New stability conditions on surfaces and new Castelnuovo-type inequalities for curves on complete-intersection surfaces

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Rebecca Tramel*)

To my family

Abstract

Let X be a smooth complex projective variety. In 2002, [Bri07] defined a notion of stability for the objects in $\mathcal{D}^b(X)$, the bounded derived category of coherent sheaves on X, which generalized the notion of slope stability for vector bundles on curves. There are many nice connections between stability conditions on X and the geometry of the variety.

In 2012, [BMT14] gave a conjectural stability condition for threefolds. In the case that X is a complete intersection threefold, the existence of this stability condition would imply a Castelnuovo-type inequality for curves on X. I give a new Castelnuovo-type inequality for curves on complete intersection surfaces of high degree. I then show how this bound would imply the bound conjectured in [BMT14] if a weaker bound could be shown for curves of lower degree.

I also construct new stability conditions for surfaces containing a curve C whose self-intersection is negative. I show that these stability conditions lie on a wall of the geometric chamber of $\mathrm{Stab}(X)$, the stability manifold of X. I then construct the moduli space $M_\sigma(\mathcal{O}_X)$ of σ-semistable objects of class $[\mathcal{O}_X]$ in $K_0(X)$ after wall-crossing.

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Contents

Chapter 1

Introduction

Suppose X is a projective variety. We can associate to X an abelian category Coh(X), the category of coherent sheaves on X. From this abelian category, we can then construct a triangulated category $\mathcal{D}^b(\mathrm{Coh(X)})$, often denoted $\mathcal{D}^b(X)$, the derived category of coherent sheaves on X. The objects in this category are complexes of coherent sheaves, and the morphisms are morphisms of complexes, with quasiisomorphisms formally inverted.

There are many relations (some conjectural) between the birational geometry of X and the category $\mathcal{D}^b(X)$, see [BO01], [Bri02], [Kaw02], [Kuz10] for some examples. We consider an example of such a relationship by studying Bridgeland stability on $\mathcal{D}^b(X)$. In [Bri07], Bridgeland defines a notion of stability for objects in $\mathcal{D}^b(X)$, which can be thought of as an extension of the idea of slope stability on vector bundles. The original motivation for this definition came from string theory. Suppose X is a Calabi-Yau threefold. There is an equivalence of the category $\mathcal{D}^b(X)$ and the category of branes in the superconformal field theory associated to X. Bridgeland stability then comes from the notion of Π-stability of Dirichlet branes given in [Dou02].

This idea of stability also has origins in algebraic geometry. In [BM02], Bridgeland and Maciocia consider varieties with elliptic fibrations $\pi\colon X \to S$. If we fix a polarization of X, we can construct a moduli space Y of Gieseker stable sheaves on the fibration π. If Y has the same dimension as X, this gives a dual fibration, $\hat{\pi}\colon Y \to S$. The authors show that in some cases there is then a derived equivalence $\phi\colon \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$. Suppose X is a threefold with a flat elliptic fibration. Then there is a choice of polarization ℓ on X so that a connected component $\mathcal{N}$ of the moduli space of torsion-free line bundles on Y is isomorphic to a connected component of the moduli space $\mathcal{M}$ of ℓ-stable torsion-free sheaves on X. This suggests that the notion of stability on X should correspond to some notion of stability on Y.

Many classical questions in algebraic geometry can be studied via Bridgeland stability. One such question is under which circumstances the classical Castelnuovo genus bounds for smooth projective curves can be improved upon. [BMT14] constructs conjectured Bridgeland stability on X a smooth projective threefold. The conjectured Bridgeland stability conditions hold if and only if a certain Bogomolov-Gieseker type inequality holds for chern classes of stable objects in $\mathcal{D}^b(X)$. This inequality, which is extended in [BMS14], predicts the existence of a bound on the genus of smooth projective curves lying in complete intersection threefolds.

Another classical question is to construct a moduli space parameterizing vector bundles on a variety of fixed numerical invariants. It is not possible in general to do this, unless we restrict to slope stable vector bundles. Similarly, we can construct moduli spaces of Bridgeland stable objects of a fixed invariant. We can further study how these moduli spaces vary as we deform the chosen stability condition. This leads to connections between Bridgeland stability and the minimal model program, particularly as studied in [Tod13] and [Tod14].

1.1 Curves on complete intersection surfaces

The original goal in the study of Bridgeland stability conditions, coming from its motivations in Physics, was to define stability on a smooth projective Calabi-Yau threefold X. Such conditions have now been constructed in [BMS14] for a certain class of Calabi-Yau threefolds, those which are quotients of abelian threefolds. More generally, the study of Bridgeland stability on smooth projective threefolds is largely open, solved only in certain cases (see citeBMT, [BMS14], [MP13a], [Sch13]). In [BMS14], the authors give a conjectured stability condition for threefolds, which holds if and only if certain two term complexes in $\mathcal{D}^b(X)$ which are "tilt-stable" satisfy a conjectured Bogomolov-Gieseker type inequality.

Conjecture 1.1.1. *[BMS14, Section 4] If* $\mathrm{NS}(X) = \mathbb{Z} \cdot H$ *and* E *is a slope-stable sheaf with* $c_1(E) = H$*, then*

$$3H^3 \mathrm{ch}_3(\mathrm{E}) \leq 2(\mathrm{Hch}_2(\mathrm{E}))^2.$$

This conjecture has been proved to hold for specific threefolds. [Mac14] shows that 1.1.1 holds for $X = \mathbb{P}^3$. Furthermore, [MP13a] shows that 1.1.1 holds for principally polarized abelian varieties under a specific choice for ω and B, and in [MP13b] for any choice of ω and B when the Picard rank is 1. The inequality was proved for smooth quadric threefolds in [Sch13].

By considering E to be the ideal sheaf of a curve C lying on a complete intersection three-fold, 1.1.1 gives the following conjectured bound on the genus of such a curve.

Conjecture 1.1.2. *[BMS14, Example 4.4] Suppose* C *is a smooth projective curve of degree* d *and genus* $g(C)$ *lying on a complete intersection threefold in* $\mathbb{P}^n$ *defined by equations of degrees* $k_1, \ldots, k_{n-3}$*. Then*

$$g(C) \leq \frac{2d^2}{3k_1 \cdots k_{n-3}} + \left(\frac{5 + 3(k_1 + \cdots + k_{n-3} - n - 1)}{6}\right) d + 1.$$

This is related to the classical Castelnuovo bounds on the genus of smooth projective curves. Castelnuovo showed that the genus g of a non-degenerate projective curve of degree d in $\mathbb{P}^n$ is bounded by $g \le (n-1)m(m-1)/2 + m\epsilon$ where $d-1 = m(n-1)+\epsilon$ and $0 \le \epsilon < n-1$ [Cas89].

We first consider curves lying on complete intersection surfaces. Our goal is to relate the degrees of the curve and the surface to the genus of the curve. This is a generalization of the results of [Har80] for curves lying on surfaces in $\mathbb{P}^3$. We are able to give the following bound on the genus g of such a curve, in terms of its degree, d and the degrees $k_1, \dots, k_{n-2}$ of the defining equations of the surface, when d satisfies:

$$d \ge k_1 \cdots k_{n-2}(k_1 + \cdots k_{n-2}). \tag{1.1}$$

Theorem 3.2.10. *Assume S is a complete intersection surface in $\mathbb{P}^n$ defined by equations of degrees $k_1, \dots, k_{n-2}$, and C is a degree d curve lying on S. Suppose the degrees $d, k_1, \dots, k_{n-2}$ satisfy (1.1). Let $\epsilon = d - k_1 \cdots k_{n-2}\lceil \frac{d}{k_1 \cdots k_{n-2}} \rceil$. Then the genus of C is bounded as follows:*

$$g(C) \le \frac{d^2}{2k_1 \cdots k_{n-2}} + \frac{1}{2}d(k_1 + \cdots + k_{n-2} - n - 1) + p(k_1, \dots, k_{n-2}, \epsilon)$$

where $p(k_1, \dots, k_{n-2}, \epsilon)$ is a degree n polynomial in $k_1, \dots, k_{n-2}$.

Our strategy is to bound the genus of the curve C by computing Hilbert functions of twists of the ideal sheaf of the set of intersection points of C with a general hyperplane. This strategy is used by Castelnuovo to achieve the classical results. Requiring that C lies on a given complete intersection surface allows us to compute these Hilbert functions directly in some cases, as in [Har80]. The main new strategy we use is to compute bounds on the Hilbert functions using a torus degeneration.

The question of bounding the genus of projective curves is also addressed in [Har82] and in [CCDG93]. Both papers address the Halphen problem, of bounding a curve in terms of the smallest degree s of a surface on which a curve lies. This question is also considered in [CCDG95], [CCDG96] and [DG08]. These papers consider curves satisfying certain flag conditions. In our case, requiring that C lies on a complete intersection surface gives a flag of smooth irreducible projective varieties $C \supseteq V_2 \supseteq \cdots \supseteq V_{n-1}$ where $V_i = Z(f_1, \dots, f_{n-i})$, such that C does not lie on any variety of dimension i with degree smaller than $k_1 \cdots k_{n-i}$. We are able to improve upon these bounds in the situation what S is a complete intersection surface.

We then consider curves lying on complete intersection threefolds. We prove that if C lies on a complete intersection threefold, it also lies on a complete intersection surface. Hence the bound in Theorem 3.2.10 would imply the bound given in Conjecture 3.3.3 if a weaker bound could be proved for curves of low degree.

1.2 Bridgeland stability on surfaces with curves of negative self-intersection

Let X be a smooth projective surface. Let $\mathcal{D}^b(X)$ be the bounded derived category of coherent sheaves on X.

Definition 1.2.1. *[Bri07, Proposition 5.3] A stability condition on X is a pair $\sigma = (Z, \mathcal{B})$ such that*

1. *$\mathcal{B}$ is a heart of a bounded t-structure in $\mathcal{D}^b(X)$.*
2. *$Z\colon K(\mathcal{B}) \to \mathbb{C}$ is a group homomorphism from the Groethendieck group of $\mathcal{B}$ to $\mathbb{C}$ whose image lies in the upper half plane unioned with $\mathbb{R}_{<0}$.*
3. *$\mathcal{B}$ has the Harder-Narasimhan property with respect to Z. That is, for every $E \in \mathcal{B}$, there is a filtration*
$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_{n-1} \hookrightarrow E_n = E$$
such that the quotients are σ-semistable of descending phase.

The full definition will follow in Chapter 2.

When X is a smooth projective surface, then the space $\mathrm{Stab}(X)$ of stability conditions on X is a manifold [Bri07, Theorem 1.2]. If we fix a class v in $K_{\mathrm{num}}(X)$, this manifold has a wall and chamber structure [Bri08, Section 9]. (This is proved in [Bri08] in particular for K3 surfaces, but the proof applies more generally to smooth projective surfaces). Within a chamber the stable objects of class v remain constant as the stability condition varies. We will fix v as the class of $\mathcal{O}_x$, the skyscraper sheaf at a point. In what is called the geometric chamber, all skyscraper sheaves $\mathcal{O}_x$ are stable. It is interesting to consider what happens as stability functions are deformed so that they cross out of the geometric chamber.

In particular, let $M^\sigma([\mathcal{O}_x])$ be the moduli space of σ-stable objects of class $[\mathcal{O}_x]$. Inside the geometric chamber, $M^\sigma([\mathcal{O}_x]) \cong X$. It is interesting to consider what $M^\sigma([\mathcal{O}_x])$ is after wall-crossing. In [Tod14], Toda shows that the minimal model program for surfaces can be achieved via wall-crosising in $\mathrm{Stab}(X)$. He shows that contractions of curves of self-intersection -1 can be realized as wall-crossing in $\mathrm{Stab}(X)$. That is, if $f\colon X \to Y$ is a birational map contracting a -1 curve on X, then there is a wall of the geometric chamber such that, after crossing, $M^\sigma([\mathcal{O}_x]) \cong Y$.

We consider a smooth projective surface X with a curve C of self-intersection $C^2 < 0$. When $C^2 < -1$, contracting such a curve would yield a singular surface. We find a wall in $\mathrm{Stab}(X)$ along which skyscraper sheaves along C become strictly semistable. In [BM14] the authors show that any stability condition σ in the geometric chamber of $\mathrm{Stab}(X)$ is associated to an ample divisor ω given by $\omega \cdot C' = \mathrm{Im}Z(\mathcal{O}_{C'})$ for all curves C' on X. Thus we construct this wall by choosing a nef divisor H on X so that $H \cdot C = 0$.

Theorem 4.2.10. *If X is a smooth projective surface containing a curve C such that $C^2 < 0$, then there is a wall in* $\mathrm{Stab}(X)$ *along which $M^\sigma([\mathcal{O}_x])$ is strictly semistable for points x on C.*

We then work to construct the moduli space of stable objects after wall-crossing.

Theorem 4.3.13. *If $C^2 = -n$, then after wall crossing, $M^\sigma([\mathcal{O}_x]) \cong X \sqcup_C \mathbb{P}^{n-1}$.*

This result generalizes the contraction of -1 curves in [Tod14] as well as the walls associated to -2 curves on K3 surfaces in [Bri08]. For $C^2 \leq -3$ it yields a reducible moduli space, the first example in the study of Bridgeland stability in which the moduli space becomes more complicated after wall crossing.

Chapter 2

Background

2.1 Bridgeland stability

Let X be a smooth projective variety, and let $\mathcal{D}^b(X)$ be the bounded derived category of coherent sheaves on X. In this section, our goal is to recall a notion of stability for objects in $\mathcal{D}^b(X)$ defined in [Bri07], and describe some properties of this definition of stability which will be important in the subsequent chapters.

First, we recall some properties of Coh(X) that distinguish it as a subcategory of $\mathcal{D}^b(X)$. We can consider Coh(X) to be a subcategory of $\mathcal{D}^b(X)$ as the set of complexes with cohomology only in degree 0. From now on we will refer to the abelian subcategory of $\mathcal{D}^b(X)$ consisting of complexes which are 0 except possibly in degree i as Coh(X)[−i].

We often view $\mathcal{D}^b(X)$ as being built out of objects in the subcategory Coh(X). All objects $E^\cdot$ in $\mathcal{D}^b(X)$ have a filtration by objects in Coh(X), called the filtration by cohomology. This filtration can be constructed as follows.

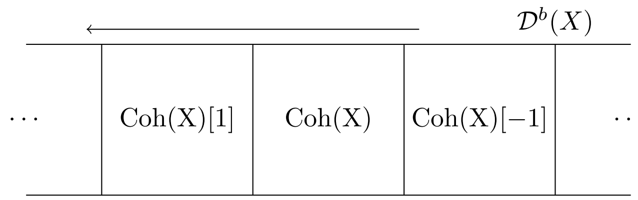
For every degree $i \in \mathbb{Z}$, we define a truncation functor $\tau_{\leq i} \colon \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ which takes any $E^\cdot \in \mathcal{D}^b(X)$ to a new complex $\tau_{\leq i}E^\cdot$ whose cohomology objects are the same as those of $E^\cdot$ for all degrees smaller than or equal to i. The terms of the complex $\tau_{\leq i}E^\cdot$ are given by

$$(\tau_{\leq i}E^\cdot)^{(j)} = \begin{cases} (E)^{(j)} & j < i \\ \ker(d_{i-1}) & j = i \\ 0 & j > i. \end{cases}$$

By construction, we get the following filtration of $E^\cdot$, where a is the smallest integer such that $H^a(E^\cdot) \neq 0$, and b is the largest integer such that $H^b(E^\cdot) \neq 0$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \tau_{\le a}E^{\cdot} & \longrightarrow & \tau_{\le a+1}E^{\cdot} & \longrightarrow \cdots \longrightarrow & \tau_{\le b-1}E^{\cdot} & \longrightarrow & \tau_{\le b}E^{\cdot} = E^{\cdot} \\
 & \nwarrow \swarrow & & \nwarrow \swarrow & & & & \nwarrow \swarrow & \\
 & H^a(E^{\cdot})[-a] & & H^{a+1}(E^{\cdot})[-(a+1)] & & & & H^bE^{\cdot}[-b] &
\end{array}
$$

Another important property of the subcategory Coh(X) in $\mathcal{D}^b(X)$ is that there are no negative extensions of objects in Coh(X). That is, If $E, F \in$ Coh(X), $\mathrm{Hom}^i(E,F) = 0$ for $i < 0$. These properties are sometimes visualized in a picture due to Bridgeland, drawn below. This picture shows how we view the category $\mathcal{D}^b(X)$ as broken into shifts of the category Coh(X). The arrow indicates that there are no morphisms from right to left.



There are other abelian subcategories of $\mathcal{D}^b(X)$ which share the properties that distinguish Coh(X) in $\mathcal{D}^b(X)$. More precisely, we could choose to take cohomology objects in other abelian subcategories of $\mathcal{D}^b(X)$. Such a subcategory is called a heart of a bounded t-structure.

Definition 2.1.1. *A heart of a bounded t-structure is a full additive subcategory $\mathcal{A}$ of $\mathcal{D}^b(X)$ satisfying*

1. $\mathrm{Hom}^i(A, B) = 0$ *for* $i < 0$ *and* $A, B \in \mathcal{A}$.

2. *Objects in $\mathcal{D}^b(X)$ have filtrations by cohomology objects in $\mathcal{A}$. That is, for all nonzero $E^{\cdot} \in \mathcal{D}^b(X)$, there is a sequence of exact triangles*

$$
\begin{array}{ccccccccc}
0 = E_0^{\cdot} & \longrightarrow & E_1^{\cdot} & \longrightarrow & E_2^{\cdot} & \longrightarrow \cdots \longrightarrow & E_{n-1}^{\cdot} & \longrightarrow & E_n^{\cdot} = E^{\cdot} \\
 & \nwarrow \swarrow & & \nwarrow \swarrow & & & & \nwarrow \swarrow & \\
 & A_1^{\cdot} & & A_2^{\cdot} & & & & A_n^{\cdot} &
\end{array}
$$

such that $A_i[-k_i] \in \mathcal{A}$ for integers $k_1 > \cdots > k_n$.

From now on, we will define the cohomology of $E^{\cdot}$ in the heart $\mathcal{A}$ to be $H^{k_i}_{\mathcal{A}}E^{\cdot} = A_i[-k_i]$.

Lemma 2.1.2. *If $\mathcal{A}$ is a heart of a bounded t-structure in $\mathcal{D}^b(X)$, then $\mathcal{A}$ is abelian.*

Definition 2.1.3. *[Bri07, Proposition 5.3] A Bridgeland stability condition is a pair $\sigma = (Z, \mathcal{A})$ where $Z\colon K_0(\mathcal{D}^b(X)) \to \mathbb{C}$ is a group homomorphism and $\mathcal{A}$ is a heart of a bounded t-structure. The pair must further satisfy that*

1. *$Z(\mathcal{A} \setminus \{0\}) \subseteq \{re^{i\pi\phi} \mid r > 0,\ 0 < \phi \leq 1\}$. Define the phase of $0 \neq E \in \mathcal{A}$ to be $\phi(E) := \phi$. We say $E \in \mathcal{A}$ is Z-semistable if for all nonzero subobjects $F \in \mathcal{A}$ of E, $\phi(F) \leq \phi(E)$. E is Z-stable if for all nonzero subobjects $F \in \mathcal{A}$ of E, $\phi(F) < \phi(E)$.*

2. *The objects of $\mathcal{A}$ have Harder-Narasimhan filtrations with respect to Z. That is, for every $E \in \mathcal{A}$ there is a unique sequence of inclusions*

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{n-1} \subseteq E_n = E$$

such that the successive quotients E_i/E_{i-1} are Z-semistable, and the phases $\phi(E_1/E_0) > \phi(E_2/E_1) > \cdots > \phi(E_{n-1}/E_{n-2}) > \phi(E_n/E_{n-1})$.

Example 2.1.4. *Let X be a smooth projective curve. Let $\mathrm{rk}(E)$ denote the rank of a sheaf $E \in \mathrm{Coh(X)}$, and let $\deg(E)$ denote the degree of E. We can define a slope function for sheaves on X as follows:*

$$\mu(E) = \begin{cases} \frac{\deg(E)}{\mathrm{rk}(E)} & E \text{ torsion} - \text{free} \\ \infty & E \text{ torsion} \end{cases}$$

There is a classical notion of slope stability for sheaves on X, defined by Mumford [Mum63]. A sheaf E is said to be slope stable if for all subsheaves $0 \neq F \subsetneq E$, $mu(F) < \mu(E)$.

We can define a stability condition in the sense of Definition 2.1.3 which extends the definition of slope stability to objects in $\mathcal{D}^b(X)$. We take the standard heart $\mathrm{Coh(X)}$ as our heart of a bounded t-structure. For an object $E^\cdot \in \mathcal{D}^b(X)$, we define the central charge $Z(E^\cdot) = -\deg(E^\cdot) + i\mathrm{rank}(E^\cdot)$. The degree and rank functions are additive on short exact sequences, and so this defines a group homomorphism $Z \colon K_0(\mathcal{D}^b(X) \to \mathbb{C}$.

The rank of a sheaf is always greater than or equal to 0. Furthermore, if $E \in \mathrm{Coh(X)}$ has rank 0, then its degree is strictly positive. Thus the image of $\mathrm{Coh(X)}$ under Z lies in the upper half plane and the negative real axis, as required. With Z defined as above, a sheaf is Z-stable if and only if it is slope stable. Thus, the existence of HN filtrations with respect to slope stability gives HN filtrations here.

In general, we will consider central charges such as the stability condition in the example, which factor through $H^*_{alg}(X, \mathbb{R})$, the chern characters of X.

2.2 Stability on surfaces

Let X be a smooth projective surface. Then it is not possible to define Bridgeland stability on the heart $\mathcal{A} = \mathrm{Coh(X)}$. To see what goes wrong, consider the following example.

Example 2.2.1. *Suppose X is a smooth projective surface containing a smooth rational curve C of nonzero self-intersection. Let $\mathcal{O}_C$ denote the pushforward of the structure sheaf of C via the* *inclusion map $C \hookrightarrow X$. Further, suppose $Z\colon K_0(X) \to \mathbb{C}$ is a group homomorphism, that is it is additive on short exact sequences.*

For any $m \in \mathbb{Z}$ there is a short exact sequence

$$0 \to \mathcal{O}_X((m-1)C) \to \mathcal{O}_X(mC) \to \mathcal{O}_C(mC) \to 0.$$

Inductively, we get that Z must satisfy the equation

$$Z(\mathcal{O}_X(mC)) = \sum_{i=1}^{m} Z(\mathcal{O}_C(iC)) + Z(\mathcal{O}_X). \tag{2.1}$$

Further, for each $x \in C$ and for each $i \in \mathbb{N}$, there is an exact sequence

$$0 \to \mathcal{O}_C((i-1)C) \to \mathcal{O}_C(iC) \to \mathcal{O}_x^{\oplus C^2} \to 0.$$

Inductively, this gives an equation for Z

$$Z(\mathcal{O}_C(iC)) = iC^2 Z(\mathcal{O}_x) + Z(\mathcal{O}_C). \tag{2.2}$$

Combining (2.1) and (2.2), Z must satisfy the quadratic equation

$$Z(\mathcal{O}_X(mC)) = Z(\mathcal{O}_X) + mZ(\mathcal{O}_C) + C^2\left(\frac{m^2+m}{2}\right) Z(\mathcal{O}_X).$$

That this equation is quadratic in m implies that the image of Coh(X) *under Z cannot lie in the upper half plane. Hence Z cannot be the central charge of a stability condition as defined in 2.1.3.*

Since we cannot define stability on Coh(X), we will have to look for a different heart of a bounded t-structure in $\mathcal{D}^b(X)$. The process by which this heart is constructed is called tilting.

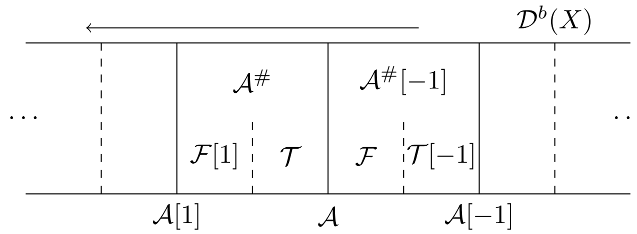
Definition 2.2.2. *A torsion pair in a heart $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full additive subcategories of $\mathcal{A}$ such that*

1. *If $T \in \mathcal{T}$ and $F \in \mathcal{F}$, then* $\mathrm{Hom}(T, F) = 0$.
2. *For all $E \in \mathcal{A}$ there is an object $T \in \mathcal{T}$ and $F \in \mathcal{F}$ so that the sequence $0 \to T \to E \to F \to 0$ is exact.*

Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, we can construct a new heart of a bounded t-structure

$$\mathcal{A}^{\#} = \{E^{\cdot} \in \mathcal{D}^b(X) \,|\, H^0_{\mathcal{A}}(E^{\cdot}) \in \mathcal{T},\, H^{-1}_{\mathcal{A}}(E^{\cdot}) \in \mathcal{F},\, H^i_{\mathcal{A}}(E^{\cdot}) = 0 \text{ for } i \neq 0, -1\}.$$

The following picture illustrates this.



We can define stability on a surface X on a tilt of the standard heart $\mathrm{Coh(X)}$. This is the tilt at slope [Bri08, Lemma 6.1]. First, we fix an ample divisor H on X. The slope of a nonzero sheaf $E \in \mathrm{Coh(X)}$ is

$$\mu_H(E) = \begin{cases} \frac{H \cdot \mathrm{ch_1(E)}}{\mathrm{ch_0(E)}} & E \text{ torsion} - \text{free} \\ \infty & E \text{ torsion} \end{cases}$$

Definition 2.2.3. *A sheaf E is μ_H-stable if for all subobjects $0 \neq F \subseteq E$, $\mu_H(F) < \mu_H(E)$. E is μ_H-semistable if for all subobjects $0 \neq F \subseteq E$, $\mu_H(F) \leq \mu_H(E)$.*

Note that it would be equivalent to define E to be μ_H-stable if for all quotients $E \twoheadrightarrow G$, $\mu_H(E) < \mu_H(G)$.

Fix a number $a \in \mathbb{R}$.

$$\mathcal{T}_H^a := \{T \in \mathrm{Coh(X)} \,|\, \mathrm{for\ all\ T \twoheadrightarrow S,\ \mu_H(S) > a}\}.$$

$$\mathcal{F}_H^a := \{F \in \mathrm{Coh(X)} \,|\, \mathrm{for\ all\ G \hookrightarrow F,\ \mu_H(G) \leq a}\}.$$

Note that all torsion sheaves and μ_H-semistable sheaves of slope greater than a lie in $\mathcal{T}^a$, and all μ_H-semistable sheaves of slope smaller than or equal to a lie in $\mathcal{F}^a$.

Lemma 2.2.4. *$(\mathcal{T}_H^a, \mathcal{F}_H^a)$ is a torsion pair in* $\mathrm{Coh(X)}$.

Proof. There are no morphisms from stable sheaves of slope greater than a to stable sheaves of slope smaller than or equal to a. This implies that $\mathrm{Hom}(\mathcal{T}_H^a, \mathcal{F}_H^a) = 0$. For any $E \in \mathrm{Coh(X)}$, we can construct a short exact sequence $0 \to T \to E \to F \to 0$ with $T \in \mathcal{T}^a$ and $F \in \mathcal{F}^a$ using the Harder-Narasimhan filtration of E with respect to μ_H. □

Proposition 2.2.5. *Choose a class $\beta \in NS_{\mathbb{R}}(X)$. The pair $\sigma_{H,\beta} = (Z_{H,\beta}, \mathcal{A}_H^0)$ is a Bridgeland stability condition on X, where $Z_{H,\beta}(E^\cdot) = -\mathrm{ch_2(E^\cdot)} + \beta \mathrm{ch_1(E^\cdot)} + \frac{\beta^2}{2}\mathrm{ch_0(E^\cdot)} + \mathrm{iHch_1(E^\cdot)}$.*

We will now consider the set of all stability conditions on X, denoted $\mathrm{Stab}(X)$. We will place a restriction on the stability conditions we consider. Recall that there is an Euler pairing on $K(X)$, defined by $\chi(E,F) = \sum_i (-1)^i \mathrm{dim\ Hom}^i(E,F)$. We will restrict to stability conditions which factor through the quotient $\mathcal{N}(X)$ of $K(X)$ by the kernel of the this pairing. These are called numerical stability conditions. The set of such stability conditions is denoted $\mathrm{Stab}_{\mathcal{N}}(X)$. The following theorem says that under this restriction, the set of stability conditions is in fact a complex manifold. Note that this theorem applies to X any smooth projective variety, without restriction on its dimension.

Theorem 2.2.6. *[Bri07, Corollary 1.3] For each connected component $\Sigma \subseteq \mathrm{Stab}_{\mathcal{N}}(X)$, there is a subspace $V(\Sigma) \subseteq \mathrm{Hom}(\mathcal{N}(X), \mathbb{C})$ and a local homeomorphism $Z\colon \Sigma \to V(\Sigma)$ which maps a stability condition to its central charge. In particular, Σ is a finite-dimensional complex manifold.*

2.3 The support property

Let X be a smooth projective surface. Given a stability condition on X, we would like to be able to deform the stability condition in $\mathrm{Stab}(X)$ and study how the set of stable objects changes and as the stability condition changes. In order to study such deformations, we will need to require that the stability conditions we study have a sort of continuity property called the support property. By [BM11, Proposition B.4], this is equivalent to the stability condtion being full, as defined in [Bri08, Definition 4.2].

Definition 2.3.1. *A stability condition $\sigma = (Z, \mathcal{A})$ satisfies the support property [KS08, Section 1.2] if there exists a constant $C > 0$ so that for all Z-stable $E^\cdot \in \mathcal{D}^b(X)$,*

$$\frac{|Z(E^\cdot)|}{||E^\cdot||} > C.$$

If σ is a stability condition satisfying the support property, and $E^\cdot$ is a σ-stable object, the argument of $Z(E^\cdot)$ does not change too much if σ is slightly deformed. In order to see this, consider a central charge W such that $||Z - W||_{op} < \epsilon$. That is, for any stable object $E^\cdot$,

$$|Z(E^\cdot) - W(E^\cdot)| < \epsilon ||E^\cdot||.$$

If Z satisfies the support property, then there exists a constant C independent of $E^\cdot$ so that

$$|Z(E^\cdot) - W(E^\cdot)| < \epsilon ||E^\cdot|| < \frac{\epsilon}{C} |Z(E^\cdot)|.$$

And so $W(E^\cdot)$ lies in a ball of radius $\frac{\epsilon}{C}|Z(E^\cdot)|$ around $Z(E^\cdot)$ [BMS14, Appendix A].

Now let us consider only stability conditions in $\mathrm{Stab}_{\mathcal{N}}(X)$ with the support property. Fix a primitive class $[E^\cdot]$ of objects in $K(\mathcal{D}^b(X))$. Then [Bri08, Section 9] shows that $\mathrm{Stab}_{\mathcal{N}}(X)$ has a wall and chamber structure. That is, $\mathrm{Stab}(X)$ decomposes into open subsets U called chambers, U, and codimension one closed submanifolds W. If σ is a stability condition in chamber U and $E^\cdot$ is a σ-stable objects of class $[E^\cdot]$, then $E^\cdot$ remains stable for all other stability conditions in U. That is, stable objects of class $[E^\cdot]$ may only destabilize along walls W.

Example 2.3.2. *Fix the class $[\mathcal{O}_x]$ of a skyscraper sheaf of a point. Then there is a special chamber U of $\mathrm{Stab}_{\mathcal{N}}(X)$ called the geometric chamber. For stability conditions $\sigma \in U$, $\mathcal{O}_x$ is stable for all $x \in X$.*

For stability conditions inside the geometric chamber of $\mathrm{Stab}(X)$, of the form given in 2.2.5, the support property comes from the classical Bogomolov-Gieseker inequality for stable sheaves [BM11]. This states that for a torsion-free slope stable sheaf E, $\mathrm{ch}_1(\mathrm{E})^2 - 2\mathrm{ch}_0(\mathrm{E})\mathrm{ch}_2(\mathrm{E}) \geq 0$. However, this inequality is no longer sufficient to prove the support property even at the walls of the geometric chamber.

In Chapter 4 we will construct stability conditions at the wall of the geometric chamber for specific surfaces. Showing that the support property holds will be a large part of the construction of these stability conditions. In Chapter 3 we will discuss the Bogomolov-Gieseker type inequality required for the support property to hold for threefolds, conjectured in [BMT14], and one of its consequences, a conjectured genus bound for curves on complete intersection threefolds.

Chapter 3

Castelnuovo-type inequality for curves on complete intersection surfaces

3.1 Introduction

It is natural to consider under what conditions the classical Castelnuovo bounds on the genus of smooth projective curves can be improved upon. We consider curves lying on complete intersection surfaces. Our goal is to relate the degrees of the curve and the surface to the genus of the curve. This is a generalization of the results of [Har80] for curves lying on surfaces in $\mathbb{P}^3$. We are able to give the following bound on the genus g of such a curve, in terms of its degree, d and the degrees $k_1, \ldots, k_{n-2}$ of the defining equations of the surface, when d is large with respect to the degree of the surface. Specifically, we will require the following:

$$d \geq k_1 \cdots k_{n-2}(k_1 + \cdots + k_{n-2}). \tag{3.1}$$

Theorem 3.2.10. *Assume S is a complete intersection surface in $\mathbb{P}^n$ defined by equations of degrees $k_1, \ldots, k_{n-2}$, and C is a degree d curve lying on S. Suppose the degrees $d, k_1, \ldots, k_{n-2}$ satisfy (3.1). Let $\epsilon = d - k_1 \cdots k_{n-2} \lceil \frac{d}{k_1 \cdots k_{n-2}} \rceil$. Then the genus of C is bounded as follows:*

$$g(C) \leq \frac{d^2}{2k_1 \cdots k_{n-2}} + \frac{1}{2} d(k_1 + \cdots + k_{n-2} - n - 1) + p(k_1, \ldots, k_{n-2}, \epsilon)$$

where $p(k_1, \ldots, k_{n-2}, \epsilon)$ is a polynomial in $k_1, \ldots, k_{n-2}, \epsilon$ given explicitly later.

Considered as a polynomial in d, the leading and linear term are sharp. The constant term, given later, is not sharp. It is a degree n polynomial in the degrees k_i of the defining equations of the surface.

3.1.1 History of the question

Our strategy is to bound the genus of the curve C by computing Hilbert functions of twists of the ideal sheaf of the set of intersection points of C with a general hyperplane. This strategy is used by Castelnuovo to achieve the classical results. Requiring that C lies on a given complete intersection surface allows us to compute these Hilbert functions directly in some cases, as in [Har80]. The main new strategy we use is to compute bounds on the Hilbert functions using a torus degeneration (see Proposition 3.2.6).

The question of bounding the genus of projective curves is also addressed in [Har82] and in [CCDG93]. Both papers address the Halphen problem, of bounding a curve in terms of the smallest degree s of a surface on which a curve lies. [Har82] gives a bound in terms of the degree d of a curve in $\mathbb{P}^r$ in terms of d, r and s when d is sufficiently large with respect to s and s is not large with respect to r. [CCDG93] extends this result to give a bound in terms of d, s and r removing the assumption about the relative sizes of r and s. These papers are able to give bounds which are sharp. Our results differ, in that we require this surface is a complete intersection surface, and we make weaker assumptions about the degree of this surface. However, in the case that the smallest degree surface is in fact a complete intersection surface satisfying the degree requirements, our bounds agree in the highest term, and our bound improves upon the bound given in [CCDG93] in the linear term.

This question is also considered in [CCDG95], [CCDG96] and [DG08]. These papers consider curves satisfying certain flag conditions. In our case, requiring that C lies on a complete intersection surface gives a flag of smooth irreducible projective varieties $C \supseteq V_2 \supseteq \cdots \supseteq V_{n-1}$ where $V_i = Z(f_1, \ldots, f_{n-i})$. It follows from Bezout's theorem, and our assumption that the degree of C is large, that the curve C cannot lie on any surface of degree less than the degree of $S = V_2$. Repeating this argument inductively, C does not lie on any i-dimensional varieties of degree smaller than the degree of V_i.

In the case when $n = 4$, [CCDG95] gives a sharp bound for curves on such a flag when $d > \max\{12(k_1k_2)^2, (k_1k_2)^3\}$. [CCDG96, Theorem 2.2] gives a bound for $n \geq 3$ when the $d > (k_1 \cdots k_{n-2})^2$ and $k_{n-2} \gg k_{n-3} \gg \cdots \gg k_1$. This bound matches ours in the quadratic term. We improve upon the linear term given in this bound, and require only (3.1), with no requirement on the relative sizes of the degrees k_i. [DG08, Inequality (2.1)] gives a bound for the genus of a curve lying on an irreducible surface without requiring it be a complete intersection surface. This result requires that $d > (k_1 \cdots k_{n-2})^2 - (k_1 \cdots k_{n-2})$, which is stronger than our assumption. The bound matches our quadratic and linear terms, while our constant term is a lower degree polynomial in the degree of the surface. [DGF12] refines this result in the case where the degree of the surface is small with respect to n.

3.1.2 Additional motivation

Our motivation for considering the genus of these curves is the study of Bridgeland stability on threefolds. [BMT14] conjectures a Bogomolov-Gieseker type inequality for stable sheaves on projective threefolds. This inequality predicts the existence of a genus bound for curves lying on complete intersection threefolds in terms of the degree of the curves, and the degrees of the defining equations of the threefold. In the final section of this paper, we show how the result of Theorem 3.2.10 could give such a bound, if it were extended to curves of low degree.

3.2 Curves on complete intersection surfaces

3.2.1 Hilbert functions and the genus of C

Our goal is to compute a bound on the genus of a curve lying on a complete intersection surface in $\mathbb{P}^n$. Our strategy will be to compute Hilbert functions of twists of a particular ideal sheaf, the ideal sheaf of points of intersection of the curve and a general hyperplane in $\mathbb{P}^n$. We will then use Riemann-Roch to arrive at a bound for the genus of the curve. This strategy follows ideas from [Har80], where he computed a bound for the case $n = 3$.

Let C be a curve of degree d in $\mathbb{P}^n$. Suppose C lies on a complete intersection surface S in $\mathbb{P}^n$. The Riemann-Roch theorem implies that if g is the genus of C, for $l \gg 0$,

$$g = dl - h^0(C, \mathcal{O}_C(l)) + 1.$$

We define α_l to be the dimension of the image of the restriction map

$$\rho_l \colon H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) \to H^0(C, \mathcal{O}_C(l)).$$

Then for $l \gg 0$, Riemann-Roch gives

$$g = dl - \alpha_l + 1. \tag{3.2}$$

Choose H to be a generic hyperplane in $\mathbb{P}^n$ and $\Gamma = H \cap C$. Let $\mathcal{I}_\Gamma$ be the ideal sheaf of Γ in $\mathbb{P}^n$. Let

$$\sigma_l \colon H^0(\mathbb{P}^n, \mathcal{I}_\Gamma(l)) \to H^0(C, \mathcal{I}_\Gamma(l)|_C)$$

be the restriction map. The spaces $H^0(C, \mathcal{O}_C(l-1))$ and $H^0(C, \mathcal{I}_\Gamma(l)|_C)$ are isomorphic via multiplication by the defining equation of H. Further, $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l-1))$ injects into $H^0(\mathbb{P}^n, \mathcal{I}_\Gamma(l))$ via the same multiplication map, call it h.

$$\begin{array}{ccc} H^0(\mathbb{P}^n, \mathcal{O}(l-1)) & \xrightarrow{\rho_{l-1}} & H^0(C, \mathcal{O}(l-1)) \\ h \big\downarrow & & \big\downarrow h|_C \wr \\ H^0(\mathbb{P}^n, \mathcal{I}_\Gamma(l)) & \xrightarrow{\sigma_l} & H^0(C, \mathcal{I}_\Gamma(l)|_C) \end{array}$$

Thus the image of ρ_{l-1} is contained in the the image of $h|_C^{-1} \circ \sigma_l$. In other words, $\alpha_{l-1} \leq \dim \mathrm{Im}\sigma_l$. Then difference $\alpha_l - \alpha_{l-1}$ is bounded from below by the difference $\alpha_l -$ dim Im σ_l. The kernel of ρ_l is $H^0(\mathbb{P}^n, \mathcal{I}_C(l))$, the sections of $\mathcal{O}_{\mathbb{P}^n}(l)$ vanishing on C. This is also the kernel of σ_l. Thus

$$\begin{aligned} \alpha_l - \alpha_{l-1} &\geq \alpha_l - \dim \mathrm{Im}\sigma_l \\ &= \big(h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) - h^0(\mathbb{P}^n, \mathcal{I}_C(l))\big) - \big(h^0(\mathbb{P}^n, \mathcal{I}_\Gamma(l)) - h^0(\mathbb{P}^n, \mathcal{I}_C(l))\big) \\ &= h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) - h^0(\mathbb{P}^n, \mathcal{I}_\Gamma(l)). \end{aligned}$$

We define β_l to be $h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) - h^0(\mathbb{P}^n, \mathcal{I}_\Gamma(l))$.

The sequence of sheaves $\mathcal{O}_{\mathbb{P}^n}(l-1) \to \mathcal{O}_{\mathbb{P}^n}(l) \to \mathcal{O}_H(l)$ is exact. Further, the sequence $\mathcal{O}_{\mathbb{P}^N}(l-1) \to \mathcal{I}_\Gamma(l) \to \mathcal{I}_\Gamma(l)|_H$ is exact. The restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) \to H^0(H, \mathcal{O}_H(l)$ is surjective, as is the restriction map $H^0(\mathbb{P}^n, \mathcal{I}_\Gamma(l)) \to H^0(H, \mathcal{I}_\Gamma(l)|_H)$. Taking cohomology of both, we see that

$$\begin{aligned} \beta_l &= h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) - h^0(\mathbb{P}^n, \mathcal{I}_\Gamma(l)) \\ &= h^0(H, \mathcal{O}_H(l)) + h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l-1)) - h^0(H, \mathcal{I}_\Gamma(l)|_H) - h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l-1)) \\ &= h^0(H, \mathcal{O}_H(l)) - h^0(H, \mathcal{I}_\Gamma(l)|_H). \end{aligned}$$

Define $\gamma_0 := \beta_0$ and $\gamma_l := \beta_l - \beta_{l-1}$ for $l \geq 1$. Let P be a generic hyperplane in $H \cong \mathbb{P}^{n-1}$ containing no points of Γ. Then the following sequence is exact:

$$0 \to H^0(H, \mathcal{I}_\Gamma(l-1)|_H) \to H^0(H, \mathcal{I}_\Gamma(l)|_H) \to H^0(P, \mathcal{O}_P(l)).$$

Define e_l to be the dimension of the image of the second map. In other words, $e_l := h^0(H, \mathcal{I}_\Gamma(l)|_H) - h^0(H, \mathcal{I}_\Gamma(l-1)|_H)$. This leads to the following relationship between γ_l and e_l:

$$\gamma_l = \binom{l+n-2}{n-2} - e_l.$$

If we can compute γ_l, this will allow us to bound the genus of C as follows. First, we claim that

$$\sum_{i=0}^{l} \beta_i = \sum_{i=0}^{l} (l - i + 1)\gamma_i. \tag{3.3}$$

This follows from the fact that $\beta_i = \sum_{k=0}^{i} \gamma_k$. Further, there is an exact sequence of sheaves

$$0 \to \mathcal{I}_\Gamma(l) \to \mathcal{O}(l) \to \mathcal{O}_\Gamma \to 0.$$

For $l \gg 0$, $H^1(\mathbb{P}^n, \mathcal{I}_\Gamma(l)) = 0$, and so $\sum_{i=0}^{l} \gamma_i = d$. Substituting this into (3.3) gives $\sum_{i=0}^{l} \beta_i = ld - \sum_{i=0}^{l}(i-1)\gamma_i$. Since $\alpha_l \geq \sum_{i=0}^{l} \beta_i$, then (3.2) gives the following bound on g in terms of γ_i.

Lemma 3.2.1. *For $l \gg 0$,*

$$g \leq \sum_{i=0}^{l} (i-1)\gamma_i + 1.$$

Our strategy will be to find contraints on the γ_i, and then to compute a function γ_i^{max} which maximizes the right hand side subject to these constraints. Substituting in γ_i^{max} to the formula above will give a bound on the genus of any such curve.

3.2.2 Calculating γ_i for curves of large degree

Let C be a curve of degree d and genus g lying on complete intersection surface S in $\mathbb{P}^n$. Say S is defined by equations $f_1, \ldots, f_{n-2}$ of degrees $k_1, \ldots, k_{n-2}$ respectively. Assume $k_1 \leq k_2 \leq \cdots \leq k_{n-2}$. We will assume that d is large in relation to the degree of S, specifically, that (3.1) holds.

Let $H \cong \mathbb{P}^{n-1}$ and $P \cong \mathbb{P}^{n-2}$ be positioned as in the previous section, with $\Gamma = C \cap H$ a set of d distinct points, and $P \subset H$ containing none of these. For small values of i, γ_i does not depend on C but only on S, and can be computed directly.

Lemma 3.2.2. *Let m be the smallest integer so that $H^0(H, \mathcal{I}_\Gamma(m))$ contains a section s not vanishing on S. For $i < m$, $\gamma_i = \binom{i+n-2}{n-2} - \dim(f_1|_P, \ldots, f_{n-2}|_P)^{(i)}$. For $i \geq m$, $\gamma_i \leq \binom{i+n-2}{n-2} - \dim(f_1|_P, \ldots, f_{n-2}|_P, s|_P)^{(i)}$.*

Proof. For $i \leq m$ all sections of $\mathcal{I}_\Gamma(i)$ lie in the ideal of S, which we then restrict to $P \cong \mathbb{P}^{n-2}$ as before, and so $e_i = \dim(f_1|_P, \ldots, f_{n-2}|_P)^{(i)}$. For $i \geq m$, the inclusion of the ideal $(f_1|_P, \ldots, f_{n-2}|_P, s|_P)$ in the ideal of Γ gives the inequality. □

Note that by Bezout's theorem, if s is a section of $\mathcal{I}_\Gamma(m)$ not vanishing on S, $S \cap H \cap Z(s)$ must be 0-dimensional subvariety of H of degree $mk_1 \cdots k_{n-2}$. And so our assumption that Γ lies in this intersection forces $m \geq m_0$ where $m_0 = \lceil \frac{d}{k_1 \cdots k_{n-2}} \rceil$.

Lemma 3.2.3. *Suppose $g_1, \dots, g_r$ form a regular sequence in $R = k[x_0, \dots, x_n]$ where k is an algebraically closed field, and the degree of g_i is d_i. Let T be the multiset of all nonzero partial sums of the d_i with elements repeated when a sum is achieved in multiple ways. For $t \in T$ define $\operatorname{sgn}(t)$ to be -1 when t is a sum of an even number of degrees d_i, and 1 otherwise. Then for $l \geq d_1 + \cdots + d_r$,*

$$\dim(g_1, \dots, g_r)^{(l)} = \sum_{t \in T} \operatorname{sgn}(t) \binom{l - t + n}{n}.$$

Proof. For $r = 1$ we can give a basis for $(g_1)^{(l)}$ by taking all monomials of degree $l - d_1$ and multiplying these by g_1. There are $\binom{l-d_1+n}{n}$ such monomials in $x_0, \dots, x_n$, and there are no relations between them, so the above formula holds.

We now proceed by induction on r. There is a short exact sequence

$$0 \to R/(g_1, \dots, g_{r-1}) \to R/(g_1, \dots, g_{r-1}) \to R/(g_1, \dots, g_r) \to 0$$

where the first map is multiplication by g_r. If we consider the degree l parts of the modules in this sequence by additivity we get the relation

$$\dim(R/(g_1, \dots, g_{r-1}))^{(l)} + \dim(R/(g_1, \dots, g_r))^{(l)} = \dim(g_r R/(g_1, \dots, g_{r-1}))^{(l)}.$$

We can compute each of these and rearrange to arrive at the following:

$$\dim(g_1, \dots, g_r)^{(l)} = \binom{l - d_r + n}{n} + \dim(g_1, \dots, g_{r-1})^{(l)} - \dim(g_1, \dots, g_{r-1})^{(l-d_r)}.$$

If the formula holds for $r - 1$ then this equation shows it holds for r. □

Now lemma 3.2.2 implies the following about the vanishing of γ_i.

Corollary 3.2.4. *Let m be the smallest integer so that $H^0(H, \mathcal{I}_\Gamma(m))$ contains a section s not vanishing on S. Then $\gamma_i = 0$ for $i \geq m + k_1 + \cdots + k_{n-2} - n + 2$.*

Proof. For $i \geq m + k_1 + \cdots k_{n-2}$, we can compute $\binom{i+n-2}{n-2} - \dim(f_1, \dots, f_{n-2}, s)^{(i)}$ directly using Lemma 3.2.3. This describes the Hilbert polynomial of a complete intersection variety in $\mathbb{P}^{n-2}$ defined by equations of degrees $k_1, \dots, k_{n-2}, m$. Since there are $n-1$ defining equations, this is the empty set, and so it is 0.

We will now consider $m + k_1 + \cdots + k_{n-2} - n + 2 \leq i < m + k_1 + \cdots + k_{n-2}$. Define $t_{max} = \sup\{t \in T \,|\, t < k_1 + \cdots + k_{n-2} + m - n + 2\}$ where T is as in Lemma 3.2.3, the multiset of partial sums of the degrees $k_1, \dots, k_{n-2}, m$. For $i > t_m$ we can compute the following bound on γ_i in a method similar to that of Lemma 3.2.3.

$$\gamma_i \leq \sum_{t\in T,\, t\leq t_{max}} \operatorname{sgn}(t)\binom{i-t+n-2}{n-2}.$$

Using the fact that the sum over all t of these binomials is 0, we can rewrite this as

$$\gamma_i \leq -\sum_{t\in T,\, t>t_{max}} \operatorname{sgn}(t)\binom{i-t+n-2}{n-2}.$$

For $k_1+\cdots+k_{n-2}+m-n+2\leq i<k_1+\cdots+k_{n-2}+m$, each binomial above is 0.

On the other hand, γ_i must be nonnegative. And so $\gamma_i=0$. □

We will now show that for $i\geq m$, γ_i is nonincreasing with i. We will first need the following lemma.

Lemma 3.2.5. *Given a hyperplane section $L\cong\mathbb{P}^{n-3}\subset P$ for which $L\cap S$ is empty, the map $H^0(H,\mathcal{I}_\Gamma(i))\to H^0(L,\mathcal{O}_L(i))$ is surjective for $i\geq k_1+\cdots+k_{n-2}$.*

Proof. Let ρ be the map from $H^0(H,\mathcal{I}_\Gamma(i))\to H^0(L,\mathcal{O}_L(i))$ restricting sections to L. The dimension of the image of ρ is $\binom{i+n-3}{n-3}-h_z(i)$ where $h_Z(i)$ is the Hilbert function $h_Z(i)$ of the variety $Z=\Gamma\cap L$. This is by construction a set of 0 points in $\mathbb{P}^{n-3}$. For i sufficiently large, $h_Z(i)=0$. Specifically, consider the ideal of $Z'=S\cap L$. This is also an ideal defining 0 points in L, and is contained in the ideal of $\Gamma\cap L$. We have $h_{Z'}(i)=0$ for $i\geq k_1+\cdots+k_{n-2}$, implying that $h_Z(i)$ vanishes for i in this range as well. □

Proposition 3.2.6. *For $i\geq k_1+\cdots+k_{n-2}$, $\gamma_{i+1}\leq\gamma_i$.*

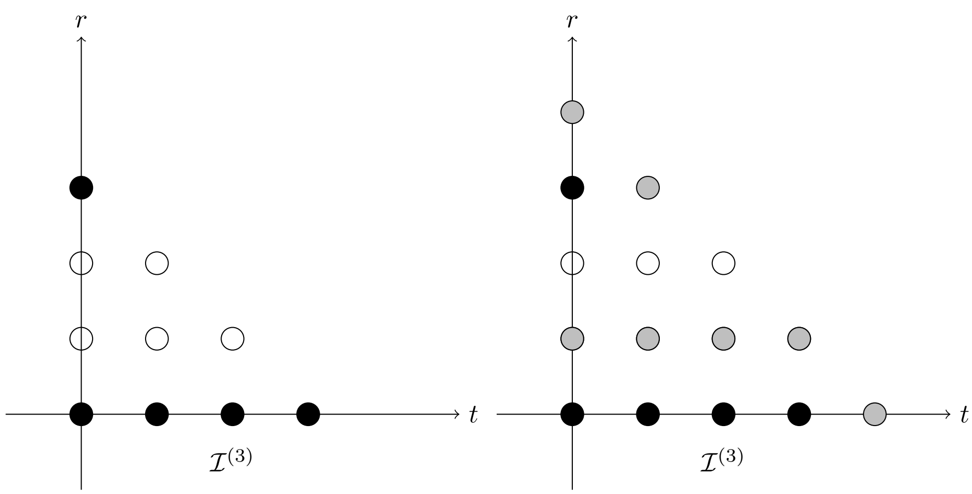
Proof. It is equivalent to show that $e_{i+1}-e_i\geq\binom{i+n-2}{n-3}$. Fix $L\cong\mathbb{P}^{n-3}$ as in Lemma 3.2.5 so that P has coordinates $x_0,\ldots,x_{n-3},y$ and L is given by the equation $y=0$ in P. Lemma 3.2.5 shows that for every monomial $x_0^{i_0}\cdots x_{n-3}^{i_{n-3}}$ such that $i_0+\cdots i_{n-3}\geq i$ there is a corresponding element $x_0^{i_0}\cdots x_{n-3}^{i_{n-3}}+yg$ in the ideal $H^0(H,\mathcal{I}_\Gamma(i))$.

Now consider the torus action on P sending $[x_0:\cdots:x_{n-3}:y]$ to $[x_0:\cdots:x_{n-3}:ty]$. Letting this act on the ideal of Γ, the limit ideal will contain $(x_0,\ldots,x_{n-3})^{(i)}$. In [HS04], the authors show that the multigraded Hilbert scheme is a projective variety. Therefore, under this degeneration of $\mathcal{I}_\Gamma$ to the new ideal $\mathcal{I}$, we do not change the Hilbert function of the ideal. This will imply that $e_i=h^0(H,\mathcal{I}(i))-h^0(H,\mathcal{I}(i-1))$.

This proof will proceed by induction. As a base case, let $n=4$. Then if a monomial m is contained in $\mathcal{I}(i)$, x_1m, x_2m and ym will be contained in $\mathcal{I}(i+1)$. Consider the embedding of $\mathcal{I}(i)$ in $\mathcal{I}(i+1)$ mapping m to x_1m. We will show that the dimension of $\mathcal{I}(i+1)$ is at least $i+2$ larger than the dimension of $\mathcal{I}(i)$ by finding $i+2$ monomials which are not in the image of this embedding.

By Lemma 3.2.5, when $i \geq k_1 + k_2$, $\mathcal{I}(i)$ contains the monomial $x_0^{i-t} x_1^t$ for each $t = 0, \ldots, i$. For each fixed value of t, we can find in $\mathcal{I}(i)$ the monomial $x_0^{i-t-r} x_1^t y^r$ for which r is maximal. Then $x_0^{i-t-r} x_1^t y^{r+1}$ is contained in $\mathcal{I}(i+1)$. Since by assumption, $x_0^{i-t-r-1} x_1^t y^{r+1}$ is not in $\mathcal{I}(i)$ we see that the dimension increases by at least $i+1$. Further, since x_1^i is assumed to be in $\mathcal{I}(i)$, x_1^{i+1} is in $\mathcal{I}(i+1)$. This gives one more new monomial in $\mathcal{I}(i+1)$, showing that the dimension increases by at least $i+2$.

The following picture illustrates this monomial counting for $n = 4$.. The filled dots in the first picture in position (t, r) represent elements $x_0^{3-t-r} x_1^t y^r$ of $\mathcal{I}(3)$. The empty dots represents monomials not contained in the ideal. In the second picture, the black circles represent the elements of $\mathcal{I}(4)$ in the image of the embedding of $\mathcal{I}(3)$ into $\mathcal{I}(4)$ via multiplication by x_0, and the gray dots represent the so-called new monomials. Note that Lemma 3.2.5 implies that the dots lying on the t-axis are filled in the first picture.



Now we will return to general n, and assume the proposition holds for $n-1$. Consider again the injection $\mathcal{I}^{(i)} \hookrightarrow \mathcal{I}^{(i+1)}$ sending monomial m to monomial $x_0 m$. We will find $\binom{i+n-2}{n-3}$ monomials in $\mathcal{I}^{(i+1)}$ which cannot be written in the form $x_0 m$ for a monomial in $\mathcal{I}^{(i)}$, showing the dimension has increased by at least this much. By Lemma 3.2.5, for $i \geq k_1 + \cdots + k_{n-2}$, $\mathcal{I}^{(i)}$ contains all monomials of degree i in the variables $x_0, \ldots, x_{n-3}$. For any fixed monomial of this form, $m = x_0^{t_0} \cdots x_{n-3}^{t_{n-3}}$, there is a maximum value of r for which $m_r = x_0^{t_0 - r} x_1^{t_1} \cdots x_{n-3}^{t_{n-3}} y^r$ lies in $\mathcal{I}^{(i)}$. Then $y m_r$ lies in $\mathcal{I}^{(i+1)}$, but $y m_r / x_0$ does not lie in $\mathcal{I}(i)$. This strategy gives $\binom{i+n-3}{n-3}$ new monomials, each containing y as a factor.

Now consider all monomials of degree i in $x_0, \ldots, x_{n-4}$. Again, by Lemma 3.2.5, all such monomials are in $\mathcal{I}^{(i)}$. Then by the inductive hypothesis, we can find $\binom{i+n-3}{n-4}$ new monomials in $\mathcal{I}^{(i+1)}$ in the variables $x_0, \ldots, x_{n-3}$. This gives a total of $\binom{i+n-2}{n-3}$ new monomials, as needed. □

We now have the following constraints for γ_i.

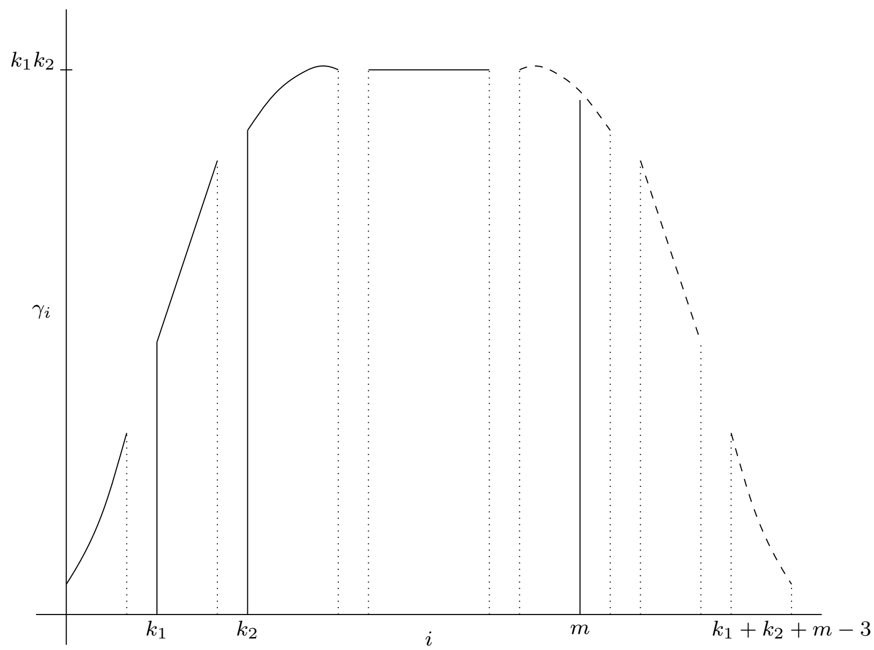
1. For $0 \leq i < m$, $\gamma_i = \binom{i+n-2}{n-2} - \dim(f_1|_P, \ldots, f_{n-2}|_P)^{(i)}$.

2. For $i \geq m$, $\gamma_i \leq \binom{i+n-2}{n-2} - \dim(f_1|_P, \ldots, f_{n-2}|_P, s|_P)^{(i)}$.

3. For $i \geq m$, γ_i is non increasing with i.

4. $\sum_{i=0}^{\infty} \gamma_i = d$.

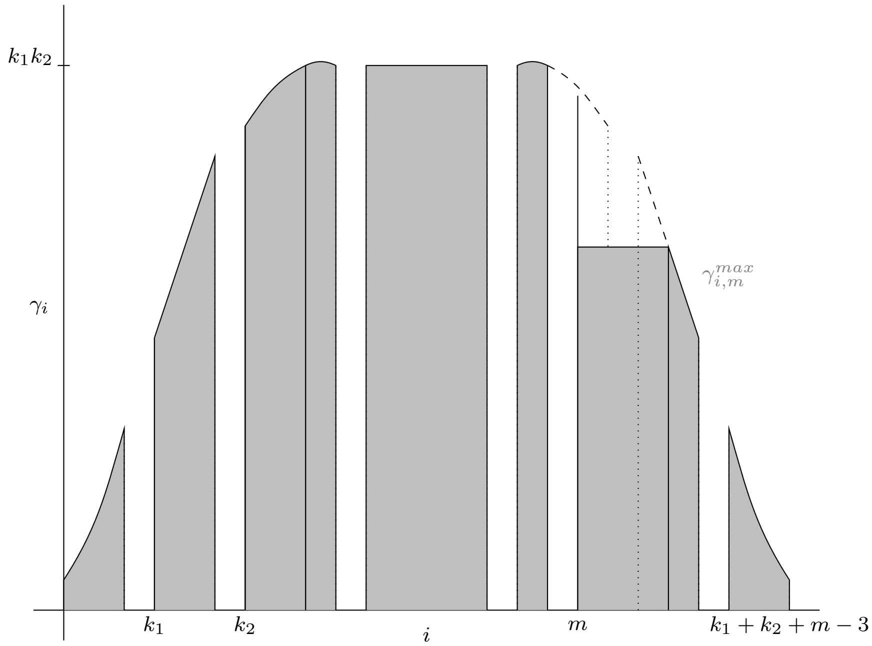
The following picture illustrates these constraints for $n = 4$. γ_i must lie along solid lines and below dashed lines. For $n > 4$ the picture would be similar, with the graph broken into as many as 2^{n-1} pieces, one for each subset of the set $\{k_1, \ldots, k_{n-2}, m\}$, and achieving a maximum height of $k_1 \cdots k_{n-2}$.



Because we know that $\sum_{i=0}^{\infty} \gamma_i = d$, we can compute

$$\sum_{i=m}^{\infty} \gamma_i = d - mk_1 \cdots k_{n-2} + \frac{1}{2} k_1 \cdots k_{n-2}(k_1 + \cdots + k_{n-2} + n - 2).$$

Given a fixed value of m, the function $\gamma_{i,m}^{max}$ will be the function γ_i satisfying this sum under the curve specified by conditions (1) and (2) which has area as far to the right as p.



This function can be calculated for each m, and then γ_i^{max} is the function $\gamma_{i,m}^{max}$ which maximizes the sum $\sum(i-1)\gamma_{i,m}^{max}$. Then the genus of the curve will be bounded by $\frac{1}{2}k_1\cdots k_{n-2}m(k_1+\cdots+k_{n-2}+m-n-1)+1-C$, where C is the weighted sum of the shaded area in the picture.

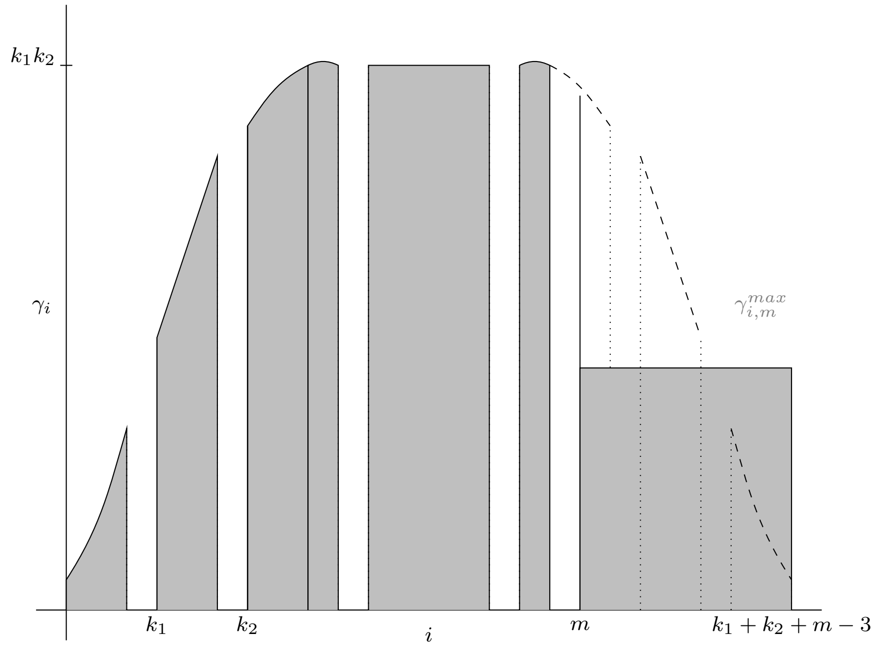
This strategy will easily give a genus bound for C given specific values of k_i and d. However, in order to compute a general bound we will relax the second constraint. For the purpose of this computation, we will replace the constraint that $\gamma_i \leq \binom{i+n-2}{n-2} - \dim(f_1|_P,,\ldots,f_{n-2}|_P, s|_P)^{(i)}$ for $i \geq m$ with the less restrictive constraint that $\gamma_i = 0$ for $i \geq m+k_1+\cdots+k_{n-2}-n+2$. That is, we will require the following of γ_i.

1. For $0 \leq i < m$, $\gamma_i = \binom{i+n-2}{n-2} - \dim(f_1|_P,\ldots,f_{n-2}|_P)^{(i)}$.
2. For $i \geq m+k_1+\cdots+k_{n-2}-n+2$, $\gamma_i = 0$.
3. For $i \geq m$, γ_i is non increasing with i.
4. $\sum_{i=0}^{\infty}\gamma_i = d$.

The function $\gamma_{i,m}^{max}$ which satisfies these constraints which maximizes $\sum(i-1)\gamma_{i,m}^{max}$ is the function which is constant for $m \leq i < m+k_1+\cdots+k_{n-2}-n+2$ and sums to d.

$$\gamma_{i,m}^{max} = \begin{cases} \binom{i+n-2}{n-2} - \dim(f_1|_P,\ldots,f_{n-2}|_P)^{(i)}, & \text{if } 0 \leq i < m \\ \frac{1}{2}k_1\cdots k_{n-2} + \frac{d-k_1\cdots k_{n-2}m}{k_1+\cdots+k_{n-2}-n+2}, & \text{if } m \leq i < m+k_1+\cdots+k_{n-2}-n+2 \end{cases}$$

The following picture illustrates $\gamma_{i,m}^{max}$ for $n = 4$.



The sum in Lemma 3.2.1 is now simple to compute for $i \geq k_1 + \cdots k_{n-2}$, where the function $\gamma_{i,m}^{max}$ is piecewise constant. This makes the following result an easy computation.

Theorem 3.2.7. *The genus g of C is bounded above by a second degree polynomial in d whose leading terms are $\frac{d^2}{2k_1 \cdots k_{n-2}} + \frac{d}{2}(k_1 + \cdots + k_{n-2} - n - 1)$.*

Proof. By Lemma 3.2.1, we can compute a bound on g by maximizing $\sum(i-1)\gamma_{i,m}^{max}$ with respect to m. Note first that for $i < k_1 + \cdots + k_{n-2}$, $\gamma_{i,m}^{max}$ contains no m terms, and so can be ignored in this computation.

We then compute $\sum_{i \geq k_1 + \cdots k_{n-2}}(i-1)\gamma_{i,m}^{max}$ and ignore any terms not involving m to get $dm - \frac{1}{2}k_1 \cdots k_{n-2}m^2$. This is maximized at $\frac{d}{k_1 \cdots k_{n-2}}$, and strictly decreasing for m greater than this. As explained before, Bezout's theorem gives a minimum value of $m_0 = \lceil \frac{d}{k_1 \cdots k_{n-2}} \rceil$ for m, and so we set $\gamma_i^{max} = \gamma_{i,m_0}^{max}$ This gives the above result. □

Note that these terms agree with the higher order terms for the genus of a complete intersection curve. In that case, the constant term would be 1. In order to compute our constant term, we need to be able to compute the sum for small i as well. Here, the function $\gamma_{i,m}^{max}$ can be computed using binomial coefficients. We will use the following binomial identity in order to compute this.

Lemma 3.2.8. *Let A and B be two nonnegative integers. Then*

$$\sum_{i=A}^{A+B-1}(i-1)\binom{i+n-2-A}{n-2}=\frac{1}{n}\binom{B+n-2}{n-1}(nA+(n-1)B-2n+1).$$

Proof. First we write

$$\begin{aligned}\sum_{i=A}^{A+B-1}(i-1)\binom{i+n-2-A}{n-2}&=(A-1)\sum_{i=0}^{B-1}\binom{i+n-2}{n-2}+\sum_{i=0}^{B-1}j\binom{i+n-2}{n-2}\\&=(A-1)\binom{B+n-2}{n-1}+\sum_{i=0}^{B-1}i\binom{i+n-2}{n-2}.\end{aligned}$$

It is then equivalent to prove that

$$\sum_{i=0}^{B-1}i\binom{i+n-2}{n-2}=\frac{1}{n}B(B-1)\binom{B+n-2}{n-2}.$$

We proceed by induction on B. For $B=1$ the claim is obvious. Now suppose it holds for $B=k$. Then

$$\begin{aligned}\sum_{i=0}^{k}i\binom{i+n-2}{n-2}&=\frac{1}{n}k(k-1)\binom{k+n-2}{n-2}+k\binom{k+n-2}{n-2}\\&=\frac{1}{n}k(k+n-1)\binom{k+n-2}{n-2}\\&=\frac{1}{n}k(k+1)\binom{k+n-1}{n-2}.\end{aligned}$$

□

Lemma 3.2.9. *Let T be the multiset of nonzero partial sums of the degrees k_i, with an element repeated in T for each way in which it can be formed as a sum of k_i. Then*

$$\sum_{t\in T}\frac{(-1)^n}{n}\operatorname{sgn}(t)t\binom{t+n-2}{n-1}=-k_1\cdots k_{n-2}\left(\frac{1}{6}\sum_{i=1}^{n-2}k_i^2+\frac{1}{4}\sum_{i=2}^{n-2}\sum_{j=1}^{i}k_ik_j+\frac{\binom{n-1}{2}}{2n}\sum_{i=1}^{n-2}k_i+\frac{(n-2)!(3n-4)\binom{n-1}{3}}{4n!}\right).$$

Proof. All terms not divisible by $k_1 \cdots k_{n-2}$ are canceled by later terms in the alternating sum above, so it suffices to determine the coefficients of terms divisible by this monomial in the product $\frac{1}{n}(k_1 + \cdots + k_{n-2})\binom{k_1+\cdots k_{n-2}+n-2}{n-1}$. That is, we need to calculate the coefficients of terms divisible by $k_1 \cdots k_{n-2}$ in $\frac{1}{n!}(k_1 + \cdots + k_{n-2})^2(k_1 + \cdots + k_{n-2} + 1) \cdots (k_1 + \cdots + k_{n-2} + n - 2)$.

For terms of degree n, this is the same as the coefficient in $\frac{1}{n!}(k_1 + \cdots + k_{n-2})^n$, which can be computed with multinomial coefficients. For the terms of degree $n-1$, the coefficient will be the product of the multinomial coefficient of $k_i k_1 \cdots k_{n-2}$ in $(k_1 + \cdots k_{n-2})^{n-1}$ multiplied with the sum $1 + \cdots + n-2$, and $\frac{1}{n!}$ giving $\frac{\binom{n-1}{2}(n-1)!}{2n!}$. Similarly, the coefficient of the degree $n-2$ term is found as the coefficient of $k_1 \cdots k_{n-2}$ in $(k_1 + \cdots + k_{n-2})^{n-2}$ multiplied by $\frac{1}{n!}$ and the sum of all products of two numbers in the list $1, \ldots, n-2$. □

We are now able to state the full genus bound for C.

Theorem 3.2.10. *Let $\epsilon = d - k_1 \cdots k_{n-2}\lceil \frac{d}{k_1 \cdots k_{n-2}} \rceil$. Then the genus of C is bounded as follows:*

$$g(C) \leq \frac{d^2}{2k_1 \cdots k_{n-2}} + \frac{1}{2}d(k_1 + \cdots + k_{n-2} - n - 1) - \frac{\epsilon^2}{2k_1 \cdots k_{n-2}} + 1 + \frac{1}{12}k_1 \cdots k_{n-2}\left(\sum_{i=1}^{n-2} k_i^2 + 3\sum_{i=2}^{n-2}\sum_{j=1}^{i} k_i k_j - 3(n-2)\sum_{i=1}^{n-2} k_i + \frac{(n-2)(3n-5)}{2}\right).$$

Proof. The bound is computed using Lemma 3.2.1, and the function $\gamma_{i,m}^{max}$. It then follows from Theorem 3.2.7 and Lemma 3.2.8 that

$$g(C) \leq \frac{d^2}{2k_1 \cdots k_{n-2}} + \frac{1}{2}d(k_1 + \cdots + k_{n-2} - n - 1) - \frac{\epsilon^2}{2k_1 \cdots k_{n-2}} + 1 - \frac{1}{4}k_1 \cdots k_{n-2}[(k_1 + \cdots + k_{n-2})^2 + (2n-7)(k_1 + \cdots + k_{n-2}) - n^2 + n] + \sum_{t \in T}(-1)^n \operatorname{sgn}(t)\frac{1}{n}\binom{t+n-2}{n-1}(nk_1 + \cdots + nk_{n-2} - t - 2n + 1).$$

Then Lemma 3.2.9 gives the constant term. □

We give this bound explicitly for small n, in order to give a sense of how large the constant term becomes.

Example 3.2.11. *In $\mathbb{P}^4$, the genus of a curve C satisfying the above conditions is bounded by the following polynomial:*

$$g(C) \leq \frac{d^2}{2k_1k_2} + \frac{1}{2}d(k_1 + k_2 - 5) - \frac{\epsilon^2}{2k_1k_2} + 1 + \frac{1}{12}k_1k_2(k_1^2 + k_2^2 + 3k_1k_2 - 6(k_1 + k_2) + 7).$$

Example 3.2.12. *In $\mathbb{P}^5$, the genus of a curve C satisfying the above conditions is bounded by the following polynomial:*

$$g(C) \leq \frac{d^2}{2k_1k_2k_3} + \frac{1}{2}d(k_1+k_2+k_3-6) - \frac{\epsilon^2}{2k_1k_2k_3} + 1 + \frac{1}{12}k_1k_2k_3(k_1^2+k_2^2+k_3^2 + 3k_1k_2+3k_1k_3+3k_2k_3-9(k_1+k_2+k_3)+15). \quad (3.4)$$

3.3 Possible future application

Theorem 3.2.10 applies to curves of large degree compared with the degree of the surface on which they lie. If this bound could be extended to curves of low degree, then we could hope to apply our result to an open problem in the study of Bridgeland stability on threefolds, which we now describe.

Suppose X is a smooth projective threefold. An important motivating question in the study of Bridgeland stability is to define a stability condition on $\mathcal{D}^b(X)$, the derived category of coherent sheaves on X. Such conditions have been defined for several types of threefolds, see [BMT14], [BMS14], [MP13a],[Sch13], but for a general threefold the question remains open. In [BMT14], the authors give a conjectured stability condition, which we describe now.

Given an ample class $\omega \in \mathrm{NS}_{\mathbb{Q}}(X)$ and a class $B \in \mathrm{NS}_{\mathbb{Q}}(X)$, we can a heart of a bounded t-structure $\mathcal{B}_{\omega,B}$ in $\mathcal{D}^b(X)$ to be a tilt of Coh(X) at a slope function depending on ω and B. We can then define a new slope function on $\mathcal{B}_{\omega,B}$ as follows. For $F \in \mathcal{B}_{\omega,B}$,

$$\nu_{\omega,B}(F) := \frac{\omega \mathrm{ch}_2^{\mathrm{B}}(\mathrm{F}) - \frac{\omega^3}{6}\mathrm{ch}_0^{\mathrm{B}}(\mathrm{F})}{\omega^2 \mathrm{ch}_1^{\mathrm{B}}(\mathrm{F})}.$$

Tilting again by this new slope function, we can define a second heart $\mathcal{A}_{\omega,B}$ in $\mathcal{D}^b(X)$. In [BMT14], the authors expect that the slope function $\nu_{\omega,B}$ defines a stability condition on $\mathcal{A}_{\omega,B}$. This would follow from the following conjectured Bogomolov-Gieseker type inequality.

Conjecture 3.3.1. *[BMT14, Conjecture 3.2.7] For any tilt-stable object $E \in \mathcal{B}_{\omega,B}$ satisfying*

$$\nu_{\omega,B}(E) = 0,$$

we have the following inequality:

$$\mathrm{ch}_3^{\mathrm{B}}(\mathrm{E}) \leq \frac{\omega^2}{18}\mathrm{ch}_1^{\mathrm{B}}(\mathrm{E}).$$

This conjecture has been proved to hold for specific threefolds. [Mac14] shows that 3.3.1 holds for $X = \mathbb{P}^3$. Furthermore, [MP13a] shows that 3.3.1 holds for principally polarized abelian varieties under a specific choice for ω and B, and in [MP13b] for any choice of ω and B when the Picard rank is 1. The inequality was proved for smooth quadric threefolds in [Sch13].

Consider now the special case in which X is a complete intersection three-fold $X = Z(f_1, \dots, f_{n-3})$ in $\mathbb{P}^n$ where f_i is a homogenous polynomial of degree k_i. Suppose now that C is a curve of degree d and genus g lying in X. Let $\mathcal{I}_C$ be the ideal sheaf of C in X.

Let H be a hyperplane section of X. There is a unique positive multiple of H, call it ω, for which $\mathcal{I}_C$ is tilt-stable with respect to $\nu_{\omega,0}$. For this value of ω, Conjecture 3.3.1 states that

$$\mathrm{ch}_3(\mathrm{E}) \leq \frac{\mathrm{t}^2\mathrm{H}^2}{18}\mathrm{ch}_1(\mathrm{E}).$$

This simplifies to the following statement as a special case of 3.3.1 (see [BMT14, Example 7.2.4] for the calculation).

Conjecture 3.3.2. *If X is a complete intersection threefold as before, and C is a degree d curve lying on X such that $d \leq \frac{1}{2}(k_1 \cdots k_{n-3})$, then*

$$g \leq \frac{d}{2}(k_1 + \cdots + k_{n-3} - n) + \frac{2d}{3} + 1.$$

This conjectured genus bound is generalized to curves of any degree lying on complete intersection three-folds in [BMS14, Section 4]. They conjecture the following Catelnuovo inequality:

Conjecture 3.3.3. *[BMS14, Example 4.4]*

$$g(C) \leq \frac{2d^2}{3k_1 \cdots k_{n-3}} + \left(\frac{5 + 3(k_1 + \cdots + k_{n-3} - n - 1)}{6}\right)d + 1.$$

Let $X \subset \mathbb{P}^n$ be a complete intersection threefold as before, and let C be a degree d curve lying on X. Consider a generic hyperplane H in $\mathbb{P}^n$ intersecting C in d distinct point. Define $\Gamma = H \cap C$. Let m_0 be the smallest integer so that for a generic choice of H, $H^0(H, \mathcal{I}_\Gamma(m_0)) \neq 0$. The following proposition argues that such a curve lies on a complete intersection surface in $\mathbb{P}^n$.

Proposition 3.3.4. *The curve C lies on a complete intersection surface in $\mathbb{P}^n$ defined by equations of degrees $k_1, \dots, k_{n-3}$ and m_0.*

Proof. Choose hyperplanes $H_1 = Z(h_1)$ and $H_2 = Z(h_2)$, such that $H^0(H_i, \mathcal{I}_{\Gamma_i}(m_0)) \neq 0$ and such that $P := H_1 \cap H_2$ does not intersect C. Then there is a pencil of hyperplanes intersecting along P in $\mathbb{P}^n$ given as $H_{\lambda_1,\lambda_2} = Z(\lambda_1 h_1 + \lambda_2 h_2)$ where $[\lambda_1 : \lambda_2] \in \mathbb{P}^1$.

Consider the blow-up $\mathrm{Bl}_P\mathbb{P}^n$ of $\mathbb{P}^n$ along P. Since C does not intersect P, $\widetilde{C} \cong C$ lies in $\mathrm{Bl}_{P\cap X}X$. Let $\widetilde{H}_{\lambda_1,\lambda_2}$ be the proper transform of H_{λ_1,λ_2}. In $\mathrm{Bl}_P\mathbb{P}^n$, the $\widetilde{H}_{\lambda_1,\lambda_2}$ are disjoint. Further, by assumption there is a nonzero section $s_{\lambda_1,\lambda_2} \in H^0(\widetilde{H}_{\lambda_1,\lambda_2}, \mathcal{I}_{\widetilde{\Gamma}_{\lambda_1,\lambda_2}}(m_0))$ for each $[\lambda_1 : \lambda_2]$ in $\mathbb{P}^1$.

Consider the map $p \colon \mathrm{Bl}_P\mathbb{P}^n \to \mathbb{P}^1$ sending $\widetilde{H}_{\lambda_1,\lambda_2}$ to $[\lambda_1 : \lambda_2]$. For a generic point of $\mathbb{P}^1$, there is an isomorphism

$$H^0([\lambda_1 : \lambda_2], p_*\mathcal{I}_{\widetilde{C}}(m_0)|_{[\lambda_1:\lambda_2]}) \cong H^0(\widetilde{H}_{\lambda_1,\lambda_2}, \mathcal{I}_{\widetilde{\Gamma}_{\lambda_1,\lambda_2}}(m_0)).$$

Thus there is a global section $s_{[\lambda_1,\lambda_2]}$ of $p_*\mathcal{I}_{\widetilde{C}}|_{[\lambda_1:\lambda_2]}$ corresponding to s_{λ_1,λ_2}. By Serre vanishing, there is a global section of $p_*\mathcal{I}_{\widetilde{C}}(l)$ restricting to this $s_{[\lambda_1,\lambda_2]}$ for l sufficiently large. This section pulls back to a global section s of $\mathcal{I}_{\widetilde{C}}$ restricting to s_{λ_1,λ_2} on $\widetilde{H}_{\lambda_1,\lambda_2}$. Set Y to be the irreducible component of $Z(s)$ in $\mathrm{Bl}_P\mathbb{P}^n$ which maps dominantly to $\mathbb{P}^1$.

Let π be the projection map $\mathrm{Bl}_P\mathbb{P}^n \to \mathbb{P}^n$. Then $S := X \cap \pi(Y)$ is a complete intersection surface in $\mathbb{P}^n$ defined by equations of degrees $k_1, \ldots, k_{n-3}$ and m_0 containing C. □

So long as m_0 is not large, so that (3.1) holds for the surface defined by equations of degrees $k_1, \cdots, k_{n-3}, m_0$, Theorem 3.2.10 and Proposition 3.3.4 imply together that that the bound in Conjecture 3.3.3 holds. In order to show that Conjecture 3.3.3 holds for all curves lying on complete intersection threefolds, a weaker bound would need to be shown for curves of lower degree lying on complete intersection surfaces.

Chapter 4

Stability on surfaces with curves of negative self-intersection

4.1 Introduction

Let X be a smooth projective surface. Let $\mathcal{D}^b(X)$ be the bounded derived category of coherent sheaves on X. As explained in Theorem 2.2.6, the space $\mathrm{Stab}(X)$ of stability conditions on X is a manifold. If we fix a class v in $K_{\mathrm{num}}(X)$, this manifold has a wall and chamber structure [Bri08, Section 9]. Within a chamber the stable objects of class v remain constant as the stability condition varies. We will fix v as the class of $\mathcal{O}_x$, the skyscraper sheaf at a point. In what is called the geometric chamber, all skyscraper sheaves $\mathcal{O}_x$ are stable. It is interesting to consider what happens as stability functions are deformed so that they cross out of the geometric chamber.

In particular, let $M_\sigma([\mathcal{O}_x])$ be the moduli space of σ-stable objects of class $[\mathcal{O}_x]$. Inside the geometric chamber, $M_\sigma([\mathcal{O}_x]) \cong X$. It is interesting to consider what $M_\sigma([\mathcal{O}_x])$ is after wall-crossing. In [Tod14], Toda shows that there is a correspondence between wall-crossing and the minimal model program. He shows that contractions of curves of self-intersection -1 can be realized as wall-crossing in $\mathrm{Stab}(X)$. That is, if $f\colon X \to Y$ is a birational map contracting a -1 curve on X, then there is a wall of the geometric chamber such that, after crossing, $M_\sigma([\mathcal{O}_x]) \cong Y$.

Our goal is to consider what else can happen when crossing walls. It is known that if $\sigma = (Z, \mathcal{B})$ is a stability condition in the geometric chamber, there is an associated ample divisor class ω [BM14]. This divisor class is defined as the class such that for any curve C in X, $\omega \cdot C = \mathrm{Im}Z(\mathcal{O}_C)$. Thus deforming to the wall of the geometric chamber, this divisor can either remain ample, or become nef.

Here we consider the situation in which the divisor becomes nef. We consider the case in which there is a curve $C \cong \mathbb{P}^1$ on X such that $C^2 = -n$ where $n \geq 2$. In Section 4.2, we construct a wall in the geometric chamber corresponding to the curve C, at which the points of C become strictly semistable.

Given a nef divisor H such that $H \cdot C = 0$ and $H \cdot C' > 0$ for all curves $C' \not\subseteq C$, and a divisor class β such that $H \cdot \beta = 0$, we construct a central charge

$$Z_{H,\beta}(E^{\cdot}) = -\mathrm{ch}_2(\mathrm{E}^{\cdot}) + \beta \cdot \mathrm{ch}_1(\mathrm{E}^{\cdot}) + \mathrm{zch}_0(\mathrm{E}^{\cdot}) + \mathrm{iH} \cdot \mathrm{ch}_1(\mathrm{E}).$$

We construct a heart of a bounded t-structure $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$ by tilting Coh(X) twice.

Theorem 4.2.10. *The pair $(Z_{H,\beta}, \mathcal{B}_{H,k}^{-\mathrm{Im}(z)})$ define a stability condition on $\mathcal{D}^b(X)$ when k is chosen so that $k + \frac{n}{2} < \beta \cdot C < k + \frac{n}{2} + 1$ and $\mathrm{Re}(z) + \frac{\mathrm{Im}(z)^2}{H^2} > -\frac{\beta^2}{2}$.*

We show that we can study wall-crossing by showing this stability condition satisfies the support property 2.3.1.

Theorem 4.2.21. *The central charge $Z_{H,\beta}$ satisfies the support property for Bridgeland semistable objects in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$.*

In Section 4.3, we study the moduli space $M_\sigma([\mathcal{O}_x])$ of stable objects of class $\mathcal{O}_x$ after crossing this wall. We show the following.

Theorem 4.3.13. *There is an isomorphism $X \sqcup_C \mathbb{P}^{n-1} \to M_\sigma([\mathcal{O}_x])$, where C is embedded in $\mathbb{P}^{n-1}$ as a rational normal curve.*

This generalizes the results of [Tod13] for $n = 1$ and [Bri08] for -2 curves on K3 surfaces. For $n \geq 3$ this space is reducible, and is the first example in the study of Bridgeland stability in which wall-crossing produces a more complicated moduli space.

4.2 Constructing a stability condition

4.2.1 Constructing a heart

For $E \in$ Coh(X) torsion-free and $a \in \mathbb{R}$ define

$$\nu_H(E) = \begin{cases} \frac{\mathrm{ch}_1(\mathrm{E}) \cdot \mathrm{H}}{\mathrm{ch}_0(\mathrm{E})} & \mathrm{ch}_0(\mathrm{E}) \neq 0 \\ -\infty & E = 0 \end{cases}$$

Further, define $\nu_H(E) = \infty$ if E is a torsion sheaf. Say E is ν_H-semistable if for all subsheaves E' of E, $\nu_H(E') \leq \nu_H(E)$. Say E is ν_H-stable if $\nu_H(E') < \nu_H(E)$ for all subsheaves E' of E. Define the following subcategories of Coh(X).

$$\mathcal{T}_H^a = \{T \in \mathrm{Coh(X)} \mid \nu_{\mathrm{H}}(\mathrm{S}) > \mathrm{a \text{ for all } T \twoheadrightarrow S}\}.$$

$$\mathcal{F}_H^a = \{F \in \mathrm{Coh(X)} \mid \nu_{\mathrm{H}}(\mathrm{G}) \leq \mathrm{a \text{ for all } G \hookrightarrow F}\}.$$

By Lemma 2.2.4 these two subcategories of Coh(X) are a torsion pair.

We will now construct a new heart of a bounded t-structure in $\mathcal{D}^b(X)$ as a tilt of Coh(X). Let

$$\mathcal{A}_H^a := \{E^{\cdot} \in \mathcal{D}^b(X) \mid H^0(E^{\cdot}) \in \mathcal{T}_H^a,\ H^{-1}(E^{\cdot}) \in \mathcal{F}_H^a,\ H^i(E^{\cdot}) = 0 \text{ if } i \neq 0, -1\}.$$

Consider now the sheaves $\mathcal{O}_C(i)$, the twists of the structure sheaf of C. These are torsion sheaves on X, and so each has slope ∞ for all choices of H. This means that all such sheaves lie in $\mathcal{T}_H^a$, and so in $\mathcal{A}_H^a$.

Recall that for $\mathcal{S} \subseteq \mathcal{D}^b(X)$, $\langle \mathcal{S} \rangle$ is notation for the extension closure of $\mathcal{S}$. That is, $\langle \mathcal{S} \rangle$ is the smallest subcategory of $\mathcal{D}^b(X)$ closed under taking extensions of objects in $\mathcal{S}$. We will now define the following subcategories of $\mathcal{A}_H^a$.

The first subcategory we define is

$$\mathcal{F}_{H,k}^a = \langle \mathcal{O}_C(i) \mid i \leq k \rangle.$$

We then define another subcategory to be the left orthogonal to $\mathcal{F}_{H,k}^a$. That is,

$$\mathcal{T}_{H,k}^a = \{E^{\cdot} \in \mathcal{A}_H^a \mid \mathrm{Hom}(E^{\cdot}, \mathcal{O}_C(i)) = 0 \text{ for } i \leq k\}.$$

Lemma 4.2.1. *If there is a sequence of inclusions in* $\mathcal{A}_H^{-\mathrm{Im}(z)}$, *say*

$$\cdots \hookrightarrow S_i^{\cdot} \hookrightarrow S_{i-1}^{\cdot} \hookrightarrow \cdots \hookrightarrow S_1^{\cdot} \hookrightarrow S_0^{\cdot}$$

whose quotients lie in $\mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$, *then for* $i \gg 0$, $S_i^{\cdot} \cong S_{i-1}^{\cdot}$

Proof. Suppose there is a sequence of inclusions

$$\cdots \hookrightarrow S_{i+1}^{\cdot} \hookrightarrow S_i^{\cdot} \hookrightarrow \cdots \hookrightarrow S_1^{\cdot} \hookrightarrow S_0^{\cdot} \tag{4.1}$$

such that for all i, $S_i^{\cdot} \in \mathcal{A}_H^{-\mathrm{Im}(z)}$, and the quotient F_i of the map $S_{i+1}^{\cdot} \hookrightarrow S_i^{\cdot}$ lies in $\mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$. First note that if we take the long exact sequence of cohomology, for every i, $H^{-1}(S_i) \cong H^{-1}(S_{i+1})$, and there is a corresponding sequence of sheaves

$$\cdots \hookrightarrow H^0(S_{i+1}^{\cdot}) \hookrightarrow H^0(S_i^{\cdot}) \hookrightarrow \cdots \hookrightarrow H^0(S_1^{\cdot}) \hookrightarrow H^0(S_0^{\cdot})$$

whose quotients are the same sheaves F_i. Hence it is enough to prove that 4.1 stabilizes when the S_i in (4.1) are sheaves in $\mathcal{T}_H^{-\mathrm{Im}(z)}$.

Furthermore, every $F_i \in \mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$ has a nonzero surjective morphism in Coh(X) to $\mathcal{O}_C(l_i)$ for some $l_i \leq k$. Let $S_i^{(1)}$ be the kernel of the composition $S_i \to F_i \to \mathcal{O}_C(l_i)$ We can see via the octahedral axiom that there is an exact sequence of sheaves

$$0 \to S_i^{(1)} \to S_i \to \mathcal{O}_C(l_i) \to 0.$$

The quotient $F_i^{(1)}$ of the map $S_{i+1} \to S_i$ fits into an exact sequence

$$0 \to F_i^{(1)} \to F_i \to \mathcal{O}_C(l_i) \to 0.$$

This implies that $F_i^{(1)} \in \mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$. Since $F_i \in \mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$, $\mathrm{ch_1(F_i) = m[C]}$ for some $m \in \mathbb{N}$. Hence $\mathrm{ch_1(F_i^{(1)}) = (m-1)[C]}$. We can now apply this process to the map $S_{i+1} \to S_i^{(1)}$ and repeat until we have a sequence

$$S_{i+1} \hookrightarrow S_i^{(m-1)} \hookrightarrow \cdots \hookrightarrow S_i^{(1)} \hookrightarrow S_i,$$

all of whose quotients are sheaves of the form $\mathcal{O}_C(l_i^{(j)})$ for some $l_i^{(j)} \leq k$. By applying this process to (4.1), we can assume each quotient F_i in (4.1) is in fact $\mathcal{O}_C(l_i)$ for some $l_i \leq k$.

Consider the exact sequence

$$0 \to S_{i+1} \to S_i \to \mathcal{O}_C(l_i) \to 0.$$

Since $l_i \leq k$, we can compute $\mathrm{Hom}(\mathcal{O}_C(l_i), \mathcal{O}_C(k)) \cong \mathbb{C}^{k-l_i+1}$. Furthermore, $\mathrm{Ext}^1(\mathcal{O}_C(l_i), \mathcal{O}_C(k)) \cong \mathcal{H}^1(X, \mathcal{O}_C(k) \otimes \mathcal{O}_C(l_i)^\vee)$. As there is an exact sequence

$$0 \to \mathcal{O}_X(-C)(l_i) \to \mathcal{O}_X(l_i) \to \mathcal{O}_C(l_i) \to 0$$

in X, we can compute $\mathcal{O}_C(l_i)^\vee$ in $\mathcal{D}^b(X)$ as the complex $\mathcal{O}_C(-l_i) \to \mathcal{O}_C(-n-l_i)$. There are no morphisms between the two sheaves in this complex, hence we have $\mathcal{H}^1(X, \mathcal{O}_C(k) \otimes \mathcal{O}_C(l_i)^\vee) \cong \mathcal{H}^1(X, \mathcal{O}_C(k-l_i)) \oplus \mathcal{H}^0(X, \mathcal{O}_C(k-l_i-n))$. Hence if $k-l_i-n \geq 0$, $\mathrm{Ext}^1(\mathcal{O}_C(l_i), \mathcal{O}_C(k)) \cong \mathbb{C}^{k-l_i-n+1}$, otherwise it is 0. By a similar calculation, if $k - l_i - n < -1$, $\mathrm{Ext}^2(\mathcal{O}_C(l_i), \mathcal{O}_C(k)) \cong \mathbb{C}^{l_i-k+n-1}$, otherwise it is zero.

In particular, this means that either $\mathrm{Ext}^1(\mathcal{O}_C(l_i), \mathcal{O}_C(k)) \cong 0$ or $\mathrm{Ext}^2(\mathcal{O}_C(l_i), \mathcal{O}_C(k)) \cong 0$. Suppose first that $\mathrm{Ext}^1(\mathcal{O}_C(l_i), \mathcal{O}_C(k)) \cong 0$. Then taking the long exact sequence of cohomology, we see there is an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{O}_C(l_i), \mathcal{O}_C(k)) \to \mathrm{Hom}(S_i, \mathcal{O}_C(k)) \to \mathrm{Hom}(S_{i+1}, \mathcal{O}_C(k)) \to 0.$$

Since $\mathrm{Hom}(\mathcal{O}_C(l_i), \mathcal{O}_C(k)) \neq 0$, this means that $\dim \mathrm{Hom}(S_i, \mathcal{O}_C(k)) > \dim \mathrm{Hom}(S_{i+1}, \mathcal{O}_C(k))$.

Now suppose that $\mathrm{Ext}^2(\mathcal{O}_C(l_i), \mathcal{O}_C(k)) \cong 0$. Then again applying $\mathrm{Hom}(-, \mathcal{O}_C(k))$ to the exact sequence

$$0 \to S_{i+1} \to S_i \to \mathcal{O}_C(l_i) \to 0$$

we see that $\mathrm{Ext}^2(S_i, \mathcal{O}_C(k)) \cong \mathrm{Ext}^2(S_{i+1}, \mathcal{O}_C(k))$ and there is an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{O}_C(l_i), \mathcal{O}_C(k)) \to \mathrm{Hom}(S_i, \mathcal{O}_C(k)) \to \mathrm{Hom}(S_{i+1}, \mathcal{O}_C(k)) \to \mathrm{Ext}^1(\mathcal{O}_C(l_i), \mathcal{O}_C(k)) \to$$
$$\to \mathrm{Ext}^1(S_i, \mathcal{O}_C(k)) \to \mathrm{Ext}^1(S_{i+1}, \mathcal{O}_C(k)) \to 0.$$

The sequence above is exact, so the alternating sum of the dimensions is 0. That is,

$$\dim \mathrm{Hom}(S_i, \mathcal{O}_C(k)) - \dim \mathrm{Hom}(S_{i+1}, \mathcal{O}_C(k)) = n - \dim \mathrm{Ext}^1(S_i, \mathcal{O}_C(k)) - \dim \mathrm{Ext}^1(S_{i+1}, \mathcal{O}_C(k)).$$

Since the map $\mathrm{Ext}^1(S_i, \mathcal{O}_C(k)) \to \mathrm{Ext}^1(S_{i+1}, \mathcal{O}_C(k))$ is surjective, we can say that $\dim \mathrm{Ext}^1(S_i, \mathcal{O}_C(k)) > \dim \mathrm{Ext}^1(S_{i+1}, \mathcal{O}_C(k))$. Hence we see that in this case as well, $\dim \mathrm{Hom}(S_i, \mathcal{O}_C(k)) > \dim \mathrm{Hom}(S_{i+1}, \mathcal{O}_C(k))$. As these dimensions decrease when i increases, we see that the sequence must terminate. □

Lemma 4.2.2. *The pair* $(\mathcal{T}^a_{H,k}, \mathcal{F}^a_{H,k})$ *form a torsion pair in* $\mathcal{A}^a_H$*.*

Proof. We must show that for any $E^{\cdot} \in \mathcal{A}^a_H$, there is an exact triangle

$$T^{\cdot} \to E^{\cdot} \to F$$

such that $T^{\cdot} \in \mathcal{T}^a_{H,k}$ and $F \in \mathcal{F}^a_{H,k}$. If $\mathrm{Hom}(E^{\cdot}, \mathcal{F}^a_{H,k}) \neq 0$, then let

$$S_1 \to E^{\cdot} \to F$$

be an exact triangle with $F \in \mathcal{F}^a_{H,k}$.

First, we show that $S_1^{\cdot} \in \mathcal{A}^a_H$. Taking the long exact sequence of sheaf cohomology, we see $H^{-1}(S_1^{\cdot}) \cong H^{-1}(E^{\cdot}) \in \mathcal{F}^a_H$. Further, there is a short exact sequence

$$0 \to H^0(S_1^{\cdot}) \to H^0(E^{\cdot}) \to F_1 \to 0.$$

Let G be any quotient of $H^0(S_1^{\cdot})$, fitting into exact sequence

$$0 \to R \to H^0(S_1^{\cdot}) \to G \to 0.$$

Then by composing the maps $R \hookrightarrow H^0(S_1^{\cdot}) \hookrightarrow H^0(E^{\cdot})$ there is a short exact

$$0 \to G \to H^0(E^{\cdot})/R \to F_1.$$

Since F_1 is supported on C, $\nu_H(G) = \nu_H(H^0(E^\cdot)/R)$. And since $H^0(E^\cdot) \in \mathcal{T}_H^a$, $\nu_H(H^0(E^\cdot)/R) > a$. Hence $S_1^\cdot \in \mathcal{A}_H^a$.

If $\mathrm{Hom}(S_1^\cdot, \mathcal{F}_{H,k}^a) \neq 0$, then we can repeat this process, and construct an exact triangle

$$S_2^\cdot \to S_1^\cdot \to F_2$$

with $F_2 \in \mathcal{F}_{H,k}^a$. If we iterate this process we get a sequence of complexes $S_i^\cdot \in \mathcal{A}_H^a$, such that $H^{-1}(S_i^\cdot) \cong H^{-1}(E^\cdot)$, and such that there is a descending chain of inclusions

$$H^0(E^\cdot) \supseteq H^0(S_1^\cdot) \supseteq \cdots \supseteq H^0(S_i^\cdot) \supseteq H^0(S_{i+1}^\cdot) \supseteq \cdots$$

in Coh(X).

By Lemma 4.2.1, this chain must terminate. That is, there exists a number n such that for $i \geq n$, $H^0(S_i^\cdot) \cong H^0(S_{i+1}^\cdot)$. It follows that $\mathrm{Hom}(S_n^\cdot, \mathcal{F}_{H,k}^a) = 0$, and

$$S_n^\cdot \to E^\cdot \to F_n$$

is the desired triangle. □

We can then perform a second tilt and define the following heart in $\mathcal{D}^b(X)$:

$$\mathcal{B}_{H,k}^a := \{E^\cdot \in \mathcal{D}^b(X) \mid H^0_{\mathcal{A}_H^a}(E^\cdot) \in \mathcal{T}_{H,k}^a,\ H^{-1}_{\mathcal{A}_H^a}(E^\cdot) \in \mathcal{F}_{H,k}^a,\ H^i_{\mathcal{A}_H^a}(E^\cdot) = 0 \text{ if } i \neq 0, -1\}.$$

Comparison with Toda's $\mathcal{B}_{f^*\omega}$

We will now explain how the heart we have constructed compares with the heart in [Tod13, Section 3.1]. This is not necessary to the construction of our stability condition, it is for the purpose of comparison. We will show that our heart and Toda's coincide when $n = 1$ and $a = 0$.

Following [Tod13, Section 3.1], let C be a curve on a smooth projective surface X such that $C^2 = -1$, and let $f\colon X \to Y$ be the map contracting this -1 curve. Let $H = f^*\omega$ be the pull-back of ample divisor ω on Y. Toda constructs a heart of a bounded t-structure in $\mathcal{D}^b(X)$ as a tilt of ${}^{-1}\mathrm{Per}(X/Y)$, the category of perverse sheaves on X. This category can itself can be constructed as a tilt of Coh(X) as in [VdB02, Lemma 3.1.1].

Let $\mathcal{C} = \{E \in \mathrm{Coh}X \mid \mathbb{R}f_*E = 0\}$. Note that the only sheaf supported on C which lies in $\mathcal{C}$ is $\mathcal{O}_C(-1)$. Consider the following torsion pair in Coh(X).

$$\mathcal{T}_{-1} = \{E \in \mathrm{Coh(X)} \mid \mathrm{R^1f_*E = 0,\ Hom(E,\mathcal{C}) = 0}\}.$$

$$\mathcal{F}_{-1} = \{E \in \mathrm{Coh(X)} \mid \mathrm{f_*E = 0}\}.$$

Then ${}^{-1}\mathrm{Per}(X/Y)$ is the tilt of $\mathrm{Coh}(X)$ at the torsion pair $(\mathcal{T}_{-1}, \mathcal{F}_{-1})$. That is,

$$^{-1}\mathrm{Per}(X/Y) = \{E^\cdot \in \mathcal{D}^b(X) \mid H^0(E^\cdot) \in \mathcal{T}_{-1},\ H^{-1}(E^\cdot) \in \mathcal{F}_{-1},\ H^i(E^\cdot) = 0 \text{ if } i \neq 0, -1\}.$$

Now define a slope function on ${}^{-1}\mathrm{Per}(X/Y)$ as we did for Coh(X). For $E^\cdot \in^{-1} \mathrm{Per}(X/Y)$, define

$$\mu_{f^*\omega}(E^\cdot) = \begin{cases} \frac{\mathrm{ch_1(E^\cdot) \cdot f^*\omega}}{\mathrm{ch_0(E^\cdot)}} & \mathrm{ch_0(E^\cdot) \neq 0} \\ \infty & \mathrm{ch_0(E^\cdot) = 0, E^\cdot \neq 0} \\ -\infty & E^\cdot = 0 \end{cases}$$

We will now tilt the category of perverse sheaves at slope, as we did for coherent sheaves before. Let

$$^{-1}\mathcal{T}_{f^*\omega} = \{T^\cdot \in^{-1} \mathrm{Per}(X/Y) \mid \mu_{f^*\omega}(S^\cdot) > 0 \text{ for all } T^\cdot \twoheadrightarrow S^\cdot\},$$

$$^{-1}\mathcal{F}_{f^*\omega} = \{F^\cdot \in^{-1} \mathrm{Per}(X/Y) \mid \mu_{f^*\omega}(E^\cdot) \leq 0 \text{ for all } E^\cdot \hookrightarrow F^\cdot\}.$$

Toda then is able to define a stability condition on the following heart, where H^i_{Per} refers to cohomology with respect to the heart ${}^{-1}\mathrm{Per}(X/Y)$:

$$\mathcal{B}_{f^*\omega} = \{E^\cdot \in \mathcal{D}^b(X) \mid H^0_{Per}(E^\cdot) \in^{-1} \mathcal{T}_{f^*\omega},\ H^{-1}_{Per}(E^\cdot) \in^{-1} \mathcal{F}_{f^*\omega},\ H^i_{Per}(E^\cdot) = 0 \text{ if } i \neq 0, -1\}.$$

Lemma 4.2.3. *For any ample divisor ω on Y, $\mathcal{F}^0_{f^*\omega,-1} = \mathcal{F}_{-1}$.*

Proof. First, since $\mathcal{O}_C(i)$ has no global sections for $i < 0$, $f_*\mathcal{O}_C(i) = 0$ when $i < 0$. Now suppose E is a sheaf in $\mathcal{F}_{-1}$, that is $f_*E = 0$. Then since $X \setminus C \cong Y \setminus P$, the support of E must be contained in C. Specifically, E must be a sheaf on C with no global sections. This implies $f_*E = 0$ and $E \in \mathcal{F}^0_{f^*\omega,-1}$. □

Lemma 4.2.4. *For any $E^\cdot \in^{-1} \mathrm{Per}(X/Y)$ such that $H^0(E^\cdot) \neq 0$, $\mu_{f^*\omega}(E^\cdot) = \nu_{f^*\omega}(H^0(E^\cdot))$.*

Proof. This follows from the fact that $\mathrm{ch(E^\cdot) = ch(H^0(E^\cdot)) - ch(H^{-1}(E^\cdot))}$. Since $H^{-1}(E^\cdot)$ is supported on C, $\mathrm{ch_0(H^{-1}(E^\cdot)) = 0}$ and $\mathrm{ch_1(H^{-1}(E^\cdot)) \cdot f^*\omega = 0}$. □

Proposition 4.2.5. *Let $E^\cdot$ be a perverse sheaf such that $H^0(E^\cdot) \neq 0$.*

1. *$H^0(E^\cdot) \in \mathcal{T}^0_{f^*\omega}$ if and only if $E^\cdot \in^{-1} \mathcal{T}_{f^*\omega}$.*
2. *$H^0(E^\cdot) \in \mathcal{F}^0_{f^*\omega}$ if and only if $E^\cdot \in^{-1} \mathcal{F}_{f^*\omega}$.*

Proof. Suppose first that $H^0(E^\cdot)$ is in $\mathcal{T}^0_{f^*\omega}$. Because perverse sheaves have cohomology only in degrees -1 and 0, for any quotient $S^\cdot$ of $E^\cdot$ in ${}^{-1}\mathrm{Per}(X/Y)$, $H^0(E^\cdot)$ surjects onto $H^0(S^\cdot)$. All quotient sheaves of $H^0(E^\cdot)$ have positive slope. This implies that $\mu_{f^*\omega}(S^\cdot) = \nu_{f^*\omega}(H^0(S^\cdot)) > 0$, and $E^\cdot$ is in ${}^{-1}\mathcal{T}_{f^*\omega}$.

Now suppose that $E^{\cdot} \in^{-1} \mathcal{T}_{f^*\omega}$. Let $H^0(E^{\cdot}) \to S$ be a surjective map of coherent sheaves. Then S is necessarily also in $\mathcal{T}_{-1}$, that is, S is a perverse sheaf. However, the map $E^{\cdot} \to H^0(E^{\cdot}) \to S$ may not be a surjection in ${}^{-1}\mathrm{Per}(X/Y)$. That is, if $P^{\cdot}$ is the kernel of the composition, fitting into exact triangle

$$P^{\cdot} \to E^{\cdot} \to S, \tag{4.2}$$

it may be that $P^{\cdot}$ is not in ${}^{-1}\mathrm{Per}(X/Y)$, since $H^0(P^{\cdot})$ need not be in $\mathcal{T}_{-1}$. We will now construct a perverse sheaf S' such that $\mu_{f^*\omega}(S') = \mu_{f^*\omega}(S)$ and $E^{\cdot} \twoheadrightarrow S'$, proving that $\mu_{f^*\omega}(E^{\cdot}) > 0$.

Since $H^0(P^{\cdot})$ is a sheaf, there exist sheaves $T \in \mathcal{T}_{-1}$ and $F \in \mathcal{F}_{-1}$ so that

$$0 \to T \to H^0(P^{\cdot}) \to F \to 0 \tag{4.3}$$

is exact. Further, since F is supported on C, $\mathrm{ch_0(F) = H \cdot ch_1(F)} = 0$ and $\nu_{f^*\omega}(T) = \nu_{f^*\omega}(H^0(P^{\cdot}))$. There is an injective map of sheaves $T \to H^0(E^{\cdot})$ composing the injective maps $T \to H^0(P^{\cdot}) \to H^0(E^{\cdot})$. Let S' be the quotient sheaf of this map, fitting into exact sequence

$$0 \to T \to H^0(E^{\cdot}) \to S' \to 0. \tag{4.4}$$

Again, S' is also necessarily a perverse sheaf. We also claim that $\mu_{f^*\omega}(S') = \mu_{f^*\omega}(S)$. The sequence (4.2) gives rise to a long exact sequence of sheaves

$$0 \to H^{-1}(P^{\cdot}) \to H^{-1}(E^{\cdot}) \to 0 \to H^0(P^{\cdot}) \to H^0(E^{\cdot}) \to S \to 0.$$

We can conclude by additivity of chern characters that

$$\mathrm{ch_0(H^0(E^{\cdot})) = ch_0(S) + ch_0(H^0(P^{\cdot}))}. \tag{4.5}$$

Sequence (4.3) shows that $\mathrm{ch_0(H^0(P^{\cdot})) = ch_0(T)}$. Thus we can rewrite equation (4.5) as

$$\mathrm{ch_0(H^0(E^{\cdot})) = ch_0(S) + ch_0(T)}. \tag{4.6}$$

But taking the long exact sequence of (4.4) we have

$$\mathrm{ch_0(H^0(E^{\cdot})) = ch_0(S') + ch_0(T)}.$$

Hence, $\mathrm{ch_0(S) = ch_0(S')}$. Note that equations (4.5) and (4.6) can also be written for $\mathrm{ch_1}$, to show that $\mathrm{ch_1(S) = ch_1(S')}$. Thus we have shown that $\mu_{f^*\omega}(S) = \mu_{f^*\omega}(S')$.

We will now show that the composition $E^{\cdot} \to H^0(E^{\cdot}) \to S'$ is surjective in ${}^{-1}\mathrm{Per}(X/Y)$. Let

$Q^{\cdot}$ be the kernel of the composition $E^{\cdot} \to H^0(E^{\cdot}) \to S'$, fitting into exact triangle

$$Q^{\cdot} \to E^{\cdot} \to S'.$$

Note that E surjects onto S' in ${}^{-1}\text{Per}(X/Y)$ if and only if $Q^{\cdot} \in^{-1} \text{Per}(X/Y)$. Taking long exact cohomology, $H^{-1}(Q^{\cdot}) \cong H^{-1}(E^{\cdot})$ which is in $\mathcal{F}_{-1}$, and $H^0(Q^{\cdot}) \cong T$ which is in $\mathcal{T}_{-1}$. Then $Q^{\cdot} \in^{-1} \text{Per}(X/Y)$, and so $E^{\cdot} \to S'$ is surjective in ${}^{-1}\text{Per}(X/Y)$. This implies $\nu_{f^*\omega}(S) = \mu_{f^*\omega}(S') > 0$.

We will now prove the second statement. Suppose $H^0(E^{\cdot}) \in \mathcal{F}^0_{f^*\omega}$, and $F^{\cdot} \to E^{\cdot}$ is an injection in ${}^{-1}\text{Per}(X/Y)$ with quotient G. There is a long exact cohomology sequence

$$\begin{array}{ccccccccccccc} 0 \longrightarrow & H^{-1}(F^{\cdot}) & \longrightarrow & H^{-1}(E^{\cdot}) & \longrightarrow & H^{-1}(G^{\cdot}) & \longrightarrow & H^0(F^{\cdot}) & \longrightarrow & H^0(E^{\cdot}) & \longrightarrow & H^0(G^{\cdot}) & \longrightarrow 0. \\ & & & & & \searrow & \nearrow & \searrow & \nearrow & & & & \\ & & & & & & K & & I & & & & \end{array}$$

Since K is a quotient of $H^{-1}(G)$ it must be supported on C. This implies that $\text{ch}_0(\text{K}) = \text{H}\cdot\text{ch}_1(\text{K}) = 0$. Then $\mu_{f^*\omega}(F^{\cdot}) = \nu_{f^*\omega}(H^0(F^{\cdot})) = \nu_{f^*\omega}(I) \leq 0$ since I is a subsheaf of $H^0(E^{\cdot})$.

Now suppose $E^{\cdot} \in^{-1} \mathcal{F}_{f^*\omega}$. Let $S \to H^0(E^{\cdot})$ be an injective morphism of sheaves. We will construct a perverse sheaf $R^{\cdot}$ which injects into $E^{\cdot}$ so that $\mu(Q^{\cdot}) = \mu(E^{\cdot})$. Since S is a sheaf, it fits into an exact sequence

$$0 \to T \to S \to F \to 0$$

where $T \in \mathcal{T}_{-1}$ and $F \in \mathcal{F}_{-1}$. Since $\text{ch}_0(\text{F}) = \text{H} \cdot \text{ch}_1(\text{F}) = 0$, $\nu_{f^*\omega}(T) = \nu_{f^*\omega}(S)$. Composing the injective maps $T \to S \to H^0(E^{\cdot})$, we get an exact sequence

$$0 \to T \to H^0(E^{\cdot}) \to Q \to 0$$

for some $Q \in \text{Coh}(X)$. $H^0(E^{\cdot})$ is a perverse sheaf, and so $Q \in \mathcal{T}_{-1}$ is a perverse sheaf.

We have morphisms $E^{\cdot} \to H^0(E^{\cdot}) \to Q$. Let $R^{\cdot}$ be the kernel of the composition, fitting into exact triangle

$$R^{\cdot} \to E^{\cdot} \to Q.$$

Taking the long exact cohomology sequence we see $H^{-1}(R^{\cdot}) \cong H^{-1}(E^{\cdot}) \in \mathcal{F}_{-1}$ and $H^0(R^{\cdot}) \cong T \in \mathcal{T}_{-1}$, so $R^{\cdot} \in^{-1} \text{Per}(X/Y)$. This means that $R^{\cdot} \to E^{\cdot}$ is an injective morphism in ${}^{-1}\text{Per}(X/Y)$. And so $0 \geq \mu_{f^*\omega}(R^{\cdot}) = \nu_{f^*\omega}(T) = \nu_{f^*\omega}(S)$.

□

Although Proposition 4.2.5 did not address perverse sheaves $E^{\cdot}$ for which $H^0(E^{\cdot}) = 0$, it is easy to see that the slope $\mu_{f^*\omega}(E^{\cdot})$ of such a perverse sheaf is ∞, and that in this case $E^{\cdot} \in^{-1} \mathcal{T}_{f^*\omega}$.

Theorem 4.2.6. $\mathcal{B}_{f^*\omega} = \mathcal{B}^0_{f^*\omega,-1}$.

Proof. It suffices to show that $\mathcal{B}^0_{f^*\omega,-1} \subset \mathcal{B}_{f^*\omega}$, since each is a heart of a bounded t-structure. Suppose $E^{\cdot} \in \mathcal{B}^0_{f^*\omega,-1}$. Then there is an exact triangle

$$F[1] \to E^{\cdot} \to T^{\cdot}$$

where F is in $\mathcal{F}^0_{f^*\omega,-1}$ and $T^{\cdot} \in \mathcal{T}^0_{f^*\omega,-1}$. By Lemma 4.2.3, $\mathcal{F}^0_{f^*\omega,-1} = \mathcal{F}_{-1}$. The sheaves in $\mathcal{F}_{-1}$ are torsion, so $\mu_{f^*\omega}(F[1]) = \infty$ and $F[1] \in^{-1} \mathcal{T}_{f^*\omega} \subset \mathcal{B}_{f^*\omega}$.

It remains to show that $T^{\cdot} \in \mathcal{B}_{f^*\omega}$. Since $T^{\cdot} \in \mathcal{T}^0_{f^*\omega,-1}$, it is contained in $\mathcal{A}^0_{f^*\omega}$. This means there is an exact triangle

$$H^{-1}(T^{\cdot})[1] \to T^{\cdot} \to H^0(T^{\cdot})$$

with $H^0(T^{\cdot}) \in \mathcal{T}^0_{f^*\omega}$ and $H^{-1}(T^{\cdot}) \in \mathcal{F}^0_{f^*\omega}$. We will now show that $H^{-1}(T^{\cdot})$ and $H^0(T^{\cdot})$ also lie in $\mathcal{B}_{f^*\omega}$.

There is an exact sequence

$$0 \to S_{-1} \to H^{-1}(T^{\cdot}) \to R_{-1} \to 0$$

with $S_{-1} \in \mathcal{T}_{-1}$ and $R_{-1} \in \mathcal{F}_{-1}$. Clearly $R_{-1}[1] \in \mathcal{B}_{f^*\omega}$. Since S_{-1} is a subsheaf of $H^{-1}(T\cdot)$, $S_{-1} \in \mathcal{T}_{-1} \cap \mathcal{F}_{f^*\omega}$, and so $S_{-1}[1] \in \mathcal{B}_{f^*\omega}$ as well. Thus $H^{-1}(T^{\cdot})[1] \in \mathcal{B}_{f^*\omega}$.

Similarly, there is an exact sequence

$$0 \to S_0 \to H^0(T^{\cdot}) \to R_0 \to 0$$

with $S_0 \in \mathcal{T}_{-1}$ and $R_0 \in \mathcal{F}_{-1}$. We know there are no nonzero maps $T^{\cdot} \to R_0$. So then if R_0 is nonzero, we get an exact triangle $H^{-1}(T^{\cdot})[2] \to C^{\cdot} \to S_0[1]$ by the octahedral axiom, where $C^{\cdot}$ is the cone of the 0 map from $T^{\cdot} \to R_0$. Taking the long exact sequence of cohomology we find that $H^0(C^{\cdot}) = 0$. But since this is the cone of the zero morphism, $H^0(C^{\cdot}) \cong R_0$. This shows that $H^0(T^{\cdot}) \cong S_0 \in \mathcal{T}_{-1} \cap \mathcal{T}_{f^*\omega}$. Thus $H^0(T^{\cdot}) \in \mathcal{B}_{f^*\omega}$. □

4.2.2 Constructing a central charge

Suppose now that C is a curve on the smooth projective surface X with $C^2 = -n$. Suppose further that there is a nef divisor H on X so that $C \in H^{\perp}$, but $H \cdot C' > 0$ for all curves $c' \subseteq X$ so that $C' \nsubseteq C$.

Let $z \in \mathbb{C}$ and let $\beta \in \mathrm{NS}_{\mathbb{R}}(X)$ so that $\beta \cdot H = 0$. We want to define a central charge

$$Z_{H,\beta}(E) = -\mathrm{ch_2(E)} + \mathrm{iHch_1(E)} + \beta\mathrm{ch_1(E)} + \mathrm{zch_0(E)}$$

on $\mathcal{D}^b(X)$. We will now show that the pair $(Z_{H,\beta}, \mathcal{B}_{H,k}^{-\mathrm{Im}(z)})$ is a stability condition if k and z satisfy $k + \frac{n}{2} < \beta \cdot C < k + \frac{n}{2} + 1$, $\mathrm{Re}(z) > 0$, and $\mathrm{Re}(z) + \frac{\mathrm{Im}(z)^2}{H^2} > -\frac{\beta^2}{2}$.

If $\beta \cdot C - \frac{n}{2}$ is an integer, then no such k will exist. However, this problem can be avoided by simply scaling the class β. In fact, so long as $\beta \cdot C \neq 0$, then by replacing β with $\left(1 - \frac{k+1}{\beta \cdot C}\right)\beta$, we can always choose k to be -1. However, we will continue in more generality.

Theorem 4.2.7 ([Bog] [Gie79]). *For any Gieseker stable sheaf E on X which is torsion-free,* $\mathrm{ch}_1(\mathrm{E})^2 \geq 2\mathrm{ch}_0(\mathrm{E})\mathrm{ch}_2(\mathrm{E})$.

Lemma 4.2.8. *The function $Z_{H,\beta}$ is a stability function on $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$, when k is chosen so that $k + \frac{n}{2} < \beta \cdot C < k + \frac{n}{2} + 1$ and $\mathrm{Re} z + \frac{\mathrm{Im} z^2}{H^2} > -\frac{\beta^2}{2}$.*

Proof. Any $E^{\cdot} \in \mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$ fits into an exact triangle

$$F[1] \to E^{\cdot} \to T^{\cdot}$$

for some $F \in \mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$ and some $T^{\cdot} \in \mathcal{T}_{H,k}^{-\mathrm{Im}(z)}$. Since we have defined $Z_{H,\beta}$ using chern characters, which are additive on exact triangles, it follows that

$$Z_{H,\beta}(E^{\cdot}) = Z_{H,\beta}(T^{\cdot}) - Z_{H,\beta}(F).$$

We have chosen H so that $H \cdot C = 0$ and so $\mathrm{Im}(Z_{H,\beta}(E^{\cdot})) = \mathrm{Im}(Z_{H,\beta}(T^{\cdot}))$. But $\mathrm{Im}(Z_{H,\beta}(T^{\cdot})) = \mathrm{Im}(Z_{H,\beta}(H^0(T^{\cdot}))) - \mathrm{Im}(Z_{H,\beta}(H^{-1}(T^{\cdot})))$. By the construction of $\mathcal{A}_{H,k}^{-\mathrm{Im}(z)}$, $\mathrm{Im}(Z_{H,\beta}(H^0(T^{\cdot}))) > 0$ and $\mathrm{Im}(Z_{H,\beta}(H^{-1}(T^{\cdot}))) \leq 0$.

Now we must show that if $\mathrm{Im}(Z_{H,\beta}(E^{\cdot})) = 0$, then $\mathrm{Re}(Z_{H,\beta}(E^{\cdot})) < 0$. Consider the following diagram of short exact sequences in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$.

$$\begin{array}{ccccc} & & & & H^{-1}(H^0_{\mathcal{A}}(E^{\cdot}))[1] \\ & & & & \downarrow \\ H^{-1}_{\mathcal{A}}(E^{\cdot})[1] & \longrightarrow & E^{\cdot} & \longrightarrow & H^0_{\mathcal{A}}(E^{\cdot}) \\ & & & & \downarrow \\ & & & & H^0(H^0_{\mathcal{A}}(E^{\cdot})) \end{array}$$

The equation $\mathrm{Im}(Z_{H,\beta}(E^{\cdot})) = 0$ holds if and only if the equations

$$\mathrm{Im}(Z_{H,\beta}(H^{-1}_{\mathcal{A}}(E^{\cdot}))) = \mathrm{Im}(Z_{H,\beta}(H^{-1}(H^0_{\mathcal{A}}(E^{\cdot}))) = \mathrm{Im}(Z_{H,\beta}(H^0(H^0_{\mathcal{A}}(E^{\cdot}))) = 0$$

also hold. Further, note that $H^{-1}_{\mathcal{A}}(E^{\cdot})) \in \mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$, $H^{-1}(H^0_{\mathcal{A}}(E^{\cdot})) \in \mathcal{F}_H^{-\mathrm{Im}(z)}$, and $H^0(H^0_{\mathcal{A}}(E^{\cdot})) \in \mathcal{T}_H^{-\mathrm{Im}(z)}$. Thus we will be proceed by showing that for any sheaves $F \in \mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$, $R \in \mathcal{T}_H^{-\mathrm{Im}(z)}$, and

$S \in \mathcal{F}_H^{-\mathrm{Im}(z)}$ such that

$$\mathrm{Im}(Z_{H,\beta}(F)) = \mathrm{Im}(Z_{H,\beta}(R)) = \mathrm{Im}(Z_{H,\beta}(S)) = 0,$$

we have that $\mathrm{Re}(Z_{H,\beta}(R)) < 0$, $\mathrm{Re}(Z_{H,\beta}(F)) > 0$, and $\mathrm{Re}(Z_{H,\beta}(S)) > 0$. This will then show that $\mathrm{Re}(Z_{H,\beta})(E^{\cdot}) < 0$.

First, suppose $F \in \mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$. Note that for each complex $\mathcal{O}_C(i)$ in $\mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$,

$$Z_{H,\beta}(\mathcal{O}_C(i)) = -i - \frac{n}{2} + \beta \cdot C.$$

Since $i \leq k$, as long as k is chosen so that $k < \beta \cdot C - \frac{n}{2}$, $\mathrm{Re}(Z_{H,\beta}(\mathcal{O}_C(i))) > 0$. Then since $Z_{H,\beta}$ is additive on exact triangles, $\mathrm{Re}(Z_{H,\beta}(F)) > 0$.

Now let $R \in \mathcal{T}_H^{-\mathrm{Im}(z)}$ be such that $\mathrm{Im}(Z_{H,\beta}(R)) = 0$. This implies that $\mathrm{ch_0(R)} = 0$. Then $Z_{H,\beta}(R) = -\mathrm{ch_2(R)} + \beta \cdot \mathrm{ch_1(R)}$. Since $\mathrm{ch_0(R)} = 0$, R must be supported on either points or curves. If R is supported at points, $\mathrm{ch_2(R)}$ will be positive and $\mathrm{ch_1(R)} = 0$, so $Z(R) < 0$. If R is supported on a curve, it must be supported on C since only $C \cdot H = 0$. In particular, R must be an extension of sheaves of the form $\mathcal{O}_C(m)$ where $m > k$, since $R \in \mathcal{T}_{H,k}^{-\mathrm{Im}(z)}$. Since $Z_{H,\beta}(\mathcal{O}_C(m)) = -m - \frac{n}{2} + \beta \cdot C$, as long as k is chosen so that $\beta \cdot C < k + 1 + \frac{n}{2}$.

Now let $S \in \mathcal{F}_H^{-\mathrm{Im}(z)}$ be such that $H \cdot \mathrm{ch_1(S)} + \mathrm{Im(z)ch_0(S)} = 0$. In this case, $\mathrm{ch_0(S)} > 0$, and so $\nu_H(S) = -\mathrm{Im}z$. Since S is an object of $\mathcal{F}_H^{-\mathrm{Im}(z)}$ of maximal possible slope, S is ν_H-semistable. And so by Theorem 4.2.7, $\mathrm{ch}_1^2\mathrm{(S)} \geq 2\mathrm{ch_0(S)ch_2(S)}$. Then

$$\begin{aligned} Z_{H,\beta}(S) &= \mathrm{Re}(z)\mathrm{ch_0(S)} + \beta \cdot \mathrm{ch_1(S)} - \mathrm{ch_2(S)} \\ &\geq \mathrm{ch_0(S)}\left(\mathrm{Re(z)} + \frac{\beta \cdot \mathrm{ch_1(S)}}{\mathrm{ch_0(S)}} - \frac{\mathrm{ch}_1^2\mathrm{(S))}}{2\mathrm{ch}_0^2\mathrm{(S)}}\right) \\ &= \mathrm{ch_0(S)}\left(\mathrm{Re(z)} - \frac{(\mathrm{ch_1(S)} - \mathrm{ch_0(S)}\beta)^2}{2\mathrm{ch}_0^2\mathrm{(S)}} + \frac{\beta^2}{2}\right). \end{aligned}$$

Since $H \cdot (\mathrm{ch_1(S)} - \mathrm{ch_0(S)}\beta) = -\mathrm{Im(z)ch_0(S)}$, we can see that

$$H \cdot \left(\mathrm{ch_1(S)} - \mathrm{ch_0(S)}\beta + \frac{\mathrm{ch_0(F)Im(z)}}{\mathrm{H \cdot H}}\mathrm{H}\right) = 0.$$

Then by the Hodge Index Theorem,

$$\left(\mathrm{ch_1(S)} - \mathrm{ch_0(S)}\beta + \frac{\mathrm{ch_0(F)Im(z)}}{\mathrm{H}^2}\mathrm{H}\right)^2 \leq 0$$

We can now rewrite

$$\begin{aligned}
Z_{H,\beta}(S) &= \mathrm{Re}(z)\mathrm{ch}_0(\mathrm{S}) + \beta\cdot\mathrm{ch}_1(\mathrm{S}) - \mathrm{ch}_2(\mathrm{S}) \\
&\geq \mathrm{ch}_0(\mathrm{S})\left(\mathrm{Re(z)} - \frac{(\mathrm{ch}_1(\mathrm{S}) - \mathrm{ch}_0(\mathrm{S})\beta)^2}{2\mathrm{ch}_0^2(\mathrm{S})} + \frac{\beta^2}{2}\right) \\
&= \mathrm{ch}_0(\mathrm{S})\left(\mathrm{Re(z)} - \frac{(\mathrm{ch}_1(\mathrm{S}) - \mathrm{ch}_0(\mathrm{S})\beta + \frac{\mathrm{ch}_0\mathrm{Im(z)}}{\mathrm{H}^2}\mathrm{H})^2}{2\mathrm{ch}_0^2(\mathrm{S})} + \frac{\mathrm{Im(z)}^2}{\mathrm{H}^2} + \frac{\beta^2}{2}\right).
\end{aligned}$$

So as long as $\mathrm{Re}(z) + \frac{\mathrm{Im}(z)^2}{H^2} > -\frac{\beta^2}{2}$, we have $Z_{H,\beta}(S) > 0$. ☐

Lemma 4.2.9. *The pair* $(Z_{H,\beta}, \mathcal{B}_{H,k})$*, with* H *chosen to be a rational class, and* $\mathrm{Im}(z) \in \mathbb{Q}$ *satisfy the HN-property.*

Proof. Following [BM11, Proposition B.2], we first show that the image of $\mathrm{Im}(Z_{H,\beta}(\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}))$ is discrete. This is clear, since the classes $\mathrm{ch}_1(\mathrm{E}^\cdot)$ lie in a lattice for all $E^\cdot \in \mathcal{D}^b(X)$. Now for $E^\cdot \in \mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$, we must show that for any sequence of inclusions

$$0 = A_0^\cdot \hookrightarrow A_1^\cdot \hookrightarrow \cdots \hookrightarrow A_j^\cdot \hookrightarrow A_{j+1}^\cdot \hookrightarrow \cdots \hookrightarrow E^\cdot$$

in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$, such that $\mathrm{Im}(Z_{H,k}(A_j^\cdot)) = 0$ for all j, the sequence $A_j^\cdot$ stabilizes.

$E^\cdot$ lies in an exact triangle

$$F[1] \to E^\cdot \to S^\cdot$$

with $F \in \mathcal{F}_{H,\beta}^{-\mathrm{Im}(z)}$ and $S^\cdot \in \mathcal{T}_{H,\beta}^{-\mathrm{Im}(z)}$. Suppose $S^\cdot$ has an HN filtration in $\mathcal{A}_H^{-\mathrm{Im}(z)}$. That is, there exists an exact triangle

$$A^\cdot \to S^\cdot \to B^\cdot \tag{4.7}$$

in $\mathcal{A}_H$, such that $\mathrm{Im}(Z_{H,\beta}(A^\cdot)) = 0$, and for all $C^\cdot \in \mathcal{A}_H$ such that $\mathrm{Im}(Z_{H,\beta}(C^\cdot)) = 0$, $\mathrm{Hom}(C^\cdot, B^\cdot) = 0$. We can take the long exact sequence of cohomology of (4.7) with respect to the heart $\mathcal{B}_{H,k}$ to get an exact sequence

$$H^0_{\mathcal{B}}(A^\cdot) \to E^\cdot \to B^\cdot \to H^1_{\mathcal{B}}(A^\cdot).$$

Let $D^\cdot$ be the cone of the morphism $H^0_{\mathcal{B}}(A^\cdot) \to E^\cdot$. Then $D^\cdot$ is automatically in $\mathcal{T}_{H,k}^{-\mathrm{Im}(z)}$, and this is an exact triangle in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$.

Further, $\mathrm{Im}(Z_{H,\beta}(H^0_{\mathcal{B}}(A^\cdot))) = 0$. Now suppose $C^\cdot$ lies in $\mathcal{B}_{H,k}$ and $\mathrm{Im}(Z_{H,\beta}(C^\cdot)) = 0$. Then $C^\cdot$ fits into an exact triangle $F^\cdot[1] \to C^\cdot \to T^\cdot$ with $F^\cdot \in \mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$, $T^\cdot \in \mathcal{T}_{H,k}^{-\mathrm{Im}(z)}$, and $\mathrm{Im}Z_{H,\beta}(F^\cdot) = \mathrm{Im}(Z_{H,\beta}(T^\cdot)) = 0$. Since $D^\cdot$ lies in $\mathcal{T}_{H,k}^{-\mathrm{Im}(z)}$ there can be no morphisms from $F^\cdot[1]$ to $D^\cdot$. There can be no morphisms $T^\cdot \to C^\cdot$ since such a morphism would imply that $\mathrm{Hom}(T^\cdot, D^\cdot) \neq 0$. Thus $\mathrm{Hom}(C^\cdot, D^\cdot) = 0$.

Now consider the morphism $E^\cdot \to D^\cdot$. The kernel of this morphism $K^\cdot$ in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$ fits into an exact triangle

$$F[1] \to K^{\cdot} \to A^{\cdot}.$$

Hence $K^{\cdot} \in \mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$ and $\mathrm{Im}(Z_{H,\beta}(K^{\cdot})) = 0$. Therefore $E^{\cdot}$ also has the HN property in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$. Therefore, it is enough to show that if $E^{\cdot} \in \mathcal{T}_{H,k}^{-\mathrm{Im}(z)}$, then $E^{\cdot}$ has an HN filtration in $\mathcal{A}_H^{-\mathrm{Im}(z)}$.

We now prove that $E^{\cdot}$ has an HN-filtration in $\mathcal{A}_H^{-\mathrm{Im}(z)}$. This proof is similar to [Bri08, Proposition 7.1], where we use the nef divisor H instead of an ample divisor ω. Suppose we have a sequence of inclusions

$$0 = A_0^{\cdot} \hookrightarrow A_1^{\cdot} \hookrightarrow \cdots \hookrightarrow A_j^{\cdot} \hookrightarrow A_{j+1}^{\cdot} \hookrightarrow \cdots \hookrightarrow E^{\cdot}$$

in $\mathcal{A}_H^{-\mathrm{Im}(z)}$, where $\mathrm{Im}(Z_{H,\beta}(A_j^{\cdot})) = 0$ for all j. Then for each j we have exact triangles

$$A_{j-1}^{\cdot} \to A_j^{\cdot} \to B_j^{\cdot} \tag{4.8}$$

$$A_j^{\cdot} \to E^{\cdot} \to C_j^{\cdot} \tag{4.9}$$

where $B_j^{\cdot}$ and $C_j^{\cdot}$ are in $\mathcal{A}_H$.

Taking the long exact sequence of cohomology of (4.8) and (4.9) yields a sequence of inclusions in $\mathrm{Coh}(X)$:

$$0 = H^{-1}(A_0^{\cdot}) \hookrightarrow H^{-1}(A_1^{\cdot}) \hookrightarrow \cdots \hookrightarrow H^{-1}(A_j^{\cdot}) \hookrightarrow H^{-1}(A_{j+1}^{\cdot}) \hookrightarrow \cdots \hookrightarrow H^{-1}(E^{\cdot}).$$

Since Coh(X) is Noetherian, this sequence stabilizes. So we can assume that $H^{-1}(A_j^{\cdot})$ is constant for all j. Then there is an exact sequence

$$0 \to H^{-1}(B_j^{\cdot}) \to H^0(A_{j-1}^{\cdot}) \to H^0(A_j^{\cdot}) \to H^0(B_j^{\cdot}) \to 0.$$

But $H^{-1}(B_j^{\cdot})$ is torsion-free, and $H^0(A_{j-1}^{\cdot})$ is a torsion sheaf, so $H^{-1}(B_j^{\cdot}) = 0$ for all j.

It remains to show that for $j \gg 0$, $H^0(B_j^{\cdot}) = 0$. The triangles (4.8) and (4.9) yield a third triangle,

$$B_j^{\cdot} \to C_{j-1}^{\cdot} \to C_j^{\cdot}. \tag{4.10}$$

The long exact sequence of cohomology of (4.9) and (4.10) together yield a sequence of surjections in $\mathrm{Coh}(X)$:

$$H^0(E^{\cdot}) \twoheadrightarrow H^0(C_1^{\cdot}) \twoheadrightarrow \cdots \twoheadrightarrow H^0(C_{j-1}^{\cdot}) \twoheadrightarrow H^0(C_j^{\cdot}) \twoheadrightarrow \cdots .$$

Since Coh(X) is Noetherian, this sequence stabilizes. So if we take $j \gg 0$, we can assume $H^0(C_j)$ are constant. Then we have an exact sequence

$$0 \to H^{-1}(C_{j-1}^{\cdot}) \to H^{-1}(C_j^{\cdot}) \to H^0(B_j^{\cdot}) \to 0. \tag{4.11}$$

Furthermore, from (4.9) we see that for $j \gg 0$, the map $H^{-1}(A^{\cdot}) \to H^{-1}(E^{\cdot})$ is constant. So there is a torsion-free sheaf Q such that for all $j \gg 0$,

$$0 \to Q \to H^{-1}(C_j^{\cdot}) \to H^0(A_j) \to 0$$

is exact. We would like to say that the sequence of inclusions

$$0 \subseteq Q \subseteq H^{-1}(C_1^{\cdot}) \subseteq \cdots \subseteq H^{-1}(C_{j-1}^{\cdot}) \subseteq H^{-1}(C_j^{\cdot}) \subseteq \cdots \tag{4.12}$$

stabilizes for $j \gg 0$

If $H^0(A_j^{\cdot})$ is supported on points for $j \gg 0$, then it follows from the argument of [Bri07, Proposition 7.1] that the sequence stabilizes for $j \gg 0$. Otherwise, $H^0(A_j^{\cdot})$ is supported on C for all j. Furthermore, since $H^0(A_j^{\cdot}) \in \mathcal{T}_{H,k}^{-\mathrm{Im}(z)}$, we can further assume that $\mathrm{Hom}(H^0(A_j^{\cdot}), \mathcal{O}_C(k)) = 0$. Also, (4.11) implies that $H^0(B_j^{\cdot})$ is the quotient $H^0(A_j^{\cdot})/H^0(A_{j-1}^{\cdot})$, and hence supported on points. □

Theorem 4.2.10. *The pair $(Z_{H,\beta}, \mathcal{B}_{H,k}^{-\mathrm{Im}(z)})$ define a stability condition on $\mathcal{D}^b(X)$ when k is chosen so that $k + \frac{n}{2} < \beta \cdot C < k + \frac{n}{2} + 1$ and $\mathrm{Re}(z) + \frac{\mathrm{Im}(z)^2}{H^2} > -\frac{\beta^2}{2}$.*

Proof. Lemmas 4.2.8 and 4.2.9 show that the pair $(Z_{H,\beta}, \mathcal{B}_{H,k}^{-\mathrm{Im}(z)})$ satisfies the properties required in Definition 2.1.3. □

In order to consider wall-crossing, we must show that when the pair $\sigma_{H,\beta} = (Z_{H,\beta}, \mathcal{B}_{H,k}^{-\mathrm{Im}(z)})$ is deformed slightly, the phases of objects do not vary too much. That is, we need to show $\sigma_{H,\beta}$ satisfies the support property, stated in Definition 2.3.1. This definition is equivalent to the following alternate definition, given in [KS08, Section 2.1].

Proposition 4.2.11. *A stability condition $\sigma = (Z, \mathcal{B})$ satisfies the support property if and only if there exists a quadratic form Q such that Q is negative definite on the kernel of the central charge Z, and for any σ-semistable objects $E^{\cdot}$ in $\mathcal{B}$, $Q(E^{\cdot}) \geq 0$.*

The proof is given in [KS08, Section 2.1] and in [BMS14, Appendix A]. We will construct such a quadratic form for a range of stability conditions σ_s we now define, by considering semistable objects in the limit as $s \to \infty$.

Definition 4.2.12. *For every $s \geq 1$ we can define a new stability condition $\sigma_{H,\beta,s} = (Z_{H,\beta,s}, \mathcal{B}_{H,k}^{-\mathrm{Im}(z)})$, where $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$ is as before, and*

$$Z_{H,\beta,s}(E^{\cdot}) = -\mathrm{ch_2(E^{\cdot})} + \beta \cdot \mathrm{ch_1(E^{\cdot})} + \mathrm{sRe(z)ch_0(E^{\cdot})} + \mathrm{i(H \cdot ch_1(E^{\cdot}) + Im(z)ch_0(E^{\cdot}))}.$$

Lemma 4.2.13. *The pair $\sigma_{H,\beta,s} = (Z_{H,\beta,s}, \mathcal{B}_{H,k}^{\mathrm{Im}(z)})$ give a stability condition on X when β and z satisfy the conditions of Lemma 4.2.8 and $\mathrm{Re}(z) > 0$.*

Proof. We need to show that the image $Z_{H,\beta,s}(\mathcal{B}^{\mathrm{Im}(z)}_{H,k})$ lies in the upper half plane for $s \geq 1$. The case $s = 1$ is shown in Lemma 4.2.8. When $s > 1$, then

$$s\mathrm{Re}(z) + \frac{\mathrm{Im}(z)^2}{H^2} > \mathrm{Re}(z) + \frac{\mathrm{Im}(z)^2}{H^2} > -\frac{\beta^2}{2},$$

and so the pair β, $s\mathrm{Re}(x) + i\mathrm{Im}(z)$ satisfy the conditions of Lemma 4.2.8, and $\sigma_{H,\beta,s}$ is also a stability condition on X. □

We will now describe what happens as s grows large.

Definition 4.2.14. *Define $\mathcal{D}$ to be the set of $E^{\cdot}$ in $\mathcal{B}^{-\mathrm{Im}(z)}_{H,k}$ such that $E^{\cdot}$ is $Z_{H,\beta,s}$-semistable for $s \gg 0$.*

Lemma 4.2.15. *If $E^{\cdot}$ is in $\mathcal{D}$ then it is of one of the following forms:*

1. *$E^{\cdot}$ is a slope semistable sheaf in $\mathcal{T}^{-\mathrm{Im}(z)}_H$.*
2. *$H^0(E^{\cdot})$ is either 0 or supported on C or on points, and $H^{-1}(E^{\cdot})$ fits into an exact sequence*

$$0 \to G \to H^{-1}(E^{\cdot}) \to F \to 0$$

where F is a slope semistable sheaf in $\mathcal{F}^{-\mathrm{Im}(z)}_H$, and $G \in \mathcal{F}^{-\mathrm{Im}(z)}_{H,k}$. Here G must be 0 unless $\nu_H(G) = -\mathrm{Im}(z)$.

Proof. Suppose that $E^{\cdot}$ is $Z_{H,\beta,s}$-semistable for $s \gg 0$. Recall that $E^{\cdot}$ fits into an exact triangle

$$G[1] \to E^{\cdot} \to T^{\cdot}$$

where $G \in \mathcal{F}^{-\mathrm{Im}(z)}_{H,k}$, and that $T^{\cdot}$ must itself fit into an exact triangle

$$F[1] \to T^{\cdot} \to S$$

where $F \in \mathcal{F}^{-\mathrm{Im}(z)}_H$ and $S \in \mathcal{T}^{-\mathrm{Im}(z)}_H$ are sheaves. Suppose first that $\mathrm{ch}_0(\mathrm{E}^{\cdot}) > 0$. Then as $s \to \infty$, $\phi_{H,\beta,s}(E^{\cdot}) \to 0$. Since $G[1]$ is fixed as s varies with phase 1, G must be 0. Further, since F is a sheaf, $\mathrm{ch}_0(\mathrm{F}[1]) < 0$. So as $s \to \infty$, $\phi_{H,\beta,s}(F[1]) \to 1$.

Note that since $\mathrm{Ext}^{-1}(F, \mathcal{O}_C(l)) = 0$ for all values of l, $F[1] \in \mathcal{T}^{-\mathrm{Im}(z)}_{H,k}$. Since $S \in \mathcal{T}^{-\mathrm{Im}(z)}_{H,k}$ as well, $F[1]$ is a subobject of $T^{\cdot}$ in $\mathcal{B}^{-\mathrm{Im}(z)}_{H,k}$. But we've assumed that $\phi_{H,\beta,s}(E^{\cdot}) \to 0$, and so $F[1] = 0$ as well.

Now $E^{\cdot} \cong S$ is a sheaf in $\mathcal{T}^{-\mathrm{Im}(z)}_H$ with $\mathrm{ch}_0(\mathrm{S}) > 0$. We can write the HN-filtration of S with respect to ν_H:

$$0 \to S_1 \hookrightarrow \cdots \hookrightarrow S_{m-1} \hookrightarrow S_m = S$$

with quotients $T_i := S_i/S_{i-1}$ which are ν_H-semistable, and with $\nu_H(T_i) > \nu_H(T_{i+1})$ for $i = 1, \ldots, m-1$. It may be that S_{m-1} is not itself in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$, but it is in $\mathcal{A}_H$, and so there is an exact triangle

$$S'_{m-1} \to S_{m-1} \to F'$$

with $S'_{m-1} \in \mathcal{T}_{H,k}^{-\mathrm{Im}(z)}$ and $F' \in \mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$. Since F' is supported on C, $\nu_H(S'_{m-1}) = \nu_H(S_{m-1})$.

Checking the long exact cohomology sequence, we can see that S'_{m-1} is a sheaf, and so we can compose maps to get an injective map of sheaves $S'_{m-1} \hookrightarrow S$. The quotient will be a sheaf of positive slope, since $S \in \mathcal{T}_H^{-\mathrm{Im}(z)}$. Further, it can have no maps to $\mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$, otherwise this would contradict that S has no such maps. And so S'_{m-1} injects into S in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$. Since $\nu_H(S'_{m-1}) > \nu_H(S)$, for s sufficiently large, $\phi_{H,\beta,s}(S'_{m-1}) > \phi_{H,\beta,s}(S)$, contradicting that S is stable. Thus S must itself be slope-semistable.

Now suppose $\mathrm{ch}_0(E^\cdot) < 0$. Then as $s \to \infty$, $\phi_{sH,\beta}(E^\cdot) \to 1$. Since S is a quotient of $E^\cdot$ in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$, it must also be that $\phi_{H,\beta,s}(S) \to 1$ as $s \to \infty$. This is possible only if $\mathrm{ch}_0(S) = 0$ and $H \cdot \mathrm{ch}_1(S) = 0$. And so S must be supported at points or along C. If $H \cdot \mathrm{ch}_1(F) < \mathrm{Im}(z)$ then G must be 0. In this case, we can use HN-filtrations in the same manner as in the previous case to show that F must be slope semistable itself. If $H \cdot \mathrm{ch}_1(F) = -\mathrm{Im}(z)$, then F is automatically slope semistable.

It remains to consider $\mathrm{ch}_0(E^\cdot) = 0$. In this case, $\mathrm{ch}_0(T^\cdot) = 0$. It is possible that $T^\cdot = 0$, since $\phi_{H,\beta,s}(G[1]) = 1$ for any value of s. If $T^\cdot \neq 0$, then first suppose $H \cdot \mathrm{ch}_1(E^\cdot) > -\mathrm{Im}(z)$. This implies that as $s \to \infty$, $\phi_{H,\beta,s}(E^\cdot) \to \frac{1}{2}$. And so $G = 0$, and F must be 0 as well. Then $E^\cdot$ is a torsion sheaf supported on a curve C' not contained in C. If $H \cdot \mathrm{ch}_1(E^\cdot) = -\mathrm{Im}(z)$, then F must again be 0, and now S must be a torsion sheaf supported on C or on points. □

We now work towards the construction of a quadratic form Q which will satisfy the requirements of Proposition 4.2.11 where the semistable objects are the objects of $\mathcal{D}$. We first need the following lemmas.

Lemma 4.2.16. *There is a positive constant C_H depending only on H so that for any sheaf E supported on a curve $C' \not\subseteq C$,*

$$H^2 \mathrm{ch}_1(E)^2 + C_H (H \cdot \mathrm{ch}_1(E))^2 \geq 0.$$

Proof. Write $C' = \alpha + lC$ with α a class in $C^\perp$. Since C' is not contained in C, $C \cdot C' \geq 0$, so $l \leq 0$. Further, for $0 < t \ll 1$, $H - tC$ is ample. This follows from the fact that H is big and nef, and that C is the only effective divisor in $H^\perp$.

Since this is an ample divisor, $C' \cdot (H - tC) > 0$. So $H \cdot \alpha > -tln \geq 0$. Then

$$\begin{aligned}(C')^2 + \frac{1}{t^2 n}(H \cdot \alpha)^2 &= \alpha^2 - l^2 n + \frac{1}{t^2 n}(H \cdot \alpha)^2 \\ &> \alpha^2.\end{aligned}$$

Further, since H is nef, the Hodge Index Theorem states that there exists some constant $C_H > 0$ depending only on H so that $H^2\alpha^2 + C_H(H \cdot \alpha)^2 \geq 0$. We then have

$$\begin{aligned}(C')^2 + \left(\frac{1}{t^2 n} + \frac{C_H}{H^2}\right)(H \cdot \alpha)^2 &= \alpha^2 - l^2 n + \left(\frac{1}{t^2 n} + \frac{C_H}{H^2}\right)(H \cdot \alpha)^2 \\ &> \alpha^2 + \frac{C_H}{H^2}(H \cdot \alpha)^2 \\ &\geq 0.\end{aligned}$$

□

Define a constant D_H as follows, where C_H is as in Lemma 4.2.16:

$$m_1 = \max\{H \cdot \mathrm{ch}_1(\mathrm{F}) \mid \mathrm{F \text{ is a slope semistable sheaf}},\ \mathrm{H \cdot ch_1(F) < -Imz,\ ch_0(F) = 1}\}.$$

$$m_2 = \max\{H \cdot \mathrm{ch}_1(\mathrm{F}) \mid \mathrm{F \text{ is a slope semistable sheaf}},\ \mathrm{H \cdot ch_1 F < -Imz,\ ch_0(F) = 2}\}.$$

$$D_H = \max\left\{\frac{\frac{3}{2}n + 2k + 3}{m_1^2}, \frac{8k + 21}{m_2^2}, C_H\right\}.$$

We now define a preliminary quadratic form.

Definition 4.2.17.

$$Q_0(E^\cdot) := \mathrm{ch_1(E^\cdot)^2 - 2ch_0(E^\cdot)ch_2(E^\cdot) + D_H(ImZ_{H,\beta}(E^\cdot))^2}.$$

Lemma 4.2.18. *$Q_0(E^\cdot) \geq 0$ for $E^\cdot$ in $\mathcal{D}$ such that* $\mathrm{Im}(Z_{H,\beta}(E^\cdot)) > 0$.

Proof. First, if $E^\cdot$ is a torsion-free sheaf or a shift of a torsion-free sheaf in $\mathcal{D}$, then by Lemma 4.2.15, this sheaf is ν_H-semistable. Thus $Q_0(E^\cdot) \geq 0$ by Theorem 4.2.7. If $E^\cdot$ is a torsion sheaf not supported on C, then it is either supported on points, in which case $Q_0(E^\cdot) = 0$, or it is supported on a curve not contained in C. In this case, $Q_0(E^\cdot) \geq 0$ by Lemma 4.2.16.

It remains to consider $E^{\cdot}$ such that there is an exact triangle

$$F[1] \to E^{\cdot} \to T$$

where T is a torsion sheaf supported on C or on points, and F is a slope semistable sheaf of slope smaller than 0. If $\nu_H(F) < \mathrm{Im}(z)$, then $\mathrm{Hom}(\mathcal{O}_C(k+1), F[1]) = 0$ since both are semistable, and $\phi_{H,\beta}(F[1]) < 1$. Further, $\mathrm{Ext}^2(\mathcal{O}_C(k+1), F[1]) = \mathrm{Ext}^3(\mathcal{O}_C(k+1), F) = 0$. Thus $\chi(\mathcal{O}_C(k+1), F[1]) \leq 0$. By Hirzebruch-Riemann-Roch, $\chi(\mathcal{O}_C(k+1), F[1]) = -C \cdot \mathrm{ch}_1(\mathrm{F}[1]) + (\mathrm{k}+2)\mathrm{ch}_0(\mathrm{F}[1])$. Combining these facts, $C \cdot \mathrm{ch}_1(\mathrm{F}[1]) \geq (\mathrm{k}+2)\mathrm{ch}_0(\mathrm{F}[1])$.

Now we have

$$\begin{aligned} Q_0(E^{\cdot}) &\geq (\mathrm{ch}_1(\mathrm{T}) + \mathrm{ch}_1(\mathrm{F}[1]))^2 - 2\mathrm{ch}_0(\mathrm{F}[1])(\mathrm{ch}_2(\mathrm{T}) + \mathrm{ch}_2(\mathrm{F}[1]) \\ &\geq \mathrm{ch}_1(\mathrm{T})^2 + 2\mathrm{ch}_1(\mathrm{F}[1])\mathrm{ch}_1(\mathrm{T}) - 2\mathrm{ch}_0(\mathrm{F}[1])\mathrm{ch}_2(\mathrm{T}). \end{aligned}$$

If T is supported on points this is clearly positive. It suffices to consider $T \cong \mathcal{O}_C(l)$ where $l > k$. Then the above inequality becomes

$$\begin{aligned} Q_0(E^{\cdot}) &\geq -n + 2C \cdot \mathrm{ch}_1(\mathrm{F}[1]) - 2\mathrm{ch}_0(\mathrm{F}[1]) \left(\mathrm{l} + \frac{\mathrm{n}}{2}\right) \\ &\geq -n + 2\left(k + 2 - l - \frac{n}{2}\right) \mathrm{ch}_0(\mathrm{F}[1]). \\ &= (\mathrm{ch}_0(\mathrm{F}) - 1)\mathrm{n} + 2\mathrm{ch}_0(\mathrm{F})(\mathrm{l} - \mathrm{k} - 2) \end{aligned}$$

For $\mathrm{ch}_0(\mathrm{F}) \geq 3$ this is necessarily positive. The only cases in which it may not be positive are $l = k+1$ and $\mathrm{ch}_0(\mathrm{F}[1]) = -1$, or $l = k+1$, $\mathrm{ch}_0(\mathrm{F}[1]) = -2$ and $n = 3$. In these cases, the choice of D_H ensures that Q_0 is positive. □

Lemma 4.2.19. *Q_0 is negative definite on the kernel of $Z_{H,\beta,s}$ as defined in Definition 4.2.12 for all $s \geq 1$.*

Proof. Suppose $Z_{H,\beta,s}(E^{\cdot}) = 0$ for some $s \geq 1$. Note that if $\mathrm{ch}^{\beta}(\mathrm{E}^{\cdot}) = \mathrm{ch}(\mathrm{E}^{\cdot})\mathrm{e}^{-\beta}$, $(\mathrm{ch}_1^{\beta})^2(\mathrm{E}^{\cdot}) - 2\mathrm{ch}_0^{\beta}(\mathrm{E}^{\cdot})\mathrm{ch}_2^{\beta}(\mathrm{E}^{\cdot}) = \mathrm{ch}_1^2(\mathrm{E}^{\cdot}) - 2\mathrm{ch}_0(\mathrm{E}^{\cdot})\mathrm{ch}_2(\mathrm{E}^{\cdot})$. Since $\beta \cdot H = 0$, $\mathrm{ch}_1^{\beta}(\mathrm{E}^{\cdot}) = \mathrm{ch}_1(\mathrm{E}^{\cdot}) - \beta\mathrm{ch}_0(\mathrm{E}^{\cdot})$ is in $H^{\perp}$, so $(\mathrm{ch}_1^{\beta})^2(\mathrm{E}^{\cdot}) \leq 0$ by the Hodge Index Theorem. Further, since $E^{\cdot}$ is in the kernel of $Z_{H,\beta,s}$, $\mathrm{ch}_2^{\beta}(\mathrm{E}^{\cdot}) = (\mathrm{s}^2\mathrm{z} + \frac{\beta^2}{2})\mathrm{ch}_0(\mathrm{E}^{\cdot})$ has the same sign as $\mathrm{ch}_0^{\beta}(\mathrm{E}^{\cdot}) = \mathrm{ch}_0(\mathrm{E}^{\cdot})$. And so $Q_0(E^{\cdot}) \leq 0$. □

Q_0 is negative on sheaves supported on C, and on their shifts. We now must adjust Q_0 to find a quadratic form which is positive on such sheaves. Note that it suffices to consider sheaves $\mathcal{O}_C(l)$ where $l > k$, and shifts $\mathcal{O}_C(m)[1]$, where $m \leq k$.

Definition 4.2.20. *Let* $m_\beta = \min\{|\beta \cdot C - k - \frac{n}{2} - 1|, |k + \frac{n}{2} - \beta \cdot C|\}$ *and* $D_\beta = \frac{n}{m_\beta^2}$. *We now define another preliminary set of quadratic forms for* $s \geq 1$*:*

$$Q_s(E^\cdot) = Q_0(E^\cdot) + D_\beta(\mathrm{Re}(Z_{sH,\beta}(E^\cdot)))^2.$$

By construction, $Q_s(E^\cdot) \geq 0$ for all $E^\cdot \in \mathcal{D}$, and Q_s is negative definite on the kernel of $Z_{H,\beta,s}$.

Theorem 4.2.21. *The central charge* $Z_{H,\beta}$ *satisfies the support property in the sense of Proposition 4.2.11 for Bridgeland semistable objects in* $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$ *with respect to the quadratic form* Q_1.

Proof. First we consider $E^\cdot \in \mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$ such that $\mathrm{Im}(Z_{H,\beta}(E^\cdot)) > 0$. The imagine of $\mathrm{Im}(Z_{H,\beta})$ is discrete, and so we may proceed by induction. Any objects for which $\mathrm{Im}(Z_{H,\beta})$ is minimal must be in $\mathcal{D}$, as any possible destabilizing subobjects must have smaller imaginary part. Lemma 4.2.18 and Lemma 4.2.19 show that the support property is satisfied for such objects.

Now suppose there is some $E^\cdot \in \mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$ which is $Z_{H,\beta}$-semistable but for which $Q_0(E^\cdot) < 0$. Assume that for any $F^\cdot$ such that $\mathrm{Im}(Z_{H,\beta}(F^\cdot)) < \mathrm{Im}(Z_{H,\beta}(E^\cdot))$, the requirements of the support property are met by Q_0. Since $E^\cdot$ is not in $\mathcal{D}$, this implies that there exists some $s > 1$ for which $E^\cdot$ is strictly $Z_{H,\beta,s}$-semistable. Let $E_1^\cdot, \ldots, E_m^\cdot$ be the Jordan-Hölder factors of $E^\cdot$. Then $\mathrm{Im}(Z_{H,\beta}(E_i^\cdot)) < \mathrm{Im}(Z_{H,\beta}(E^\cdot))$ for all $i = 1, \ldots, m$. And so by the inductive hypothesis, $Q_0(E_i) \geq 0$.

The quadratic form Q_0 divides $K(\mathcal{D}^b(X))$ into a positive and negative cone. For any pair $E_i^\cdot$ and $E_j^\cdot$ of Jordan Hölder factors of $E^\cdot$, these lie on the same ray in the image of $Z_{H,\beta,s}$. And so there is some $a > 0$ for which $Z_{H,\beta,s}(E_i^\cdot) - aZ_{H,\beta,s}(E_j^\cdot) = 0$. The restriction of Q_0 to the kernel of $Z_{H,\beta,s}$ is negative definite, and so this combination $[E_i^\cdot] - a[E_j^\cdot]$ must lie in the negative cone of Q_0. This implies that any positive linear combination of $[E_i^\cdot]$ and $[E_j^\cdot]$ lies in the positive cone of Q_0. Since this is true for any pair $E_i^\cdot$ and $E_j^\cdot$, it follows that $Q_0(E^\cdot) \geq 0$.

We have shown that Q_0 satisfies the requirements of the support property for semistable objects of strictly positive imaginary part. We can now use Q_1 from Definition 4.2.20 which will satisfy the support property for all $Z_{H,\beta}$-semistable objects. □

The above shows that $(Z_{H,\beta}, \mathcal{B}_{H,k}^{-\mathrm{Im}(z)})$ is a stability condition with the support property when H and $\mathrm{Im}(z)$ are rational. We need to extend this results to real H and $\mathrm{Im}(z)$.

Theorem 4.2.22. *The pair* $(Z_{H,\beta}, \mathcal{B}_{H,k}^{-\mathrm{Im}(z)})$ *is a stability condition with the support property for* H *and* $\mathrm{Im}(z)$ *real.*

Proof. By Theorem 4.2.21 this holds for H and $-\mathrm{Im}(z)$ in $\mathbb{Q}$. Then we can deform these stability conditions to have stability conditions on $\mathbb{R}$. It remains to show that this is well-defined. This holds by an argument similar to those in [Tod14, Section 5] and [BMS14, Appendix B].

For each stability condition $\sigma_{H,\beta} = (Z_{H,\beta}, \mathcal{B}_{H,k}^{-\mathrm{Im}(z)})$ with $\mathrm{Im}(z)$ and H rational, we can obtain an open subset of the space of stability conditions by deforming $\sigma_{H,\beta}$ [BMS14, Proposition A.5]. This gives a cover of the wall of the geometric chamber.

If σ_{H_1,β_1} and σ_{H_2,β_2} are two such stability conditions, and U_1 and U_2 are the corresponding open subsets, it remains to show that deforming σ_{H_1,β_1} in U_1 and σ_{H_2,β_2} in U_2 gives the same stability conditions in $U_1 \cap U_2$. It would suffice to show that there exists a stability condition $\sigma_{H,\beta} \in U_1 \cap U_2$ where this holds. But $U_1 \cap U_2$ contains stability conditions in the geometric chamber of $\mathrm{Stab}(X)$. Since this holds inside the geometric chamber, it thus holds on the wall. $\square$

4.3 Wall-crossing and the construction of the moduli space of stable objects

We now consider a stability condition τ across the wall constructed in the previous section. We will construct a moduli space $M_\tau([\mathcal{O}_x])$ for stable objects of class $[\mathcal{O}_x]$.

Lemma 4.3.1. *$\mathcal{O}_C(k+1)$ and $\mathcal{O}_C(k)[1]$ are simple objects in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$.*

Proof. Suppose $A^\cdot$ is a subobject of $\mathcal{O}_C(k)[1]$ in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$. Then there is an exact triangle $A^\cdot \to \mathcal{O}_C(k)[1] \to B^\cdot$ for some $B^\cdot \in \mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$. Since we assume $A^\cdot$ and $B^\cdot$ are in the heart $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$ we know that $A^\cdot$ and $B^\cdot$ have cohomology only in degrees -1 and 0. Hence we get the following long exact sequence by taking cohomology in Coh(X).

$$0 \to H^{-1}(A^\cdot) \to \mathcal{O}_C(k) \to H^{-1}(B^\cdot) \to H^0(A^\cdot) \to 0.$$

Further, since $B^\cdot$ is a quotient of $\mathcal{O}_C(k)[1]$, which lies in $\mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$, we have $B^\cdot \in \mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$.

We see from the sequence above that $H^{-1}(A^\cdot)$ is a sheaf which injects into $\mathcal{O}_C(k)$. This leaves only a few possiblities for which sheaf $H^{-1}(A^\cdot)$ can be. If $H^{-1}(A^\cdot)$ is a proper subsheaf of $\mathcal{O}_C(k)$, then $H^{-1}(A^\cdot) \cong \mathcal{O}_C(l)$ for some $l < k$. The quotient $H^{-1}(A^\cdot) \to \mathcal{O}_C(k)$ is then supported on points. But such a quotient could not inject into $H^{-1}(B^\cdot)$, since all sheaves supported on points lie in $\mathcal{T}_{H,k}^{-\mathrm{Im}(z)}$.

This leaves only the possiblity that $H^{-1}(A^\cdot)$ is not a proper subsheaf of $\mathcal{O}_C(k)$. That is, $H^{-1}(A^\cdot)$ is 0 or $\mathcal{O}_C(k)$. If $H^{-1}(A^\cdot) \cong \mathcal{O}_C(k)$, then $H^{-1}(B^\cdot) \cong H^0(A^\cdot)$. This implies that these sheaves are both 0, and $A^\cdot \cong \mathcal{O}_C(k)[1]$. If $H^{-1}(A^\cdot) \cong 0$ then since $H^{-1}(B^\cdot)$ lies in $\mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$, $H^0(A^\cdot) \cong 0$ and $B^\cdot \cong \mathcal{O}_C(k)[1]$.

Now suppose $A^\cdot$ is a subobject of $\mathcal{O}_C(k+1)$ and fits into an exact triangle $A^\cdot \to \mathcal{O}_C(k+1) \to B^\cdot$. Again, taking cohomology with respect to $\mathcal{A}_H^{-\mathrm{Im}(z)}$ and $\mathrm{Coh}(X)$ separately, we can deduce that $A^\cdot$ is a sheaf supported on C, and that there is an exact sequence

$$0 \to H^{-1}(B^\cdot) \to A^\cdot \to \mathcal{O}_C(k+1) \to H^0(B^\cdot) \to 0.$$

Further $H^{-1}(B^\cdot) \in \mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$, and $H^0(B^\cdot)$ is supported on C or points.

If $H^0(B^\cdot)$ were supported on points, then the kernel of the map $\mathcal{O}_C(k+1) \to H^0(B^\cdot)$ would be a sheaf in $\mathcal{F}_{H,k}^{-Imag(z)}$, from which $H^0(A^\cdot)$ could have no morphisms. And so $H^0(B^\cdot)$ can be only $\mathcal{O}_C(k+1)$ or 0. In the first case, $A^\cdot \cong 0$ and $B^\cdot \cong \mathcal{O}_C(k+1)$. In the second case, $A^\cdot \cong \mathcal{O}_x$ and $B^\cdot \cong 0$. □

Lemma 4.3.2. *If $x \in X \setminus C$, then $\mathcal{O}_x$ is $\sigma_{H,\beta}$-stable. If $x \in C$, $\mathcal{O}_x$ is strictly $\sigma_{H,\beta}$-semistable, destabilized by the exact triangle*

$$\mathcal{O}_C(k+1) \to \mathcal{O}_x \to \mathcal{O}_C(k)[1].$$

Proof. Since $\mathcal{O}_x$ is stable inside the geometric chamber, it is either $\sigma_{H,\beta}$-stable or it is $\sigma_{H,\beta}$-semistable. Suppose it is semistable. Then there is an exact triangle

$$A^\cdot \to \mathcal{O}_x \to B^\cdot$$

in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$ destabilizing $\mathcal{O}_x$. Taking cohomology, we see that $A^\cdot$ is a sheaf, and that $H^0(B^\cdot)$ is either 0 or $\mathcal{O}_x$. In the latter case, $H^{-1}(B^\cdot) \cong H^0(A^\cdot) \cong 0$, so $B^\cdot \cong \mathcal{O}_x$.

In the first case, we see $A^\cdot$ must be a torsion sheaf supported on C or points, and $B^\cdot \cong F[1]$ for some sheaf $F \in \mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$. Such a sequence can only exist when $x \in C$, so otherwise $\mathcal{O}_x$ is stable. For points x on C, the sequence $\mathcal{O}_C(k+1) \to \mathcal{O}_x \to \mathcal{O}_C(k)[1]$ destabilizes $\mathcal{O}_x$. □

Lemma 4.3.3. *Suppose $E^\cdot$ is of class $[\mathcal{O}_x]$ and is $\sigma_{H,\beta}$-semistable, then the only possible Jordan-Hölder factors of $E^\cdot$ are $\mathcal{O}_C(k+1)$ and $\mathcal{O}_C(k)[1]$, or $\mathcal{O}_x$ for some $x \notin C$.*

Proof. The Jordan-Hölder factors of $\mathcal{O}_x$ must lie in the saturation of the lattice generated by $\mathcal{O}_C$ and $\mathcal{O}_x$ in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$. The objects $\mathcal{O}_C(k+1)$, $\mathcal{O}_C(k)[1]$, and $\mathcal{O}_x$ are simple objects in this lattice. Suppose there is another simple object $E^\cdot$ in this lattice. Note that $\mathrm{ch}_0(\mathrm{E}^\cdot) = 0$, and $H \cdot \mathrm{ch}_1(\mathrm{E}^\cdot) = 0$.

We know $E^\cdot$ fits into an exact triangle $F[1] \to E^\cdot \to T^\cdot$ where $F \in \mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$ and $T^\cdot \in \mathcal{T}_{H,k}^{-\mathrm{Im}(z)}$. So one of F and $T^\cdot$ must be 0. If $E^\cdot = F[1]$, and is simple, we claim $E^\cdot \cong \mathcal{O}_C(k)[1]$. To see this, note that since $\mathcal{F}_{H,k}^{-\mathrm{Im}(z)}$ was constructed as the extension closure of the set of objects of the form $\mathcal{O}_C(l)$ for some $l \leq k$, all objects in $\mathcal{F}_{H,k}^{-\mathrm{Im}(z)}[1]$ have a morphism to $\mathcal{O}_C(l)[1]$ for some $l \leq k$ which is surjective in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$. Since $E^\cdot = 0$, we see $E^\cdot \cong \mathcal{O}_C(l)[1]$ for this l. But then if $l \neq k$, there is an exact triangle in $\mathcal{B}_{H,k}^{-\mathrm{Im}(z)}$,

$$T \to \mathcal{O}_C(l)[1] \to \mathcal{O}_C(k)[1]$$

where T is a sheaf supported on points of length $k - l$. Hence $k = l$.

If $E^\cdot = T^\cdot$, then since $H \cdot \mathrm{ch}_1(\mathrm{E}^\cdot) = \mathrm{ch}_0(\mathrm{E}^\cdot) = 0$, $E^\cdot$ must be a sheaf supported on C or on points. If $E^\cdot$ is supported on points and simple, then $E^\cdot$ is a skyscraper sheaf $\mathcal{O}_x$ where $x \notin C$. If $E^\cdot$ is supported on C, and $E^\cdot \in \mathcal{T}_{H,k}^{-\mathrm{Im}(z)}$, then $E^\cdot$ has $\mathcal{O}_C(k+1)$ as a subobject. Hence since $E^\cdot$ is simple, $E^\cdot \cong \mathcal{O}_C(k+1)$. □

We will now consider a new stability condition τ across the wall we have constructed, obtained by deforming $\sigma_{H,\beta}$. We will study the moduli space of τ-stable objects of class $[\mathcal{O}_x]$. In order to study objects of this class, we will look at a local model and study a neighborhood of the curve C in X. Let $\mathcal{D}^b_C(X)$ denote the subcategory of $\mathcal{D}^b(X)$ of objects supported on C. Let $\widehat{X}$ be the completion of X at C.

Lemma 4.3.4. $\mathcal{D}^b_C(X) \cong \mathcal{D}^b_C(\widehat{X})$.

Proof. By Proposition 1.7.11 in [KS90], $\mathcal{D}^b_C(X) \cong \mathcal{D}^b(\mathrm{Coh_C(X)})$ and $\mathcal{D}^b_C(\widehat{X}) \cong \mathcal{D}^b(\mathrm{Coh_C(\widehat{X})})$. It remains to show that $\mathrm{Coh_C(X)} \cong \mathrm{Coh_C(\widehat{X})}$. Any sheaf $\mathcal{F} \in \mathrm{Coh_C(X)}$ is supported in a finite-order neighborhood C_n of C in X. The embedding $\mathrm{Coh(C_n)} \to \mathrm{Coh_C(X)}$ is fully faithful. Similarly for $\widehat{X}$, any sheaf in $\mathrm{Coh_C(\widehat{X})}$ is supported on a finite-order neighborhood of C, isomorphic to C_n by construction. Since $\mathrm{Coh(C_n)} \to \mathrm{Coh_C(\widehat{X})}$ is also a fully faithful embedding, it follows that $\mathrm{Coh_C(X)} \cong \mathrm{Coh_C(\widehat{X})}$. □

Lemma 4.3.5. *$\widehat{X}$ is isomorphic to the completion of* Tot $\mathcal{O}_{\mathbb{P}^1}(-n)$ *at the* 0*-section.*

Proof. The curve C is contractible. Up to isomorphism, there is a unique local singularity to which $\widehat{X}$ contracts. Further, the completion of Tot $\mathcal{O}_{\mathbb{P}^1}(-n)$ at the 0-section is another $-n$-curve, and hence it must contract to the same singularity. This local singularity has a unique minimal resolution, and so $\widehat{X}$ and the completion of Tot $\mathcal{O}_{\mathbb{P}^1}(-n)$ at the 0-section must be isomorphic. □

We will now construct a family of τ-semistable objects of class $[\mathcal{O}_x]$ in $K_0(X)$, with the goal of constructing a universal family over $M_\tau([\mathcal{O}_x])$. We will do this by considering stable objects of the form $\mathcal{O}_x$ for some $x \in X \setminus C$ and stable objects of the form $\eta(y)$ for some $y \in \mathbb{P}\mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1])$ separately, and then gluing along C.

Inside the geometric chamber of $\mathrm{Stab}(X)$, the stable objects of class $[\mathcal{O}_x]$ are the skyscraper sheaves $\mathcal{O}_x$ themselves. Hence a family is given by the object $\mathcal{O}_{\Delta_X}$ in $\mathcal{D}^b(X \times X)$. However, along the wall we have constructed we will construct a new family of τ-stable objects via semistable reduction.

Consider the following diagram.

$$\begin{array}{ccc} C \times X & \xrightarrow{j} & X \times X \\ \downarrow & & \downarrow \\ C & \longrightarrow & X \end{array}$$

There is an exact triangle in $\mathcal{D}^b(C \times X)$ as follows:

$$\mathcal{O}_{C\times C}(-k-2, k) \to \mathcal{O}_{C\times C}(-k-1, k+1) \to \mathcal{O}_{\Delta_C}.$$

Define $\mathcal{E}$ by

$$\mathcal{E} \to \mathcal{O}_{\Delta_X} \to j_* \mathcal{O}_{C\times C}(-k-2,k)[1] \tag{4.13}$$

where the second map is given by the composition of the map coming from the exact triangle and the restriction map $\mathcal{O}_{\Delta_X} \to j_* \mathcal{O}_{\Delta_C}$.

First, note that $\mathcal{E}$ is a sheaf. It fits into the exact sequence

$$0 \to j_* \mathcal{O}_{C\times C}(-k-2,k) \to \mathcal{E} \to \mathcal{O}_{\Delta_X} \to 0$$

on $X \times X$. $\mathcal{E}$ is supported on $(C \times C) \cup_{\Delta_C} \Delta_X$. Using the octahedral axiom, we can say further that $\mathcal{E}$ fits into the exact sequence

$$0 \to \mathcal{O}_{\Delta_X}(-C) \to \mathcal{E} \to j_* \mathcal{O}_{C\times C}(-k-1,k+1) \to 0.$$

We can see from this that $\mathcal{E} \cong \mathcal{O}_S$, where S is the surface $(C \times C) \cup_{\Delta_C} \Delta_X$.

For any point $x \in X$ there is an inclusion map $j_x \colon x \times X \hookrightarrow X \times X$. If we consider the pullback of 4.13 via j_x, we obtain the exact triangle

$$\mathbb{L}j_x^* \mathcal{O}_{C\times C}(-k-2,k) \to \mathbb{L}j_x^* \mathcal{O}_S \to \mathcal{O}_{x\times x}.$$

If $x \in X \setminus C$, $\mathbb{L}j_x^* \mathcal{O}_{C\times C}(-k-2,k) \cong 0$. This shows that $\mathbb{L}j_x^* \mathcal{O}_S \cong \mathcal{O}_{x\times x}$, the skyscraper sheaf of the point $x \times x \in X \times X$. On the other hand, if $x \in C$, $\mathbb{L}j_x^* \mathcal{O}_{C\times C}(-k-2,k) \cong \mathcal{O}_{x\times C}(k)[1] \oplus \mathcal{O}_{x\times C}(k)$. Hence $\mathbb{L}j_x^* \mathcal{O}_S$ fits into an exact sequence

$$\mathcal{O}_{x\times C}(k)[1] \to \mathbb{L}j_x^* \mathcal{O}_S \to \mathcal{O}_{x\times x} \to \mathcal{O}_{x\times C}(k)[1].$$

The kernel of the map $\mathcal{O}_{x\times x} \to \mathcal{O}_{x\times C}(k)[1]$ is $\mathcal{O}_{x\times C}(k+1)$, and so this shows that $\mathbb{L}j_x^* \mathcal{O}_S$ is isomorphic to a class in $\mathbb{P}\mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1]$, if this class in nontrivial. That is, $\mathcal{O}_S$ defines a family of τ-stable objects of class $[\mathcal{O}_x]$.

Lemma 4.3.6. *There is a $\mathbb{P}^{n-1}$ parametrizing τ-stable objects of class $[\mathcal{O}_x]$ which are not isomorphic to $\mathcal{O}_x$ for any $x \in X$.*

Proof. Suppose $E^\cdot$ is a τ-stable object of class $\mathcal{O}_x$ for some $x \in C$. Then $E^\cdot$ must be strictly $\sigma_{H,\beta}$-semistable. By Lemma 4.3.3 the Jordan-Hölder factors of $E^\cdot$ must be $[\mathcal{O}_C(k+1)]$ and $[\mathcal{O}_C(k)[1]]$.

We will now work in the local model described in Lemma 4.3.5. Since the sheaves $\mathcal{O}_x$ were destabilized by the triangle

$$\mathcal{O}_C(k+1) \to \mathcal{O}_x \to \mathcal{O}_C(k)[1]$$

we know that $\phi_\tau(\mathcal{O}_C(k+1)) > \phi_\tau(\mathcal{O}_C(k)[1])$. Hence since $E^\cdot$ is τ-stable, it must fit into an exact triangle

$$\mathcal{O}_C(k)[1] \to E^{\cdot} \to \mathcal{O}_C(k+1).$$

This means that the new τ-stable objects $E^{\cdot}$ of class $[\mathcal{O}_x]$ are parameterized by $\mathbb{P}\mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1])$.

We can calculate the dimension $\mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1])$ as the dimension of $H^2(X, \mathcal{O}_C(k) \otimes \mathcal{O}_C(k+1)^\vee)$. The sheaf $\mathcal{O}_C(k+1)$ is quasiisomorphic to the complex $\mathcal{O}_X(-C)(k+1) \to \mathcal{O}_X(k+1)$. Then $\mathcal{O}_C(k+1)^\vee$ is quasiisomorphic to the complex $\mathcal{O}_X(-k-1) \to \mathcal{O}_X(C)(-k-1)$. Tensoring with $\mathcal{O}_C(k)$, we now want to calculate $H^2(X, \mathcal{O}_C(-1) \to \mathcal{O}_C(-n-1))$. Note that $n > 0$, and so there are no morphisms from $\mathcal{O}_C(-1)$ to $\mathcal{O}_C(-n-1)$. Hence we must compute $H^2(X, \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-n-1)[-1])$ This is the direct sum $H^2(X, \mathcal{O}_C(-1)) \oplus H^1(X, \mathcal{O}_C(-n-1)) \cong \mathbb{C}^n$. □

Now we will show that the extension class $\mathcal{E}$ is nonzero. Further, we will study the map $i\colon C \to \mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1])$ induced by $\mathcal{E} = \mathcal{O}_S$. We will do computations on the local model described in Lemma 4.3.4 and Lemma 4.3.5.

Lemma 4.3.7. *The degree of the map $i\colon C \hookrightarrow \mathbb{P}^{n-1}$ is $n-1$.*

Proof. The family $\mathcal{O}_S$ induces a map from C to $\mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1])$, which we can see via the following cohomology argument. We can compute the cohomology of the pullback $\mathbb{L}j^* j_* \mathcal{O}_{C\times C}(-k-2, k)$, using the fact that j is the inclusion of a divisor in $X \times X$. $H^0(\mathbb{L}j^* j_* \mathcal{O}_{C\times C}(-k-2, k)) = \mathcal{O}_{C\times C}(-k-2, k)$, and $H^{-1}(\mathbb{L}j^* j_* \mathcal{O}_{C\times C}(-k-2, k)) = \mathcal{O}_{C\times C}(n-k-2, k)$. This shows that $H^0(\mathbb{L}j^*\mathcal{E}) = \mathcal{O}_{C\times C}(-k-1, k+1)$ and $H^{-1}(\mathcal{O}_{C\times C}(n-k-2, k))$.

$\mathcal{E}$ is then a class in $\mathrm{Ext}^1(\mathcal{O}_{C\times C}(-k-1, k+1), \mathcal{O}_{C\times C}(n-k-2, k)[1])$. This space is isomorphic to $H^0(\mathcal{O}_C(n-1)) \otimes \mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1])$. The map that C induces to $\mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1])$ comes from a section of $\mathcal{O}_C(n-1)$. As long as this section is nonzero, this map has degree $n-1$. We will now show this section is nonzero.

Let c be a point on C. Consider the inclusion $i_c\colon c \times X \to X \times X$. We will now show that $i_c^*\mathcal{E}$ is a nonsplit extension of $\mathcal{O}_C(-k-1, k+1)$ and $\mathcal{O}_C(-k-2, k)[1]$. Lemma 4.3.4 shows we can do this computation on the local model.

By Lemma 4.3.5, we can see that the coordinate ring of $\widehat{X} \times \widehat{X}$ is the completion of the ring $R = \mathbb{C}[x_1, y_1, w_1, x_2, y_2, w_2]$ with respect to w_1 and w_2, where w_1 and w_2 are the equations of the curve in each component, and have degree $(-n, 0)$ and $(0, -n)$ respectively. The degree of x_1 and y_1 will be $(1, 0)$, and the degree of x_2 and y_2 will be $(0, 1)$.

Using the description of $\mathcal{E} \cong \mathcal{O}_S$, where $S = (C \times C) \cup_{\Delta_C} \Delta_{\widehat{X}}$, we can write down a free resolution of $\mathcal{E}$, which we will then pull back via i_C. S is defined in $\widehat{X}$ by the ideal $(w_1(x_1y_2 - x_2y_1), w_2(x_1y_2 - x_2y_1), x_1^n w_1 - x_2^n w_2, \ldots, y_1^n w_1 - y_2^n w_2)$. The resolution of this ideal is

$$R^{\oplus n-1} \to R^{\oplus 2n+1} \to R^{\oplus n+3} \to R.$$

Pulled back to $c \times X$, and considering degrees, this gives a resolution of $\mathbb{L}i_c^*\mathcal{E}$ as follows.

$$\mathcal{O}(k)^{\oplus n-1} \xrightarrow{M_3} \mathcal{O}(k+n) \oplus (\mathcal{O}(k) \oplus \mathcal{O}(k+1))^{\oplus n} \xrightarrow{M_2} \mathcal{O}(k) \oplus \mathcal{O}(k+n) \oplus \mathcal{O}(k+1)^{\oplus n+1} \xrightarrow{M_1} \mathcal{O}(k+1)$$

The maps in this sequence are in terms of x_2, y_2, w_2. x_1 and y_1 are fixed. The first map M_1 is

$$M_1 = \begin{pmatrix} 0 & w_2(x_1y_2 - x_2y_1) & x_2^n w_2 & \cdots & y_2^n w_2 \end{pmatrix}.$$

The next map M_2 is given by

$$M_2 = \begin{pmatrix} -w_2 & x_1^{n-1} & 0 & \cdots & y_1^{n-1} & 0 \\ 0 & 0 & x_2^{n-1} & \cdots & 0 & y_2^{n-1} \\ 0 & -y_2 & -y_1 & \cdots & 0 & 0 \\ 0 & x_2 & x_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -y_2 & -y_1 \\ 0 & 0 & 0 & \cdots & x_2 & x_1 \end{pmatrix}.$$

The last map M_3 is given by

$$M_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -y_1 & 0 & \cdots & 0 \\ y_2 & 0 & \cdots & 0 \\ x_1 & -y_1 & \cdots & 0 \\ -x_2 & y_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_1 \\ 0 & 0 & \cdots & -x_2 \end{pmatrix}.$$

Let $F^{\cdot}$ be the resolution described above. Recall the notation $\tau_{\leq a}$ given in Section 2.1. There is an exact triangle

$$\tau_{\leq -1}F^{\cdot} \to F^{\cdot} \to H^0(F^{\cdot}).$$

Since $\mathrm{Hom}(\mathcal{O}_C(k)[1], H^0(F^{\cdot})) = 0$, and $\mathrm{Hom}(\mathcal{O}_C(k)[1], H^0(F^{\cdot})[-1]) = 0$, we know from the long exact Hom sequence applied to the triangle that $\mathrm{Hom}(\mathcal{O}_C(k)[1], F^{\cdot}) \cong \mathrm{Hom}(\mathcal{O}_C(k)[1], \tau_{\leq -1}F^{\cdot})$.

Similarly, there is an exact triangle

$$\tau_{\leq -2}F^{\cdot} \to \tau \leq -1F^{\cdot} \to H^{-1}(F^{\cdot})[1].$$

By degree arguments, $\mathrm{Hom}(\mathcal{O}_C(k)[1], \tau_{\leq -2}F^{\cdot}) = 0$. It remains to compute $\mathrm{Hom}(\mathcal{O}_C(k)[1], H^{-1}(F^{\cdot})[1])$.

We know that $H^{-1}(F^{\cdot}) \cong \mathrm{Ker}(M_1)/\mathrm{Im}(M_2)$. Looking at the maps M_2 and M_1 explicitly, we see this quotient is supported on C, in degree higher than k. Hence, there are no morphisms from $\mathcal{O}_C(k)[1]$ to this resolution, and so the sequence $0 \to \mathcal{O}_C(k)[1] \to \mathbb{L}i_C^*\mathcal{E} \to \mathcal{O}_C(k+1) \to 0$ is nonsplit. □

Proposition 4.3.8. *Let* $\eta\colon \mathbb{P}^{n-1} \to \mathbb{P}\mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1])$ *be the isomorphism described in Lemma 4.3.6. There is a bijection* $\gamma\colon (X - C) \cup \mathbb{P}^{n-1} \to M_\tau([\mathcal{O}_x])$ *defined as follows:*

$$\gamma(y) = \begin{cases} \mathcal{O}_y, & \text{if } y \in X \setminus C \\ \eta(y), & \text{if } y \in \mathbb{P}^{n-1}. \end{cases}$$

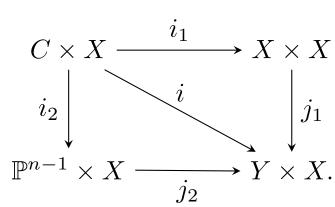
Proof. This follows from Lemma 4.3.2 and Lemma 4.3.6. □

Proposition 4.3.9. *Let* $Y = (X \setminus C) \cup \mathbb{P}^{n-1}$. *Then there is a family* $\mathcal{U}_\tau$ *of* τ*-stable objects over* Y *such that the induced map* $Y\colon M_\tau([\mathcal{O}_x])$ *induces the injection in Proposition 4.3.8 on points.*

Proof. We have constructed a family on X, the object $\mathcal{O}_S$ in $\mathcal{D}^b(X \times X)$. We also have a family on $\mathbb{P}^{n-1}$ given by the universal extension of $\mathcal{O}_C(k+1)$ and $\mathcal{O}_C(k)[1]$ [LP97, p.118]. Consider the projections $p_1\colon \mathbb{P}^{n-1} \times X \to \mathbb{P}^{n-1}$ and $p_2\colon \mathbb{P}^{n-1} \times X \to X$. By [LP97, Proposition 4.2.2], the object $\mathcal{E}xt^1(p_2^*\mathcal{O}_C(k+1), p_1^*\mathcal{O}_C(k)[1])$ in $\mathcal{D}^b(\mathbb{P}^{n-1} \times X)$ is isomorphic to $H^0(\mathcal{O}_{\mathbb{P}^{n-1}} \otimes \mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1]))$. If we consider the element $E^{\cdot} \in \mathrm{Ext}^1(p_2^*\mathcal{O}_C(k+1), p_1^*\mathcal{O}_C(k)[1])$ in $\mathcal{D}^b(\mathbb{P}^{n-1} \times X)$ corresponding to the identity map $\mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$, then $E^{\cdot} \in \mathcal{D}^b(\mathbb{P}^{n-1} \times X)$ is a universal family on $\mathbb{P}^{n-1}$ parameterizing extensions $\mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1])$.

We now claim that these two objects can be glued along C to constructa family $\mathcal{U}_\tau$ over $Y = (X \setminus C) \sqcup \mathbb{P}^{n-1}$ inducing the injection in Proposition 4.3.8.

Consider the following diagram of inclusions:



By construction, $i_1^*\mathcal{O}_S \cong i_2^*E^{\cdot}$. Let $L^{\cdot} = i_1^*\mathcal{O}_S$. Further, if we consider the isomorphisms, $i_1^*\mathcal{O}_S \to L^{\cdot}$ and $i_2^*E^{\cdot} \to L^{\cdot}$, then via adjunction and push-forward we get morphisms $(j_1)_*\mathcal{O}_S \to i_*L^{\cdot}$ and $(j_2)_*E^{\cdot} \to i_*L^{\cdot}$. Define $P^{\cdot}$ to be the object fitting into the exact triangle

$$P^{\cdot} \to (j_1)_*\mathcal{O}_S \oplus (i_2)_*E^{\cdot} \to i_*L^{\cdot}.$$

We will now show that $P^{\cdot}$ is the desired family $\mathcal{U}_\tau$.

First, suppose x is a point in $X \setminus C$. Then restricting the triangle above to $\{x\} \times X$, $E^{\cdot}$ and $L^{\cdot}$ become 0, so $P^{\cdot}|_{\{x\}\times X} \cong \mathcal{O}_S|_{\{x\}\times X}$. Similarly, if we choose a point $y \in \mathbb{P}^{n-1} \setminus C$, we find $P^{\cdot}|_{\{y\}\times X} \cong E^{\cdot}|_{\{y\}\times \mathbb{P}^{n-1}}$. What remains is to show that $P^{\cdot}|_{C\times X} \cong L^{\cdot}$. In fact, we will show that this is true in a formal neighborhood of a point $x \in C$. That is, we will look at the exact triangle

$$P^{\cdot} \otimes^{\mathbb{L}} i_*\mathcal{O}_C \to ((j_1)_*\mathcal{O}_S \oplus (j_2)_*E^{\cdot}) \otimes^{\mathbb{L}} i_*\mathcal{O}_C \to i_*L^{\cdot} \otimes^{\mathbb{L}} i_*\mathcal{O}_C$$

and show that in a formal neighborhood of a point $x \in C$, $P^{\cdot} \otimes^{\mathbb{L}} i_*\mathcal{O}_C \cong i_*L^{\cdot}$. This will show via the projection formula that near x, $i_*i^*P^{\cdot} \cong i_*L^{\cdot}$ so that $i^*P^{\cdot} \cong L^{\cdot}$.

We will now describe a formal neighborhood of a point $x \in C$. Along C, we can look at an affine patch $\mathbb{A}^{n-1}$ of $\mathbb{P}^{n-1}$ and $\mathbb{A}^2$ of X, glued along the affine patch $\mathbb{A}^1$ of C. The coordinate ring of this space is $R = k[x, y, z_1, \ldots, z_{n-2}]/(yz_1, \ldots, yz_{n-1})$, where $C = \mathrm{Spec}(k[x])$, $\mathbb{A}^2 = \mathrm{Spec}(k[x, y])$ and $\mathbb{A}^{n-1} = \mathrm{Spec}(k[x, z_1, \ldots, z_{n-2}])$. The formal neighborhood of x in Y is the completion $k[[x, y, z_1, \ldots, z_{n-2}]]/(yz_1, \ldots, yz_{n-2})$ of this ring R. Since the inclusion of this neighborhood in Y is flat, we may restrict any complexes to this neighborhood. Here on we will use i_1, i_2, j_1, j_2, and i to describe these maps after base change.

The resolution of C in the ring R is the resolution of the ideal $(y, z_1, \ldots, z_{n-1})$. This resolution is given by the complex $R^{\cdot}$ below.

$$\cdots \xrightarrow{d_2} R^{a_2} \xrightarrow{d_1} R^{a_1} \xrightarrow{d_0} R^{a_0}.$$

We can see that $a_0 = 1$ and $a_1 = n-1$, as the first differential, d_0 is given by multiplication by the equations $y, z_1, \ldots, z_{n-1}$ describing C. The next differential, d_1, describes the relations between these. The relations are given by the $2(n-2)$ products of z_i with y (which is 0 in this ring) and the first step in the Koszul complex for $z_1, \ldots, z_{n-2}$, call it $K^{\cdot}$. This gives $a_3 = 2(n-2) + \binom{n-2}{2} = \frac{(n-2)(n+1)}{2}$ factors of R at the third step in the resolution. For example, when $n = 4$, the degree -2 to 0 terms of $R^{\cdot}$ are

$$\cdots \xrightarrow{\quad d_2 \quad} R^5 \xrightarrow{\begin{pmatrix} 0 & z_2 & 0 & z_1 & 0 \\ -z_2 & 0 & 0 & 0 & y \\ z_1 & 0 & y & 0 & 0 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} y & z_1 & z_2 \end{pmatrix}} R.$$

Let m_i be the rank of R in the ith term of the resolution. The ith differential will consist of linear terms $y, z_1, \ldots, z_{n-2}$ which multiply with the $i-1$th differential to give relations of the form yz_i or relations in the Koszul complex of $z_1, \ldots, z_{n-2}$. For every summand R of R^{a_i}, the total number of the maps coming into R from d_i and the maps coming out of R from d_{i-1} will be $n-1$, with each linear map $y, z_1, \dots, z_{n-2}$ appearing exactly once.

We now compute the tensor product $(j_1)_*\mathcal{O}_S \otimes^{\mathbb{L}} i_*\mathcal{O}_C$, obtained by tensoring $R^{\cdot}$ with $(j_1)_*\mathcal{O}_S$. The maps z_i are 0 on $\mathcal{O}_S$, since $\mathcal{O}_S$ is supported on $X \times X$ and z_i are coordinated on $\mathbb{P}^{n-1}$. Furthermore, any term of the form $\left((j_1)_*\mathcal{O}_S \xrightarrow{y} (j_1)_*\mathcal{O}_S\right)$ is isomorphic to $i_*L^{\cdot}$. This follows from the fact that $(i_1)_*L^{\cdot} \cong \mathcal{O}_S \otimes^{\mathbb{L}} (i_1)_*\mathcal{O}_C \cong \left((j_1)_*\mathcal{O}_S \xrightarrow{y} (j_1)_*\mathcal{O}_S\right)$ via the map y. Therefore $i_*L^{\cdot} \cong (j_1)_*\mathcal{O}_S \to (j_1)_*\mathcal{O}_S$ via the map y.

Every copy of $(j_1)_*\mathcal{O}_S$ in the complex $(j_1)_*\mathcal{O}_S \otimes^{\mathbb{L}} i_*\mathcal{O}_C$ given by

$$\cdots \xrightarrow{d_2} (j_1)_*\mathcal{O}_S^{a_2} \xrightarrow{d_1} (j_1)_*\mathcal{O}_S^{a_1} \xrightarrow{d_0} (j_1)_*\mathcal{O}_S^{a_0}.$$

will occur either at the end of an incoming map y or at the beginning of an outgoing map y. Therefore, this complex is isomorphic to

$$(j_1)_*\mathcal{O}_S \otimes^{\mathbb{L}} i_*\mathcal{O}_C \cong i_*L^{\cdot} \oplus i_*L^{\cdot}[1]^{\oplus b_1} \oplus i_*L^{\cdot}[2]^{\oplus b_2} \oplus \cdots$$

where b_i is the number of incoming y maps in the $-i$th degree term of $R^{\cdot}$.

Now, we compute the the tensor product of $(j_2)_*E^{\cdot} \otimes^{\mathbb{L}} i_*\mathcal{O}_C$. The maps y are 0 on $(j_2)_*E^{\cdot}$, since $E^{\cdot}$ is supported on $\mathbb{P}^{n-1} \times X$ and y is a coordinate on X. Further, the Koszul complex $K^{\cdot}$ of $z_1, \dots, z_{n-2}$ tensored with $(j_2)_*E^{\cdot}$ is isomorphic to $i_*L^{\cdot}$. This is because $(i_2)_*L^{\cdot} \cong E^{\cdot} \otimes i_*\mathcal{O}_C \cong E^{\cdot} \otimes K^{\cdot}$. Hence $i_*L^{\cdot} \cong (j_2)_*E^{\cdot} \otimes K^{\cdot}$.

Consider the complex $(j_2)_*E^{\cdot} \otimes^{\mathbb{L}} i_*\mathcal{O}_C$ given by

$$\cdots \xrightarrow{d_2} (j_2)_*(E^{\cdot})^{\oplus a_2} \xrightarrow{d_1} (j_2)_*(E^{\cdot})^{\oplus a_1} \xrightarrow{d_0} (j_2)_*E^{\cdot}.$$

For each copy of $(j_2)_*E^{\cdot}$ which occurs in degree $-i$ in this complex, and occurs at the end of a complex of the form $(j_2)_*E^{\cdot} \otimes K^{\cdot}$, we get a direct summand of $i_*L^{\cdot}[i]$ in $(j_2)_*E^{\cdot} \otimes^{\mathbb{L}} i_*\mathcal{O}_C$. Since the maps y are now 0, all nonzero maps in this complex occur as part of some shift of $(j_2)_*E^{\cdot} \otimes K^{\cdot}$, hence $(j_2)_*E^{\cdot} \otimes^{\mathbb{L}} i_*\mathcal{O}_C$ is a direct sum of shifts of $i_*L^{\cdot}$. We must now count these terms to determine the complex $(j_2)_*E^{\cdot} \otimes^{\mathbb{L}} i_*\mathcal{O}_C$.

If we now consider the complex $R^{\cdot}$, we note that if a summand R of R^{a_i} has an outgoing map y, then it must have $n-2$ incoming maps $z_1, \dots, z_{n-2}$, since $yz_i = 0$ is a relation in R. We have seen that the differentials in $R^{\cdot}$ all come from relations yz_i or from the differentials in $K^{\cdot}$, A summand R with $n-2$ incoming maps, one for each z_i, must then occur at the end of a Koszul complex $K^{\cdot}$. Therefore, if we let c_i be the degree of $i_*L^{\cdot}[i]$ in $(j_2)_*E^{\cdot} \otimes^{\mathbb{L}} i_*\mathcal{O}_C$, we see $c_i = a_i - b_i$ for $i > 0$.

Now consider $i_*L^\cdot \otimes i_*\mathcal{O}_C$. The maps $y, z_1, \ldots, z_{n-2}$ are all 0 on C. Hence this complex will be a direct sum of terms $i_*L^\cdot$, of the form $i_*L^\cdot \oplus i_*L^\cdot[1]^{\oplus n-1} \oplus i_*L^\cdot[2]^{\oplus a_1} \oplus \cdots$. The exact triangle

$$P^\cdot \otimes^{\mathbb{L}} i_*\mathcal{O}_C \to ((j_1)_*\mathcal{O}_S \oplus (j_2)_*E^\cdot) \otimes^{\mathbb{L}} i_*\mathcal{O}_C \to i_*L^\cdot \otimes^{\mathbb{L}} i_*\mathcal{O}_C$$

is now locally given by

$$P^\cdot \otimes^{\mathbb{L}} i_*\mathcal{O}_C \to (i_*L^\cdot \oplus i_*L^\cdot[1]^{\oplus b_1} \oplus \cdots) \oplus (i_*L^\cdot \oplus i_*L^\cdot[1]^{\oplus a_1 - b_1} \oplus \cdots) \to i_*L^\cdot \oplus i_*L^\cdot[1]^{\oplus a_1} \oplus \cdots.$$

We can then see that $P^\cdot \otimes^{\mathbb{L}} i_*\mathcal{O}_C \cong i_*L^\cdot$, which completes our proof that $P^\cdot$ is a family over Y which is obtained by gluing $E^\cdot$ and $\mathcal{O}_S$ along C. Since $E^\cdot$ and $\mathcal{O}_S$ induce the isomorphism in Proposition 4.3.8 over $\mathbb{P}^{n-1}$ and X respectively, and agree on C, the glued object $P^\cdot$ we have constructed will induce the map $Y \to M_\tau([\mathcal{O}_x])$ in Proposition 4.3.8. $\square$

We now will describe the tangent space $\mathrm{Ext}^1(E, E)$ for $E \in M_\tau([\mathcal{O}_x])$. In particular, we will show that γ induces an isomorphism of tangent spaces between $M_\tau([\mathcal{O}_x])$ and $X \sqcup_C \mathbb{P}^{n-1}$, where C is embedded as a rational normal curve in $\mathbb{P}^{n-1}$. In the course of this argument we will specifically describe the image of C in $\mathbb{P}\mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1])$ as the locus of extensions where the tangent space jumps in dimension.

Lemma 4.3.10. *The map $\gamma\colon X \sqcup_C \mathbb{P}^{n-1} \to M^\tau([\mathcal{O}_x])$ induces a isomorphism of tangent spaces.*

Proof. If $E \in M^\tau([\mathcal{O}_x])$ is the class of a stable object $\mathcal{O}_x$ for some $x \in X \setminus C$ then $\mathrm{Ext}^1(E, E) \cong T_xX$. We will now consider E a stable object of class $[\mathcal{O}_x]$ for some $x \in C$. We will show that $\mathrm{Ext}^1(E, E) \cong \mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1]) \cong \mathbb{C}^{n-1}$ except for E lying on a copy of C in $\mathbb{P}^{n-1}$, where $\mathrm{Ext}^1(E, E) \cong \mathbb{C}^n$.

There is a morphism $\mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1]) \to \mathrm{Ext}^1(E, E)$ given by composing $\mathrm{Ext}^1(\mathcal{O}_C(k+1), \mathcal{O}_C(k)[1]) \to \mathrm{Ext}^1(E, \mathcal{O}_C(k)[1]) \to \mathrm{Ext}^1(E, E)$. We would like to show this morphism is surjective. Applying Hom to the triangle $\mathcal{O}_C(k)[1] \to E \to \mathcal{O}_C(k+1)$, we see this is equivalent to showing that $\mathrm{Ext}^1(E, \mathcal{O}_C(k+1)) \to \mathrm{Ext}^2(E, \mathcal{O}_C(k)[1])$ is injective.

Consider the commutative diagram of exact sequences:

$$\begin{array}{ccc}
\mathrm{Ext}^2(\mathcal{O}_C(k+1),\mathcal{O}_C(k+1)) & \longrightarrow & 0\\
\lambda\uparrow & & \uparrow\\
\mathrm{Hom}(\mathcal{O}_C(k),\mathcal{O}_C(k+1)) & \xrightarrow{\beta} & \mathrm{Ext}^2(\mathcal{O}_C(k),\mathcal{O}_C(k))\\
\uparrow & & \uparrow\\
\mathrm{Ext}^1(E,\mathcal{O}_C(k+1)) & \xrightarrow{\alpha} & \mathrm{Ext}^2(E,\mathcal{O}_C(k)[1])\\
\uparrow & & \uparrow\\
0 & \longrightarrow & 0
\end{array}$$

In order to show that the map f is injective, we show that $\ker(\beta)$ and $\ker(\lambda)$ intersect nontrivially in $\mathrm{Hom}(\mathcal{O}_C(k),\mathcal{O}_C(k+1))$.

Let $\Delta \in \mathrm{Hom}(\mathcal{O}_C(k+1),\mathcal{O}_C(k)[2])$ be the class of E. For $f \in \mathrm{Hom}(\mathcal{O}_C(k),\mathcal{O}_C(k+1))$, $\beta(f) = \Delta \circ f$. By Serre duality, we have isomorphisms ϕ so that the following square is commutative.

$$\begin{array}{ccc}
\mathrm{Hom}(\mathcal{O}_C(k+1),\mathcal{O}_C(k)[2]) & \xrightarrow{f} & \mathrm{Hom}(\mathcal{O}_C(k),\mathcal{O}_C(k)[2])\\
\phi\downarrow & & \downarrow\phi\\
\mathrm{Hom}(\mathcal{O}_C(k)[2],\mathcal{O}_C(k+n-1)[2])^* & \xrightarrow{F} & \mathrm{Hom}(\mathcal{O}_C(k)[2],\mathcal{O}_C(k+n-2)[2])^*
\end{array}$$

The map f composes a class Δ with f. For any $g \in \mathrm{Hom}(\mathcal{O}_C(k)[2],\mathcal{O}_C(k+n-2)[2])$ and functional $\xi \in \mathrm{Hom}(\mathcal{O}_C(k)[2],\mathcal{O}_C(k+n-1)[2])^*$, $F(\xi)(g) = \xi(f[2]\circ g)$, where $f[2]$ is now viewed as lying in $\mathrm{Hom}(\mathcal{O}_C(k+n-2)[2],\mathcal{O}_C(k+n-1)[2])$. The commutativity of this square shows that $\phi(\Delta\circ f)(g) = \phi(\Delta)(f[2]\circ g)$.

Similarly, $\lambda(f) = f\circ\Delta$. Using Serre duality, we see that for $h \in \mathrm{Hom}(\mathcal{O}_C(k+1),\mathcal{O}_C(k+n-1))$, $\phi(\lambda(f))(h) = \phi(f\circ\Delta)(h) = \phi(\Delta)(h\circ f[2])$. We now see that $\ker(\lambda) = \ker(\beta)$, and is given by the condition that f must be such that $\phi(\Delta)$ vanishes on the image of the map $\mathrm{Hom}(\mathcal{O}_C(k)[2],\mathcal{O}_C(k+n-2)[2]) \to \mathrm{Hom}(\mathcal{O}_C(k)[2],\mathcal{O}_C(k+n-1)[2])$ given by multiplication by $f[2]$. For general Δ, no such f will exist, and both β and λ will be injective. In this case, there is a surjection $\mathrm{Ext}^1(\mathcal{O}_C(k+1),\mathcal{O}_C(k)[1]) \twoheadrightarrow \mathrm{Ext}^1(E,E)$ with 1-dimensional kernel, and $\mathrm{Ext}^1(E,E)\cong\mathbb{P}^{n-1}$.

For each point c on a rational curve C, there is a $\Delta_c \in \mathrm{Hom}(\mathcal{O}_C(k+1),\mathcal{O}_C(k)[2])$ which is dual to $\delta_c \in \mathrm{Hom}(\mathcal{O}_C(k)[2],\mathcal{O}_C(k+n-1)[2])$, the shift by 2 of the map $\mathcal{O}_C(k)\to\mathcal{O}_C(k+n-1)$ given with cokernel supported at c. For this Δ_c, the kernel of β and the kernel of λ is one-dimensional, and $\mathrm{Ext}^1(E,E)\cong\mathbb{P}^n$. $\square$

We would now like to say that the map γ is an isomorphism, and $M_\tau([\mathcal{O}_x]) \cong X \sqcup_C \mathbb{P}^{n-1}$. We have shown in Proposition 4.3.8 that γ is a bijection on points, and in Lemma 4.3.10 that γ induces an isomorphism of tangent spaces. Were $X \sqcup_C \mathbb{P}^{n-1}$ smooth, then following [Har80, Corollary 14.10] this would be enough to show that γ is an isomorphism.

Of course, $X \sqcup_C \mathbb{P}^{n-1}$ is not smooth. It is in fact reducible, singular along the curve C where the two varieties X and $\mathbb{P}^{n-1}$ meet. Hence [Har80, Corollary 14.10] is enough to show only that γ is an isomorphism away from C. We will show now that a slight generalization of [Har80, Corollary 14.10] applies to $X \sqcup_C \mathbb{P}^{n-1}$. That is, $X \sqcup_C \mathbb{P}^{n-1}$ is still a nice enough space for the conditions that γ is a bijection on points and an isomorphism of tangent spaces to imply that γ is an isomorphism.

Lemma 4.3.11. *Let X be a projective scheme, and let $f\colon X \to Y$ be a finite morphism. Further assume X and Y are of finite type over a field k. Then if f is a bijection of points, and induces an isomorphism of tangent spaces, then f is a closed embedding.*

Proof. We will follow the proof of [Har92, Theorem 14.9] to show it applies more generally to this case. We may reduce to the case where $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine, and the rings A and B are local. Let $\mathfrak{m}$ be the maximal ideal of A and let $\mathfrak{n}$ be the maximal ideal of B.

Since f induces an injection of tangent spaces, the map $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$ is surjective. This implies that the image $\mathfrak{m}B$ of $\mathfrak{m}$ in B is exactly $\mathfrak{n}$ in $B/\mathfrak{n}^2$. Hence $\mathfrak{m}B + \mathfrak{n}^2 = \mathfrak{n}$. By Nakayama's Lemma, this implies that $\mathfrak{m}B = \mathfrak{n}$ in B. But then $B/A \otimes A/\mathfrak{m} \cong B/(\mathfrak{m}\mathfrak{B} + A)$ is isomorphic to $B/(\mathfrak{n} + A)$. This is 0, so Nakayama's Lemma shows $B/A = 0$. Hence f induces a surjection $A \to B$, and so f is a closed immersion. □

Lemma 4.3.12. *Suppose X is a projective scheme of finite type over a field k which has no nilpotents, and $f\colon X \to Y$ is a morphism such that f is a bijection on points and f induces a isomorphism of Zariski tangent spaces on closed points. Then Y has no nilpotents.*

Proof. First we restrict to affine open subsets, $U \subseteq X$ and $V \subseteq Y$, where $U = \mathrm{Spec}(B)$ and $V = \mathrm{Spec}(A)$. Suppose $f\colon A \to B$ is a bijection of points which induces an isomorphism of tangent spaces. Suppose B has no nilpotent elements. Let $\mathfrak{n}$ be the Jacobson radical of A. Since A is a finitely generated k-algebra, $\mathfrak{n}$ is also the nilradical of A. Then if f is a bijection of points, we must have that $B \cong A/\mathfrak{n}$.

Suppose $\mathfrak{n}$ is nonzero. Then there is some nonzero $a \in A$ such that for every maximal ideal $\mathfrak{m}$ of A, $a \in \mathfrak{m}$. But for every maximal ideal $\mathfrak{m}$ of A, f induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}'/\mathfrak{m}'^2$ where $\mathfrak{m}'$ is the unique maximal ideal of B containing $f(\mathfrak{m})$. And the nilradical of B is zero. Hence $a \in \mathfrak{m}^2$ for each maximal ideal $\mathfrak{m}$ of A.

Fix a maximal ideal $\mathfrak{m}_0$, and consider the localization $A_{\mathfrak{m}_0}$. Clearly, $\mathfrak{n}\mathfrak{m}_0 A_{\mathfrak{m}_0} \subseteq \mathfrak{n}A_{\mathfrak{m}_0} = \bigcap \mathfrak{m}^2 A_{\mathfrak{m}_0}$. On the other hand, $\bigcap \mathfrak{m}^2 A_{\mathfrak{m}_0} \subseteq \mathfrak{n}\mathfrak{m}_0 A_{\mathfrak{m}_o}$. Therefore, $\mathfrak{n}\mathfrak{m}_0 A_{\mathfrak{m}_0} = \mathfrak{n}A_{\mathfrak{m}_0}$. Nakayama's Lemma then implies $\mathfrak{n}A_{\mathfrak{m}_o} = 0$. Since this is true for each maximal ideal $\mathfrak{m}_0$, we must have that $\mathfrak{n} = 0$. Therefore A has no nilpotents.

□

Theorem 4.3.13. *The map γ induces an isomorphism $X \sqcup_C \mathbb{P}^{n-1} \to M_\tau([\mathcal{O}_x])$, where C is embedded in $\mathbb{P}^{n-1}$ as a rational normal curve.*

Proof. By Proposition 4.3.8, γ is a bijection on points. Since f is a bijection on points, it is quasifinite. f is also projective, and so by [GD60a, Theorem 5.5.3] it is proper. Then by [GD60b, Theorem 8.11.1] f is finite. By Lemma 4.3.10, we also know that γ induces an isomorphism of tangent spaces. Hence, by Lemma 4.3.11, f is a closed immersion. Then applying Lemma 4.3.12, we see γ is an isomorphism. □

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