Robustness of irrationality in financial markets

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Abstract

Recent research in financial economics has suggested that models incorporating agents whose expectations are not fully rational may help us to understand and explain financial market behaviour. Using a noise trader framework this thesis presents a theoretical appraisal of the robustness of such irrational beliefs and the prescriptions of noise trader models. In contrast to previous work we show that the effects of noise trading are not a short-term phenomena. As agents horizons increase noise trader induced risk becomes more important, particularly if their beliefs are persistent. Various proposed policy measures to reduce speculative behaviour are considered. We demonstrate that noise trading is robust to the imposition of both linear and non-linear taxes transaction taxes, but could be reduced by direct market intervention. A new mixed policy of tax financed intervention is shown to reduce the destabilising effects of noise traders. We also consider models in which all agents are endowed with adaptive learning rules. Conditions for convergence of the adaptive model to the rational expectations solution and expectational stability are derived. An evolutionary version of the model is investigated through computer experiments. Asset prices converge to a neighbourhood of the rational expectations equilibrium but exhibit exaggerated movements consistent with empirical evidence. The experimental evidence accords with the analytical results on the robustness of noise trader beliefs and their effects, but suggests that noise traders will not come to dominate the market.
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Declaration

The material contained in this thesis is my own work, has not been published elsewhere, and is an original composition.

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Introduction

This thesis presents a theoretical appraisal of the robustness of irrational beliefs in asset markets. Since the mid 1980's there has been a continued interest in models of heterogeneous agents with a view to explaining observed asset market behaviour. A variety of such models has been proposed with a correspondingly large range of results. We focus on one of the more popular frameworks in which two groups of agents hold differing beliefs. The first group of agents has beliefs which conform to the rational expectations hypothesis while the second group exhibit a systematic bias from rational expectations in their expectations formation. Following the convention in the literature we term the first group of agents rational traders and the second group noise traders.

The main contribution of this thesis is to examine in the context of a simple model and its extensions, the robustness of the second groups beliefs in different circumstances. Alternatively, the thesis could be viewed as examining the fragility of such beliefs. As there is little agreement in the literature on a standard set up for such models or their focus, the issue of robustness is a neglected one. Although the thesis also contributes to other areas this is the main theme that permeates our work.

If noise traders are to matter for finance and economics in general and not just the sub-field of behavioural finance then the robustness of noise trader beliefs must be scrutinised. The noise trader paradigm with some irrational agents is a potential competitor to the efficient markets hypothesis with all rational agents. For noise trading to be taken seriously it must first be determined that the paradigm is a robust rather than a convenient assumption.
As discussed in chapter 1 there is some evidence to suggest that financial markets are less than fully efficient when measured against accepted benchmarks. There is no agreement at present as to the likely cause(s) of the difference between observed efficiency and the prescription of benchmark models. We are thus left to name the residual. Although not exhaustive, it is possible to split explanations for the residual into rational and irrational headings. Rational headings include bubbles, sunspots, Peso problems, stochastic discount factors and time varying risk premia. Irrational headings include fads, noise traders, market psychology, learning and herding. Ultimately the most plausible 'story' is a matter of personnel choice. Aesthetically we prefer the irrational explanations story and specifically examine the plausibility of the noise trader paradigm.

If noise traders cause the efficiency residual and their beliefs are robust then there may be policy consequences. As a rule we speak of informational efficiency in financial markets not allocative efficiency. Even more rare is any discussion of the effects of financial market performance and efficiency on the general macroeconomy. The most obvious channel for noise traders activities to effect the macroeconomy is through investment. If noise traders distort the returns on financial assets relative to productive capital then the size and/or the type of productive capital investment may be detrimentally effected. In such a case governments may be required to act to restore a more optimal balance between financial market investment and productive capital investment. We note that the effects of noise traders on the real economy are far from clear cut however. We provide an analysis of potential policy measures which might be used to curb noise trading activities in chapters 3 and 5 and also examine how robust noise trader beliefs are in such circumstances. If noise traders and their effects are not robust then perhaps any effect on the macroeconomy is purely transitory and offsetting.
policies are not required. As our analysis is conducted in a partial equilibrium context we are unable to speak to the likely effects of noise traders on the real economy.

The thesis is set out as follows. In chapter 1 we discuss the motivation behind noise trader models and attempt to define noise trading. Using this definition we distinguish noise trading from the related concepts of chartism and liquidity traders, and provide a survey of the main papers in the literature. We then give a detailed analysis of arguably the most notable and influential model due to De Long, Shleifer, Summers and Waldman (1990a), (DSSW hereafter) to illustrate the richness of the noise trader approach. Variants of their approach form the starting point for all of the analysis in this thesis. In the final section of the chapter this model is extended to include heterogeneous noise trader beliefs to capture the notion that noise traders will likely trade both with rational traders and amongst themselves.

Chapter 2 asks what effect the horizon of agents has on asset price behaviour and the robustness of noise trader beliefs. While there are some static models that employ horizons of greater than two periods, there are few existing descriptions of multiperiod dynamic models. DSSW discuss some possible effects of extending agents horizon but do not explicitly model this. The results in chapter 2 validate some of their conjectures but cast doubt on the view that noise traders have a lesser effect on prices and perform worse than rational traders as the horizon is extended. We show that traders horizons do matter, particularly if noise traders misperceptions are persistent. There is a complex interplay between agents pricing of conditional return and risk which is not present in the shortest horizon version of the model. Most importantly, the performance of noise traders, measured in terms of conditional returns, need not decrease as the horizon is extended. As agents horizon is increased noise trader induced risk becomes more important.
In chapter 3 we ask how robust are noise traders to different forms of policy intervention which have been advocated to deter allegedly destabilising speculation in financial markets. The results contribute to the topical and contentious debate on the proposed use of Tobin taxes in financial market transactions. We present an analysis of simple linear taxes for models with different time horizons and different specifications of noise trader beliefs. Following this we analyse non-linear taxes to examine the effects of a more progressive tax regime. For proponents of Tobin taxes our results are not encouraging but accord with some of the results from the empirical literature on transactions taxes. We then explore direct government intervention as an alternative policy. Surprisingly we find that intervention in the opposite direction to (same direction as) noise trader beliefs benefits (damages) the relative performance of noise traders. While all results are model dependent this is a beguilingly simple policy prescription, albeit unconventional. The chapter finishes by analysing an adjustment tax on agents holdings and a mixed intervention policy using a transactions tax to finance government purchases of the risky asset.

Chapter 4 breaks with the rational expectations analysis of the first three chapters to examine the limiting behaviour of the noise trader model when agents form expectations adaptively. The primary purpose of the chapter is to determine whether the previous results with rational expectations are upheld when agents form their price expectations using learning algorithms. We show that our model with heterogeneous agents will converge to the rational expectations solution under a given learning scheme and thus the previous results hold under learning, described as expectationally stable. The results of the chapter show that noise trading is robust under learning. At a more general level the chapter contributes to the learning literature by providing a specific example of an economic model in which an economy with heterogeneous
adaptive agents converges to the equilibrium path of the rational expectations economy. The vast majority of papers in the learning literature employ a representative agent framework or a single learning agent, there are currently very few which look at a specific model with heterogeneous learning rules.

In the last chapter we again look at learning but use simulations to study the behaviour of more complicated environments. Of all the models considered in the thesis the final chapter contains arguably the most realistic. However, even though the models are relatively simple the dynamics are complex and we must forsake closed-form analytical solutions for the pricing equations and resort to simulation evidence. At the heart of this chapter is the desire to free the basic model by endogenising some of its assumptions. Specifically, we endogenise the proportion of agents according to relative past performance and also endogenise the degree to which agents change their expectations. After outlining the structure we proceed to use the model to examine the effects of the mixed intervention policy presented in chapter 3.

Lastly we provide a conclusion which reiterates the main results and points to directions for future or omitted research which are suggested by the results in the thesis.
1. A model of noise trading

1.1 Introduction

There is an inextricable link between the theory of efficient capital markets in finance and rational expectations equilibrium in macroeconomics, (see inter alia Fama (1976) and Melino (1987)). The origins of the academic interest in models of irrational beliefs lie in theoretical and empirical work which points to failures of the theory of efficient capital markets. Such evidence is not of great relevance to this thesis or research on noise trading per se. More importantly, the evidence helped to create a climate in which theories of irrational behaviour would be more acceptable to economists. The market efficiency debate is a moot point. Parallel work on non-rational behaviour in macroeconomic models has illustrated the usefulness of retreating from full rationality in order to explain actual behaviour, (see for example Akerlof and Yellen (1985), Haltiwanger and Waldmann (1985, 1989), Sethi and Franke (1995)). The debate over alleged excess volatility of asset prices during the 1980's was instrumental in the creation of a climate in which models positing irrational beliefs could be developed with some likelihood of acceptance. Models of irrationality and noise trading can be viewed as an attempt to explain evidence anomalous to the prescriptions of fully rational models. The seminal papers in the volatility debate, Shiller (1981) and LeRoy and Porter (1981), differed in their approach and interpretation of the evidence but were responsible for a vast number of related studies. The residual difference between actual price and warranted price has been interpreted in different ways. Some authors have proposed a rational bubble argument (for discussion see Flood and Garber (1994)), a fads explanation (Shiller (1989)), time varying risk premia (see Cuthbertson (1996)), or learning effects
(Timmermann (1995)). While many economists seem loathe to admit irrational explanations for observed asset market behaviour the evidence against rationality has not diminished since the 1980’s. An example of a recent study favouring irrationality is McQueen, Pinegar and Thorley (1996). The authors find that the cross-autocorrelation puzzle cannot be explained by either traditional theories or time-varying risk premia. Much to their disappointment the authors conclude in favour of the ‘heretic’ theories of bubbles, fads and noise traders.

What is a noise trader?

Cootner (1964) states that securities prices “are typically very sensitive, responsive to all events, both real and imagined”. In the simplest possible definition noise traders are those agents who focus on, and manifest, the imagined determinants of asset prices. There are many possible realizations of this noise. Basic irrationality of agents is one explanation. In economics and finance rationality is synonymous with the rational expectations hypothesis. The constrained optimality and consistency requirements of rational expectations deliver models of great elegance but typically trivialise interaction and heterogeneity of agents. To incorporate heterogeneity requires endowing agents with different information sets, preferences or introducing irrational agents. The latter option, while arguably the most interesting, is still the least preferred.

Black (1986) replaced the derisory ‘i-word’ with noise thereby sanitising irrationality and rendering it more palatable to the narrow minded majority. In Black’s paper such traders may be acting on the basis of pseudo-signals, enjoy trading in itself, believe that noise is in fact relevant information, or trade on the basis of information that has already been reflected in the market price. He further notes that “noise trading is
essential to the existence of liquid markets” and “provides the essential missing ingredient”\(^1\).

Smith (1983) outlines four orientations or distinct types of investor trading on the New York Stock Exchange which are useful for distinguishing between the different varieties of noise trader currently populating the academic literature:

1. Fundamentalist/Economic - which can be summarised by the phrase ‘market values reflect economic values’.
2. Insider/Influence - conceives of market events in interpersonal terms; supply and demand factors resulting from the activities of powerful institutions and individuals.
3. Chartist/Cyclist - involves the belief that ‘the market has a life of its own and that it cannot be explained by reference to outside factors’.
4. Trader/Market action - stresses that understanding the market involves a ‘feel’ for the ‘mood’ of the market and this involves both intellectual and emotional skills.

The first orientation accords with EMH, ‘Following Fama (1970) . . . ’ literature whereby agents endowed with rational expectations discount the future stream of expected dividends and stock prices reflect all available relevant information with respect to a defined information set.

In the market microstructure literature (see for example, Kyle (1985), Admati and Pfleiderer (1988) and Snell and Tonks (1995), and O’Hara (1995)) noise trading is associated with liquidity trading. Noise is incorporated into these models through the demands of the liquidity traders as a convenient method of obfuscating the no-trade

\(^1\) Black furthermore states “If my conclusions are not accepted, I will blame it on noise.”.
scenarios of Tirole (1985) and Milgrom and Stokey (1982). The intuition is that some investors have to liquidate their portfolios for reasons not justified by the underlying fundamentals. These investors provide liquidity and depth within the market but are not behaving rationally with regard to fundamental values. Thus they are not trading for reasons that can be explained in terms of the future payoffs of the financial assets they are trading. Their motives derive from outwith the market itself\(^2\). An example is a financial institution whose trading relates to a clients liquidity needs or simply their own portfolio rebalancing needs. The agent may well be behaving rationally with respect to their liquidity and portfolio requirements but not from the perspective of market fundamentals. In much of the literature the emphasis is on analysing the behaviour of informed traders and/or market makers. The oversimplistic specification of noise/liquidity trader behaviour as simply random means that they would be better described as liquidity traders, as the focus of such research is not on the behaviour of noise traders per se. An exception to this convenient specification of liquidity traders is Admati and Pfleiderer (1988). In their model there are two types of noise/liquidity traders: the first type have no discretion over the timing of their trades; the second type choose the timing of their trades strategically. This latter type engage in the phenomena of rational hiding whereby they attempt to trade when the market is ‘thick’ so that their trading has the smallest impact on market prices and trade tends to cluster within such periods. The view embodied in the microstructure literature most closely accords with the second orientation described above.

\(^2\) Given that agents investment resources are limited it would be rational to liquidate an asset if another investment opportunity was more favourable. Typically theoretical models contain only a single risky asset and thus theories of ‘hot’ assets cannot be analysed.
In the international finance literature noise trading is closely associated with chartist behaviour. Frankel and Froot (1986, 1990) argue forcefully that the use of chartist techniques are a source of changes in the demand for foreign exchange and that large exchange rate movements may take place with little basis in macroeconomic fundamentals. Further evidence of the elusiveness of defining fundamentals is provided in Flood and Rose (1995) who argue that any potentially valid exchange rate fundamental determinant must also experience a dramatic increase in conditional volatility when a previously fixed exchange rate is floated. Empirically they cannot find macroeconomic variables to satisfy this and conclude that the most critical determinants of exchange rate volatility are not macroeconomic. Hallwood and Macdonald (1994) also equate noise trading with chartist behaviour and cite the noise trader paradigm as a possible successor to the efficient markets hypothesis. Thus the international finance literature has concentrated on the third of Smith’s orientations. Allen and Taylor (1989) use expectation survey evidence for the London Forex market to show that chartist techniques are widely used and particularly prevalent at short horizons. They also found that while there was no evidence of bandwagon expectations the most successful chartists showed evidence of adaptive expectations.

In the macroeconomics and finance literature noise trading is synonymous with the work of De Long, Shleifer, Summers and Waldmann (1987, 1989, 1990a, 1990b, 1991). This work accords with Smith’s fourth orientation and the approach is outlined clearly in Shleifer and Summers (1990):

“Our approach rests on two assumptions. First, some investors are not fully rational and their demand for risky assets is affected by their beliefs or sentiments that are not
fully justified by fundamental news. Second, arbitrage - defined as trading by fully rational investors not subject to such sentiment - is risky and therefore limited."

The authors believe that their approach is in many ways superior to the conventional efficient markets paradigm, and cite Keynes (1936, Ch 12) extensively to attest to the long lineage of the noise trader approach. The psychology literature on over confidence in predictions and the correlation of agents mistakes is cited as an additional means of rationalizing noise trader behaviour. Such a specification of trader also accords with the views expressed by some market practitioners:

"Ninety percent of what we do is based on perception. It doesn't matter if that perception is right or wrong or real. It only matters that other people in the market believe it. I may know it's crazy, I may think it's wrong. But I lose my shirt by ignoring it. This business turns on decisions made in seconds. If you wait a minute to reflect on things, you're lost. I can't afford to be five steps ahead of everybody else in the market. That's suicide."4

In common with Black (1986) these authors stress the practical difficulty of distinguishing between rational traders and noise traders. Extending this reasoning it is plausible that there is no strict delineation between noise and rational traders and that there exist noise and rational strategies with agents behaving as both noise trader and rational trader at different times. Perhaps the orientations of Smith would be better thought of as a set of strategies which can be used independently or interactively. An example of this is Frankel and Froot (1986) who present a model of

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3 See the references in DSSW (1990), Shleifer and Summers (1990), and Shiller (1984, 1997).
   -- in Froot, Scharfstein and Stein (1992)
chartists, fundamentalists and fund managers to explain expectation formation and price fluctuations in foreign exchange markets. The fund managers expectations are given by a weighted average of the forecasts from the other two groups, where the weight given to each group is an increasing function of relative past performance.

The main difference between the microstructure treatment of liquidity traders and the macro treatment of noise traders is their effect on rational traders. In the former liquidity traders exert a positive externality on rational traders, while in the latter they exert a negative externality.

**Noise trading literature**

In order to make this review manageable we do not cover 'noise' trading models form the market microstructure literature. Instead we first describe the work of DSSW and the approach outlined by Shleifer and Summers (1990). Following this we survey a selection of results from other work on noise trading.

DSSW published four papers on noise trading, DSSW (1989, 1990a, 1990b, 1991). The most influential of these, DSSW (1990a), is discussed and extended in this and the following chapters. In this section we briefly describe the significance of their other contributions.

DSSW (1989) study the welfare consequences of noise trading using an OLG framework very similar to that used throughout this thesis. Capital is provided by risk-neutral entrepreneurs who are prohibited from trading in the asset market for their capital good. Noise traders make consumption more volatile but their general effect on welfare is ambiguous. If noise traders are bullish on average then they increase the capital stock by reducing the associated cost of capital. Assuming that there are
positive spillovers from having a higher capital stock then noise trader activity may be beneficial for the economy as a whole.

DSSW (1990b) analyses the issue of price stabilisation or destabilisation of asset prices by rational traders. There exists a sizeable literature on this issue (see DSSW (1990b), Admati (199X) and O'Hara (199X) for references). From a somewhat cynical perspective it appears that findings of stabilising or destabilising properties of rational speculators is largely model dependent, and that this paper is no exception. In this model noise traders follow a positive feedback strategy in forming their demands for risky assets, (i.e. buy when prices increase and sell when prices decrease)

Rational traders destabilise prices in the model by manipulating noise traders future behaviour, knowing that noise traders follow a positive feedback strategy:

“Rational speculators jump on the bandwagon when they anticipate positive feedback trading, . . .” DSSW (1990b) p392.

While the argument is plausible it is not without defects. Firstly, the static nature of the model and the stylised decision structure make the robustness of the results highly questionable. Secondly, the specification of all noise traders as positive feedback traders is likely to be misleading. For example, it is equally plausible that noise traders might follow a contrarian or negative feedback strategy. In a model with both types it is not clear that the DSSW results would hold.

DSSW (1991) analyse whether noise traders are likely to survive in an asset market, by looking at the long run distribution of noise trader and rational trader wealth. The analysis is simplified and differentiated from DSSW (1990a) because noise traders do

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5 It is worth noting that Allen and Taylor (1989) found no significant evidence of extrapolative or bandwagon expectations on the part of foreign exchange market participants within their sample of chartists expectations on the London foreign exchange market.


7 In DSSW (1990a) the emphasis was on noise trader and rational trader relative expected returns.
not affect prices. Traders are specified as being infinitely lived with CRRA utility functions. A continuum of noise traders, indexed by i, misperceive the expected return on the corresponding ith asset in the market. This means that individually noise traders hold inefficient portfolios. Noise traders are assumed to hold rational expectations over the remaining risky assets. If the degree of belief is constant over their lifetime and each misperceives the variance of their respective ith stock by the same measure, then aggregating over all noise traders it is shown that as a group noise traders hold an efficient portfolio. The main result of the paper is that if traders coefficients of risk aversion are greater than implied by log-utility (i.e. \( \gamma > 1 \)) then noise traders can have a faster degree of wealth accumulation than rational traders. The 1991 paper thus strengthens the previous argument (DSSW (1990a) that noise traders will not be arbitraged out of the market by risk averse rational traders and may indeed come to dominate. Furthermore, an economy consisting solely of rational traders is not ‘evolutionary stable’ in the sense that it is not robust to the introduction of noise traders.

An important extension to the work of DSSW is provided by Palomino (1996). In a model of imperfect competition it is shown that noise traders can earn higher expected utility than risk averse rational traders. The intuition is that noise traders play non-optimal strategies relative to rational traders, and this may damage the reward to being rational more than to the noise traders themselves. Palomino shows that if noise traders are moderately bullish and their misperception variance is small relative to fundamental dividend variance, then in equilibrium noise traders are present in greater number than rational traders. The results are best interpreted in terms of specific markets which are less competitive: emerging stock markets, markets for small stocks; closed-end discount funds.
Biais and Shadur (1993) also show that noise traders can earn higher utility than their rational counterparts. They consider a pairwise matching model composed of rational and irrational agents. Crucially, they assume that irrational sellers of stock are over-optimistic, while irrational buyers are over-pessimistic. This assumption provides irrational sellers with bargaining power and contrasts with the DSSW (1990a) assumption of constant noise trader bullishness. The long-term convergence of this system is given by an exogenous Darwinian system for updating of the population structure. The authors show that in the long-run both noise traders and rational traders coexist. As noise traders distort the degree of risk sharing, social welfare is reduced by their presence. As mentioned earlier the importance of Biais and Shadur, and Palomino, is in illustrating that the presence of noise traders is robust to updating based on past utility levels and not just returns, as studied in DSSW (1990a).

Of the few empirical applications of noise trading most have attempted to explain the closed end discount fund anomaly from a noise trading perspective. A notable and important exception is Campbell and Kyle (1993) who estimate a model of smart money (rational traders) and noise traders to account for the volatility and predictability of U.S. stock returns. The specification of their underlying theoretical models differs from standard treatments. Rational traders are infinitely lived with CARA utility functions, but noise traders are more akin to the liquidity type traders discussed earlier. In particular, noise traders are assumed to trade randomly and hold an aggregate position which follows an Ornstein-Uhlenbeck process. The population of both types increases at a constant exogenous rate equal to the time rate of dividend payments. This assumption ensures that returns per unit of stock are stationary and facilitates equilibrium. One interesting result is the existence of a positive relation

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8 See Lee, Shleifer and Thaler (1991) and Pontiff(1997) and references therein.
between the interest rate and the effects of noise. If the interest rate is low (<4%) then stock prices movements in the U.S. can be accounted for without recourse to a noise explanation. However, if the interest rate is sufficiently high (>5%) then 'noise is extremely important in moving the stock market'. Unfortunately the data is unable to decide between the low or high interest rate scenario.

9 A continuous time version of a stationary autoregressive Gaussian process.
1.2 Basic model — 2 period horizon

In this section we outline the model of De Long, Shleifer, Summers, and Waldmann (1990a), hereafter simply DSSW, which to our knowledge has only been previously extended by Palomino (1996). The purpose of their model is to analyse the effects of a group of irrational agents, called noise traders, on the behaviour of rational traders and asset prices. The model is embedded in a standard overlapping generations framework with agents living for 2-periods. Agents problem is to form a portfolio in the first period which maximises expected utility of wealth in the second period.

There is no first period consumption, no labour supply decision and no bequests. The resources that agents have to invest are exogenous. There are two assets in the economy. The safe asset, denoted s, pays a fixed real dividend of r, is in perfectly elastic supply, and trades at a permanently fixed price of one. The unsafe asset, denoted u, pays an identical dividend of r but is not in elastic supply. The quantity of u is restricted and fixed and its price in period t is pt.

If the price of each asset was equal to the net present value of its associated dividend stream then the two assets would be perfect substitutes, and would be traded at the identical price of one in all periods.

The population of agents is split into two groups. There are sophisticated or rational investors, denoted i, and noise traders, denoted n. The two groups are present in proportions \((1 - \mu)\) and \(\mu\) respectively. In essence this means that we have a model with two agents, and that trade is between noise traders and their rational counterparts. All agents live for two periods. Sophisticated investors have rational expectations while noise traders have rational expectations perturbed by an i.i.d. random misperception term, denoted \(\rho_i\):
\[ \rho_t \sim N(\rho^*, \sigma^2_{\rho}) \]  

(1.1)

The mean is a measure of noise traders' average "bullishness", and the variance is a measure of their misperception of the expected return per unit of the risky asset. Noise traders thus have an erroneous belief that the distribution of the price of \( u \) next period has mean \( \rho_t \) above its true value. This beguilingly simple augmentation of noise trader expectations allows the model to be solved relatively simply\(^\text{11}\). It is crucial for the main results of the paper that noise traders mean misperception is non-zero. Throughout the analysis it is typically assumed positive to ensure that on average noise traders are bullish about stocks.

All agents construct a portfolio consisting of assets' \( u \) and \( s \) in their first period of life from their exogenous labour income, which is common to both types and denoted \( L \). Each agent seeks to maximise their expectation of utility, where utility is a Constant Absolute Risk Aversion (CARA) function of wealth when old:

\[ u = -e^{-(2\gamma)w} \]  

(1.2)

where \( \gamma \) is the coefficient of absolute risk aversion \( \gamma \geq 0 \). This has the desirable properties of being both tractable and popular (see Le Baron (1995)), but the

\(^{10}\) The use of 'i' for intelligent and 'n' for noise is unfortunate, given that they are normally used for indexing, we follow the original authors in their use of this notation.

\(^{11}\) A more realistic specification would be for heterogeneous noise traders to share a correlated misperception. For their results to hold qualitatively, DSSW's model requires only that the form of noise traders misperceptions is such that they act with some degree of homogeneity. If their expectations are completely unrelated then their effect is idiosyncratic. In this case they only cause non-systematic risk which would not be priced. For noise traders to 'succeed' in this model their demands must be correlated to cause noise trader risk. A further reason for modelling noise trader demands as being loosely correlated is that this would allow noise traders to trade amongst themselves, and not simply with rational traders. This would incorporate the notion of 'churning'. See section 1.4.
downside is that there are no wealth effects, and that at zero wealth the marginal utility of wealth is infinite. Assuming that returns to holding a unit of the risky asset are normally distributed, and thus defined by the first two moments of the distribution, then maximising the expected value of (1.2) is equivalent to maximising

\[ \bar{w} - \gamma \sigma^2_w \]  

(1.3)

where \( \bar{w} \) is expected final wealth and \( \sigma^2_w \) is the one period ahead variance of wealth.

With initial wealth exogenously given as \( L \), each agents portfolio is formed from the budget constraint

\[ L = v_i^n + \lambda_i^n (p_t) \]  

(1.4)

\[ \Rightarrow v_i^n = L - \lambda_i^n p_t \]

where \( v \) denotes holdings of asset s, and \( \lambda \) denotes holdings of asset u. A sophisticated investor chooses the amount \( \lambda_i \) of the risky asset held to maximise

\[ E(U) = \bar{w} - \gamma \sigma^2_w \]  

(1.5)

where,

\[ \bar{w} = E(w) = L(1+r) + \lambda_i [r + p_{i+1} - p_i (1+r)] \]
where we have used (1.4) to substitute out agents demands for the safe asset. The variance of wealth and conditional price variance, respectively, are given by

\[ \sigma^2_w = \lambda^2_i \sigma^2_{p_{i+1}} \]

\[ i \sigma^2_{p_{i+1}} = E_i \left\{ \left( p_{i+1} - E_i(p_{i+1}) \right)^2 \right\} \]

The expected utility of a rational trader born at time t is thus given by,

\[ E(U)^r = L(1 + r) + \lambda^r_i [r + p_{i+1} - p_t (1 + r)] - \gamma \left( \lambda^r_i \right)^2 i \sigma^2_{p_{i+1}} \]

(1.6)

DSSW denote \( L(1+r) \) by \( c_o \). Similarly, noise traders choose the amount \( \lambda^n_i \) of the risky asset held to maximise,

\[ E(U)^n = \bar{w} - \sigma^2_w \]

\[ = L(1 + r) + \lambda^n_i [r + p_{i+1} - p_t (1 + r)] - \gamma \left( \lambda^n_i \right)^2 \sigma^2_{p_{i+1}} + \lambda^n_i (\rho_t) \]

(1.7)

The only difference between (1.6) and (1.7) is the final term on the right hand side of (1.7) which measures the effect of the noise trader misperception of the expected return from holding \( \lambda^n_i \) units of the risky asset on noise traders expected utility.

Differentiating (1.6) and (1.7) with respect to \( \lambda \), setting to zero, and rearranging we obtain expressions for the two agents holdings of asset u:
\[
\lambda_t' = \frac{r^t p_{t+1} - (1+r)p_t}{2\gamma\left(t, \sigma^2_{p_t} \right)} \tag{1.8}
\]
\[
\lambda_t'' = \frac{r^t p_{t+1} - (1+r)p_t + \rho_t}{2\gamma\left(t, \sigma^2_{p_t} \right)} \tag{1.9}
\]

Equilibrium prices are calculated by noting that agents in the second period of life sell all of their asset holdings to the new agents. Such a restrictive market clearing condition means that we can say nothing about transactions volume.

\[
(1 - \mu)\lambda_t' + \mu\lambda_t'' = 1 \tag{1.10}
\]

Substituting the first-order conditions (1.8) and (1.9) into (1.10) yields

\[
(1 - \mu) \left[ \frac{r^t p_{t+1} - (1+r)p_t}{2\gamma\left(t, \sigma^2_{p_t} \right)} \right] + \mu \left[ \frac{r^t p_{t+1} - (1+r)p_t + \rho_t}{2\gamma\left(t, \sigma^2_{p_t} \right)} \right] = 1 \tag{1.11}
\]

Solving (1.11) for \(p_t\) yields the following structural expression for the equilibrium price of asset \(u\),

\[
p_t = \frac{1}{1+r} \left[ r^t p_{t+1} - 2\gamma\left(t, \sigma^2_{p_t} \right) + \mu \rho_t \right] \tag{1.12}
\]

Invoking rational expectations allows us to solve out the endogenous price term on the right hand side of (1.12). To find the rational expectations equilibrium price
distribution we use the method of undetermined coefficients. Conjecture a solution of the form

\[ p_t = \phi_0 + \phi_1 \rho_t \]  

(1.13)

\[ \Rightarrow p_{t+1} = \phi_0 + \phi_1 \rho_{t+1} \]

\[ \Rightarrow p_{t+1} = \phi_0 + \phi_1 \rho^* \]

Inserting this into (1.12) yields

\[ \phi_0 + \phi_1 \rho_t = \frac{1}{1+r} \left[ r + \phi_0 + \phi_1 \rho^* - 2\gamma \sigma^2_{p_{t+1}} + \mu p_t \right] \]

If (1.12) is the correct solution form then the associated coefficient values are given by

\[ \phi_0 = 1 + \frac{\mu \rho^*}{r(1+r)} \quad \frac{2\gamma \sigma^2_{p_{t+1}}}{r} \quad \phi_1 = \frac{\mu}{1+r} \]

Substituting these values back into our conjectured solution yields a reduced form pricing equation,

\[ p_t = 1 + \frac{\mu \rho^*}{r(1+r)} + \frac{\mu p_t}{1+r} - \frac{2\gamma \sigma^2_{p_{t+1}}}{r} \]
With a little manipulation we can reexpress this to include a term with misperception mean deviations

$$ p_t = 1 + \frac{\mu(p_t - p^*)}{1 + r} + \frac{\mu p^*}{r} - \frac{2\gamma}{r} \left( \sigma^2_{\rho_t} \right) $$

(1.14)

From (1.14) the only stochastic element of the pricing function is $\mu \rho_t / (1 + r)$, and thus the conditional one step ahead variance of price is given by

$$ \sigma^2_{P, t+1} = \sigma^2_{P, t+1} = \frac{\mu^2 \sigma^2_{\rho}}{(1 + r)^2} $$

(1.15)

Note from (1.15) that for given risk-free rate, $r$, and misperception distribution variance, $\sigma^2_{\rho}$, the conditional price variance can change only if the proportion of noise traders, $\mu$, is allowed to vary. This is a clear failing of such models, although not central to the original results. Using (1.15) in (1.14) we obtain

$$ p_t = 1 + \frac{\mu(p_t - p^*)}{1 + r} + \frac{\mu p^*}{r} - \frac{(2\gamma) \mu^2 \sigma^2_{\rho}}{r(1 + r)^2} $$

(1.16)

In this final form of the pricing rule, the price of asset $u$ is determined by exogenous parameters of the model and noise traders misperceptions. The final three terms on the right hand side of (1.16) show the effect of noise traders on the price of the risky asset. As mentioned earlier, if there are no noise traders then the safe asset and the

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12 We relax the exogeneity of proportions in Chapter 5.
risky asset are perfect substitutes and (1.16) reduces to \( p_t = 1 \), as we would expect. The second term can be interpreted as a per-period misperception effect. If \( p_t > p^* > 0 \), then the price of the risky asset is bid up because the current generation of noise traders is more optimistic than the average generation. If \( p_t < p^* \) then the second term is negative and the price is bid down by a generation of noise traders who are more pessimistic than the average.

The third term in (1.16) shows the effect of the mean misperception common to all generations of noise traders. It is assumed in the model that the mean misperception is a non-zero constant. If \( \rho^* \) is positive then this ‘price pressure’ effect bids up the price of the risky asset. In this case noise traders bear a larger than normal share of the price risk which they themselves create. As the noise trader share of price risk increases, the sophisticated trader share must decrease. For any investor as their share of risk falls so does the expected return required to make them hold the asset. Thus when \( \rho^* \) is high, sophisticated investors require a lower expected return on their holdings of the risky asset and are willing to pay a higher price for that asset.

DSSW describe the final term in (1.16) as ‘the heart of the model’. They call this term the create space effect of noise traders. It is this create space effect which deters rational arbitrage and causes the persistent deviations of the actual price of the risky asset from its fundamental value of one:

“At the margin, the returns from enlarging one’s position in an asset that everyone agrees is mispriced (but different types think is mispriced in different directions) is offset by the additional price risk that must be run. Noise traders thus ‘create their own space’: the uncertainty over what next periods noise traders will
believe makes the otherwise riskless asset u risky and drives its price down and its return up."

DSSW, p712.

In the next chapter it is shown that as we extend agents horizons the main effect on equilibrium prices is through associated changes in the create space effect.
Trader Returns

The standard argument against the existence of noise or irrational traders is that an economic form of natural selection favours rational agents, and that irrational agents will be driven out by the actions of these rational arbitrageurs. Friedman (1953) argued that irrational agents who have some effect on equilibrium prices must earn lower returns than their rational counterparts and will thus become extinct. A more recent argument along these lines is provided by Dybvig and Ross (1987):

"One appeal of results based on the absence of arbitrage is the intuition that absence of arbitrage is more primitive than equilibrium, since only relatively few rational agents are needed to bid away arbitrage opportunities, even in the presence of a sea of agents driven by 'animal spirits'."

In the model presented and extended in this thesis, and in the adaptive model presented in chapter's four and five, this is not always the case. It can be shown that there exist certain circumstances under which noise traders have higher expected returns than rational traders. Furthermore, if proportions are a function of past return performance then strategic complementarity in the MW of noise traders means that effective arbitrage requires more than a 'few' rational agents. In the adaptive model even allowing for the traders proportions to evolve according to their past returns there are circumstances

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1 A simple gains from trade argument suggests that consumers will always be better off when they are allowed to trade with speculators than when they do not have this option. Given this, in a less partial analysis, it may not be optimal for rational traders to get rid of their noisy counterparts. The only source of risk in this model is noise trader induced, and rational traders benefit from its existence as they would have no trade possibilities.

2 This attack on Friedman by DSSW is perhaps unfair as it is doubtful that the environment which they describe is the one intended by Friedman 37 years earlier. I am grateful to Thomas Sargent for pointing this out. However, by using a model structure as close as possible to a full rational expectations model the power of any criticisms is increased.
where the noise trader share is non-zero. Thus the standard argument does not always hold.

Noise traders who are optimistic bear more risk than their rational counterparts. In simple terms this bearing of a higher risk portfolio leads to a higher expected return (assuming the existence of a positive risk-return trade off). Given that the rate of return on the safe asset is fixed at r, any difference in the portfolio returns of noise traders and rational traders must result from differences in their holdings of the risky asset.

The excess return of asset \( u \) is,

\[
\Delta R_{u,s} = [r + p_{t+1} - p_t(1+r)]
\]  

(1.17)

If an agents expectation, conditional on information at time \( t \), of (1.17) is negative then they will take a short position (i.e. if \([r + p_{t+1}] < p_t(1+r)\) then agents sell in the first period of life and buy in the second period ), subject to the market clearing condition being satisfied. A problem arises here because \( \mu \) and \((1-\mu)\) are both non-negative by definition. Given (1.10) this implies that either \( \lambda_i^r \) or \( \lambda_o^r \) can be negative but not both in the same period. Furthermore conditioned on the proportions of traders, a negative holding must be exactly offset by a positive holding such that (1.10) is strictly satisfied.

The difference in returns to the two types of agents is,

\[
\Delta R_{n-i} = (\lambda_o^r - \lambda_i^r)\left[r + p_{t+1} - p_t(1+r)\right]
\]  

(1.18)

From (1.11) the difference between the two types' demands for the risky asset is,

\footnote{As noted in the introduction to this chapter it is also possible (in different setups) for noise traders to}
\[ \lambda_t^* - \lambda_t^i = \frac{\rho_t}{2\gamma \sigma_{p_{t+1}}^2} = \frac{(1+r)^2 \rho_t}{(2\gamma) \mu^2 \sigma_p^2} \]  

(1.19)

From (1.12) the expected value of this excess return on asset u, at time t, is

\[ \int [r + p_{t+1} - p_t(1+r)] = 2\gamma \left( \sigma_{p_{t+1}}^2 \right) - \mu \rho_t = \frac{(2\gamma) \mu^2 \sigma_p^2}{(1+r)^2} - \mu \rho_t \]  

(1.20)

The expected excess return on the risky asset is positive if

\[ \Rightarrow \frac{2\gamma \mu^2 \sigma_p^2}{(1+r)^2} - \mu \rho_t > 0 \quad \text{i.e. if} \quad \rho_t < \frac{2\gamma \mu \sigma_p^2}{(1+r)^2} \]

With some manipulation we derive,

\[ \int (\Delta R_{n-1}) = \rho_t - \frac{(1+r)^2 \rho_t^2}{(2\gamma) \mu \sigma_p^2} \]  

(1.21)

If \( \rho_t \) is positive then (1.19) is positive, while if the risky asset has a current price below fundamental price then (1.20) is also positive i.e \( r + p_{t+1} > p_t(1+r) \). Therefore for noise traders to have a positive excess total return they must be optimistic and the price of u must be below the fundamental value. Rewriting (1.21) as an inequality restriction on \( \rho_t \) makes the analysis easier for future comparison,

receive higher utility than rational traders (see for example Palomino (1996)).
\[(\Delta R_{n-1}) > 0 \text{ if } \rho_t \geq \frac{(1+r)^2 \rho_t^2}{(2\gamma)\mu \sigma_r^2} \Rightarrow 0 < \rho_t < \frac{(2\gamma)\mu \sigma_r^2}{(1+r)^2}\]

This gives us bounds on the value of the noise trader misperception term which gives positive values of (1.21). We define this bounded area as the 'misperception window' of noise traders, MW hereafter. Sethi and Franke (1995) use the concept of an 'expectational corridor' which defines the region in the state space where non-optimisers can outperform the group of optimisers. They argue that this is a useful device in helping to understand the circumstances that may prove favourable for one group or the other.

The unconditional expectation of (1.21) is,

\[
E(\Delta R_{n-1}) = \rho^* - \frac{(1+r)^2(\rho^*)^2 + (1+r)^2 \sigma_r^2}{(2\gamma)\mu \sigma_r^2}
\]

This shows that for noise traders to have higher expected returns they must be optimistic about future stock returns, $\rho^* > 0$. If they are pessimistic then they demand less of the risky asset relative to rational traders, bear less risk and receive a commensurately lower expected return. DSSW interpret the first term in the numerator of (1.22) as a 'price pressure' effect. As noise traders become more optimistic they bid up the price of the risky asset making it closer to fundamental value and reducing their relative excess expected return. The second term is described as a 'Friedman effect'. The random nature of noise traders misperceptions means that they cannot time their
trades and instead purchase the most of the risky asset when other noise traders are doing likewise and thus are most likely to suffer a capital loss\textsuperscript{4}.

As we would expect the expected excess relative return of noise traders is increasing in the common risk aversion coefficient and in the proportion of noise traders. Both of these allow noise traders more room by further deterring rational traders from effective arbitrage.

\textsuperscript{4} Rewriting (1.22) as

\[
E(\Delta R_{n,t}) = \rho^* - \frac{(1+r)^2(\rho^*)^2}{2\gamma\mu\sigma^2_\rho} - \frac{(1+r)^2}{2\gamma\mu}
\]

we note that the 'Friedman effect' disappears?
1.3 Heterogeneous misperceptions

We noted in section 1.2 that the homogeneity of noise traders misperceptions was a weakness in the analysis. This assumption simplifies the model immensely as in effect there are only two traders whose net demands must clear the market. We now relax this assumption and attempt to analyse a ‘more’ realistic environment.

If we are to view noise traders as non-rational speculators it is puzzling that there are no trades between noise traders allowed. With homogeneity within type there are a limited number of questions we cannot ask. Can heterogeneous noise traders deter rational arbitrage more than homogeneous? Do heterogeneous noise traders damage other noise traders returns? This modelling anomaly also afflicts the microstructure literature: market makers are assumed to collude; liquidity traders have identical liquidity needs in each period.

In this section we model noise traders misperceptions within each cohort as having both a homogeneous component, denoted $\rho_j$, and a heterogeneous component denoted $\psi_j$ which is unique to the jth noise trader. We assume that there are is a continuum of noise traders indexed by j, (j=1, . . . , J), located on a compact set $\Omega$. Let $\varphi$ denote a measure over $\Omega$ with $\varphi(\Omega)=1$. We take $\Omega$ to be the interval $|\mu| = \mu - 0$, and identical rational traders to be located over the interval $|1-\mu|$, where $\mu \in [0,1]$. Thus all agents are located on the unit interval and in aggregation we can draw comparison with the proportions in the previous sections. The number of noise

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5 I am grateful to Robert Waldman, Andrei Shleifer and an anonymous referee for encouraging the analysis of this section.

6 Following Judd (1985) we note that for a continuum of random variables an appropriate measure may not exist, $\Omega$ may be non-measurable, and a law of large numbers may be unintelligible. However, Judd shows that there exist measures such that almost all paths are measurable and that a law of large numbers holds. We assume that $\varphi$ is such a measure. See also Feldman and Gilles (1985).
traders holding the jth misperception is assumed to be constant over time. All noise traders have rational expectations perturbed by a composite misperception term which we denote \( \varepsilon^j_t \),

\[
\varepsilon^j_t = \rho_t + \psi^j_t
\]  

(1.23)

The following assumptions are made

A.1 \( \rho_t \sim N(\rho^*, \sigma^2_\rho) \)

A.2 \( \psi_j \sim N(0, \sigma^2_\psi) \), \( \psi_t = (\psi^1_t, ..., \psi^J_t) \)

A.3 \( E(\rho_t, \psi_j) = E(\rho_t)E(\psi_j) \quad \forall j \)

A.4 \( E(\psi_1^j, ..., \psi^J_t) = E(\psi_1^j) ... E(\psi^J_t) \)

Given (1.23) and A.1 - A.4 the expected value and variance of the misperception term are given by

\[
E(\varepsilon^j_t) = E(\rho_t) + E(\psi^j_t) = \rho^*
\]  

(1.23a)

\[
\text{var}(\varepsilon^j_t) = \text{var}(\rho_t) + \text{var}(\psi^j_t) + 2\text{cov}(\rho_t, \psi^j_t)
\]

\[
= \sigma^2_\rho + \sigma^2_\psi
\]  

(1.23b)

Noise traders thus have a common belief in \( \rho_t \), as before, but each individual noise trader is differentiated by an individual belief, \( \psi^j_t \). We assume that the common misperception, \( \rho_t \), is a normally distributed random variable with non-zero mean, that each individual misperception, \( \psi^j_t \), is drawn from a multivariate normal distribution with zero mean and constant variance, and that \( \rho_t \) and \( \psi^j_t \) are independent for all \( j \).

We also assume that the individual misperceptions are independent of each other, i.e.

\[ \text{var}(\varepsilon^j_t) \]

7 The interval \( |\mu| \) corresponds to the proportion of noise traders in section 1.2. We assume that the number of noise traders is uniformly distributed over the interval for ease of aggregation.
there is no homogeneity of belief at the individual level, other than the shared belief in $\rho_t$.

This formulation of noise traders beliefs makes the distinction between noise traders, chartists and liquidity traders more tenuous. In section 1.1 we argued that chartists were heterogeneous in their beliefs and thus would not behave as the noise traders in DSSW, and that liquidity traders were typically just a lubricant in market microstructure models. From the orientations outlined in section 1.1 the composite misperception (1.23) would encompass both noise traders and chartists. We also believe the inclusion of an idiosyncratic component is one way of modelling churning by speculators and captures the notion that noise traders would take positions against each other as well as against rational traders\(^8\).

As before each rational trader chooses their first period demands to maximise the expected utility of terminal wealth given by,

$$E(U)^I = L(1 + r) + \lambda'_i (r + p_{t+1} - p_t (1 + r)) - \gamma \left( \lambda'_i \right)^2, \sigma^2_{p_t}$$

(1.24)

Similarly, the jth noise trader chooses his demand to maximise the expected utility of terminal wealth given by,

$$E(U)^\nu = L(1 + r) + \lambda''_j (r + p_{t+1} - p_t (1 + r) + \epsilon^t_j) - \gamma \left( \lambda''_j \right)^2, \sigma^2_{p_t}$$

(1.25)

We continue to assume that all agents have identical degrees of risk aversion. Agents respective demands are thus given by,

$$\lambda'_i = \frac{r + p_{t+1} - p_t (1 + r)}{2 \gamma \sigma^2_{p_t}}$$

(1.26)

for rational traders, and

\(^8\) Allen and Gorton (1993) present a model in which fund managers deviate from rational trading by churning their clients portfolios in the hope of a speculative profit.
\begin{equation}
\lambda_i^n = \frac{r+s_p^{t+1} - p_t(1+r) + \epsilon_i}{2\gamma\sigma_{\rho_t}^2} = \frac{r+s_p^{t+1} - p_t(1+r) + \rho_t + \psi_i}{2\gamma\sigma_{\rho_t}^2}
\end{equation}

(1.27)

for the jth noise trader. To find aggregate noise trader demands we integrate (1.27) over j

\begin{align*}
\lambda_i^n = & \int_{\Omega} \lambda_i^n \varphi(dj) \\
= & \frac{r+s_p^{t+1} - p_t(1+r) + \rho_t + \int_{\Omega} \psi_i \varphi(dj)}{2\gamma\rho_t^2} \\
= & \frac{r+s_p^{t+1} - p_t(1+r) + \rho_t}{2\gamma\rho_t^2}
\end{align*}

(1.28)

Given our assumptions on the distribution law for \( \psi_t \), aggregate noise trader demand is structurally identical to the homogeneous case: aggregation washes out the heterogeneity of misperceptions.

The market clearing condition becomes

\begin{equation}
(1 - \mu)\lambda_i^n + \int_{\Omega} \lambda_i^n \varphi(dj) = (1 - \mu)\lambda_i^n + \mu\lambda_i^n = 1
\end{equation}

(1.29)

In forming (1.29) we exploit the consistency of the rational traders. Inserting (1.26) and (1.28) into (1.29) yields a structural pricing equation,

\begin{align*}
p_t = & \frac{1}{1+r} \left[ r+s_p^{t+1} + \int_{\Omega} \epsilon_i \varphi(dj) - 2\gamma\sigma_{\rho_t}^2 \right] \\
= & \frac{1}{1+r} \left[ r+s_p^{t+1} + \mu\rho_t - 2\gamma\sigma_{\rho_t}^2 \right]
\end{align*}

(1.30)

Given our assumptions on the distribution of \( \psi_t \), this pricing equation is structurally identical to (1.12).

The rational expectations equilibrium is given by

\begin{equation}
p_t = 1 + \frac{\mu(\rho_t - \rho^*)}{1+r} + \frac{\mu\rho^*}{r} - \frac{2\gamma\mu^2(\sigma_{\rho}^2 + \sigma_{\psi}^2)}{r(1+r)^2}
\end{equation}

(1.31)
In (1.31) we see that the create space effect is increased relative to the homogeneous belief equilibrium (1.16). The ability of noise traders to deter rational arbitrage is 'greater' by the amount $2\gamma\mu^2\sigma^2_v / r(1+r)^2$, and the expected price of the risky asset is decreased by the same amount.

The conditional expected excess relative return of the jth noise trader is

$$t(\Delta R_{n_{-1}}) = \rho_t + \psi_t' - \frac{(1+r)^2 \mu \rho_t^2}{2\gamma \mu^2 \left(\sigma^2_\rho + \sigma^2_v\right)}$$

(1.32)

On average the conditional expected excess return of a cohort of noise traders will increase for $\rho_t > 0$. The addition of the heterogeneous variance in the denominator of the second term on the r.h.s. of (1.32) increases the conditional expected excess return of noise traders compared to (1.21). For noise traders as a whole there is a corresponding increase in the associated MW. The conditional expected excess return over all noise traders is

$$t(\Delta R_{n_{-1}}) = \rho_t - \frac{(1+r)^2 \mu \rho_t^2}{2\gamma \mu^2 \left(\sigma^2_\rho + \sigma^2_v\right)}$$

(1.33)

and the corresponding MW is

$$0 < \rho_t < \frac{2\gamma \mu \left(\sigma^2_\rho + \sigma^2_v\right)}{(1+r)^2}$$

(1.34)

From (41) the MW of the jth noise trader is

$$0 < \rho_t < \frac{2\gamma \mu \left(\sigma^2_\rho + \sigma^2_v\right)}{(1+r)^2} \left[1 + \frac{\psi_t'}{\rho_t}\right]$$

(1.35)

for $\rho_t > 0$ the jth noise traders MW is greater (less) than the average MW (1.34) for $\psi_t' > 0 (<0)$. As we would expect noise traders with a positive heterogeneous misperception have a higher MW than their homogeneous and negative heterogeneous
misperception counterparts. Because noise traders who are more optimistic than the average \( \psi'_i \) take on more risk than noise traders who are more pessimistic than the average \( \psi''_i \), such optimistic noise traders perform relatively worse when the common per-period sentiment is negative \( \rho_t < 0 \),

\[
\left( \Delta R_{\psi'_i - \psi''_i} \right) = \psi'_i \left( 1 + \frac{(1+r)^2 \rho_t}{2\gamma \mu (\sigma^2 + \sigma^2_w)} \right) - \psi''_i \left( 1 + \frac{(1+r)^2 \rho_t}{2\gamma \mu (\sigma^2 + \sigma^2_w)} \right)
\]  

(1.36)

Note that when \( \rho_t > 0 \) (1.36) is positive, but for \( \rho_t < 0 \) (1.36) is negative. Noise traders who are less optimistic than average do better than optimistic noise traders, when the group as a whole is pessimistic. The relative benefit of being more (less) optimistic is obviously increasing (decreasing) in \( \rho^* \).

1.4 Conclusion

In this chapter we motivated the study of irrationality in financial markets in the light of the recent theoretical and empirical finance literature. A working definition and clarification of noise traders within related literatures was also provided. We defined noise traders as having a systematic bias which was correlated across all such traders. On this basis we argued that chartists and in particular liquidity traders do not act as noise traders.

The model of DSSW (1990a) was presented in detail and their basic analysis is extended in subsequent chapters. Two simple extensions to their model were considered. Firstly, we modelled the proportion of rational traders as being stochastic to show that the increase in uncertainty benefited noise traders in terms of relative expected returns. Secondly, we extended the noise trader misperception to include a
common belief and an individual belief to capture trade between noise traders. This again caused an increase in uncertainty. As a result, the ability of noise traders to deter rational arbitrage, the create space effect, was increased. The range in state space over which noise traders expected returns outperformed those of rational traders was also increased. Interestingly we found that depending on the sign of the common misperception term noise traders would wish to hold beliefs at either extreme of the distribution of heterogeneous beliefs.

Increases in uncertainty even if not driven by noise traders thus make noise trader effects more robust in equilibrium. Put differently, as the limits to arbitrage become greater noise traders have a larger effect on market prices because their demands tend on average to be less affected by both noise induced and fundamental risk, (see also Shleifer and Vishny (1997)).

In future work it would be interesting to allow for transactions volume to see if noise trading can explain correlations and the temporal behaviour of volume and volatility, (see Campbell, Grossman and Wang (1993), and Brock and Le Baron (1995)). The analysis in section 1.3 could be extended to loosely correlated beliefs which are an endogenous function of past belief performance in generating returns⁹.

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⁹ Useful suggestions for such a strategy are contained in Brock (1993, 1996) and Brock and Hommes (1995).
2. Noise and Horizon Effects

2.1 Introduction

In this chapter we extend the analysis to a multi-period setting, to examine if the effects of noise traders are horizon specific. Our initial motivation is to find out how robust the results and conjectures of DSSW are to changes in the horizon of agents. In common with virtually all analysis of overlapping generations models DSSW restrict themselves to the two period case. Although DSSW provide some discussion of the likely effects of extending agents trading horizon they do not explicitly model this. Here we are concerned with the effects of short-horizon investment versus long-horizon investment in an environment composed of rational and irrational traders.

Le Baron writes,

"A final question that few, if any, of the models have addressed is the speed of learning, and the horizon that agents look at."


Froot, Scharfstein, and Stein pose the following question,

"How do speculators trading horizons affect the nature of asset prices?"


One purpose of the present chapter is to contribute to this literature by analysing trader demands and asset price behaviour at different horizons. The conventional response to trader horizon is that it is irrelevant. By a simple application of backward induction a sequence of short horizon trades results in the same price expectation as warranted by the long run fundamentals. Since both short run traders and long run investors are both assumed to be conditioning expectations on fundamentals the
specific horizon of the agents determining prices is deemed irrelevant. A sequence of short run traders serve the same purpose as a single long run trader.

DSSW (1987) argue that the horizon of agents does matter in the context of their model:

"If agents live for more than two periods the equilibrium will be closer to the "fundamental" equilibrium than if agents live for two periods. ... Having a longer horizon allows one to engage in self-insurance by taking advantage of the fact that the two period-ahead variance is no greater than the one period-ahead price variance. For the longer the holding period, the smaller the excess rate of return necessary to compensate for a given amount of price risk and the greater the chance to earn additional profits from market timing. Changing the maximum 'horizon' of agents in the model has real effects on the behaviour of equilibrium prices because returns compound from period to period while price risk does not." p29.

In contrast to DSSW we find that the equilibrium need not be closer to the 'fundamental' equilibrium, and that price risk is compounded from period to period. We also explore the effects of different types of intergenerational overlap and the effects of i.i.d. and AR(1) misperceptions.

While there has been comparatively little work done on economies with intermediate horizons the results obtained suggest that research in this area is very worthwhile. Huffman (1987) presents a dynamic general equilibrium model in which agents lifetimes are arbitrary but finite. This economy exhibits 'some interesting features when agents live for more than two periods'. Huffman notes that two period OLG

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1 It is interesting to note that short horizon agents are typically termed traders or speculators while long horizon agents are awarded the term investor.
models (Huberman (1984)), and infinite horizon representative agent models (Lucas (1978)), can yield the same descriptions of asset price behaviour. However, his model with intermediate planning horizons of length \( j, 2 < j < \infty \), yields different and more complex patterns of asset price behaviour. Leach (1991) studies an OLG economy in which agents trade a single asset. The model illustrates that price bubbles are admissible in a stationary, rational expectations equilibrium. Leach finds that in a two period economy prices follow an i.i.d. process, but that in a multi-period model prices follow a Markov process. It is argued that multi-period OLG models give agents some degree of choice over their actions and that the standard two period assumption 'is, in this context, not a simplifying assumption but a critical restriction'.

Our own initial motivation for studying agents horizons is the belief that agents with heterogeneous expectations would have endogenous and heterogeneous horizons. Financial market participants have varied planning horizons, ranging from intra-day for spot traders to one year and beyond for certain classes of fund manager (see Crossland and Moizer (1995)). Lakonishok, Shleifer and Vishny (1993) write:

"Another important factor is that most investors have shorter time horizons than are required for value strategies to consistently pay off. Many individuals look for stocks that will earn them high abnormal returns within a few months, rather than 4 percent over the next five years. Institutional money mangers often have even shorter time horizons." p29.

One would also expect institutional and individual investors to exhibit different investment horizons\(^2\). While each types 'normal' investment horizon is determined by

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\(^2\) For example. institutional investors will be privy to more short-run and market induced signals.
such factors as role in the market, frequency of appraisal, need for liquidity, and speed of information acquisition, uncertainty over current and future market conditions can act to reduce or increase agents horizons. We conjecture that during 'turbulent' periods the distribution of horizons collapses, such that a disproportionate number of investors take short term positions thereby propagating and accentuating fluctuations in asset prices and returns.

Although this thesis fails to develop such a theory of temporal heterogeneity we believe that this is an important omission from current approaches to endogenous fluctuations and economic behaviour.

A recent contribution in this direction is Holden and Subrahmanyam (1996). They analyse a model, based on Kyle (1985), in which informed traders choose endogenously whether to specialise in short-term or long-term information. It is found that high variances of liquidity trading can result in more informed traders choosing a short term strategy. This highlights the role of uncertainty in influencing agents choice of horizon. Shleifer and Vishny (1990) note that arbitrage is cheaper for short term assets than for long term assets, as the former can only be mispriced for a shorter period. In a perfect capital market there is no difference in the relative cost of arbitraging long term assets as opposed to short term assets.

The chapter is organised as follows. Section 2 extends the horizon to 3, 4 and n-period settings. By allowing generations to overlap only at the beginning and end of the horizon we find that the price series is non-stationary. Section 3 recasts this analysis by allowing agents to be born in each period of the horizon and a stationary price series results. Section 4 looks at the effects of noise traders having AR(1) as opposed to iid misperceptions. Section 5 concludes and offers our interpretation of the results.
2.2 Non-stationary prices

In this section we present multi-period models in which generations only overlap in birth/death periods. There is no overlap in retrade periods and we show that this causes the price series to be non-stationary. One interpretation of this is that outside of birth/death periods no new agents enter the market and trade is between the original cohorts. While this is a nonstandard analysis for OLG models there is no compelling reason to assume that generations overlap in all periods, and new agents are constantly entering the market. We begin with a description of three and four period models and then analyse the behaviour in a \( j \)-period model.

A three period lived rational trader chooses \( \lambda_t, \lambda_{t+1} \) to maximise their expected utility of terminal wealth given by,

\[
E(U) = L(1+r)^2 + (1+r)\lambda_t(r+P_{t+1} - P_t(1+r)) + \\
\lambda_{t+1}(r+P_{t+2} - P_{t+1}(1+r)) - \gamma(\sigma_w)^2
\]  

Figure 1. Horizon structure for stationary and non-stationary OLG economies.
Similarly, a 3-period lived noise trader chooses \( \lambda_{t_1}, \lambda_{t+1} \) to maximise their expected utility of terminal wealth given by,

\[
E(U)^n = L(1+r)^2 + (1+r)\lambda_{t_1}^n (r+p_{t+1} - p_t(1+r) + \rho_t) + \lambda_{t+1}^n (r+p_{t+2} - p_{t+1}(1+r) + \rho_{t+1})
- \gamma(\sigma_w^2)^n 
\]  
\[
(2.2)
\]

To write (2.1) and (2.2) in a more compact form define

\[
R_{t+1} = r + p_{t+1} - p_t(1+r)
\]

\[
\Omega_{t+2}^i = \lambda_{t_1}^i (R_{t+2})
\]

\[
\Omega_{t+2}^n = \lambda_{t_1}^n (R_{t+2} + \rho_{t+1})
\]

Then the expected utility of rational traders and noise traders can be written as

\[
E(U)^i = L(1+r)^2 + (1+r)\lambda_{t_1}^i (r, R_{t+1}) + \Omega_{t+2}^i - \gamma(\sigma_w^2)^i 
\]  
\[
(2.3)
\]

\[
E(U)^n = L(1+r)^2 + (1+r)\lambda_{t_1}^n (r, R_{t+1} + \rho_t) + \Omega_{t+2}^n - \gamma(\sigma_w^2)^n 
\]  
\[
(2.4)
\]

The variance of terminal wealth for three period lived agents is given by,

\[
(\sigma_w^2)^i = (1+r)^2 (\lambda_t^i)^2 \text{var}(R_{t+1}) + \text{var}(\Omega_{t+2}^i) + 2(1+r)\lambda_t^i \text{cov}(R_{t+1}, \Omega_{t+2}^i) 
\]  
\[
(2.5)
\]

for rational traders, and

\[
(\sigma_w^2)^n = (1+r)^2 (\lambda_t^n)^2 \text{var}(R_{t+1}) + \text{var}(\Omega_{t+2}^n) + 2(1+r)\lambda_t^n \text{cov}(R_{t+1}, \Omega_{t+2}^n) 
\]  
\[
(2.6)
\]
for noise traders.

The solution procedure is to work back from the retrade period so that agents retrade demands can be used to solve for initial period demands. These retrade demands are structurally identical to agents demands in a 2-period model given by (1.8) and (1.9).

To find agents retrade period demands differentiate (2.3) and (2.4) for $\lambda_{t+1}^l$ and $\lambda_{t+1}^n$ respectively, to yield

\[ \lambda_{t+1}^l = \frac{R_{t+2}}{2\gamma_{t+1}\sigma_{P,2}^2} \]  

(2.7)

for rational trader retrade demands, and

\[ \lambda_{t+1}^n = \frac{R_{t+2} + P_{t+1}}{2\gamma_{t+1}\sigma_{P,2}^2} \]  

(2.8)

for noise traders retrade demands. An anterior subscript again denotes the dating of the information set on which expectations are conditioned. As would be expected (2.7) and (2.8) are identical to (1.8) and (1.9) lead by one period. The demands in DSSW thus obtain as penultimate period demands as we extend agents trading horizon.

As in the previous 2-period analysis of chapter 1 we continue to impose market clearing in each period. In the retrade period market clearing is given by,

\[ (1 - \mu)\lambda_{t+1}^l + \mu\lambda_{t+1}^n = 1 \]  

(2.9)
With the horizon structure of this section, agents in a retrade period rebalance portfolios which were formed in the birth/death period from the sales of the old agents. Here there are no old agents in a retrade period. Therefore, (2.9) should not be interpreted as the old selling all of their holdings to the new young. Instead existing agents rebalance their portfolios between each other by recalculating their relative holdings to give new demand functions. Inserting (2.7) and (2.8) into (2.9) and solving for $p_{t+1}$ yields

$$ \begin{align*}
(1 - \mu) \left[ \frac{t+1 R_{t+2}}{2 \gamma_{t+1} \sigma_{p_{t+2}}} \right] + \mu \left[ \frac{t+1 R_{t+2} + \rho_{t+1}}{2 \gamma_{t+1} \sigma_{p_{t+2}}} \right] &= 1 \\

p_{t+1} &= \frac{1}{1+r} \left( r + \rho_{t+1} p_{t+2} - 2 \gamma_{t+1} \sigma_{p_{t+2}}^2 + \mu \rho_{t+1} \right) 
\end{align*} $$

(2.10)

From (2.10) we see that the conditional one step ahead variance of the price of asset $u$ is identical to that of the 2-period case, i.e.

$$ t+1 \sigma_{p_{t+2}}^2 = t \sigma_{p_{t+1}}^2 = \frac{\mu^2 \sigma_p^2}{(1+r)^2} $$

Using (2.10) we can rewrite the expected excess return on asset $u$ over asset $s$ as,

$$ t+1 R_{t+2} = r_{t+1} p_{t+2} - p_{t+1} (1+r) = 2 \gamma_{t+1} \sigma_{p_{t+2}}^2 - \mu \rho_{t+1} $$

(33)
Inserting (2.11) into (2.7) and (2.8) we obtain reduced form expressions for the retrade demands of rational traders and noise traders respectively,

\[ \lambda^r_{t+1} = \frac{2\gamma_{t+1} \sigma^2}{2\gamma_{t+1} \sigma^2_{Pr2}} - \mu \rho_{t+1} = 1 - \frac{(1+r)^{2}}{2\gamma \mu^2 \sigma^2_{\rho}} \]  

(2.12)

\[ \lambda^n_{t+1} = \frac{2\gamma_{t+1} \sigma^2}{2\gamma_{t+1} \sigma^2_{Pr2}} + (1-\mu) \rho_{t+1} = 1 + \frac{(1+r)^{2}(1-\mu)}{2\gamma \mu^2 \sigma^2_{\rho}} \]  

(2.13)

Note that both types demands are independent of prices and that noise traders demand more (less) of the risky asset than rational traders if the per-period misperception is positive (negative). As in Chapter 1 both types demands are a function of the risk-free rate, the proportion of noise traders, their per-period misperception and its variance, and the coefficient of absolute risk aversion.

Having solved for retrade demands (2.12-13) and the retrade structural pricing equation we now repeat the analysis for the birth/death period. Differentiating (2.3) and (2.4) with respect to \( \lambda^r_t, \lambda^n_t \) yields

\[ \lambda^r_t = \frac{R_{t+1}}{2\gamma(1+r) \var{R_{t+1}}} - \frac{\cov_t(R_{t+1}, \Omega^r_{t+2})}{(1+r) \var{R_{t+1}}} \]  

(2.14)

for rational traders, and
\[ \lambda_t^r = \frac{\mu R_{t+1} + \rho_t}{2\gamma(1+r)\text{var}_t(R_{t+1})} - \frac{\text{cov}_t(R_{t+1}, \Omega_{t+2}^r)}{(1+r)\text{var}_t(R_{t+1})} \]  

(2.15)

for noise traders. Because of the birth death process the old will sell all of their holdings to the new young. The market clearing condition is given by

\[ (1 - \mu)\lambda_t^r + \mu\lambda_t^n = 1 \]

Inserting (2.14) and (2.18) into the market clearing condition above, yields

\[ (1 - \mu) \left[ \frac{\mu R_{t+1}}{2\gamma(1+r)\text{var}_t(R_{t+1})} - \frac{\text{cov}_t(R_{t+1}, \Omega_{t+2}^r)}{(1+r)\text{var}_t(R_{t+1})} \right] + \]

\[ \mu \left[ \frac{\mu R_{t+1} + \rho_t}{2\gamma(1+r)\text{var}_t(R_{t+1})} - \frac{\text{cov}_t(R_{t+1}, \Omega_{t+2}^r)}{(1+r)\text{var}_t(R_{t+1})} \right] = 1 \]

\[ \downarrow \]

\[ \frac{\mu R_{t+1} + \rho_t}{2\gamma(1+r)\text{var}_t(R_{t+1})} - \left[ \frac{(1 - \mu)\text{cov}_t(R_{t+1}, \Omega_{t+2}^r) + \mu\text{cov}_t(R_{t+1}, \Omega_{t+2}^n)}{(1+r)\text{var}_t(R_{t+1})} \right] = 1 \]

To simplify this further observe that although \( \Omega_{t+2}^r \neq \Omega_{t+2}^n \), both rational and noise traders know that the market will clear in the retrade period, i.e.

\[ (1 - \mu)\lambda_{t+1}^r + \mu\lambda_{t+1}^n = 1 \]

\[ \downarrow \]

\[ (1 - \mu)\Omega_{t+2}^r + \mu\Omega_{t+2}^n = R_{t+2} \]
\[
\frac{R_{t+1} + \mu_{t+1}}{2\gamma(1+r)\text{var}_t(R_{t+1})} - \frac{\text{cov}_t(R_{t+1}, R_{t+2})}{(1+r)\text{var}_t(R_{t+1})} = 1
\] (2.16)

Solving (2.16) for \( p_t \), we obtain the structural pricing equation for asset \( u \) in the retrade period,

\[
p_t = \frac{1}{1+r} \left( r + p_{t+1} + \mu_{t+1} - 2\gamma(1+r)\text{var}_t(R_{t+1}) - 2\gamma \text{cov}_t(R_{t+1}, R_{t+2}) \right)
\] (2.17)
Inspection of (2.17) and (2.10) shows that the conditional price risk is compounded in the initial period. The covariance of the expected excess returns of the unsafe asset over the safe asset in the first and retrade period will only affect agents demands in the first period.

We now derive the covariance term in (2.17). Recall that both rational traders and noise traders are endowed with rational expectations, i.e.

\[ p_{t+1} - p_{t+1} = \zeta_{t+1} \]

where

\[ E[\zeta_{t+1}] = 0 \]

\[ E[\zeta_t \zeta_s] = \sigma^2 \zeta \] if \( t = s \)

\[ E[\zeta_t \zeta_s] = 0 \] \( \forall t \neq s \)

The random variable \( \zeta_t \) is therefore a standard rational expectations “white-noise” error term. The value of the covariance term in (2.17) is given by

\[ \text{cov}_t (R_{t+1}, R_{t+2}) = E_t \left\{ (R_{t+1} - E(R_{t+1}))(R_{t+2} - (E(R_{t+2}))) \right\} \]

where

\[ R_{t+1} - E(R_{t+1}) = p_{t+1} - p_{t+1} = \zeta_{t+1} \]

\[ R_{t+2} - E(R_{t+2}) = \mu \rho - \mu \rho_{t+1} - \zeta_{t+2} \]

Inserting these expressions into the covariance formula we have

\[ \text{cov}_t (R_{t+1}, R_{t+2}) = E_t \left\{ \left( \zeta_{t+1} \right) \left( \mu \rho - \mu \rho_{t+1} - \zeta_{t+2} \right) \right\} \]
Combining the structural pricing equation for the birth/death period (2.17) and the retrade period (2.10) yields

\[
p_t = \frac{1}{1+r} \left[ r + p_{t+1} + \mu \rho_t + \kappa_1 + \delta_t \kappa_2 \right]
\]
(2.19)

where,

\[\kappa_1 = -2\gamma (1+r) \text{var}(R_{t+1}) - 2\gamma \text{cov}(R_{t+1}, R_{t+2}) = 0\]

\[\kappa_2 = -\kappa_1 - 2\gamma \sigma_{\rho_{t+2}}^2\]

\[\delta_t = \begin{cases} 1 & \text{retrade} \\ 0 & \text{birth/death} \end{cases}\]

To solve for the rational expectations equilibrium we conjecture a solution of the form

\[
p_t = \phi_0 + \phi_1 \rho_t + \phi_2 \delta_t + \phi_3 \delta_{t+1}
\]
(2.20)

\[\Rightarrow p_{t+1} = \phi_0 + \phi_1 \rho_{t+1} + \phi_2 \delta_{t+1} + \phi_3 \delta_{t+2=1}\]

\[\Rightarrow p_{t+1} = \phi_0 + \phi_1 \rho^* + \phi_2 \delta_{t+1} + \phi_3 \delta_{t+2=1}\]

If (2.20) is correct then the following equality must hold true

\[
\phi_0 + \phi_1 \rho_t + \phi_2 \delta_t + \phi_3 \delta_{t+1} = \frac{r}{1+r} + \frac{\kappa_1 + \delta_t \kappa_2}{1+r} + \frac{\phi_0 + \phi_1 \rho^* + \phi_2 \delta_{t+1} + \phi_3 \delta_t}{1+r} + \frac{\mu \rho_t}{1+r}
\]
and the associated values for the $\phi$ terms are given by,

$$\phi_0 = 1 + \frac{\mu p^*}{r(1+r)} + \frac{k_1}{r}, \quad \phi_1 = \frac{\mu}{1+r}, \quad \phi_2 = \frac{(1+r)\kappa_2}{(1+r)^2 - 1},$$

$$\phi_3 = \frac{\kappa_2}{(1+r)^2 - 1}$$

Substituting these coefficient values into (2.20) and after some manipulation we obtain the rational expectations price solution

$$p_t = 1 + \frac{\mu (\rho_t - \rho^*)}{1+r} + \frac{\mu p^* + k_1}{r} + \frac{\delta_t(1+r)\kappa_2}{(1+r)^2 - 1} + \frac{\delta_t \kappa_2}{(1+r)^2 - 1}$$

(2.21)

From (2.21) the birth/death and retrade equilibrium pricing functions are given by

$$p_{t-b/d} = 1 + \frac{\mu (\rho_t - \rho^*)}{1+r} + \frac{\mu p^*}{r} + \frac{k_1}{r} + \frac{\kappa_2}{(1+r)^2 - 1}$$

$$= 1 + \frac{\mu (\rho_t - \rho^*)}{1+r} + \frac{\mu p^*}{r} + \frac{2\gamma \mu^2 \sigma_p^2}{r(1+r)^2(2+r)}$$

(2.22)

for the birth/death period, and

$$p_{t-r} = 1 + \frac{\mu (\rho_t - \rho^*)}{1+r} + \frac{\mu p^*}{r} + \frac{k_1}{r} + \frac{(1+r)\kappa_2}{(1+r)^2 - 1}$$
for the retrade period. From (2.22) and (2.23) we can calculate the difference in expected birth/death and retrade period prices

$$E(p_{t=b/d}) - E(p_{t=r}) = \frac{r^2\mu^2 \sigma_p^2}{(1+r)^2(2+r)} > 0, \forall r>0.$$  

(2.24)

On average, price in a birth/death period is greater than price in a retrade period by an amount equal to the change in the create space effect between birth/death and retrade periods. From (2.24) this difference is increasing in the number of noise traders (i.e. the greater is the proportion of noise traders, the greater the switch in mean prices), the variance of their misperceptions and the risk aversion coefficient. It is decreasing in the interest rate.

A fuller discussion of this is deferred until later in the chapter but a possible interpretation, based on the analysis so far, is that the difference constitutes a positive choice premium which makes the risky asset more attractive in the first period of life. A position taken in the retrade period (or a portfolio rebalancing) must be closed in the following period. Thus a choice premium will only exist if the current period is not the penultimate period. An additional explanation of the difference is a dividend effect. There is a greater dividend return from a birth/death investment than from a subsequent retrade investment as the investor has a claim to two dividend payouts rather than one. This would also make first period investment more attractive than second period investment because the capital gain/loss component in total returns diminishes relative to the dividend component as the trading horizon increases. This latter point is made in DSSW (1990) when they argue that long-horizon investors derive insurance from dividends.
2.2.1 2 Retrade Periods

In a model with four periods rational traders choose $\lambda^i_{t+j}$, $j = 0,1,2$, to maximise

$$E(U)^i = L(1+r)^3 + (1+r)^2 \Omega^i_{t+1} + (1+r)\Omega^i_{t+2} + \Omega^i_{t+3} - \gamma \sigma^2_w$$

and noise traders choose $\lambda^n_{t+j}$, $j = 0,1,2$, to maximise

$$E(U)^n = L(1+r)^3 + (1+r)^2 \Omega^n_{t+1} + (1+r)\Omega^n_{t+2} + \Omega^n_{t+3} - \gamma \sigma^2_w$$

Making use of the results derived for the three period case we need only solve for the structural price in the first of the four periods. The variance of terminal period wealth is given by,

$$\sigma^2_w = (1+r)^4 \left( \lambda^i_{t+1} \right)^2 \text{var}(R_{t+1}) + (1+r)^2 \text{var}(\Omega^i_{t+2}) + \text{var}(\Omega^i_{t+3}) + 2(1+r) \text{cov}(\Omega^i_{t+2}, \Omega^i_{t+3}) + 2(1+r)^2 \lambda^i_t \text{cov}(R_{t+1}, \Omega^i_{t+2})$$

for a rational trader. The equivalent expression for a noise trader can be obtained by replacing $i$ with $n$ throughout. Inserting the appropriate form of (2.27) into (2.25) and (2.26) yields the respective first period, first-order conditions for rational traders and noise traders respectively,

$$\lambda^i_t = \frac{(1+r)^2 (R_{t+1}) - 2\gamma (1+r)^3 \text{cov}(R_{t+1}, \Omega^i_{t+2}) - 2\gamma (1+r)^2 \text{cov}(R_{t+1}, \Omega^i_{t+3})}{2\gamma (1+r)^4 \text{var}(R_{t+1})}$$

$$\lambda^n_t = \frac{(1+r)^2 (R_{t+1} + \rho_t) - 2\gamma (1+r)^3 \text{cov}(R_{t+1}, \Omega^n_{t+2}) - 2\gamma (1+r)^2 \text{cov}(R_{t+1}, \Omega^n_{t+3})}{2\gamma (1+r)^4 \text{var}(R_{t+1})}$$

Imposing market clearing and solving for price yields,

$$p_t = \frac{1}{1+r} \left[ r + p_{t+1} + \mu \rho_t - 2\gamma (1+r)^2 \text{var}(R_{t+1}) - 2\gamma (1+r) \text{cov}(R_{t+1}, \Omega_{t+2}) - 2\gamma \text{cov}(R_{t+1}, \Omega_{t+3}) \right]$$

(2.30)
Combining (12), (32) and (52) produces a single structural pricing equation,

\[ p_t = \frac{1}{1+r} \left[ r + p_{t+1} + \mu_i + k_1 + \delta_t k_2 + \alpha_i k_3 \right] \quad (2.31) \]

where,

\[ k_1 = -2\gamma (1+r)^2 \text{var}(R_{t+1}) - 2\gamma (1+r) \text{cov}(R_{t+1}, \Omega_{t+2}) - 2\gamma \text{cov}(R_{t+1}, \Omega_{t+3}) = 0 \]

\[ k_2 = -k_1 - 2\gamma (1+r) \text{var}(R_{t+1}) - 2\gamma \text{cov}(R_{t+1}, \Omega_{t+2}) \]

\[ k_3 = -k_1 - 2\gamma (t+2) \sigma^2_{n,t} \]

\[ \text{cov}(R_{t+1}, \Omega_{t+2}) = \frac{-\mu^2 \sigma^2_p}{1+r}, \quad \text{cov}(R_{t+1}, \Omega_{t+3}) = 0 \]

\[ \delta_t = \begin{cases} 1 & \text{retrade}(1) \\ 0 & \text{otherwise} \end{cases} \quad \delta_{t+1} = \begin{cases} 1 & \text{retrade}(2) \\ 0 & \text{otherwise} \end{cases} \]

To find the rational expectations equilibrium solution for (2.31) conjecture a solution of the form,

\[ p_t = \phi_0 + \phi_1 \rho_t + \phi_2 \delta_t + \phi_3 \delta_{t+1} + \phi_4 \delta_{t+2} \quad (2.32) \]

Moving forward one period we have,

\[ p_{t+1} = \phi_0 + \phi_1 \rho_{t+1} + \phi_2 \delta_{t+1} + \phi_3 \delta_{t+2} + \phi_4 \delta_{t+3} \]

Taking expectations conditional on time t information yields,

\[ E[p_{t+1}] = \phi_0 + \phi_1 \rho^* + \phi_2 \delta_{t+1} + \phi_3 \delta_{t+2} + \phi_4 \delta_{t+3} \]

If (2.32) is correct then the following equality must hold

\[ \phi_0 + \phi_1 \rho_t + \phi_2 \delta_t + \phi_3 \delta_{t+1} + \phi_4 \delta_{t+2} = \frac{r}{1+r} + \frac{k_1 + \delta_t k_2 + \delta_{t+1} k_3}{1+r} \]

\[ + \frac{\phi_0 + \phi_1 \rho^* + \phi_2 \delta_{t+1} + \phi_3 \delta_{t+2} + \phi_4 \delta_{t+3}}{1+r} + \frac{\mu \rho_t}{1+r} \]

which gives the following values for the \( \phi \) terms
\[
\phi_0 = 1 + \frac{k_1}{r(1+r)} + \frac{\mu \rho^*}{r}, \quad \phi_1 = \frac{\mu}{1+r}, \quad \phi_2 = \frac{(1+r)^2 k_2 + k_3}{(1+r)^3 - 1}, \\
\phi_3 = \frac{(1+r)k_2 + (1+r)^2 k_3}{(1+r)^3 - 1}, \quad \phi_4 = \frac{(1+r)k_2 + (1+r)^2 k_3}{(1+r)^3 - 1}.
\]

Substitution of these coefficient values into (2.32) yields

\[
p_t = 1 + \frac{\mu (\rho_t - \rho^*)}{1+r} + \frac{\mu \rho^*}{r} + \frac{k_1}{(1+r)^3 - 1} + \frac{\delta_t k_3}{(1+r)^3 - 1} + \frac{\delta_{t+1} (1+r)^2 k_3}{(1+r)^3 - 1} + \frac{\delta_{t+2} (1+r) k_3}{(1+r)^3 - 1}
\]

(2.33)

The individual period prices are then given by

\[
p_{t=bd} = 1 + \frac{\mu (\rho_t - \rho^*)}{1+r} + \frac{\mu \rho^*}{r} - \frac{(1+r)2\gamma \mu^2 \sigma^2}{(1+r)^3 - 1}
\]

(2.34)

\[
p_{t=r(1)} = 1 + \frac{\mu (\rho_t - \rho^*)}{1+r} + \frac{\mu \rho^*}{r} - \frac{2\gamma \mu^2 \sigma^2}{(1+r)^3 - 1}
\]

(2.35)

\[
p_{t=r(2)} = 1 + \frac{\mu (\rho_t - \rho^*)}{1+r} + \frac{\mu \rho^*}{r} - \frac{(1+r)^2 2\gamma \mu^2 \sigma^2}{(1+r)^3 - 1}
\]

(2.36)

The above equilibrium pricing equations again show that the per-period misperception effect and the hold more effect are unchanged as the horizon is increased. As before, the change in expected price behaviour is a result of changes in the create space effect. We note that the mean price series over the agents horizon is not monotonic. If we consider the expected prices in (2.34-36), we find that

\[
E[p_{t=r(1)}] > E[p_{t=bd}] > E[p_{t=r(2)}]
\]

(2.37)

In simple terms, the mean price increases between the birth\death period and the first retrade period and then falls, below the birth\death level, between the first and second retrade periods. In the three period case the expected birth/death price was higher than the expected retrade price. In our four period context this translates into the first
retrade price being greater than the second retrade price in expected value. For the DSSW conjecture to hold - price approaching fundamental value as the horizon of agents increases - would require that instead of (2.37) we have

$$E[p_{t=bi|d}] > E[p_{t=r(1)}] > E[p_{t=r(2)}]$$  \hspace{1cm} (2.38)

If (2.38) held then DSSW would be correct and as the horizon increased successive mean birth/death prices would be higher, which in turn would cause price to approach fundamental value. Inspection of (2.37) suggests that successive mean birth/death prices decrease for horizons greater than three periods. This implies that as the horizon of agents increases price does not approach fundamental value but diverges from it.

Simply extending the model to a three period horizon would lead us to believe that the DSSW conjecture was correct. From the results in the four period case we know that either this argument does not hold, or only holds in the restricted case described by DSSW. Recall that they consider the case of agents who live for three periods but are forbidden to trade in the middle period of life. This assumption makes the asset more like a derivative contract which may only be exercised in the terminal period (expiry date). By allowing agents to rebalance their positions we believe that the results presented here are more general and more pertinent to positions in underlying assets such as securities.

We now investigate the behaviour of prices in an n-period model and explain the apparently anomalous switch in price from 3 to 4 periods. Consider the birth/death demands of rational and noise traders. Letting \( j \) denote the horizon, \( j \geq 0 \), these can be written as
\[ \lambda'_j = \frac{r_{t+1}}{2\gamma \text{var}(R_{t+1})} \quad \text{for } j=2. \quad (2.39) \]

\[ \lambda''_j = \frac{r_{t+1}}{2\gamma (1+r)^{-j-2} \text{var}(R_{t+1})} - \frac{\text{cov}(R_{t+1}, \Omega''_{t+j})}{(1+r) \text{var}(R_{t+1})}, \quad \text{for } j>2 \quad (2.40) \]

\[ \lambda''''_j = \frac{r_{t+1} + \rho_t}{2\gamma \text{var}(R_{t+1})}, \quad \text{for } j=2 \quad (2.41) \]

\[ \lambda''''''_j = \frac{r_{t+1} + \rho_t}{2\gamma (1+r)^{-j-2} \text{var}(R_{t+1})} - \frac{\text{cov}(R_{t+1}, \Omega''''''_{t+j})}{(1+r) \text{var}(R_{t+1})}, \quad \text{for } j>2 \quad (2.42) \]

Clearly the first terms on the right hand side of (2.40) and (2.42) are decreasing in \( j \) while the second terms are constant. Given that the behaviour of an infinite series typically differs from the limiting behaviour of a finite series the following remarks are made tentatively. From (2.40) and (2.42) as \( j \to \infty \) the first terms on the right hand side tend to zero and demands would then become non-stochastic\(^1\). The weight that both types of trader attach to the variance of prices (\( \text{var}(R_{t+1}) \)) swamps any dividend effects or choice premium. In a sense DSSW would be correct because asset demands of both types converge to their fundamental level of one. However, this is not the interpretation of long horizon behaviour offered by DSSW. Note also that, in effect, there would be no net trade as the demands do not offset each other. As a corollary the result suggests that agents in asset markets must have ‘short’ horizons if

---

\(^1\) For AR(1) misperceptions the second term on the right hand side of (2.40) and (2.42) is positive. Therefore agents birth/death demands, and equilibrium prices, are decreasing in the length of the horizon.
we are to account for differential trading positions and volume, which are not simply
due to intergenerational motives.

This explains the difference between the intra-horizon behaviour of the three and four
period models. In the three period case the price variance effect was not sufficient to
reduce agents price demands. In the four period case this effect causes the expected
birth/death price to fall below the expected first retrade price. Despite being able to
rebalance their portfolios, if the horizon is 'long' but not infinite then agents early
demand prices are decreasing in \( j \) and this causes the equilibrium price to fall in early
periods\(^2\).

2.3 Stationary Prices

We now extend the horizon using a different structure for the intergenerational
overlap. Previously we viewed different cohorts as only overlapping in birth/death
periods. This was interpreted as the same group of traders trading solely between
themselves as no new traders enter the market. In this section new agents enter the
market every period and cohorts at different stages of their life-cycle interact to
determine the market price. One of the main differences caused by this is that we no
longer have non-stationary intra-horizon prices. Instead we have a unique stationary
price for each horizon, hence we differentiate what follows by terming it the
stationary case. In the non-stationary case we were concerned with intra-horizon price
behaviour. In the analysis of stationary prices presented in this section we are
concerned with inter-horizon price behaviour.

\(^2\) We note that this is likely an artefact of maximising the utility of terminal wealth. Terminal wealth as
an argument becomes less reasonable as the horizon increases. Agents would then be most likely to
maximise with respect to epochs, e.g. every quarter over a year time frame.
The derivation of the equilibrium prices is similar to that for the non-stationary case so we present only the key expressions. Cohorts are overlapping in each period and we assume for simplicity that all noise traders receive the same misperception signal regardless of age. If the first cohort is born at time $t$ and the second at time $t+1$, then they will both trade to determine the price of the risky asset at time $t+1$. We therefore solve for the price at $t+1$. The respective demand functions are virtually identical to (2.7), (2.8), (2.14) and (2.15), except that we must differentiate agents by their birth date. The demands of rational traders and noise traders young in period $t$, at time $t+1$ are given by

$$\lambda_{t+1}^R = \frac{R_{t+2} - \sigma_{P_{t+2}}}{2(1+r)\text{var}(R_{t+2})}$$  \hspace{1cm} (2.46)$$

for rational traders, and

$$\lambda_{t+1}^n = \frac{R_{t+2} + \rho_{t+1} - \sigma_{P_{t+2}}}{2(1+r)\text{var}(R_{t+2})}$$  \hspace{1cm} (2.47)$$

for noise traders. The demands of rational traders and noise traders young in period $t+1$, at time $t+1$, are given by,

$$\lambda_{t+1}^R = \frac{R_{t+2} - \sigma_{P_{t+2}}}{2(1+r)\text{var}(R_{t+2})} - \frac{\text{cov}(R_{t+2}, \Omega_{t+3})}{r(1+r)\text{var}(R_{t+2})}$$  \hspace{1cm} (2.48)$$

for rational traders, and

$$\lambda_{t+1}^n = \frac{R_{t+2} + \rho_{t+1} - \sigma_{P_{t+2}}}{2(1+r)\text{var}(R_{t+2})} - \frac{\text{cov}(R_{t+2}, \Omega_{t+3})}{r(1+r)\text{var}(R_{t+2})}$$  \hspace{1cm} (2.49)$$

for noise traders. For purposes of comparison with the DSSW two period model of Chapter 1 we alter the market clearing condition, which is now given by,

$$\frac{(1-\mu)}{2}(\lambda_{t+1}^R + \lambda_{t+1}^n) + \frac{\mu}{2}(\lambda_{t+1}^R + \lambda_{t+1}^n) = 1$$  \hspace{1cm} (2.50)$$
As there are four different agents demands to aggregate we have to scale the proportions by 1/2 to avoid spuriously introducing population effects into the analysis.\(^3\)

Inserting (2.46), (2.47), (2.48), and (2.49) into the market clearing condition and solving for \(p_{t+1}\) yields

\[
p_{t+1} = \frac{1}{1+r} \left[ r^{+} p_{t+2} + \mu \rho_{t+1} - \frac{2\gamma \text{cov}(R_{t+2}, R_{t+3})}{(2+r)} - \frac{4\gamma (1+r) \text{var}(R_{t+2})}{(2+r)} \right]
\]

(2.51)

where the covariance term is given by (40). The rational expectations equilibrium price distribution is given by,

\[
p_{t} = 1 + \frac{\mu}{1+r} \left( \rho_{t} - \rho^{*} \right) + \frac{\mu \rho^{*}}{r} - \frac{2\gamma \mu^{2} \sigma_{\rho}^{2}}{r(2+r)(1+r)}
\]

(2.52)

As in the non-stationary case the per-period misperception effect and the hold more effect are the same. Note also that in this stationary case the variance of prices, on which agents condition their demands, is also unchanged and thus the conjecture by DSSW that the variance of prices would be unchanged by the horizon of agents is correct. The effect on prices from a horizon extension is shown in the create space effect. Taking expectations of (2.52) and its two period analogue (1.14), and subtracting shows that the create space effect, "the heart of the model" is reduced by the amount

\[
\frac{2\gamma \mu^{2} \sigma_{\rho}^{2}}{r(2+r)(1+r)^{2}}
\]

which parallels our findings for the non-stationary case.

We now consider a four period case for comparison with the results derived in Section 2.2. If agents have a four period investment horizon then in each period there will be three cohorts trading to determine that periods price. One cohort will be forming their

\(^{3}\) Equivalently we could double the fixed stock of \(u\) to be traded.
initial portfolio while the two other cohorts will be rebalancing their portfolios. In
Section 2.2 we found that the mean price behaviour was no longer monotonic for a
four period horizon due to a price variance effect. This feature and its interpretation
carries over to stationary price equilibrium, but is lessened by averaging over cohorts.
Assuming that our analysis begins at time $t$ the three cohorts overlap in period $t+2$ and
we therefore solve for $p_{t+2}$. Each cohorts expected utility and variance of terminal
wealth is given by the appropriate lead of (2.25), (2.26) and (2.27) respectively. The
initial demands of a four period rational trader, young in period $t+3$, is given by,

$$
\lambda^{4}_{t+3} = \frac{(1+r)^2(R_{t+3}) - 2\gamma(1+r)^3 \text{cov}(R_{t+3}, \Omega_{t+4}) - 2\gamma(1+r)^2 \text{cov}(R_{t+3}, \Omega_{t+5})}{2\gamma(1+r)^4 \text{var}(R_{t+3})}
$$

(2.53)

and for a noise trader born in period $t+3$ we have,

$$
\lambda^{n}_{t+3} = \frac{(1+r)^2(R_{t+3} + \rho_{t+2}) - 2\gamma(1+r)^3 \text{cov}(R_{t+3}, \Omega_{t+4}) - 2\gamma(1+r)^2 \text{cov}(R_{t+3}, \Omega_{t+5})}{2\gamma(1+r)^4 \text{var}(R_{t+3})}
$$

(2.54)

To avoid population effects the market clearing condition becomes,

$$
\frac{1}{3} \left[ \lambda^{4}_{t+2} + \lambda^{n}_{t+2} + \lambda^{n}_{t+2} \right] + \frac{\mu}{3} \left[ \lambda^{4}_{t+2} + \lambda^{n}_{t+2} + \lambda^{n}_{t+2} \right] = 1
$$

(2.55)

Setting the time subscript to $t$ and solving for price we obtain the following structural
pricing equation,

$$
p_t = \frac{1}{1+r} \left[ r + \mu \rho_t - \Theta \right]
$$

(2.56)

where,

$$
\Theta = \frac{6\gamma(1+r)^2 \text{var}(R_{t+1}) + (1+r)4\gamma \text{cov}(R_{t+1}, R_{t+2})}{(1+r)^2 + 2 + r}
$$

The four period rational expectations price distribution is then given by,
$$p_t = 1 + \frac{\mu (\rho_t - \rho^*)}{1 + r} + \frac{\mu \rho^*}{r} - \frac{2\gamma \mu^2 \sigma^2}{r(3(1+r)+r^2)} \tag{2.57}$$

Taking unconditional expectations and comparing (2.57) with its three period equivalent (2.52) shows that the expected four period price is greater than the expected three period price,

$$E(p_t(3)) < E(p_t(4)) \quad \forall r > 0.$$

but by a lower amount than $E(p_t(3)) - E(p_t(2))$. Averaging over cohorts causes the price series to appear monotonic and to conform to the conjecture of DSSW. However, given the pattern of successive demands as $j$ increases, (2.40-.42), the above inequality will be reversed for some $j$.

We now turn to an analysis of AR(1) misperceptions where the horizon behaviour is more clear cut than in the ‘normal’ case outlined above.
2.4 AR(1) Misperceptions

A number of studies have documented mean reversion in asset prices and return series, and interpreted this as evidence of the existence of predictable structure. Fama and French (1988) and Poterba and Summers (1988) find evidence of significant mean reversion, negative serial correlation, in security prices at 'long' horizons of eight to ten years. This contrasts with the short horizon results of Lo and MacKinlay (1988) and others, who find that at frequencies of one month and less, index returns exhibit significant positive serial correlation. Although the findings of Fama and French and Poterba and Summers have been questioned (see inter alia, Kim, Nelson and Startz (1988) Richardson and Stock (1989), and Campbell (1993)), DSSW used the original results to motivate modelling the noise trader misperception as following a stationary AR(1) process. It should be stressed that DSSW do not state that asset prices follow an AR(1) process as some authors have wrongly alleged. In the original paper this analysis forms a minor part of the overall results, and is merely included to illustrate that AR(1) misperceptions strengthen the create space effect of noise traders and their ability to deter rational arbitrage.

In this section we derive the two period case and then look at a multi-period example for comparison with the i.i.d. misperception results discussed in the previous sections.

For convenience we begin the analysis by rewriting the two period structural pricing equation (1.12)

\[
p_t = \frac{1}{1+r} \left[ r + p_{t+1} + \mu \rho - 2 \lambda \sigma^2 \right]
\]

where, the noise trader misperception is modelled as a stationary first order autoregressive process, (AR(1)),

58
\[ \rho_t = \alpha \rho_{t-1} + \theta + \eta_t \]  
(2.59a)

| \alpha | < 1, \eta_t \sim N(0, \sigma^2) , \theta \neq 0

The constant term \( \theta \) is included to ensure that noise traders average misperception is non-zero,

\[ \rho^* = \frac{\theta}{1 - \alpha} \]  
(2.59b)

To find the rational expectations price equilibrium conjecture a solution of the form,

\[ p_t = \phi_0 + \phi_1 \rho_t \]  
(2.60)

\[ tP_{t+1} = \phi_0 + \phi_1 (\alpha \rho_t + \theta) \]

If (2.60) is correct then the following equality must hold

\[ \phi_0 + \phi_1 \rho = \frac{1}{1+r} \left[ r + \phi_0 + \phi_1 \alpha \rho_t + \phi_1 \theta + \mu \rho_t - 2 \gamma \sigma^2_{\rho_{t+1}} \right] \]

implying values for the coefficient terms

\[ \phi_0 = 1 + \frac{\mu \theta}{r(1+r-\alpha)} - \frac{2 \gamma \sigma^2_{\rho_{t+1}}}{r} \]

\[ \phi_1 = \frac{\mu}{1+r-\alpha} \]

Substituting these coefficient values into (2.60) yields,

\[ P_t = 1 + \frac{\mu \rho_t}{1+r-\alpha} + \frac{\mu \theta}{r(1+r-\alpha)} - \frac{2 \gamma \sigma^2_{\rho_{t+1}}}{r} \]  
(2.61)

Using (2.59) and noting that the conditional price variance is given by

\[ \sigma^2_{\rho_{t+1}} = \frac{\mu^2 \sigma^2_{\eta}}{(1+r-\alpha)^2} \]

we can write the final form of the pricing rule as \(^1\)

\(^1\) The associated reduced form demand equations are
\[
p_t = 1 + \frac{\mu (\rho_t - \rho^*)}{1 + r - \alpha} + \frac{\mu \rho^*}{r} - \frac{2 \gamma \mu^2 \sigma_n^2}{r(1 + r - \alpha)^2}
\]  
(2.62)

Inspection of (2.62) shows that for autoregressive misperceptions the per period misperception effect and the create space effect are both altered from the i.i.d case described in chapter 1. The hold more effect is unchanged as we have written the expression. The per-period misperception effect has a greater (lesser) effect on the price of the risky asset if the autoregressive parameter is positive (negative). This is also the case for the create space effect assuming that the earlier \( \sigma^2_p \) is equal to \( \sigma^2_n \) in (2.63). Similarly the conditional price variance is greater (lesser) if the autoregressive parameter is positive (negative). Thus if noise traders misperceptions exhibit positive serial correlation then they have a greater effect on prices and the variance of prices. This translates into a greater deterrence of rational arbitrage through an increased create space effect.

We would expect an increased create space effect to result in higher expected excess returns for noise traders assuming that their misperception is positively serially correlated. The conditional expected excess return of noise traders is given by

\[
\gamma_t(\Delta R_{n,t}) = \rho_t - \frac{(1 + r - \alpha)^2 \rho_t^2}{2 \gamma \mu \sigma_n^2}
\]  
(2.63)

which implies a misperception window which is greater than for ‘normal’ misperceptions,

\[
0 < \rho_t < \frac{2 \gamma \mu \sigma_n^2}{(1 + r - \alpha)^2}
\]

for \( \sigma_n^2 = \sigma_p^2 \) and \( 0 < \alpha < 1 \).

Comparing with the ‘normal’ MW we have
\[
\frac{2 \gamma \mu \sigma_n^2}{(1 + r - \alpha)^2} \geq \frac{2 \gamma \mu^2 \sigma_p^2}{(1 + r)^2}
\]

In the case of positive serial correlation\(^2\) of noise trader misperceptions the range in state space over which noise traders have higher expected returns than rational traders is greater than in the ‘normal’ misperception case. Thus we would expect noise traders to be more likely to survive and win converts in this case. The unconditional expectation of (2.63) is

\[
E(\Delta R_{n,t}) = \rho^* - \frac{(1 + r - \alpha)^2 \rho^{**}}{2 \gamma \mu \sigma_n^2} + \frac{(1 + r - \alpha)^2}{2 \gamma \mu (1 - \alpha^2)}
\]

which is equation (32) in DSSW.

We now repeat the horizon analysis in section 2.2 for ‘normal’ misperceptions with the AR(1) misperceptions outlined above. To avoid unnecessary repetition we present only the key expressions. Assuming that agents live for three periods and that there are no birth/deaths in the retrade period then in equilibrium the pricing rules are

\[
P_{t,bid} = 1 + \frac{\mu (\rho_t - \rho^*)}{(1 + r - \alpha)} + \frac{\mu \rho^*}{r} - \frac{2 \gamma}{(1 + r)^2} \left[ \frac{(1 + r)^2 + 1}{(1 + r - \alpha)^2} \right] \left( \frac{\mu^2 \sigma_n^2}{(1 + r - \alpha)^2} + \frac{\alpha \mu^2 \sigma_n^2}{1 - \alpha^2} \right)
\]

(2.64)

for birth/death periods, and

\[
P_{t,ur} = 1 + \frac{\mu (\rho_t - \rho^*)}{(1 + r - \alpha)} + \frac{\mu \rho^*}{r} - \frac{2 \gamma}{(1 + r)^2} \left[ \frac{2(1 + r) \mu^2 \sigma_n^2}{(1 + r - \alpha)^2} + \frac{\alpha \mu^2 \sigma_n^2}{1 - \alpha^2} \right]
\]

(2.65)

for retrade periods. The first three terms on the right hand side of (2.64) and (2.65) are identical to those for two period lived agents. This is analogous to the results in section 2.2. Again we find that as the horizon is extended the price of the risky asset

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\(^2\) We concentrate on positive serial correlation of misperceptions as negative correlation would imply that successive misperceptions would tend to be of opposite sign. We consider such a scenario to be both unlikely and unsatisfactory.
changes through the create space effect. The covariance between adjacent periods expected excess return on the risky asset is given by,

$$\text{cov}(R_{t+1}, R_{t+2}) = \frac{\alpha \mu^2 \sigma_n^2}{1 - \alpha^2}$$  \hspace{1cm} (2.66)

which for $\alpha > 0$ contrasts with the negative value in the 'normal' misperception case. Taking expectations of (2.64) and (2.65) and subtracting we find that that the expected retrade price is greater than the expected birth/death price

$$E(p_{t-r}) - E(p_{t-bid}) = \frac{2\gamma}{(1+r)^2 - 1} \left[ \frac{r^2 \mu^2 \sigma_n^2}{(1 + r - \alpha)^2} \right] > 0$$  \hspace{1cm} (2.67)

In the case of normal misperceptions we found some anomalous results between the behaviour of prices for a three period horizon and a four period horizon. For AR(1) misperceptions the expected price series increases monotonically as we approach the terminal period of the horizon. This gives justification to the argument that if noise traders misperceptions are autoregressive then rational arbitrage is further deterred by the anticipation that misperceptions may not revert to the mean for long periods, thus making rational traders more likely to suffer a loss.

The conditional expected excess return of noise traders in the initial period is

$$t(\Delta R_{n,t}) = \rho_t + \frac{\alpha p_t (1 + r - \alpha)^2}{(1+r)(1-\alpha^2)} - \frac{(1+r-\alpha)^2 \rho_t^2}{2\gamma(1+r)\mu \sigma_n^2}$$  \hspace{1cm} (2.68)

Comparing (2.68) with (2.63) noise traders have higher (lower) conditional expected excess returns in the initial period if $\rho_t > 0$ ($\rho_t < 0$), with the retrade value given by (2.63). Thus noise traders expected excess returns are more variable in a three period model than in a two period model. The first period MW is given by

$$0 < \rho_t < 2\gamma \mu \sigma_n^2 \left[ \frac{1+r}{(1+r-\alpha)^2} + \frac{\alpha}{(1-\alpha^2)} \right]$$  \hspace{1cm} (2.69)
which is clearly greater than the two period MW (2.63) for all positive \( \alpha \). DSSW used the AR(1) example to illustrate how unconditional price variance grows rapidly if misperceptions are serially correlated.

Thus as the horizon is extended the expected excess relative returns of noise traders is greater than in the two period case provided their misperception is positive. There is some ambiguity in comparing 2-period and 3-period prices. The expected birth/death price in the 3-period case will be greater than its two period analogue if

\[
\frac{2\gamma(1-r)}{r(1+r-\alpha)^2} < \frac{\alpha}{(1-\alpha)^2}
\]

For noise traders misperceptions which are highly persistent, \( \alpha \) close to one, then the inequality will be satisfied for admissible parameter values.

### 2.5 Conclusion

In this chapter we looked at the behaviour of an economy with noise traders when the horizon of all agents was extended beyond two periods. It was shown that the DSSW conjecture of price approaching fundamental value with increasing horizons need not hold, particularly if noise traders misperceptions follow an autoregressive process.

If the horizon structure is such that generations overlap only in birth/death periods then the price series is non-stationary. For a j-period horizon there will be j-1 expected prices. Within the horizon the price series will exhibit mean-reverting behaviour even if misperceptions themselves are ‘normal’. If generations overlap in all periods then the price series is stationary. The conditional price variance was shown to be constant across all horizons as was conjectured by DSSW.

The results show that the issue of trader horizon is certainly not irrelevant. In common with Leach (1991) we find that the most interesting behaviour occurs for intermediate
horizons, greater than two periods but less than infinity. Within this range there is a complex interplay between dividend effects, choice premiums and agents treatment of risk. The key feature being that as the horizon grows agents weight the variance of returns more than the corresponding conditional mean return in forming their demands. In future work we hope to analyse this more thoroughly by endogenising horizon according to belief type.
3. Throwing Sand at Noise Traders

3.1 Introduction

The notion of controlling the behaviour of financial markets by trading taxes is an important and contentious issue. A recent Policy Forum in the Economic Journal (1995) was devoted to the consequences and viability of throwing “Sand in the Wheels of International Finance”. The main proponent of a trading tax has been James Tobin, (1978) who argues that trading taxes be imposed to deter damaging speculation, encourage rational trading and thereby reduce volatility and bring prices more in line with fundamental value. Eichengreen, Tobin and Wyplosz (1995) argue that

“Transactions taxes are one way to throw sand in the wheels of super-efficient financial vehicles.”

Writing in the same issue Garber and Taylor (1995) are pessimistic about the use of such measures which they regard as ill-advised:

“A policy of throwing sand in the wheels of international finance would very likely amount to little more than a futile, Canutian attempt to command the tides of international capital flows.”

Transactions taxes have been proposed for securities markets. Summers and Summers (1989) p231 write,
"Such a tax would have the beneficial effect of curbing instability introduced by speculation, reducing the diversion of resources into the financial sector of the economy, and lengthening the horizon of corporate managers."

There seems to be general agreement amongst economists that asset prices exhibit excess volatility and that actual prices deviate significantly from fundamental value. However, there is considerable disagreement whether or not trading taxes are viable or will yield the desired outcome. Much of the literature is very discursive and amounts to little more than a roundtable debate. A recent example can be found in the paper by Obstfeld (1995) and subsequent discussion. Obstfeld believes that such a tax would discourage both stabilising and destabilising trades and thereby aggravate short-term and possibly even long-term volatility. He argues that without a complete understanding of exchange rate determination we cannot predict how a Tobin tax would work in practice. By contrast, in the discussion, Dornbusch urges us to "... try a Tobin tax". He believes a Tobin tax would stabilise medium term prices at the expense of a possible increase in short-term volatility. Finally, Sims doubts that a Tobin tax would decrease volatility because it would amplify deviations in price caused by speculative runs. The above shows the difficulty in even agreeing on the set of possible events, let alone their direction.

Existing theoretical studies of the effects of transactions costs (including trading taxes, information costs etc) have tended to focus on different implications than the 'Tobin tax' debate. Mayshar (1981) shows that incorporation of transactions costs in a CAPM framework leads to a major reformulation of the basic asset pricing equation. Magill and Constantinides (1976), using a continuous time model, find
that in the presence of transactions costs investors will only seek to make use of trading opportunities at randomly spaced instants of time. Milne and Smith (1980) look at the effects of proportional transactions costs and capital asset pricing. Their main finding is that individuals hold different portfolios of risky assets in the presence of transactions costs. Agents will only change their portfolio holdings if the benefits of either increased expected future consumption or reductions in the variance of future consumption outweigh the costs paid in terms of future consumption from incurring transactions costs.

Campbell and Froot (1994) provide a survey of international experiences with securities transactions taxes. A reduction in overall trading and a migration of trading into offshore markets are the main responses which they find. Furthermore, if the demands of the stabilising traders are reduced along with the demands of destabilising traders then the overall effect on excess volatility of a reduction in trading volume is ambiguous.

In this chapter we explore the issue of trading taxes using the basic noise trader model of DSSW outlined in chapter 1 and the horizon extensions in chapter 2. This provides a simple means of analysing the effects of Tobin taxes on the variance of asset prices, the level of asset prices and the relative returns of noise traders (speculators) and rational traders. Throughout what follows the destabilising speculators whose activities are meant to be curbed by "sand" are modelled as noise traders.

The main result of the chapter is that a simple Tobin tax has largely neutral effects. It has no effect on the variance of prices or on traders relative returns, but reduces the mean price of risky assets. We find that the imposition of a simple tax does not
make noise trader strategies less attractive. These findings are robust with respect to the horizon of all traders in the model. Extending the trading horizon of agents does not alter the limited effects of a tax. This provides a theoretical background to the empirical findings of Umlauf (1993):

“Volatility did not decline in response to the introduction of taxes although stock price levels and turnover did.”

If the tax is imposed as a non-linear function of agents demands then the results are less neutral. In this case all agents demands are less variable and, in terms of expected returns, noise traders may benefit relative to rational traders from the imposition of a tax. In this sense the effects of a non-linear tax are thus mixed.

If the policy objective is to reduce the attractiveness of following a noise trader strategy, thereby encouraging agents to follow a rational strategy, then the best policy is not a tax but direct government intervention. The intervention policy advocated requires the government to estimate the average “mood” of the noise traders, and then take a position which either follows this mood or offsets it. It is found that the best policy is to follow or intervene in the same direction as the noise traders average mood.

The chapter is organised as follows. Section 1 provides a brief description of the DSSW model and analyses trading taxes where noise traders have i.i.d misperceptions. Section 2 looks at the case where noise traders misperceptions follow an autoregressive process. Section 3 analyses the effect of direct

1 With respect to the revenue which might be raised by such a scheme, they argue that a tax along British lines, if implemented in the USA, would lead to major changes in the behaviour of investors, making estimated figures of
intervention by the government or regulatory authority. Section 4 analyses transactions taxes in a three period setting. Section 5 looks at the effects of imposing a non-linear tax. Section 6 provides concluding comments and an Appendix contains analysis of a three period model with an adjustment tax, and a mixed sand policy of tax-financed government intervention.

3.2 Basic model

The model of DSSW analyses the effects of a group of irrational agents, called noise traders, on the behaviour of rational traders and asset prices. Their main results are that noise traders who significantly effect prices, can earn higher returns than rational traders and will not necessarily be driven out of the market by rational traders. The analysis is embedded in a standard 2-period overlapping generations framework. Agents problem is to form a portfolio in the first period of life to maximise expected utility of wealth in the second period. There is no consumption in the first period, no bequests and no labour supply decision. Agents initial wealth endowments are exogenous, and denoted $L$. Portfolios are composed of two assets. The safe asset, denoted $s$, yields a fixed real dividend of $r$, is in perfectly elastic supply and trades at a fixed price of one in every period. The unsafe or risky asset, denoted $u$, yields the same dividend as asset $s$ but its quantity is fixed in all periods. The quantity of $u$ is restricted to one unit and the price in period $t$ is denoted $p_t$.

The heterogeneous population is split into two groups. There are sophisticated investors, denoted $i$, and noise traders, denoted $n$. The two types are present in

$10 billion in revenue for a 0.5% tax too optimistic.

* In this section we repeat much of the analysis of chapter 1 in order to make this chapter as self contained as possible.
proportion \((1 - \mu)\) and \(\mu\) respectively. In the basic DSSW model there are no evolutionary dynamics and the proportions are simply treated as exogenous constants. Both noise traders and sophisticated investors are endowed with a rational expectations technology. However, the expectations of noise traders are perturbed by a stochastic misperception term, denoted \(\rho_t\). In this section we follow the basic DSSW model and assume that this misperception is a normally distributed random variable with non-zero mean,

\[ \rho_t \sim N(\rho^*, \sigma^2_\rho) \]  

(3.1)

This misperception is common to all noise traders of a given generation. The mean misperception is a measure of noise traders average optimism or pessimism, while the variance measures their misperception of the expected return per unit of asset \(u\).

The description of noise traders used in this paper follows that of DSSW and is purposefully general. In 1964 Paul Cootner stated that securities prices “are typically very sensitive, responsive to all events, both real and imagined”. In the simplest possible definition noise traders are those agents who focus on, and manifest, the imagined determinants of asset prices. There are many possible realisations of this noise. Basic irrationality of agents is one explanation. Fischer Black (1986) replaced the derisory ‘i-word’ with noise thereby sanitising irrationality and rendering it more palatable. In Black’s paper such traders may be acting on the basis of pseudo-signals, enjoy trading in itself, believe that noise is in fact relevant information, or trade on the basis of information that has already been
reflected in the market price. With characteristic insight he notes that “noise trading is essential to the existence of liquid markets” and “provides the essential missing ingredient”3. This macro approach to noise trading is further outlined in Shleifer and Summers (1990):

“Our approach rests on two assumptions. First, some investors are not fully rational and their demand for risky assets is affected by their beliefs or sentiments that are not fully justified by fundamental news. Second, arbitrage - defined as trading by fully rational investors not subject to such sentiment - is risky and therefore limited.”

In the present paper we associate destabilising speculation with the activities of our noise traders subject to erroneous stochastic misperceptions. Specifically, noise traders have both a per-period misperception and an average misperception. The former is a stochastic variable which allows noise traders to deter rational arbitrage by creating future uncertainty, while the latter is a constant term which causes noise traders to bear either more or less risk on average. These beliefs are common to all noise traders.

We are interested in whether or not an advocated policy can bring price in line with fundamental value, reduce price volatility and/or demand volatility, and most importantly reduce the attractiveness of being a noise trader. The success of a policy is judged on these criteria and we are not concerned with the efficacy of implementation.

3 Black furthermore states “If my conclusions are not accepted, I will blame it on noise.”
All agents have constant absolute risk aversion utility functions and construct their portfolios to maximise the expected utility of terminal wealth. The first period budget constraint is

\[ L = v_t + \lambda_t p_t + \lambda_t \chi \]  

(3.2)

Where \( v_t \) and \( \lambda_t \) are first period demands, or holdings, of assets \( s \) and \( u \) respectively, and \( \chi \) is the trading tax. With \( \chi = 0 \) our model reduces to that of DSSW. The trading tax is specified as being proportional to the quantity demanded of the risky asset. With the budget constraint specified as in (3.2) we must assume that agents demands are positive in all periods. The precise conditions for this to hold are detailed later. Although not innocuous the assumption is found to be reasonable, dependent on the values of the variance of noise traders misperception and agents degree of risk aversion. It is possible to solve the model using squared demands and this is given in Section 5. It is not possible for both types to have negative demands in the same period, and thus (3.2) is the most consistent and tractable formulation

From this first period investment traders have terminal wealth of

\[ w = L(1+r) + \lambda_t \left( r + p_{t+1} - p_t(1+r) - \chi(1+r) \right) \]  

(3.3)

Both noise traders and sophisticated investors choose their \( \lambda_t \) to maximise

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4 The vast majority of positions taken in securities markets are long positions, and short positions usually entail margin restrictions and stock borrowing. We feel that the positive demands assumption is therefore realistic.
\[ E(U)' = L(1+r) + \lambda_i'(r_t + p_{t+1} - p_t(1+r) - \chi(1+r)) - \gamma \sigma_w^2 \]  
\[ E(U)'' = L(1+r) + \lambda_i''(r_t + p_{t+1} - p_t(1+r) + \rho_t - \chi(1+r)) - \gamma \sigma_w^2 \]

Where \( r_t p_{t+1} \) denotes the rational expectation, formed at time \( t \), of the price of asset \( u \) in period \( t+1 \), \( \gamma \) is the coefficient of absolute risk aversion \( (\gamma \geq 0) \), and \( \sigma_w^2 \) is the variance of terminal wealth. To find agents optimal holdings of \( u \), differentiate (4) and (5) w.r.t. \( \lambda_i \),

\[
\lambda_i' = \frac{r_t p_{t+1} - p_t(1+r) - \chi(1+r)}{2\gamma \sigma_w^2_{p_{t+1}}} \]
\[
\lambda_i'' = \frac{r_t p_{t+1} - p_t(1+r) + \rho_t - \chi(1+r)}{2\gamma \sigma_w^2_{p_{t+1}}} \]

The only difference between the two types demands is the addition of the misperception term in the demand equation for noise traders. Notice that the trading tax has an identical effect on both types. It can be shown that these demands are independent of prices\(^5\). It is assumed that in each period the old sell all of their holdings to the new young. Thus the model is demand driven and with a fixed supply of asset \( u \) there can be no consideration of the effect of a trading tax on transactions volume. The market clearing condition is simply

\[(1 - \mu)\lambda_i' + \mu \lambda_i'' = 1 \]
Inspection of (3.8) shows that both types demands cannot be negative in the same period for the market to clear. Inserting agents demands into (3.8) and solving for price yields the structural pricing equation

\[ p_t = \frac{1}{1+r} \left[ r_t + p_{t+1} - \lambda (1+r) + \mu \rho_t - 2\gamma \rho \sigma_{\rho t}^2 \right] \]  

(3.9)

where \( \sigma_{\rho t}^2 \) is the conditional expectation of the one step ahead price variance of asset \( u \). From (3.9) and using (3.6) and (3.7) we can derive conditions for agents demands to be positive in all periods. Specifically it can be shown that

\[ \lambda_t, \lambda_t^* > 0 \quad \forall \rho_t < 0, \]

and if \( \rho_t > 0 \), then

\[ \lambda_t, \lambda_t^* > 0 \quad \text{if} \quad \rho_t < \frac{2\gamma \mu \sigma^2}{(1+r)^2}, \quad \text{for} \quad \mu = 1 - \mu \]

If we assume that the variance of noise traders misperception is 'large' then the likelihood of a positive realised value of \( \rho_t \), causing demands to be negative can be made arbitrarily small. The model could be calibrated to ensure that for the

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5 Simply insert (9) into (6) and (7).

6 From (9) it is easily seen that

\[ \sigma_{\rho t}^2 = \frac{\mu^2 \sigma^2}{(1+r)^2} \]
majority of realisations of $\rho$, rational and noise trader demands would be positive. As this is the most tractable formulation we continue with the positive demand assumption while noting that it will not hold for all values of the misperception term. A more robust formulation is presented in Section 5.

The rational expectations equilibrium price is derived as follows. Conjecture a solution of the form

$$p_t = \phi_0 + \phi_1 \rho_t$$

then

$$\phi_0 = 1 + \frac{\mu \rho^*}{(1+r)r} - \frac{\chi(1+r)}{r} - \frac{2\gamma \mu^2 \sigma^2}{r(1+r)^2}$$

$$\phi_1 = \frac{\mu}{(1+r)}$$

Substituting these coefficient values into our conjectured solution and simplifying yields the final form of the pricing equation

$$p_t^x = 1 + \frac{\mu(\rho_t - \rho^*)}{1+r} + \frac{\mu \rho^*}{r} - \frac{\chi(1+r)}{r} - \frac{2\gamma \mu^2 \sigma^2}{r(1+r)^2}$$

(3.10)

The DSSW analogue is given by
\[
p^D_t = 1 + \frac{\mu(\rho_t - \rho^*)}{1+r} + \frac{\mu \rho^*}{r} - \frac{2\mu^2 \sigma_r^2}{r(1+r)^2}
\]

(3.11)

and the posterior superscript D denotes a result from DSSW. Using the terminology of DSSW (3.10) and (3.11) are interpreted as follows. The price of asset \( u \) is determined by exogenous parameters of the model and the misperceptions of noise traders. The final three terms on the right hand side of (3.11) illustrate the effect of noise traders on the price of the risky asset. Note firstly that if there are no noise traders then the safe asset and the risky asset are perfect substitutes and trade at the identical price of one. The second term should be thought of as the per-period misperception effect of noise traders. If \( \rho_t > \rho^* > 0 \), then the price of the risky asset is bid up because the current generation of noise traders is more optimistic than the average generation. If \( \rho_t < \rho^* \), then the second term is negative and the price is bid down by a generation of noise traders who are more pessimistic than the average. The third term shows the effect of the mean misperception held by all generations of noise traders. It is assumed in the model that the mean misperception is a non-zero exogenous constant. If \( \rho^* \) is positive then this 'price pressure' effect bids up the price of the risky asset. In this case noise traders are bearing a larger share than rational traders of the price risk which they themselves have created. As the noise trader share of price risk rises then clearly the rational trader share is falling. For any investor as their exposure to risk falls so does the commensurate level of expected return required to cause them to hold the asset. This means that when \( \rho^* \) is high, sophisticated investors require a lower expected return on holdings of asset
and will pay a higher price for the asset. DSSW describe the final term as 'the heart of the model'. They call this term the 'create space' effect of noise traders. It is this term that deters rational arbitrage and causes the persistent deviations of the actual price of the risky asset from its fundamental value of one.

The difference between the two prices is given by

\[ p_t^p - p_t^z = \frac{x(1+r)}{r} \]  \hspace{1cm} (3.12)

Clearly the introduction of a trading tax will result in a lower expected price for the risky asset. The trading tax effect works in the same direction as the create space effect of noise traders. We might therefore expect the tax effect to benefit noise traders relative to sophisticated investors. The tax does not affect the variance of prices in the model which is still a function of the proportion of noise traders, the variance of their misperception and the interest rate. These results mirror those of Umlauf (1993) in his empirical study of the Swedish Stock Market. He found that the level of prices was reduced and that volatility remained constant. As mentioned earlier given the form of the market clearing condition we cannot make explicit comments about the effect of a tax on trading volume.

We now analyse the effect of a trading tax on traders relative returns and compare this with the results of DSSW to find out whether the tax benefits noise traders or rational traders. The difference in returns to the two types is simply given by the

\[ p_t^p - p_t^z = \frac{x(1+r)}{r} \]

\[ \text{It can be shown that the variance of prices remains constant even if we extend the horizon of all agents.} \]
difference in their holdings of the risky asset multiplied by the excess return of the risky asset over the safe asset.

\[ \Delta R_{n-i} = (\chi_i^* - \chi_i^d)(r + p_{t+i} - p_t(1+r) - \chi(1+r)) \]  

(3.13)

In the DSSW case the conditional excess relative return of noise traders is given by

\[ t(\Delta R_{n-i}) = \rho_t - \frac{(1+r)^2(\rho_t)^2}{2\gamma \mu \sigma_p^2} \]  

(3.14)

and the unconditional expectation is given by

\[ E(\Delta R_{n-i}) = \rho^* - \frac{(1+r)^2(\rho^*)^2 + (1+r)^2 \sigma_p^2}{2\gamma \mu \sigma_p^2} \]  

(3.15)

Solving analogously for the trading tax case yields identical equations, as the tax does not affect demands or the expected excess return of the unsafe asset over the safe asset,

\[ t(\Delta R_{n-i})^\tau = \rho_t - \frac{(1+r)^2(\rho_t)^2}{2\gamma \mu \sigma_p^2} \]  

(3.16)

\[ E(\Delta R_{n-i})^\tau = \rho^* - \frac{(1+r)^2(\rho^*)^2 + (1+r)^2 \sigma_p^2}{2\gamma \mu \sigma_p^2} \]  

(3.17)
In (3.14) we see that for noise traders to have higher relative returns than rational traders requires that their misperception that period is positive, and that the risky asset has a current price below fundamental price \((r^+, p_{t+1} > p_t(1+r))\)\(^8\). From (3.14) and (3.16) we have the following inequality restrictions on \(\rho_t\), for both the no tax and the tax case,

\[
\begin{align*}
(\Delta R_{n-1})^D > 0 & \quad \text{if} \quad \rho_t > \frac{(1+r)^2 \rho_i^2}{(2\gamma) \mu \sigma_p^2} \\
\text{MW} \Rightarrow 0 < \rho_t < \frac{(2\gamma) \mu \sigma_p^2}{(1+r)^2}
\end{align*}
\]

This gives us bounds on the range of realised values of the noise trader misperception term which give positive values of (3.14) and (3.16) respectively. I define this bounded area as the ‘misperception window’ of noise traders, denoted MW hereafter. Sethi and Franke (1995) use the concept of an ‘expectational corridor’ which defines the region in the state space where non-optimisers can outperform the group of optimisers. They argue that this is a useful device in helping to understand the circumstances that may prove favourable for one group or the other. The misperception window is identical in the trading tax case, and shows that the tax does not reduce the relative attractiveness of following a noise trader strategy as opposed to a rational strategy.

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\(^8\) Note that the excess relative return of noise traders illustrates strategic complementarity in the model. Haltiwanger and Waldmann (1989) explain this concept as follows:

“To put it simply, an environment exhibits strategic complementarity if the higher is the total number of agents who choose a particular behaviour, the higher is the return to agent \(i\) from using that behaviour.”
It should be remembered that noise traders will always have lower utility than rational traders. However, as argued by DSSW relative returns and not utility is a more satisfactory criterion for comparing the two types performances given the nature of financial markets. Agents are appraised on their relative returns in financial markets and successful noise traders will have conspicuously higher returns and higher consumption than their rational counterparts.

We now look at the possibility of constructing the "tax" such that the expected price of the risky asset is equal to its fundamental value of one. The expected prices in DSSW and the trading tax case are given by

\[
E(p_t^D) = 1 + \frac{\mu \rho^*}{r} - \frac{2\gamma u^2 \sigma_p^2}{r(1+r)^2} \tag{3.18}
\]

\[
E(p_t^S) = 1 + \frac{\mu \rho^*}{r} - \frac{\chi (1+r)}{r} \frac{2\gamma u^2 \sigma_p^2}{r(1+r)^2} \tag{3.19}
\]

For \(E(p_t) = 1\) requires \(\rho^*\) to take the following values

\[
(D) \quad \rho^* = \frac{2\gamma u \sigma_p^2}{(1+r)^2} \quad (x) \quad \rho^* = \frac{\chi (1+r)}{\mu} + \frac{2\gamma u \sigma_p^2}{(1+r)^2} \tag{3.20}
\]

In DSSW the mean price of asset \(u\) is determined by noise trader's bear/bullishness as all of the other terms on the r.h.s. of (3.18) are constants. There is no instrument

They also find, in their (1985) article, that in an environment characterised by strategic complementarity, it is the agents with a limited expectations technology who will have a
available to offset the mood of noise traders and thereby bring the price of asset \( u \) closer to its fundamental value. In the trading tax case this is possible by choosing an optimal “tax”, denoted \( \chi \), which satisfies the following

\[
\chi = \frac{\mu \rho^*}{(1+r)} - \frac{2\gamma \mu^2 \sigma_p^2}{(1+r)^3} \quad \Rightarrow E(p_t^*) = 1 \tag{3.21}
\]

This value for the transaction cost causes sufficient friction for the expected value of the risky asset to equal its fundamental value. For \( \chi \) to be positive (a tax and not a subsidy) requires

\[
\frac{\mu \rho^*}{(1+r)} > \frac{2\gamma \mu^2 \sigma_p^2}{(1+r)^3} \tag{3.22}
\]

Thus we want \( \gamma \) and \( \sigma_p^2 \) to be “small”, \( \rho^* \) to be “large”, and \( \mu \) to be close to 0 or 1.

For plausible parameter values the likely sign of \( \chi \) is negative i.e. to bring expected price in line with fundamental value requires a trading subsidy! The intuition behind this result is very simple. For expected price to approach fundamental value traders relative demands must increase. In this way agents bid up the price of the risky asset. Assuming all other factors are held constant then a negative value of \( \chi \) in (3.6) and (3.7) clearly increases traders relative demands and thus actual prices would on average be closer to fundamental value. The unconditional MW can be written in a form analogous to (3.22)
\[
0 < \frac{\mu_0^*}{(1+r)} < \frac{2\mu_0^2\sigma_\rho^2}{(1+r)^3}
\]

Inspection of the two expressions makes clear that the optimal tax level results in noise traders unconditional expected excess relative return being negative.

In summary it has been shown that introducing a trading tax into a market with noise traders has no effect on the variance of prices, reduces the average level of prices and has no discernible effect on the relative returns of the two types. If the tax does not succeed in making a noise trading strategy less attractive then it is difficult to believe that it will curb such activities.

### 3.3 AR(1) Misperceptions

Following the analysis in chapter 2 we now analyse tax effects when the misperceptions of noise traders is determined as a first order autoregressive process. A large and ever increasing literature has documented the mean reversion of aggregate equity prices and shown that prices exhibit autoregressive patterns, (see inter alia, Poterba and Summers (1988), Fama and French (1988), and the excellent survey by Leroy (1989)).

The structural pricing equation is unchanged and given by

\[
p_t = \frac{1}{1+r} [r_+ \rho_{t+1} - \chi (1+r) + \mu_0 - 2\gamma_0 \sigma_\rho^2] 
\]

(3.23)

The misperception now follows an autoregressive process with non-zero mean

\[
\rho_t = \alpha \rho_{t-1} + \theta + \eta_t
\]
\[ |\alpha| < 1, \; \eta_t \sim N(0, \sigma^2_\eta), \; \theta \neq 0, \; \rho^* = \frac{\theta}{1-\alpha}. \]

Conjecture a solution of the form

\[ p_t = \phi_0 + \phi_1 \rho_t \]

\[ \Rightarrow p_{t+1} = \phi_0 + \phi_1 \rho_{t+1} = \phi_0 + \phi_1 (\alpha \rho_t + \theta + \eta_{t+1}) \]

\[ \Rightarrow p_{t+1} = \phi_0 + \phi_1 (\alpha \rho_t + \theta) \]

If the conjectured solution is correct then the following must hold

\[ \phi_0 + \phi_1 \rho_t = \frac{1}{1+r} \left[ r + \phi_0 + \phi_1 \alpha \rho_t + \phi_1^2 \rho - \chi(1+r) + \mu \rho_t - 2 \gamma \sigma^2_{\eta_{t+1}} \right] \]

The coefficient values are thus given by

\[ \phi_0 = (1+r)^{-1} \left( \phi_1 \theta - \chi(1+r) - 2 \gamma \sigma^2_{\eta_{t+1}} \right) \]

\[ \phi_1 = \frac{\mu}{1+r-\alpha} \]

The rational expectations solution is then given by

\[ p^*_t = 1 + \frac{\mu (\rho_t - \rho^*)}{1+r-\alpha} + \frac{\mu \rho^*}{r} - \frac{\chi(1+r)}{r} - \frac{2 \gamma \mu^2 \sigma^2_\rho}{r(1+r-\alpha)^2} \]  \hspace{1cm} (3.24)
The expected price is given by

\[
E(p^*_t) = 1 + \frac{\mu \rho^*}{r} \frac{\chi (1 + r)}{r} - \frac{2\gamma \mu^2 \sigma^2_y}{r(1 + r - \alpha)^2} \\
= 1 + \frac{\mu \theta}{r(1 - \alpha)} \frac{\chi (1 + r)}{r} - \frac{2\gamma \mu^2 \sigma^2_y}{r(1 + r - \alpha)^2}
\]

(3.25)

In DSSW the AR(1) case is again considered without a trading tax and is used to show the increased effect of noise traders on prices, and the increased likelihood of their survival, when their misperceptions follow an autoregressive process. There is a marked increase in the create space effect and the ability of noise traders to deter rational traders, (see chapter 2). The results in DSSW obtain when we set the trading tax equal to zero. With \( \chi = 0 \), we have the following equations for the conditional expected relative excess return of noise traders and the associated MW:

\[
\rho_i < \rho^* < \frac{(1 + r - \alpha)^2}{2\gamma \mu \sigma^2_y} 
\]

(3.26)

\[
0 < \rho_i < \frac{2\gamma \mu \sigma^2_y}{(1 + r - \alpha)^2}
\]

(3.27)

If a trading tax is imposed then we again derive identical expressions. As before agents relative returns are unaffected by the tax and thus noise traders are again not curbed by such a policy.

With a trading tax the expected price of asset \( u \) is given by
\[
E(p_t)x = 1 + \frac{\mu\rho^*}{r} - \frac{\chi(1+r)}{r} - \frac{2\gamma\mu^2\sigma^2_n}{r(1+r-\alpha)^2}
\]

If it is the aim of the regulator to impose the trading tax such that the mean price of the risky asset is equal to its fundamental value, then we can derive the associated value of the trading tax in terms of the parameters of the model.

For \(E(p_t)x = 1\) requires that

\[
\frac{\mu\rho^*}{r} - \frac{\chi(1+r)}{r} - \frac{2\gamma\mu^2\sigma^2_n}{r(1+r-\alpha)^2} = 0
\]

Denoting the required trading tax as \(\hat{\chi}\), we have the following result

\[
\text{sign}(\hat{\chi}) = \text{sign}\left(\frac{\mu\rho^*}{r} - \frac{2\gamma\mu\sigma^2_n}{(1+r)(1+r-\alpha)^2}\right)
\]

\[
\Rightarrow \hat{\chi} > 0 \text{ if } \frac{\mu\rho^*}{(1+r)} > \frac{2\gamma\mu\sigma^2_n}{(1+r)(1+r-\alpha)^2}, \quad \hat{\chi} < 0 \text{ if } \frac{\mu\rho^*}{(1+r)} < \frac{2\gamma\mu\sigma^2_n}{(1+r)(1+r-\alpha)^2}
\]

\[
\Rightarrow \frac{\mu\rho^*}{r} > \frac{2\gamma\mu^2\sigma^2_n}{r(1+r-\alpha)^2}, \quad \frac{\mu\rho^*}{r} < \frac{2\gamma\mu^2\sigma^2_n}{r(1+r-\alpha)^2}
\]

In the final form above the sign of the trading tax is determined by the relative sizes of the create space effect and the hold more effect. If the hold more effect is greater (less) than the create space effect then \(\hat{\chi}\) is a tax (subsidy). For given mean misperception and if \(\sigma^2_n = \sigma^2_p\), then the create space effect is always greater in the case of AR(1) misperceptions and hence the likelihood of \(\hat{\chi}\) being negative (i.e. a subsidy) is also greater.
3.4 Government Intervention

We have seen in the above that imposing a trading tax in the noise trader model of DSSW is unlikely to deliver the results conjectured by Tobin when he argued for taxes on speculative assets. An alternative policy measure is for the regulator or government body to actively participate in the market for the speculative asset. In this section the government enters the model as a third group and is denoted $g$. The government does not have an informational advantage over the other two groups. The government is endowed with rational expectations and attempts to offset the noise trader group by taking either opposite or identical positions on average. It acts as if it also has a mean misperception that is equal in absolute value to the noise trader misperception, but of the opposite or the same sign.

Such a policy has a clear advantage over alternatives such as defending an exchange rate, or any asset, at some perceived fundamental value. Defence of a publicly known exchange rate target leads to defeat by speculators and eventual revaluation in most cases. It also causes currency reserve losses, deterioration of credibility and expectations of future realignments (so called Peso problems). Furthermore, it is difficult to think of any actual asset for which the fundamental value can be calculated. In what follows I propose a more feasible measure whereby the government offsets speculators in the market by estimating the optimism or pessimism of the speculators. This is clearly far simpler than trying to adhere to a target, which is most likely to be wrong. If the government had unlimited resources with which to pursue an interventionist strategy, then trivially, it could act to set the price equal to its fundamental value and thereby reduce the noise traders "space" to zero. We believe that such analysis is both unrealistic and uninteresting.
It is assumed that although the government takes positions in the market its interest is in containing the effects of noise traders and not maximising utility per se, in the manner of the other two groups. For simplicity it is assumed that all groups are present in equal proportion and that these proportions sum to one

$$\mu^i + \mu^n + \mu^g = 1$$  \hspace{1cm} (3.29)

Realistically the government would not intervene in all periods. Although outwith the scope of the present analysis it is plausible that a credible threat of intervention might deter new traders from entering the market and following noise trader strategies, assuming that such intervention damages noise trader returns.

There is no trading tax in the model and we consider the case where noise traders have i.i.d. misperceptions. The demands of the three types are given by the following equations,

$$\lambda^i_t = \frac{r+\pi_{t+1} - p_t(1+r)}{2\gamma \sigma^2_{\pi_{t+1}}}$$  \hspace{1cm} (3.30)

$$\lambda^n_t = \frac{r+\pi_{t+1} - p_t(1+r) + \rho_t}{2\gamma \sigma^2_{\pi_{t+1}}}$$  \hspace{1cm} (3.31)

$$\lambda^g_t = \frac{r+\pi_{t+1} - p_t(1+r) + \kappa}{2\gamma \sigma^2_{\pi_{t+1}}}$$  \hspace{1cm} (3.32)
Where \( \kappa \) is the misperception of the government\(^1\), and is initially defined \( \kappa = -\rho^* \).

The market clearing condition is changed to

\[
\mu^i \lambda^i + \mu^n \lambda^n + \mu^x \lambda^x = 1
\]

(3.33)

Solving for the rational expectations price of asset \( u \) gives

\[
p_t = 1 + \mu^n (\rho_i - \rho^*) + \frac{\mu^n \rho^*}{r} + \frac{\mu^x \kappa}{r} - \frac{2\gamma \mu^x \sigma^2}{r(1+r)^2}
\]

(3.34)

If the proportions of all types are equal then this reduces to

\[
p_t = 1 + \frac{\mu^n (\rho_i - \rho^*)}{1+r} - \frac{2\gamma \mu^x \sigma^2}{r(1+r)^2}
\]

(3.35)

In this form the price of the risky asset is unaffected by the hold more effect of noise traders. The per-period misperception effect and the create space effect cause the deviation of price from its fundamental value in this case. Clearly both have a reduced effect on the price of the risky asset given that \( \mu^n < \mu \). To discuss differences in pricing equations would therefore be misleading as the reduced proportion of noise traders leads to trivial comparison. However, the variance of the price of the risky asset is clearly lower in this case.

---

\(^1\) We are assuming that the government chooses its demands to maximise

\[
E(U)_i^x = \lambda^x (r+p_{i+1} - p_i(1+r)+\kappa) - \gamma \sigma^2_w
\]
The expected price is given by

\[ E(p_t) = 1 - \frac{2\gamma \mu^2 \sigma^2_p}{r(1+r)^2} \]  

which once again illustrates that the mean price will only be affected by the create space effect, which acts to drive the price of the risky asset below its fundamental value, and below the mean price in DSSW.

It is more interesting and meaningful to compare traders relative returns in the model. Although we do not specify an evolutionary framework for updating of the proportions the most rational criteria would be past returns. Therefore traders with higher relative returns are more likely to survive through imitation or evolution over time. The returns of the groups are calculated as follows. The expected excess return on the risky asset is given by

\[ R_{e,x} = \left[ r + p_{t+1} - p_t (1+r) \right] = 2\gamma \sigma^2_{p,t+1} - \mu^x \rho_t - \mu^x \kappa \]  

Comparing noise traders and sophisticated investors expected relative returns yields

\[ \rho_t - \frac{(1+r)^2 (\rho_t^2 + \kappa \rho_t)}{2\gamma \mu^x \sigma^2_p} = \rho_t - \frac{(1+r)^2 (\rho_t^2)}{2\gamma \mu^x \sigma^2_p} + \frac{(1+r)^2 \rho_t \rho^*}{2\gamma \mu^x \sigma^2_p} \]  

and that the government has the same degree of risk aversion as the two types of trader. These assumptions are made for tractability and ease of exposition.
\[ MW, \quad 0 < \rho_t < \frac{2\gamma \mu^\ast \sigma^2_{\rho}}{(1+r)^2} + \rho^* \]  

(3.38a)

and the unconditional expectation is

\[ E(\Delta R_{n-i}) = \rho^* - \frac{(1+r)^2\left((\rho^*)^2 + \sigma^2_{\rho} - (\rho^*)^2\right)}{2\gamma \mu^\ast \sigma^2_{\rho}} = \rho^* - \frac{(1+r)^2\sigma^2_{\rho}}{2\gamma \mu^\ast \sigma^2_{\rho}} \]  

(3.39)

The above results show that if the government intervenes, in the manner suggested, then noise traders benefit compared to the no intervention case. Their conditional and unconditional returns are both higher, if the usual assumptions are satisfied, and the MW is correspondingly greater.

The conditional expected excess return of noise traders relative to government is given by

\[ E_t(\Delta R_{n-i}) = \rho_t + \rho^* + \frac{(1+r)^2(\rho^*)^2}{2\gamma \mu^\ast \sigma^2_{\rho}} - \frac{(1+r)^2(\rho^*)^2}{2\gamma \mu^\ast \sigma^2_{\rho}} \]  

(3.40)

The above result is more easily interpreted if we take the unconditional expectation, which is given by

\[ E(\Delta R_{n-i}) = 2\rho^* - \frac{\sigma^2_{\rho}}{2\gamma \mu^\ast} \]  

(3.41)

This will be positive if
\[ \rho^* > \frac{\sigma^2}{4\gamma \mu^*} \]

Which is easily satisfied for admissible parameter values and merely requires that noise traders are sufficiently bearish on average and that the variance of their misperception is not too "large".

Lastly we can compare the relative expected returns of government and rational traders. The conditional expected excess return of government over rational traders is given by

\[
\Delta R_{g-i} = \frac{(1+r)^2 \rho^* \rho_i}{2\gamma \mu^* \sigma^2} - \frac{(1+r)^2 (\rho^*)^2}{2\gamma \mu^* \sigma^2} - \rho^* \tag{3.42}
\]

For given parameters of the model this is likely to be negative. For the expression to be positive requires

\[ \rho_i > \frac{2\gamma \mu^* \sigma^2}{(1+r)^2} + 1 \]

Which is unlikely to be satisfied. The expression states that rational traders will have higher expected returns than government, unless the noise trader misperception is greater than the fundamental value of the asset.

Thus government intervention to offset the mean effect of noise traders will result in noise traders having higher expected returns and the mean price of the risky asset being driven further away from its fundamental value. The intuition behind this is
that the government intervention drives the price of the risky asset further below its fundamental value. As has already been shown this type of intervention will tend to raise noise traders returns. Noise traders purchase more of the asset than any other group in this case, thereby accepting more price risk, and benefit more from the reduction in the expected price.

An alternative intervention policy would be to “fight fire with fire” and have the government intervene in the same direction as noise traders mean misperception. If we now assume \( \kappa = \rho^* > 0 \), then we would expect the mean price of the risky asset to be higher and therefore the excess return of noise traders to be lower. If we make this change to the direction of government intervention then we have the following results. The mean price of asset \( u \) is altered from (3.36) and is now given by

\[
E(p_t) = 1 + \frac{2\mu^n \rho^*}{r} - \frac{2\gamma \mu^n^2 \sigma^n^2}{r(1+r)^2}
\]  

As expected this form of intervention increases the mean price of the risky asset above the mean in the DSSW case and thus expected price will be closer to fundamental value. The conditional expected excess return of noise traders relative to rational traders and correspondingly the MW fall compared with the case of no intervention\(^2\)

\(^2\) The reduction in the MW of noise traders is increasing in \( \kappa \). It is plausible that the government should then intervene more aggressively. However, if we consider both the expected excess return of noise traders over government, and also government over rational traders we see that the situation is more complex. See p. 18.
$$i(\Delta R_{n,i}) = \rho_i - \frac{(1+r)^2(\rho_i^2)}{2\mu'\sigma^2} - \frac{(1+r)^2\rho_i^*}{2\mu'\sigma^2} \quad (3.44)$$

$$\text{MW} \quad 0 < \rho_i < \frac{2\mu'\sigma^2}{(1+r)^2} - \rho^* \quad (3.45)$$

Comparing (3.45) with (3.38a) it is clear that the MW is smaller, in the case of \( \kappa = +\rho^* \), by the amount \( 2\rho^* \). The intuition for this is simply that government intervention of this type means that the price of the risky asset will be higher on average and the probability of price being below fundamental value is correspondingly lower. Recall that one of the requirements for noise traders to have higher relative returns was that current price had to be below fundamental value. This type of intervention means that noise traders earn a smaller share of the rewards to risk bearing simply because the government is now bearing some of this risk. The above result seems strange and is certainly at odds with real-world government intervention in foreign exchange markets that nearly always tries to buck the current trend rather than accentuate it. Interestingly the result that intervention should follow the noise trader “bullishness” is also at odds with an argument of DSSW (1987) p.25, (perhaps I misinterpret them)

"The story told here provides a rationale that might justify government intervention in foreign exchange markets. Government actions to offset noise and reduce noise trader created risk can raise social welfare."
The expected excess relative returns of noise traders over government and government over rational traders are given by,

\[ t\left(R_{n-g}\right) = \rho_t - \rho^* + \frac{(1+r)^2\left((\rho^*)^2 - (\rho^t)^2\right)}{2\gamma\mu^2\sigma^2_p} \]  

(3.46)

\[ t\left(R_{g-i}\right) = \rho^* - \frac{(1+r)^2\left(\rho_t\rho^* + (\rho_t^2)\right)}{2\gamma\mu^2\sigma^2_p} \]  

(3.47)

The expected returns of noise traders in (3.46) are lower than in (3.40) for all \( \kappa \geq \rho^* > 0 \).

From (3.47) we see that if (3.45) does not hold then rational traders have higher expected relative returns when compared with noise traders and government. Thus the smaller MW of noise traders will benefit rational traders relative to both noise traders and government in this case. Intervention in the direction of noise traders average misperception thus reduces the returns to following a noise trader strategy while increasing the returns to following a rational strategy.
Summary

The results above clearly show that if the government wants to “offset noise” then it should “fight fire with fire” and not try to “buck the trend”. Also note that this policy does not require a fixed fundamental value or target exchange rate to be used by the government. The policy's strength therefore lies in its simplicity. It clearly damages the expected returns of speculators relative to the DSSW no intervention case. In a model of imitation based on recent returns then the noise trader share would clearly fall in this case. This specification also raises the expected value of the risky asset and brings it closer to its fundamental value. Although outwith the scope of the model the threat of such action by the government could conceivably cause noise traders to temper their actions. A corollary is that the government would not need to intervene in all periods. By following such a strategy the government would reduce the expected returns of noise traders and therefore the persistence of speculative strategies in such situations.

3 This suggests that “fire with fire” might be a profitable trading rule for agents in the market. However, consider the average strategy choice of a new agent entering the market. If the government is following a “fire with fire” policy then new agents would choose to be rational on average.
3.5 Extending the horizon

This section analyses a multi-period model with trading taxes to see if a tax is successful in deterring short-term speculation when agents have the opportunity to retrade in the risky asset. In a two period model it is not possible for agents to have long horizons. It may be argued that the earlier two period model gave unduly harsh results for the introduction of a Tobin tax. Eichengreen, Tobin and Wyplosz (1995) write,

"The hope that transactions taxes will diminish excess volatility depends on the likelihood that Keynes speculators have shorter horizons and holding periods than market participants engaged in long-term foreign investment and otherwise oriented towards fundamentals. If so it is speculators who are more deterred by the tax."

In DSSW (1987) we find a more measured view,

"It is nevertheless unclear that increasing the difficulty of transactions by imposing transactions taxes, and thus removing from the market those with short horizons is a good idea. Transactions taxes do penalise those with short horizons. But such taxes also reduce the liquidity of each individuals investment. There are two wedges between the market price of capital goods and the fundamental value of their quasi-rents: first, capital sells at a discount because it is subject to noise trader generated price risk; second capital sells at a discount because it is not as liquid as cash. It is not clear whether transactions push q towards its fundamental value, for they would tend to reduce the first wedge and increase the second, as Keynes (1936) noted."

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In what follows a tax is imposed in multi-period settings. This is the only difference made to the model outlined in section 1. When we extend the overlapping generations model beyond two periods we have to decide on the structure of the overlap. Following chapter 2 we consider two distinct types of overlap. Firstly, we can assume that all agents are allowed to retrade in a middle period but that there are no births or deaths in this middle period. This results in a non-stationary price distribution with \( j-1 \) mean prices, where \( j \) is the length of the horizon. We term this the non-stationary case. Secondly, we can assume that generations overlap in all periods, including retrade periods. In this stationary case the expected price is the same in all periods and we do not get mean switching of expected prices. In both cases the population of agents is identically equal to one in all periods and there are no distortionary population or demographic effects. The stationary structure is used in this section while an example of the non-stationary case with an adjustment tax is given in the Appendix.

DSSW (1989), in a similar type of model, do not analyse transactions taxes or a multiperiod model because they believe their model is unsuitable for such an analysis. One of the key differences with the model presented here is that they endogenise the supply of the risky asset through the introduction of risk neutral entrepreneurs. Furthermore they argue that trading in the model is for life-cycle purposes. This view is eschewed in the present analysis and we think of the model as simply describing a single position (2-period) or a position with rebalancing (3-period).

Kupiec (1995) uses a multi-period framework but makes different assumptions in order to derive his results. In Kupiec's model there are rational traders and noise traders in a four-period overlapping generations model with generations overlapping in all periods.

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\(^4\) DSSW (1987) contains a brief discussion of a three period model in which agents are forbidden from trading in the middle period. In the present paper we are interested in rebalancing by agents in the
To simplify his model he makes the crucial assumption that noise traders are not irrational for life. Specifically, all traders are born rational and a proportion mutates into noise traders in the first retrade and then mutate back in the second retrade. This assumption drives the results and is difficult to rationalise otherwise. The exogenous mutation causes excess rebalancing by agents which is interpreted as excess trading volume. Kupiec finds that a securities transaction tax can reduce price volatility and transactions volume, but will increase the volatility of the risky asset’s return. The unconditional volatility of prices decreases with the tax for low rates and then increases again (i.e. parabolic relationship) for higher values of the tax.

In the model presented here agents who are born noise traders do not mutate during their life and stay as noise traders. The proportion of each type remains an exogenous constant in all periods. Furthermore we explicitly analyse the effect of a tax on the relative returns of both types of trader in the model and thus the likelihood of a tax deterring noise trader or rational strategies, or both. The cause of excess volatility in this model is due to noise traders expectational difference of opinion. Unless the prescribed policy reduces the returns to following a misperception strategy then it cannot reduce this noise trader induced excess volatility and an alternative type of sand, such as direct intervention, must be sought. In the three period model which follows transactions taxes do nothing to deter noise trader strategies.

3.5.1 Three Period Model with Stationary Prices

The model presented in this section uses the stationary structure. New agents are born in each period and all agents have a three period horizon. Noise traders are subject to a random misperception in each period, common to all noise traders in that period, and retrade periods and thus ignore this specification of the horizon.
the proportions of agents present is an exogenous constant. In any given period there
will be two generations either forming or rebalancing their portfolio and a third
generation liquidating their portfolio. In keeping with the previous analysis we assume
that agents demands are positive in each period to make the analysis as tractable as
possible.

Terminal wealth in this three period model is given by

\[ w_3 = L(1+r)^2 + (1+r)\lambda_t (r + p_{t+1} - p_t(1+r) - \chi(1+r)) + \alpha_{t-1} \left( r + p_{t+2} - p_{t+1}(1+r) - \chi(1+r) \right) \]

(3.46)

To simplify the presentation we use the notation from chapter 2, define

\[ R_{t+1} = r + p_{t+1} - p_t(1+r) - \chi(1+r) \]
\[ \Omega_{t+2}^i = \lambda_t (R_{t+2}) \]
\[ \Omega_{t+2}^n = \lambda_t (R_{t+2} + \rho_{t+1}) \]

This allows us to write agents expected utility, more succinctly, as

\[ E(U)^i = L(1+r)^2 + (1+r)\lambda_t (R_{t+1}) + \Omega_{t+2}^i - \gamma \sigma_{w_3}^2 \]
\[ E(U)^n = L(1+r)^2 + (1+r)\lambda_t (R_{t+1} + \rho_t) + \Omega_{t+2}^n - \gamma \sigma_{w_3}^2 \]
\[ E(U)^{i+1} = L(1+r)^2 + (1+r)\lambda_t (R_{t+2}) + \Omega_{t+3}^{i+1} - \gamma \sigma_{w_3}^2 \]
\[ E(U)^{n+1} = L(1+r)^2 + (1+r)\lambda_t (R_{t+2} + \rho_{t+1}) + \Omega_{t+3}^{n+1} - \gamma \sigma_{w_3}^2 \]

5 If noise traders and rational traders had different horizons then it is possible that a tax would have
some effect. However, this temporal heterogeneity would have to be endogenous and not simply
assumed.
The first two expressions show the expected utility of rational traders and noise traders born in period \( t \). The final two expressions show the expected utility of rational traders and noise traders born in period \( t+1 \). The final term in each expression is the coefficient of absolute risk aversion times the expected variance of wealth. This expression is constant over time for a given type. The variance of wealth for agents born in period \( t \) is given by,

\[
\left( \sigma_{w,t}^2 \right)^b = (1 + r)^2 \left( \lambda_t^b \right)^2 \text{var}(R_{t+1}) + \text{var}\left( \Omega_{t+2}^b \right) + 2(1+r) \lambda_t^b \text{cov}(R_{t+1}, \Omega_{t+2}^b) \quad (3.47)
\]

\[
\left( \sigma_{w,t}^2 \right)^n = (1 + r)^2 \left( \lambda_t^n \right)^2 \text{var}(R_{t+1}) + \text{var}\left( \Omega_{t+2}^n \right) + 2(1+r) \lambda_t^n \text{cov}(R_{t+1}, \Omega_{t+2}^n) \quad (3.48)
\]

The analogous expression for period \( t \) young is easily found by leading (3.47) and (3.48) by one period. To solve for price in this case we must first obtain the demands of the period \( t \) young for period \( t+1 \), and the demands of the period \( t+1 \) young for period \( t+1 \). The retrade demands of agents born in period \( t \) are given by

\[
\lambda_{t+1}^b = \frac{r_{t+1}\mathcal{R}_{t+2}}{2\gamma \text{var}_{t+1}(\mathcal{R}_{t+2})} \quad (3.49)
\]

\[
\lambda_{t+1}^n = \frac{r_{t+1}\mathcal{R}_{t+2} + r_{t+1}}{2\gamma \text{var}_{t+1}(\mathcal{R}_{t+2})} \quad (3.50)
\]

the initial demands of agents born in \( t+1 \) are given by
\[
\lambda_{t+1}^{m+1} = \frac{t_{t+1}R_{t+2} - 2\gamma \text{cov}(R_{t+2}, \Omega_{t+3})}{2\gamma(1+r)\text{var}_{t+1}(R_{t+2})}
\]

\[
\lambda_{t+1}^{n+1} = \frac{t_{t+1}R_{t+2} + \rho_{t+1} - 2\gamma \text{cov}(R_{t+2}, \Omega_{t+3}^n)}{2\gamma(1+r)\text{var}_{t+1}(R_{t+2})}
\]

The market clearing condition is given by\(^6\)

\[
(1-\mu)(\lambda_{t+1}^{l} + \lambda_{t+1}^{m}) + \mu(\lambda_{t+1}^{n} + \lambda_{t+1}^{n+1}) = 2
\]

Inserting agents demands into the market clearing condition and solving for price yields a rational expectations price solution of

\[
p_{t+1} = 1 + \frac{\mu(\rho_{t+1} - \rho^*)}{1+r} + \frac{\mu\rho^*}{r} - \frac{\chi(1+r)}{r} \frac{2\gamma\mu^2\sigma^2}{r(1+r)(2+r)}
\]

The tax thus decreases the expected price of the risky asset. The tax has no effect on the per-period misperception effect, the hold more effect or the create space effect. The variance of price remains constant. Although the expected price is closer to the fundamental value, compared to the two period case in 3.2, this is due to horizon effects and not the tax. The horizon effect is a composite of a dividend effect and a choice premium. As the horizon increases the dividend component in total returns increases relative to the capital gain component. Agents with a long horizon gain from

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\(^6\) This altered specification of the market clearing condition means that there are no demographic or population effects distorting the analysis, and allows comparison with the two period results presented in Section F.
the compounding of their dividends over the horizon. This insurance from dividends is termed the dividend effect. In addition agents have more opportunities to liquidate their position but face a constant level of price risk. This spreading of risk over more than one period gives rise to a choice premium.

To analyse the effect of the transactions tax on agents relative returns we solve for agents born in period \( t \). The conditional excess return of noise traders born in period \( t \) is

\[
\Delta R_{t+1} = \frac{(1+r)\rho_{t+1}}{2+r} - \frac{(1+r)^2(\rho_{t+1})^2}{2\gamma\mu\sigma_p^2}
\]

This expression is positive if the misperception is positive, but even in a multi-period setting the tax has no effect on traders relative expected returns. The MW in this case, although reduced relative to a two period model, is unaffected by the tax as we would expect. The reduction is due to the previously described horizon effects.

\[
0 < \rho_{t+1} < \frac{2\gamma\mu\sigma_p^2}{(2+r)(1+r)}
\]

### 3.6 Non-linear taxes

In the preceding sections agents trading tax liability was a simple linear function of their respective demands. In order to solve the model we had to assume that demands were always positive in each period. This is only valid if noise traders have negative misperceptions or sufficiently “small” positive misperceptions. In what follows this assumption is unnecessary as we specify the tax liability as being a function of agents squared demands. In this section we consider a two period model using this non-linear
specification and simple iid noise trader misperceptions. This avoids any objection to assuming positive demands in all periods, penalises agents with relatively large demands and gives less neutral results than the linear tax case.

With our new specification of the taxing scheme agents have terminal wealth given by

$$w_2 = L(1+r) + \lambda_t (r + p_{t+1} - p_t (1+r)) - (\lambda_t)^2 \chi(1+r)$$  \hspace{1cm} (3.57)

The change does not affect the variance of terminal wealth, but changes the denominator in agents respective demand functions:

$$\lambda'_{t} = \frac{r + p_{t+1} - p_t (1+r)}{2\chi(1+r) + 2\gamma \sigma^2_{\rho_{t+1}}} = 1 - \frac{\mu \rho_t}{\Lambda}$$  \hspace{1cm} (3.58)

$$\lambda''_{t} = \frac{r + p_{t+1} - p_t (1+r) + \rho_t}{2\chi(1+r) + 2\gamma \sigma^2_{\rho_{t+1}}} = 1 + \frac{(1-\mu) \rho_t}{\Lambda}$$  \hspace{1cm} (3.59)

where,

$$\Lambda = 2\gamma \sigma^2_{\rho_{t+1}} + 2\chi(1+r)$$

Given the increase in the denominator, we find that that relative to DSSW and the linear tax case, the variance of all types demands are reduced. A non-linear tax does has an effect on agents demand behaviour unlike the linear tax case. A non-linear tax penalises more heavily those agents with higher demands and therefore both agents smooth their demands. We cannot make any statements about the effect on trading level volumes given the nature of the market clearing condition, but if agents demands
are less variable than we might expect trading volume to be less variable. A reduction in demand volatility could be a policy objective if demand volatility is thought to be linked to price volatility.

Imposing market clearing and solving for price yields,

$$p_t = \frac{1}{1 + r} \left[ r + p_{t+1} + \mu \rho_t - 2 \chi (1 + r) - 2 \gamma \sigma^2_{p_{t+1}} \right]$$

Solving for the rational expectations equilibrium price of the risky asset yields,

$$p_t = 1 + \frac{\mu (\rho_t - \rho^*)}{1 + r} + \frac{\mu \rho^*}{r} \frac{2 \chi (1 + r)}{r} \frac{2 \gamma \mu^2 \sigma^2_p}{r(1 + r)^2}$$

The only difference is that the tax has a slightly increased effect, by a factor of two, in decreasing the expected price of the risky asset. Trivially, the value of the tax which causes expected price to equal fundamental value will be lower, but for admissible parameter values this optimal tax will still be negative. We note that the decrease in demand volatility does not translate into a decrease in price volatility. The conditional one step ahead variance of the price of asset $u$ is still given by,

$$\sigma^2_{p_{t+1}} = \frac{\mu^2 \sigma^2_p}{(1 + r)^2}$$

We now analyse the effect of a non-linear tax on the expected relative returns of the two types. We want to multiply the difference in the agents' holdings of asset $u$ and the
expected excess return of asset u over asset s. The expected excess relative return of noise traders is given by,

\[
\Delta(R^u-t) = \lambda^u_t \left( r + p_{t+1} - p_t(1+r) - \lambda^s_t \right) \chi(1+r) - \lambda^l_t \left( r + p_{t+1} - p_t(1+r) - \lambda^l_t \right) \chi(1+r)
\]

\[
= \left( \lambda^u_t - \lambda^l_t \right) \left( r + p_{t+1} - p_t(1+r) \right) + \left( \lambda^u_t - \lambda^l_t \right)^2 \chi(1+r)
\]

(3.62)

To solve (3.62) we make use of the following

\[
\left( \lambda^u_t - \lambda^l_t \right) = \frac{\rho_t}{2 \chi(1+r) + 2 \gamma \sigma^2_{p_t}} = \frac{\rho_t}{\Lambda}
\]

\[r + p_{t+1} - p_t(1+r) = \Lambda - \mu \rho_t\]

\[
\left( \lambda^u_t \right)^2 - \left( \lambda^l_t \right)^2 = \left( 1 - \frac{\mu \rho_t}{\Lambda} \right)^2 - \left( 1 + \frac{(1-\mu) \rho_t}{\Lambda} \right)^2
\]

\[= \frac{2 \rho_t}{\Lambda} - \frac{(1-2\mu) \rho_t^2}{\Lambda^2}\]

Using the above we can rewrite (3.62) as

\[
\Delta(R^u-t) = \frac{\rho_t}{\Lambda} (\Lambda - \mu \rho_t) + \left( \frac{2 \rho_t}{\Lambda} - \frac{(1-2\mu) \rho_t^2}{\Lambda^2} \right) \chi(1+r)
\]

(3.63)
If we assume that the proportions of the two types are equal then we can simplify this further to

\[
\Delta(R^{n+1}) = \rho_t - \frac{\mu \rho_t^2}{\Lambda} + \frac{2\rho_t \chi(1+r)}{\Lambda}
\]  

(3.64)

Using (3.64) rather than (3.63) makes the analysis easier. The final term in (3.63) is of the same sign as \(\text{sgn}[\mu - (1 - \mu)]\). Given the likely magnitude of the denominator relative to the numerator, in this term, the effect will be very small in any case. In the linear tax case (and in DSSW) the analogous term was given by

\[
\Delta(R^{n+1}) = \rho_t - \frac{\mu \rho_t^2}{2\gamma \sigma^2_{p_{21}}}
\]  

(3.65)

Solving (3.64) explicitly is messy and unnecessary given that the MW is directly related. The MW of noise traders is given by

\[
\text{MW: } 0 < \rho_t < \frac{2\gamma \mu \sigma^2_{\nu}}{(1+r)^2} + \frac{4\chi(1+r)}{\mu}
\]  

(3.66)

The analogous MW for both the linear tax case and the DSSW case is found by deleting the last term on the right hand side of (3.66). Clearly the non-linear tax increases the MW of noise traders (the range in state space in which they outperform the rational traders in terms of expected return). Comparing (3.64) with (3.65) it can be shown that that the non-linear pricing scheme gives noise traders higher expected relative returns than in the linear tax case for most values of their misperception i.e.
As an example, if the proportion of traders is equal, then noise traders have higher expected returns in the non-linear tax case provided their misperception is greater than minus two. Given that the fundamental value is only equal to one then (3.64) will be greater than (3.65) in nearly all realisations.

In summary, a non-linear tax scheme has the effect of decreasing expected price further below fundamental value, has no effect on the variance of prices, reduces the variance of demands, and if noise traders misperception is positive, will benefit noise traders relative to rational traders (i.e. noise traders will have higher expected relative returns as evidenced by the increase in their MW) when compared with the linear tax results or the no tax results (DSSW).

3.7 Conclusion
In this chapter we analysed the effects of imposing linear and non-linear trading taxes, and direct government intervention in an asset market composed of rational traders and speculative traders. Where noise traders were categorised as the destabilising speculators whose activities should be curbed by such a tax. It was found that a linear tax caused expected prices to decrease and move away from fundamental value rather than towards it. The expected relative returns of noise traders and rational traders was unaffected when a tax was imposed and thus noise traders are just as likely to survive and win converts in such a situation. These results held for both two and three period
models. If the government wishes price to be closer to fundamental value then it has to use sand in the form of a subsidy and not a tax. These results held for both i.i.d. innovations in prices, and prices that followed a predictable autoregressive process, although the results were stronger in the latter case. In the case of a non-linear tax the results were less neutral. The demands of all agents were less variable, expected price was further below fundamental price and the variance of price was unaffected. Furthermore, if noise traders misperception was positive then their expected excess relative return was increased by the non-linear tax.

As an alternative two forms of government intervention were proposed. This intervention did not require calculation and transmission of a target value for the risky asset but merely required estimation of the “mood” of the market. This in itself is a clear advantage over policies that advocate setting a target rate, which is likely to vary over time and be impossible to calculate anyway. A policy of offsetting this “mood” was found to benefit noise traders and to drive expected price further away from fundamental value. By contrast a policy that follows the “mood”, or “fights fire with fire”, is the most successful for bringing price in line with fundamental value and damaging the returns of noise traders. The results clearly advise caution over the implementation of a trading tax on risky assets. In a market composed of rational and speculative traders the results point to the use of direct intervention to meet government objectives regarding asset market behaviour as opposed to transactions taxes.
APPENDIX

A3.1 Non-stationary Prices

All agents are assumed to live for three periods. There are no births or deaths in the middle period and the generations only overlap in the initial and terminal periods of their horizon. Rather than repeat the analysis in section 4 we look at the imposition of an adjustment or rebalancing tax.

Two possible forms for an adjustment tax are

\[ \chi \left( \lambda_{t+1} - \lambda_t \right) \]  \hspace{2cm} (A3.1)

and

\[ \chi \left( \lambda_{t+1} - \lambda_t \right)^2 \]  \hspace{2cm} (A3.2)

Where \( \chi \) is the tax, \( \chi > 0 \), and \( \lambda_{t+1} \) and \( \lambda_t \) are retrade and first period holdings of the risky asset respectively. The tax therefore penalises rebalancing by agents of their portfolios in the middle period. The tax is paid in the terminal period when agents liquidate their portfolios and consume all of their terminal wealth. There is no consumption in the retrade period and noise traders expectations are subject to a random misperception in the first period and the retrade period.

For simplicity we analyse the first description of the taxing scheme and leave the quadratic case till later. Terminal wealth, denoted \( w_3 \), is given by the following

\[ w_3 = L(1+r)^2 + (1+r)\lambda_t \left( r + p_{t+1} - p_t(1+r) \right) + \lambda_{t+1} \left( r + p_{t+2} - p_{t+1}(1+r) \right) - \chi \left( \lambda_{t+1} - \lambda_t \right) \]  \hspace{2cm} (A3.3)

Again we assume that all agents act to maximise their utility of terminal wealth. For ease of exposition we use the notation,

\[ R_{t+1} = r + p_{t+1} - p_t(1+r) \]
\[
\Omega_{t+2}^r = \lambda_{t+1}^r \left( r + p_{t+2} - p_{t+1}(1+r) \right)
\]
\[
\Omega_{t+2}^n = \lambda_{t+1}^n \left( r + p_{t+2} - p_{t+1}(1+r) + p_{t+1} \right)
\]

This allows us to write the expected utility of both types more succinctly as

\[
E(U)^r = L(1+r)^2 + (1+r)\Omega_{t+1}^r + \Omega_{t+1}^r - \chi \left( \lambda_{t+1}^r - \lambda_{t+1} \right) - \gamma \sigma_{w_2}^2
\]

for rational traders, and

\[
E(U)^n = L(1+r)^2 + (1+r)\Omega_{t+1}^n + \Omega_{t+1}^n - \chi \left( \lambda_{t+1}^n - \lambda_{t+1} \right) - \gamma \sigma_{w_2}^2
\]

for noise traders. Where \( \gamma \) is the coefficient of absolute risk aversion and \( \sigma_{w_2}^2 \) is the expected variance of terminal wealth conditioned on the first period information set and given by

\[
\begin{align*}
\left( \sigma_{w_2}^2 \right)^r &= (1+r)^2 \left( \lambda_{t+1}^r \right)^2 \text{var}(R_{t+1}) + \text{var}(\Omega_{t+2}^r) + \chi^2 \text{var}(\lambda_{t+1}^r) \\
&+ 2(1+r)\lambda_{t+1}^r \text{cov}(R_{t+1}, \Omega_{t+2}^r) - 2(1+r)\lambda_{t+1}^r \chi \text{cov}(R_{t+1}, \lambda_{t+1}^r) - 2\chi \text{cov}(\Omega_{t+2}^r, \lambda_{t+1}^r)
\end{align*}
\]

(A3.6)

for sophisticated investors, and

\[
\begin{align*}
\left( \sigma_{w_2}^2 \right)^n &= (1+r)^2 \left( \lambda_{t+1}^n \right)^2 \text{var}(R_{t+1}) + \text{var}(\Omega_{t+2}^n) + \chi^2 \text{var}(\lambda_{t+1}^n) \\
&+ 2(1+r)\lambda_{t+1}^n \text{cov}(R_{t+1}, \Omega_{t+2}^n) - 2(1+r)\lambda_{t+1}^n \chi \text{cov}(R_{t+1}, \lambda_{t+1}^n) - 2\chi \text{cov}(\Omega_{t+2}^n, \lambda_{t+1}^n)
\end{align*}
\]

(A3.7)

for noise traders. The solution procedure is to solve for the retrade period and then insert expected retrade demands into the first period maximisation problem. Solving (A3.4) and (A3.5) for retrade demands yields,

\[
\lambda_{t+1}^r = \frac{t+1 R_{t+2} - \chi}{2 \gamma \text{var}_{t+1}(R_{t+2})}
\]

(A3.8)

\[
\lambda_{t+1}^n = \frac{t+1 R_{t+2} + p_{t+1} - \chi}{2 \gamma \text{var}_{t+1}(R_{t+2})}
\]

(A3.9)

Imposing market clearing and solving for \( p_{t+1} \) gives the structural form of the pricing equation,
\[ p_{t+1} = \frac{1}{1+r} \left[ r_{t+1} p_{t+2} + \mu \rho_{t+1} - 2\gamma \text{var}_{t+1}(R_{t+2}) - \chi \right] \]  
(A3.10)

If we insert (A3.10) into (A3.8) and (A3.9) then we find that trading taxes and prices do not enter into the demand equations,

\[ \hat{\lambda}^*_t = 1 - \frac{(1+r)^2 \mu \rho_{t+1}}{2\gamma \mu^2 \sigma_p^2} \]  
(A3.11)

\[ \hat{\lambda}^n_t = 1 + \frac{(1+r)^2 (1-\mu) \rho_{t+1}}{2\gamma \mu^2 \sigma_p^2} \]  
(A3.12)

From (A3.11) and (A3.12) the demands of noise traders will be higher than those of rational traders when their misperception is positive, and lower if their misperception is negative.

Differentiating (A3.4) and (A3.5) for first period holdings yields

\[ \hat{\lambda}'_t = \frac{(1+r)(t R_{t+1}) + \chi}{2\gamma (1+r)^2 \text{var}(R_{t+1})} - \frac{\text{cov}_t(R_{t+1}, \Omega_{t+2})}{(1+r) \text{var}(R_{t+1})} \]  
(A3.13)

\[ \hat{\lambda}''_t = \frac{(1+r)(t R_{t+1}) + \chi}{2\gamma (1+r)^2 \text{var}(R_{t+1})} - \frac{\text{cov}_t(R_{t+1}, \Omega_{t+2}^n)}{(1+r) \text{var}(R_{t+1})} \]  
(A3.14)

Imposing first period market clearing, simplifying, and then solving for the structural pricing equation we have,

\[ \frac{(1+r)(t R_{t+1}) + \chi}{2\gamma (1+r)^2 \text{var}(R_{t+1})} - \frac{\text{cov}_t(R_{t+1}, R_{t+2})}{(1+r) \text{var}(R_{t+1})} = 1 \]  
(A3.15)
\[ p_t = \frac{1}{1+r} \left[ r + p_{t+1} + \chi - 2\gamma \text{cov}_t(R_{t+1}, R_{t+2}) - 2\gamma(1+r)\text{var}_t(R_{t+1}) \right] \quad \text{(A3.16)} \]

The conditional variance and covariance are given by.

\[ \text{var}_t(R_{t+1}) = \text{var}_{t+1}(R_{t+2}) = \frac{\mu^2 \sigma_p^2}{(1+r)^2} \]

\[ \text{cov}_t(R_{t+1}, R_{t+2}) = -\frac{\mu^2 \sigma_p^2}{1+r} \]

To solve for the rational expectations equilibrium combine (A3.10) and (A3.16) to produce a single price equation

\[ p_t = \frac{1}{1+r} \left[ r + p_{t+1} + \mu \rho_t + k_1 + \delta_t k_2 \right] \quad \text{(A3.17)} \]

where \( k_1 = -2\gamma(1+r)\text{var}_t(R_{t+1}) - 2\gamma \text{cov}_t(R_{t+1}, R_{t+2}) + \chi = \chi \)

\[ k_2 = -k_1 - 2\gamma \text{var}_t(R_{t+2}) - \chi \]

\[ \delta_t = \begin{cases} 1 & \text{retrade} \\ 0 & \text{birth/death} \end{cases} \]

The solution to (A3.15) is found by undetermined coefficients. The conjectured solution is

\[ p_t = \phi_{t_0} + \phi_1 \rho_t + \phi_2 \delta_t + \phi_3 \delta_{t-1} \quad \text{(A3.18)} \]
which implies that

\[ tP_{t+1} = \phi_0 + \phi_1 \rho^* + \phi_2 \delta_{t+1} + \phi_3 \delta_{t+2} \]

If (A3.18) is correct then the following equality must hold

\[ \phi_0 + \phi_1 \rho + \phi_2 \delta + \phi_3 \delta_{t+1} = \frac{1}{1+r} \left( r + k_1 + \delta_1 k_2 + \phi_0 + \phi_1 \rho^* + \phi_2 \delta_{t+1} + \phi_3 \delta_1 + \mu \rho_1 \right) \]

which gives the following values for the \( \phi \) terms,

\[ \phi_0 = 1 + \frac{\mu \rho^*}{r(1+r)} + \frac{k_1}{r} \], \quad \phi_1 = \frac{\mu}{1+r}, \quad \phi_2 = \frac{(1+r)k_2}{(1+r)^2 - 1}, \]

\[ \phi_3 = \frac{k_2}{(1+r)^2 - 1} \]

Substitution of these coefficient values into (A3.18) yields,

\[ p_t = 1 + \frac{\mu(\rho_t - \rho^*)}{1+r} + \frac{\mu \rho^*}{r} + \frac{k_1}{r} + \frac{\delta_1(1+r)k_2}{(1+r)^2 - 1} + \frac{\delta_{t+1}k_2}{(1+r)^2 - 1} \]  \hspace{1cm} (A3.19)

The specific retrade and birth/death prices are as follows,
\[ p_{t=b/d} = 1 + \frac{\mu (p_i - p^*)}{1+r} + \frac{\mu p^*}{r} + \frac{k_1}{r} + \frac{k_2}{(1+r)^2 - 1} \]

\[ = 1 + \frac{\mu (p_i - p^*)}{1+r} + \frac{\mu p^*}{r} + \frac{\chi}{2+r} - \frac{2\gamma \mu^2 \sigma_p^2}{r(1+r)^2(2+r)} \]  

(A3.20)

\[ p_{t=r} = 1 + \frac{\mu (p_i - p^*)}{1+r} + \frac{\mu p^*}{r} + \frac{k_1}{r} + \frac{(1+r)k_2}{(1+r)^2 - 1} \]

\[ = 1 + \frac{\mu (p_i - p^*)}{1+r} + \frac{\mu p^*}{r} + \frac{\chi}{2+r} - \frac{2\gamma (1+r)\mu^2 \sigma_p^2}{r(1+r)^2(2+r)} \]  

(A3.21)

From (A3.20) and (A3.21) it is clear that the adjustment tax increases the expected price in the initial period and decreases the expected price in the retrade period. The tax therefore amplifies the mean-switching effect which gives rise to the non-stationary prices.

If the government wishes to set a tax such that the mean price of the risky asset is equal to its fundamental value of one then this optimal value, denoted \( \hat{\chi} \), is given by

\[ \hat{\chi}_{t=b/d} = \frac{2\gamma \mu^2 \sigma_p^2}{r(1+r)^2} - \frac{(2+r)\mu p^*}{r} \]  

(A3.22)

\[ \hat{\chi}_{t=r} = \frac{(2+r)\mu p^*}{r} - \frac{2\gamma \mu^2 \sigma_p^2}{r(1+r)^2} \]  

(A3.23)
Inspection shows that the optimal tax levels in the birth/death and retrade periods are equal in absolute terms but have opposite signs. For admissible parameter values the birth/death value is negative while the retrade value is positive\(^1\).

In summary, the specification chosen here increases the expected birth/death price, decreases the expected retrade price, has no effect on traders demands and requires astronomical levels of tax/subsidy for the expected value of the risky asset to equal its fundamental value. Although informative, the specification gives disappointing results.

\(^1\) A numerical example illustrates this more clearly and indicates the size of the tax subsidy required. Choose \(\mu=.5, r=.04, \gamma=2, \sigma^2_\rho = .09\), and \(\rho^* = .1\). Then the birth/death period requires a tax of -23.5 and the retrade period a tax of 23.5.
A3.2 Tax financed government intervention

We now present a model involving the use of a mixed sand policy. In the previous analysis we gave no indication as to where the resources for government intervention would come from. In this section we propose that revenue from a transactions tax is used to fund the government position. Although the model remains a partial equilibrium model it is more self-contained and we can avoid possible fiscal implications. In what follows we sketch out the rational expectations version of this mixed sand policy, while in Chapter 5 we present simulation results for this proposal in an artificial economy with learning.

While some advocates of Tobin taxes have cited the revenue generating potential as a valid reason for introducing a tax we are not aware of any previous discussion of a mixed sand policy. The analysis therefore complements the previous material of this chapter and gives a new slant to the Tobin tax debate.

Define $\mu', \mu^n, \mu^g$ as the proportions of rational traders, noise traders, and government respectively. To reduce notation and simplify the analysis we assume that all types are present in equal proportion. As before we assume that all agents have rational expectations, with noise trader expectations being perturbed by their per-period misperception. Denoting government demands for the risky asset by $\lambda^g_t$, the market clearing condition is given by

$$\mu' \lambda^r_t + \mu^n \lambda^n_t + \mu^g \lambda^g_t = 1 \quad (A3.24)$$

We assume that government positions are also subject to the transactions tax. Denoting trading tax revenue in period $t$ by $\Gamma_t$, the government budget constraint is

$$\Gamma_t = \chi(\lambda^r_t + \lambda^n_t + \lambda^g_t) \quad (A3.25)$$
Tax revenue is increasing in agents demands and thus intervention is highest in the periods when rational and/or noise traders are most bullish. This has the welcome feature that government purchases will bid up the price of the risky asset, and thereby damage the relative returns of noise traders when noise traders have high demands. While this will also be the case for bullish rational traders, we recall that the demands of noise traders have the greater variance, and for $\rho^* > 0$ noise traders average demands are higher. Thus on average noise traders will incur higher transactions taxes and contribute more to $\Gamma_t$ than rational traders.

In section 3.2 we noted that it was possible for traders first period demands to be negative, in which case the tax would act as a first period subsidy. Similar reasoning would suggest that trading tax revenue as a function of aggregate demands could be negative. Fortunately this is not the case as the market clearing condition dictates that aggregate demands must be positive in all periods.

From (A3.25) the government's budget constraint at the beginning of period $t$ is

$$\Gamma_t = v^{gs} + \lambda^{gs}_t p_t + \lambda^{gs}_t \chi.$$  

Where we continue to assume that the government will purchase quantities of both the safe and the unsafe asset. The governments expected utility is then given by

$$E(U^g) = \chi (1+r)(\lambda^{gs}_t + \lambda^{gs}_t) + \lambda^{gs}_t (r_t + p_{t+1} + \kappa - p_t (1+r)) - \gamma (\lambda^{gs}_t)^2 \sigma^{2}_p.$$  

The demands of each type are given by

$$\lambda^{gs}_t = \frac{r_t + p_{t+1} - (1+r)(p_t + \chi)}{2\gamma \sigma^{2}_p}. \quad (A3.26)$$  

for rational traders,

---

2 Alternatively we could model the government as only purchasing the unsafe asset. While this might be more realistic we conjecture that the thrust of the results would not be changed. What is important is that the government have demands based on a price expectation rule $p^{gs}_{t+1} = p_{t+1} + \kappa$. 
\[ \lambda^T_t = \frac{r + \rho_{t+1} - (1+r)(p_t + \chi) + \rho_t}{2\gamma \sigma^2_{\rho_{t+1}}} \]  
(A3.27)

for noise traders, and

\[ \lambda^G_t = \frac{r + \rho_{t+1} - (1+r)p_t + \kappa}{2\gamma \sigma^2_{\rho_{t+1}}} \]  
(A3.28)

for government. Inserting these demand equations into the market clearing condition and solving for \( p_t \) yields the structural pricing equation

\[ p_t = \frac{(1+r)}{r} \left( r + \rho_{t+1} + \mu^T \rho_t + \mu^G \kappa - (1 - \mu^G)\chi(1+r) - 2\gamma \sigma^2_{\rho_{t+1}} \right) \]  
(A3.29)

From (A3.29) if \( \kappa > 0 \) \( (< 0) \) then government intervention will offset (amplify) the negative effect of the tax on the expected price of asset \( u \). The previous analysis of government intervention suggested that to deter noise traders required \( \kappa > 0 \). We therefore assume that the government 'misperception' is indeed positive and thus offsets the negative tax effect.

The rational expectations equilibrium pricing function is given by

\[ p_t = 1 + \frac{\mu^U (\rho_t - \rho^*)}{1+r} + \frac{\mu^U \rho^*}{r} - \frac{2\gamma \mu^U \sigma^2_{\rho}}{r(1+r)^2} - \frac{(1 - \mu^G)\chi(1+r)}{r} + \frac{\mu^G \kappa}{r} \]  
(A3.30)

The first four terms on the right hand side of (A3.30) have the usual interpretation. The final terms measure the relative effect of intervention and of the tax, on the price of the risky asset. Given our assumption on \( \kappa \), the terms in \( \kappa \) and \( \chi \) are of opposite sign and thus have an offsetting effect. Taking unconditional expectations of (A3.30) yields

\[ E(p_t) = 1 + \frac{\mu^U \rho^*}{r} - \frac{2\gamma \mu^U \sigma^2_{\rho}}{r(1+r)^2} + \frac{(1 - \mu^G)\chi(1+r)}{r} + \frac{\mu^G \kappa}{r} \]

which will be greater than the no-sand case if the final term on the r.h.s. is positive. Assuming that \( \kappa \) is set equal to \( \rho^* \), and is therefore dependent on noise trader beliefs,
rather than under government control, we solve the condition for the tax rate \( \kappa \), which gives

\[
\kappa < \frac{\mu^s \kappa}{(1 + r)(1 - \mu^s)}.
\]

As an example assume a mean misperception of 0.1, interest rate of 0.06, and the government proportion set to 1/3, then the tax rate must be less than 0.0472. Typically transactions taxes are suggested in the range .5-2%. For a tax of 0.02 the increase in the expected price would be 0.147. Such low taxes are suggested in the literature as it is believed that higher taxes would have a real impact on liquidity, the depth of the market and international trade flows. In the present case such a low tax is a requirement of the model if the policy is to have the desired effect.

Solving for the conditional expected excess relative return of noise traders over rational traders is given by

\[
\left( \Delta R^*_{n/r} \right) = \rho - \frac{(1 + r)^2 \mu^r \rho^2}{2\gamma (\mu^r)^2 \sigma_{\rho}^2} - \frac{(1 + r)^2 \mu^s \rho \kappa}{2\gamma (\mu^s)^2 \sigma_{\rho}^2} \tag{A3.31}
\]

which for \( \rho^* = \kappa \) is identical to (3.44). The conditional expected excess relative return expressions involving rational traders and government are analogous to (3.46) and (3.47).

In Chapter 5 we present simulation evidence on this mixed sand policy. For the moment we note that such a policy makes the analysis far more self-contained and also gives an interesting theoretical rationale for setting a ‘low’ tax rate.
4. Learning rules and noise trading

4.1 Introduction

In this chapter our treatment of learning adheres closely to equilibrium strictures and the macro approach to learning described in Sargent (1993) and Evans and Honkapohja (1995a, 1997). This will become self-evident as we discuss some of the main issues and survey a small part of the bourgeoning literature on learning and boundedly rational modelling. In the next chapter more consideration is given to transition dynamics as we attempt to endogenise much of the analysis through simulated economies.

In this and the subsequent chapter we drop the assumption of rational expectations for both types of agent. Retaining the horizon structure, and all other features, provides us with an interesting adaptive analogue to DSSW and the extensions of the earlier chapters. We focus exclusively on the case of AR(1) noise trader misperceptions. Firstly, because the AR(1) case yields more interesting behaviour. Secondly, because previous chapters have dealt almost exclusively with the 'normal' misperception case. Lastly, because the AR(1) case gives the agents a more complex learning environment and is closer to the type of structure studied in the learning literature on macroeconomics and finance.

In recent years there has been a resurgence of interest in learning and boundedly rational modelling. For some authors the motivation has been to use learning as a selection device in rational expectations models with indeterminate outcomes, multiple equilibria or a continuum of equilibria. For others there is a deeper
motivation to design models which will usurp rather than complement previous theories, and in which the model builder is on a par, in the sense of computation and information, with the fictitious agents and not inferior.

In the previous chapters we used rational expectations to close the model in each case. Rational expectations models comprise the received benchmarks in much of modern economic theory. As was noted in the first chapter rational expectations macroeconomics and the theory of efficient capital markets in finance are closely related. Fama (1976) modifies his earlier definition of market efficiency to include rational expectations. The assumption of rational expectations thus became a maintained hypothesis virtually all work in macroeconomics and finance from the 1970's onwards. It is interesting that rejections of models employing rational expectations were rarely interpreted as rejection of rational expectations, rather they were seen as failures due to preferences or the structural makeup of the model in question. Endowing agents with learning technologies is an obvious way of trying to understand and recreate the behaviour of financial markets, particularly the perceived excess volatility findings and rejections of rational expectations efficient markets models.

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1 For a current example witness the growing literature on stochastic discount factors and non time separable utility functions currently used in an attempt to provide a rational explanation for asset price behaviour, and the Mehra - Prescott (1985) equity premium puzzle in particular.

2 Shiller (1989), Leroy (1989) and Cuthbertson (1996) argue the anti-efficient markets case. Cochrane (1991) interprets the volatility debate as being a moot point. Campbell, Lo and MacKinlay (1997) note that testing of an idealised benchmark is of limited interest. They argue that measuring relative efficiency across markets and temporally, within markets, is a more relevant research agenda.
Learning

Rational expectations continues despite the unreasonable computational and informational demands which it places on our fictitious agents. Frank Hahn, quoted in Waldrop (1992), asks the question “Where do you set the dial of rationality?”. Exponents of rational expectations ignore such considerations by setting the dial at its maximum. In this way the computational onus is shifted from the researcher to the fictitious agents in the model. The means by which agents arrive at the beliefs and abilities to form rational expectations are typically not specified.

The consistency condition of rational expectations expresses the outcome of a process in which agents have chosen their perceptions optimally. It is generally assumed that there are unexploited utility or profit opportunities if the consistency condition is not met. In much of the literature on learning by economic agents' economists seek to find out if an adaptive scheme will converge to this consistency condition and deliver the rational expectations equilibrium (see for example Evans and Honkapohja (1995b)). However, a preoccupation with a substantive equilibrium approach has the disadvantage that we do not allow sufficient space for transition dynamics. As Sargent (1993) notes the bounded rationality literature has yet to fully address this important issue.

Marimon (1996) cites four main non-mutually exclusive motivations behind the study of learning in economics: bounded rationality; equilibrium justification; equilibrium selection; and observed non-equilibrium behaviour. With respect to bounded rationality Marimon argues that a choice theoretical model of learning is of limited interest if its only contribution is to make our assumptions more realistic. Bounded rationality must contribute more than increased "realism", as "'realism' can not be the object of economic modelling".
For equilibrium justification an observed social outcome must be learnable by agents. If learning justifies an equilibrium then we should focus on the equilibrium and not learning itself. Thus for learning to be interesting and worthwhile in its own right, it must go beyond justification of equilibrium.

Learning can act as a powerful tool for discriminating between multiple equilibria. However, as Marimon notes, stability criteria used to select any equilibria are not independent of the learning rules with which the agents are endowed.

Marimon states that learning models should allow agents to gain experience such that "the 'bound' on rationality should be displaced away as agents learn through experience". This is a substantive argument which the present author cannot fully agree with. To displace the bound requires at the very least a stationary environment. Why should such a requirement be met and why should we always seek to 'kill' the system such that a substantive analysis can be applied?

To make this point more clearly we borrow from the geneticist Richard Leowontin who distinguishes between two types of scientist. The first type see the world as typically being in equilibrium. If a system is out of equilibrium at any point then the trick is to put it back into equilibrium again. Such scientists are termed 'Platonists' after the Athenian philosopher who stated that the messy, imperfect objects around us are simply reflections of perfect 'archetypes'. The second type view the world in terms of flow and change, such that it is in a state of constant flux or perpetual novelty.

These scientists are termed 'Heraclitians' after the Ionian philosopher who observed that 'Upon those who step into the same rivers flow other and yet other waters'. In this chapter we follow the mainstream learning literature and are very much Platonists. In the next chapter we attempt to inject a little Ionian spirit and let the model breathe.

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3 Plato would have liked the projection facility which we describe below.
In discussing the motivation to explain observed non-equilibrium behaviour Marimon cites three experimental facts that learning may help us to understand. Firstly, there is a 'sensitivity to marginal payoffs', i.e. agents actions are affected by the relative performance of their actions. Secondly, there is 'experience or expectations across environments', which affects agents behaviour. Experience and/or expectations from other environments can produce 'non-equilibrium' patterns resulting from agents who attempting act rationally over different environments. Thirdly, and of particular relevance to financial market analysis, equilibria can be stable in the large but unstable in the local sense. As an example of this Marimon cites instances where equilibria exist for low frequency data but we have persistent volatility in high frequency data.

**Related Finance Literature**

An early and seminal contribution to the learning literature is contained in Bray (1982). Bray uses an infinitely repeated version of the Grossman-Stiglitz (1980) asset market model, to study convergence and stability of rational expectations equilibrium under learning. Although agents can learn about the realtionships between variables, based on their current beliefs, revision of the beliefs changes the relationship between variables. Following on from this, one of Bray's main contributions is to explain that outside of a rational expectations equilibrium "it is not usually rational to use estimation techniques which are based on a correct specification of the rational expectations equilibrium". Such estimation rules are thus described as being 'reasonable'.
In common with the analysis presented here Bray's model contains heterogeneous agents. However, Bray endows the informed traders with rational expectations and only models learning by the uninformed traders. By contrast, we endow both rational traders and noise traders with learning rules. For Bray's analysis convergence to a rational expectations equilibrium is dependent only on convergence to rational expectations by the uninformed traders, whose estimation procedure is mis-specified outside of a rational expectations equilibrium. For convergence to obtain Bray shows that the coefficient on the price expectation term must be less than one. If this necessary condition is fulfilled then the system will converge to a rational expectations equilibrium with probability one. In our analysis we find a similar condition but use a simpler method of convergence and stability analysis. Of particular interest in the present context Bray also finds that if the number of informed is relatively large then learning tends to generate instability.

Hussman (1992) looks at the time series properties and efficiency of a securities market in which disparately informed traders hold rational expectations and extract signals from the market price. Market efficiency and inefficiency emerge as special cases of the model. When fully rational traders have imperfect private information the rational expectations equilibrium exhibits strong market efficiency and zero trading volume. In contrast, when there is a small degree of noise in the net supply of the asset available to rational traders, then market efficiency fails to hold. For such 'noisy' equilibria Hussman's model exhibits mean reversion, 'excess' volatility, trading volume, and correlation between dividend yields and subsequent returns.

Borrowing from Sargent (1991), equilibrium in the model is defined as the fixed point of a finite-dimensional operator that maps perceived laws of motion to actual laws of
motion. A notable feature of the model is that in equilibrium differentially informed traders hold identical expectations about future returns. Following Marcet and Sargent (1989) the basis for Hussman's fixed point approach is the convergence property of OLS learning. The model obviates the Grossman-Stiglitz paradox (discussed in chapter 1) because for any market which is some small distance away from the efficient rational expectations equilibrium, then profits are available for traders who act on their information. As traders degrees of risk aversion increase the variance of excess market returns drops, and return forecasts are less able to predict agents future returns. The intuition would appear to be that as risk aversion increases less private information is revealed in the equilibrium price. Agents forecast on the basis of their private information. Therefore if less of their private information is impounded into prices then the associated forecasts will have less predictive power.

An empirical analysis of learning in financial markets is described in Timmermann (1994). The main result of the paper is that convergence of learning crucially depends on the prior information 'agents' impose on the learning process. When agents learning feeds back on the actual law of motion of the economy, convergence of their learning rule to a rational expectations equilibrium is not guaranteed. If agents had tried to learn on the long-run dynamics of the model then they could not have learned to form rational expectations. However, if agents have strong initial priors such that they impose a unit root on the model then recursive learning may eventually lead to a rational expectations equilibrium.

In empirical work it is standard to assume that agents have had rational expectations throughout the sample. Timmermann argues that agents will only learn to form

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4 Hussman does not use learning explicitly in the paper, instead the perceived to actual mapping framework is used on the basis of convergence of OLS learning to rational expectations equilibrium.
5 Hussman does not offer an explanation.
6 For strong initial priors we should perhaps read strong initial coaxing by the author.
rational expectations over time\textsuperscript{7}. The type of learning behaviour specified is adaptive rather than optimal because agents ignore feedback from the learning rule to the actual law of motion. Bray and Kreps (1987) call such learning rules 'reasonable'. However, we would argue that if the economy is competitive and composed of many agents with heterogeneous learning rules then these rules would be 'more reasonable'. Individual agents learning rules would then have a negligible influence on the actual law of motion of the economy. Timmermann's framework is that of a representative agent, and the theoretical analysis follows that of Marcet and Sargent (1989a) and Evans and Honkapohja (1995b) described below\textsuperscript{8}.

Using 70 years of annual data on a portfolio of 30 UK stocks, Timmermann finds that agents could not arrive at a point of convergence if they attempt to learn the long run dynamics, but could converge if they instead focussed on the short-run dynamics. The results of Marcet and Sargent (1995) may help to explain this. They find that convergence to rational expectations equilibrium can be sufficiently slow that the standard asymptotic distributions used in classical econometrics will not obtain. Furthermore, convergence occurs more slowly when agents have more information, as the mapping from perceived to actual law of motion is more informative when there is hidden information. Timmermann concludes that learning causes fluctuations in price en route to equilibrium which are consistent with findings of high volatility.

\subsection*{4.2 Simple learning schemes}

The basis of a stochastic learning problem is to specify a time map

\footnote{Perhaps pedantically, we can turn this around to note that agents could conceivably have learned to form rational expectations prior to the beginning of the econometricians sample period. If this is the case then the within sample objections to rational expectations become less objectionable.}

\footnote{Timmermann uses the ordinary differential equation stochastic approximation approach, and in common with other authors in the area employs a projection facility. This ensures that agents forecasts on the path to equilibrium are bounded to within a neighbourhood of the equilibrium.}
\[ p_t = f\left(t, p^e_{t+1}\right) + \eta_t \]  

where \( p^e_{t+1} \) denotes an expectation, possibly non-rational, formed at time \( t \) of price in period \( t+1 \), and \( \eta_t \) is an i.i.d. random variable which is sometimes taken to have compact support\(^9\). The mapping given by (i) connects the state variable \( p_t \) with its expected value \( p^e_{t+1} \) one period ahead. An adaptive learning rule of the general form

\[ p^e_{t+1} = g(p_t, p_{t-1}, \ldots, \eta_t, \eta_{t-1}, \ldots) \]  

(ii)

formalises the method by which forecasts are made on past realisations of price and the random variable \( \{\eta_t\} \). Guesnerie and Woodford (1991) specify (ii) as

\[ p^e_{t+1} = p^e_{t+1-k} + \theta[p_{t+1-k} - p^e_{t+1-k}] \]  

(iii)

for a fixed parameter \( \theta \in [0,1] \) and \( k \geq 1 \). For this learning rule errors in agents forecasts are updated with a \( k \)-period lag because it is assumed that agents know the equilibrium is of period \( k \). Alternatively agents could run an ordinary least squares regression of \( p_t \) on \( p_{t-1} \),

\[ p_t = b + \alpha p_{t-1} \]  

(iv)

on realisations to date. The expectations scheme is then said to converge if the slope coefficient \( \alpha \) converges to zero and the intercept term \( b \) converges to the fixed point of the \( f \) - mapping as the number of realisations tends to infinity.

A further specification of (ii) is the so called statistical learning rule of Bray (1982), Lucas (1986)\(^10\), Evans and Honkapohja (1990), and Evans Honkapohja and Marimon

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\(^9\) If \( x \) is a random variable and \( f(x) \) is its density function then the support of \( x \) is the range of values of \( x \) such that \( f(x) \neq 0 \). The density and distribution of \( x \) are concentrated on an interval \( I = \overline{ab} \), such that for all \( x \not\in I \), \( f(x) = 0 \). In which case the interval \( I \) is said to be the support of the density function. It is often assumed in the learning literature that the support of any forcing random variable is small or compact so that agents cannot be 'forced off' a given learning path due to a large realisation of the random variable.

\(^10\) Lucas's view of the learning debate is somewhat narrow and distinctly substantive:

"Technically, I think of economics as studying decision rules that are steady states of some adaptive process, decision rules that are found to work over a range of situations and hence are no longer revised..."
where forecasts of the state variable are obtained from the sample mean of all past realisations using the rule

\[ p_{t+1}^e = \frac{1}{t} p_t^e + \frac{t}{t} (p_t - p_t^e) \]  

In (5) as \( t \to \infty \), forecasts converge to a value which is independent of forecasts made at the beginning of time.

appreciably as more experience accumulates. ... From this point of view, the question whether people are in general 'rational' or 'adaptive' does not seem to me worth arguing over." p218

"The models studied by Smith and Bray have unique equilibria. Their results, experimental and theoretical, have the effect of making us feel more comfortable with the predictions of certain theoretical models but do not lead to modifications of or improvements in the prediction of these models (though I think they have the potential for doing so)." p240

The above argument seems only to fit in a situation where individuals are confronted with an environment which is stationary or essentially static, in the sense that revision can be turned off as the 'inevitable' endpoint is reached. It is interesting to contrast Lucas's view with that of Sargent (1993) who sees the learning literature as providing the tools and models for the analysis of transition dynamics and non-recurrent systems.
4.3 E-stability

In this section we analyse the stability of the rational expectations equilibrium (REE) derived in chapters 1 and 2. In particular, we seek to show whether or not the REE is expectationally stable, E-stable, in the sense of Evans and Honkapohja (1995a,b). Their general definition of E-stability can be described as follows.

Let a rational expectations equilibrium be described by given values of a finite set of parameters. Denote the parameters by the vector $\theta$ and the rational expectations equilibrium by $\theta^*$. Suppose that agents use a perceived law of motion, based on a particular value of $\theta$, to make their forecasts of decision variables. Placing the forecast into the structural equations yields an associated actual law of motion, denoted $T(\theta)$, for the given perceived law of motion.

This gives a mapping $\theta \rightarrow T(\theta)$ from the perceived to actual law of motion. If the permissible perceived laws of motion nest the rational expectations equilibrium of interest then, in at least a local sense, $\theta$ and $T(\theta)$ will be in the same space. In this case the rational expectations equilibrium is a fixed point of the $T$-mapping, $T(\theta^*) = \theta^*$. If they are not in the same space then we face difficulties with mis-specified laws of motion and nonconvergence.

The rational expectations equilibrium $\theta^*$ is said to be E-stable if the associated ordinary differential equation

$$\frac{d\theta}{d\tau} = T(\theta) - \theta \quad (4.1)$$

is locally asymptotically stable at $\theta^*$. In (4.1) the differential is defined in notional time denoted by $\tau$, also known as virtual or fictitious time. One of the most powerful features of the E-stability approach is that it also governs the convergence of associated learning rules described in real time. We illustrate this link in the next
section. In simple terms (4.1) measures the difference (distance) between the path of
the actual law of motion and the agents perceived law of motion for the economy.
Heuristically, it describes the limiting behaviour of an adjustment process.
Evans and Honkapohja also distinguish between weak and strong E-stability on the
basis of the degree of parameterisation of the PLM compared to the rational
expectations solution. Stability of a given equilibrium can be affected by the specific
learning scheme which is adopted. Weak and strong E-stability is a method of
analysing the sensitivity of a particular equilibrium under adaptive learning. Strong
E-stability requires robustness under overparameterisation of the Minimum State
Variable (MSV) solution or 'fundamentals' solution. Under weak E-stability our PLM
is of the same parametric form as the MSV solution. For strong E-stability we might
add further lagged or lead endogenous variables or forcing terms (e.g. bubble or
sunspot terms). Under McCallum's MSV criteria solutions based on such additions are
ruled out and thus strong E-stability over parameterises the PLM with respect to the
MSV solution.
Following Evans and Honkapohja (1995a,b) we illustrate the concept of E-stability
using a linear model with a unique rational expectations equilibrium. Let the structure
of the economy be given by
\[
y_t = \mu + A E_{t-1}^* y_t + C w_t \\
w_t = S w_{t-1} + v_t
\]
(4.2) (4.3)
where, \(y\) is an \(n\times1\) endogenous vector, \(w\) is an observed \(p\times1\) vector of exogenous
variables and \(v\) is a \(p\times1\) vector of white noise shocks. To ensure stationarity of the
\(w_t\) process all eigenvalues of the \(p\times p\) matrix \(S\) are assumed to lie within the unit

\(^1\) The MSV criteria for selecting among multiple rational expectations equilibria is given in McCallum
(1983).
circle. $E_{t-1}^*y_t$ denotes the expectation, possibly non-rational, of $y_t$ held by agents at $t-1$ based upon the agents perceived law of motion for the economy. The parameters $\mu$, $A$, $C$, and $S$ are assumed to be conformable and $I-A$ is invertible.

The rational expectations solution to the economy described by (4.2) and (4.3) is derived as follows. Conjecture a solution of the form

$$y_t = \phi_0 + \phi_1w_{t-1} + \phi_2v_t$$

(4.4)

$$t-1y_t = \phi_0 + \phi_1w_{t-1}$$

(4.5)

Inserting (4.4) and (4.5) into (4.2) yields

$$\phi_0 + \phi_1w_{t-1} + \phi_2v_t = \mu + A(\phi_0 + \phi_1w_{t-1}) + CSw_{t-1} + Cv_t$$

Equating coefficients yields

$$\phi_0 = \mu + A\phi_0 = (I-A)^{-1}\mu$$

$$\phi_1 = A\phi_1 + CS = (I-A)^{-1}CS$$

$$\phi_2 = C$$

Thus the model defined in (4.2) and (4.3) has the unique rational expectations solution

$$y_t = \bar{a} + \bar{b}w_{t-1} + \eta_t$$

(4.6)

where $\bar{a} = (I-A)^{-1}\mu$, $\bar{b} = (I-A)^{-1}CS$, and $\eta_t = Cv_t$.

EH then ask whether or not this solution is E-stable. To answer this they consider perceived laws of motion of the form

$$y_t = a + bw_{t-1} + \eta_t$$

(4.7)

where $a$ and $b$ are arbitrary $n\times1$ vectors and $n\times p$ matrices respectively. The expectation function associated with (4.6) is $E_{t-1}^*y_t = a + bw_{t-1}$. Inserting this expectation function into (4.2) yields

$$y_t = \mu + A(a + bw_{t-1}) + Cv_t$$
\[ = \mu + A(a + bw_{t-1}) + CS w_{t-1} + C v_t \]

Equating with the conjectured solution form and solving for \( y_t \), we obtain the actual law of motion (ALM) corresponding to the perceived law of motion (4.7)

\[ \phi_0 + \phi_1 w_{t-1} + \phi_2 v_t = \mu + A(a + bw_{t-1}) + CS w_{t-1} + C v_t \]

\( \Rightarrow \phi_0 = \mu + Aa, \; \phi_1 = Ab + CS, \; \phi_2 = C \)

\( \Rightarrow \; y_t = (\mu + Aa) + (Ab + CS) w_{t-1} + C v_t \)

\[ = (\mu + Aa) + (Ab + CS) w_{t-1} + \eta_i \] \hspace{1cm} (4.8)

Comparing (4.7) and (4.8) it follows that the mapping from the PLM to the ALM is given by

\[ T(a, b) = (\mu + Aa, Ab + CS) \]

Let \( \theta' = (a, b) \), then E-stability of the REE \( \tilde{\theta}' = (a, b) \) is determined by the associated ordinary differential equations \( da/d\tau \) and \( db/d\tau \)

\[ \frac{da}{d\tau} = T(a) - (a) = (\mu + Aa) - a = \mu + (A - I)a \]

\[ \frac{db}{d\tau} = T(b) - (b) = CS + Ab - b = CS + (A - I)b \]

From the final equalities in both expressions the system will be locally asymptotically stable if and only if all eigenvalues of \( A \) have real parts less than 1.

We now consider the E-stability of the rational expectations equilibrium with autoregressive misperceptions discussed previously in chapter 2. Noise traders misperceptions are assumed to follow a simple first-order stationary autoregressive process

\[ \rho_t = \alpha \rho_{t-1} + \theta + \eta_t \] \hspace{1cm} (4.9)
where, \(|\alpha| < 1, \eta_t \sim N(0, \sigma^2_\eta), \theta \neq 0, E(\rho_t) = \rho^* = \theta/(1 - \alpha)\). The structure of the economy is given by

\[ p_t = u + A_t p_{t+1} + C \rho_t \tag{4.10} \]

and (4.9). The parameters in (4.10) are \(u = (1 + r)^{-1}(r - 2\gamma \sigma^2_{\rho_{t+1}}), \quad A = (1 + r)^{-1}, \quad C = \mu\). From chapter 2 we know that the rational expectations solution to (4.9) and (4.10) is

\[ p_t = a + b \rho_t \tag{4.11} \]

where,

\[
\bar{a} = 1 + \frac{\mu \theta}{r(1 + r - \alpha)} - \frac{2\gamma \sigma^2_{\rho_{t+1}}}{r} \quad \bar{b} = \frac{\mu}{1 + r - \alpha}
\]

Let agents posit a combined PLM of the form²

\[ p_t = a + b \rho_t \tag{4.12} \]

The associated expectations function is

\[ p_{t+1} = a + b(\alpha \rho_t + \theta) \]

Inserting this into (4.10) yields

\[ p_t = u + A(a + b(\alpha \rho_t + \theta)) + C \rho_t \]

Equating with the conjectured solution form gives

\[ \phi_0 + \phi_1 \rho_t = u + A(a + b(\alpha \rho_t + \theta)) + C \rho_t \]

and coefficient values of

\[ \phi_0 = u + Aa + Ab \theta, \quad \phi_1 = Ab \alpha + C \]

The actual law of motion corresponding to the perceived law of motion (4.11) is thus

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² We can model our system with heterogeneous agents with a single PLM because the proportions are time invariant and the weighting of the noise trader misperception in (4.12) by each type is itself a constant. In a sense the aggregation in (4.12) is akin to the aggregation in the structural pricing function.
which gives us the following form for the T-mapping from the PLM to the ALM

\[
T(a, b) = (u + Aa + Ab\theta, Ab\alpha + C)
\]  

(4.13a)

The associated ordinary differential equations are given by

\[
\frac{da}{d\tau} = T(a) - a = u + Aa + Ab\theta - a = u + Ab\theta + (A - I)a
\]

\[
\frac{db}{d\tau} = T(b) - b = Ab\alpha + C - b = C + (A\alpha - I)b
\]

(4.13b)

The system will be locally asymptotically stable if and only if all eigenvalues of A and A\alpha have real parts less than 1\(^3\). For our problem this is particularly simple given that A is equal to \((1 + r)^{-1}\) and \(|\alpha| < 1\) by assumption. For the system to be weakly E-stable we therefore only require that the identical dividend paid on the risk free and the risky asset be greater than zero. This requirement that the coefficient of the future price expectation be strictly less than one is virtually identical to that of Bray (1982)\(^4\). If the system is not E-stable then adaptive learning will converge to the rational expectations solution with probability zero.

Note that the model is adaptive in the sense that the T-mapping is derived based on agents knowing with certainty that the correct law of motion for the economy is time invariant and given by the PLM (4.12). The model does not involve fully optimal behaviour as the agents assume that the law of motion is time invariant, and this assumption is only true in the limit of an equilibrium.

\(^3\) We have used the notation I for the identity matrix despite the fact that the matrix in question is a scalar. This is done to conform with the treatment in the literature and the example presented earlier, it is by no means necessary.

\(^4\) See Sargent (1993) p.85 for a description of Bray's model and a link to the stochastic approximation approach to stability under learning. Bray used martingale convergence theorems to prove the results rather than the differential equation approach. We note that the ordinary differential equation approach utilised above is considerably simpler.
4.4 Real-time convergence

Having shown that the system is E-stable we now turn to the associated question of convergence of a given algorithm or class of algorithms, to our E-stable rational expectations solution in real-time\(^5\). One of the reasons for using the differential equation approach (4.1) is that, following from the results of Marcet and Sargent (1989a), convergence of real time learning algorithms are also governed by the approach in (4.1)\(^6\). We will demonstrate this in the context of the noise trader model with autoregressive misperceptions from the previous section.

This powerful result has led to the widespread use of Ljung's algorithm in a number of macroeconomic and game theory learning papers (see inter alia, and in addition to the papers cited above, Marcet and Sargent (1989b, 1989c), Woodford (1990), and Fudenberg and Kreps (1993)).

In this section we assume that agents update their price expectations using recursive least squares to forecast next period's price\(^7\). From the structure of the economy represented by (4.9) and (4.10) we postulate that at time \(t-1\) agents have the following perceived law of motion

\[
p_t = a_{t-1} + b_{t-1} \rho_{t-1} + \eta_t
\]  
(4.14)

which they use to construct the expectation \(E_{t-1} \star p_t\)\(^8\). Note that, in contrast to (4.11) the perceived law of motion is now time dependent. In conjunction with the exogenous variables \(\rho_t\), this rule will generate the value of \(p_t\). We assume that agents

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\(^5\) Recall that the E-stability analysis is conducted in virtual or notional time, denoted \(\tau\).

\(^6\) Marcet and Sargent utilised the results of Ljung (1977) to show that convergence of the PLM and ALM to one another was dependent on the behaviour of the associated ordinary differential equation. Textbook expositions can be found in Ljung and Soderstrom (1983), and Ljung, Pflug, and Walk (1992).

\(^7\) In the next chapter agents use the so-called statistical learning rule (4.5).

\(^8\) As before we simply state a single PLM given that the proportions of noise and rational traders, and their expectations functions are time invariant and thus (4.14) is an aggregate which is determined through the market clearing condition.
act as econometricians, updating the parameters \((a_t, b_t)\) over time by least squares regressions of \(p_t\) on an intercept and \(p_{t-1}\). To use Ljung’s results in this context we use recursive least squares to describe the coevolution of \((a_t, b_t)\) over time. Recursive least squares is a simple example of stochastic approximation.

**Recursive Least Squares**

Following Harvey (1981, 1990) recursive least squares involves updating on successive observations using least squares, but without having to repeatedly invert the cross-product matrix. Consider the classical linear regression model

\[
y_t = x_t' \beta + \epsilon_t
\]

where \(x_t\) is a \(k \times 1\) vector of exogenous explanatory variables and \(\beta\) is the associated \(k \times 1\) vector of unknown parameters. Assuming that we have calculated an estimator of \(\beta\) in (i) using only the first \(t-1\) observations then we can use the \(t^{th}\) observation to recursively update our estimator for \(\beta\). Let \(X_t = (x_1, \ldots, x_t)'\), then the recursive least squares updating formulae are (see Harvey (19xx) p.99)

\[
b_t = b_{t-1} + (x_t' X_{t-1})^{-1} x_t (y_t - x_t' b_{t-1}) / f_t
\]

where

\[
(x_t' x_t)^{-1} = (x_{t-1}' x_{t-1})^{-1} - (x_{t-1}' x_{t-1})^{-1} x_t x_t' (x_{t-1}' x_{t-1})^{-1} / f_t
\]

\[
f_t = 1 + x_t' x_t^{-1} x_t \quad t = k + 1, \ldots, T
\]

To compute an OLS estimator of \(\beta\) we require at least \(k\) observations. When the final observation is reached the estimator \(b_t\) will be identical to the standard OLS estimator calculated over the entire sample of length \(T\).
Continuing with our analysis of real-time convergence, let \( \beta_t' = (\alpha_t, h_t), \ z_t' = (1, \rho_t) \)
then the recursive least squares formulation for (4.9) and (4.10) is
\[
\beta_t' = \beta_{t-1}' + t^{-1}R_{t-1}^{-1}z_{t-1}'e_t'
\]
(4.15)
\[
R_t = R_{t-1} + t^{-1}(z_{t-1}'z_{t-1}' - R_{t-1})
\]
(4.16)
where the definition of \( R_t \) is analogous to (iii) above, and the least squares error is defined as
\[
e_t = p_t - \beta_{t-1}z_{t-1}
\]
(4.17)
Recall that in real-time learning the PLM is time-dependent. Agents are updating the parameters of their PLM \( \beta_t \) using the latest forecast errors. As before we can calculate the ALM for \( p_t \) corresponding to the PLM and (4.9) and (4.10), thus the determination of \( p_t \) is given by
\[
p_t = T(\beta_{t-1}z_{t-1} + C\eta_t)
\]
(4.18)
where the T-mapping in (4.18) is given by (4.13).
To derive the ODE note from the assumptions on (4.9) that \( Ez_tz_t' = M \) for some positive definite matrix \( M \). Change the timing for the system governing \( R_t \) by setting \( S_{t-1} = R_t \), to fit the general form of Ljung ((iv) below)\(^9\),

\(^9\) Ljung considers recursive stochastic algorithms of the form
\[
\begin{align*}
x_t &= x_{t-1} + \gamma_t Q(t, x_{t-1}, z_t) \\
z_t &= F(x_{t-1})z_{t-1} + G(x_{t-1})\nu_t
\end{align*}
\]
(iv)
where \( x_t \) is a vector of parameter 'estimates', \( z_t \) is a vector of observed state variables, \( \nu_t \) is a vector of white noise shocks, and \( \gamma_t \) is a non-stochastic non-increasing sequence of gains which converge to zero at an appropriate rate
\[
\lim_{t \to \infty} \gamma_t = 1 \quad \sum_{t=1}^{\infty} \gamma_t = \infty \quad \sum_{t=1}^{\infty} \gamma_t^2 < \infty
\]
In our example \( \gamma_t = 1/t \), and thus from (4.15), (4.16) and (4.18) our system satisfies the form above.
Furthermore, given the assumptions on \( \rho_t, \ \eta_t \), and the linear structure of the system, the moment and regularity conditions required for Ljung's approach are easily seen to be satisfied.
The algorithm is often modified to include a projection facility which is designed to ensure that trajectories which leave a subspace of the stable solution are projected back into that subspace if they
\[ S_t = S_{t-1} + t^{-1}(z_t z'_t - S_{t-1}) \]

Substituting in for \( \varepsilon_t \) from (4.17) yields

\[ \varepsilon_t = T(\beta_{t-1}) z_{t-1} + C \eta_t - \beta_{t-1} z_{t-1} \]

\[ \beta'_t = \beta'_{t-1} + t^{-1} S_{t-1}^{-1} z_{t-1} \left( T(\beta_{t-1}) z_{t-1} + C \eta_t - \beta_{t-1} z_{t-1} \right) \]

\[ = \beta'_{t-1} + t^{-1} S_{t-1}^{-1} z_{t-1} z'_t \left( T(\beta_{t-1}) - \beta_{t-1} \right) + t^{-1} S_{t-1}^{-1} z_{t-1} C \eta_t \]

If we take limits and expectations then the associated ordinary differential equation is

\[ \frac{d\beta'}{d\tau} = R^{-1} M \left( T(\beta) - \beta \right) \]  \hspace{1cm} (4.19a)

\[ \frac{dR}{d\tau} = M - R \]  \hspace{1cm} (4.19b)

Note from the second equation that \( \lim_{\tau \to \infty} R(\tau) = M \), and thus asymptotically the behaviour of the first equation is akin to that of the 'small' ODE (13b), i.e. the local stability condition

\[ \frac{d\beta}{d\tau} = \frac{d}{d\tau} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} u + Ab \theta + (A - I)a \\ C + (A\alpha - I)b \end{bmatrix} \]

From the previous section we know that this ODE is locally asymptotically stable. Equation (4.19b) does not depend on \( \beta \) and is thus globally asymptotically stable. As mentioned earlier the E-stability condition also governs convergence of real-time learning. For the economy described by (4.9) and (4.10) if noise and rational traders learn by RLS then they will converge to the rational expectations equilibrium given leave that subspace. More formally, let \( D_1 \) and \( D_2 \) be subsets of the parameter space of possible values of \( x \) in (iv). Where \( D_1 \) is open, \( D_2 \subset D_1 \) is closed, and \( x \) (the fixed point of the mapping) lies in an open ball in \( D_2 \). The projection facility is then invoked if \( x_t \) is determined by (iv) for \( x_t \) in \( D_1 \), but projected to a point in \( D_2 \) if \( x_t \) should leave \( D_1 \) as determined by (iv). The projected point is sometimes taken to be the REE which must be in \( D_2 \) from the outset. Although controversial and without any economic rationale it does conform with the idea of agents having a reference set of
by (4.11). Furthermore, given that \(|\alpha| < 1\) then from Corollary 2 of Marcet and Sargent (1989a) \(P(\beta_i \to \beta_f) = 1\) for all parameter values. Convergence with probability one does however require the use of an appropriately large finite projection facility (See Proposition in Evans and Honkapohja (1995a) p109, and Propositions 1-3 of Marcet and Sargent (1989a) which apply in our case). The rational expectations equilibrium is the only possible limit point of the least squares learning algorithm.

4.5 Conclusion

In this chapter we applied the stochastic approximation methodology from the recent macro-learning literature to the noise trader model presented in chapter 1. We showed that the rational expectations solution is stable under learning, E-stable (weakly), and that if agents were endowed with a recursive least squares learning algorithm the system would converge to the rational expectations equilibrium. This extends the earlier analysis on the robustness of noise trader results to economies where agents are learning rather than being endowed with rational expectations from the outset. Thus the results, particularly of chapter 1 (section 1.2) and chapters 2 and 3 are robust under learning in the sense outlined in this chapter. In contrast to earlier work on heterogeneous learning, Bray (1982) and Evans, Honkapohja and Sargent (1993) parameter values and does ensure that single large realisations of the forcing variable cannot cause non-convergence of the algorithm.

10 Modified to the present case Evans and Honkapohja’s relevant proposition is as follows:

Assume that \(A\) has no eigenvalues with real parts equal to unity. Consider the least squares learning algorithm (4.13a), (4.15)-(4.18) with an appropriate arbitrarily large projection facility. Then

\[
(a_i, b_i) \to (\tilde{a}, \tilde{b}) \text{ with probability 1 if the RE solution } (\tilde{a}, \tilde{b}) \text{ to the model (4.9), (4.10) is E-stable.}
\]

If \((\tilde{a}, \tilde{b})\) is not E-stable then \((a_i, b_i) \to (\tilde{a}, \tilde{b})\) with probability 0 for any nontrivial projection.
where only one type of agent in the economy is learning, we modelled both types of agent as having an adaptive learning rule (albeit of the same general form). This is a notable contribution given that there has been little research on heterogeneous learning\textsuperscript{11}. In the next chapter we relax some of the assumptions needed to meet the conditions of the stochastic approximation approach to analyse the effects of learning in simulated economies. In this chapter by adhering to equilibrium strictures we end with nothing to say on the question of transition dynamics. It could be argued that our analysis of convergence and stability thus merely serves as an asymptotic benchmark.

In common with the other literature we only model agents learning on the first moment of the price distribution. In effect agents are learning on asset price levels and ignore (because we have primed them to do so) the variance of prices en route to the rational expectations equilibrium. Given that the trade off between risk and return is a central tenet of asset pricing theory this is an obvious omission, which we hope to correct in future work. Would learning on the second moment of the variable of interest lead to faster rates of convergence, or instability and non-convergence? For example, we conjecture that some form of GARCH learning would be appropriate\textsuperscript{12}. Given the bewildering array of models within the ARCH class (countably infinite?) it may be possible to select specific ARCH models using stability criteria and an under/over parameterisation approach (weak/strong E-stability).

\textsuperscript{11} Evans and Honkapohja (1995a) write: 'A related important issue concerns heterogeneity: how are the stability properties affected by different agents having different perceived laws of motion or using different econometric algorithms?' p122.

\textsuperscript{12} See Bollerslev, Engle and Nelson (1993) for a recent survey of ARCH methodology and some applications.
5. Artificial Noise Trader Economies

5.1 Introduction

In the previous chapter we studied long memory Ljung-style learning rules and showed that in the noise trader framework of this thesis the rational expectations solution was stable under an econometric learning rule. The noise trader model is robust under this style of learning in the sense that the limiting behaviour of the learning economy is the same as in the rational expectations economy. In this chapter we extend the analysis to consider different types of learning rule which need not converge to the rational expectations equilibrium. At best these learning rules will converge to a neighbourhood of the REE. We also introduce evolutionary effects through a simple updating rule for the proportion of each type based on past return performance. This allows us to analyse under what circumstances, if any, noise traders can come to dominate the market by being present in greater number.

A central theme of this chapter is to endogenise as many of the key assumptions as possible from the purely analytical closed-form solutions obtained in previous chapters. Specifically, we look to endogenise agents relative proportions and their gain rule, in order to allow the system to 'breathe' and to yield dynamics which are less preordained and more complex. To completely endogenise this is clearly impossible as the researcher will always have specified the reward structure and the updating process, (for examples and discussion of this problem see Holland (1992)). We therefore attempt to choose simple updating rules to make the analysis more robust. Through simulations we also have some scope to analyse transition dynamics, out of equilibrium behaviour, and persistence in the model. The previous chapters have all focussed on closed-form equilibrium solutions which amounts to looking at a
steady-state endpoint. A significant proportion of financial economics research is devoted to explaining out of equilibrium behaviour such as “bubbles”, excess volatility, over-reaction, and market crashes. We believe that without an analysis of the behavioural evolution of a system such questions cannot be answered.

To retain a focus we concentrate on the effectiveness of the mixed sand policy outlined in chapter 3. The experiments suggest that the mixed sand policy is very successful in reducing the volatility of prices by reducing the proportion of noise traders in equilibrium.

The chapter proceeds as follows. In section 2 we discuss some related recent literature to give our results a context and to show the variety of methods and findings which already exist. In section 3 we discuss some candidate learning rules and the reasoning behind our final model choice. In section 4 we outline our adaptive model for the benchmark case and also for the mixed-sand economy described in chapter 3. Results of our experiments are contained in section 5 along with discussion and interpretation. Section 6 contains some further observations on the behaviour of the system and the conclusion s given in section 7. Related plots, summary statistics and sample code are contained in an appendix.

5.2 Related literature

In this section we discuss some of the recent work on heterogeneous beliefs and simulated asset markets that are relevant to this chapter. We make no attempt to survey the entire literature and instead focus on selected papers that contain the most interesting findings and complementary approaches. The series of papers by William Brock and coauthors (Brock (1993), Brock and de Lima (1995), Brock and Hommes

1 All of the simulation code is written in Matlab 4.2c, as M-files, and are available from the author on request. Calculation of the sample moments requires the Matlab Statistics toolbox.
(1997), Brock and Le Baron (1996), and Le Baron (1996)) are far more comprehensive. The purpose of this section is to explain where the results of this chapter fit into this literature and to provide further motivation for studying artificial asset market models. In addition, we point out the differences between this chapter and other studies.

Arguably the most ambitious work in this area is Arthur et al (1996) who propose a theory of asset pricing based on heterogeneous agents who continually adapt their expectations to the market which embodies their aggregate expectations. Relaxation of rational expectation restrictions is achieved by modelling agents as adjusting both their model of the economy and the parameters of the model, (this feature is also used in Darley and Kauffman (1997)). In contrast, we model the relaxation of rational expectations using agents who have a fixed model and only adjust the parameters of this single model.

Their model is explained using the Santa Fe Artificial Stock Market. Surprisingly, they find that the practitioners view (technical trading, psychology, and feedback effects), and the academics view (rationality and efficient markets) coexist within their framework. When traders have a low rate of adaptation/experimentation of their expectations then the market settles to the rational expectations equilibrium predicted by efficient markets theory. If the rate of expectational experimentation is sufficiently high then the market exhibits much more complex behaviour. In particular asset prices and trading volume exhibit GARCH like behaviour. The model used in this chapter delivers a similar result. Specifically we find that if the reward for past period performance is high (low) then there will be more (less) noise traders than rational traders. Our model does not appear to exhibit this behaviour with respect to variations in the 'experimentation' parameter.
Of particular interest to this thesis Arthur et al criticise the noise trader view which we have used in the previous chapters on two of its alleged assumptions. “Thus noise trader theories, while they explain much, are not robust.”


Firstly noise traders do not learn over time that their forecasts are mistaken and thus correct them. Secondly, the rational traders have knowledge of their beliefs and of the noise trader beliefs and thus the model is not robust. In this chapter we counter the first of these criticisms by allowing the relative proportions of noise and rational traders to change according to past performance. However we deviate from Arthur et al’s criticism by using realised returns rather than price forecast accuracy as the updating criteria. We do not introduce any asymmetric information effects and the second criticism remains valid. A further difference is that agents in the Arthur et al model choose from a set of rules each period whereas we model agents as updating parameters. Our approach is probably more commonplace and is similar to the macro learning approach of the previous chapter and the references cited therein.

In Arthur et al’s model the main feature is that agents forecasts co-evolve in an economy which they co-create. Traders endowed with heterogeneous beliefs are randomly clustered around the homogeneous expectation REE, and then a computational solution method is sought. The analysis and methods are more complex than we consider in this chapter but the common features and the authors criticism of noise trading models make a comparison worthwhile.

They further argue that given agents expectational differences there is no logical way to arrive at expectations. In the economies presented in this chapter we get around this problem by modelling heterogeneous expectations with only two fixed types, one of which nests the other. Noise trader expectations are an augmented form of rational
trader expectations as in the previous chapters. Thus our agents do not face the co-
ordination problem inherent in Arthur et al\(^2\).

Whereas Arthur et al rely heavily on the simulation of their economy to yield results
Brock and Le Baron (1996) pursue a more analytical approach and then attempt to
match real market data. The objective is to use an asymmetric information model
whose simulated behaviour is consistent with the stylised time series facts of stock
returns. The second main feature of their model is the use of adaptive belief formation
to generate the required dynamics. In this sense their approach is similar to our model
but with greater complexity as they also consider time series volume and volatility
behaviour. Their model contains many features similar to our own including
heterogeneous agents and disparate beliefs. One interesting feature of their model is
that the time scale of agent's belief updating is slower than the time scale of the price
process. This use of an epoch or T-slab updating system is primarily responsible for
the strength of their results. In future work we hope to incorporate both the epoch
argument and the mean field dynamics used to derive the analytical results.

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\(^2\) Much is made of the need for sufficient agent heterogeneity in such models however it is not clear
whether a large number of agents are needed, or few agents and a large number of beliefs. For example
Arthur et al endow each agent with a finite set of predictors from which to choose each periods realised
or invoked belief. It is not clear whether we need many different agents or if a small number of agents
with a large number of predictors would yield similar results.
5.3 Candidate learning rules

In this section we discuss our choice of a learning scheme for the simulated economies. In chapter 4 we used a recursive least squares algorithm to illustrate convergence of the system to the unique rational expectations equilibrium. In this chapter we consider the use of a more general stochastic approximation algorithm by agents in forming their price expectations. We explain why we use possibly time-varying gain specifications rather than the commonly used decreasing gain estimators.

An obvious candidate expectations scheme to replace the rational expectations of the previous analysis is a 'relaxation algorithm', Sargent (1993)

\[ X_K^* = X_{K-1}^* + \beta (X_K - X_{K-1}^*) \]  

(5.1)

where \( \beta \in [0,1] \) is the relaxation parameter and \( X_K^* \) is the agent's estimate of the value of the considered variable at the kth iteration. Progressive iterations of the algorithm adjust the expected value of \( X \) toward its actual value by an increment that depends on the relaxation parameter. In some circumstances stability of the learning scheme is taken to be the requirement that the scheme converges. If we replace \( K \) with \( t \) then we have the standard adaptive expectations formula\(^3\)

\[ X_t^* = (1 - \beta)X_{t+1}^* + \beta X_t, \quad \beta \in [0,1] \]  

(5.2)

Agents are simply revising their expectations by a fraction of the forecast error, \( p_{t+1} - p_t \). Special cases are \( \beta=1 \), where there is complete myopia, with agents never revising their expectations, and \( \beta=0 \), which would be the case when \( p_{t+1} = p_t \).

---

\(^3\) This is equivalent to the postulate that expectations are formulated as

\[ p_{t+1} = \beta p_t + \beta (1 - \beta) p_{t+1} + \beta (1 - \beta)^2 p_{t+2} + \ldots \]

This says that the one-period ahead forecast is formulated as a weighted sum of the past actual price, where the weight structure is that of a geometric lag.
In what follows both sophisticated investors and noise traders are endowed with such a learning scheme.4

Consider the following simple forms of relaxation algorithm

\( tP_{t+1}^1 = tP_t^1 + \alpha_t \left( p_{t-1} - tP_t^1 \right) \) (i)

\( tP_{t+1}^n = tP_t^n + \alpha_t \left( p_{t-1} - tP_t^n \right) + \rho_t \) (ii)

\( tP_{t+1}^n = tP_t^1 + \rho_t \)

\( = tP_t^1 + \alpha_t \left( p_{t-1} - tP_t^1 \right) + \rho_t \) (iii)

\( tP_{t+1}^n = tP_t^1 + \alpha_t \left( p_{t-1} - tP_t^1 + \rho_t \right) \) (iv)

The specification for rational traders is straightforward and given by (i). For noise traders we have a variety of possible specifications, each of which has a non-trivial impact on the behaviour of prices. For the moment we assume that the gain parameter for noise traders and rational traders is identical. We relax this assumption in the next section. If we use (i) and (ii) then rational and noise traders expectations will be more heterogeneous. Combining (i) and (iii) is more akin to the original rational expectations model in chapter 1, in that noise traders expectations are identical to those of rational traders, except for the addition of the stochastic misperception term.

The final specification, combining (i) and (iv), has the gain parameter operating on that periods noise trader misperception term. This would capture a situation of noise traders getting 'smarter' over time if the gain parameter is monotonically decreasing.

We can see what the differences between the combinations would mean for price by solving each case. For (i) and (ii) the pricing equation is given by

---

4 Given that agents only live for two periods in the basic model, the reader may argue that learning cannot take place because the horizon is too short. Evans and Honkapohja (1993) suggest the following reasoning: "In considering learning rules we are straining the overlapping generations interpretation of the model. Implicitly we are assuming that agents inherit forecast rules from their parents which they then update."
\[ p_t = \frac{1}{1+r} \left( r + \mu \rho_t - 2\gamma \sigma_{p_t}^2 + (1-\alpha_t) \left[ (1-\mu)_{t-1} p_t^i + \mu_{t-1} p_t^* \right] \right) \]  

(5.3)

If the relaxation parameter is of the decreasing gain type, \( \alpha_t = 1/t \), then the limit of (5.3) as \( t \to \infty \) is

\[ \lim_{t \to \infty} p_t = \frac{1}{1+r} \left( r + \mu \rho_t - 2\gamma \sigma_{p_t}^2 + (1-\mu)_{t-1} p_t^i + \mu_{t-1} p_t^* \right) \]  

(5.4)

where the limiting price expectations for each type of trader is given by

\[ \lim_{t \to \infty} p_{t+1}^i = \lim_{t \to \infty} p_t^i \]

for rational traders, and

\[ \lim_{t \to \infty} p_{t+1}^n = \lim_{t \to \infty} p_t^n + \rho_t \]

for noise traders. In the limit rational traders price expectations converge to a constant, with no further revisions being made. The limiting noise traders expectations continue to be stochastic because the gain parameter does not operate on their misperception.

If rational traders expectations are determined by (i) and noise traders expectations are determined by (iii) then the model is considerably simplified. In this case the pricing function is given by

\[ p_t = \frac{1}{1+r} \left( r + \mu \rho_t - 2\gamma \sigma_{p_t}^2 + \alpha_t \left[ (p_{t-1}^i - p_t^i) \right] \right) \]  

(5.5)

and the price is effectively determined by a single expectations function, \( t_{-1} p_t^i \). If the relaxation parameter is modelled as a decreasing gain then the limiting pricing function is given by,

\[ \lim_{t \to \infty} p_t = \frac{1}{1+r} \left( r + \mu \rho_t - 2\gamma \sigma_{p_t}^2 \right) \]  

(5.6)

where the limit price expectations are given by

\[ \lim_{t \to \infty} p_{t+1}^i = \lim_{t \to \infty} p_t^i \]
for rational traders, and
\[
\lim_{t \to \infty} P_{t+1}^n = P_t^r + \rho = r - P_t^r + P_t^r
\]
for noise traders. In the limit rational traders price expectations tend to a constant, while noise traders price expectations tend to the limiting rational trader expectation plus their per period misperception. Thus noise trader expectations are stochastic even in the limit.

A further alternative is to combine (i) and (iv), where in the latter noise traders gain parameter operates on their per-period misperception term. In this case the pricing function is given by
\[
P_t = \frac{1}{1+r} \left( r + t_p^r + \alpha, \mu \rho_t - 2 \gamma \sigma^2_{\hat{p}_t} + \alpha_r \left( P_{t-\hat{p}_t} - t_x \hat{p}_t \right) \right)
\]
(5.7)

If the relaxation parameter is a decreasing gain then the limit pricing function will be
\[
\lim_{t \to \infty} P_t = \frac{1}{1+r} \left( r + t_p^r - 2 \gamma \sigma^2_{\hat{p}_t} \right)
\]
(5.8)

Because the gain parameter operates directly on the noise trader misperception the limit distribution of the price function converges to a constant. The limiting price expectations are given by
\[
\lim_{t \to \infty} P_t = t_x P_t^r
\]
for rational traders, and
\[
\lim_{t \to \infty} P_{t+1}^n = \lim_{t \to \infty} P_{t+1}^r = t_x P_t^r
\]
for noise traders. To model the limiting noise trader expectation as identical to rational traders expectation is unsatisfactory as all agents will then have static expectations. The main source of dynamics will have been removed from the model.

The rational expectations version has noise trader expectations equal to rational expectations plus a misperception which is still present in the limit. It is therefore
important that the gain parameter should operate on lagged misperceptions but not on the current periods misperception. In this way the main source of the models dynamics is present even in the limit.

Experiments involving decreasing gain estimators of the form $\alpha_t = 1/t^\delta$, $\delta \in [0,1]$, converge to the level of the REE distribution but exhibit uninteresting dynamics. In particular, models with decreasing gain estimators leave no room for evolutionary dynamics or analysis of transition dynamics. We therefore use a more complicated design of learning algorithm for each type of agent, which exhibits greater heterogeneity. Specifically we assume that each group of agents forms their expectations according to a more general algorithm

$$P_{t+1}^i = \beta_t^i (p_{t-1}^i - P_t^i) + \beta_t^j P_t^i.$$ 

This specification allows agents expectations to differ through differences in their previous period expectation, $p_{t-1}^i$, or differences in their gain parameter, $\beta_t^i$. The purpose of a non-decreasing gain estimator is to track changes in the underlying system and the variable of interest. While the vast majority of the learning literature (chapter 4 included) uses decreasing gain estimators it seems an unjustifiable assumption in an asset market model with trader heterogeneity and evolutionary behaviour. Benveniste et al (1990) note that decreasing gain estimators are the most commonly used in the literature on stochastic approximation, but that algorithms of constant gain or asymptotic constant gain are "almost the only ones used in practice", (p29). On page 14 they state the 'First message' of stochastic approximation: "The main reason for using adaptive algorithms is to track temporal system variations".

Given the complexity of the noise trader economies it would be meaningless to endow agents with decreasing gain estimators. The system will not be constant, and thus agents require algorithms capable of tracking temporal variation. The most common
method of selecting the size of the gain parameter is to examine the mean squared error associated with different values of constant gain parameters over a number of experiments. Given that agents only observe one realisation of time series data this approach seems unreasonable. In contrast, we believe that our agents already have sufficient priming. We therefore model agents as setting their step-size in real-time based on past return performance.

5.4 The model

In this section, we present the main features of the artificial economy used to generate the plots and summary statistics in the Appendix. As mentioned earlier we have simulated other variants of the model but concentrate on this version because it is the most endogenous and for the sake of brevity. This version of the model also created the most interesting dynamics through the interaction of the forcing variable (noise traders misperception), the gain modification rule, heterogeneity in the sequence of agents expectations, and the proportion updating rule. We first describe the simple no tax economy, which then forms the basis for the mixed sand economy.
5.4.1 No tax economy

The noise trader misperception is modelled as following a stationary first order autoregressive process,

\[ p_t = \delta + \alpha p_{t-1} + \eta_t \]  

(5.9)

where \( \eta_t \sim N(0, \sigma^2) \), \( |\alpha| < 1 \), \( \delta > 0 \). For the experiments we set \( \alpha = .8 \), which gives a half-life of just over three periods. Setting the value higher to increase the persistence of their beliefs does not materially affect any of the results, but does make the system more volatile.

Following the discussion in the previous section, we model each agents price expectations as being formed from a recursive learning algorithm. Each agent uses a relaxation algorithm with a possibly time varying gain denoted \( \beta^i_j \), \( j = i, n \). At time \( t \) rational traders form their expectation of the price in period \( t+1 \) using the learning rule

\[ p_{t+1}^i = p_t^i + \beta^i_t (p_{t-1}^i - p_t^i) \]  

(5.10)

where \( \beta^i_t \) is the gain parameter of a rational trader in period \( t \), \( \beta^i_t \in [0,1] \), and the term in parenthesis contains \( p_{t-1} \) rather than \( p_t \) to avoid simultaneity. The simultaneity problem arises because agents would otherwise be using the current price in their one period led expectations which in turn would be one of the variables determining the current price. For simulation purposes the simultaneity problem is not operationally significant.

Similarly, noise traders form their price expectation using the learning rule

\[ p_{t+1}^n = p_t^n + \beta^n_t (p_{t-1}^n - p_t^n) + \rho_t \]  

(5.11)

where \( \beta^n_t \) is the gain parameter of a noise trader in period \( t \), \( \beta^n_t \in [0,1] \), which does not operate on this periods misperception. This ensures that the effect of the
misperception on noise traders expectations is not damped over time. We restrict agents gain parameters to be positive but note that this can be relaxed, and the system will still be stable for certain parameter values. In (5.10) and (5.11) both types of agent are modelled as updating using the realised price, \( p_{t-1} \), and their own price expectation, \( \epsilon_i - 1 \). Each agent is therefore unaware of the content of the other types price expectation series (except through the realised price series observed with a lag).

Inserting these rules into the agents respective demand functions yields

\[
\lambda_{i}^t = \frac{r_{i}^{t-1} p_{i}^t + \epsilon_{i-1}^t - p_{i}^t (1 + r)}{2\gamma \sigma_{\epsilon_{i-1}}^2}
\] (5.12)

for rational traders, and

\[
\lambda_{n}^t = \frac{r_{n}^{t-1} p_{n}^t + \epsilon_{n-1}^t - p_{n}^t (1 + r) + \rho_{i}}{2\gamma \sigma_{\epsilon_{n-1}}^2}
\] (5.13)

for noise traders. The reader will note that we continue to assume that agents beliefs differ with respect to conditional mean price expectations but are identical for the one-step-ahead conditional variance expectation. We have therefore primed our agents with the knowledge of current trader proportions. We make this assumption for tractability.

At the cost of a considerable increase in complexity we could have also specified a procedure for estimating proportions. While this is clearly not an entirely realistic assumption, it is plausible that conditional mean beliefs are more likely to differ than conditional variance beliefs (at least for short-horizons). To motivate this we appeal to the recent results of Nelson (1992) and Nelson and Foster (1994, 1995) on ARCH filtering and forecasting. Nelson (1992) states

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1 Allowing gains to be negative admits 'extrapolative' behaviour. In some of our experiments with negative gains noise and rational traders take positions of opposite sign for long periods. Economies
"... ARCH models are remarkably robust to certain types of misspecification. In particular, if the process is well approximated by a diffusion, broad classes of ARCH models provide consistent ... estimates of the conditional covariances. The ARCH models may seriously misspecify both the conditional mean of the process and the dynamic behaviour of the conditional variances. In fact, the misspecification can be so severe that the ARCH models would make no sense as data-generating processes and may make terrible medium and long-term forecasts – *without affecting the consistency of the one-step-ahead conditional covariance estimates.*” (Emphasis added).

That a misspecified model can yield consistent one-step-ahead conditional variance estimates suggests that assuming homogeneous expectations in this regard and priming the agents with the necessary proportion information may not be a large omission. Furthermore, Nelson and Foster (1994) show that misspecifying conditional means adds only trivially to the measurement error of ARCH conditional variance estimates. Thus assuming that agents conditional mean beliefs are heterogeneous while their conditional variance beliefs are homogeneous is not inconsistent. Alternatively, recall that DSSW interpret the risky asset u as aggregate equities not an individual equity. Our assumption is that heterogeneous agents beliefs about the future variance of aggregate equities may be more homogeneous than their beliefs about the future conditional mean. Perhaps agents devote more of their efforts to outguessing each other about future price levels, rather than future price dispersion\(^2\). Merton (1980) makes the more general point that "the variance of returns can be estimated far more accurately from the available time series of realised returns than can the expected returns", (p355).

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<table>
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<tr>
<th>with negative gains also exhibit more striking periods of quiescence and turbulence.</th>
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<td>(^2) We hope to pursue this issue in future work.</td>
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Using the demand equations in the market clearing condition yields the structural pricing equation that is used in the experiments to determine the realised adaptive price in each period.

\[ p_t = (1 + r)^{-1} \left[ r + (1 - \mu_t)(p_{t-1} - p_t) + \beta_t \left( p_{t-1} - p_t \right) + \gamma_t \left( p_{t-1} - p_t \right) + \rho_t \right] - 2\gamma \sigma_{p_{t-1}}^2 \]  

(5.14)

In previous chapters we always assumed that the proportion of noise traders and rational traders was an exogenously fixed constant. We relax this assumption and allow traders proportions to be determined by evolutionary forces. The proportion of noise traders in period \( t \), denoted \( \mu_t^n \), is a function of lagged realised relative returns. Letting \( \Delta R_{t-1}^{n,i} \) denote the difference between noise trader and rational trader realised returns in period \( t-1 \),

\[ \mu_t^n = f(\Delta R_{t-1}^{n,i}, \mu_{t-1}^n) \]  

(5.15)

The proportion of each type is now a random variable. At time \( t \) noise traders are present in proportion \( \mu_t^n \) while rational traders are present in proportion \( 1 - \mu_t^n \). Given that \( \rho_t \) has full support, \((-\infty, +\infty)\), large realisations of \( \Delta R_{t-1}^{n,i} \) could cause the proportion of one type to exceed one, while making the other types proportion negative. We therefore seek a suitable rule such that values of \( \Delta R_{t-1}^{n,i} \) in the range \(-\infty\) to \(+\infty\), are mapped onto the unit interval in a 'smooth' manner.

We consider two different methods of achieving bounded proportions regardless of the value of \( \Delta R_{t-1}^{n,i} \). The experimental results show that our choice of proportion updating rule has a non-trivial impact on the effectiveness of the mixed sand policy. This is important because an economy with our first updating rule does not respond to the mixed sand policy. In our second economy with a different updating rule the mixed
sand policy works remarkably well. The primary difference between the two updating schemes is the degree of persistence of the resulting trader proportion series.

The first rule we consider exhibits very little, if any, persistence in the evolution of trader types. We will call this rule the sigmoid rule. Define

\[ v = \frac{1}{1 + e^{-(\psi \Delta R^n_{t-1})}} \quad q = \frac{1}{1 + e^{-(\psi \Delta R^n_{t-1})}}, \]  

(5.16a)

where \( trr_{t-1} \) denotes the hyperbolic tangent of \( \Delta R^n_{t-1} \),

\[ trr_{t-1} = \tanh(\psi \Delta R^n_{t-1}). \]  

(5.16b)

The hyperbolic tangent maps values of the excess realised return of noise traders over rational traders onto the closed interval \([-1, +1]\). The inverse temperature parameter in (5.16b), \( \psi \), controls the steepness of the sigmoid curve, with a value less than one flattening the curve, and a value greater than one steepening the curve\(^3\).

The proportion of each type present at time \( t \) is then determined by the rule

\[ \mu_t^n = \frac{v}{v + q}, \quad \mu_t^r = \frac{q}{v + q}, \]  

(5.16c)

which by definition will give proportions which sum to one for all \( t \). If the returns of noise traders and rational traders are identical in period \( t-1 \), (\( trr_{t-1} = 0 \)), then both types are present in equal proportion in period \( t \) (\( v = q = 0.5 \)). If \( \Delta R^n_{t-1} \) is negative then \( v < q \), and the proportion of noise traders is decreased relative to rational traders. If \( \Delta R^n_{t-1} \) is positive then \( v > q \), and the proportion of noise traders is increased relative to rational traders. Furthermore, the proportion of noise traders (rational traders) is increasing (decreasing) in \( \Delta R^n_{t-1} \).

\(^3\) Hertz, Krogh, and Palmer (1991) contain a useful discussion of the properties of sigmoid functions.
In (5.16a) the constant $\theta$ is an inverse temperature parameter, which affects the curvature (sensitivity) of $V$ and $q$ to $trr_{t-1}$. The effect of the temperature parameter is more pronounced at the point of inflection of the sigmoid curve where $\Delta R^{\mu\nu}_{t-1} = 0$. For $\theta$ greater than one the shape of the sigmoid curve becomes more extreme. In particular the shape around the origin becomes much steeper. As $\theta \to 0$, the sigmoid curve becomes much flatter, and the resulting values of $\mu_t\mu$ and $\mu_t\nu$ exhibit less variation over a much wider range of values of $\Delta R^{\mu\nu}_{t-1}$. In the simulations we set $\psi < 1$, to smooth out the behaviour of $\Delta R^{\mu\nu}_{t-1}$, and $\theta > 1$ to ensure that agents proportions are separated and do not simply cluster around .5. This means that proportions will go between 0 and 1 smoothly as $\Delta R^{\mu\nu}_{t-1}$ goes from $-\infty$ to $+\infty^4$. This updating rule will only be persistent to the extent that the system exhibits strategic complementarity.

The second proportion updating rule which we consider is based on an AR(1) representation for $\mu_t\mu$ with innovations given by $trr_{t-1}$,

$$
\mu_t\mu = \xi + \theta \mu_{t-1}\mu + u_{trr_{t-1}} \tag{5.17}
$$

$$
\mu_t\nu = 1 - \mu_t\mu.
$$

The value of the constant, $\xi$, is set such that the theoretical (AR(1)) mean $\xi/(1-\theta) = .5$. This ignores the fact that $\{trr\}$ is not a zero mean white noise process$^5$.

$^4$ For very low values of the proportion of noise traders the model becomes unstable due mainly to the create space effect. We therefore set a lower bound of .01 and an upper bound of .99 for noise trader proportions in the experiments. Instability arises in terms, such as demands, which involve the square of the noise trader proportion in the denominator.

$^5$ In the experiments the mean of the process is marginally negative, e.g. $-0.02$. This in itself gives further motivation for using a proportion updating rule which exhibits memory. The $\{trr\}$ series is also heavily skewed towards negative values.
Experiments involving the AR(1) updating rule, (5.17) are termed economy 1. Experiments involving the sigmoid rule (5.16) are termed economy 2.

By allowing the proportion of noise traders to vary over time the model can now exhibit time variation in the conditional variance via the presence of more or less noise traders in the population. In the AR(1) misperception RE model of chapter 2 the conditional one-step-ahead variance of the risky asset was given by

$$\sigma_{n+1}^2 = \frac{\mu^2 \sigma_\rho^2}{(1 + r - \alpha)^2}.$$

As the proportion of noise traders increases (decreases), the conditional variance of the risky asset increases (decreases). For a proportion updating rule which causes cyclical, or mean reverting, behaviour in the proportion of noise traders, the model will then exhibit serial correlation in conditional variance or ARCH type effects. There are very few theoretical models in the asset pricing literature which give rise to ARCH effects (see for example, the discussion in Diebold and Lopez (1995)). Of these, most have modelled the information flow as having an ARCH data generating process. In the present case ARCH effects need not be modelled from the outset but can be created endogenously by variation in beliefs within the economy.

Determining the gain

It has been common practice in the learning literature to use a decreasing gain estimator, (see the discussion in the previous section). Unfortunately, this practice essentially kills any dynamics in the model in a short number of iterations and seems eminently unsuitable for economic analysis unless we are primarily concerned with asymptotics. While convergence is virtually guaranteed (under suitable conditions), the
system becomes lifeless very quickly. In early simulations we experimented with decreasing gains of order $O(T)$ and $O(\sqrt{T})$. We found that the system would converge to a fixed point (near the rational expectations equilibrium) after about fifty iterations for $O(T)$ and about 300 iterations for $O(\sqrt{T})$. We interpret this as indicating that our model with a decreasing gain estimator was simply not complex enough to be meaningful, and that in the circumstances a decreasing gain estimator was mis-specified. This is similar to the results of Sargent (1993) where it was found that a constant gain estimator was required to obviate a no-trade result.

In the present model each agent's step size is endogenously determined as a function of its own lagged values and the previous period's realised return to that trader type. Agents are operating in a non-stationary environment. We deliberately keep the step size away from zero to allow 'tracking'. Recall that agents' objective is to maximise terminal wealth. The competitive evolutionary frameworks described by (5.16) and (5.17) require that agents choose their gain sequence to yield positive relative returns. One common approach to choosing the 'optimal' gain size is to run the system with different fixed gain parameters and then choose the gain size with the smallest associated mean squared error. In any realistic setting it seems highly unlikely that agents could repeat the experiment with sufficient precision and control to train themselves to perform 'well'. In this chapter we eschew the Groundhog Day scenario and endow agents with a simple scheme for choosing their gains. Kushner and Yin (1997, Ch 11) describe iterate averaging and Polyak methods for selecting good step sizes which might be used to extend the current work. In the present analysis we

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6 We discuss this point further in the conclusion when we point to directions for future research.
7 These experiments were conducted using a slightly different model and were run using Shazam rather than Matlab.
choose a simpler heuristic method by specifying an autoregressive representation for agents step sizes, where the innovation term is given by a function of lagged realised returns. We do not claim that the rule is optimal in any sense, but would argue that it is reasonable.

For agents to have a positive return they must choose \( t_{i-1} p_i^j \), \( j = i, n \), such that \( \lambda_{i-1}^j > 0 \) (\( < 0 \)) when \( R_i^{U-S} = r + p_i - p_{i-1}(1 + r) > 0 \) (\( < 0 \)). Implicitly this structure rewards agents forecasting accuracy, as adaptive expectations of return will be closest to the true return when \( t_{i-1} p_i^j = p_i \). We therefore use the following adaptive algorithm for each agents step size

\[
\beta_t^j = \tau^j + \omega \beta_{t-1}^j + \varepsilon (rr_{t-1})
\]

(5.18)

where, \( 0 < \omega < 1 \), \( \tau^j \) is set such that the mean gain \( \tau^j/(1 - \omega) \) is at a 'stable' level, and \( \varepsilon \) is a small constant. The algorithm (5.18) has the properties that if \( rr_{t-1} = 0 \), then on average \( \beta_t^j \leq \beta_{t-1}^j \), and if \( R_{i-1}^j < 0 \) (\( > 0 \)) then \( \beta_t^j > \beta_{t-1}^j \) (\( \beta_t^j < \beta_{t-1}^j \)). From (5.10) and (5.11), when agents experience a negative return they increase their subsequent gain. This has the effect of agents giving less weight to their own expectation and more weight to the market price when forming next periods expectation. Following Kushner and Yin (1997) we constrain the gain sequence to lie in the interval \([\beta_t^j, \beta_t^{j+1}]\) where \( 0 < \beta_t^j < \beta_t^{j+1} \). In practical applications \( \beta_t^j \) can be set to zero and the inequality need not be strict. Kushner and Yin also recommend setting \( \beta_t^j \) at a small value initially and then increasing this value if the sequence of gains hovers around the upper limit and the algorithm is well behaved. In the present model this can be achieved by setting the

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8 Groundog Day refers to the film of the same title in which Bill Murray's character repeatedly re-lives the same day over and over again with progressively better 'performance'.
mean gain at a stable level. The motivation for setting the mean gain at a stable level follows Kushner and Yin (1997, p327)

"The basic stochastic algorithm tends to be more robust with a larger step size in that it is less likely to get stuck at an early stage and more likely to have a faster initial convergence."

We discuss this further in section 5.6.

We now have an autoregressive rule (5.18) imposed upon the price expectation algorithms (5.10) and (5.11). Taken together with the proportion updating rules (5.16) and (5.17), these equations are responsible for generating endogenous dynamics in the system. This forms the benchmark case that is used for comparison in judging the effectiveness of the mixed sand policy.

5.4.2 Mixed sand economy

The rational expectations equilibrium with a mixed sand policy was outlined in chapter 3. We now describe the mixed sand equilibrium under adaptive learning analogous to the discussion in the previous section. In the mixed sand economy the government now enters as a third group. Government demands in period t, \( \lambda^g_t \), are constrained by revenue, \( \Gamma_t \), obtained from taxing all agents demands

\[
\Gamma_t = \lambda^g_t P_t = \chi(\lambda^r_{t-1} + \lambda^n_{t-1} + \lambda^g_{t-1}).
\]  

(5.19)

To avoid simultaneity revenue is equal to tax imposed on demands in the previous period, although this has not been found operationally significant. Noise trader price expectations and rational trader expectations are given by (5.13) and (5.12) respectively.
From chapter 3 eqn. (A.29) the structural pricing equation in a mixed sand economy with rational expectations is

\[ p_t = (1 + r)^{-1} (r + p_{t-1} + \mu^p r_t + \mu^s \kappa - (1 - \mu^s) \chi (1 + r) - 2\gamma \sigma^2_{\varepsilon_t}), \]  

(5.20)

We assume that the proportion of all agents are initially equal and fix $\mu^s = 1/3$, but allow the proportions of noise traders and rational traders to evolve according to modified versions of (5.16) and (5.17). For economy 1 the proportion updating rule becomes

\[ \mu_t^n = (1 - \mu^s) (g + \theta \mu_{t-1} + \nu \text{tr}_{t-1}) \]

\[ \mu_t^r = (1 - \mu^s) - \mu_t^n. \]

For economy 2 the proportion updating rule becomes

\[ \mu_t^n = (1 - \mu^s) (v/v + q) \]

\[ \mu_t^r = (1 - \mu^s) (q/v + q). \]

As before, to give stability of the system we bound the proportion of noise traders between 0.01 and 2/3. At the risk of a small explosion in notation we could set the proportion of government traders smaller than 1/3. We conjecture that this would not qualitatively affect our results. A potential improvement to the model would follow the suggestion in chapter 3 that the government taxes every period but only intervenes periodically. In such circumstances it would be more reasonable to model the government proportion being as large as 1/3, in those periods where the government actively intervenes.

For the experiments the rational expectations price is determined by the pricing function

\[ p_t = 1 + \frac{\mu^p (\rho_{t-1} - \rho^*)}{1 + r - \alpha} + \frac{\mu^p \rho^*}{r} + \frac{\mu^s \kappa - (1 - \mu^s) \chi (1 + r)}{r} - \frac{2\gamma \mu^s \sigma^2_{\varepsilon_t}}{r(1 + r - \alpha)^3} \]  

(5.21)
and the adaptive price series is given by

\[ p_t = (1 + r)^{-1} \left( r + \left( \mu_1^t + \mu_2^t \right) \left( p_{t-1}^n + \beta_t^t \left( p_{t-1} - p_t^t \right) \right) + \mu_3^t \kappa + \mu_4^t \left( 1 - p_{t-1} \right) \right) \] 

(5.22)

As before in both the rational expectations case (5.21) and the adaptive (5.22) we are priming agents with knowledge of the variance of noise traders misperception, the common coefficient of absolute risk aversion, and the current proportion of each type of agent. The latter assumption is perhaps the most unappealing, and future work should consider agents estimating proportions at a given point in time. It should be noted that in calculating the RE price the proportions that enter are those determined by the active updating rule (5.16) or (5.17). Given that the RE price is derived under the assumption of fixed proportions we view this as the simplest method of getting around the problem. In common with Arthur et al (1996, Appendix) the RE pricing equations should be regarded as quasi RE or approximations. One reason to favour the AR(1) updating rule (5.17) is that such an approximation is likely to be better if proportions change smoothly and the variance is small.

5.5 The experiments

The next two sections describe the results for the simulated economies based on the models described above. For both the no-sand and sand economies the only changes involve equations (5.21), (5.22), and the modified versions of (5.16) and (5.17).
5.5.1 Economy 1: AR(1) updating

For economy 1 and 2 the benchmark parameter values are: $\gamma = 2$, $r = .06$, $\sigma_{\eta}^2 = .02$, $\psi = .01$, $\alpha = .8$ (implied half-life of three periods), $\rho^* = .05$, $E(\beta^t) = E(\beta^*) = .1$, $\beta = 1.8$, $\nu = .1$, $\theta = .99$ (implied half-life of 69 periods).

The experiments last for 50000 periods and we use the final 20000 observations on each series to calculate the sample moments reported in the tables. This allows the system to embed, and is designed to ensure that these calculations are free from initial variations. Changes in the initial values given in the sample code were not found to affect the results. We found that our simulated economies do not display any dependence on initial conditions.

We now discuss figures 1-14 which correspond to the results in tables 1 and 2.

In figure 1 we plot the histogram of noise traders misperceptions (the forcing variable) for the benchmark parameters given above. The per-period misperception ranges from $-0.6$ to $+0.7$ and the sample higher moments confirm that the pseudo-random number generator in Matlab produces normally distributed misperceptions (these and other higher moment results are available on request).

The relation between $p$ and rep for the final 1000 periods of economy1 with no sand are plotted in figure 2. The rep series appears stationary around the fundamental value of 1, while the adaptive series, $p$, fluctuates widely above and below the rep series. This corroborates the usefulness of learning models in explaining deviations from fundamentals in asset markets. A histogram of both series is provided in figure 3. The rep distribution is heavily peaked and concentrated around 1 while the $p$ series shows far greater dispersion with a pronounced negative skew (-0.7190). While the $p$ series
may be argued to have converged to a neighbourhood of the rep series the
neighbourhood appears very large.

The corresponding figures for the mixed sand economy are figures 8 and 11. Inspection of figure 8 shows that the p series is now far more in line with the rep series. We still find that the adaptive price will overshoot the rational expectations price but the difference between the two is far less marked. This is made more clear in figure 9 where we plot the price difference for economy 1 with no sand and economy 1 with mixed sand. The mixed sand price difference shows significantly less volatility than the no sand series as can also be seen from the sample moments in Tables 1 and 2. The histogram of p and rep for the mixed sand case is provided in figure 11. In this instance we can be far more confident in stating that the adaptive series has converged to a neighbourhood of the rational expectations distribution.

Given the above we would argue that the mixed sand policy is very successful at curbing destabilising speculation, and reducing the variance of speculative prices.

The relation between noise trader price expectations and rational trader price expectations is illustrated in figures 5 and 12, for the no sand and mixed sand economies respectively. For the no sand economy rational trader expectations appear to follow those of noise traders. Alternatively, we could think of rational expectations as a smoother imitation of noise expectations. (See the discussion in section 5.6). For the mixed sand economy, because there are less noise traders, the expectations of rational traders appear far smoother while noise expectations still fluctuate. The continued volatility of the noise trader expectations is due to both their ubiquitous misperception and the fact that their mean step size increases as their proportion drops.
This latter effect is clearly visible in figures 4 and 11 and the histograms in figures 7 and 14. For the no sand economy both traders step sizes fluctuate about the mean level of 0.1 with noise traders above and rational traders below. Given the value of mean trr this is as we would expect. In the mixed sand case the poor performance of the noise strategy causes noise traders to have step sizes which spend a lot of time toward the upper bound of 0.3. Similarly, because of their very good relative performance in a mixed sand economy rational traders significantly reduce their step size.

In the remaining figures we illustrate the relation between the proportion of noise traders and realised relative returns. For the no sand economy figure 6 shows that un is essentially a smoothed response to trr with the autoregressive component in un causing it to wander from its mean and exhibit persistence. In the mixed sand economy, figure 14, we find that the trr fluctuates wildly between its bounds, (-1,+1), but the noise trader proportion is very stable around its mean of 0.0247. The fluctuation in trr under a mixed sand policy is due to the increase in both the mean and variance of noise traders step size. It appears that their strategy and/or their expectations rules are insufficient to combat the mixed sand policy.

Given the strong nature of the results in Tables 1 and 2, and the associated figures discussed above, we now present the results of selected parameter sensitivity analysis.

Mean step sizes

From Table 2 when the mixed sand policy is implemented traders mean step sizes appear to diverge from the theoretical value of 0.1. We therefore ran another simulation for the mixed sand economy with the mean step sizes set to their values in Table 2, for both noise traders and rational traders. The moment statistics for this are presented in Table 3. Inspection of Table 3 shows that the results are very similar and
that allowing differential mean step sizes does not reduce the effectiveness of the mixed sand policy. The adaptive, p, and rational expectations, rep, price series has a lower variance as does the step size of both agents. The proportion of each type is stable with the proportion of noise traders remaining at around 0.025. The moderate decrease in variance for these variables validates the setting of differential mean step sizes but the results indicate that the mixed sand policy results are robust. We therefore set the mean step size for each type of trader at the same level (0.1) for the following experiments.

*Changing the mean misperception*

For the benchmark case of economy 1 we set noise traders mean misperception to 0.05. In our discussion of the expected excess return of noise traders in chapter 1 we noted that if noise traders were overly optimistic then this would damage their returns. In tables 4-9 we examine the results of our model and the mixed sand policy to changes in the mean misperception of noise traders. For the no sand and mixed sand economies all other parameter values are set at the benchmark values for Tables 1 and 2 respectively. We set values of the mean misperception to 0.1, 0.2, and 0.01. From tables 4 and 6 increasing the mean misperception causes and increase in both the adaptive mean price and the rational expectations mean price. The increase in the adaptive mean price is is greater howeverever, and the increase in the mean price difference (dp) reflects this. As mentioned above overly optimistic noise tradersean damage their own return. This theory is borne out by the decreasing values of mean trr (-0.0088, -0.1330, -0.4657) as the mean misperception increases from 0.05 to 0.1 to 0.2. As noise traders excess relative return falls so their mean proportion falls from 0.4318 in Table 1 to 0.1561 in Table 6.
Another notable feature, which is common to virtually all of the experimental economies, is the behaviour of the agents mean step sizes as the mean proportions change. As the proportion of noise traders decreases (increases) the mean step size of rational traders falls (rises), and vice versa. While we would expect this result from the design of the step-size rules it confirms that the system is well behaved.

For the mixed sand economy 1 the effect on mean prices, p and rep, is more muted. In contrast to the no sand economy the mean adaptive price increases by proportionately less relative to the mean of the RE series. The more muted response of p to parameter changes compared with rep seems to be a common feature of the mixed sand economies, The opposite would appear to be true in the no sand economies. We consider this encouraging for the use of a mixed sand policy.

In Tables 8 and 9 we report the results of reducing the mean misperception to 0.01. The most dramatic effect in the no sand economy is the reduction in mean p from 2.5533 to 0.0337, while mean rep falls from 1.2502 to 0.9563. The slightly negative value of mean trr in Table 8 explains the increase in the proportion of noise traders. The relatively large mean proportion of noise traders, 0.4597, causes both the variance of p, and the variance of ip and np to be large. Note that both agents step sizes were stable, (var(bi)=0.0002 and var(bn)=0.0007), around the hypothetical mean value of 0.1.

Again the mixed sand policy was demonstrated to be effective with noise traders mean proportion falling from 0.4597 to 0.0286, as trr fell from −0.0134 to −0.1263. As in the other mixed sand economies both price series exhibit a dramatic decrease in variance. In the case of var(rep) by a factor of 41 and in the case of var(p) by a factor of 300. Similar reductions hold for noth traders price expectation variance.
Coefficient of risk aversion

Tables 10 to 13 present results for mixed and no sand economies where we have changed the common risk aversion coefficient \( \gamma \). This is perhaps one of the most theoretically interesting parameters. For our benchmark we chose \( \gamma = 2 \). In Arthur et al (1996) value of \( \gamma = .5 \) is used. Campbell et al (1997) note that using the Hansen-Jagannathan methodology in a consumption based asset pricing model fit to US data, a coefficient of about 20 is required. We therefore checked the robustness of our results with values set to 0.5 and 4 for both the no sand and mixed sand economies.

For our model and Arthur et al the coefficient is important because by increasing \( \gamma \) the economy should be more accommodating to those agents who have expectations which deviate from full rationality, or contain some bias.

In the noise trader framework the coefficient appears in the numerator of the create space effect and therefore any increase should depress mean prices and further deter rational arbitrage. It also appears in the denominator of agents demand functions. Therefore, we would expect an increase (decrease) in \( \gamma \) to decrease(increase) demands.

Lastly, we have shown earlier that the misperception window of noise traders is increasing in \( \gamma \).

We mention that two possible improvements to our model would be to model time varying risk aversion (see for example Campbell et al (1993)), and wealth accumulation. We note that the former would likely require the latter.

Turning first to the case \( \gamma = .5 \), we find that the mean proportion of noise traders decreases from 0.4318 to 0.3575. In line with the arguments given above, for \( \gamma = 4 \) the mean proportion of noise traders increases from 0.4318 to 0.4766. Again the changes in proportion behaviour concur with the behaviour of trr. The mixed sand policy
results are again shown to be robust. In particular, for $\gamma=0.5$, the mean price difference $(dp)$ is 0.0006. All of the other values reported for both mixed sand economies appear to be insensitive to a change in the coefficient of risk aversion.

**Reward weighting parameter**

What we have termed the reward weighting parameter, $\psi$ in (5.16b), has a very strong effect on the mean proportion of each type and the system in general. Our benchmark case has $\psi$ set to 0.01, This ensures that for all but the largest movements in $rr$, $trr$ will be 'small' in absolute value. This is important for encouraging a degree of smoothness in the system. When we increase $\psi$ the innovation in agents step size rules and the proportion updating rule has a stronger effect (relative to the autoregressive component). We find that for large enough values of $\psi$ this effect is sufficiently strong to cause noise traders to be the majority group. The other parameters of the model do not seem capable of giving this result.

We therefore repeated our experiments with $\psi$ increased from 0.01 to 0.6 (basically making the mapping in (5.16b) steeper). The results are presented in tables 14 and 15. Form Table 14 the most obvious change is the increase in mean $un$ from 0.4318 to 0.5987. The variance of noise trader and in particular rational trader price expectations increases greatly. This reflects the fact that as $un$ increases so does the variance of $ip$ and $np$. This increase in expectational volatility feeds through into the adaptive and rational expectations price series. We discuss this point further in section 5.6, see also Figures 17 and 18. Despite what appears as a favourable environment for noise traders, when we implement the mixed sand policy mean $p$ and mean $rep$ are closer, $un$ falls to
0.05, and trr falls to -0.2925. The mixed sand policy is therefore still effective although we note that the results are not as strong as in Table 2.

*The Tobin tax rate*

In the appendix to chapter 3 we showed that the mixed sand policy would be most effective if the tax rate was 'small' (see page 119). We therefore repeated our experiments for the mixed sand version of economy 1 with χ set to 0.0025, 0.05, and 0.1. Inspection of Tables 16-18 shows that for χ=0.0025 the results are very similar to the benchmark case where χ=0.02, although they are slightly weaker. However, for χ=.05 (table 17) and χ=.1 (table 18) we find that these tax rates are too high. As the tax rate increases the mean price difference (dp) as does the variance. For χ=.1 the mean rep figure is only 0.1280. We also note that the effect of increasing the tax rate on trr is not monotonic, although the proportion of noise traders in these economies is still between 2 and 3%. These results agree with the condition given at the top of page 119. For the parameters we have used the required tax should be less than 0.0235.

**5.5.2 Economy 2: Sigmoid updating**

The results for economy 1 discussed above for both the benchmark and the sensitivity analysis are very strong. We felt that the results were worryingly strong and therefore sought a counter-example. The counter-example is given by economy 2 where proportion updating follows the sigmoid rule. Recall that by design the sigmoid rule does not exhibit persistence in proportions. With no memory the rule causes proportions to 'bounce' between the bounds set by the respective temperature parameters. This rule could be viewed as the analogy of a market in which the
evolution traders proportions exhibit complete myopia. We concentrate less on these results than for economy 1 and seek only to show that a mixed sand policy in such an economy is of little, if any, effect. The results for economy 2 are shown in Tables 19-22. Tables 19 and 20 report the results for economy 2 with the same benchmark parameters as economy 1. Immediately obvious is the very large difference in mean prices (dp) and the lack of any variance in traders proportions (mean trr is only 0.0004 in this case). The latter effect can be changed if we change the temperature parameters, but only slightly. Trader proportions are bunched around 0.5 for all experiments involving economy 2 without sand. The mixed sand policy in this case, Table 20, has no effect on separating agents proportions. The adaptive price series and the rational expectations price series are closer together but both have an increased rather than reduced variance. In order to try and induce some variation in the proportions we increased the reward weighting parameter to 0.3. These results are reported in tables 21 and 22. In this case the mixed sand policy may be deemed to be slightly effective in that mean dp is reduced from 4.4202 to 0.4409. However, the variance of pa and rep is still higher under the mixed sand policy. We prefer interpret any apparent benefit from the mixed sand policy for economy 2 to be due to the enforced reduction of noise traders (recall that government proportion is fixed at 1/3) rather than the policy itself.

5.6 Further observations

In this section we present some further analysis of the behaviour of pa and rep with respect to un, and the behaviour of ip and np with respect to un. These experiments were conducted in the framework of economy 1 and the results are plotted in figures 15-18.

We earlier argued that the behaviour of the adaptive price series was more muted and also less clear cut than the rational expectations price series. An interesting example of
this is given by the no sand Economy 1 with the reward weighting parameter increased to 0.3. We increase the parameter to this level to ensure that un goes between its full range of 0.01 to 0.99. Figure 15 plots rational expectations price series against the proportion of noise traders for this economy. The figure shows that when un is very small the rational expectations series exhibits far less variance and is concentrated around its theoretical mean of 1. As the un increases the volatility of rep appears to grow linearly as we would expect. For the adaptive price series plotted in Figure 16 the situation is more complicated. An initial impression would be that the adaptive price series is little affected by un, but that there is slight increase in variance as un increases. For un<4 we see that the there is a band around 0 to 1 in which there is no adaptive price. One interpretation of this would be that for un<.4 we have two price regimes – one above the mean RE price and one below. The vertical slices around un=0.1, 0.2, 0.3 are currently a puzzle, although they could be due to the greater impact of rational trader price expectations on the adaptive price series for low un. The correlation coefficients for (un,rep) and (un,p) are 0.2346 and -0.2744 respectively.

This latter point is more clearly seen in figures 17 and 18. Here we have plotted noise trader expectations for un<.5 against rational trader expectations for un<.5. One interesting aspect of the plot is the series of vertical slices indicating that for a wide range of noise trader expectations rational traders use the same price expectation. This can be most clearly seen for the extreme values of the joint relationship. In figure 18 we plot both traders price expectations for un>.5. Comparison of the two figures suggests that traders expectations are now more homogenous. We conjecture that as the proportion of noise traders increases the behaviour of rational traders causes their beliefs to 'imitate' those of the noise traders. This can also be seen from examination of
figures 5 and 12. For example, the correlation coefficient between np1 and ip1 is 0.88 while for ip2 and np2 it is 0.96°.

5.7 Conclusion
In this chapter we analysed an artificial economy in which noise and rational traders were endowed with learning rules based on stochastic approximation algorithms. We introduced proportion updating based on past relative return performance to capture evolutionary effects. Two types of rule were considered. The first rule exhibited little or no persistence while the second rule exhibited a high degree of proportion persistence. It was shown that the effectiveness of the mixed sand policy was dependent on which form of updating rule was chosen. For the first rule we found that the mixed sand policy had no effect on either the volatility of prices or the proportion of noise traders, and concluded that in such circumstances a mixed sand policy was useless.

The bulk of our results and analysis relates to the second proportion updating rule. For the second rule our mixed sand policy works surprisingly well. The variance of prices, and the proportion of noise traders are dramatically reduced. The system is well behaved and adaptive and rational expectations price distributions are very similar.

Sensitivity tests of these results with respect to mean step size, risk aversion, mean misperception, tax rate, and a reward parameter all supported our initial conclusion that a mixed sand policy can be very effective in achieving price stability.

° We note that the correlation coefficient will only pick up a linear association and that the relationship is most likely highly non-linear.
We also found that it was relatively hard to find reasonable parameter values for which the proportion of noise traders would be greater than the proportion of rational traders. From this we conjecture that noise traders will not evolve to dominate a given market under most conditions.

Interestingly, we found that the behaviour of rational traders imitated that of noise traders if noise traders had recently been successful and the proportion of noise traders was high. It was also shown that the tax rate should be set at a low level if the mixed sand policy is to perform at its best. This result strengthened an analytical result provided in chapter 3.

Although the results are obviously model dependent we conclude by cautiously recommending a mixed sand policy and advocating the development of further artificial economies with heterogeneous traders.
Appendix

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>No sand</td>
<td>Mean</td>
<td>2.5533</td>
<td>1.2502</td>
<td>-1.3031</td>
<td>0.5682</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>2.4249</td>
<td>0.0776</td>
<td>2.3896</td>
<td>0.0044</td>
</tr>
<tr>
<td></td>
<td>bi</td>
<td>0.0925</td>
<td>0.1085</td>
<td>2.5600</td>
<td>3.0202</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0001</td>
<td>0.0002</td>
<td>2.3105</td>
<td>3.7030</td>
</tr>
</tbody>
</table>

Table 1: Economy 1 with sand, results as above should be compared with table 2 below which is the mixed sand analogue.

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixed sand</td>
<td>Mean</td>
<td>1.0427</td>
<td>1.0571</td>
<td>0.0143</td>
<td>0.6419</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0082</td>
<td>0.0016</td>
<td>0.0077</td>
<td>0.0008</td>
</tr>
<tr>
<td></td>
<td>bi</td>
<td>0.0374</td>
<td>0.2245</td>
<td>1.0443</td>
<td>1.2322</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0012</td>
<td>0.0031</td>
<td>0.0072</td>
<td>0.7187</td>
</tr>
</tbody>
</table>

Table 2: Economy 1 with mixed sand. Parameter values identical to Table 1.

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>No sand</td>
<td>Mean</td>
<td>3.1156</td>
<td>1.4260</td>
<td>-1.6897</td>
<td>0.7059</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.3103</td>
<td>0.0798</td>
<td>0.4085</td>
<td>0.0185</td>
</tr>
<tr>
<td></td>
<td>bi</td>
<td>0.0560</td>
<td>0.1847</td>
<td>3.1494</td>
<td>3.6900</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0007</td>
<td>0.0029</td>
<td>0.2680</td>
<td>0.9637</td>
</tr>
</tbody>
</table>

Table 4: Economy 1 with no sand. Mean misperception increased to 0.1.

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixed sand</td>
<td>Mean</td>
<td>1.1416</td>
<td>1.3463</td>
<td>0.2047</td>
<td>0.6465</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0023</td>
<td>0.0009</td>
<td>0.0031</td>
<td>0.0005</td>
</tr>
<tr>
<td></td>
<td>bi</td>
<td>0.0176</td>
<td>0.2707</td>
<td>1.1329</td>
<td>1.4816</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0005</td>
<td>0.0013</td>
<td>0.0016</td>
<td>0.4814</td>
</tr>
</tbody>
</table>

Table 5: Economy 1 with mixed sand. Mean misperception increased to 0.1.

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>No sand</td>
<td>Mean</td>
<td>3.6027</td>
<td>1.4783</td>
<td>-2.1244</td>
<td>0.8439</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0615</td>
<td>0.2285</td>
<td>0.2620</td>
<td>0.0227</td>
</tr>
<tr>
<td></td>
<td>bi</td>
<td>0.0244</td>
<td>0.2564</td>
<td>3.6783</td>
<td>4.3650</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0005</td>
<td>0.0018</td>
<td>0.0442</td>
<td>0.4365</td>
</tr>
</tbody>
</table>

Table 6: Economy 1 with no sand. Mean misperception increased to 0.2.
<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixed sand</td>
<td>Mean</td>
<td>1.7469</td>
<td>1.9308</td>
<td>0.1839</td>
<td>0.6481</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0011</td>
<td>0.0037</td>
<td>0.0056</td>
<td>0.0006</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>0.0090</td>
<td>0.2886</td>
<td>1.7347</td>
<td>2.4133</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0003</td>
<td>0.0005</td>
<td>0.0009</td>
<td>0.4142</td>
</tr>
</tbody>
</table>

Table 7: Economy 1 with mixed sand. Mean misperception increased to 0.2

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>No sand</td>
<td>Mean</td>
<td>0.0337</td>
<td>0.9563</td>
<td>0.9226</td>
<td>0.5403</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>3.7892</td>
<td>0.0913</td>
<td>3.6476</td>
<td>0.0064</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>0.0930</td>
<td>0.1116</td>
<td>0.0298</td>
<td>0.1691</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0002</td>
<td>0.0007</td>
<td>3.6863</td>
<td>5.5390</td>
</tr>
</tbody>
</table>

Table 8: Economy 1 with no sand. Mean misperception reduced to 0.01

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixed sand</td>
<td>Mean</td>
<td>0.3741</td>
<td>0.8258</td>
<td>-0.1483</td>
<td>0.6380</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0127</td>
<td>0.0022</td>
<td>0.0114</td>
<td>0.0011</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>0.0488</td>
<td>0.2102</td>
<td>0.9789</td>
<td>1.0944</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0015</td>
<td>0.0036</td>
<td>0.0115</td>
<td>0.9281</td>
</tr>
</tbody>
</table>

Table 9: Economy 1 with mixed sand. Mean misperception reduced to 0.01

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>No sand</td>
<td>Mean</td>
<td>2.3187</td>
<td>1.2714</td>
<td>-1.0473</td>
<td>0.6425</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.6967</td>
<td>0.0791</td>
<td>0.7518</td>
<td>0.0269</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>0.0610</td>
<td>0.1818</td>
<td>2.3482</td>
<td>2.6075</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0008</td>
<td>0.0030</td>
<td>0.6300</td>
<td>1.4155</td>
</tr>
</tbody>
</table>

Table 10: Economy 1 with no sand. Coefficient of risk aversion reduced to 0.5

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixed sand</td>
<td>Mean</td>
<td>1.0638</td>
<td>1.0644</td>
<td>0.0006</td>
<td>0.6335</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0123</td>
<td>0.0033</td>
<td>0.0119</td>
<td>0.0015</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>0.0369</td>
<td>0.2406</td>
<td>1.0655</td>
<td>1.2653</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0014</td>
<td>0.0030</td>
<td>0.0109</td>
<td>0.6934</td>
</tr>
</tbody>
</table>

Table 11: Economy 1 with mixed sand. Coefficient of risk aversion reduced to 0.5

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>No sand</td>
<td>Mean</td>
<td>0.5643</td>
<td>1.1277</td>
<td>0.5634</td>
<td>0.5234</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>4.4350</td>
<td>0.0933</td>
<td>4.3248</td>
<td>0.0039</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>0.0967</td>
<td>0.1052</td>
<td>0.5729</td>
<td>1.0696</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0001</td>
<td>0.0004</td>
<td>4.3586</td>
<td>6.1119</td>
</tr>
</tbody>
</table>

Table 12: Economy 1 with no sand. Coefficient of risk aversion increased to 4
<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixed sand</td>
<td>Mean</td>
<td>1.0681</td>
<td>1.0552</td>
<td>-0.0129</td>
<td>0.6450</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0103</td>
<td>0.0013</td>
<td>0.0097</td>
<td>0.0006</td>
</tr>
<tr>
<td></td>
<td>bi</td>
<td>0.0404</td>
<td>0.2106</td>
<td>1.0710</td>
<td>1.3139</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0013</td>
<td>0.0035</td>
<td>0.0094</td>
<td>0.8848</td>
</tr>
</tbody>
</table>

Table 13: Economy 1 with mixed sand. Coefficient of risk aversion increased to 4

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>No sand</td>
<td>Mean</td>
<td>1.0181</td>
<td>1.2300</td>
<td>0.2120</td>
<td>0.4013</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>4.6081</td>
<td>0.2038</td>
<td>4.5092</td>
<td>0.0962</td>
</tr>
<tr>
<td></td>
<td>bi</td>
<td>0.0915</td>
<td>0.1483</td>
<td>1.0698</td>
<td>1.3751</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0022</td>
<td>0.0047</td>
<td>4.5143</td>
<td>5.6480</td>
</tr>
</tbody>
</table>

Table 14: Economy 1 with no sand. Reward weighting parameter increased to 0.6

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixed sand</td>
<td>Mean</td>
<td>1.1697</td>
<td>1.0723</td>
<td>-0.0974</td>
<td>0.6164</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.3138</td>
<td>0.0075</td>
<td>0.3100</td>
<td>0.0034</td>
</tr>
<tr>
<td></td>
<td>bi</td>
<td>0.0504</td>
<td>0.2315</td>
<td>1.1786</td>
<td>1.3767</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0034</td>
<td>0.0048</td>
<td>0.3338</td>
<td>1.5543</td>
</tr>
</tbody>
</table>

Table 15: Economy 1 with mixed sand. Reward weighting parameter increased to 0.6

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixed sand</td>
<td>Mean</td>
<td>1.1130</td>
<td>1.2591</td>
<td>0.1461</td>
<td>0.6549</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0050</td>
<td>0.0009</td>
<td>0.0047</td>
<td>0.0005</td>
</tr>
<tr>
<td></td>
<td>bi</td>
<td>0.0340</td>
<td>0.2262</td>
<td>1.1100</td>
<td>1.2998</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0011</td>
<td>0.0031</td>
<td>0.0039</td>
<td>0.6925</td>
</tr>
</tbody>
</table>

Table 16: Economy 1 with mixed sand. Tax rate reduced to 0.0025

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixed sand</td>
<td>Mean</td>
<td>1.0100</td>
<td>0.7086</td>
<td>-0.3014</td>
<td>0.6433</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0140</td>
<td>0.0019</td>
<td>0.0315</td>
<td>0.0007</td>
</tr>
<tr>
<td></td>
<td>bi</td>
<td>0.0501</td>
<td>0.1832</td>
<td>1.0204</td>
<td>1.3649</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0015</td>
<td>0.0035</td>
<td>0.0136</td>
<td>0.9700</td>
</tr>
</tbody>
</table>

Table 17: Economy 1 with mixed sand. Tax rate increased to 0.05

<table>
<thead>
<tr>
<th>Economy 1</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixed sand</td>
<td>Mean</td>
<td>0.7965</td>
<td>0.1280</td>
<td>-0.6685</td>
<td>0.6372</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0505</td>
<td>0.0027</td>
<td>0.0505</td>
<td>0.0008</td>
</tr>
<tr>
<td></td>
<td>bi</td>
<td>0.0650</td>
<td>0.1771</td>
<td>0.8163</td>
<td>1.2195</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0025</td>
<td>0.0049</td>
<td>0.0532</td>
<td>1.3269</td>
</tr>
</tbody>
</table>

Table 18: Economy 1 with mixed sand tax rate increased to 0.1
### Table 19: Economy 2, benchmark as for Economy 1, with no sand.

<table>
<thead>
<tr>
<th>Economy 2</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>No sand</td>
<td>Mean</td>
<td>-3.130</td>
<td>1.1262</td>
<td>4.4392</td>
<td>0.5000</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.1797</td>
<td>0.2004</td>
<td>0.2804</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>0.1004</td>
<td>0.0996</td>
<td>-3.3130</td>
<td>-3.2601</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.1719</td>
<td>0.2227</td>
</tr>
</tbody>
</table>

### Table 20: Economy 2, benchmark values as for Economy 1, with mixed sand.

<table>
<thead>
<tr>
<th>Economy 2</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixed sand</td>
<td>Mean</td>
<td>1.5913</td>
<td>1.1909</td>
<td>-0.4004</td>
<td>0.3361</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>2.8010</td>
<td>0.0896</td>
<td>20717</td>
<td>0.0001</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>0.0955</td>
<td>0.1045</td>
<td>1.5947</td>
<td>2.0834</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0000</td>
<td>0.0000</td>
<td>2.6634</td>
<td>5.3362</td>
</tr>
</tbody>
</table>

### Table 21: Economy 2 with no sand. Reward weighting parameter increased to 0.3.

<table>
<thead>
<tr>
<th>Economy 2</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>No sand</td>
<td>Mean</td>
<td>-3.2970</td>
<td>1.1232</td>
<td>4.4202</td>
<td>0.5000</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.1903</td>
<td>0.1969</td>
<td>0.2869</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>0.1129</td>
<td>0.0871</td>
<td>-3.2970</td>
<td>-3.2422</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.1840</td>
<td>0.2308</td>
</tr>
</tbody>
</table>

### Table 22: Economy 2 with mixed sand. Reward weighting parameter increased to 0.3.

<table>
<thead>
<tr>
<th>Economy 2</th>
<th>p</th>
<th>rep</th>
<th>dp</th>
<th>ui</th>
<th>un</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixed sand</td>
<td>Mean</td>
<td>0.7004</td>
<td>1.1503</td>
<td>0.4499</td>
<td>0.3662</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>1.5241</td>
<td>0.1234</td>
<td>1.4463</td>
<td>0.0352</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>0.0720</td>
<td>0.1709</td>
<td>0.7411</td>
<td>1.0323</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>0.0018</td>
<td>0.0042</td>
<td>1.5039</td>
<td>3.3419</td>
</tr>
</tbody>
</table>
Figure 1: Histogram of noise trader misperception for economy 1. 50000 iterations. $\rho_t = \delta + \alpha \rho_{t-1} + \eta_t$. $\rho^* = .05$, $\alpha = .8$, $\sigma_\eta^2 = .01$. 
Figure 2: Adaptive price (feint) and rational expectations price (dark) in economy 1. 50000 iteration last 1000 plotted.
Figure 3: Histogram of Adaptive (feint) and RE (dark) price series from economy 1. 50000 iterations. 50 bins.
Figure 4: Noise, bn, and rational trader, bi, stepsizes for economy 1. 50000 iterations.
Figure 5: Noise, np, and rational, ip, trader price expectations in economy 1. Time 49000 to 50000.
Figure 6: Noise trader proportion, un, and relative returns, trr. Last 1000 periods of economy 1.
Figure 7: Histogram of noise, $bn$, and rational, $bi$, trader step sizes in economy 1.
Figure 8: Adaptive, p, and rational expectations, rep, price series in economy 1 with mixed sand. 50000 iterations, last 1000 plotted.
Figure 9: Price difference, $dp(t) = p(t) - rep(t)$, for economy 1 with no sand, (feint), and mixed sand, (dark). 50000 iterations, last 1000 plotted.
Figure 10: Histogram of adaptive, (feint), and RE, (dark), price series in economy 1 with mixed sand. 50000 iterations.
Figure 11: Noise, bn, and rational, bi, trader step sizes in economy 1 with mixed sand. 50000 iterations, 40000-50000 plotted.
Figure 12: Noise, np, and rational, ip, trader price expectations in economy 1 with mixed sand. Iterations 49000-50000.
Figure 13: Noise proportion, un, and tanh relative returns, trr. Last 1000 periods of economy 1 with mixed sand.
Figure 14: Histogram of noise, bn, and rational, bi, trader step sizes. Economy 1 with mixed sand. 50000 iterations.
Figure 15: Rational expectations price against proportion of noise traders
Figure 16: Adaptive price against proportion of noise traders
Figure 17: Noise expectations against rational for un<.5
Figure 18: Noise expectations against rational for un > .5
% C. SAMPLE MATLAB CODE

% Final Version Economy 1 without mixed sand,
% Number of iterations
T=500000;
% constants, interest rate, coefficient of risk aversion, temperature parameters
r=0.06;
g=2; % lower is g the greater the variance of p and switching of ui un;
e=.001; %very important parameter;
y=.01;
sig=.01;
% tanh weighting
d=.01;
% autoregressive parameters, for noise trader misperception
m=.05; a=.0; h=m*(1-a);
% variables, vector initialised to increase speed
rep=zeros(T,1); p=zeros(T,1); np=zeros(T,1); ip=zeros(T,1); np=zeros(T,1);
id=zeros(T,1); nd=zeros(T,1); id=zeros(T,1);
un=zeros(T,1); ui=zeros(T,1); bn(1)=.001; bi(1)=.001; cr(1)=.1;
rr(1)=.1; p(1)=1; rep(1)=1; nd(1)=1; id(1)=1;
% loop commands
for t=2:T;
    % noise trader misperception
    d(t)=h+a*d(t-1)+randn*sqrt(sig);
    %bn(t)=.2; ^constant gain case.
    bn(t)=(.001+.99*bn(t-1)+de*(trr(t-1)));
    if bn(t)<0;
        bn(t)=.001;
    elseif bn(t)>.3;
        bn(t)=.3;
    end;
    %bi(t)=.2; ^constant gain case.
    bi(t)=(.001+.99*bi(t-1)+de*(trr(t-1)));
    if bi(t)<0;
        bi(t)=.001;
    elseif bi(t)>.3;
        bi(t)=.3;
    end;
    % proportion updating formula with AR(1) representation
    un(t)=.005+.99*un(t-1)+.1*trr(t-1);
    if un(t)<.01;
        un(t)=.01;
    elseif un(t)>.99;
        un(t)=.99;
    end;
    ui(t)=1-un(t);
    % create space term (common knowledge)
    cr(t)=(2*g*un(t)^2*sig)/((1+r-a)^2);
    % pricing equations
    p(t)=(r+ui(t)*((p(t-1)+bi(t)*(p(t-1)-ip(t-1)))+un(t)*np(t-1)+bn(t)*(p(t-1)-np(t-1)))+un(t)*d(t)-cr(t))/(1+r);
    rep(t)=1+(un(t)*d(t)-m)/(1+r-a)+(un(t)*m)/r-cr(t);
% price difference
dp(t) = rep(t) - p(t);
diffp(t) = p(t) - p(t-1);
rus(t) = (r + p(t) - p(t-1)*(1+r));

% pricing algorithms
np(t) = np(t-1) + bn(t)*(p(t-1) - np(t-1)) + d(t);
ip(t) = ip(t-1) + bi(t)*(p(t-1) - ip(t-1));

% demands
dn(t) = (r + np(t-1) - p(t-1))/cr(t-1);
id(t) = (r + ip(t-1) - p(t-1))/cr(t-1);

% returns
nr(t) = nd(t-1)*(r + p(t) - p(t-1)*(1+r));
ir(t) = id(t-1)*(r + p(t) - p(t-1)*(1+r))
rr(t) = nr(t) - ir(t);
trr(t) = tanh(.3*rr(t)); %benchmark is .02;

end

% moment commands
j = 30000; J = 50000;
avgp = mean(p(j:J)); avgrep = mean(rep(j:J)); avgbn = mean(bn(j:J));
avgbi = mean(bi(j:J));
avguk = mean(un(j:J));
advip = mean(ip(j:J)); avgnp = mean(np(j:J));
varp = var(p(j:J)); varrep = var(rep(j:J)); varbi = var(bi(j:J)); varbn = var(bn(j:J));
vartrr = var(trr(j:J));
varri = var(ip(j:J)); varun = var(un(j:J)); vardp = var(dp(j:J));
vard = var(d(j:J));
varcr = var(cr(j:J));

kurtp = kurtosis(p(j:J)); kurtrep = kurtosis(rep(j:J)); kurtbi = kurtosis(bi(j:J));
kurtbn = kurtosis(bn(j:J)); kurtui = kurtosis(un(j:J));
kurtip = kurtosis(ip(j:J));
kurtd = kurtosis(d(j:J));
kurttrr = kurtosis(trr(j:J));

disp([-avgp avgrep avgdp avgv np avgbn avgd avgbi avgun avgrep avgun avgdp avgv nj J np cr dp ip trr']);

% plot commands
title('No sand');
subplot(2,2,1); plot([p rep], xlab('Time'), ylab('Price'), title('Adaptive(...) and RE(-) price series'))
subplot(2,2,2); plot([bn bi], xlab('Time'), ylab('Adaptive step sizes'), title('Noise(...) and rational(-) step sizes'))
subplot(2,2,3); plot([np ip], xlab('Time'), ylab('Price expectations'), title('Noise(...) and rational(-) expected prices'))
subplot(2,2,4); plot([un ui], xlab('Time'), ylab('Trader proportions'), title('Evolution of u(...) and ui(...)'))
% simulation using AR(1) proportion updating with mixed sand, % Mixed sand Equivalent of a51
% number of iterations.
% Number of iterations
T=50000;
% constants, interest rate, coefficient of risk aversion, temperature parameters
r=0.06;
g=2; % lower is g the greater the variance of p and switching of ui un;
e=.001; %very important parameter;
y=.01;
sig=.02;
ug=1/3;
% tanh weighting
de=.01;
% autoregressive parameters, for noise trader misperception
m=.05; a=.8; h=m*(1-a); k=m;
% transactions tax benchmark is .01?
ki=.1;
% variables, vector initialised to increase speed
rep=zeros(T, 1); p=zeros(T, 1); ip=zeros(T, 1); np=zeros(T, 1);
t=zeros(T, 1);
id=zeros(T, 1);
nd=zeros(T,1); rid=zeros(T,1); rnd=zeros(T,1);
un=zeros(T,1);
ui=zeros(T,1);
d=zeros(T,1);
ir=zeros(T,1);
rn=zeros(T,1);
rr=zeros(T,1);
dp=zeros(T,1);
bn=zeros(T,1);
bi=zeros(T,1);

% initial values
un(1)=.5; ui(1)=.5; bn(1)=.001; bi(1)=.001; cr(1)=.1;
rr(1)=.1; p(1)=1; rep(1)=1; nd(1)=1; id(1)=1;

% loop commands
for t=2:T;
% noise trader misperception
   d(t)=h+a*d(t-1)+randn*sqrt(sig);
% step size algorithms
% set default to mean
   %bn(t)=l/t; %decreasing gain;
   %bn(t)=.2; %constant gain
   bn(t)=(.001+.99*bn(t-1)-de*(trr(t-1))); %
   if bn(t)<0;
      bn(t)=.001;
   elseif bn(t)>.3;
      bn(t)=.3;
   end;
   %bi(t)=l/t; %decreasing gain;
   %bi(t)=.2; %constant gain case,
   bi(t)=(.001+.99*bi(t-1)+de*(trr(t-1))); %
   if bi(t)<0;
      bi(t)=.001;
   elseif bi(t)>.3;
      bi(t)=.3;
   end;
% proportion updating formula with AR(1) representation
   un(t)=(1-ug)*(.005+.99*un(t-1)+.1+trr(t-1));
   if un(t)<.01;
      un(t)=.01;
   elseif un(t)>1-ug;
      un(t)=1-ug;
   end;
   ui(t)=2/3-un(t); % ok!
% create space term (common knowledge)
cr(t)=(2*g*un(t)^2*sig)/((1+r-a)^2);
% pricing equations
   p(t)=(r+(ui(t)+ug)*(ip(t-1)+bi(t)*(p(t-1)-ip(t-1)))+ug*k+un(t)*np(t-1)+bn(t)*(p(t-1)-np(t-1))+un(t)*d(t)-(l-ug)*ki*(1+r)-cr(t))/(1+r);
\[
\text{rep}(t) = 1 + (\text{un}(t) \cdot (d(t) - m))/(1 + r - a) + (\text{un}(t) \cdot m)/r + (ug \cdot k - (1 - ug) \cdot ki \cdot (1 + r))/r - \text{cr}(t);
\]

\% price difference
\[\text{dp}(t) = \text{rep}(t) - p(t);\]
\[\text{diffp}(t) = p(t) - p(t-1);\]
\[\text{rus}(t) = (r + p(t) - p(t-1) \cdot (1 + r));\]

\% pricing algorithms
\[\text{np}(t) = \text{np}(t-1) + \text{bn}(t) \cdot [p(t-1) - \text{np}(t-1)] + d(t);\]
\% Alternative form puts \(d(t)\) inside the brackets
\[\text{ip}(t) = \text{ip}(t-1) + \text{bi}(t) \cdot [p(t-1) - \text{ip}(t-1)];\]

\% demands
\[\text{nd}(t) = (r + p(t-1) - p(t-1) \cdot (1 + r));\]
\[\text{id}(t) = (r + \text{ip}(t-1) - p(t-1) \cdot (1 + r));\]

\% returns
\[\text{nr}(t) = \text{nd}(t-1) \cdot (r + p(t) - p(t-1) \cdot (1 + r));\]
\[\text{ir}(t) = \text{id}(t-1) \cdot (r + p(t-1) - p(t-1) \cdot (1 + r));\]
\[\text{rr}(t) = \text{nr}(t) - \text{ir}(t);\]
\[\text{trr}(t) = \tanh(0.01 \cdot \text{rr}(t));\]

\% moment commands
\[j = 30000; J = 50000;\]
\[\text{avgp} = \text{mean}(p(j:J)); \text{avgrep} = \text{mean}(\text{rep}(j:J)); \text{avgbi} = \text{mean}(\text{bi}(j:J)); \text{avgbn} = \text{mean}(\text{bn}(j:J));\]
\[\text{avgui} = \text{mean}(\text{ui}(j:J)); \text{avgip} = \text{mean}(\text{ip}(j:J));\]
\[\text{avgun} = \text{mean}(\text{un}(j:J)); \text{avgdp} = \text{mean}(\text{dp}(j:J)); \text{avgd} = \text{mean}(d(j:J)); \text{avgcr} = \text{mean}(\text{cr}(j:J));\]
\[\text{avgtrr} = \text{mean}(\text{trr}(j:J)); \text{avgnp} = \text{mean}(\text{np}(j:J));\]
\[\text{varp} = \text{var}(p(j:J)); \text{varrep} = \text{var}(\text{rep}(j:J)); \text{varbi} = \text{var}(\text{bi}(j:J)); \text{varbn} = \text{var}(\text{bn}(j:J));\]
\[\text{varui} = \text{var}(\text{ui}(j:J)); \text{varun} = \text{var}(\text{un}(j:J)); \text{vardp} = \text{var}(\text{dp}(j:J)); \text{vard} = \text{var}(d(j:J));\]
\[\text{varcr} = \text{var}(\text{cr}(j:J)); \text{vartrr} = \text{var}(\text{trr}(j:J));\]
\[\text{skewp} = \text{skewness}(p(j:J)); \text{skewrep} = \text{skewness}(\text{rep}(j:J)); \text{skewbi} = \text{skewness}(\text{bi}(j:J));\]
\[\text{skewnp} = \text{skewness}(\text{np}(j:J)); \text{skewui} = \text{skewness}(\text{ui}(j:J)); \text{skewun} = \text{skewness}(\text{un}(j:J));\]
\[\text{skewdp} = \text{skewness}(\text{dp}(j:J)); \text{skewd} = \text{skewness}(d(j:J)); \text{skewcr} = \text{skewness}(\text{cr}(j:J));\]
\[\text{kurtrep} = \text{kurtosis}(\text{rep}(j:J)); \text{kurtbi} = \text{kurtosis}(\text{bi}(j:J));\]
\[\text{kurtnp} = \text{kurtosis}(\text{np}(j:J)); \text{kurtui} = \text{kurtosis}(\text{ui}(j:J)); \text{kurtun} = \text{kurtosis}(\text{un}(j:J));\]
\[\text{kurtnp} = \text{kurtosis}(\text{np}(j:J)); \text{kurtip} = \text{kurtosis}(\text{ip}(j:J));\]
\[\text{disp}([''P REP DP UI UN BI BN D IP NP CR TRR''])\]
\[\text{disp}([\text{avgp} \text{avgrep} \text{avgdp} \text{avgui} \text{avgun} \text{avgbi} \text{avgbi} \text{avgdp} \text{avgd} \text{avgcr} \text{avgtrr} \text{varp} \text{varrep} \text{vardp} \text{varui} \text{varun} \text{varbi} \text{varbn} \text{varcr} \text{vartrr} \text{skewp} \text{skewrep} \text{skewui} \text{skewun} \text{skewbi} \text{skewbn} \text{skewdp} \text{skewd} \text{skewcr} \text{kurtrep} \text{kurtbi} \text{kurtui} \text{kurtun} \text{kurtcr} \text{kurttrr}]);\]

\% plot commands
\[\text{title('No sand');}\]
\[\text{subplot(2,2,1)}\];
\[\text{plot([p rep]), xlabel('Time'), ylabel('Price'), title('Adaptive(..) and RE(-) price series')};\]
\[\text{subplot(2,2,2)}\];
\[\text{plot([bn bi]), xlabel('Time'), ylabel('Adaptive step sizes'), title('Noise(..) and rational(-) step sizes')};\]
\[\text{subplot(2,2,3)}\];
\[\text{plot([np ip]), xlabel('Time'), ylabel('Price expectations'), title('Noise(..) and rational(-) expected prices')};\]
\[\text{subplot(2,2,4)}\];
\[\text{plot([un ui]), xlabel('Time'), ylabel('Trader proportions'), title('Evolution of u(..) and ui(-)')};\]
6. Conclusion

This thesis presented a theoretical appraisal of the robustness of irrational beliefs in asset markets. We focussed on one of the more popular frameworks in which two groups of agents hold differing beliefs. The first group of agents has beliefs which conform to the rational expectations hypothesis while the second group exhibit a systematic bias from rational expectations in their expectations formation. Following the convention in the literature we term the first group of agents rational traders and the second group noise traders.

The main contribution of this thesis is to examine in the context of a simple model and its extensions, the robustness of the second groups beliefs in different circumstances. Alternatively, the thesis could be viewed as examining the fragility of such beliefs. As there is little agreement in the literature on a standard set up for such models or their focus, the issue of robustness is a neglected one. Although the thesis also contributes to other areas this is the main theme that permeates our work.

If noise traders are to matter for finance and economics in general and not just the subfield of behavioural finance then the robustness of noise trader beliefs must be scrutinised. The noise trader paradigm with some irrational agents is a potential competitor to the efficient markets hypothesis with all rational agents. For noise trading to be taken seriously it must first be determined that the paradigm is a robust rather than a convenient assumption. Our general conclusion would be that noise trader beliefs are robust and probably more robust than has previously been thought.
In chapter 1 we discussed the motivation behind noise trader models and attempted to define noise trading. Using this definition we distinguished noise trading from the related concepts of chartism and liquidity traders, and provide a survey of the main papers in the literature. It is particularly important to realise the difference between the liquidity traders of market microstructure who have pure random exogenous demands, and the noise traders we have discussed, who have more conventional demands based on a well specified expectation of future prices, albeit with a bias. One direction that we intend to pursue is to inject the more life-like noise traders into a microstructure framework, such as Kyle (1985) and its offspring, to see what the effects might be. We would conjecture that the microstructure results would be less robust than the noise trader results.

We proceeded to give a detailed analysis of arguably the most notable and influential model due to De Long, Shleifer, Summers and Waldman (1990a),(DSSW hereafter) to illustrate the richness of the noise trader approach. Variants of their approach form the starting point for all of the analysis in this thesis. In the final section of the chapter we extended the model to include heterogeneous noise trader beliefs in order to capture the notion that noise traders will likely trade both with rational traders and amongst themselves. The results suggest that this is a useful modelling strategy although we still found that the basic noise trader effects would be preserved. Future work on this should attempt to get around the measure problem. A possible solution would be to treat each group of heterogeneous noise traders beliefs as an exchangeable random variable, and then attach the same weight to each belief set.
Chapter 2 asks what effect the horizon of agents has on asset price behaviour and the robustness of noise trader beliefs. DSSW discuss some possible effects of extending agents horizon but do not explicitly model this. The results in chapter 2 validate some of their conjectures but cast doubt on the view that noise traders have a lesser effect on prices and perform worse than rational traders as the horizon is extended. We show that traders horizons do matter, particularly if noise traders misperceptions are persistent. There is a complex interplay between agents pricing of conditional return and risk which is not present in the shortest horizon version of the model. Most importantly, the performance of noise traders, measured in terms of conditional returns, need not decrease as the horizon is extended. As agents horizon is increased noise trader induced risk becomes more important.

In chapter 3 we ask how robust are noise traders to different forms of policy intervention which have been advocated to deter allegedly destabilising speculation in financial markets. The results contribute to the topical and contentious debate on the proposed use of Tobin taxes in financial market transactions. In contrast to the discursive nature of this literature to date we attempted to analyse this problem in an analytical fashion. The variety of forms which we consider suggests that some pruning is in order. We presented an analysis of simple linear taxes for models with different time horizons and different specifications of noise trader beliefs. Following this we analysed non-linear taxes to examine the effects of a more progressive tax regime. For proponents of Tobin taxes our results are not encouraging but accord with some of the results from the empirical literature on transactions taxes. We then explore direct government intervention as an alternative policy. Surprisingly we find that intervention in the opposite direction to (same direction as) noise trader beliefs benefits (damages) the relative performance of noise traders. While all results are model dependent this is
a beguilingly simple policy prescription, albeit unconventional. The chapter finishes by analysing an adjustment tax on agents holdings and a mixed intervention policy using a transactions tax to finance government purchases of the risky asset. This final policy is found to be very successful in the simulations reported in chapter 5.

Chapter 4 departed from the rational expectations analysis of the first three chapters to examine the limiting behaviour of the noise trader model when agents form expectations adaptively. The primary purpose of the chapter is to determine whether the previous results with rational expectations are upheld when agents form their price expectations using learning algorithms. We show that our model with heterogeneous agents will converge to the rational expectations solution under a given learning scheme and thus the previous results hold under learning, described as expectationally stable or E-stable. The results of the chapter show that noise trading is robust under learning. At a more general level, the chapter contributes to the learning literature by providing a specific example of an economic model in which an economy with heterogeneous adaptive agents converges to the equilibrium path of the rational expectations economy. The vast majority of papers in the learning literature employ a representative agent framework or a single learning agent, there are currently very few which look at a specific model with heterogeneous learning rules. A further extension of this work would be to integrate the heterogeneity presented in chapter 1. From an aesthetic point of view the exclusive focus in the chapter on convergence is somewhat disappointing and hollow. However, if the model were altered to admit multiple equilibria then the analysis would be ultimately more worthwhile.

In the last chapter we again look at learning but use simulations to study the behaviour of more complicated environments. Of all the models considered in the thesis the final chapter contains arguably the most realistic and, at the same time, the most arbitrary.
In the author's opinion this work is also the most enjoyable and the most promising. Although the models are relatively simple the dynamics are complex and we must forsake closed-form analytical solutions for the pricing equations and resort to simulation evidence. At the heart of this chapter is the desire to free the basic model by endogenising some of its assumptions. Specifically, we endogenise the proportion of agents according to relative past performance and also endogenise the degree to which agents change their expectations. The model and its mixed sand variant perform well and hopefully shows that artificial economies are not a research wilderness.

For a proportion updating rule which exhibits persistence the mixed sand policy of simultaneously imposing a transactions tax and directly intervening in the market is very successful. When the proportion updating rule is myopic (near Markov) the policy does not work. Any discernible improvements are interpreted as coming from the imposition of the government trading proportion. We believe that the persistent rule is more realistic and regard the myopic rule as a useful and important counter example.

The final chapter also shows that when noise strategies perform well there is a tendency for rational traders to imitate. This blurs the distinction between what is a noise trader and what is a rational trader. As a result it would be interesting to redo the analysis in a model where the distinction between different types of trader is less clear cut (as in Arthur et al (1996)).

For the majority of our experiments it was found that noise traders would not come to dominate the market being present in greater proportion – this could happen only for sub-periods. Taken together with the results in the previous chapters we argue that noise traders beliefs and effects while robust, are not dominant.
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