_2(F^s_2 - x_2) = -e^{-\frac{a_1\lambda_1}{a_1+\alpha_2}} \left( \frac{a_1\lambda_1}{a_2\lambda_2} \right)^{\frac{a_2}{1+\alpha_2}}
\]

\[
u_1(F^f_1 - x_1) = -e^{-\frac{a_1\lambda_1}{a_1+\alpha_2}} \left( \frac{a_1\lambda_1}{a_2\lambda_2} \right)^{\frac{-a_1}{1+\alpha_2}}
\]

\[
u_2(F^f_2 - x_2) = (-1) e^{-\frac{a_1\lambda_1}{a_1+\alpha_2}} \left( \frac{a_1\lambda_1}{a_2\lambda_2} \right)^{\frac{-a_2}{1+\alpha_2}}.
\]

It is worth mentioning that \( x_1 \) and \( x_2 \) enter only through \( \Psi \). The intuition that
the production function is additive in their investments, and that the preferences are quasi-linear – i.e. the fact that return/loss less investment, all goes inside the utility function. The S.P. chooses $\Psi$ to maximize

$$\lambda_1 \left[ -pe^{-\frac{a_1 \lambda_1}{a_2 \lambda_2}} \left( \frac{a_1 \lambda_1}{a_2 \lambda_2} \right)^{-a_1} e^{-\frac{a_1 \lambda_1 a_2 \Psi}{a_1 + a_2} - (1 - p) e^{-\frac{a_1 \lambda_1 (\beta - 1) \Psi}{a_1 + a_2}} \left( \frac{a_1 \lambda_1}{a_2 \lambda_2} \right)^{-a_1}} \right] +$$

$$\lambda_2 \left[ -pe^{-\frac{a_1 \lambda_1}{a_2 \lambda_2}} \left( \frac{a_1 \lambda_1}{a_2 \lambda_2} \right)^{-a_2} e^{-\frac{a_1 \lambda_1 a_2 \Psi}{a_1 + a_2} - (1 - p) e^{-\frac{a_1 \lambda_1 (\beta - 1) \Psi}{a_1 + a_2}} \left( \frac{a_1 \lambda_1}{a_2 \lambda_2} \right)^{-a_2}} \right]. \quad (3)$$

We then get:

$$\arg \max(3) = \Psi^* = \begin{cases} \ln\left( \frac{rp}{(1 - p)(1 - \beta)} \right) \cdot \frac{a_1 + a_2}{a_1 + a_2(1 - \beta + r)} & \text{for } rp \geq (1 - p)(1 - \beta) \\ 0 & \text{otherwise} \end{cases}$$

$\Psi^*$ is independent of $\lambda_i$ as the quasi-linear structure implies that surplus is maximised independently of $\lambda_i$ but the Pareto weights determine the split of surplus.

Note that $\Psi^*$ is the total investment level in the two agent $Game^{Prop}$. Intuitively, if the condition $rp \geq (1 - p)(1 - \beta)$ holds, on average the project yields a nonnegative return, and when this is strict, a small positive investment will yield a positive first-order gain, while the increase in variance will be second-order, so investment must be positive.

**Remark 1.** One should notice that the P.O. allocation might yield utility levels different from those of the competitive game under PROP. With 2 agents. For them to be equal,

$$\left( \frac{a_1 \lambda_1}{a_2 \lambda_2} \right)^{a_2/a_1} + a_2 = \left( \frac{a_1 \lambda_1}{a_2 \lambda_2} \right)^{-a_1/a_1} + a_2 = 1$$

must hold. Since the S.P. is concerned with sum of utilities, he exploits the fact
that more risk-averse agents are more productive in the case of success and vice versa.

In order to have a feasible allocation, no agent should receive a negative payment in any state of the world. The allocation is feasible if it satisfies the following:

1. $0 \leq F_s^1 \leq (1 + r)\Psi$,
2. $F_s^1 + F_s^2 = (1 + r)\Psi$,
3. $0 \leq F_f^1 \leq \beta\Psi$
4. $F_f^1 + F_f^2 = \beta\Psi$

One can check that the four conditions above hold for a very large portion of the parameter space.

**Proposition 3.** If there are $N$ agents playing the investment game and the S.P. chooses investment levels as well as the allocations in both states of the world by respecting the individual rationality constraints, then the total investment $\Psi^* = 0$ if $pr < (1 - p)(1 - \beta)$. Otherwise,

$$\Psi^* = \frac{\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) \cdot \sum_j 1/a_j}{1 - \beta + r}$$

i.e. the level of investment under $Game^{PROP}$ is preserved. However, depending on the S.P.'s preferences, $PROP$ is not the only P.O allocation. Moreover, individual levels of investment do not matter.

### 3.2.2 Social Planner’s Problem for $N$ Agents

$$\max_{F_s^i, F_f^i} \sum_i \lambda_i u_i(F_s^i - x_i) + (1 - p) \sum_i \lambda_i u_i(F_f^i - x_i)$$

where $\sum_i F_s^i = (1 + r) \sum_i x_i$, $\sum_i F_f^i = \beta \sum_i x_i$ and $F_j^j \geq 0$, $j = s, f$. For all $i \in N$, the last set of constraints, however, might not be binding. Therefore, we
ignore them at this point and check whether the condition is satisfied after the maximization.

Solving for the first order conditions, we find:

\[
F_s^i = x_i - \frac{\ln \left( \frac{\alpha_1}{p \lambda_i a_i} \right)}{a_i}
\]

and

\[
F_f^i = x_i - \frac{\ln \left( \frac{\alpha_2}{(1 - p) \lambda_i a_i} \right)}{a_i}
\]

for all \( i \in N \) where

\[
\alpha_1 = \frac{1}{\sum_i} \sqrt{e^{-r} \sum_i x_i \prod_i (p \lambda_i a_i)^{\frac{1}{a_i}}}
\]

and

\[
\alpha_2 = \frac{1}{\sum_i} \sqrt{e^{(1 - \beta) \sum_i x_i \prod_i ((1 - p) \lambda_i a_i)^{\frac{1}{a_i}}}}
\]

As in the two agent case, the utilities depend on \( \Psi = \sum_i x_i \) and not on individual investments levels.

Hence, the S.P. will choose the social utility maximising level of \( \Psi \), i.e.,

\[
\arg \max \sum_i \lambda_i \left( -p \cdot e^{\frac{-r \Phi}{\sum_j} + \sum_k \ln(p \lambda_k a_{ki}) \frac{1}{\sum_j a_{kj}} \ln(p \lambda_{ki}) - (1 - p) \cdot e^{\frac{(1 - \beta) \Psi}{\sum_j} + \sum_k \ln((1 - p) \lambda_k a_{ki}) \frac{1}{\sum_j a_{kj}}} - \ln(p \lambda a_i) \right)
\]

which yields the optimal level of total investment:

\[
\Psi^* = \frac{\ln(\frac{pr}{(1-p)(1-\beta)}) \sum_j 1/a_j}{1 - \beta + r}
\]

which is equal to the total value of investments in the proportional game.

Note: Notice that if \( pr < (1 - p)(1 - \beta) \) then \( \Psi^* \) is not determined. This is
because the investment will have a negative expected value in this case and it is optimal not to invest anything at all.

Since the agents are borrowing from outside, S.P. only decides how much each agent will lose or gain. The payment scheme depends on the individual investment so that the earnings don’t depend on individual investments but on the sum of investments. (Moreover, there are no externalities in Game\textsuperscript{PROP} so the S.P. is effectively maximizing each agent’s utility separately, that’s why our result holds and Kıbrıs and Kıbrıs (2013) find a dominant strategy equilibrium).

**Proposition 4.** Among the class of bankruptcy rules, PROP is P.O, but it is not unique.

Another important issue is whether PROP is a P.O. rule and whether it is unique. In order to see if that’s the case, we are going to consider all possible allocation schemes in the case of a failure. i.e. we will not be interested in how the agents share the surplus in the case of success as in that scenario there is going to be a higher income to share than the amount invested in the venture. Translating this to the language of bankruptcy, the endowment to be distributed will exceed the sum of the claims, hence it will be possible to jointly honour all the claims. Nevertheless, we need to make an assumption on how the success profits will be divided since agents’ utilities also depend on this variable. One may expect to see different levels of losses for each agent depending on what the S.P. assigns to each of them in the case of success. Since we have assumed that the production technology is quasi-linear, we may start by assuming that each agent will receive a profit equal to ”r” portion of her investment. In other words, we are going to impose proportional allocation of gains. If we do so, however, we can intuitively anticipate proportional division in the case of failure as the ability of S.P. to allocate will be limited to the failure event. i.e. even if one of the agents promise a higher marginal utility than the other if the S.P. transfers an infinitesimal unit from the latter to the former when the venture is successful.
and vice versa in the failure case, since transfers in the case of success won’t be allowed, an allocation different than PROP. might lead to a marginal utility discrepancy. Thus, it appears to be reasonable to predict that the S.P. will want to make marginal utilities of agents match with each other and choose proportional allocation as the utilities are CARA.

One other thing is that the feasibility restrictions have to be tightened up. Instead of requiring that no agent receives more than what’s available in the case of success, we confine each agent’s maximum earning to a proportion of her investment. The constraints defined for the failure case, however, will remain the same. Therefore, we revisit our baseline model with 2 agents. The S.P.’s problem is:

\[
\max_{F_1^f,F_2^f} \lambda_1[pu_1(rx_1) + (1-p)u_1(F_1^f - x_1)] + \lambda_2[pu_2(rx_2) + (1-p)u_2(F_2^f - x_2)]
\]

such that

\[F_1^f + F_2^f = \beta(x_1 + x_2)\quad (1)\]

and

\[0 \leq F_1^f \leq \beta(x_1 + x_2).\quad (2)\]

Plugging \(F_2^f = \beta(x_1 + x_2) - F_1^f\) in the equation, we write the Lagrangian:

\[
L = \lambda_1[-p.e^{-a_1rx_1} + (1-p)(-e^{-a_1(F_1^f - x_1)}] + \\
\lambda_2[-p.e^{-a_2rx_2} + (1-p)(-e^{-a_2(\beta(x_1 + x_2) - F_1^f - x_2))] + \\
\mu_1F_1^f + \mu_2(\beta(x_1 + x_2) - F_1^f)
\]

which gives optimality and complementarity conditions;
\[ a_1 \lambda_1 (1 - p) (-e^{-a_1(F_1^f - x_1)} - a_2 \lambda_2 (1 - p) e^{-a_2(\beta(x_1 + x_2) - F_1^f - x_2)} + \mu_1 - \mu_2 = 0 \] (1)

\[ \mu_1 F_1^f = 0 \] (2)

\[ \mu_2 (\beta(x_1 + x_2) - F_1^f) = 0 \] (3)

\[ -F_1^f \leq 0 \] (4)

\[ F_1^f - \beta(x_1 + x_2) \leq 0 \] (5)

\[ \mu_1, \mu_2 \leq 0 \] (6)

Then, from (2) and (3), one of the following must hold;

\[ \mu_1 = \mu_2 = 0 \] (1)

\[ \beta(x_1 + x_2) - F_1^f = \mu_1 = 0 \] (2)

\[ \beta(x_1 + x_2) - F_1^f = F_1^f = 0 \] (3)

\[ F_1^f = \mu_2 = 0 \] (4)

(3) can be satisfied only if the S.P. decides to make no investment at all. Since it may occur only depending on the parameters of the production technology and once we have such parameters, there is not going to be any investment regardless of the risk aversion coefficients or the allocation rule used. Therefore, we rule out (3). If (2) or (4) holds, then we have a corner solution and the S.P. allocates all the remaining endowment to the first agent in case (2), and to the second agent in case (4). Since we have fixed the returns in the success case and for each agent the return cannot be greater than her investment, as there is the risk losing all the investment, it yields negative expected returns for one of the agents and violates individual rationality. With such a rule, regardless of whether the S.P. chooses the investment levels or the agents do, the optimal (or the chosen) investment
would be 0 for one of the agents. (Actually, it would be 0 for the more risk averse investor) Hence, we have $\mu_1 = \mu_2 = 0$. Plugging $\mu_1 = \mu_2 = 0$ in (1) and solving for $F_1^f$ we find:

$$F_1^f = \frac{\ln \left( \frac{a_1 \lambda_1}{a_2 \lambda_2} \right) + a_2 \beta (x_1 + x_2) - a_2 x_2 + a_1 x_1}{a_1 + a_2}$$

which is the same equation we found without fixing the success returns. The reason is that the S.P. treats the two different events separately. Above equation also shows that one can find $\lambda_1, \lambda_2$ such that PROP is a P.O. rule but it is not unique.

Note that the same result also applies to the case with an arbitrary number of agents.

### 3.2.3 Difficulties in Applying the P.O. Allocation

In order to apply the P.O. outcome in this setting, the S.P. has to have complete knowledge about the agents’ preferences over the lotteries generated by this risky venture. In a realistic marketplace where there are several potential investors, it is too optimistic to say that it is a realistic assumption. Therefore, in order to apply these levels of investments and allocations, the S.P. has to come up with a mechanism which will reveal the preferences correctly.

A solution to this problem might be a free market mechanism where agents can trade their potential gains and potential losses. Since at a given allocation, it is possible to see different marginal utilities, even each agent is receiving exactly the same payments in each state of the world, agents with different risk attitudes might benefit from such trade.
3.2.4 Trading the Different Payments in Different States of the World

Once the investment levels and the corresponding allocation scheme is determined, all agents know how much they are going to be paid in two different outcomes. Regarding the type of utility functions they have, one can view them as traders in a marketplace with two goods as endowment: the payment in the case of “success” and the payment in the case of “failure”. Once these two are allowed to be exchanged, we will end up with a pure exchange economy after the allocation is determined. Hence, we will be able to apply two renowned theorems in order to justify this method.

Definition 18. (Pure Exchange Economy) A pure exchange economy $\varepsilon$ with consumption space $X$ and set of agents $N$, is

$$\varepsilon = \{e_i, >_i\}_{i \in N},$$

where $e_i \in X$ and $>_i \in X \times X$ denote the agent $i$’s initial endowment and preferences, respectively.

Theorem 3. (First Theorem of Welfare Economics) For any exchange economy $\varepsilon$, where agents preferences are given by continuous, complete, preorders satisfying local non-satiation $CE(E) \subseteq PO(E)$.

Definition 19. (Local Non-Satiation) A preference relation $>_i \in X \times X$ is locally non-satiated if for all $x \in X$ and all $\varepsilon > 0$ there exists $y \in X$ with $\|y - x\| < \varepsilon$ and $y > x$.

Since the agents in our setting have monotonic preferences and local non-satiation is implied by monotonicity, one of the requirements to be able to apply the theorem is automatically satisfied.

Theorem 4. (Social Planner’s Problem and Pareto Weights) If every agent $i$’s preferences are represented by a continuous, strictly increasing and concave utility function $u_i : X \rightarrow \mathbb{R}(0 \in X)$, and $e_i >> 0$ then an allocation $x^* \in X^n$ is P.O. if
and only if there exists $\theta^* \in \Delta(N)$ and $x^*$ solves the social planner’s problem for $\varepsilon$ at $\theta^*$ given by $\max_{x \in F(\varepsilon)} \sum_{i \in N} \theta^*_i u_i(x_i)$.

It is easy to verify that $u_i(\cdot)$ is continuous and strictly increasing in both arguments. Thus, it is enough to check for concavity.

**Theorem 5.** (Checking for Concavity) A twice continuously differentiable function $f$ is concave if the Hessian matrix $H(x)$ is negative semi-definite at all points $x$.

As a result, we need to check for the eigenvalues of the matrix $H(s, f) = \frac{\partial^2 u_i}{\partial s^2} \frac{\partial^2 u_i}{\partial f \partial s}$

$\frac{\partial^2 u_i}{\partial s \partial f} \frac{\partial^2 u_i}{\partial f^2}$

Where the endowments in the case of “success” and “failure” are denoted by $s$ and $f$, respectively.

Since the eigenvalues are all negative and the utility functions are concave, then we conclude that the preferences satisfy the desired properties and, thus, the theorems hold. As a result, the price mechanism is efficient and will correct any error the S.P. makes when allocating the liquidated firm.

### 3.3 Conclusion

The model presented in Kibrís and Kibrís (2013) does not account for some of the situations which occur in real life. Namely, they do not allow for any investor to receive more than her claim. Moreover, they conduct their analysis for only the 4 most common rules in the literature. However, these are the most favoured rules in the literature. All of them are studied and suggested by various authors on both axiomatic and game theoretic grounds. In order to adopt a more realistic approach to risky investments, we assume that investors’ possible gains are not determined by individual contributions. Yet, we assume the same sort of utility function which is also used in the analysis of Kibrís and Kibrís (2013).
leads to the result that the socially optimal level of welfare is not dependent on individual levels of investment, although the investors have different attitudes towards risk. We get this result by exploiting the fact that the production function is additive in investors’ contributions and that the preferences are quasi-linear.

On a modified version of the model, we impose that each agent’s profit is a fixed fraction of her investment in the case of success. On the other hand, we leave the constraints for the failure case unchanged and solve the S.P.’s problem. By doing so, not only do we show that PROP. is P.O. among the class of bankruptcy rules but we also establish that it is a P.O. way of distributing the ex-post welfare even if the S.P. is not restricted to give each agent at most as much as she invests. This may explain why PROP. is also used in real ventures. However, it is noteworthy that PROP. is not the only P.O. way of allocation. In fact, we have an abundance of P.O. allocations. Moreover, since different risk aversion coefficients require different Pareto weights, given the Pareto weights of the S.P., a different allocation needs to be chosen since no allocation is P.O. for all levels of risk-aversion.

In that case, S.P. encounters the problem of having incomplete information about agents’ preferences. In order to cope with the problem, we suggest a market mechanism where agents can trade their potential gains and potential losses freely. By making use of some well-known results, we conclude that the mechanism ends up with a P.O. allocation.
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