BANACH ALGEBRAS WITH COMPLEMENTED IDEALS

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Complemented Banach algebras were introduced as successive generalisations by Saworotnow [10] and Tomiuk [12] of the $H^*$-algebras of Ambrose (Trans. Amer. Math. Soc. 65 (1949)). The algebras which we shall consider are those that are called complemented in [12]: an $H^*$-algebra is a Banach algebra that is also a Hilbert space and in which the involution is intimately related to the inner product. A complemented algebra is not assumed to be a Hilbert space, nor to have an involution, but there is a mapping defined on its sets of closed one-sided ideals that satisfies the important structural properties of orthogonal complementation. If such a mapping can be defined only on, say, the set of all closed right ideals then the algebra is said to be right complemented. In an $H^*$-algebra the orthogonal complement of a closed right ideal is equal to the set of adjoints of the left annihilator of the ideal. The starting point of this investigation was the projected result of Tomiuk that, in the special case of a right complemented algebra that was also a topologically simple complex $H^*$-algebra, it was possible to induce a subsidiary involution with respect to which the complement of any closed right ideal was again the set of adjoints of its left annihilator. This proves to be true if the algebra has no minimal one-sided ideals of finite dimension. In order to isolate the extra property that is satisfied in this infinite dimensional case we introduce the concept of a continuous complementor. Chapter 2 is a study
of continuous complementors in a $B^*$-algebra. The main result is that, if a $B^*$-algebra has no minimal one-sided ideals that are two-dimensional, then a necessary and sufficient condition that a complementor can be exhibited in the required form is that it be continuous. The contents of sections 2 and 3 and 4 (as far as Theorem 2.4.4) is shortly to be submitted for publication in a joint paper with B.J. Tomiuk, but this portion of the paper is essentially the work of the present author.

The definition of continuity of a complementor given in Chapter 2 is essentially a relative one; a complemented $B^*$-algebra is dual and such an algebra always has at least one natural complementor of the required form defined by the given involution in the algebra. We have defined a general complementor to be continuous if a certain mapping between two sets of minimal idempotents corresponding to the two complementors is continuous. This does not admit generalisation to the non-$B^*$ case. However, by examining the projection operators associated with a right complementor, we are able to give an equivalent definition that can be extended to any right complementor in any Banach algebra. Thus in Chapter 3 we study continuous right complementors in a general semi-simple Banach algebra. In this case the algebra need not be an annihilator algebra, and so discussion of the question previously formulated is only possible when this additional assumption is made. However, even then we find that, while an involution can be defined on a subalgebra, it is not in general possible to define it on the whole algebra.
Thus our main emphasis in this Chapter is on what was shown in the $B^*$-case to be an equivalent problem; viz the construction of a Hilbert space representation through which the complementor is naturally exhibited. Perhaps the most important result in this Chapter is that any semi-simple Banach algebra with no minimal left ideals of dimension two and with a continuous right complementor has a faithful continuous representation on a Hilbert space; further the complement of any closed right ideal $R$ is the set of elements of the algebra whose images under the representation map the Hilbert space into the orthogonal complement of the subspace generated by the images of the elements of $R$. Thus we see that in the general semi-simple case we obtain results that are very similar to those that can be obtained in the $B^*$-case. Another result of this kind is that, if a semi-simple Banach algebra with a continuous right complementor satisfies the weakest of the annihilator conditions — it is either a left or a right annihilator algebra — then it is already an annihilator algebra and in the topologically simple case it is dual. Again in the topologically simple case we give a complete characterisation of those Banach algebras of operators on a Hilbert space that have a continuous right complementor associated with orthogonality in the Hilbert space.

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NOTATION. The reader is assumed to be familiar with the basic elements of functional analysis; the general reference books that have been used are Riesz–Nagy [9] and Rickart [8]. Any terminology not explained in the text is thought to be well-known or self explanatory. A few points should be mentioned here.

If $\{I_\lambda : \lambda \in \Lambda\}$ is a family of ideals of a Banach algebra then $\sum_{\lambda \in \Lambda} I_\lambda$ will denote the set of all finite linear combinations of elements of the $I_\lambda$. The closure of this set is referred to as the topological sum and written $\text{cl}(\sum_{\lambda \in \Lambda} I_\lambda)$.

In general we shall use either the usual $\overline{S}$ or $\text{cl}(S)$ to denote the closure of a set $S$ in a Banach space.

If $S$ is a subset of a Banach algebra we shall denote by $S_1$ and $S_r$ the left and right annihilators of $S$.

It should perhaps be mentioned that a Banach algebra is a left annihilator algebra if every proper closed right ideal in it has a non-zero left annihilator. A right annihilator algebra is analogously defined.

In the course of the text we introduce several notations—references to these will be found in the index on page 96.

Throughout we consider only Banach algebras that are over the complex field; thus a Banach algebra means a complex Banach algebra. Similarly a complemented or right complemented algebra will always be a Banach algebra.

The following reference system is used: $(\mathbf{z}, y, x, \ldots)$ refers to object $x$ of section $y$ of chapter $z$; since there is only one section in Chapter 1 we omit the $y$ in this case.
CHAPTER 1. PRELIMINARIES.

Let \( \mathfrak{A} \) be a complex Banach Algebra and let \( L_r \) denote the set of all closed right ideals of \( \mathfrak{A} \). Following [12] we shall say that \( \mathfrak{A} \) is a RIGHT COMPLEMENTED BANACH ALGEBRA if there is a one-to-one mapping \( p : R \rightarrow R^D \) of \( L_r \) onto itself that has the following properties:

- \( C(i) \) \( R \cap R^D = \{0\} \) (\( R \in L_r \));
- \( C(ii) \) \( R \cap R^D = \mathfrak{A} \) (\( R \in L_r \));
- \( C(iii) \) \( (R^D)^P = R \) (\( R \in L_r \));
- \( C(iv) \) if \( R_1 \subset R_2 \) then \( R_1^D \supset R_2^D \) \((R_1, R_2 \in L_r)\).

We shall denote by \( C(ii)' \), \( C(iii)' \), \( C(iii)'' \) the weaker properties:

- \( C(ii)' \) \( \text{cl}(R \cap R^D) = \mathfrak{A} \) (\( R \in L_r \));
- \( C(iii)' \) \( (R^D)^P \supset R \) (\( R \in L_r \));
- \( C(iii)'' \) \( (R^D)^P \subset R \) (\( R \in L_r \)).

The mapping \( p \) is called a RIGHT COMPLEMENTOR on \( \mathfrak{A} \).

Analogously we define a LEFT COMPLEMENTOR and a LEFT COMPLEMENTED BANACH ALGEBRA. A Banach algebra that is both left complemented and right complemented is said to be BI-COMPLEMENTED.

**Lemma 1.1.** \( C(iii) \) may be replaced by \( C(iii)' \), or by \( C(iii)'' \) in the definition of a right complemented Banach Algebra.

**Proof.** \( R^D \cap R = \mathfrak{A} ; R^D \cap (R^D)^P = \mathfrak{A} \). Hence one inclusion between \( R \) and \( (R^D)^P \) will give equality.
LEMMA 1.2. If \( \mathfrak{A} \) is a Banach algebra with right complementor \( p \) and \( \{ R_\lambda : \lambda \in \Lambda \} \) is a family of closed right ideals of \( \mathfrak{A} \), then
\[
\left( \bigcap \lambda \ R_\lambda^P \right)^P = \text{cl} \left( \bigcup \lambda \ R_\lambda \right).
\]

PROOF. Since, for each \( \lambda \in \Lambda \), \( R_\lambda^P \supseteq \bigcap \lambda \ R_\lambda^P \) we have by \( (iv) \), \( (iii) \),
\[
\left( \bigcap \lambda \ R_\lambda^P \right)^P \supseteq R_\lambda \text{ and hence } \left( \bigcap \lambda \ R_\lambda^P \right)^P \supseteq \text{cl} \left( \bigcup \lambda \ R_\lambda \right). \]
However, \( \text{cl} \left( \bigcup \lambda \ R_\lambda \right) \supseteq R_\lambda \) for each \( \lambda \in \Lambda \); therefore, by a further application of \( (iv) \) \( \left( \text{cl} \left( \bigcup \lambda \ R_\lambda \right) \right)^P \subseteq \bigcap \lambda \ R_\lambda^P \), and consequently
\[
\text{cl} \left( \bigcup \lambda \ R_\lambda \right) = \left( \bigcap \lambda \ R_\lambda^P \right)^P. \]
Therefore \( \text{cl} \left( \bigcup \lambda \ R_\lambda \right) = \left( \bigcap \lambda \ R_\lambda^P \right)^P. \)

LEMMA 1.3. If \( \mathfrak{M} \) is a right complemented Banach algebra and if \( \omega \mathfrak{M} = 0 \) implies \( a = 0 \ (a \in \mathfrak{M}) \), then \( a \omega = \omega a \) for all \( a \) in \( \mathfrak{M} \).

PROOF. Let \( R = a \omega \). Then \( a \) has a unique expression
\[
a = a_1 + a_2, \quad a_1 \in R, \quad a_2 \in R^P.
\]
Now for any \( b \) in \( \mathfrak{M} \): \( a_2 b = ab - a_1 b \). However, \( a_2 b \in R^P \) and \( ab, a_1 b \in R \); thus \( a_2 b \in R^- \). Therefore \( a_2 \mathfrak{M} = 0 \), and so \( a_2 = 0 \) and \( a \in R = a \omega \).

DEFINITION. An idempotent \( e \) in a Banach algebra \( \mathfrak{M} \) is said to be minimal if \( e \mathfrak{M} e \) is a division algebra. An equivalent condition is that \( e \mathfrak{M} (e \mathfrak{M}) \) be a minimal right (left) ideal of \( \mathfrak{M} \) \((\mathfrak{M} \) is semi-simple).

We now prove a theorem that will enable us to establish the existence of minimal idempotents in certain Banach algebras, on whose one-sided ideals complementors are defined.
THEOREM 1.4. If \( \mathcal{A} \) is a Banach algebra, \( R \) a modular closed right ideal of \( \mathcal{A} \) and \( R' \) another closed right ideal of \( \mathcal{A} \) and \( R \odot R' = \mathcal{A} \)' then there is a unique idempotent \( e \) of \( \mathcal{A} \) that satisfies \( e \mathcal{A} = R' \), \( (1-e) \mathcal{A} = R \).

PROOF. The proof that there is an idempotent \( e \) satisfying \( e \mathcal{A} = R' \), \( (1-e) \mathcal{A} = R \) is a consequence of Lemma 2 in [12]. However we shall give a proof here as the result is basic.

Let \( j \) be a modular identity for \( R \); then \( j = d + e \), where \( d \in R \) and \( e \in R' \) by hypothesis. Now, for any \( a \) in \( \mathcal{A} \), \( a - ea = (a - ja) + da \); the first term on the right hand side of this expression is an element of \( R \) since \( j \) is a modular identity for \( R \) and the second term is also in \( R \) since \( d \) is in \( R \). Thus \( e \) is also a modular identity for \( R \). Now \( R \odot R' = \mathcal{A} = (1-e) \mathcal{A} \odot e \in R \) and \( e \mathcal{A} \subseteq R' \), \( (1-e) \mathcal{A} \subseteq R \) from which it is clear that \( R = (1-e) \mathcal{A} \) and \( R' = e \mathcal{A} \). Also \( e \) is idempotent for \( e^2 \in R \cap R' = (0) \).

We now show that \( e \) is unique: suppose that there is an idempotent \( k \) of \( \mathcal{A} \) with \( k \mathcal{A} = R' \), \( (1-k) \mathcal{A} = R \). Then \( kR = (0) \) and so \( k \in R_1 = \mathcal{A} \) since \( k = ke \). But \( (1-k) \mathcal{A} = R \) and \( k, e \in R' \); therefore \( ke-e \in R \cap R' = (0) \) and so \( ke = e \). Now \( k = ke = e \).

COROLLARY 1.5. If \( \mathcal{A} \) is a Banach algebra with right complementor \( p \) and \( R \) is any maximal modular right ideal of \( \mathcal{A} \), then there is a unique minimal idempotent \( e \) of \( \mathcal{A} \) that satisfies \( (1-e) \mathcal{A} = R \), \( e \mathcal{A} = Rp \).

PROOF. It is easily seen that \( Rp \) is minimal from \( C(iv) \) and the fact that \( p \) is one-to-one. The existence of an idempotent \( e \) in \( Rp \) satisfying \( (1-e) \mathcal{A} = R \), \( e \mathcal{A} = Rp \) is immediate from the theorem and, since \( Rp \) is minimal, \( e \) is a minimal idempotent.
DEFINITION 1.6. If e is any minimal idempotent of \( \mathfrak{A} \) that satisfies \(((1-e)A)^p = e \mathfrak{A}\) then e is said to be a \(p\)-PROJECTION.

It is clear from corollary 1.5 that any non-radical right complemented Banach algebra contains a \(p\)-projection.

The following Proposition is probably well known but we have been unable to find an explicit statement of it in the literature.

PROPOSITION 1.7. If \( \mathfrak{X} \) is a Banach space and \( S, S' \) closed subspaces of \( \mathfrak{X} \) such that \( S \oplus S' = \mathfrak{X} \), then there is a bounded linear projection \( L \) of \( \mathfrak{X} \) onto \( S \) whose null space is \( S' \).

PROOF. Since any element \( x \) of \( \mathfrak{X} \) has a unique decomposition \( x = x_1 + x_2 \) where \( x_1 \in S \), and \( x_2 \in S' \), the operator \( L \) is well defined by \( Lx = x_1 \). It is also linear. We show that it is closed:

Let \( x_n \in \mathfrak{X} \), \( x_n \to x \), \( Lx_n \to y \); then

\[ x = Lx + (1-L)x \quad (i) \quad x_n = Lx_n + (1-L)x_n \quad (ii) \]

Now in (ii)

\( \{x_n\} \) and \( \{Lx_n\} \) converge and therefore \( \{(1-L)x_n\} \) converges.

Also \( Lx_n \in S \) for all \( n \) and as \( S \) is closed, the limit \( y \) is in \( S \).

Similarly \( (1-L)x_n \in S' \) for all \( n \) and its limit \( z \) belongs to \( S' \). Thus letting \( n \to \infty \) in (ii) we see that \( x = y + z \) where \( y \in S \) and \( z \in S' \). Now from the uniqueness of the decomposition \( (i) \) \( Lx = y \). Thus \( L \) is closed and it follows from the closed graph theorem that it is continuous.

The following lemma is due to Tomiuk [12].

[see Theorem 4.8.3. in [11].]
LEMMA 1.8. Let \( \mathcal{A} \) be a Banach algebra with no two-sided ideals whose square is zero. Then if \( \mathcal{A} \) is right complemented and \( \mathcal{J} \) is any closed two-sided ideal of \( \mathcal{A} \), \( \mathcal{J}^\mathcal{P} = \mathcal{J}_1 - \mathcal{J}_r \).

PROOF. \( \mathcal{J}^\mathcal{P} \mathcal{J} \subseteq \mathcal{J} \mathcal{J}^\mathcal{P} = (0) \). Thus \( \mathcal{J}^\mathcal{P} \subseteq \mathcal{J}_1 \). Also if \( x \) is in \( \mathcal{J}_1 \) then \( x = y + z \) where \( y \in \mathcal{J} \) and \( z \in \mathcal{J}^\mathcal{P} \); now \( x \mathcal{J} = (0) \), \( z \mathcal{J} = (0) \) and therefore \( y \mathcal{J} = (0) \). However \( \{ y : y \in \mathcal{J}, y \mathcal{J} = (0) \} \) is a nil ideal and is zero by hypothesis; thus \( y = 0 \) and \( x \) is in \( \mathcal{J}^\mathcal{P} \). Thus \( \mathcal{J}_1 \subseteq \mathcal{J}^\mathcal{P} \) and equality follows. The proof that \( \mathcal{J}_r = \mathcal{J}^\mathcal{P} \) is similar, since it is now clear that \( \mathcal{J}^\mathcal{P} \) is a two-sided ideal.

APPENDIX.

The following is the work of F.F. Bonsall and B.J. Tomiuk. It is to be contained in a joint paper by the author and Tomiuk, but as this is not yet published and the result is necessary in the next chapter, we are reproducing it here.

PROPOSITION 1.9. If \( \mathcal{A} \) is a \( B^* \)-algebra and \( p \) a right complementor on \( \mathcal{A} \), then for each maximal closed right ideal \( M \) of \( \mathcal{A} \) there exists a unique minimal idempotent \( e \) such that \( M^\mathcal{P} = e \mathcal{A} \), \( M = (1 - e) \mathcal{A} \).

PROOF. By 2.9.5 (ii) in [2] \( M \) is a modular right ideal, and so by corollary 1.5. the idempotent \( e \) exists.

THEOREM 1.10. A \( B^* \)-algebra is right complemented if and only if it is dual.

PROOF. Suppose that \( \mathcal{A} \) is dual. Let \( R \) be any proper closed right ideal of \( \mathcal{A} \); define \( p : R \to R^\mathcal{P} \) by \( R^\mathcal{P} = (R_1)^\mathcal{P} \). Then \( R^\mathcal{P} \) is a
closed right ideal of $\mathcal{A}$. By Theorem 3 in [1], $p$ satisfies $C(i)$. Also $R^*(R_1)^* = (0)$ and therefore $R \subset [(R_1)^*]^* = R^{DP}$ Thus $p$ satisfies $C(iii)$. $C(iv)$ is trivial and it follows that $p$ is a right complementor from lemma 1.1.

Now suppose that $p$ is a right complementor on $\mathcal{A}$. Let $R$ be a proper closed right ideal of $\mathcal{A}$. Then by 2.9.5 (ii) in [2], $R$ is equal to the intersection of all the maximal modular right ideals of $\mathcal{A}$ in which it is contained. In particular it is contained in a maximal closed right ideal of $\mathcal{A}$ and so, by proposition 1.9, $R \subset (1-e)\mathcal{A}$ for some minimal idempotent $e$ of $\mathcal{A}$. Then $\mathcal{A}e \subset R_1$ and it follows that $R_1 \neq (0)$. Using the involution it can easily be seen that every proper closed left ideal of $\mathcal{A}$ has a non-zero right annihilator. Thus $\mathcal{A}$ is an annihilator algebra and Theorem 4.10.26 in [8] completes the proof.
II. COMPLEMENTED $B^*$-ALGEBRAS.

INTRODUCTION. If $\mathfrak{M}$ is a $B^*$-algebra it is easy to see that there is a one-to-one correspondence between left complementors and right complementors on $\mathfrak{M}$. Thus we shall refer to a complemented $B^*$-algebra as one that has a left or right complementor defined upon it. In a complemented $B^*$-algebra we shall invariably consider the right complementor and for brevity shall call this a complementor. However, it is clear that analogous results can be deduced for the associated left complementor ($L \rightarrow L^{\ast P}$ where $L$ is a closed left ideal of $\mathfrak{M}$ and $p$ is the complementor in $\mathfrak{M}$). In particular, if $q$ denotes the left complementor and $e$ is any $p$-projection, then $(\mathfrak{M}e)q = \mathfrak{M}(1-e^*)$.

Our main purpose in this chapter will be to see how general in the case of a $B^*$-algebra is the right complementor $R \rightarrow (R_1)^\ast$. We seek conditions on a complementor $p$ that will ensure that it can be expressed in this form for a subsidiary involution $\ast$ — it will be shown that in this case there is an equivalent norm in $\mathfrak{M}$ that satisfies the $B^*$ condition for $\ast$. In order to do this we introduce in §1 the concept of a continuous complementor by examining the set of all $p$-projections in $\mathfrak{M}$ and the set of all self-adjoint minimal idempotents of $\mathfrak{M}$. In §1 we show that a complemented $B^*$-algebra is (asymptotically $\ast$-isomorphic with) the $B^*(\omega)$ sum of a family (viz: the family of all minimal closed two-sided ideals of $\mathfrak{M}$) of topologically simple complemented $B^*$-algebras; further the complementors in
each \( \mathcal{J}_\lambda \) (where \( \mathcal{J}_\lambda : \lambda \in \Lambda \) denotes the family of all minimal closed two-sided ideals of \( \mathfrak{A} \)) are naturally induced by the complementor in \( \mathcal{A} \). And are continuous if and only if the complementor in \( \mathcal{O} \) is continuous. In §3 we show that, if \( R \) is any closed right ideal of \( \mathcal{O} \), then the images of \( R \) and \( R^D \) in \( (\Sigma \mathcal{I}_\lambda)_{0} \) are the \( B^*-(\infty) \) sums of the intersections \( R_\lambda \) of \( R \) with \( \mathcal{J}_\lambda \) and of the complements in \( \mathcal{J}_\lambda \) of the \( R_\lambda \) respectively. Thus the complement \( R^D \) of \( R \) may be obtained from a knowledge of the complements of its components in a topologically simple complemented \( B^* \)-algebra. Because of this result we study in detail in §2 complementors in a topologically simple \( B^* \)-algebra. Our main result is that, if any minimal closed left ideal of \( \mathcal{O} \) is of linear space dimension at least three, then a necessary and sufficient condition that a complementor in \( \mathcal{O} \) is expressible in the above form is that it be continuous. In §3 we extend this result to the case of a general \( B^* \)-algebra.

In §4 we examine other consequences of the continuity condition both because of their intrinsic interest and also as a motivation to the extension of the definition to semi-simple right complemented Banach algebras that are not assumed to be \( B^* \). We show that, if a complementor \( p \) is continuous, then so is the associated left complementor.

In §5 we give counter examples to show that the dimension restrictions cannot be removed and that \( C(11) \) cannot be relaxed.
1. CONTINUOUS COMPLEMENTORS.

DEFINITION 2.1.1. Let \( \mathfrak{A} \) be a \( B^* \)-algebra with a complementor \( p \). Let \( \xi \) denote the set of all self-adjoint minimal idempotents of \( \mathfrak{A} \) and \( \xi_p \) the set of all \( p \)-projections of \( \mathfrak{A} \). Elements of \( \xi \) will be referred to as \( ' \)-projections; it will be seen later (2.2.4) that this is in accordance with definition 1.6, if \( ' \) denotes the complementor \( R \rightarrow (R)^* = R' \). Now for any \( e \) in \( \xi \) let \( P(e) \) be the unique element of \( \xi_p \) that satisfies \( P(e) \xi = e \xi \). The mapping \( P \) is called the \( p \)-DERIVED MAPPING. Clearly \( P \) maps \( \xi \) onto \( \xi_p \). The complementor \( p \) is said to be CONTINUOUS if \( P \) is continuous with respect to the relative metric topologies of \( \xi, \xi_p \) induced by the norm in \( \mathfrak{A} \).

THEOREM 2.1.2. Let \( \mathfrak{A} \) be a \( B^* \)-algebra with a complementor \( p \) and let \( \mathfrak{J} \) be a closed two-sided ideal of \( \mathfrak{A} \). Then \( p \) induces a complementor \( p_{\mathfrak{J}} \) in the \( B^* \)-algebra \( \mathfrak{J} \). If \( p \) is continuous then \( p_{\mathfrak{J}} \) is continuous.

PROOF. \( \mathfrak{J} \), being a closed two-sided ideal of a \( B^* \)-algebra, is \( B^* \) (Theorem 4.9.2. in [8]). From Lemma 1.8. \( \mathfrak{J} = \mathfrak{J}^P \).

Now let \( R \) be any closed right ideal of the algebra \( \mathfrak{J} \); \( R \xi = R(\xi + \xi_r) = R \xi \subseteq R \) and so \( R \) is also a closed right ideal of \( \mathfrak{A} \). Thus for any such \( R \) we may define \( R^P \) to be \( R^P \). Clearly \( R^P \) is a closed right ideal of \( \mathfrak{J} \) and it is easy to verify that \( p_{\mathfrak{J}} : R \rightarrow R^P \) satisfies \( C(I), C(iii)', \) and \( C(iv) \); also if \( a \) is an element of \( \mathfrak{J} \) then \( a = a_1 + a_2 \) where \( a_1 \in R \subseteq \mathfrak{J} \), and \( a_2 \in R^P \); from this it is clear that \( a_2 \in \mathfrak{J} \).
and, therefore, that \( a_2 \in R^P_0 \) which establishes \( C_{11} \). Thus \( p_J \) is a complementor in the \( B^\star \)-algebra \( J \).

Now suppose that \( p \) is continuous. Let \( E^J \) and \( E^J_p \) be the sets of \(^1\)-projections and \( p_J \)-projections in \( J \), and let \( P_J \) be the \( p_J \)-derived mapping of \( E^J \) onto \( E^J_p \). Since minimal closed right ideals of \( J \) are also minimal closed right ideals of \( M \), it follows that an idempotent \( e \) in \( E \) is minimal in \( J \) if and only if it is minimal in \( M \). It is now easy to verify that \( E^J = E \cap J \) and \( E^J_p = E_p \cap J \) and \( P_J = \text{P}^J \) (the restriction of \( P \) to \( E^J \)). Hence the continuity of \( P \) will imply that of \( P_J \) and the proof is complete.

**COROLLARY 2.1.3.** If \( M \) is a \( B^\star \)-algebra with a complementor \( p \), then \( M \) is isometrically *-isomorphic with the sum \( (E^J_\lambda)_0 \) of its family \( \{ J_\lambda : \lambda \in \Lambda \} \) of minimal closed two-sided ideals each of which is a simple \( B^\star \)-algebra with an induced complementor \( p_\lambda \). The complementor \( p \) is continuous if and only if each complementor \( p_\lambda \) is continuous.

**PROOF.** By Theorem 1.10, \( M \) is dual. Then Theorem 4.16.14 in \([3] \) gives an isometric *-isomorphism between \( M \) and \( (E^J_\lambda)_0 \). Now Theorem 2.1.2 gives the definition and existence of the induced complementors \( p_\lambda \). Also from Theorem 2.1.2 continuity of \( p \) will imply that of each \( p_\lambda \). To prove the converse we observe that any element \( e \) of \( E \) is contained in a minimal closed two-sided ideal of \( M \) (i.e. one of the \( J_\lambda \)) and hence that \( e \) is a \(^1\)-projection of the algebra \( J_\lambda \). The result follows, since if \( e_1, e_2 \in E \) but are elements of distinct minimal closed two-sided ideals \( J_\lambda, J_\mu \), then \( J_\lambda J_\mu = (0) \) and \( \| e_1 \| = \| e_1 (e_1 - e_2) \| < \| e_1 \| \| e_1 - e_2 \| \) and therefore \( 1 < \| e_1 - e_2 \| \).
This result enables us to restrict our attention for the moment to simple complemented $B^\sigma$-algebras. Such an algebra, being dual, is isometrically $\ast$-isomorphic with the algebra $K(H)$ of all compact operators on a Hilbert space $H$ (Theorem 4.10.20. in [8]). In the next section we shall study these.

2. COMPLEMENTORS ON $K(H)$.

Throughout this section $H$ will denote a fixed complex Hilbert space with inner product $(\,\, ,\,)$, and $\mathcal{B}$ the algebra $K(H)$. If $x$ and $y$ are elements of $H$ then $x \otimes y$ will denote the operator on $H$ defined by the relation $(x \otimes y)(h) = (h, y)x$ for all $h$ in $H$. For any $x$ in $H$, $[x]$ denotes the linear subspace of $H$ spanned by $x$. If $T$ is a linear operator on $H$ $T(H)$ will denote the RANGE of $T$—i.e. $T(H) = \{ y \in H : y = Tx \text{ for some } x \in H \}$. Let $\langle , \rangle$ denote any inner product in $H$ that is consistent with the topology of $H$ ( $\| x \| = \langle x, x \rangle^{1/2}$ is a norm in $H$ that is equivalent to the original norm ). We say that $\langle , \rangle$ is an EQUIVALENT INNER PRODUCT in $H$. Also $\perp$, will denote orthogonality with respect to $\langle , \rangle$; if $T$ is a bounded linear operator on $H$ then $T^\ast$ will be the adjoint of $T$ with respect to $\langle , \rangle$; we shall normally write $\perp$ for $\perp_{\langle , \rangle}$ and $\ast$ for $\ast_{\langle , \rangle}$. It is clear that $K(H)$ is self-adjoint with respect to $\ast_{\langle , \rangle}$ for any equivalent inner product $\langle , \rangle$. For each $a$ in $K(H)$ we write $\| a \|_{\langle , \rangle}$ for the operator norm of $a$ with respect to the norm $\| \, \|_{\langle , \rangle}$ in $H$ $\| a \|_{\langle , \rangle} = \sup_{h \in H} \frac{\langle ah, ah \rangle^{1/2}}{\langle h, h \rangle^{1/2}}$; $\| \|_{\langle , \rangle}$ is the norm $\| \|$. Clearly $\| a^\ast a \|_{\langle , \rangle} = \| a \|_{\langle , \rangle}^2$ for all $a$ in $K(H)$. If $S$ is any subset of $K(H)$ then $S(H) = \{ h : h \in H, h = sh' \text{ for some } s \in S \}$. $h' \in H \}$. 
NOTATION. For every closed subspace $S$ of $H$ we write

$$
j(S) = \{ a \in \mathcal{A} : a(H) \subseteq S \};$$

and for every closed right ideal $R$ of $\mathcal{A}$, $j(R)$ is the smallest closed subspace of $H$ that contains the range $a(H)$ of each operator $a$ in $R$.

The following lemma was suggested by Tomiuk; the proof given here is similar to his proof.

**Lemma 2.2.1.** For every closed right ideal $R$ of $\mathcal{A}$, $R = j(R)$, and for every closed subspace $S$ of $H$, $j(S)$ is a closed right ideal of $\mathcal{A}$ and $S = j(j(S))$.

**Proof.** By the duality of $\mathcal{A}$ and Lemma 2.8.24, in [8] we see that for all closed right ideals $R$ of $\mathcal{A}$, $R = j(R)$. Now let $S$ be a closed subspace of $H$; it is clear that $j(S)$ is a closed right ideal of $\mathcal{A}$ and so by the first part $j(S) = j(j(S))$.

Suppose that $m \in S$; then $m \in j(S)$ and therefore $m \in j(j(S))$, from which it is clear that $m \in j(S)$. Thus $S \subseteq j(j(S))$.

However the reverse inclusion is clear since $S$ is a closed subspace of $H$ containing the range of each operator in $j(S)$.

We now show that the inner product in $H$ defines a complementor in $\mathcal{A}$ in a natural way.

**Theorem 2.2.2.** Let $R$ be any closed right ideal of $\mathcal{A}$; define $R^P$ to be

$$j\left[\left(\frac{1}{j(R)}\right)^P\right].$$

Then $R^P$ is a complementor in $\mathcal{A}$ and $R^P = (R_1)^P$.

**Proof.** By the proof of Theorem 1.10 it is clear that $R \rightarrow (R_1)^*$ is a complementor in $\mathcal{A}$.

Now for any $a$ in $\mathcal{A}$ let $N(a)$ be its null space. Clearly

$$R_1 = \left\{ a : a \in \mathcal{A}, N(a) \cap j(R) \subseteq \end{array} \right\}.$$ 

However, $cl(a(H)) = (N(a^*))^\perp$. Thus

$$(R_1)^* = \left\{ a : a \in \mathcal{A}, a(H) \in j(R)^\perp \right\} = R^P.$$ 

This completes the proof.
It is clear that this result may be generalised to any equivalent inner product in $H$.

**COROLLARY 2.2.3.** If $\langle , \rangle$ is any equivalent inner product in $H$ and $R_{\langle , \rangle}$ denotes the closed right ideal $\left\{ (d(R), \langle , \rangle) \right\}$ of $\mathfrak{N}$, then $P_{\langle , \rangle}$ is a complementor on $\mathfrak{M}$; and $R_{\langle , \rangle} = (R_{\langle , \rangle})_{\langle , \rangle}$.

**COROLLARY 2.2.4.** If $e$ is a $P_{\langle , \rangle}$-projection then $e_{\langle , \rangle} = e$, and all minimal idempotents of $\mathfrak{N}$ that are self-adjoint with respect to $\langle , \rangle$ are $P_{\langle , \rangle}$-projections.

**PROOF.** Suppose that $e$ is a $P_{\langle , \rangle}$-projection; let $R = e \mathfrak{M}$; then $R_{\langle , \rangle} = (1-e) \mathfrak{M} = (e \mathfrak{M})_{\langle , \rangle} = (e \mathfrak{M} (1-e))_{\langle , \rangle} = (1-e) \mathfrak{M}$ and consequently $e (1-e) \mathfrak{M} = 0$. Since $\mathfrak{M}$ is semi-simple, $e (1-e)_{\langle , \rangle} = 0$ so that $e = e_{\langle , \rangle}$ and, applying the involution $\langle , \rangle$, we see that $e_{\langle , \rangle} = e_{\langle , \rangle}$. Hence $e_{\langle , \rangle} = e$.

Conversely, if $e$ is a minimal idempotent and $e_{\langle , \rangle} = e$, we have $(e \mathfrak{M})_{\langle , \rangle} = (1-e) \mathfrak{M}$ and therefore $(e \mathfrak{M})_{\langle , \rangle} = (1-e) \mathfrak{M}$.

Theorem 2.2.2. and the first part of Corollary 2.2.3. are originally due to Tomiuk. The present proofs, unlike his, use Theorem 1.10 and are consequently shorter.

**NOTE.** In future we shall always use $P_{\langle , \rangle}$ to denote the complementor given in Corollary 2.2.3.

As we mentioned previously it seemed probable that the complementor $P_{\langle , \rangle}$ is the most general complementor in $\mathfrak{M}$, and, indeed, it will be shown that this is true if $H$ is infinite dimensional. However it will be shown by a counter-example in §5 that this is not true if $H$ is finite dimensional. It
will be shown that for \( H \) of dimension at least three
continuity of the complementor is a necessary and sufficient
condition that it be of this form, and that the necessity
holds without the dimension restriction. The case of a
space of dimension two is not very interesting as all
proper closed one-sided ideals are minimal; \( G_{(iv)} \) then
becomes redundant and any involutory mapping of the set of
minimal right ideals onto itself will satify the defining
axioms. In §5 we give an example of a continuous complem-
entor on \( K(H) \) (\( H \) is a Hilbert space of dimension two) that
is not of the form \( p \cdot \) for any equivalent inner product \( \cdot \).

We now let \( p \) be any complementor in \( \mathcal{O} \). We shall examine
the properties of \( p \)-projections in more detail. Assume in
future that, unless the contrary is explicitly stated, then the
dimension of \( H \) is at least three.

**Lemma 2.2.5.** Let \( e_i \) (\( i=1, 2, \ldots, n \)) be minimal idempotents
in \( \mathcal{O} \). Then:

(i) \( \left[ e_i^*(H) \right]^+ = (1-e_i)(H) \);
(ii) \( \int (e_i(H)) = e_i \mathcal{O} \); \( \int (e_i \mathcal{O}) = e_i(H) \);
(iii) \( \int \left[ \left( \sum_{i=1}^n e_i(H) \right) \right] = cl \left( \sum_{i=1}^n e_i \mathcal{O} \right) \); \( \int (\sum_{i=1}^n e_i \mathcal{O}) = \sum_{i=1}^n e_i(H) \);
(iv) \( \int [(1-e_i)(H)] = (1-e_i) \mathcal{O} \);
(v) \( \int \left[ \left( \sum_{i=1}^n (1-e_i)(H) \right) \right] = \sum_{i=1}^n (1-e_i \mathcal{O}) \); \( \int (\sum_{i=1}^n (1-e_i)(H)) = \sum_{i=1}^n (1-e_i)(H) \).

**Proof.** First observe that \( S_1 \subset S_2 \Leftrightarrow \int (S_1) \subset \int (S_2) \).
(i) is a simple consequence of the property \( \left[ \text{range } T^* \right] = \text{null } T \)
(null denotes the null space) for any continuous linear
operator \( T \) on \( H \).
(ii): Since $\mathcal{O}$ is strictly irreducible on $H$ we have $\mathcal{O}(e_1) = e_1(H)$ and by Lemma 2.2.1. $e_1\mathcal{O} = \mathcal{O}(e_1\mathcal{O}) = \mathcal{O}(e_1(H))$.

(iii): the range of $\sum_{i=1}^{n} e_i\mathcal{O}$ is contained in $\sum_{i=1}^{n} e_i(H)$ and, since the latter is closed, it follows that the range of $\text{cl}(\sum_{i=1}^{n} e_i\mathcal{O})$ is contained in it. Thus $\mathcal{O}(\text{cl} \sum_{i=1}^{n} e_i\mathcal{O}) \subset \sum_{i=1}^{n} e_i(H)$; the strict irreducibility of $\mathcal{O}$ gives the reverse inclusion. The second part follows from Lemma 2.2.1.

(iv) and (v) are easily proved by similar methods.

**THEOREM 2.2.6.** Let $p$ be any complementor in $\mathcal{O}$. Let 
\[ \{ e_\lambda = x_\lambda y_\lambda : \lambda \in A \} \]
be a family of $p$-projections. Then the set 
\[ \{ x_\lambda : \lambda \in A \} \]
is linearly independent if and only if the set 
\[ \{ y_\lambda : \lambda \in A \} \]
is linearly independent.

**PROOF.** Let \( \{ x_1; i=1...n \} \) be any finite subset of \( \{ x_\lambda \} \) and let \( \{ y_1; i=1...n \} \) be the corresponding elements of \( \{ y_\lambda \} \).

First observe that $x_n$ depends linearly on $x_1...x_{n-1}$ if and only if $e_n(H) \subset e_1(H) + ... + e_{n-1}(H)$, and $y_n$ depends linearly on $y_1,...,y_{n-1}$ if and only if $e_n^*(H) \subset e_1^*(H) + ... + e_{n-1}^*(H)$.

The proof follows from the following chain of inclusions each of which is easily seen to be equivalent to its predecessor:

\[ e_n(H) \subset e_1(H) + ... + e_{n-1}(H); \quad e_n \cap \text{cl} \sum_{i=1}^{n} e_i\mathcal{O} \quad \text{(using lemma 2.2.5)} \]

\[ (e_n\mathcal{O})^P \supset \cap_{i=1}^{n} (e_i\mathcal{O})^P = \cap_{i=1}^{n} (1-e_i)\mathcal{O} \quad \text{(using lemma 1.2)} \]

\[ (1-e_n)(H) \supset \cap_{i=1}^{n} (1-e_i)(H) \quad \text{(using lemma 2.2.5. (iv) and (v))} \]

\[ (1-e_n)(H) \subset \sum_{i=1}^{n} (1-e_i)(H) \subset e_n^*(H) \subset e_1^*(H) + ... + e_{n-1}^*(H) \]

If $e$ is a $p$-projection then $e$ is of the form $x \otimes y$ but either $x$, or $y$ may be varied by any scalar multiplier (the other is then determined). In order to examine continuous
complementors more closely we shall need to specify a fixed form for these elements. We do this now.

Let $x$ be any non-zero element of $H$. Then we shall denote by $e_x$ the unique $p$-projection that is contained in $f((x^*)^*)$. Thus there is a unique $y \in H$ such that $x^*y = e_x y$. Also there is a unique $p$-projection $g_x$ contained in $f((x^*)^*)$. Then $(1-g_x)(x^*) = 0$ and hence $(1-g_x^p)(H) = [x]$ or $e_x^p(H) = [x]$. Thus $e_x = z^x$ for some $z$ in $H$. Also by reversing this argument it can be seen that, if for some $z$ in $H$ $z^x$ is a $p$-projection, then $z^x$ is $g_x$. Thus there is a unique element $z$ in $H$ such that $z^x e_x = e_x^p$.

Now suppose $x_{i_1} y_{j_1} (i=1,2) \in \mathcal{E}$. Then $(x_{i_1}, y_{j_1}) = 0 \iff (y_{j_1}, x_{i_1})(h, x_{j_1}) = 0$ for all $h$ in $H$. This is equivalent to $e_{x_{i_1} y_{j_1}} e_{x_{i_1} y_{j_1}}(h, x_{j_1}) = 0 \iff e_{x_{i_1} y_{j_1}} e_{x_{i_1} y_{j_1}}(h, x_{j_1}) = 0 \iff e_{x_{i_1} y_{j_1}} e_{x_{i_1} y_{j_1}} = 0$. Thus from $\mathcal{C}(iii), \mathcal{C}(iv)$ we have $(x_{i_1}, y_{j_1}) = 0 \iff (x_{j_1}, y_{i_1}) = 0$.

Now fix an element $x_0 \not= 0$ in $H$. We shall refer to it as the INITIAL POINT. Let $y, z_0$ satisfy $x_0 y = z_0$, $z_0 x_0 \in \mathcal{E}$; let $\mathcal{X}$ be $x_0$ and $\mathcal{Y}$ be $z_0$. Then if $x \in \mathcal{X}$ and $xy \in \mathcal{E}$ we have $y \in \mathcal{Y}$. Also $y \not= z_0$, for $y_0 \not= 0 \iff x_0 \not= 0$.

Let $x \in \mathcal{X}$; $e_x = xy_0$, $e_x x = (x_0 + x) y_0$. Then $\{y_0, y_0\}$ and $\{y_0, y_0\}$ are linearly independent but $\{y_0, y_0\}$ is linearly dependent. Thus $u = \alpha y_0 + \beta y$ where $\alpha, \beta$ are non-zero finite scalars. Also this decomposition is unique. We can thus make the following DEFINITION 2.2.7. for any non-zero element $x$ of $\mathcal{X}$, $Tx$ is the unique element of $\mathcal{Y}$ satisfying $(x_0 + x) y_0 + Tx = \lambda e_x x_0 + x$ for some non-zero $\lambda \in \mathcal{E}$. To = 0.

Then we see from the above discussion that $\alpha y_0 + \beta y = \lambda (x_0 + x)$ and hence that $e_x$ is a scalar multiple of $x^*Tx$. Since $e_x$ is idempotent it follows that $(x, Tx) \neq 0$ and $e_x = x^*Tx/(x, Tx)$. $T$ is the $p$-REPRESENTING OPERATOR of $\mathcal{X}$ into $\mathcal{Y}$. 
THEOREM 2.2.8. The operator $T$ has the following properties:

(i) $T(x_1+x_2) = Tx_1 + Tx_2$ for all $x$ in $\mathcal{F}$;
(ii) $T(ax) = a'Tx$ where $a \to a'$ is a 1-1 mapping of $\mathcal{C}$ into itself; and is independent of $x$ in $\mathcal{F}$;
(iii) $T$ is one-to-one and maps $\mathcal{F}$ onto $\mathcal{Y}$;
(iv) $a \to a'$ is an automorphism of the complex field.

PROOF. We first prove (i) for $x_1, x_2$ linearly independent:
Let $x_3 = x_1 + x_2$; then for $i = 1, 2, 3$ there are scalars $\kappa_i, \lambda_i$ satisfying

$$e_{x_0 + x_1} = \kappa_1 (x_0 + x_1) + (y_0 + Tx_1); \quad e_{x_1} = \lambda_1 x_1 \otimes Tx_1.$$  

Now, since $\{x_0 + x_3, x_0 + x_1, x_2\}$ is linearly dependent, we see by Theorem 2.2.6. that $\{y_0 + Tx_3, y_0 + Tx_1, Tx_2\}$ is linearly dependent. That is:

$$y_0 + Tx_3 = a(y_0 + Tx_1) + \beta_1 Tx_2$$

for some scalars $\alpha, \beta_1$; we may immediately put $\alpha = 1$ since $\{x_1, x_2, y_0, y\}$ is linearly independent. Thus:

$$Tx_3 = Tx_1 + \beta_1 Tx_2.$$  \hfill (1)

Similarly by considering $\{x_0 + x_3, x_1, x_0 + x_2\}$ we may show that

$$Tx_3 = \beta_2 Tx_1 + Tx_2$$

(for some $\beta_2 \in \mathcal{C}$). \hfill (2)

A further application of Theorem 2.2.6. gives the linear independence of $\{Tx_1, Tx_2\}$ and it follows from (1) and (2) that $\beta_1 = \beta_2 = 1$, and hence

$$T(x_1 + x_2) = Tx_1 + Tx_2.$$  

We shall complete the proof of (i) later.

(ii): Let $x$ be an arbitrary element of $\mathcal{F}$ and $x'$ any other element of $\mathcal{F}$ such that $\{x, x'\}$ is linearly independent ($x'$ exists since the dimension of $\mathcal{H}$ is at least three). Then, using arguments similar to those above and the case of (i) already established, it is easy to see that for some $\alpha_1, \kappa_1 \in \mathcal{C}$
\[ e_{x_0} + a(x+x') = \kappa_1 (x_0 + a(x+x')) \oplus (y_0 + a_1 T(x+x')) \]
\[ = \kappa_1 (x_0 + a(x+x')) \oplus (y_0 + a_1 Tx + a_1 Tx') \]. But also:
\[ e_{x_0} + a(x+x') = \kappa_2 (x_0 + a(x+x')) \oplus (y_0 + a_{2} T(ax+ax')) \]
\[ = \kappa_2 (x_0 + a(x+x')) \oplus (y_0 + a_2 Tx + a_2 Tx') \] (\(\kappa_2 \in \mathbb{C}\))

The linear independence of \(\{x_0, x, x'\}\) and Theorem 2.2.6. now give \(\kappa_1 = \kappa_2, a_1 = a_2, a_1 = a_2\). Denote the common value of the \(a_1\) by \(a'\). Then we have shown that
\(T(az) = a'Tz\) for \(z = x, x'\) satisfying \(\{z, x\}\) linearly independent; it remains only to show that \(T(a(\lambda x)) = a'T(\lambda x)\) for any \(\lambda \in \mathbb{C}\). But this is easy since, if \(\{x, x'\}\) is linearly independent, then \(\{\lambda x, x'\}\) is also linearly independent, and we may repeat the above argument for these two elements.

\(a \rightarrow a'\) is 1-1: suppose \(a_1' = a_2'\) i.e. \(T(a_1 x) = T(a_2 x)\) \((x \in \mathbb{X})\).

Then \(e_{x_0} + a_1 x = \kappa_1 (x_0 + a_1 x) \oplus (y_0 + T(a_1 x))\) \((i = 1, 2, \kappa_1 \in \mathbb{C})\).

The set \(\{y_0 + T(a_1 x), y_0 + T(a_2 x)\}\) is linearly dependent and so by Theorem 2.2.6. the set \(\{x_0 + a_1 x, x_0 + a_2 x\}\) is linearly dependent. However \(\{x, x_0\}\) is linearly independent and it follows that \(a_1 = a_2\).

(i): It is trivial that \(0' = 0\); also for any \(x\) in \(\mathbb{X}\) \(T(x) = T(1x) = T(1x')\) and therefore \(1' = 1\); again \(T(x) = T((-1)(-1)x) = ((-1)'^2T(x)\) and, since \(x \rightarrow a'\) is 1-1, we see that \((-1)' = -1\).

Let \(x\) be a given element of \(\mathbb{X}\) and let \(x'\) be any element of \(\mathbb{X}\) that is linearly independent of \(x\). Now for \(\lambda \neq 1:\)
\(T(x + \lambda x) = T[(x + x') + (\lambda x - x')]\), and by the part of (i) already established this is
\(T(x + x') + T(\lambda x - x') = Tx + Tx' + T(\lambda x) + T((-1)'x') = Tx + T(\lambda x)\) (since \((-1)' = -1\)); in the case \(\lambda = -1\) we have
\(T(x + \lambda x) = T(x - x) = T(0x) = 0 = Tx - Tx = Tx + T(-x)\). This establishes (i).
It is clear from Theorem 2.2.6. that if \( \{x_1, x_2\} \) is linearly independent then \( Tx_1 \neq Tx_2 \); that \( T \) is 1-1 is now an immediate consequence of the fact that \( a \rightarrow a' \) is 1-1.

\( T \) maps \( X \) onto \( Y \): let \( u \in Y \); then there exists a unique element \( z \) in \( H \) so that \( z \notin (y_0 + u) \in \mathcal{L}^0 \). Now \( z = \mathbf{x}(x_0 + x) \) where \( x \in \mathcal{X} \); it can then be seen that \( Tx = u \).

\( a \rightarrow a' \) is onto: let \( \mu \in \mathbb{C} \); choose arbitrary \( x \in \mathcal{X} \); then there is an element \( u \) in \( \mathcal{X} \) satisfying \( Tu = \mu Tx \); from Theorem 2.2.6 we see that \( \{u, x\} \) is linearly dependent. Thus there is some \( \lambda \) in \( \mathbb{C} \) such that \( u = \lambda x \); then it is clear that \( \mu = \lambda' \).

\( a \rightarrow a' \) preserves the structure of \( \mathcal{G} \): for this we observe that
\[
(a_1 + a_2)'Tx = T((a_1 + a_2)x) = T(a_1 x + a_2 x) = T(a_1 x) + T(a_2 x) = (a_1 ' + a_2 ')Tx
\]
and
\[
(a_1 a_2 ' y)Tx = T((a_1 a_2 x) = T(a_1 (a_2 x)) = a_1 ' T(a_2 x) = a_1 a_2 ' Tx.
\]
This completes the proof of the theorem.

We are now in a position to extend the definition of \( T \) to the whole of \( H \).

DEFINITION 2.2.9. The \( p \)-REPRESENTING OPERATOR ON \( H \) (initial point \( x_0 \)) \( T \) \(_\mathbb{C} \) is defined by:

\[
T \mathbb{C}(x) = a'_y + Tx'
\]
where \( x = ax_0 + x' \) is the unique decomposition of \( x \in H \) into its components in \( [x_0] \) and \( \mathcal{X} \). Since \( T \) \(_\mathbb{C} \) is an extension of \( T \) it will in future be denoted by \( T \).

Clearly \( T \) is a one-to-one mapping of \( H \) onto itself.
REMARK. For any $x$ in $H$ write $x = ax + x'$ where $x' \in H$. If $a = 0$ then $x \in \mathcal{E}$ and $e_x = (x \otimes Tx) / (x, Tx)$ by the remark following Definition 2.2.7.

If $a \neq 0$ then $a' \neq 0$ and $e_x = e_{x_o + (1/\alpha)x} = (y_o + T(x'/\alpha)) (x_o + T(x'/\alpha)) = \alpha e_{x_o + x'} (x_o + x') = \alpha e_{x_o + x'} (x \otimes Tx)$

Thus once again $e_x$ is a finite scalar multiple of $x \otimes Tx$ and since it is an idempotent, we have $(x, Tx) \neq 0$ and therefore $e_x = (x \otimes Tx) / (x, Tx)$.

NOTATION. For any $x$ in $H$ we write $f_x$ for the unique element $e$ of $\mathcal{E}$ satisfying $e(H) = [x]$.

LEMMA 2.2.10. If $p$ is continuous then the $p$-representing operator is linear.

PROOF. We show first that the map $a \rightarrow a'$ is continuous.

First observe that for any $x, x'$ in $H$: $\| f_{x'} - f_x \| = \sup_{\| h \| = 1} \| (x', h) x' - (x, h) x \|$

Since the maps $x \rightarrow (x, h)$ (fixed $h$ in $H$) and $x \rightarrow (x, x)$ of $H$ into $C$ are continuous it is clear that the map $x \rightarrow f_x$ of $H$ into $\mathcal{E}$ is continuous with respect to the norm in $H$ and the induced metric topology in $\mathcal{E}$. Then, since by hypothesis the $p$-derived map $P: f_x \rightarrow e_x$ of $\mathcal{E}$ into $\mathcal{E}_p$ is continuous, the composite map $x \rightarrow e_x$ is continuous (taking the induced metric topology on $\mathcal{E}_p$).

Now fix a non-zero $x$ in $\mathcal{E}$ and denote by $e_a$ the $p$-projection $e_{x_o + ax}$ ($a \in \mathcal{E}$); let $y = Tx$. Then it is clear that the
map $a \mapsto e_a$ will be continuous (with the usual topology on $C$).

Then $(y_0 + a'y) \perp (1 - e_a)(H)$ and therefore $(y_0 + a'y, h - e_a h) = 0$ for all $h$ in $H$. Subtracting two such equations for a fixed $a_0$ and a variable $a$, we obtain on simplification:

$$(a' - a_0')(y, h + e_a h) + (y_0 + a_0'y, (e_a - e_a)h) = 0$$  \hspace{1cm} (1)

However, $y \not\in \mathcal{J}$, and so $y$ is not contained in $[y_0 + a_0'y] = [(1 - e_{a_0})(H)]$. Therefore there is an element $h_0$ in $H$ such that $(y, h_0 + e_{a_0} h_0) = 1$. Now given $\varepsilon > 0$ by the continuity of $a \mapsto e_a$, there exists $\delta > 0$ such that:

$$\|e_a - e_{a_0}\| \leq \min \left\{ \frac{1}{2} \|h_0\| \|y\|, \varepsilon / \left( \|h_0\| \|y + a_0'y\| \right) \right\}$$  \hspace{1cm} (2)

for all $a$ satisfying $|a - a_0| < \delta$.

Then $(y, h_0 + e_{a_0}h_0) = (y, h_0 + e_{a_0}h_0) + (y, (e_a - e_{a_0})h_0)$ and it is easily seen from this that if $|a - a_0| < \delta$ then:

$$|y, h_0 + e_{a_0}h_0| > \frac{1}{2}$$  \hspace{1cm} (3)

Now substituting in (1) $h = h_0$, and using (2), (3) we see that $|a' - a_0'| < 2\varepsilon$ whenever $|a - a_0| < \delta$. Thus $a \mapsto a'$ is continuous.

Since by [5] the only continuous automorphisms of the complex field are the identity and the conjugacy, we need only show that $a' = \bar{a}$ is impossible. So suppose that $a' = \bar{a}$.

Let $x$ be any element of $X$ and $y = Tx$. Consider the equation

$$(x_0 + ax, y_0 + \bar{y}) = 0$$  \hspace{1cm} (4)

Clearly (4) has a solution $a_0 \not= 0$; thus $(x_0 + a_0x) \perp (y_0 + \bar{y})$ and hence $e_{x_0 + a_0x}(H) \subseteq e_{x_0 + a_0x}(H) \perp e_{x_0 + a_0x}(H) = (1 - e_{x_0 + a_0x})(H)$. This implies that

$$e_{x_0 + a_0x} \mathcal{C}(1 - e_{x_0 + a_0x}) = (e_{x_0 + a_0x} \mathcal{C})^p$$ which is the
LEMMA 2.2.11. If $X$ is any Banach space and $f$ is a mapping from $X$ into $C$ that satisfies:

(i) $f(x_1 + ax_2, y) = f(x_1, y) + af(x_2, y)$; $f(x, y_1 + by_2) = f(x, y_1) + bf(x, y_2)$
for all $x_1, x_2, x, y_1, y_2, y$ in $X$;
(ii) $f(x, y) = 0$ if and only if $f(y, x) = 0$;
(iii) $f(x, x) \neq 0$ unless $x = 0$, and there is an element $x_0$ in
such that $f(x_0, x_0)$ is real and positive; then $f$ also satisfies
(iv) $f(x, y) = \overline{f(y, x)}$ for all $x, y$ in $X$;
(v) $f(x, x)$ is real and positive for all $x$ in $X, x \neq 0$.

PROOF. (iv) holds trivially if either term is zero by (ii), so assume that both are non-zero. Now there exist non-zero scalars $\alpha_1$ such that

$$\begin{align*}
\alpha_1 f(x, x) + f(x, y) &= 0 \\
\alpha_2 f(y, y) + f(x, y) &= 0.
\end{align*}$$

From these we may deduce

$$\begin{align*}
\alpha_1 f(x, x) + \overline{f(y, x)} &= 0 \\
\alpha_2 f(y, y) + \overline{f(x, y)} &= 0.
\end{align*}$$

We give the deduction of (2a) from (1a):

$$\begin{align*}
0 &= \alpha_1 f(x, x) + f(x, y) = f(x, \overline{a_1 y}) + f(x, y)
= f(x, \overline{a_1 x + y}) = f(\overline{a_1 x + y}, x) \\
&= \overline{\alpha_1 f(x, x) + f(y, x)} = \alpha_1 f(x, x) + \overline{f(y, x)}.
\end{align*}$$

Now from (1a, 1b) $\alpha_1 f(x, x) = \alpha_2 f(y, y)$, and from (2a, 2b) $\alpha_1 f(x, x) = \alpha_2 f(y, y)$.

Since none of these expressions is zero it follows that

$$\frac{f(x, x)}{f(x, x)} = \frac{f(y, y)}{f(y, y)}.$$ 

Now we already know that

$f(x_0, x_0) = 0$ we have $f(x, x)$ real. Again if $f(x, x_0) = 0$ then $f(x_0, x_0 + x) = 0$.
and therefore \( f(x_o+x, x_o+x) \) is real. It follows that \( f(x, x) \) is real. Now (1a) and (2a) give (iv).

It remains only to prove that \( f(x, x) \) is positive for all \( x \). Suppose there is an element \( x \) of \( S \) satisfying \( f(x, x) < 0 \).

The equation
\[
\lambda^2 f(x, x) + \lambda f(x, x) + f(x, x) + f(x, x) = 0
\]
has a real root \( \lambda _0 \). Thus \( f(\lambda _0 x + x_0) = 0 \), and so \( x_0 = -\lambda _0 x \). But this yields the contradiction: \( 0 < f(x, x) = \lambda _0 f(x, x) < 0 \)

and establishes (v).

The proof of the above Lemma is essentially contained in the proof of Theorem 1 in [4]. It has been stated in full generality as it will be required in the next chapter. What is really required for our present purposes is contained in the following Corollary.

**COROLLARY 2.2.1.2.** If \( p \) is a complementor in \( \mathcal{L} = K(H) \) and the \( p \)-representing operator \( T \) is linear then

(i) \( (x_1, Tx_2) = 0 \) if and only if \( (x_2, Tx_1) = 0 \) \( (x_1, x_2 \in H) \);
(ii) \( (x_1, Tx_2) = (x_2, Tx_1) \) \( (x_1, x_2 \in H) \);
(iii) \( (x, Ty) > 0 \) \( (x \in H, x \neq 0) \);
(iv) \( T \) is a bounded positive hermitian linear operator with bounded inverse.

**PROOF.** (i) is immediate if either \( x_1 \) or \( x_2 \) is zero so assume that both are non-zero. Let \( e_{x_1} = x_1 \otimes y_1 \) and \( R_1 = e_{x_1} \otimes \mathcal{L} \) \((i=1, 2)\).

Then \( e_{x_1} = \lambda_1 (x_1 \otimes Tx_1) \) for appropriate non-zero scalars \( \lambda_1 \);

thus \( y_1 = \lambda_1 Tx_1 \) and \( (x_1, y_j) = 0 \iff (x_1, Tx_j) = 0 \) \((j=1, 2)\). Now \( (x_i, y_j) = 0 \iff (y_j, x_i)(h, x_j) = 0 \) for all \( h \) in \( H \). This is equivalent to \( e_{x_1}^* e_{x_1}^*(h) = 0 \iff e_{x_1}^* e_{x_1}^*(h) = 0 \)

\( \iff \mathcal{L}(1-e_{x_1}^*) \).
\( e_x \in (1 - e_{x_j})_{i} \Leftrightarrow e_{x_i} = (1 - e_{x_j})_{i} \Rightarrow (1 - e_{x_j})_{i} \in \mathbb{R} \).

(i) is now clear from \( C_{(iv)} \).

Now \((x, Tx) \neq 0 \) from the remark following definition 2.2.9.

Also \((x_0, Tx_0) = 1\). Therefore the form \((x, Ty)\) satisfies the conditions of Lemma 2.2.11. and the results (i), (ii), (iii) follow immediately.

(iv): \( T \) is linear and defined everywhere on \( H \) and by (ii) it is hermitian. It is therefore bounded \((\) by the Theorem on page 296 in \([9]\)). Moreover \( T \) is one-to-one with range equal to \( H \) and so \( T^{-1} \) exists. It will also be hermitian and everywhere defined and hence bounded.

**Lemma 2.2.13.** Let \( p \) be any complementer in \( \mathfrak{N} \) and \( T \) the \( p \)-representing operator (initial point \( x_0 \)). Then, if \( T \) is a bounded hermitian linear operator with bounded inverse \( T^{-1} \), the complementer is continuous.

**Proof.** Let \( x \in H \) and \( \lambda \in \mathfrak{C} \); then \( e_x = e_{\lambda x}, f_x = f_{\lambda x} \) and thus in any discussion of the continuity of \( F : f_x \to e_x \) we may assume that \( ||x|| = 1 \).

Now let \( x, x' \) be two elements of \( H \) with unit norm.

Then \( ||e_x - e_{x'}|| = \sup_{||h|| = 1} ||(x, Tx) - (x', T x')|| \)

\( \leq \sup_{||h|| = 1} \left\| \left[ \frac{x}{(x, Tx)} - \frac{x'}{(x', T x')} \right] \right\| \leq \left\| \frac{x}{(x, Tx)} - \frac{x'}{(x', T x')} \right\| ||T|| \)

\( = \left\| \frac{f_x}{(x, Tx)} - \frac{f_{x'}}{(x', T x')} \right\| ||T||. \) \hspace{1cm} (1)

Now from Corollary 2.2.12 \( \langle x, y \rangle = (x, Ty) \) defines an inner product in \( H \) and, since \( T, T^{-1} \) are bounded, the associated norm \( ||x||_\langle \rangle \) is equivalent to the original norm in \( H \). Then if \( \otimes' \) denotes the tensor product with respect
to $\langle , \rangle$ (i.e. $(x \otimes y) = \langle h, y \rangle x$ for all $x, y, h$ in $H$) it can easily be seen that $x \otimes Tx = x \otimes x$ ($x \in H$). Then, if $\| \cdot \|$ is the operator norm in $M$ associated with the norm $\langle \cdot , \cdot \rangle$ in $H$, we have $\| x \otimes \| \langle T, T^\dagger x \rangle \|_x \leq \| T \| \| T^\dagger \| \| x \|_S$, and therefore $\| x \otimes \| \langle T, T^\dagger x \rangle \|_x \leq \| T \| \| T^\dagger \| \| x \|_S$ (x \in H)$. It can now be seen that $\| a \| \leq \| T \| \| T^\dagger \| \| a \|_S \leq \| T \| \| T^\dagger \| \| a \|_S$ (a \in M).

Now $\| x \otimes x \|_S \leq \| x \|_S = (x, Tx)$; therefore $\| (x, Tx) - (x', Tx') \|_S \leq \| x \otimes Tx - x' \otimes Tx' \|_S \leq \| T \| \| T^\dagger \| \| x \otimes x' - x' \otimes x' \|_S$.

Now using (1):

$$\| f_x - e_x \|_S \leq \| T \| \left[ \| f_x - f_x \|_S + \| f_x \|_S \right] \leq \| T \| \left[ \| T \| \| T^\dagger \| \right] \left( 1 + \| T \| \| T^\dagger \| \right)$$

since $\| x \| = \| x' \| = \| f_x \| = 1$.

It is now clear that $P: f_x \rightarrow e_x$ is continuous and, therefore, that the complementor $p$ is continuous.

**COROLLARY 2.2.14.** Under the conditions of the Lemma the $p$-derived mapping is uniformly continuous.

**PROOF.** It is clear that the proof of the lemma establishes this stronger result.

We summarise the results of the recent Lemmas in the following Theorem:

**THEOREM 2.2.15.** Let $p$ be any complementor in $M$ and $T$ the $p$-representing operator (initial point $x_0$). Then the following are equivalent:

(i) $p$ is continuous; (ii) $T$ is linear; (iii) $T$ is a positive bounded hermitian operator with continuous inverse $T$. 
We shall now examine the representing operator of a complementor \( p \), where \( \langle \cdot, \cdot \rangle \) is an equivalent inner product in \( H \).

**Lemma 2.2.16.** Let \( \langle \cdot, \cdot \rangle \) be an equivalent inner product in \( H \); then \( \langle x, y \rangle = (x, Qy) \) \( \langle x, y \rangle \in H \) for some positive hermitian linear operator \( Q \) on \( H \). If \( T \) is the \( p_{\langle \cdot, \cdot \rangle} \)-representing operator (initial point \( x_0 \)) then \( T \) is a continuous linear operator on \( H \) and \( T = aQ \) for some scalar \( a \).

**Proof.** The existence of \( Q \) is a well known result of Hilbert space theory. Let \( x \) be an arbitrary element of \( H \) and let \( R \) be \( \left\{ [x] \right\} \). Then \( R^x \) is a maximal closed right ideal of \( H \) and \( e_\alpha \) satisfies \( (1-e_\alpha)R^x = R^z \); \( e_\alpha R^z = R \). By Corollary 2.2.4 \( e_\alpha \langle \cdot, \cdot \rangle = e_\alpha \) and it is then easy to see that \( e_\alpha = x^\alpha (x, x) / \langle x, x \rangle \) ( \( \otimes \) denotes the tensor product with respect to \( \langle \cdot, \cdot \rangle \) as before). However, we have seen in the remark following definition 2.2.9. that \( e_\alpha = x^\alpha T / \langle x, T x \rangle \). Thus for all \( h \) in \( H \):

\[
\langle h, Qx \rangle = \langle h, x \rangle \langle x, T x \rangle = (h, T x) / \langle x, x \rangle \quad \text{and hence } Qx = \left[ \frac{\langle x, x \rangle}{\langle x, T x \rangle} \right] T x
\]

for all \( x \) in \( H \). In particular, \( [Qx] = [T x] \).

Now let \( \lambda \otimes x \) be the automorphism of \( H \) associated with the semi-linear operator \( T \). Then \( \lambda' = 0 \). Suppose that \( x_1, x_2 \) are arbitrary linearly independent elements of \( H \); then \( Qx_1 = \lambda_1 T x_1 \), \( Qx_2 = \lambda_2 T x_2 \), \( Q(x_1 + x_2) = \lambda_3 T(x_1 + x_2) \), \( \lambda_1' \neq 0 \). Therefore

\[
Q(x_1 + x_2) = \lambda_3 T(x_1 + x_2) = \lambda_3 (T x_1 + T x_2) = \lambda_3 / \lambda_1 Qx_1 + (\lambda_3 / \lambda_2) Q x_2,
\]

and, since \( Q \) is linear, we have \( \lambda_1 / \lambda_3 = \lambda_2 / \lambda_3 \) and consequently \( \lambda_1 = \lambda_2 \).

Now suppose \( \lambda \neq 0 \); then \( \{ x_1, x_1 + \lambda x_2 \} \) is linearly independent and so there exists \( \lambda(\neq 0) / \lambda \) such that \( Qx_1 = \lambda T x_1 \), \( Qx_2 = \lambda T x_2 \), \( Q(x_1 + \lambda x_2) = \lambda T(x_1 + \lambda x_2) \). However, \( T(x_1 + \lambda x_2) = T(x_1 + \lambda' T x_2) \) and so we have \( Q(x_1 + \lambda x_2) = \lambda Q x_1 + \lambda' Q x_2 \). It follows that \( \lambda Q x_2 = \lambda' Q x_2 \). Since \( Q x_2 \neq 0 \) we have \( \lambda = \lambda' \). Thus \( T \) is linear.
Also we have shown that, if \( x \) is fixed in \( H \) and \( Qx = \alpha x \), then \( Qx' = \alpha x' \) for all \( x' \) in \( H \) that are linearly independent of \( x \). By considering \( \{ \lambda x, x' \} \) it can be shown that \( Q(\lambda x) = \alpha^r(\lambda x) \). Thus \( T = \frac{1}{\alpha} \) for some \( \alpha \in \ell \) as required.

**COROLLARY 2.2.17.** If a complementor \( p \) in \( \mathcal{L} \) is equal to \( p = \langle \cdot, \cdot \rangle \) for some equivalent inner product \( \langle \cdot, \cdot \rangle \) in \( H \) then the \( p \)-representing operator \( T \) is independent of the initial point up to a multiplicative scalar constant. \( T \) may by suitable choice of the initial point be taken to satisfy \( \langle x, y \rangle = (x, Ty) \).

We are now able to prove the main result of this section.

**THEOREM 2.2.18.** If \( p \) is a complementor in \( \mathcal{L} \) the following are equivalent:

(i) \( p \) is continuous;

(ii) there is an equivalent inner product \( \langle \cdot, \cdot \rangle \) in \( H \) such that \( p = p = \langle \cdot, \cdot \rangle \); 

(iii) \( \mathcal{L} \) can be expressed as \( \langle R_1 \rangle^* \) for a subsidiary involution in \( \mathcal{L}(R \in L) \);

(iv) \( \mathcal{L} \) can be expressed as \( \langle R_1 \rangle^* \) for a subsidiary involution \( \ast \) in \( \mathcal{L} \) and there is an equivalent norm in \( \mathcal{L} \) that satisfies the \( B^* \) condition for \( \ast \).

**PROOF.** (i) \( \Rightarrow \) (ii): let \( T \) be the \( p \)-representing operator. Then from Lemma 2.2.10 \( T \) is linear and by Corollary 2.2.12 \( \langle x, y \rangle = (x, Ty) \) \( (x, y \in H) \) defines an equivalent inner product in \( H \). Thus from Corollary 2.2.3. \( p = \langle \cdot, \cdot \rangle \) is a complementor in \( \mathcal{L} \). We prove that \( p = p = \langle \cdot, \cdot \rangle \). First let \( R \) be any maximal closed right ideal of \( \mathcal{L} \) and \( e \) the \( p \)-projection
that is contained in $\mathbb{R}^P$ (e exists and is unique by Proposition 1.9.). Then, by the remark following definition 2.2.9, $e = (x \otimes Tx)/(x,Tx)$ and, using this, it is easy to verify that, for any $x,y$ in $H$, $\langle ex,y \rangle = \langle x,ey \rangle$. Thus $e$ is self-adjoint with respect to $\langle \rangle$. However, the $p_\langle \rangle$-projection in $\mathbb{R}^P$ is the unique $\langle \rangle$-self-adjoint idempotent in $\mathbb{R}^P$ (Corollary 2.2.4.) and therefore $e$ is also a $p_\langle \rangle$-projection. Thus $(\mathbb{R}^P)_{\perp} = (1-e)\mathcal{L} = (\mathbb{R}^P)_{\perp}$; applying the complementor $p_\langle \rangle$ to this we see that $\mathbb{R}^P_{\perp} = \mathbb{R}^P$.

Now let $R$ be any closed right ideal of $\mathcal{L}$. By 2.9.5(i) in [2] $R$ is equal to the intersection of all those maximal modular right ideals of $\mathcal{L}$ in which it is contained; we shall denote the set of all such maximal modular right ideals by $\mathcal{M}$. Then by Lemma 1.2:

$$R^P = \text{cl}(\sum_{m \in \mathcal{M}} m^P) = \text{cl}(\sum_{m \in \mathcal{M}} m^P_{\perp}) = R^P_{\perp}.$$  

(ii) $\Rightarrow$ (iv): is immediate for we may take $\star$ to be $\langle \rangle$.

(iv) $\Rightarrow$ (iii): is trivial.

(iii) $\Rightarrow$ (ii): we show first that $a^{\star}a = 0 \Rightarrow a = 0$ for all $a$ in $\mathcal{L}$. Let $R = \text{cl}(a\mathcal{L})$; then $a^{\star}a = 0$, $a^{\star} \in (a\mathcal{L})_{\perp}$, and hence $a \in R^P$. But $a \in R$, by Lemma 1.3, and thus $a \in R^P = (0)$.

Now let $e$ be any $p$-projection. By the proof of Corollary 2.2.4, it can be seen that $e^{\star} = e$. Thus $(\mathcal{L}e)^{\star} = e\mathcal{L}$.

Now $e$ is an operator of rank one on $H$ and so we may choose $u,v$ in $H$ such that $e = u \otimes v$. Then

$$\mathcal{L}e = \left\{ h \otimes v : h \in H \right\} \quad \text{and} \quad e\mathcal{L} = \left\{ u \otimes h : h \in H \right\}.$$  

Thus for any $h$ in $H$ $(h \otimes v)^{\star} = (u \otimes h')$ for some $h'$ in $H$.

Define an operator $Q$ on $H$ by $Qh = h'$. It is easily verified that $Q$ is one-to-one, onto, and linear. Also
(x, Qy)e = (u & Qy)(x & v) = (y & v)'(x & v) = \left[(x & v)'(y & v)\right]'
= \left[(u & Qx)(y & v)\right]' = \left[(y, Qx)e\right]' = (y, Qx)e = (Qx, y)\xi;

it follows that Q is hermitian.

Again: (Qx, x) = 0 \Rightarrow (x & v)'(x & v) = 0 \Rightarrow (x & v) = 0 \Rightarrow x = 0.

Finally, since Qu = v and (u, v)e = e = e, we see that (Qu, u) > 0 and thus
we may apply Lemma 2.2.11 to deduce that Q is positive

definite. Q is a one-to-one hermitian operator of H onto
itself, and so is bounded and has a bounded inverse Q'.

Thus for x, y in H we may define \langle x, y \rangle = (x, Qy); it is
easy to verify that \langle, \rangle is an equivalent inner product in H.

Also for any a in \mathcal{A}:

\langle ax, y \rangle e = (u & Qy)(ax & v) = (u & Qy)a(x & v) = (y & v)'a(x & v)
= (y(x & v))'(x & v) = (y & v)'x(x & v) = \langle x, a'y \rangle e.

Hence \langle ax, y \rangle = \langle x, a'y \rangle, for all x, y in H and all a in \mathcal{A}. It follows that a' = a'y
for all a in \mathcal{A}, so that a' = a'. Thus \mathbf{R}_P = (R_1)^{a'} = (R_1)^{a'}.

Finally, by Corollary 2.2.3, (R_1)^{a'} = \mathbf{R}_P. Thus p = p', proving (ii).

COROLLARY 2.2.19. If R \rightarrow (R_1)^{a'} is a complementor in any
\mathbf{B}^*-algebra \mathcal{A}, where \ast is any involution in \mathcal{A}, then \ast a = 0
if and only if a = 0.

Proof It is easily seen that the proof of this result
given in the proof of the Theorem requires only the
hypotheses of this Corollary.

As a second corollary we deduce the following result
concerning a second involution in a \mathbf{B}^*-algebra. It
is a variant of the fundamental isomorphism Theorem for
primitive Banach algebras (Theorem 2.5.19 in [8]) in
the special \mathbf{B}^* case.
COROLLARY 2.2.20 If $\ast'$ is a second involution in $K(H)$ ($\ast$ being the defining involution and the dimension of $H$ being at least three) and for all closed right ideals $R$ of $K(H)$ $R\cap (R_1)^{\ast'} = 0$, then there exists a positive bi-continuous hermitian operator $Q$ of $H$ onto itself such that for all $A$ in $K(H)$ $A^{\ast'} = Q^1 A^* Q$.

PROOF As in the proof of (iii)$\Rightarrow$(ii) of the Theorem we may establish the existence of $Q$. Then for $x, y$ in $H$:

$$<x, A^* y> = <Ax, y> = (QAx, y) = (x, A^{\ast'} Qy) = (Qx, Q^1 A^* Qy) = <x, Q^1 A^* Qy>.$$

We shall now consider separately the cases in which $H$ has a specified dimension. In particular $H$ infinite dimensional where we shall obtain stronger results than hitherto, and $H$ of dimension two which has been omitted from our discussion. We shall require the following result of [4].

LEMMA 2.2.21. Let $T$ be a one-to-one semi-linear transformation of an infinite dimensional complex normed linear space $\mathcal{E}$ onto a second such $\mathcal{Y}$. Then, if $T$ carries maximal closed subspaces into maximal closed subspaces, the automorphism involved in the semi-linearity of $T$ is a continuous one.

THEOREM 2.2.22. If $H$ is infinite dimensional then any complementar in $\mathcal{H}$ is continuous.

PROOF Let $p$ be any complementor in $\mathcal{H}$, and let $T$ be the $p$-representing operator, initial point $x_0$.  

Let \( X \) be a maximal closed subspace of \( H \). Then \( S = \left[ \mathcal{d}[\mathcal{d}(x)](x) \right]^L \) is a proper closed subspace of \( H \). Also \( Y = TX = \text{lin} \{ x \in H : x \in X \} \subseteq S \). We show that \( Y \) is maximal closed.

First suppose that \( Y \) is not closed: then there exists \( y \) in \( \overline{Y \setminus Y} \); let \( T^{-1}y = x \). Then \( X \{ x \} = H \) since \( X \) is maximal closed and \( X \{ x \} \) is closed. Therefore \( T(X \{ x \}) = Y \{ y \} \) = \( H \). Thus \( Y \) is dense in \( H \) which is a contradiction since \( Y \subseteq S \subseteq H \).

Now suppose that \( Y \) is not maximal closed. Then there exists \( y_1 \notin Y \) such that \( Y \{ y_1 \} \neq H \); hence there exists \( y_2 \notin (Y \{ y_1 \}) \). Let \( x_i = T^{-1}y_i \) \( (i = 1, 2) \). Then \( \{ x_1, x_2 \} \) is linearly independent and \( \{ x_1, x_2 \} \backslash X = 0 \) which contradicts the maximality of \( X \).

Now let \( \lambda \to \lambda' \) be the automorphism of \( \mathcal{O} \) obtained in Theorem 2.2.8; then, by Lemma 2.2.21, \( \lambda \to \lambda' \) is continuous, and from the proof of Lemma 2.2.10, \( \lambda = \lambda' \). Theorem 2.2.15 now completes the proof.

**Corollary 2.2.23.** If \( H \) is infinite dimensional and \( p \) any complementor in \( \mathcal{O} \), then \( p = \mathcal{P} \), for some equivalent inner product \( \langle , \rangle \) in \( H \).

**Proof.** Immediate from the Theorem and Theorem 2.2.18.
NOTE that had we been interested only in this particular result it could have been deduced very much more easily using Theorem 1 of [1]. It will be shown by a counter-example in §5 that the dimension restriction in Theorem 2.2.22 and Corollary 2.2.23 is necessary.

We now show that, if the dimension of $H$ is two, then one conclusion of Theorem 2.2.18 holds. A counter-example in §5 shows that the whole Theorem cannot be extended to include this case.

**THEOREM 2.2.24.** If $H$ is a Hilbert space of dimension two and $\langle \cdot, \cdot \rangle$ an equivalent inner product in $H$, then $p_{\langle \cdot, \cdot \rangle}$ is a continuous complementor in $K(H)$ and, for any closed right ideal $R$ of $(\mathcal{B}(H))$, $R_{\langle \cdot, \cdot \rangle} = (R_{\langle \cdot, \cdot \rangle})^*$.  

**PROOF.** That $p_{\langle \cdot, \cdot \rangle}$ is a complementor in $H$ and that $R_{\langle \cdot, \cdot \rangle} = (R_{\langle \cdot, \cdot \rangle})^*$ is clear. Now if $x, x' \in H$ and $\otimes$ denotes the tensor product with respect to $\langle \cdot, \cdot \rangle$ we have $e_x = x \otimes x / \langle x, x \rangle$, $f_x = x \otimes x / \langle x, x \rangle$, and similar expressions hold for $x'$. Now as in Lemma 2.2.16 there is a positive hermitian linear operator $Q$ that is bi-continuous and satisfies $\langle x, y \rangle = (x, Qy)$ $(x, y \in H)$. Then
\[
\|e_x - e_{x'}\| = \|x \otimes x - \otimes x' \otimes x'\| = \|x \otimes Qx - \otimes x' \otimes Qx'\|,
\]
and substitution of $Q$ for $T$ in the proof of Lemma 2.2.13 establishes the result.
3. COMPLEMENTORS IN A GENERAL $B^\#$-ALGEBRA.

Let $\mathfrak{A}$ be a complex $B^\#$-algebra with a complementor $\rho$. Then $\mathfrak{A}$ is dual and is, therefore, isometrically *-isomorphic with the sum $(\sum \mathfrak{J}_\lambda)_0$ of its family $\{\mathfrak{J}_\lambda: \lambda \in \Lambda\}$ of minimal closed two-sided ideals. We shall identify $\mathfrak{A}$ with $(\sum \mathfrak{J}_\lambda)_0$. Thus every element $a$ of $\mathfrak{A}$ may be considered to be of the form $\left\{a_{\lambda}^\perp: (a_{\lambda} \in \mathfrak{J}_\lambda)\right\}$; also $a^* = \left\{a_{\lambda}^*\right\}$ and $\|a\| = \sup_{\lambda} \|a_{\lambda}\|$. By Theorem 2.1.2 we see that $\rho$ induces a complementor $\rho_{\lambda}$ in $\mathfrak{J}_\lambda$ and that $\rho_{\lambda}$ is continuous for each $\lambda$ in $\Lambda$ if and only if $\rho$ itself is continuous.

We shall frequently assume that $\mathfrak{A}$ does not have any minimal one-sided ideals of linear space dimension less than three; an equivalent assumption is that each $\mathfrak{J}_\lambda$ has an isometric *-representation as the algebra $K(H_\lambda)$ of all compact operators on a Hilbert space $H_\lambda$ whose dimension is at least three.

We see from (2) that continuity of $\rho_{\lambda}$ is a necessary and sufficient condition that there should be a subsidiary involution $\cdot ^*$ in $\mathfrak{J}_\lambda$ satisfying $\mathcal{R}^{\rho_{\lambda}} = (\mathcal{R}_1)^{\cdot ^*}$. Our main result in this section will be an extension of this. In order to do this we obtain results concerning the decomposition of closed right ideals and their complements into the sum of components in the $\mathfrak{J}_\lambda$ which are of independent interest.

At first we do not make any assumptions about the dimension of minimal one-sided ideals of $\mathfrak{A}$ or the continuity of $\rho$. 
LEMMA 2.3.1. If $R$ is a closed right ideal of $\mathfrak{M}$ and is contained in $\mathfrak{J}_\lambda$, then $R^P$ contains $\mathfrak{J}_\mu$ for all $\mu \in \chi, \mu \neq \lambda$.

PROOF. It is clearly sufficient to show that $\mathfrak{J}_\lambda^P \supset \mathfrak{J}_\mu$ ($\mu \neq \lambda$).

Now by Lemma 1.8, $\mathfrak{J}_\lambda^P = (\mathfrak{J}_\lambda)_\lambda^P$ and so is a closed two-sided ideal of $\mathfrak{M}$. Thus $\mathfrak{J}_\lambda^P \cap \mathfrak{J}_\mu$ is a closed two-sided ideal of $\mathfrak{M}$ that is contained in $\mathfrak{J}_\mu$; it must, therefore, be $(0)$ or $\mathfrak{J}_\mu$. Suppose that there is some $\mu \in \chi, \mu \neq \lambda$ such that $\mathfrak{J}_\lambda^P \cap \mathfrak{J}_\mu = (0)$. Then if $x, x$ are elements of $\mathfrak{J}_\mu$, put $x = y + z$ where $y \in \mathfrak{J}_\lambda, z \in \mathfrak{J}_\lambda^P$; then $xx = xy + zx$; however, $yx = 0$ since it is contained in $\mathfrak{J}_\mu \cap \mathfrak{J}_\lambda$ and $zx = 0$ by our choice of $\mu$. It follows that $xx = 0$ and thus that $(\mathfrak{J}_\mu)^2 = (0)$, which contradicts the semi-simplicity of $\mathfrak{M}$.

NOTATION. We shall denote the closed right ideal $R \mathfrak{J}_\lambda$ of $\mathfrak{J}_\lambda$ by $R_\lambda$ ($R \in L_\lambda$).

We can now give the decomposition theorem.

THEOREM 2.3.2. If $R$ is a closed right ideal of $\mathfrak{M}$, then, in the identification of $\mathfrak{M}$ with $(\sum \mathfrak{J}_\lambda)_0$:

(i) $R = (\sum R_\lambda)_0$; (ii) $R^P = (\sum R_\lambda^P)_0$; (iii) $R_1 \otimes \mathfrak{J}_\lambda = ((R_1)_1^i \otimes \mathfrak{J}_\lambda)_0$; (iv) $(R_1)_1^i = (\sum ((R_1)_1^i))_0$, ($R \in L_\lambda$) and $1_\lambda$ denotes left annihilation in $\mathfrak{J}_\lambda$.

PROOF. (i) Let $y \in R; y = \sum y_\lambda$. Now, since $(yz)_\lambda = y_\lambda z_\lambda$ for any $y, z$ in $\mathfrak{M}$, it is easily seen that $y \mathfrak{J}_\lambda = y_\lambda \mathfrak{J}_\lambda$ and thus $y \mathfrak{J}_\lambda \subset R_\lambda$. Now $y_\lambda = y_1 + y_2$ where $y_1 \in R_\lambda, y_2 \in R_\lambda^P$; then $(y_1 + y_2) \mathfrak{J}_\lambda \subset R_\lambda$ and so $y_2 \mathfrak{J}_\lambda \subset R_\lambda$; hence $y_2 \mathfrak{J}_\lambda \subset R_\lambda \cap R_\lambda^P = (0)$. Now by the semi-simplicity of $\mathfrak{J}_\lambda$ we see that $y_2 = 0$ and so $y_1 \in R_\lambda$. 

Thus \( R \subseteq (\Sigma R^\lambda) \), and since the reverse inclusion is immediate we have equality.

(ii): From (i) \( R^p = (\Sigma R^p \cap J^\lambda) \). However, by Lemma 2.3.1,
\[ R + J^\lambda \subseteq (R \cap J^\lambda) + J^\lambda \]
and so by Lemma 1.2,
\[ (R \cap J^\lambda)^p \cap J^\lambda = [\text{cl}(R \cap J^\lambda + J^\lambda)^p] = [\text{cl}(R + J^\lambda)^p] = R^p \cap J^\lambda. \]
Therefore \( R^p \cap J^\lambda = R^\lambda \cap J^\lambda \) and hence \( R^p = (\Sigma R^\lambda)^p \).

(iii): That \( R_1 \cap J^\lambda \subseteq (R \cap J^\lambda)^\lambda_1 \) is clear.

Now suppose that \( x \in (R^\lambda)^\lambda_1 \). Let \( y \in R_1 \); then by (i)
\[ y = \{ y^\lambda_\mu \} \text{ where } y^\lambda_\mu \in R^\lambda_\mu. \]
Now if \( \mu \neq \lambda \), \( (xy)_\mu = x^\mu y^\lambda_\mu = 0 \) since \( x^\mu = 0 \); also \( (xy)^\lambda = xy^\lambda = 0 \) by hypothesis. Therefore \( x \in R_1 \) and \( (R \cap J^\lambda)^\lambda_1 \subseteq R_1 \cap J^\lambda \). The result follows since
\[ J^\lambda = J^\lambda. \]

(iv): Take \( p \) in (ii) to be given by \( R^p = (R_1)^* \). Then
\[ (R_1)^* = \left( \Sigma ((R^\lambda)^* \cap J^\lambda) \right) = \left( \Sigma (R^\lambda)^* \right)^p \] by (iii).

We now examine the projection operators of \( \mathcal{O} \) into itself associated with the closed right ideals of \( \mathcal{O} \) and their complements. This will later enable us to show that the complementors \( p^\lambda \) are in a sense equi-continuous.

**Theorem 2.3.3.** If \( R \) is any closed right ideal of \( \mathcal{O} \), then there is a bounded projection \( E_R \) of \( \mathcal{O} \) into \( R \) whose null space is \( R^p \), \( R = E_R(\mathcal{O}) \), and \( R^p = (1 - E_R)(\mathcal{O}) \).

**Proof.** This is an immediate consequence of Proposition 1.7.
COROLLARY 2.3.4. In the identification of \( \mathcal{A} \) with \( (\Sigma \mathcal{J}_\lambda)_o \): if \( E_{R_\lambda} \) is the bounded projection of \( \mathcal{J}_\lambda \) onto itself satisfying \( R_\lambda = E_{R_\lambda}(\mathcal{J}_\lambda) \), \( P_\lambda = (1 - E_{R_\lambda})(\mathcal{J}_\lambda) \) and \( E_R \) is the bounded projection on \( \mathcal{A} \) given in the Theorem then \( E_R \) satisfies: \( (E_R(a))_\lambda = E_{R_\lambda}(a_\lambda) \) for all \( a \) in \( \mathcal{A} \).

PROOF. We have \( a = E_R(a) + (1 - E_R)(a) \) and thus \( a_\lambda = (E_R(a))_\lambda + ((1 - E_R)(a))_\lambda \), by Theorem 2.3.2, \( (E_R(a))_\lambda \in R_\lambda \) and \( ((1 - E_R)(a))_\lambda \in R_\lambda^{P_\lambda} \). But also: \( a_\lambda = E_{R_\lambda}(a_\lambda) + (1 - E_{R_\lambda})(a_\lambda) \) and \( E_{R_\lambda}(a_\lambda) \in R_\lambda^{P_\lambda} \). Now by the uniqueness of the decomposition of \( a_\lambda \) into its components in \( R_\lambda \) and \( R_\lambda^{P_\lambda} \) we see that \( E_{R_\lambda}(a_\lambda) = (E_R(a))_\lambda \).

COROLLARY 2.3.5. If \( \left\{ e_\lambda : \lambda \in \Lambda \right\} \) is a set of \( p \)-projections with \( e_\lambda \in \mathcal{J}_\lambda \) then \( \sup \left\{ \| e_\lambda \| : \lambda \in \Lambda \right\} < \infty \).

PROOF. Let \( R = (\sum e_\lambda a_\lambda)_o \). Then by Corollary 2.3.4, \( (E_R(a))_\lambda = a_\lambda \) for all \( a \) in \( \mathcal{J}_\lambda \) \( (\lambda \in \Lambda) \). Let the bound of \( E_R \) be \( M < \infty \). Then \( \| e_\lambda a \| \leq M \| a \| \) for all \( a \) in \( \mathcal{J}_\lambda \) \( (\lambda \in \Lambda) \). The result follows since \( \mathcal{J}_\lambda \) has a bounded approximate identity (Theorem 4.8.14 in [8]).

LEMMA 2.3.6. Suppose \( \left\{ H_\lambda : (\lambda \in \Lambda) \right\} \) is a family of Hilbert spaces all of whose dimensions are at least three; let \( \mathcal{J}_\lambda \) be \( \mathcal{K}(H_\lambda) \) and \( \mathcal{A} \) be \( (\sum \mathcal{J}_\lambda)_o \). If \( p_\lambda \) is any complementor in \( \mathcal{J}_\lambda \) such that \( p: R \to R^p = (\sum R_\lambda^{P_\lambda})_o \) \( (R \) any closed right ideal of \( \mathcal{A} \) \) is a continuous complementor in \( \mathcal{J}_\lambda \) then

(1) \( \sup \left\{ \| e \| : e \in E_\rho \right\} < \infty \);

(ii) \( p_\lambda \) -representing operators \( T_\lambda \) may be found that satisfy \( 1 \leq T_\lambda \leq M \) \( \lambda \leq T_\lambda \leq M \) \( (M \) finite, independent of \( \lambda) \).
PROOF. Since \( p \) is continuous by Theorem 2.1.3 each \( p_\lambda \)
is continuous. Therefore by Theorem 2.2.15 any \( p_\lambda \)-representing operator \( T_\lambda \) is a bounded positive hermitian linear operator and the inverse \( T_\lambda^{-1} \) exists and is of the same type (i.e., positive, bounded, hermitian). Thus there are constant scalars \( M_\lambda, m_\lambda \) both finite and strictly positive such that

\[
m_\lambda I \leq T_\lambda \leq M_\lambda I.
\]

Now it is clear that, if \( T_\lambda \) is a \( p_\lambda \)-representing operator, then so also is \( aT_\lambda \) for any positive real scalar \( a \).

Therefore by suitable scalar multiplication we may select the operators \( T_\lambda \) to satisfy \( \|T_\lambda^{-1}\| = \|p_\lambda^{-1}\| = 1 \). Thus \( T_\lambda \geq I \) for all \( \lambda \).

Suppose that with this choice of \( T_\lambda \) it is not possible to find a constant \( M \) such that \( T_\lambda \leq M I \) for all \( \lambda \in \mathbb{A} \). Then there exists a sequence \( \{\lambda_n\} \subset \mathbb{A} \) such that \( T_{\lambda_n} \notin n^2 I \). Write \( T_n \) for \( T_{\lambda_n} \), \( H_n \) for \( H_{\lambda_n} \) etc. Now since \( \|T_n^{-1/2}\| = \|T_n^{-1/2}\| = 1 \), there are elements \( x_{on} \) in \( H_n \) such that \( \|x_{on}\| = 1 \), \( (T_n x_{on}, T_n x_{on}) \), \( (T_n x_{on}, x_{on}) \leq 2 \); but also by hypothesis there are elements \( x_n \) in \( H_n \) such that \( (x_n, x_{on}) = 0 \) and \( (T_n x_n, x_n) = \kappa^2 (x_n, x_n) = \kappa_n^2 > n^2 \) (it is clear that we may include the condition \( (x_n, x_{on}) = 0 \) since \( H_n = [x_{on}] + x_{on}^\perp \)). Now let \( y_n = (1/K_n) x_n + x_{on} \) and consider the element \( e_n \) of \( H_n \); we shall denote it by \( e_n \). Then

\[
\|e_n\| > \|p_n x_n\| \quad \text{and since} \quad \|x_{on}, T_n x_n\| = \|T_n x_{on}, x_n\| = \|T_n x_{on}, x_{on}\| \leq 2,
\]

we have:

\[
\|e_n\| > 1 \frac{(x_n, T_n y_n)}{(y_n, T_n y_n)} \|y_n\| > \frac{(\kappa_n - \overline{\kappa})(1 + \frac{1}{\kappa_n^2})}{1 + \frac{n^2}{2} + \frac{1}{2} \frac{n}{2}} > \frac{\kappa_n - \overline{\kappa}}{\overline{\kappa} + 2} \frac{n^2}{2} \geq \frac{n^2}{2} \geq \frac{n^2}{2^1 + 2^1}. \]
However this contradicts the conclusion of Corollary 2.3.5. Thus (ii) is proved.

(i): Let $e \in \mathcal{E}$; then from the discussion in the proof of Corollary 2.1.3, $e$ is contained in $\mathcal{E}_\lambda$ for some $\lambda$. Thus $e = x \mathcal{T}_\lambda x / (x, T_\lambda x)$ for some $x$ in $H_\lambda$, and $(x, x) = 1$. Now $\|e\| = \|x\|/\mathcal{T}_\lambda x \mathcal{T}_\lambda / \mathcal{T}_\lambda / \|x\|^2 < M^\infty$.

(ii) of this Lemma gives the equi-continuity of the complementors $p_\lambda$ that is required for the extension to the general case of Theorem 2.2.18; this will be given in the next Theorem. Part (i) of the Lemma will not be used subsequently but was inserted for its intrinsic interest.

**Theorem 2.3.7.** Let $\mathfrak{A}$ be a $B^*$-algebra with no minimal closed one-sided ideals of linear space dimension less than three; let $p$ be any complementor in $\mathfrak{A}$. Then $p$ is continuous if and only if there is a subsidiary involution $\#^\prime$ in $\mathfrak{A}$ satisfying $R^p = (R_1)^{\#^\prime}$ for all closed right ideals $R$ of $\mathfrak{A}$. If there is an involution $\#^\prime$ with this property then there is an equivalent norm $\|\|$ in $\mathfrak{A}$ that satisfies the $B^*$-condition for $\#^\prime$.

**Proof.** Suppose there exists an involution $\#^\prime$ satisfying $R^p = (R_1)^{\#^\prime}$. Then by Corollary 2.2.19 $a^{\#^\prime} a = 0 \Rightarrow a = 0$ for all $a$ in $\mathfrak{A}$. Hence we may deduce from Theorem 4.10.13 in [8] that $J_\lambda^{\#^\prime} = J_\lambda$ for all $\lambda \in \mathfrak{A}$ ( $\{J_\lambda : \lambda \in \mathfrak{A}\}$ is the family of all minimal closed two-sided ideals of $\mathfrak{A}$). Now consider the induced complementor $p_\lambda$ in $J_\lambda$:

$$R^{p_\lambda} = R^p \cap J_\lambda = (R_1)^{\#^\prime} \cap J_\lambda = (R_1 \cap J_\lambda)^{\#^\prime} = (R_1)^{\#^\prime}$$

for all closed
right ideals $R$ of $J_\lambda$. Thus by Theorem 2.2.18 we see that $p_\lambda$ is continuous; now from Corollary 2.1.3. $p$ itself is continuous.

Conversely if $p$ is continuous then each $p_\lambda$ is continuous. Since $\mathfrak{D}$ is isometrically $\ast$-isomorphic to the $B^\infty(\infty)$ sum of its family of minimal closed two-sided ideals, each of which is isometrically $\ast$-isomorphic with $K(H_\lambda)$ (where $H_\lambda$ is a Hilbert space of dimension not less than three) it is clearly sufficient to prove the result for $\mathfrak{D}=(\sum K(H_\lambda))_0$.

Then by Theorem 2.2.18 $p_\lambda=p_\lambda^\sim$, where $\langle.,\rangle$ is an equivalent inner product in $H_\lambda$, and for all closed right ideals $R$ of $K(H_\lambda)$: $R^\lambda=(R_\lambda^\lambda)^\sim$. We must now use the involutions $\star$ in $K(H_\lambda)$ to define an involution on the whole of $\mathfrak{D}$, and Lemma 2.3.6 (ii) will enable us to do this. We choose $p_\lambda$-representing operators $T_\lambda$ to satisfy

$$(x,x) \leq (T_\lambda x,x) \leq M^2(x,x) \text{ for all } x \text{ in } H_\lambda \text{ and all } \lambda \text{ in } \Lambda.$$  

It is clear that in each $H_\lambda$ we may take $\langle.,\rangle$ to be given by $\langle x,y \rangle = \langle x,T_\lambda y \rangle$ for all $x,y$ in $H_\lambda$ (i.e. this is an equivalent inner product in $H_\lambda$ with the required properties).

Let $\|a\|_\lambda$ denote the inner product norm $\|a\|_\lambda \langle x,x \rangle ^{1/2}$ in $H_\lambda$ and the associated operator norm in $K(H_\lambda)$. Then:

$$\|x\|_{\langle x,x \rangle} \leq M\|x\|, \quad (1/M)\|a\|_{\langle x,x \rangle} \leq \|a\|_{\langle x,x \rangle} \leq M\|a\|_{\langle x,x \rangle} \text{ for all } x \in H_\lambda, \text{ and } a \in K(H_\lambda).$$

Thus $\|a\|_{\langle x,x \rangle} \leq M\|a\|_{\langle x,x \rangle} = M\|a\|_{\langle x,x \rangle} \leq M^2\|a\|_{\langle x,x \rangle}$ for all $a$ in $K(H_\lambda)$ ($\lambda \in \Lambda$).

Now we may define $\ast'$ in $\mathfrak{D}$ by:

$$(a^\sim)_\lambda = (a_\lambda^\sim)^{\langle x,x \rangle}; \text{ let } \|a\|_{\langle x,x \rangle} = \sup \{\|a\|_{\langle x,x \rangle} : \lambda \in \Lambda\}.$$  

Then it is clear that $\|\|,\|$ are equivalent norms on $\mathfrak{D}$ and that $\ast'$ is an involution defined on the whole of $\mathfrak{D}$; also $(\mathfrak{A},\mathfrak{x},\|\|,\ast')$ is a dual $B^\infty$-algebra and $R \to (R_\lambda^\sim)^{\langle x,x \rangle}$ ($R \in L_\lambda$) is a
complementor in it. Hence \( R \rightarrow (R^\perp)^\perp \) is a complementor in \( \mathcal{H} \). That it is identical with \( p \) follows from the fact that \( p^\perp = p \) and Theorem 2.3.2.

We may remove the dimension restriction in the sufficiency case.

**COROLLARY 2.3.8.** If \( \mathfrak{A} \) is a \( B^* \)-algebra and \( * \) a second involution in \( \mathfrak{A} \) such that \( R \rightarrow (R^\perp)^\perp \) is a complementor on \( \mathcal{H} \), then the complementor is continuous.

**PROOF.** The result is immediate from the proof here and Theorem 2.2.23.

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We conclude this section with a partial result on the inheritance of complementors under sub-direct sums.

**THEOREM 2.3.9.** Suppose that \( \{ H_\lambda : \lambda \in \Lambda \} \) is a family of Hilbert spaces of dimension at least three and \( p_\lambda \) is a complementor in \( K(H_\lambda) \). Let \( \mathcal{A} \) be \( (\sum K(H_\lambda)) \) and for any closed right ideal \( R \) of \( \mathcal{A} \) define \( R^D \) to be \( (\sum R_\lambda)^D \) where \( R_\lambda \) is the intersection of \( R \) with the image of \( K(H_\lambda) \) in \( \mathcal{A} \).

Then \( p \) is a continuous complementor in \( \mathcal{A} \) if and only if \( p_\lambda \) is continuous and there exist \( p_\lambda \)-representing operators \( T_\lambda \) satisfying \( \ast \leq M \leq T_\lambda \leq M^* \) (\( M \) independent of \( \lambda \)).

**PROOF.** Suppose that \( p \) is a continuous complementor; then Lemma 2.3.6 gives the continuity of \( p_\lambda \) and the existence of the \( T_\lambda \).

Conversely, suppose the \( p_\lambda \) are continuous and \( T_\lambda \) exist with the given properties. Then it is clear that \( p \) satisfies \( C(i), C(ii), C(iii), C(iv) \) (using (i), (iii), (iv) of Theorem 2.3.2), since \( \mathcal{A} \) is a dual \( B^* \)-algebra and therefore
complemented). We show that $R + R^p$ is closed; but this is clear for we may define $\| \cdot \|$ in $\mathfrak{A}$ to be $\| \cdot \| = \sup \{ \| a \cdot e \| : \lambda \in \lambda \}$, and, as in Theorem 2.3.7, this is equivalent to the given norm in $\mathfrak{A}$; however it is easy to see that $R + R^p$ is closed with respect to $\| \cdot \|$. 
4. FURTHER CONTINUITY PROPERTIES.

In this section we shall examine in more detail the continuity properties of a continuous complementor. We show that if \( p \) is a continuous complementor in a \( B^* \)-algebra, then \( 'p' \) is also a continuous complementor, and we identify its associated family of minimal idempotents and its derived mapping, which is closely related to the adjoint \( p \)-derived mapping. We also show that the left complementor \((L \rightarrow L^*_{P^*})\) is continuous in a natural extension of the definition. We show further that both the \( p \)-derived map and the adjoint \( p \)-derived map are uniformly continuous and are homeomorphisms. Finally we define strong continuity of a complementor and show that this is equivalent to continuity; this will enable us to define a continuous complementor in an algebra that is not known to be \( B^* \) or, more importantly, not known to have one complementor of the kind in which we are interested already defined on it.

**DEFINITION 2.1.** If \( e \) is any \( \gamma \)-projection in \( \mathfrak{H} \), then \((1-e)\mathfrak{H}\) is a maximal modular right ideal of \( \mathfrak{H} \) and so there exists a unique element \( f \) of \( \mathfrak{E}_p \) such that \((1-e)\mathfrak{E} = (1-f)\mathfrak{E} \). We can, therefore, define the **ADJOINT \( p \)-DERIVED MAPPING** \( P' \) of \( \mathfrak{E} \) onto \( \mathfrak{E}_p \) by:

\[ P'(e) \text{ is the unique element of } \mathfrak{E}_p \text{ that satisfies } (1-e)\mathfrak{E} = (1-P'(e))\mathfrak{E}. \]

**LEMMA 2.1.2.** If \( \mathfrak{H} \) is a Hilbert space and \( S, S' \) two closed subspaces of \( \mathfrak{H} \) such that \( S \oplus S' = \mathfrak{H} \), then \( S^\perp \oplus S'^\perp = \mathfrak{H} \).
PROOF. It is clear that $cl(S^+ + S'^-) = (S \cap S')^\perp = H$, and so it is only necessary to prove that $S^+ + S'^-$ is closed. Now by proposition 1.7. there is a bounded projection $U$ on $H$ that satisfies $UH = S$, $(1-U)H = S'$. Now suppose that $a_n \in S^+$, $b_n \in S'^-$ and $a_n + b_n \to c$. Let $h$ be an arbitrary element of $H$; then

$$(a_n, h) = (a_n, (1-U)h) = (a_n + b_n, (1-U)h),$$

and letting $n \to \infty$ we see that

$$(a_n, h) \to (c, (1-U)h) = ((1-U)c, h).$$

Thus $\{a_n\}$ converges weakly to $(1-U)c$ and, since $a_n \in S^+$ which, being a closed subspace of a Hilbert space, is itself a Hilbert space, it follows that $(1-U)c \in S^+$. Similarly it can be shown that $U^*c \in S'^-$ and so

$$c = (1-U)c + U^*c \in S^+ + S'^-.$$

THEOREM 2.4.3. If $A$ is any $B^*$-algebra with a complementor $p$ and $\,^*$ denotes the natural complementor, then $q: R \to R' = R^*p'$ is a complementor. $\mathcal{E} = (\mathcal{E}_p)^*$ and the $q$-derived mapping $Q$ satisfies $Q(e) = (P'(e))^* (e \in \mathcal{E})$.

PROOF. It is clear that $q$ satisfies $C(iii)$ and $C(iv)$.

$C(i)$: $a \in R \cap R^p' \Rightarrow a \mathcal{O}cR \cap R^p' \Rightarrow (a \mathcal{O}D)^p \subset R^p \cap R' = (0)$; thus $a \mathcal{O} = (0)$ and by the semi-simplicity of $\mathcal{A}$ we have $a=0$.

$C(ii)$: clearly $R + R^q$ is dense in $\mathcal{A}$ and so we need only show that it is closed. Now it is clearly sufficient to show that for all closed right ideals $R$ of $\mathcal{A}$, $R + R^p$ is closed (since $\,^*$ is 1-1, onto). Let $\mathcal{A}_0 = R' + R^p$. Then if $a \in \mathcal{A}_0$, $a$ has a unique decomposition to the sum of elements in $R'$, and $R^p$ (since $R' \cap R^p = (0)$). Let $r(a)$ denote
its component in R. Now \( \mathfrak{A} \) has an isometric \( * \)-representation \( a \to T_a \) on a Hilbert space \( H \). Denote by \( T_R \) the image of \( R \) under this representation \( (R \mathfrak{A} \mathfrak{L}_R) \); also \( T_o \mathfrak{L}(H) \) is dense in \( H \). Then let \( S = \text{cl}(T_R H) \) and \( S^D = \text{cl}(T_R^D H) \) for any given fixed \( R \) in \( L_R \). Then, using Theorem 2.3.3, it is easy to see that \( S \ominus S^D = H \); therefore we may apply Lemma 2.4.2 to obtain \( S^\perp \oplus S^D = H \). Since it is clear that \( S^\perp \ominus S^D = (0) \) there is a bounded projection \( U \) of \( H \) onto itself satisfying \( U H = S^\perp \), \( (1-U)H = S^D \). Then for any \( a \) in \( \mathfrak{A}_0 \): \( a = r(a) + (1-r)(a) \), and therefore \( T_a h = Tr(a) h + T(1-r)(a) h \) \( (h \in H) \), and in this expression \( T_r(a) h \in S^\perp \) and \( T(1-r)(a) h \in S^D \). But also:

\[ T_a h = UT_a h + (1-U)T_a h, \quad \text{and} \quad UT_a h \in S^\perp \text{, (}1-U)T_a h \in S^D. \]

Therefore \( T_r(a) h = UT_a h \) for all \( h \) in \( H \), and consequently \( T_r(a) = UT_a \) for all \( a \) in \( \mathfrak{A}_0 \). Now suppose that \( \{a_n\} \) is a Cauchy sequence in \( \mathfrak{A}_0 \); let \( a \) be its limit in \( \mathfrak{A} \). Let \( M \) be the bound of \( U \) as an operator on \( H \). Then

\[ \| a_n - a_m \| = \| T_r(a_n) - T_r(a_m) \| = \| UT_a - UT_{a_n} \| \leq M \| T_a - T_{a_n} \| = M \| a_n - a_m \|. \]

Thus \( \{a_n\} \) is Cauchy and, since it is contained in the closed right ideal \( R' \), it converges to an element of \( R' \). It is now easy to see that \( \{(1-r)(a_n)\} \) converges to an element of \( R^D \), and thus that \( a \) is contained in \( R' + R^D \). This completes the proof that \( q \) is a complementer.

Now let \( e \) be any element of \( \mathcal{E} \):

\[ (P'(e)^* \mathfrak{A})^P' = (1-P'(e)) \mathfrak{A}^P' = (P'(e) \mathfrak{A})^P = (1-P'(e)^*) \mathfrak{A}; \]

thus \( \mathcal{E}_P^* \subseteq \mathcal{E}_q \); again if \( f \) is any element of \( \mathcal{E}_q \)

\[ (f \mathfrak{A})^P' = (1-f) \mathfrak{A} \Rightarrow [1-(1^*) \mathfrak{A}]^P = f^* \mathfrak{A} \Rightarrow f^* \in \mathcal{E}_P. \]

Thus \( \mathcal{E}_P^* = \mathcal{E}_q \).
Finally, for any \( e \) in \( \mathcal{E} \):
\[
e \sigma l = ((1-e) \sigma l)' = ((1-P'(e)) \sigma l)' = (P'(e))^* \sigma l; \quad \text{and it is clear from this that} \quad q(e) = (P'(e))^*.
\]

If \( s \colon L \to L^s \) is a left complementor on \( \sigma l \), then we may define a \( s \)-projection to be a minimal idempotent \( g \) that satisfies \( \sigma l g^s = \sigma l(1-g) \). The \( s \)-derived mapping \( S \) is defined by: for any \( e \) in \( \mathcal{E} \), \( S(e) \) is the unique \( s \)-projection that is contained in the left ideal \( \sigma l e \). If the \( s \)-derived mapping is continuous, then the left complementor \( s \) is said to be continuous. All these are natural analogues of the definitions for complementors. Now, if \( p \) is a given complementor in \( \sigma l \), we have already seen that \( L \to L^p_{\sigma l} \) is a left complementor; since for any closed right ideal \( R \)
\[
R_{\sigma l}^* = R_{P}^* \quad \text{it follows from the above Theorem that} \quad L \to L^p_{\sigma l}
\]
is also a left complementor in \( \sigma l \). It is a corollary to the following Theorem that the continuity of either of these left complementors is equivalent to that of \( p \).

**Theorem 2.4.4.** If \( \sigma l \) is a \( B^\ast \)-algebra with a complementor \( p \) and \( \sigma l \) has no two-sided ideals of dimension less than three, then the following are equivalent:

(i) \( p \) is continuous;

(ii) \( q = p \) is continuous;

(iii) \( P, P' \) (the derived maps of \( p \)) are uniformly continuous;

(iv) \( P \) and \( P' \) are homeomorphisms.
PROOF. (i) $\Leftrightarrow$ (ii): Since $p = q$, and $q =^* p$, it is clearly sufficient to show that $(i) \Rightarrow (ii)$, and in view of Corollary 2.1.3, we may restrict attention to $\mathcal{A} = \mathcal{K}(H)$. Let $T$ be the $p$-representing operator on $H$. Then, since $p$ is continuous, by Theorem 2.2.15, $T$ and its inverse $T^*$ are bounded positive hermitian operators. Thus $\{x, y\} = (x, T^* y)$ is an equivalent inner product on $H$. Also $T^*$ is a $p$-representing operator and, therefore, by a further application of Theorem 2.2.15 $\mathcal{P}^*$ is continuous. However, if $R$ is a closed right ideal of $\mathcal{A}$, $R^p = (R^p)^*$, hence $R^p$, $R^p = \{Ta^*: a \in R^p\}$. Also $R^{\mathcal{P}} = \{T^* a^*: a \in R^{\mathcal{P}}\}$. Thus $R^p = R^p$ and, since $R^p = R^{\mathcal{P}}$ we have equality. Therefore $q$ is continuous.

(i) $\Rightarrow$ (iii): First consider $\mathcal{K}(H)$. In Lemma 2.2.13, we showed that for $x, x' \in H$

$$\|e_x - e_{x'}\| \leq \|f_x - f_{x'}\| + \|T\| [\|f_x\| + \|T\| |T^*|] + 1 + \|T\| |T^*|.$$ 

Thus, if $f_1, f_2 \in C^*$, $\|P(f_1) - P(f_2)\| \leq \alpha(T)\|f_1 - f_2\|$, where $\alpha(T)$ is a monotonic increasing function of $\|T\|$ and $\|T^*\|$.

For general $\mathcal{A}$, we may consider the isometric $*$-representations as $\mathcal{K}(H_\Lambda)$ of its minimal closed two-sided ideals $\{J_\Lambda : \Lambda \in \mathcal{L}\}$ as in the proof of Theorem 2.3.7, taking the representing operators $T_\Lambda$ of the induced complementors $p_\Lambda$ in $\mathcal{K}(H_\Lambda)$ to satisfy $I \leq T_\Lambda \leq MI$. Then it is clear that there is a constant $\alpha$ such that $\alpha(T_\Lambda) \leq \alpha$ for all $\Lambda$. That $P$ is uniformly continuous follows, since $f_1, f_2 \in C$, $\|f_1 - f_2\| < \varepsilon \Rightarrow f_1$, and $f_2$ are elements of the same $J_\Lambda$. By considering $q$ we see that $\alpha(P'(e))^*$ is uniformly continuous, and therefore $P'$ is uniformly continuous since * is isometric.
(iii) or (iv) ⇒ (i) are trivial.

(i) ⇒ (iv): If p is continuous, then there is a second involution * in $\mathcal{A}$ and equivalent norm $\|\|'$ such that $\sigma' = (\sigma, \mathcal{A}, \|\|')$ is $\mathbb{B}_e^*$ and $p$ is the natural complementor in $\sigma'$, and $\mathcal{E}_p$ is the set of all self-adjoint minimal idempotents in $\sigma'$. Thus, by considering the complementor $R \mapsto (R_1)^*$ in $\sigma'$, it can be seen that the map $F^1: \mathcal{E}_p \to \mathcal{E}$ is continuous. To prove the corresponding results for $P'$ again consider $q$.

**Corollary 2.4.5.** If $s, t$ denote the left complementors $L \mapsto L^{*P*}$, $L \mapsto L^P_1$ respectively, then $(\mathcal{E}_p)^*$ is the set of $s$-projections and $\mathcal{E}_p$ is the set of $t$-projections; further, if $S, T$ are the $s, t$-derived maps, then $S(e) = (P(e))^*$ and $T = P'$. Also one of $p, q, s, t$, continuous implies that all are continuous.

**Proof.** The first part is easy verification, the second then follows immediately from the Theorem.

Now let $\mathcal{M}$ be the set of all minimal right ideals of a $B'$-algebra $\mathcal{A}$ and let $p$ be any complementor in $\mathcal{A}$.

**Definition 2.4.6.** If $R$ is any closed right ideal of $\mathcal{A}$ and $a$ any element of $\mathcal{A}$, then $a = a_1 + a_2$ where $a_1 \in R$, $a_2 \in \mathcal{A}$, and this expression is unique. We call $a_1$ the $p$-component of $a$ in $R$.

**Definition 2.4.7.** Let $R$ be a closed right ideal of $\mathcal{A}$ and $E_R$ the projection on $\mathcal{A}$ that maps each $a \in \mathcal{A}$ into its $p$-component in $R$. A sequence $\{R_n\}_{n \in \mathcal{N}}$ is said to be $p$-convergent to $R \in \mathcal{M}$ if $E_{R_n}$ converges uniformly to $E_R$.

**Definition 2.4.8.** A complementor $p$ is said to be strongly continuous if, for all sequences $\{a_i : i = 0, 1, \ldots, a_i \in \mathcal{A}\}$ that satisfy $a_n \to a_0$ as $n \to \infty$, $a_n \in \mathcal{A}$ is $p$-convergent to $a_0 \in \mathcal{A}$. 

THEOREM 2.4.9. For a complementor $p$ in a $B^\star$-algebra $\mathcal{A}$, continuity and strong continuity are equivalent.

PROOF. Suppose that $p$ is strongly continuous:

if $p$ is not continuous, then the $p$-derived map $P$ is not continuous at some point $e$ of $\mathcal{E}$. Thus there are elements $e_n$ of $\mathcal{E}$ and $K>0$ such that $\|e-e_n\|\leq\frac{1}{n}$ but $\|P(e)-P(e_n)\|>K$.

Now $\{e_n\}$ converges to $e$ and so, by the strong continuity, $\{P(e_n)\}$ converges to $P(e)$, since multiplication on the left by these elements corresponds to the projections of $a\in\mathcal{A}$ into its $p$-components in $e_n\mathcal{A}$ and $e\mathcal{A}$ respectively.

This is a contradiction, and thus $p$ is continuous.

Conversely, suppose that $p$ is continuous.

Let $\{e_n\}\subset\mathcal{M}$ and $R_0\subset\mathcal{M}$, and suppose there exist $\alpha_n\in R_n$, $\alpha_0\in R_0$ such that $\eta_n\rightarrow\alpha_0$. Then each $\alpha_n$ is contained in some $\mathcal{J}_\lambda$ (where $\{\mathcal{J}_\lambda : \lambda\in\Lambda\}$ is the family of all minimal closed two-sided ideals of $\mathcal{A}$) and let $\mathcal{J}_0$ be the one in which $\alpha_0$ is contained. Then it can easily be verified that there is an $N$ such that $\alpha_n\in\mathcal{J}_0$ for all $n>N$. Thus it suffices to prove the result in the simple case (since, if $R_0\subset\mathcal{J}_\lambda$, then the norm of $E_{R_0}$ as an operator on $\mathcal{A}$ will be equal to the norm of its restriction to $\mathcal{J}_\lambda$). So suppose that $\mathcal{J}=K(H)$; $\alpha_n=x_n\otimes y_n$; $\alpha_0=x\otimes y$; and we may assume that $\|x_n\| = \|x\| = 1$. It is clear that $E_{R_0} a = e_x \alpha$ for all $a$ in $\mathcal{J}$ and a similar expression holds for $E_{R_n}$. Thus we wish to prove that $\|e-x\| \rightarrow 0$ as $n \rightarrow \infty$, and, since $p$ is continuous, it is sufficient to show that $\|f_{x_n} - f_x\| \rightarrow 0$. Now for all $h$ in $H$:

$\|x_n \otimes y_n(h) - x \otimes y(h)\| \rightarrow 0$, and putting $h=y$ we have:

$\|x_n \otimes y_n(y) - x \otimes y(y)\| \rightarrow 0$ or $\|(y, y_n) x_n - (y, y)x\| \rightarrow 0$. Therefore
\[(y, y_n) - (y, y) = (y, y_n) - (y, y) \| x_n \| \to 0; \text{ thus, since } (y, y) \text{ is real and positive, we obtain } (y, y_n - y) \to 0. \text{ Now}
\]

\[(y, y)\|x_n - x\| = (y, y_n - (y, y)) \| x_n \| + (y, y_n) x_n - (y, y) x \|
\]

It is clear from this that \(\|x_n - x\| \to 0 \text{ as } n \to \infty. \) Therefore \(\|x_n x - x x\| \to 0\) i.e. \(\|f_n - f\| \to 0\). Thus \(p\) is strongly continuous.
5. SOME COUNTER-EXAMPLES.

EXAMPLE 2.5.1. The dimension restriction in Theorem 2.2.21 cannot be removed; if \( H \) is a Hilbert space whose dimension is finite but greater than two then there are complementors in \( K(H) \) that are not continuous. This example is motivated by an example in [4].

Let \( H = \mathbb{C}^n \) and \( \mathcal{A} = K(H) \); then \( H \) is a Hilbert space with inner product \( \left\langle \lambda_1, \mu_1 \right\rangle = \sum_1 \lambda_1 \overline{\mu_1} \) and \( K(H) \) is a \( \mathcal{B}^* \)-algebra.

Let \( a \) be any automorphism of the complex field other than \( a(\lambda) = \lambda, \ a(\lambda) = \overline{\lambda} \ (\lambda \in \mathbb{C}) \); such an \( a \) exists from \([5]\).

Now for any \( \lambda \) in \( \mathbb{C} \) define \( \lambda' \) to be \( a^{-1}(\overline{a(\lambda)}) \). Then, if \( \sum_1 \lambda_1 \lambda_1' = 0 \), we have \( \sum a(\lambda_1) a(\lambda_1') = 0 \), and thus \( \sum a(\lambda_1) = 0 \), from which it follows that \( a(\lambda_i) = 0 \) for \( i = 1, \ldots, n \); therefore \( \lambda_i = 0 \) for \( i = 1, \ldots, n \).

Now for \( x = \{\lambda_1\}, y = \{\mu_1\} \) in \( H \) define \( \left\langle x, y \right\rangle \) to be \( \sum_1 \lambda_1 \mu_1' \).

Then \( \left\langle x, y \right\rangle = 0 \Leftrightarrow \left\langle y, x \right\rangle = 0 \), for \( \left\langle y, x \right\rangle = 0 \Rightarrow a^{-1}(\sum_1 \mu_1 a(\lambda_1)) = 0 \).

For any subspace \( S \) of \( H \) define \( S^p \) by

\[ S^p = \{ x : \left\langle x, y \right\rangle = 0 \text{ for all } y \text{ in } S \} \]

Then \( S \cap S^p = (0), \ S_{pp} = S \), and if \( S_1 \subseteq S_2 \) then \( S_1 \subseteq S_2^p \). Since all elements of \( \mathcal{A} \) are of finite rank there is a one-to-one correspondence \( R \to \mathcal{F}(R) \) between right ideals of \( \mathcal{A} \) and subspaces of \( H \). Given a right ideal \( R \) of \( \mathcal{A} \) define \( R^p \) to be \( \mathcal{F}(\mathcal{F}(R))^p \). Then we shall show that \( p \) is a complementor in \( \mathcal{A} \); it is clear that \( p \) satisfies \( C(i), C(iii), C(iv) \). Also, for any subspace \( S \) of \( H \), \( S + S^p \) is a subspace of \( H \) and \( (S + S^p)^C \subseteq S^p \cap S = (0) \). It is easily seen from this that \( S + S^p \) contains all elements of \( H \) of the form \( \{\mu_1\} \) with \( \mu_1 = 1, \mu_1 = 0 \) and hence that \( S + S^p = H \). It follows that \( p \) also satisfies \( C(ii) \).
We now show that \( p \) cannot be the natural complementor associated with an equivalent inner product on \( H \).

For if there is an equivalent inner product such that \( R^p = \mathcal{J}(d(R)^{-1}) \), then for any subspace \( S \) of \( H \), \( S^p = S^\perp \). We show that this yields a contradiction.

First observe that \( 1' = 1 \) since \( c(1) = 1 \).

Now let \( x_1 = \{\lambda_1\} \lambda_1 = 1, \lambda_1 = 0 \) if \( 1 \neq 1 \); \( x_2 = \{\mu_1\} \mu_2 = 1, \mu_1 = 0 \) if \( 2 \).

Then \( \langle x_1, x_2 \rangle = 0 \) and therefore \( \langle x_1, x_2 \rangle = 0 \). Let \( K_1 = [x_1, x_1] \).

Now let \( a \) be real; consider \( x = \{\lambda_i\}, x' = \{\mu_i\} \) where \( \lambda_1 = \lambda_2 = 1, \lambda_2 = -1, \lambda_1 = \mu_1 = 0 \) (if \( 1 \neq 1, 2 \)).

Then \( [x, x'] = (K_1 a) K_2 + (-K_2 a) K_1 = 0 \) and therefore \( \langle x, x' \rangle = 0 \); therefore \( a = (K_1 / K_2) a' \) and, since \( 1' = 1 \), we see that \( (K_1 / K_2)' = 1 \).

Thus \( a' = a \) for all real \( a \).

In particular, this implies \( K_1 = K_2 \), and we may clearly suppose that they are equal to 1.

Now again, if \( a \) is real, \( [x_1 + i a x_2, i a x_1 + x_2] = 0 \) and therefore \( \langle x_1 + i a x_2, i a x_1 + x_2 \rangle = 0 \); and consequently \( (i a)' + i a = 0 \); hence \( i' = -i \). Thus for all \( \lambda \) in \( C \), \( \lambda' = \overline{\lambda} \), which is the required contradiction. Thus \( p \) cannot be exhibited concretely by means of an inner product, and so by Theorem 2.2.18 it is not continuous.

**Example 2.5.2.** If \( H \) is a Hilbert space of dimension 2, then there exists a continuous complementor in \( K(H) \) that is not the natural complementor corresponding to an equivalent inner product in \( H \). Thus the dimension restriction in Theorem 2.2.18 cannot be removed.
Let \( x, y \) be an orthonormal basis for \( H \). For \( \lambda \in \mathbb{C} \) define \( \lambda' \frac{\lambda}{\bar{\lambda}} \)

Now if \( S \) is any one-dimensional subspace of \( H \), \( S \) will be \( [x+\lambda y] \) for some \( \lambda \in \mathbb{C} \) or \( S=[y] \). Define \( p \) by \( [x+\lambda y]^p=[x+\lambda'y] \), \( [x]^p=[y] \), \( [y]^p=[x] \). For any proper right ideal \( R \) of \( K(H) \) define \( R^p \) to be \( \int (\langle R \rangle)^p \). Let \( (0)^p=K(H) \), and \( (K(H))^p=(0) \). Then it can be seen that \( p \) is a complementor on \( K(H) \). Also, since \( [x+\lambda y]^p=[x+\lambda|\lambda|^2 y] \) and \( [y]^p=[y] \), also \( e_x+\lambda y=\frac{(x+\lambda y)}{v_x+\lambda|\lambda|^2 y} \), \( e_y=v \). \( e_y \) is continuous, it can be seen that \( x \mapsto e_x \) is continuous, and hence that \( p \) is continuous.

Suppose there is an inner product \( \langle \cdot, \cdot \rangle \) in \( H \) such that \( \langle x, y \rangle = 0 \), and for \( \lambda \neq 0 \)

\[
\langle x+\lambda y, x+\lambda y \rangle = \langle x, x \rangle - \frac{\lambda}{|\lambda|^2} \langle y, y \rangle = 0.
\]

This cannot hold for all \( \lambda \) and is the required contradiction.

**EXAMPLE 2.5.3.** \( C(i) \) cannot be replaced by \( C(i') \). This example shows clearly how essential the strong form \( C(i) \) was for the extension of Theorem 2.2.18 to Theorem 2.3.7.

Let \( H \) be a three dimensional Hilbert space with an orthonormal basis \( x, y, z \); let the inner product in \( H \) be \( \langle \cdot, \cdot \rangle \). Define an operator \( T_n \) on \( H \) by \( T_n(ax+\beta y+\gamma z)=a\gamma x+\beta y+\gamma z; \) \( a, \beta, \gamma \) any elements of \( \mathbb{C} \). Then \( T_n \) is a bounded, positive, hermitian operator on \( H \), and its inverse is defined and is an operator of the same kind. Thus we may define an equivalent inner product \( \langle \cdot, \cdot \rangle_n \) in \( H \) by \( \langle u, v \rangle_n=(T_n u, v) \). Let \( a \) be \( K(H) \).

Define \( p_n \) in \( a \) to be \( p \langle \cdot, \cdot \rangle_n \). Now let \( a_0=(\Sigma a_i) \), \( i \in \mathbb{Z}, \). Let \( a=\nu \).

\[ \nu = -i/(\lambda^2 - 5). \]
Then $\mathbf{A}_q$ is a dual $B^*$-algebra, and so by Theorem 2.3.2, if $R$ is any closed right ideal of $\mathbf{A}_q$, then $R=(\sum R_1)_o$, where $R_1$ is the intersection of $R$ with the image of $A_1$ in $\mathbf{A}_o$. Define $p$ on $\mathbf{A}_o$ by $R^p=(\sum R_1 P_1)_o$. Then it can be verified that $p$ satisfies $C(i), C(ii), C(iii), C(iv)$. Suppose that $p$ satisfies $C(ii)$. Then it is a complementor and, since each $p_1$ is continuous, $p$ is continuous. Therefore by Lemma 2.3.6, if we select $p_1$-representing operators $T_1'$ to satisfy $\|T_1'\|_o=1$, then there is a constant $M$ such that $T_1' \equiv MI$. However $T_1$ is a $p_1$-representing operator and $\|T_1^{-1}\|_o=1$; thus there exists $M$ such that $T_1 \equiv MI$; but this is a contradiction since $(T_n x, x)=n(x, x)$. 
3. SEMI-SIMPLE RIGHT COMPLEMENTED ALGEBRAS.

1. INTRODUCTION.

We have shown in Chapter 2 that in the case of a right complemented Banach algebra that is known to be $B^*$ the complementor, if continuous, can be expressed in two simple ways: viz. as the natural complementor associated with a subsidiary involution in the algebra, and, via a Hilbert space representation, as the natural complementor associated with an equivalent inner product in the representing space. Conversely, complementors of this form are continuous. The main emphasis in this Chapter will be on an investigation of the representation theory of a semi-simple right complemented algebra that is not known to be $B^*$ to see how closely we can approach the results available in the $B^*$ case. We give a definition of continuity of a right complementor which, because of Theorem 2.4.9, is equivalent to the original definition in the $B^*$ case. In §3 we show that every primitive Banach algebra (none of whose minimal two-sided ideals is of dimension less than three) with a continuous complementor admits a Hilbert space representation, and that orthogonality in the Hilbert space is related to the right complementor in the expected way. Further, we give necessary and sufficient conditions on a primitive Banach algebra of operators on a Hilbert space for it to have a right complementor naturally induced by the inner product in the Hilbert space. We show that the complementor will then be continuous. In §4 we apply these results to a semi-simple right complemented algebra and again obtain a faithful continuous representation on a Hilbert space through which the complementor can be exhibited.
We discuss the second form of exhibition of the right complementor in §5 when we are considering annihilator algebra algebras.

While it is true that this constitutes the main emphasis in this Chapter, we do not restrict ourselves only to this. In the $B^*$ case the structure of the algebras concerned was well known and they were already known to be dual. These questions are examined here. In §2 we give an account of the structure of semi-simple right complemented Banach algebras. In §5 we discuss situations in which such algebras are annihilator algebras and the consequences of their being so; in particular, we show that a primitive right complemented Banach algebra that is either a left or right annihilator algebra is dual, and that a semi-simple bi-complemented Banach algebra with the one-sided annihilator properties is dual.

Our discussion in this Chapter does in some cases overlap with [12]. For completeness of the present account we have not avoided this; however, it is explicitly stated when this does occur.

We have only considered right complementors (which are sometimes referred to as complementors) but it is clear that analogous results could have been obtained for left complementors.

In §6 we give some counter-examples. In particular we show that a primitive Banach algebra with a continuous right complementor need not be a one-sided annihilator algebra.

As before we shall consider only complex algebras.
2. STRUCTURE THEOREMS.

In this section we examine the structure of a semi-simple right complemented algebra \( \mathcal{A} \). We begin by showing that the socle is dense in \( \mathcal{A} \) and that \( \mathcal{A} \) is the direct topological sum of its minimal closed two-sided ideals, each of which has a naturally induced complementor. We then show that the complementor in \( \mathcal{A} \) can be constructed from a knowledge of the induced right complementors, for any closed right ideal of \( \mathcal{A} \) is the topological sum of its intersections with the minimal closed two-sided ideals, and its complement is the topological sum of the induced complements of these intersections. We show finally that a complementor is induced in the quotient algebra of \( \mathcal{A} \) with any closed two-sided ideal.

Let \( p \) be the complementor in \( \mathcal{A} \); we begin with two results of Tomiuk [12].

**Lemma 3.2.1.** The socle \( \mathcal{J} \) is dense in \( \mathcal{A} \).

**Proof.** Let \( I = \text{cl}(\sum_{\lambda} \mathcal{P}_{\lambda} \mathcal{A}) \) where \( \{\mathcal{P}_{\lambda} : \lambda \in \Lambda\} \) is the set of all minimal idempotents in \( \mathcal{A} \). If \( x \in I^D \), then, since \( I \) is a two-sided ideal, and by Lemma 1.8, \( x \in I \), and therefore \( \mathcal{P}_{\lambda} x = 0 \) for all \( \lambda \). Thus \( x \in (\mathcal{P}_{\lambda} - 1) \mathcal{A} \) and is therefore (using Corollary 1.5) contained in the intersection of all the maximal modular right ideals of \( \mathcal{A} \).

Since \( \mathcal{A} \) is semi-simple, \( x = 0 \). Thus \( I^D = (0) \) and \( I = A \).

**Theorem 3.2.2.** \( \mathcal{A} \) is the direct topological sum of its family of minimal closed two-sided ideals.

**Proof.** Let \( K \) be the topological sum of the minimal closed two-sided ideals of \( \mathcal{A} \). Let \( R \) be a minimal closed right ideal of \( \mathcal{A} \); then, by the proof of Lemma 2.8.8. in [9], the smallest closed two-sided ideal \( \mathcal{J} \) of \( \mathcal{A} \) that contains \( R \) is minimal.
closed. That $K=\sigma$ now follows from Lemma 3.2.1. By the proof of Theorem 6 in [1] $K$ is a direct topological sum.

We now show that, if $\{J_{\lambda}: \lambda \in \Lambda\}$ is the family of all minimal closed two-sided ideals of $\sigma$, then a right complementor $p_{\lambda}$ may be induced in $J_{\lambda}$ in a natural way. Here we are following a similar course to [12] (see Lemma 1).

**LEMMA 3.2.3.** If $R$ is a closed right ideal in $J_{\lambda}$, then $R$ is a closed right ideal in $\sigma$.

**PROOF.** Certainly $R$ is closed in $\sigma$. From Lemma 1.8, we see that $J_{\lambda}^P$ is a two-sided ideal of $\sigma$. Now let $y$ be an element of $\sigma, y=r(a_1+a_2)$ where $r \in R, a_1 \in J_{\lambda}, a_2 \in J_{\lambda}^P$; thus $y=r_1+ra_2$ where $r_1 \in R$; however $ra_2 \in J_{\lambda} \cap J_{\lambda}^P=(0)$, and therefore $y=r_1$ and $R\sigma \subset R$ as required.

**THEOREM 3.2.4.** A right complementor $p_{\lambda}$ can be induced in $J_{\lambda}$.

**PROOF.** For any closed right ideal $R$ of $J_{\lambda}$ define $R^P$ to be $R \cap J_{\lambda}$. It is clear that this is a right complementor in $J_{\lambda}$ (the proof is identical to the corresponding part of Theorem 2.1.2).

Now this discussion and [12] diverge.

**PROPOSITION 3.2.5.** (i) $J_{\mu}^P \subset J_{\lambda}^P \subset J_{\lambda}$; (ii) if $R \sigma L$ and $R \subset J_{\lambda}$, then $(R^P \cap J_{\lambda})^P = R \cap J_{\lambda}^P$; (iii) if $R \sigma L$ and $R \subset J_{\lambda}$, then $R^P \cap J_{\lambda}^P = R^P$.

**PROOF.** (i) is immediate from the proof of Lemma 2.3.1.

(ii): By Lemma 1.2, we have $(R^P \cap J_{\lambda})^P = cl(R \cap J_{\lambda}^P)$.

By the properties of $p_{\lambda}$ we see that $(R^P \cap J_{\lambda})^P \cap J_{\lambda} = R$. 

\[\text{insert: and let } y=ra, r \in R; \text{ then we can write } a=a_1+a_2,\]
Now suppose that \( x \in (R^P \cap J_\lambda)^P \) but \( x \notin R \cap J_\lambda^P \). Put \( x = t_1 + t_2 \) where \( t_1 \in J_\lambda, t_2 \in J_\lambda^P \). Therefore \( x - t_2 = t_1 \in (R^P \cap J_\lambda)^P \setminus (R \cap J_\lambda^P) \) since \( t_2 \in J_\lambda^P \cap J_\lambda^P \setminus R \). However \( t_1 \in J_\lambda \) and therefore \( t_1 \in J_\lambda^P \cap (R^P \cap J_\lambda)^P \) \( = \mathbb{R} \). This is a contradiction.

(iii): Apply the result of (ii) to the closed right ideal \( R^P \) of \( J_\lambda \). Then \( R^P \cap J_\lambda^P = (R^P \cap J_\lambda^P)^P = (R^P \cap J_\lambda^P)^P = R^P \).

NOTATION. For any closed right ideal \( R \) of \( \mathcal{A} \) denote by \( R_\lambda \) the closed right ideal \( R \cap J_\lambda \) of \( \mathcal{A}_\lambda (\lambda \epsilon \Lambda) \).

THEOREM 3.2.6. If \( R \) is a closed right ideal of \( \mathcal{A} \), then \( R \) is equal to \( \text{cl}(\Sigma R_\lambda: \lambda \epsilon \Lambda) \), and \( R^P \) is equal to \( \text{cl}(\Sigma R_\lambda^P: \lambda \epsilon \Lambda) \).

PROOF. Let \( a \in R; \) then by Lemma 1.3. \( a \in a\overline{\mathcal{A}} \). Then using Theorem 3.2.2., it can be seen that, given any \( \epsilon > 0 \), there exist \( \{ \lambda_1: i = 1, \ldots, n \} \subset \Lambda \) and elements \( b_\lambda \) of \( J_\lambda \) such that

\[
\| a - \sum b_\lambda \| < \epsilon.
\]

Thus \( R \subset \text{cl}(\Sigma R_\lambda) \).

However, \( R^P \subset R \cap J_\lambda \) and it follows that \( R \subset \text{cl}(\Sigma R \cap J_\lambda) \).

Conversely, \( R \supset R \cap J_\lambda \) for all \( \lambda \epsilon \Lambda \), and therefore \( R \subset \text{cl}(\Sigma R_\lambda) \).

Consequently, we have \( R = \text{cl}(\Sigma R_\lambda) \).

Now applying this result to \( R^P \) we see that

\[
R^P = \text{cl}(\Sigma R^P \cap J_\lambda^P) \subset \text{cl}(\Sigma R_\lambda^P \cap J_\lambda^P) = \text{cl}(\Sigma R_\lambda^P).
\]

Also, since \( R = \text{cl}(\Sigma R_\lambda) \) and \( J_\lambda^P \supset J_\mu \), it is easily seen that

\[
R \subset \text{cl}(R_\lambda + J_\lambda^P).
\]

Therefore \( R^P \supset (\text{cl}(R_\lambda + J_\lambda^P))^P = R_\lambda^P \cap J_\lambda = R_\lambda^P \), hence \( R^P \supset \text{cl}(\Sigma R_\lambda^P) \). Combining this with the previous inclusion we have equality.

We conclude this section with a discussion of circumstances in which a right complementor in a Banach algebra is inherited.
THEOREM 3.2.7. Let \( \mathcal{A} \) be a right complemented Banach algebra and \( J \) be a closed two-sided ideal of \( \mathcal{A} \). Then, if \( \mathcal{A}' \) is the quotient algebra \( \mathcal{A}/J \), it is possible to induce a right complementor in \( \mathcal{A}' \).

PROOF. Let \( R' \) be any closed right ideal of \( \mathcal{A}' \); let \( q \) be the map \( \alpha \rightarrow \alpha': a \rightarrow [a+J] \). Then \( R'=q^{\dagger}(R') \) is a closed right ideal of \( \mathcal{A} \) and thus its complement \( R^D \) is defined. Define \( p' \) on the closed right ideals of \( \mathcal{A}' \) by \( R^D'=e_{\mathcal{A}}(q(q^{\dagger}(R'))^D) \). It is easily seen that this is \( q(cl(q^{\dagger}(R')+J))=q((q^{\dagger}(R')+J)^D) \).

It is clear that \( p' \) satisfies \( C(i) \).

\( C(i) \): we show that \( R \cap cl(R^D+J)=J \). Let \( x \) be in this intersection, then put \( x=y+z \) where \( y \in J, z \in J^D \); since \( x, y \) are in \( R \) it follows that \( z \in R \). But also \( x \in cl(R^D+J) \) and, since \( y \in J \), it follows that \( z \in cl(R^D+J) \). Thus \( z \in cl(R^D+J) \cap R \cap J^D \). However \( (R \cap J^D)^D=cl(R^D+J) \) and therefore \( z=0 \) and \( x \in J \).

\( C(ii) \): let \( a' \) be any element of \( \mathcal{A}' \). Let \( a \) be any element of \( \mathcal{A} \) that satisfies \( q(a)=a' \). Then put \( a=b+c \) where \( b \in R, c \in R^D \).

Then \( a'=q(b)+q(c) \) and \( q(b) \in R', q(c) \in q((q^{\dagger}(R'))^D) \), and it follows that \( \mathcal{A}'=R' \cup R^D' \). and that \( q((q^{\dagger}(R'))^D) \) is closed.

\( C(iii) \): \( R^D'=q(q^{\dagger}(R'))^D \); thus \( q^{\dagger}(R^D') \supset q^{\dagger}(R') \).

Therefore \( q(q^{\dagger}(R^D')) \subset R \).

Then by Lemma 1.1. \( p' \) is a right complementor.

A natural question is, whether, if a Banach algebra \( \mathcal{A} \) is the direct topological sum of a family of its closed two-sided ideals and each of these is right complemented, then \( \mathcal{A} \) is itself right complemented. An answer has been given to this in the special case in Theorem 2.3.9. In this case \( \mathcal{A} \) was

\[ \text{This is closed by [41] page 44.} \]
already known to have an (unrelated) complementor \((R \to R_1^*)\) defined upon it, and thus to satisfy the first part of Theorem 3.2.6 (in the stronger form of \(B^*(\infty)\) sums). Thus to obtain any general result it would seem to be necessary to include in the hypothesis the condition that any closed right ideal of \(\pi\) can be expressed in this form. It would not therefore be very interesting.

Theorems 3.2.2 and 3.2.6 effectively reduce the study of semi-simple right complemented Banach algebras to the primitive case. We study these in the next section.

3. PRIMITIVE RIGHT COMPLEMENTED BANACH ALGEBRAS.

We are considering here a primitive complex Banach algebra which, being right complemented, has minimal right ideals. Thus [8] 2.4.8 to 2.4.19 is relevant. We shall make frequent use of the results contained here. In the discussion in [8] a continuous representation is obtained on a pair of Banach spaces in normed duality. In our particular case we shall show that one of these is a Hilbert space. The representation in question is the left regular representation on a minimal left ideal. We give a more general definition of continuity of a right complementor and show that this is precisely the condition required to obtain the required representation. We begin, however, by giving conditions on an algebra of operators on a Hilbert space that will ensure that it is right complemented.
Let $\mathcal{A}$ be an irreducible algebra of operators on a Hilbert space $H$, and suppose that $\mathcal{A}$ is a Banach algebra with respect to a norm, $\| \cdot \|$, that majorises the operator norm, $\| \cdot \|$.

**Lemma 3.3.1.** Let $S$ be a closed subspace of $K(H)$. If $E_\mathcal{A} a = 0$ for all $E$ in $S$, then left multiplication of $\mathcal{A}$ by $E$ is a bounded linear operator of $\mathcal{A}$ into itself. Further, there exists a constant $k_a$ such that $\|E a\| \leq k_a \|a\|$ for all $a$ in $\mathcal{A}$, $E$ in $S$.

**Proof.** We first prove that each $E$ in $S$ is a continuous operator on $\mathcal{A}$; to do this we show that $E$ is closed.

Suppose $\|a_n - a\| \to 0$, $\|E a_n - b\| \to 0$; then $\|a_n - a\| \to 0 \Rightarrow \|a_n - a\| \to 0 \Rightarrow \|E a_n - E a\| \to 0$ and $\|E a_n - b\| \to 0 \Rightarrow \|E a_n - b\| \to 0$; therefore $E a = b$. Hence by the closed graph theorem $E$ is continuous.

We now prove that the map $U_a$ of $S$ into $\mathcal{A}$ defined by $U_a E = E a$ is continuous for each $a$ in $\mathcal{A}$. Again it is sufficient to show that it is closed.

Suppose $\|E_n - E\| \to 0$ (i) and $\|E_n a - b\| \to 0$ (ii) ($E_n, E \in S$).

Then from (i) $\|E_n a - E a\| \to 0$, and from (ii) $\|E_n a - b\| \to 0$.

Therefore $b = U_a E$ and the operator $U_a$ is closed. Therefore it is continuous, and so there exists a constant $k_a$ such that $\|E a\| \leq k_a \|E\|$ ($E \in S$). The result now follows using the uniform boundedness theorem.

**Insert:** left multiplication by
We can now show that a very general algebra of operators on a Hilbert space is right complemented.

**Theorem 3.3.2.** If $H$ is a Hilbert space and $\mathcal{A}$ a strictly dense algebra of operators on $H$ that is a Banach algebra under a norm $\|\|$, then $\mathcal{A}$ is right complemented if it has a dense socle and either of the following conditions is satisfied:

(i): $E\mathcal{A} \subseteq \mathcal{A}$ for all orthogonal projections $E$ on $H$;

(ii): $E\mathcal{A} \subseteq \mathcal{A}$ for all $E \in \mathcal{K}(H)$.

**Proof.** We first observe that, if $E$ is an orthogonal projection of rank one on $H$, then $E \mathcal{A} \subseteq \mathcal{A}$ and so $\mathcal{A}$ contains elements of rank one; it follows that $\mathcal{A}$ contains minimal one-sided ideals. Then from Theorem 2.4.14, in [8] it follows that $\|\|$ majorises the operator norm $\|\|$. Now for any closed right ideal $R$ of $\mathcal{A}$ define $R^p$ to be

$$\{a:a \in \mathcal{A}, ah \subseteq (RH)^{\perp}\}.$$ We show that this is closed:

suppose $a_n \in R^p, \|a_n-a\| \to 0$; then $\|a_n-a\|^2 \to 0$ and therefore, since $a_n \in RH$, it follows that $ah \subseteq RH$ and hence $a \in R^p$. Thus $p$ maps the set of all closed right ideals of $\mathcal{A}$ into itself. Also it is clear that $p$ satisfies $C(1), C(iv)$.

We now prove $C(ii), C(iii)$ in the special case of $R$ minimal. Let $E$ be the orthogonal projection of $H$ onto $RH$. We show that $R=ER$: let $a \in R$; then certainly $Eah=ah (h \in H)$ and therefore $Ea=a$. Thus $ER=R$ and so $E\mathcal{A} \subseteq R$. Now let $Ea$ be any element of $E\mathcal{A}$; then $Ea$ is an element of $\mathcal{A}$ of finite rank whose range is contained in $RH$, and so, by Theorem 2.4.18, in $[8], E \in R$. Thus $E\mathcal{A} \subseteq R$ and, combining this with our previous inclusion, we have equality.

$C(ii)$: Now $R^p=\{a:ah \subseteq (RH)^{\perp}\}$ and since $(1-E)ah \subseteq (RH)^{\perp}$, we see
that \((1-E)\alpha \in \mathbb{R}^\mathbb{P}\). However \(\alpha = E\alpha + (1-E)\alpha\) and \(E\mathbb{R} = \mathbb{R}, (1-E)\alpha \in \mathbb{R}^\mathbb{P}\), and \(\mathbb{R} \cap \mathbb{R}^\mathbb{P} = (0)\), from which it follows that \(\mathbb{R}^\mathbb{P} = (1-E)\alpha\), and hence \(\alpha = \mathbb{R} + \mathbb{R}^\mathbb{P}\).

C(iii)': \(\mathbb{R}^\mathbb{P} = \{a : a \in \mathbb{R}^\mathbb{H}, \exists R^\alpha \in \mathbb{R}^\mathbb{H}\}\) \(\cap \{a : a \in \mathbb{R}^\mathbb{H}\} \supset \mathbb{R}\).

This proof of C(iii)' holds for general \(\mathbb{R}\); in this particular case we see that \(\mathbb{H}^\mathbb{P}\) is \((1-E)\mathbb{H}\) and its orthogonal complement is \(\mathbb{E}^\mathbb{H}\), and thus we have equality throughout.

We now suppose that \(\mathbb{R}\) is a closed right ideal of \(\alpha\) that is the topological sum of a family \(\{\mathbb{R}_\lambda : \lambda \in \Lambda\}\) of minimal right ideals of \(\alpha\). Since the proof of Lemma 1.2. uses only C(iv) and C(iii)', we have \(\mathbb{R}^\mathbb{P} = \bigcup \mathbb{R}_\lambda^\mathbb{P}\).

We now consider \(\mathbb{R} + \mathbb{R}^\mathbb{P}\). It is clear that \(\text{cl}(\mathbb{R} + \mathbb{R}^\mathbb{P})\mathbb{H}\) contains \(\mathbb{R}\mathbb{H}\) and \((\mathbb{R}\mathbb{H})^\perp\) and is, therefore, the whole of \(\mathbb{H}\). Thus, since elements of finite rank are dense in \(\alpha^\uparrow\) and using Theorem 2.4.18 in [8] we see that \(\text{cl}(\mathbb{R} + \mathbb{R}^\mathbb{P}) = \alpha\). We now consider the two cases separately to show that \(\mathbb{R} + \mathbb{R}^\mathbb{P}\) is closed.

(i): Let \(\{a_n + b_n\}\) be a Cauchy sequence in \(\mathbb{R} + \mathbb{R}^\mathbb{P}\) that converges to \(c\) in \(\alpha(a_n \in \mathbb{R}, b_n \in \mathbb{R}^\mathbb{P})\). Let \(E_R\) be the orthogonal projection of \(\mathbb{H}\) onto \(\mathbb{R}\). Then \(\|a_n + b_n - c\| \to 0\) and, since \(E_R\) is a bounded linear operator of \(\alpha\) into itself (a consequence of the first part of the proof of Lemma 3.3.1.), we have \(\|E_R a_n + E_R b_n - E_R c\| \to 0\).

However, \(E_R a_n h = a_n (h \in \mathbb{H})\) and so \(E_R a_n = a_n\) \(E_R b_n = 0\) \((h \in \mathbb{H})\) and so \(E_R b_n = 0\). Thus \(\|a_n - E_R c\| \to 0\). Thus \(\{a_n\}\) is a Cauchy sequence, and, since it is contained in \(\mathbb{R}\), it converges to an element of \(\mathbb{R}\). Thus \(E_R c \in \mathbb{R}\). It is now clear that \(\{b_n\}\) is Cauchy, and, being contained in \(\mathbb{R}^\mathbb{P}\), it converges to an element \(d\) of \(\mathbb{R}^\mathbb{P}\).

Now \(\{a_n + b_n\}\) converges to \(E_R c + d\) and therefore \(c = E_R c + d \in \mathbb{R} + \mathbb{R}^\mathbb{P}\).

Thus \(\mathbb{R} + \mathbb{R}^\mathbb{P}\) is closed.

The socle of \(\alpha\) coincides with the set of elements of finite rank ([8] page 65).
(ii): Again let \( \{a_n + b_n\} \) be a Cauchy sequence in \( R + R^p \) and let its limit be \( c \) \((a_n \in R, b_n \in R^p)\). Let \( K \) be the maximum of 1 and the constant obtained in Lemma 3.3.1. Now given \( \varepsilon > 0 \); there exists \( n, c \in \mathbb{R} \) such that \( \|a_n - c\| < \varepsilon / 2K \); \( a_n \in \mathbb{R} + \cdots + \mathbb{R} \) where \( c = c(n, \varepsilon) \); also there is an \( N \) such that \( \|a_n + b_n - c\| < \varepsilon / K \) for all \( n > N \). Let \( E_n \) be the orthogonal projection onto \( (\mathbb{R}_1 + \cdots + \mathbb{R}_p)(H) \). Then \( a_n = E_n a_n \) and consequently
\[
\|a_n - E_n a_n\| \leq \|a_n - a\| + \|E_n a_n - a\| \leq (\varepsilon / 2K) + (\varepsilon / 2) < \varepsilon.
\]
Now let \( F \) be the orthogonal projection onto any other finite dimensional subspace of \( \mathbb{R} \), and denote by \( (F \cup E_n \varepsilon) \) the orthogonal projection of \( H \) onto \( F + E_n \varepsilon \). Then
\[
\|(F \cup E_n \varepsilon) a_n - a_n\| \leq \|E_n a_n + (F \cup E_n \varepsilon)(1 - E_n \varepsilon)a_n - a_n\|
\leq \|E_n a_n - a_n\| + K \|a_n - E_n a_n\| \leq \varepsilon (1 + K).
\]
Now take \( F \) to be \( E_m \varepsilon \):
\[
\|(E_n \varepsilon \cup E_m \varepsilon) a_n - (E_n \varepsilon \cup E_m \varepsilon)c\| = \|(E_n \varepsilon \cup E_m \varepsilon)(a_n + b_n - c)\| < \varepsilon (n > N).
\]
Then:
\[
\|a_n - a_m\| \leq \|(E_n \varepsilon \cup E_m \varepsilon)a_n - a_m\| + \|(E_n \varepsilon \cup E_m \varepsilon)a_m - a_m\| + \|(E_n \varepsilon \cup E_m \varepsilon)(a_m - c)\|
\]
\[ + \|(E_n \varepsilon \cup E_m \varepsilon)(a_m - c)\| < 2\varepsilon (1 + K) + 2\varepsilon (n, m > N).
\]
Thus \( \{a_n\} \) is Cauchy and as before converges to an element \( a \) of \( R \); then \( \{b_n\} \) is also Cauchy and converges to an element \( b \) of \( R^p \), and \( c = a + b \). Thus \( R + R^p \) is closed.

C(iii) is now immediate in both cases since we already have C(iii)' . We complete the proof by showing that every closed right ideal of \( \mathcal{O} \) is the topological sum of all the minimal right ideals that it contains. Let \( R \) be any closed right ideal of \( \mathcal{O} \) and \( \{R_\lambda : \lambda \in \Lambda\} \) the family of all minimal right ideals of \( \mathcal{O} \) that are contained in \( R \).
Now let a be any element of R: put \( a = b + c \) where \( b \in \cl (\Sigma R_\lambda) \) and \( c \in (\cl (\Sigma R_\lambda))^\perp = \bigcap R_\lambda \). Since \( R \supset \cl (\Sigma R_\lambda) \), \( b \in R \) and thus \( c \in R \cap \bigcap R_\lambda \).

Now let \( h \in H \) and let \( E \) be the orthogonal projection of \( H \) onto \( \{ch\} \). Note that, since \( c \in R \), \( \{ch\} \subset R^H \). Now, by hypothesis, \( E \in \mathcal{A} \). Also \( E \mathcal{A} \) is a minimal right ideal of \( \mathcal{A} \) since each of its elements has rank one (see [8] page 65). Also \( E \mathcal{A} \subset R \); in fact \( E \mathcal{A} = \{ch\} \cap (\mathcal{A}(R)) \), and hence \( E \mathcal{A} \subset R \) by Theorem 2.4.18 in [8]. It follows that \( E \mathcal{A} \) for some \( \lambda \in I \). Now \( c \in R^\perp \), so that \( chR_\lambda H = E \mathcal{A} \subset \{ch\} \). In particular \( \{ch, ch\} = 0 \) and hence \( ch = 0 \). Since \( h \) was arbitrary in \( H \), we have \( c = 0 \). It follows that \( R = \cl (\Sigma R_\lambda) \).

This completes the proof that \( p \) is a right complementor in \( \mathcal{A} \).

In order to obtain a representation of this form for a general semi-simple right complemented algebra we shall extend the definition of continuity to include the general case.

**DEFINITION 3.3.3.** A sequence of minimal right ideals \( \{R_n\} \) of a right complemented Banach algebra \( \mathcal{A} \) is said to be \( p \)-convergent to a minimal right ideal \( R \) if the associated projection operators \( E_{R_n}, E_R \) (notation of definition 2.4.7) are such that \( E_{R_n} \) converges uniformly to \( E_R \) on any minimal left ideal of \( \mathcal{A} \).

It can be seen that, if \( \mathcal{A} \) is \( B^* \), then this definition is equivalent to definition 2.4.7.

**DEFINITION 3.3.4.** A right complementor \( p \) in a Banach algebra is continuous if, for every convergent sequence \( \{a_n\} \) of elements of rank one of \( \mathcal{A} \), the ideals \( a_n \) are \( p \)-convergent to \( a \), where \( a \) is the limit of \( \{a_n\} \).
Again, we see that if $\mathcal{A}$ is $B^*$ then this definition is equivalent to strong continuity, which was shown to be equivalent to the definition of continuity given in Chapter 2.

We shall now proceed by using results of §2 of Chapter 2 to obtain a Hilbert space representation for a primitive Banach algebra that has a continuous right complementor. Such an algebra has minimal one-sided ideals and a dense socle.

We first prove a lemma that is quoted in [4] as a natural extension of the real case which is proved in [6].

**Lemma 3.3.5.** Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are complex linear spaces with dimension greater than two and $A \to A'$ is a one-to-one correspondence between their one-dimensional subspaces that preserves linear independence. Then there exists a one-to-one semi-linear transformation $T$ of $\mathcal{F}_1$ onto $\mathcal{F}_2$ such that $[T(x)] = [x]'$ ($x \in \mathcal{F}_1$).

**Proof.** Fix linearly independent elements $x_0, x_1$ in $\mathcal{F}_1$, and $y_0$ in $[x_0]'$. If the semi-linear mapping $T$ exists, then $T(x_0) = \lambda y_0$ and without loss in generality we can take $\lambda = 1$. Then, if $x + \beta x_0$ \([x]', [x_0]' [x_0-x]'\) are linearly independent and $[x]', [x_0]', [x_0-x]'$ are linearly dependent. Thus $y_0 = y + z$ uniquely where $y \in [x]', z \in [x_0-x]'$. Define $Tx = y$ for such $x$. In particular $Tx_1 = y_1$ is defined, and a similar process using $x_1$ now gives $T(\beta x_0)$.

Let $M$ be a three dimensional subspace of $\mathcal{F}_1$ containing $x_1, x_0, x_1$, and let $M'$ be the three dimensional subspace of $\mathcal{F}_2$ spanned by $\{A': A \subset M \text{ and is 1-dimensional}\}$. The lemma is known for three dimensional subspaces (a collineation between two complex projective planes may be represented analytically by

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*In fact, we use Theorem 1 of [4] and thus render this lemma redundant.*
means of a semi-linear transformation... [7] pg 111-112). Thus there exists a semi-linear transformation $T'$ taking $M$ onto $M'$ and satisfying $[T'x]=[x]'$. But from the uniqueness of $T$ shown above it follows that $T$ must be a scalar multiple of $T'$; thus $T$ is semi-linear on $M$. In particular, the associated automorphism of $C$ is obtained, either by considering $T(\lambda x_o)$, or $T(\lambda x)$ where $x$ was an arbitrary element of $x_1$. Now, taking $M$ to be $\text{lin}(x_o,x,u)$ (arbitrary $x,u$ in $x_1$), we shall obtain the same semi-linear transformation. Thus, for all $x,u$ in $x_1, \lambda, \mu \in C$, $T(\lambda x+\mu u)=\lambda'Tx+\mu'Tu$.

NOTE that use of the Theorem concerning collineations between complex projective planes could have slightly shortened the first part of §2 of Chapter 2. It was not used there as it seemed desirable to exhibit the behaviour of p-projections more clearly.

NOTATION. If $\mathcal{X}$ is a Banach space and $S,T$ two closed subspaces that satisfy $S+T=\mathcal{X}, S\cap T=(0)$, then we shall call the bounded projection of $\mathcal{X}$ into $S$ taking $x=y+z$ ($y\in S, z\in T$) into $y$ the projection by $(S;T)$ of $\mathcal{X}$ onto $S$.

In the case of a $B^*$-algebra that was complemented we used the complementor to induce a new notion of orthogonality into the representing Hilbert space; in order to generalise this we shall define a linear space complementor. A primitive Banach algebra with right complementor has a faithful strictly dense representation on a Banach space. We shall show that a linear space complementor is induced in this space.
DEFINITION 3.3.6. Suppose $\mathcal{X}$ is a Banach space and $S \rightarrow S^p$ is a one-to-one mapping of the set $\mathcal{J}$ of closed subspaces of $\mathcal{X}$ onto itself that satisfies:

L(i): $S \cap S^p = (0)$ ($s \in \mathcal{J}$);
L(ii): $S + S^p = \mathcal{X}$ ($s \in \mathcal{J}$);
L(iii): $S^{pp} = S$ ($s \in \mathcal{J}$);
L(iv): if $S_1 \subseteq S_2$ then $S_1^p \subseteq S_2^p$ ($S_1, S_2 \in \mathcal{J}$);

then $p:S \rightarrow S^p$ is a LINEAR SPACE COMPLEMENTOR (LSC) on $\mathcal{X}$.

A sequence $\{A_n\}$ of one-dimensional subspaces of $\mathcal{X}$ is said to be $p$-CONVERGENT to a closed subspace $A$ if the sequence $\{E_n\}$ is uniformly convergent to $E$, where $E_n$ denotes the projection by $(A_n; A_n^p)$ of $\mathcal{X}$ onto $A_n$ and $E$ denotes the projection by $(A; A^p)$ of $\mathcal{X}$ onto $A$. An (LSC) $p$ is CONTINUOUS if $[x_n]$ is $p$-convergent to $[x]$ whenever $x_n$ converges to $x$ ($x_n, x \in \mathcal{X}$).

We shall now use Theorem 2 of [12]:

THEOREM 3.3.7. If $\mathcal{X}$ is a Banach space that has an LSC defined on it, then an inner product can be introduced in $\mathcal{X}$. The inner product norm is equivalent to the given norm in $\mathcal{X}$.

We do not in fact require the full force of this Theorem, but only that, if $\mathcal{X}$ has a continuous LSC, then $\mathcal{X}$ is reflexive. However, an independent proof of this is not conspicuously short, so it is not given.

Now, if $\mathcal{A}$ is a primitive right complemented algebra, it contains a minimal idempotent $e$. Thus $\mathcal{A}e$ is a minimal left ideal of $\mathcal{A}$. We shall denote it by $\mathcal{L}$. We shall construct an LSC on it in the next Theorem; part of the proof is contained in Lemma 6 of [12]. Note that $\mathcal{L}$ is closed ( [8], Lemma 2.1.10).
NOTATION. For any closed subspace $S$ of $\mathcal{L}$ write $\mathcal{J}(S) = \{a \in \mathcal{L} : a \cdot S \subseteq S\}$, and for any closed right ideal $R$ of $\mathcal{L}$ write $\mathcal{J}(R)$ for $R$.

THEOREM 3.3.8. If $\mathfrak{M}$ is a right complemented primitive Banach algebra and $\mathcal{L} = \mathfrak{M}$ a fixed minimal left ideal of $\mathfrak{M}$, then:

(i): $R \to \mathcal{J}(R)$ defines a 1-1 mapping of the set of all closed right ideals of $\mathfrak{M}$ onto the set of all closed subspaces of $\mathcal{L}$; $\mathcal{J}(\mathcal{J}(S)) = S$; $\mathcal{J}(\mathcal{J}(R)) = R$;

(ii): $\mathcal{J}(\cdot)$, the right complementor in $\mathfrak{M}$, induces an LSC $q$ in $\mathcal{L}$ defined by $S^q = \mathcal{J}(\mathcal{J}(S))$;

(iii): if $\mathcal{J}(\cdot)$ is continuous then $q$ is continuous.

PROOF. (i): it is clear that $\mathcal{J}(S)$ and $\mathcal{J}(R)$ are closed for all $S, R$.

(a) $\mathcal{J}(R) = R \cap \mathcal{L}$: that $\mathcal{J}(R) \subseteq R \cap \mathcal{L}$ is clear; also $x \in R \cap \mathcal{L} \iff x = x \cdot 1 \in R$.

(b) $\mathcal{J}$ is onto: given any closed subspace $S$ of $\mathcal{L}$ consider the closed right ideal $\overline{S \mathfrak{M}}$ of $\mathfrak{M}$. $S = S \cap \mathcal{L}$ and so $\overline{S \mathfrak{M}} \cap \mathcal{L} = S \cap \mathcal{L} = S$; if $z \in \overline{S \mathfrak{M}} \cap \mathcal{L}$ then $z = \lim z_n$, $z_n \in S \cap \mathcal{L}$ and therefore $z = \lim z_n \in S \cap \mathcal{L}$; and therefore $z \in S$ and $\mathcal{J}(\overline{S \mathfrak{M}}) \subseteq S$. Since $S = S \cap \mathcal{L}$, the reverse inclusion is clear and thus $\mathcal{J}(\overline{S \mathfrak{M}}) = S$.

(c) $\mathcal{J}$ is 1-1: suppose $\mathcal{J}(R_1) = \mathcal{J}(R_2) = S$ ($R_1, R_2$ closed right ideals of $\mathfrak{M}$). Then if $a \in R_2$, $a \in S$ and we may write $a = a_1 + a_2$ where $a_1 \in R_1, a_2 \in R_1 \cap \mathcal{L}$. Let $x \in \mathcal{L}$; $ax = a_1 x + a_2 x$, but $ax \in S$ ($a \in R_2$) and $a_1 x \in S$ ($a_1 \in R_1$), and therefore $a_2 x \in R_1 \cap \mathcal{L}$. Thus $a_2 x \in R_1 \cap \mathcal{L} = (0)$. It follows that $a_2 = (0)$ and, since $\mathfrak{M}$ is topologically simple and semi-simple, we have $a_2 = 0$. Thus $R_2 \subseteq R_1$ and the argument is clearly symmetric.

(d) $\mathcal{J}(\mathcal{J}(R)) = R$: for any closed right ideal $R$, $\mathcal{J}(\mathcal{J}(R)) = \mathcal{J}(R \cap \mathcal{L}) \cap \mathcal{L}$; also, if $a \in \mathcal{J}(\mathcal{J}(R))$, write $a = a_1 + a_2$ where $a_1 \in R, a_2 \in R \cap \mathcal{L}$. Then $a_1 = a_1 + a_2 1$ (in $\mathcal{L}$) and, as above, $a_2 = 0$ and therefore $a_2 = 0$.

(e) $\mathcal{J}(\mathcal{J}(S)) = S$: for any closed subspace $S$ of $\mathcal{L}$ there is a
closed right ideal $R$ of $\mathfrak{A}$ such that $R \subseteq \mathcal{F}(R)$, and the result follows applying (a) to $R$.

This completes the proof of (i).

(ii): $L(i)$. If $x \in S \subseteq S^q$, then $f([x]) \subseteq f(S) \cap f(S^q) = \mathcal{F}(S) \cap f(S)^P = \emptyset$.

$L(ii)$. For any $x \in \mathcal{L}$, $x = x_1 + x_2$ where $x_1 \in f(S)$, $x_2 \in f(S)^P$, and this decomposition is unique. Since $x = xe$, we have also $x = x_1 e + x_2 e$; and, since $x_1 \in f(S)$, $x_2 \in f(S)^P$, it follows that $x_1 = x_1 e$, $x_2 = x_2 e$. Thus $x_1 \in S$ and $x_2 \in S^q$.

$L(iii)$ can be verified immediately using (d), (e) above and $L(iv)$ is trivial.

(iii): suppose that $p$ is continuous. Let $\{x_n\}$ be a sequence $(x_n + e)$ in $\mathcal{L}$ that converges to an element $x$ in $\mathcal{L}$; let $E_x$ be the projection by $([x]; [x]^q)$ of $\mathcal{L}$ onto $[x]$, and let $E_{x_n}$ be similarly defined. Then $x_n \mathfrak{A}, x \mathfrak{A}$ are minimal right ideals of $\mathfrak{A}$ and we shall denote them by $R_n, R$. Then let $E_{R_n}, E_R$ be the projections by $(R_n; R_n^P)$ and $(R; R^P)$ respectively of $\mathfrak{A}$ onto $R_n, R$. By Lemma 1.3, $x_n \in R_n, x \in R$ and therefore, since $p$ is continuous, $\{E_{R_n}\}$ converges to $E_R$ uniformly on any minimal left ideal of $\mathfrak{A}$; in particular it converges uniformly on $\mathcal{L}$. Now for any $z$ in $\mathcal{L}$, $z = z_1 + z_2$ where $z_1 = E_R z \in R$, $z_2 = (1 - E_R) z \in R^P$, and from the proof of $L(ii)$ in (ii) $z_1 \in f(R)$, $z_2 \in f(R^P) = [x]^q$. Thus $E_x$ is equal to the restriction of $E_R$ to $\mathcal{L}$, and it can be shown similarly that $E_{x_n}$ is the restriction of $E_{R_n}$ to $\mathcal{L}$.

It follows immediately that $q$ is continuous.

We shall now consider also the algebra $\mathcal{F}$ that is the closure in the operator norm of the algebra of all operators of finite rank on $\mathcal{L}$. We shall write $\mathcal{F}(R)$ for $\overline{\mathcal{L}}(R)$ any closed right
right ideal of $\mathcal{J}$) and $\mathcal{J}_f^{(3)} = \{ A \in \mathcal{J} : ALc_3 \}$ (for any closed subspace $c$).  

**Theorem 3.3.9.** If $\mathcal{M}$ has a right complementor $p$ then $\mathcal{J}$ is dual.

**Proof.** By Theorems 3.3.8 and 3.3.7 we see that $\mathcal{L}$ is a Hilbert space under an equivalent norm, and thus $\mathcal{J}$ is a $B^*$-algebra under an equivalent norm. The result is immediate.

**Theorem 3.3.10.** If $\mathcal{M}$ has a continuous right complementor $p$, then $\mathcal{J}$ has a continuous right complementor $p_e$ that is a natural extension of $p$.

**Proof.** We see from Theorem 3.3.8 that $p$ induces a continuous LSC. $q$ in $\mathcal{L}$. Now for any closed right ideal $R$ of $\mathcal{J}$ define $R^e$ to be $\{ a; a \in \mathcal{J}, aLC_j(R)^q \}$. This is certainly a closed right ideal of $\mathcal{J}$. Also it is clear that $p_e$ satisfies $C(i)$ and $C(iv)$. For a given closed right ideal $R$ of $\mathcal{J}$ let $E$ be the projection by $(\mathcal{J}(R); \mathcal{J}(R)^q)$ of $\mathcal{L}$ onto $\mathcal{J}(R)$. Suppose that $a$ is an arbitrary element of $\mathcal{J}$. Then $a = \lim a_n$ where $\{ a_n \}$ is a sequence of elements of finite rank. Thus $Ea_n$ is operator on $\mathcal{L}$ of finite rank and therefore $Ea_n \in \mathcal{J}$. Thus $\{ E_{a_n} \}$ is a Cauchy sequence in $\mathcal{J}$ and it converges to an element $b$ of $\mathcal{J}$; also, from Theorem 18 in $[1]$, it can be seen that $Ea_n \in R$ and, since $R$ is closed, it follows that $b \in R$. Now it is clear that $\{(1-E)a_n\}$ is a Cauchy sequence, and by the same Theorem, it is contained in $R^e$ and so converges to a limit $c$ in $R^e$. Thus $a = b + c \in R + R^e$. Finally, by another application of Theorem 18 in $[1]$, $R = \mathcal{J}(\mathcal{J}(R))$, and it is then clear that $p_e$ satisfies $C(iii)$. 
It remains to show that \( p_e \) is continuous. Suppose that \( \{a_n\} \)
is a sequence of elements of rank one on \( \mathcal{L} \) and \( a_n \to a \), another
element of rank one on \( \mathcal{L} \). Then the closed right ideals
\( a_n \mathcal{J}, a \mathcal{J} \) are minimal and we wish to show that \( \{a_n \mathcal{J}\} \) is \( p_e \)-convergent to \( a \mathcal{J} \). Every maximal closed right ideal of \( \mathcal{J} \) is
modular (since \( \mathcal{J} \) is dual), and so by Corollary 1.5, every
minimal right ideal contains a unique \( p_e \)-projection. Let
\( f_n, f \) be \( p_e \)-projections satisfying \( f_n \mathcal{J} = a_n \mathcal{J}, f \mathcal{J} = a \mathcal{J} \). It is
clear that multiplication on the left by \( f_n, f \) gives rise to
the projections by \( (a_n \mathcal{J}; (a_n \mathcal{J})^p_e) \), and \( (a \mathcal{J}; (a \mathcal{J})^c) \) respectively
of \( \mathcal{J} \) onto \( a_n \mathcal{J}, a \mathcal{J} \). Thus it will be sufficient to show that
\( |f_n - f| \to 0 \). However, it can also be seen that \( f_n, f \) are the
projections by \( (a_n \mathcal{L}; (a_n \mathcal{L})^q) \) and \( (a \mathcal{L}; (a \mathcal{L})^q) \) of \( \mathcal{L} \) onto \( a_n \mathcal{L}, a \mathcal{L} \).
Theorem 3.3.8 gives the continuity of \( q \) and therefore \( |f_n - f| \to 0 \),
which completes the proof.

This Theorem shows that \( \mathcal{J} \), under an equivalent norm, is a \( B^* \)
-algebra with a continuous right complementor. Thus we may
apply the results of Chapter 2 to \( \mathcal{J} \).

**THEOREM 3.3.11.** If the right complementor \( p \) in \( \mathcal{L} \) is continuous
and the dimension of \( \mathcal{L} \) is at least three, then an inner
product \( \langle, \rangle \) can be introduced into \( \mathcal{L} \) such that
\( p_e = p \langle, \rangle \).

**PROOF.** This is an immediate consequence of Theorem 2.2.18.

**COROLLARY 3.3.12.** The LSC \( q \) in \( \mathcal{L} \) corresponds to orthogonal
complementation with respect to \( \langle, \rangle \).
We can now state the main Theorem of this section.

**THEOREM 3.3.13.** If \( \mathfrak{A} \) is a primitive Banach algebra\(^{†} \) with a continuous right complementor \( p \), then \( \mathfrak{A} \) has a continuous, faithful, strictly dense representation on a Hilbert space \( H \). For any closed right ideal \( R \) of \( \mathfrak{A} \), \( R^p = \{ \sum \alpha_i T_a H : \alpha_i \in H, heT_a H; a \in R \} \) (\( a \rightarrow T_a \) denotes the representation). The socle of \( \mathfrak{A} \) consists of all elements of \( \mathfrak{A} \) whose image under the representation is of finite rank on \( H \); this image is generated by the set of all operators of the form \( x \otimes y \) where \( x \) ranges through \( H \) and \( y \) ranges through a dense subspace \( H_0 \) of \( H \). Further, if \( E \) is any orthogonal projection on \( H \), then for any \( a \) in \( \mathfrak{A} \) there is an element \( b \) in \( \mathfrak{A} \) such that \( ET_a = T_b \).  

**PROOF.** Let \( e \) be a fixed minimal idempotent of \( \mathfrak{A} \) and \( \mathcal{L} \) the minimal left ideal \( e \mathfrak{A} \) of \( \mathfrak{A} \). Then by Theorem 3.3.8 \( e \mathfrak{A} \) induces a continuous LSC in the Banach space \( \mathcal{L} \). Now by Corollary 3.3.12 there is an inner product \((,\)\) in \( \mathcal{L} \) such that the LSC \( q \) corresponds to orthogonal complementation with respect to \((,\)\) and the inner product norm in \( \mathcal{L} \) is equivalent to the original norm. \( H \) is the Hilbert space whose elements are those of \( \mathcal{L} \) and whose inner product is \((,\)\). Let \( a \rightarrow T_a \) be the representation of \( \mathfrak{A} \) on \( H \) corresponding to the left regular representation on \( \mathcal{L} \). Now from Theorem 2.4.12 in [3] the left regular representation of \( \mathfrak{A} \) on \( \mathcal{L} \) is continuous and faithful, so it follows that \( a \rightarrow T_a \) is continuous and faithful. Also the socle of \( \mathfrak{A} \), which is dense from Lemma 3.2.1, maps onto the subalgebra of \( B(H) \) generated by the elements of the form \( x \otimes y \) where \( x \) ranges through \( H \) and \( y \) ranges through a subspace of \( H \). This subspace, which we shall denote by \( H_0 \), is \( \{ y \in H : x \otimes y = T_a \text{ for some } a \in \mathfrak{A} \} \). Now we see from Theorem 3.3.8 (i), (ii)

\( ^{†} \) Insert: (whose minimal left ideals have dimension at least three)
that \( R^P = \{ \mathcal{S}(R)^q \} = \{ \mathcal{S}(R)^P \} \) as required.

In order to simplify the notation we shall prove the remainder of the Theorem for an algebra of operators of this form on a Hilbert space. It is clearly sufficient to do this. We show first that, if \( \mathcal{A} \) is the given algebra and \( H \) the Hilbert space and \( R \) any orthogonal projection on \( H \), then \( \mathcal{E}(\mathcal{A}). \) Let \( S \) be the range of \( R \); and let \( R = \mathcal{J}(S) \); then \( R^P = \mathcal{J}(S^P). \)

Then for any \( a \) in \( \mathcal{A} : a = a_1 + a_2 \) where \( a_1 \in R, a_2 \in R^P \), and also ah = a_1 h + a_2 h (\( h \in H \)), and \( a_1 h \in S, a_2 h \in E \); but also \( ah = Eah + (1 - E)ah \) and therefore \( Ea = a_1 \in \mathcal{A}. \)

We now prove that \( H_0 \) is dense in \( H \). Let \( S = \alpha_1 (H_0) \). If \( S \neq H \) then there exists an element \( x^+ \) in \( S^P \). Now any operator of \( \mathcal{A} \) of finite rank is of the form \( \sum_{i=1}^{n} (x_i \otimes y_i) \) and we can assume that the set \( \{ x_i \} \) is linearly independent. Let \((1 - E) \) be the orthogonal projection of \( H \) onto the subspace generated by \( \{ x_j : j = 1, 2, \ldots, n \} \). Then \( \mathcal{E}(\sum_{i=1}^{n} (x_i \otimes y_i)) \in \mathcal{A}, \) i.e. \( \mathcal{E} x_1 \otimes y_1 \in \mathcal{A} \) and thus \( y_1 \in H_0 \). Therefore \( \sum_{i=1}^{n} (x_i \otimes y_i) x = 0. \) However operators of finite rank are dense in \( \mathcal{A} \) with respect to \( \| \| \); thus, given \( a \in \mathcal{A}, \) there exists a sequence \( \{ a_n \} \) of elements of finite rank such that \( \| a - a_n \| < \epsilon; \) thus \( \| a - a_n \| < \epsilon \) and, since \( a_n x = 0 \) for all \( n, \) it follows that \( ax = 0 \) for all \( a \) in \( \mathcal{A} \) which is a contradiction.

Therefore \( S = H. \)

This Theorem shows that every primitive Banach algebra with a continuous right complementor has a continuous representation as an algebra of the kind given in Theorem 3.3.2 (1); and that this representation preserves the right complementor. In the next Theorem we give a converse to this. Condition (ii) of Theorem 3.3.2 will be discussed in §5, §6.
We first give as a Corollary an alternative characterisation of these algebras. \( \mathcal{O}_0 \) is the completion of \( \mathcal{O} \) with respect to \( \| \cdot \| \).

**Corollary 3.3.14.** If \( \mathcal{O} \) is a primitive Banach algebra with a continuous right complementor, then \( \mathcal{O} \) may be embedded in a \( B^\mathcal{O} \)-algebra \( \mathcal{O}_0 \). The image \( I(\mathcal{O}) \) is a dense subalgebra of \( \mathcal{O}_0 \), and, if \( ^* \) denotes the natural complementor of \( \mathcal{O}_0 \) and \( R \) is any closed right ideal of \( \mathcal{O} \), then \( \text{cl}I(R)' \cap I(\mathcal{O}) = I(R^P) \).

**Theorem 3.3.14.** Let \( \mathcal{O} \) be a strictly irreducible algebra of operators on a Hilbert space \( H \) that is a Banach algebra under a norm \( \| \cdot \| \) that majorises the operator norm \( \| \cdot \| \). Suppose also that operators of finite rank are dense in \( \mathcal{O} \), and that \( \mathcal{O} \subseteq \mathcal{A}(H) \) for any orthogonal projection \( E \) on \( H \). Then the map \( p : R \to R^P = \{ [x] (R)^P \mid x \in \mathcal{O}(H) \} \) is a continuous right complementor on \( \mathcal{O} \).

**Proof.** That \( p \) is a right complementor is immediate from Theorem 3.3.2. We show that it is continuous. Let \( \{ R_n : n = 0, 1, \ldots \} \) be a set of minimal right ideals of \( \mathcal{O} \) and suppose there are elements \( a_n \in R_n \) such that \( a_n \to a \neq 0 \). Then we must show that \( \{ R_n \} \) is \( p \)-convergent to \( R_0 \). Let \( E_n \) be the projection by \( (R_n; R_n^P) \) of \( \mathcal{O} \) onto \( R_n \). Now, since the norm \( \| \cdot \| \) majorises the operator norm, we see that \( \| a_n - a \| \to 0 \). Let \( \mathcal{F} \) be the algebra \( K(H) \); then the natural complementor in \( \mathcal{F} \) is continuous by Theorem 2.4.8. Now let \( S_n \) be the closed subspace \( \mathcal{O}(R_n) \) of \( H \) and \( P_n \) the orthogonal projection onto \( S_n \). Now it can easily be verified that multiplication on the left by \( P_n \) gives rise to the operator \( E_n \) on \( \mathcal{O} \) and to the projection by \( (a_n \mathcal{F}, (a_n)' \) \) of \( \mathcal{F} \) onto \( a_n \mathcal{F} \); then, by the continuity of \( \cdot \) in \( \mathcal{F} / P_n E/P \),

Let \( \mathcal{L} \) be a minimal left ideal of \( \mathcal{O} \); then \( \mathcal{L} = \{ x \in \mathcal{A}(H) \mid x \mathcal{E} \text{ fixed and } x \in \mathcal{H} \text{ variable} \} \) and, by Lemma 2.4.13 in [3], the map
x \& g \mapsto x is a bi-continuous isomorphism of \( L \) onto \( H \); also 
\( E_n(x \& g) \mapsto P_n x \). It is now clear that \( \{E_n\} \) converges uniformly 
to \( E_0 \) on any minimal left ideal. Thus \( p \) is continuous.

We now specialise to the infinite dimensional case.

**Theorem 3.3.16.** If \( \mathcal{A} \) is a primitive Banach algebra whose 
minimal left ideals are infinite dimensional, then if \( p \) is a right complementor on \( \mathcal{A} \), \( p \) is continuous.

**Proof.** Construct a LSC \( q \) in \( L \) and a right complementor \( p_e \) 
in \( \mathcal{A} \) as in Theorems 3.3.8 and 3.3.10 (the relevant part of 
the proof of Theorem 3.3.10 does not use the continuity of \( p \)).

Now, since \( L \) is infinite dimensional, \( p_e \) is continuous. Then 
let \( \{a_n\} \) be a sequence of elements of \( \mathcal{A} \) that are contained 
in minimal right ideals \( R_n \); suppose \( \{a_n\} \) converges to \( a \) and that \( a \mathcal{A} \) is a minimal right ideal \( R \). Let \( S = \mathcal{A}(R), S_n = \mathcal{A}(R_n) \).

Let \( E, E_n \) be the projections by \( (S; S^q), (S_n; S_n^q) \) of \( L \) onto 
\( S, S_n \). Then multiplication on the left by \( E, E_n \) in \( B(L) \) gives 
in \( \mathcal{A} \) the projections by \( (J_f(S), (J_f(S))^p), (J_f(S_n), (J_f(S_n))^p) \) 
of \( \mathcal{A} \) onto \( J_f(S), J_f(S_n) \); also the left regular representation 
gives a mapping \( a \mapsto \mathcal{A} \) of \( \mathcal{A} \) into \( \mathcal{A} \) and this is continuous. Thus 
\( T_{a_n} \in J_f(S_n) \) and \( T_{a} \in J_f(S) \) and \( |T_{a_n} - T_{a}| \rightarrow 0 \); then, by the continuity 
of \( p_e \), we see that \( |E_n - E| \rightarrow 0 \). However, \( E_n, E \) are easily seen to 
be equal to the restrictions to \( L \) of the projections by 
\( (R_n, R_n^p) \) and \( (R, R^p) \) of \( \mathcal{A} \) onto \( R_n, R \); now, since \( |E_n - E| \rightarrow 0 \), we 
see that these projections are uniformly convergent on \( L \).

The result follows since \( L \) is arbitrary.
4. REPRESENTATION THEOREMS IN THE SEMI-SIMPLE CASE.

We begin this section by showing that, if \( A \) is a semi-simple Banach algebra with a right complementor \( p \) and \( \{ f_{\lambda} : \lambda \in \Lambda \} \) is the family of all minimal closed two-sided ideals of \( A \), then \( p \) is continuous if and only if all the induced right complementors \( p_\lambda \) in \( \mathfrak{F}_\lambda \) are continuous. We then proceed to extend the results for the primitive case to this general case by methods analogous to, and suggested by, \( \mathfrak{F}_3 \) of Chapter 2.

The situation here is more complicated for, if \( E_\lambda \) denotes the projection by \( (f_{\lambda} : f_{\lambda}^p) \) of \( A \) onto \( \mathfrak{F}_\lambda \), we knew in the \( B^* \) case that the norm of \( E_\lambda \) as an operator on \( A \) was 1; in the present case, however, we are unable to give an upper bound to these norms. Equally, we cannot give an example of a case in which no such bound exists.

**Theorem 3.4.1.** The right complementor \( p \) is continuous if and only if the right complementor \( p_\lambda \) is continuous for each \( \lambda \).

**Proof.** Suppose \( \{ R_n : n=0,1, \ldots \} \) is a set of minimal right ideals of \( A \). Then each \( R_n \) is contained in a minimal closed two-sided ideal of \( A \) (proof of Lemma 2.3.3 in \( [8] \)); denote by \( \mathfrak{F}_n \) this minimal closed two-sided ideal. Suppose also that \( \{ a_n \} \) is a sequence in \( A \) that satisfies \( a_n \in R_n \), \( a_n \to a \) as \( n \to \infty \).

We show that there exists \( N \) such that, for \( n>N \), \( a_n \in \mathfrak{F}_n \); suppose the contrary, if possible. Then, taking a sub-sequence if necessary, we have a sequence \( a_n \in \mathfrak{F}_n \), \( a_n \to a \), \( a_n = a_n \) as \( n \to \infty \).

Let \( t \in \mathfrak{F}_n \); \( \| ta \| = \| t(a-a_n) \| < \| t \| \| a-a_n \| \); and letting \( n \to \infty \) we have \( ta = 0 \). Thus \( \mathfrak{F}_0 a = (0) \) and hence, by the topological simplicity

\[ a_n \neq 0 \]
of \( J_0 \), we have \( a = 0 \), which is the required contradiction. Thus we can assume that the sequence \( \{ R_n \} \) is contained in \( J_\lambda \) for some \( \lambda \) in \( A \); then, if \( p_\lambda \) is continuous, \( \{ R_n \} \) is \( p_\lambda \)-convergent, and hence \( p_\lambda \)-convergent to \( R_0 \) (if \( L \) is a minimal left ideal of \( \mathfrak{a} \) that is not contained in \( J_\lambda \), then the projection by \( (R, R^D) \) of \( \mathfrak{a} \) onto \( R \) is zero on \( L \) for \( R \subset J_\lambda \), and, if \( L \subset J_\lambda \), then the restrictions to \( L \) of the projection by \( (R, R^D) \) of \( \mathfrak{a} \) onto \( R \) and the projection by \( (R, R^D) \) of \( J_\lambda \) onto \( R \) are equal). It follows that \( p_\lambda \) is continuous if each \( p_\lambda \) is continuous. The converse is clear, since a sequence of minimal right ideals of \( J_\lambda \) is also a sequence of minimal right ideals of \( \mathfrak{a} \).

NOTATION. Let \( R \) be any closed right ideal of \( \mathfrak{a} \). Then \( R_\lambda \) will denote \( R \cap J_\lambda \); \( E_R \) will denote the projection by \( (R; R^D) \) of \( \mathfrak{a} \) onto \( R \); \( E_\lambda \) will denote \( E_\lambda (R, R^D) \).

THEOREM 3.3.2. For any \( a \) in \( \mathfrak{a} \): \( E_\lambda (E_R a) = E_{R_\lambda} a \).

PROOF. We have \( a = E_R a + (1 - E_R) a \), and thus \( E_\lambda a = E_\lambda (E_R a) + E_\lambda (1 - E_R) a \). Now by Theorem 3.2.6 \( E_R \) is a dense subspace of \( E_\lambda \mathfrak{a} \); and by proposition 3.2.5 (i) \( E_\mu (\Sigma R_\lambda) = R_\mu \). Thus, since \( E_\lambda \) is continuous and \( R_\lambda \) is closed, it is clear that \( E_\lambda (E_R \mathfrak{a}) = R_\lambda \).

Similarly, since \( \Sigma R_\lambda \) is a dense subspace of \( (1 - E_R) \mathfrak{a} \) and \( E_\mu (\Sigma R_\lambda) = R_\mu \), we see that \( E_\lambda (1 - E_R) \mathfrak{a} = R_\lambda \).

Thus \( E_\lambda a = E_\lambda E_R a + E_\lambda (1 - E_R) a \) and \( E_\lambda a = E_{R_\lambda} E_\lambda a + (1 - E_{R_\lambda}) E_\lambda a \), and in these expressions \( E_\lambda E_R a, E_{R_\lambda} E_\lambda a \in R_\lambda \); \( E_\lambda (1 - E_R) a, (1 - E_{R_\lambda}) E_\lambda a \in R_\lambda \).

Thus, by the uniqueness of the decomposition \( p_\lambda \), we have
\[ E_R a = E_R a = E_R (a - (1 - E_R) a) = E_R a. \]

**COROLLARY 3.4.3.** Let \( \mathcal{A} \) be any subset of \( \Lambda \). For each \( \lambda \in \mathcal{A} \) let \( S_\lambda \) be a minimal closed right ideal of \( \mathcal{A} \) contained in \( \mathcal{J}_\lambda \). Let \( U_\lambda \) be the projection by \((S_\lambda; S_\lambda^D)\) of \( \mathcal{A} \) onto \( S_\lambda \); and let \( \|U_\lambda\| \) denote the operator norm of the restriction of \( U_\lambda \) to \( \mathcal{J}_\lambda \). Then \( \sup \{\|U_\lambda\| : \lambda \in \mathcal{A} \} < \infty. \)

**PROOF.** Let \( R = \text{cl}(\Sigma U_\lambda \mathcal{A}) \). Then \( R \) is a closed right ideal of \( \mathcal{A} \) and \( E_R \) is a bounded operator; let \( M \) be its bound. Then, if \( \alpha \in \mathcal{J}_\lambda \), \( E_R a = E_R a \) and \( E_\lambda (1 - E_R) a = (1 - E_R) a. \)

Also \( a = E_R a + (1 - E_R) a \) and \( E_\lambda a E_R c R; \) we shall show that \((1 - E_R) a \in \mathcal{J}_\lambda \), from which we can deduce that \( E_R a = E_R a. \)

Now \((1 - E_R) a = E_\lambda (1 - E_R) a = E_\lambda (1 - E_R) a \in \mathcal{J}_\lambda \cap R \mathcal{A} \subset R \mathcal{A} \subset R \mathcal{A} \).

Also \( E_R = U_\lambda \) since \( E_R \mathcal{A} = E_\lambda E_R \mathcal{A} = E_\lambda (\text{cl}(\Sigma U_\lambda \mathcal{A})) = U_\lambda \mathcal{A} \), and \( U_\lambda = E_\lambda \mathcal{A} \).

Therefore \( E_R \mathcal{J}_\lambda = U_\lambda \mathcal{J}_\lambda \), and it follows that the operator norm of \( U_\lambda \) on \( \mathcal{J}_\lambda \) is \( \leq M. \)

Now assume any 1-sided ideal of \( \mathcal{A} \) has dimension \( \geq 2. \)

**LEMMA 3.4.4.** If \( p_\lambda \) is any right complementor in \( \mathcal{J}_\lambda \) and \( R \rightarrow \text{cl}(\Sigma P_\lambda) \) is a continuous right complementor in \( \mathcal{J}_\lambda \), then we can select minimal left ideals \( L_\lambda \) in \( \mathcal{J}_\lambda \) and induce inner products \( (\cdot, \cdot)_\lambda \) in \( L_\lambda \) such that \( p_\lambda = p(\cdot, \cdot)_\lambda \) and there is a finite constant \( M \) such that \( \|x\|^2 (x, x)_\lambda \leq M^2 \|x\|^2 \) for all \( x \) in \( L_\lambda \) and all \( \lambda \) in \( \Lambda. \)

**PROOF.** Each \( p_\lambda \) is continuous by Theorem \( 3.4.1 \) and, since the dimension of any minimal left ideal in \( \mathcal{J}_\lambda \) is at least three, we may take any minimal left ideal \( L_\lambda \) in \( \mathcal{J}_\lambda \) and induce an inner product \( (\cdot, \cdot)_\lambda \) in \( L_\lambda \) such that \( p_\lambda = p(\cdot, \cdot)_\lambda \) and the inner product norm is equivalent to the original norm, (from Theorem \( 3.3.11 \)).
Also this situation will be unaltered by multiplication of \((,)_\lambda\) by any positive real constant. So choose \((,)_\lambda\) to satisfy \(\|<|_\lambda\| = \|<\|\) where \(\|\|_\lambda\) is the inner product norm. Suppose that with this choice of inner products it is not possible to find the required constant \(M\). Then there exists a sequence \(\{\lambda_n\}\) such that \(\|\lambda_n\| \leq n\). Let \(x_n\) be an element of \(L_{\lambda_n}\) such that \(\|x_n\| = 1\) and \((x_n, x_n)_\lambda = k_n^2 > n^2\). Also there are elements \(x_{on}\) in \(L_{\lambda_n}\) such that \(\|x_{on}\| = 1\), \((x_{on}, x_{on})_\lambda = 2\). Now \(x_{on} = \lambda x_n + x_n'\) where \(x_n'\) is \(x_n\); then \((x_{on}, x_{on})_\lambda = |\lambda|^2 k_n^2 + (x_n', x_n')_\lambda\), and therefore \((x_n', x_n') < 2\) and \(\|x\| < \sqrt{2}/k_n < \sqrt{2}/n\); thus \(\|x_n'\| > \|x_{on}\| - |\lambda| \|x_n\| > 1 - \sqrt{2}/n\). (We omit the subscript \(\lambda_n\) where no ambiguity can arise.)

We shall consider the subspace of \(L_{\lambda_n}\) generated by \(y_n = \frac{1}{k_n} x_n + x_n'\). Let \(R_n\) be the closed right ideal \(\langle (Ly_n) \rangle\) of \(\mathcal{J}_{\lambda_n}\); then \(R_n\) is also a closed right ideal of \(\mathcal{J}\) and we shall denote by \(U_n\) the projection by \((R_n; R_n^\perp)\) of \(\mathcal{J}\) onto \(R_n\). Then, if \(\|U_n\|_n\) denotes the operator norm of the restriction of \(U_n\) to \(\mathcal{J}_{\lambda_n}\), we see from Corollary 3.4.3. that \(\sup_n \|U_n\|_n = M\) (I).

From the work in Chapter 2 it is clear that \(U_n\) restricted to \(L_{\lambda_n}\) is equal to the operator \(y_n \otimes y_n / (y_n, y_n)\n\). Then \(\|U_n\|_n \geq \|U_n x_n\| = \|y_n \otimes x_n / (y_n, y_n)\| = \|(x_n, y_n)\|_n / (y_n, y_n)\) .

However \(y_n = (1/k_n) x_n + x_n'\), and so \((y_n, y_n) \leq 3\); \((x_n, y_n) = k_n > n\); and \(\|y_n\| > k_n' \|1/k_n x_n\| > 1 - \sqrt{2}/n - 1/n^2\). Therefore \(\|U_n\|_n > (n/3)(1 - 2/n - 1/n^2)\), which contradicts (I).
We are now able to give our representation theorem.

**Theorem 3.4.5.** If \(\mathcal{A}\) is a semi-simple Banach algebra each of whose minimal one-sided ideals has dimension at least three and if \(p\) is a continuous right complementor on \(\mathcal{A}\), then \(\mathcal{A}\) has a faithful continuous representation \(a \rightarrow T_a\) on a Hilbert space \(H\). For any closed right ideal \(R\) of \(\mathcal{A}\):

\[
\{h: h \in T_aH, a \in R\} = \overline{\{h: h \in T_aH, a \in R\}}; \quad R^p = \{a: T_aH \subseteq \{h: h \in T_aH, a \in R\}\}
\]

**Proof.** Let \(\{\mathcal{J}_\lambda: \lambda \in \Lambda\}\) be the family of all minimal closed two-sided ideals of \(\mathcal{A}\). Select minimal left ideals \(L_\lambda\) in \(\mathcal{J}_\lambda\) as in Lemma 3.4.4, and induce inner products \((,)_\lambda\) in \(L_\lambda\) as in the same Lemma. Then each \(L_\lambda\) is a minimal left ideal of \(\mathcal{A}\). We shall construct the direct sum of the left regular representations \(a \rightarrow T_a^\lambda\) of \(\mathcal{A}\) on \(L_\lambda\). Let \(H\) be the sum \(\sum (L_\lambda, (,)_\lambda)\) in the notation of [8], page 197. \(H\) is the collection of all elements \(f\) of the direct sum of the Hilbert spaces \(L_\lambda\) that satisfy \(|f|^2 = (\sum_{\lambda \in \Lambda} |f(\lambda)|^2)_\infty\). \(H\) is a Hilbert space with inner product \((f, g) = \sum_{\lambda \in \Lambda} (f(\lambda), g(\lambda))_\lambda\). Now for each \(a\) in \(\mathcal{A}\) define the operator \(T_a\) by:

\[
(T_a f)(\lambda) = T_a^\lambda f(\lambda).
\]

Now \(\|T_a^\lambda f(\lambda)\| \leq \|f(\lambda)\| \leq M a \|f(\lambda)\|_{\lambda}\); and thus for any \(f\) in \(H\)
\[
\Sigma |(T_a f)(\lambda)|^2 \leq M^2 \Sigma \|T_a f(\lambda)\|_{\lambda}^2 \leq M^2 a \|f(\lambda)\|_{\lambda}^2 \infty; \quad \text{and thus \(T_a f \in H\).}
\]

The direct sum of the representations \(a \rightarrow T_a^\lambda\) is defined. Also \(\|T_a\| \leq M \|a\|\), so that the representation \(a \rightarrow T_a\) is continuous.

To prove the remainder we first observe that \(T_a^\lambda = T_{E_\lambda} a^\lambda\).

Also, if \(R \subseteq \mathcal{J}_\lambda\), then \(\{R^p\lambda\} = \{R\} \cap L_\lambda\), where \(\{R\}\) denotes \(\overline{\{h: h \in T_aH, a \in R\}}\). Now let \(R\) be any closed right ideal of \(\mathcal{A}\). From Theorem 3.2.6 \(R = \overline{\{R^p\lambda: \lambda \in \Lambda\}}\) and \(R^p = \overline{\{R^p\lambda: \lambda \in \Lambda\}}\):
and since the operator norm of $T_a$ is majorised by $||a||$, it follows that \( \{ h : \alpha T_a H, \text{ } a \in \Sigma R_\lambda \} \) and \( \{ h : e \in T_a H, \text{ } a \in \Sigma R_\lambda H \} \) are dense subspaces of $J(R)$, $J(R^P)$ respectively.

Thus $J(R) = \{ h : h(\lambda_1) \in J(\Sigma R_\lambda), h(\lambda) = 0, \lambda \neq \lambda_1 \}; \text{ all finite sets } \lambda_1, \text{ } \lambda_2 \}
= \{ h : h(\lambda) \in J(\Sigma R_\lambda) \}
= \{ h : h(\lambda) \in J(\Sigma R_\lambda) \}
= \text{cl} \{ h : h \in H, \text{ } a \in \Sigma R_\lambda \}
= J(R^P).

We now show that the representation is faithful:
suppose $a \in \mathfrak{d}$ and $T_a = 0$; then by Lemma 1.3, $a \in \Sigma R_\lambda = R$, and it can be verified that $R = E_\lambda a_\lambda$. Now, since $R = \text{cl}(\Sigma R_\lambda)$, given any $\varepsilon > 0$, there is a set \( \{ \lambda_1, \ldots, \lambda_n(\varepsilon) \} \) and elements $b_i \in \Sigma R_\lambda$ such that $||a - (b_1 + \ldots + b_n)|| < \varepsilon/2$; also $b_i \in E_\lambda a_\lambda$, and so there exist elements $c_i$ such that $||b_i - E_\lambda a_\lambda c_i|| < \varepsilon/2n$.

Now $||a - (E_\lambda a_\lambda c_1 + \ldots + E_\lambda a_\lambda c_n)|| < ||a - (b_1 + \ldots + b_n)|| + \sum ||b_i - E_\lambda a_\lambda c_i|| < \varepsilon.$

However, $E_\lambda a_\lambda = 0$, since $T_{E_\lambda a_\lambda} = 0$ and the restriction to $\mathcal{J}_\lambda$ of $a \to T_a \lambda$ is faithful.

Therefore $||a|| < \varepsilon$; thus $a = 0$ as required.

Finally we show that if $T_a \in J(R^P)$, then $a \in R^P$. Suppose there is some $a$ for which this is not so; then put $a = a_1 + a_2$ where $a_1 \in \mathfrak{d}$, $a_2 \in R^P$; then, for any $h \in H$, $T_{a_1} - T_a h^+ h + T_{a_2} h$, and thus,

since $J(R^P) = J(R^P)$, $T_{a_1} h \notin J(R^P) \cap J(R^P) = 0$; it follows that $T_{a_1} = 0$ and, since the representation is faithful, $a_1 = 0$ and $a = a_2 \in R^P$. 

In the next Theorem we show that, if a right complementor in a semi-simple Banach algebra can be expressed in this form, then it is continuous.

**THEOREM 3.4.6.** Let $\mathfrak{A}$ be a semi-simple algebra of operators on a Hilbert space $H$ that is a Banach algebra under a norm $\|\|$ that majorises the operator norm $\|\|$. Suppose $R \to R^p = \{a: aH \subseteq RH\}^+$ is a right complementor in $\mathfrak{A}$. Then it is continuous.

**PROOF.** Let $\{\mathfrak{J}_\lambda: \lambda \in \Lambda\}$ be the family of all minimal closed two-sided ideals of $\mathfrak{A}$. Then $\mathfrak{A}$ is equal to the topological sum of the ideals $\mathfrak{J}_\lambda$ and in each of these a right complementor $p_\lambda$ is induced. By Theorem 3.4.1 $p$ is continuous if each $p_\lambda$ is continuous. Let $\mathfrak{J}_\lambda H = H$. Let $R$ be any closed right ideal of $\mathfrak{J}_\lambda$:

then $R^p \cap \mathfrak{J}_\lambda = \{a: aH \subseteq RH\}^+$. However, if $aH \subseteq RH_\lambda$, put $a = a_1 + a_2$, $a_1 \in H_\lambda$, $a_2 \in \mathfrak{J}_\lambda^p$; then $a_2H_\lambda \cap H_\lambda^+ = (0)$, and thus $a_2 = 0$ and $a \in \mathfrak{J}_\lambda$.

Therefore, if $\mathfrak{J}_\lambda$ denotes orthogonality in the Hilbert space $E_\lambda$, $R^p = \{a: aH \subseteq RH\}^+$, and then it can be seen from the proof of Theorem 3.3.14 that $p$ is continuous.

**THE OPERATORS $E_\lambda$.**

It remains an open question whether there is a constant $M$ such that $|E_\lambda| \leq M$ for all $\lambda$ in $\Lambda$. From Theorem 3.4.5 we see that for any $a$ in $\mathfrak{A}$ there is only a countable set $\{\lambda_i\}_i < \Lambda$ for which $E_\lambda a \neq 0$. An equivalent question to the above is whether each $a$ may be expressed in the form $a = \sum_{i} E_{\lambda_i} a$. It is clear that if this limit is in $\mathfrak{A}$, then it is equal to $a$, but in general we can only say that there is a set $a_{ij} \in E_{\lambda_i} \mathfrak{A}$ such that $\|\sum_{j=1}^{k_j} a_{ij} - a\| < 1/j$ and $a_{ij} \to E_{\lambda_i} a$ as $j \to \infty$. 
5. ANNIHILATOR CONDITIONS.

We begin this section by examining the form the representation of Theorem 3.3.13 takes if the algebra $\mathcal{A}$ is an annihilator algebra. We show that a simple algebra of compact operators on a Hilbert space that is a Banach algebra under a norm $\|\|$ that majorises the operator norm $\|\|$ and has a continuous right complementor $p$ is an annihilator algebra if and only if it contains all operators of finite rank on $H$. We show further that, if such an algebra is either right or left annihilator, then it is an annihilator algebra; moreover it will then be dual. Some of these results are capable of complete extension to the semi-simple case; a semi-simple Banach algebra with a continuous right complementor $p$ is an annihilator algebra if it is either a left annihilator or a right annihilator algebra; in order to obtain the stronger result concerning its duality we must include in the hypothesis the condition that some left complementor can also be defined on $\mathcal{A}$. There is a natural candidate for a left complementor in a right complemented annihilator algebra: $L \to L_p^D$ for any closed left ideal of $\mathcal{A}$. At the end of this section we examine the related questions of whether this is a left complementor and whether $\mathcal{A}$ is self-adjoint with respect to the partial involution defined in $\mathcal{A}$ by the involution in the representing algebra $\mathcal{B}(H)$.

Now let $\mathcal{A}$ be a simple Banach algebra with a continuous right complementor $p$. Let $L=\mathcal{O}$ be a minimal left ideal of $\mathcal{O}$ and $e$ a $p$-projection. Then an inner product can be induced in $L$ so that $L$ is a Hilbert space, and it is clear that by suitable scalar multiplication we can have $(e,e)=1$.

In future we shall assume that the dimension of $L$ is at least three.
LEMMA 3.5.1. If \( a \mapsto T_a \) denotes the representation of \( \sigma \) on the Hilbert space \( L \) given by the left regular representation of \( \sigma \) on \( L \) as in Theorem 3.3.13, then \( T_a = a \otimes e \) for all \( a \) in \( L \).

PROOF. If \( a \in L \) and \( b \) is an arbitrary element of \( L \), then \( ab = aebe = \lambda a \) for some \( \lambda \in F \), and thus \( T_a \) is an operator of rank one; let \( p \in L \) satisfy \( T_a = a \otimes p \). Now, since \( L \) is a minimal left ideal of \( \sigma L \), it follows that \( \{ T_a : a \in L \} = \{ b \otimes p : b \in L \} \).

Now \( ae = Ta = (a \otimes p)e = (e, p)ae \) for all \( a \) in \( L \), and thus \( (e, p) = 1 \).

Now let \( R = \sigma L \), \( R^0 = (1 - e) \sigma L \); then \( z \in R^0 \cap \sigma e \) \( \gamma (e, p)z = ez = 0 \) and therefore \( (v, z) = 0 \).

Then write \( p = f + g \) where \( f \in Re = \{ e \}, g \in R^0 \sigma e = e^+ \); \( (e, p) = (e, f) \) and so \( (e, f) = 1 \) and \( f = e \); also \( g \in R^0 \sigma e \) and so \( (v, g) = 0 \); and therefore \( (e, f) = (g, p) - (g, f) = 0 \) and so \( g = 0 \) and \( p = e \).

PROPOSITION 3.5.2. The following are equivalent:

(i) the image of \( \sigma \) contains all operators of finite rank on \( L \);

(ii) there is a scalar \( \lambda \) such that, if \( x, y \in L \) and there is an \( a \) in \( \sigma L \), \( T_a = x \otimes y \), then \( \| a \| < \lambda \| x \| \| y \| \); 

(iii) \( \sigma \) is left annihilator.

PROOF. (ii) \( \Rightarrow \) (i): Let \( \sigma L = \{ x \in L : x \otimes y = T_a \} \) for some \( a \in \sigma L \), all \( x \) in \( L \), and for \( y \) in \( L^2 \), define \( \| y \|_* = \| a \| \) where \( T_a = e \otimes y \).

Then it can be seen from Lemma 2.4.11 in [8] that \( \sigma L \) is complete under the norm \( \| . \|_* \). Write \( \| T_a \| \) for \( \| a \| \).

Then for any \( x \) in \( L \), \( (x \otimes x)(x \otimes x) = (x, x)(x \otimes x) \), and therefore \( \| x \|^2 = \| x \otimes x \| = \| (x \otimes e)(e \otimes x) \| < \| x \| \| x \|_x \). However, the inner product norm \( \| . \| \) is equivalent to the original norm in \( L \), and therefore there is some scalar constant \( \beta \) such that \( \| x \| \| x \| < \beta \| x \| \) \( (x \in L) \).

Also by hypothesis, \( \| x \otimes y \| < \lambda \| x \| \| y \| \), and putting \( x = e \), we have \( \| y \| < \lambda \| y \| \). Thus \( \| . \| \) and \( \| . \|_* \) are equivalent on \( \sigma L \). Therefore
\[ \mathcal{L}^\omega \] is complete with respect to \( \| \cdot \| \) and dense in \( \mathcal{L} \) with respect to \( \| \cdot \| \); it follows that \( \mathcal{L}^\omega \subseteq \mathcal{L} \) and that the image of \( \mathcal{O} \) contains all operators of finite rank.

(i) \( \Rightarrow \) (ii): If \( \mathcal{L}^\omega \subseteq \mathcal{L} \) we consider the identity map from \( \mathcal{L}^\omega \) onto \( \mathcal{L} \). From the first part we have \( \| \cdot \| \leq \beta \| \cdot \| \) and now the closed graph theorem gives a \( \mu \in C \) such that \( \| \cdot \| \leq \mu \| \cdot \| \). The result follows.

(i) \( \Rightarrow \) (iii): Let \( R \) be any closed right ideal of \( \mathcal{O} \). Then \( J(R^\perp) \) is a proper closed subspace of \( \mathcal{L} \); therefore there exists an element \( x \) of \( \mathcal{L}^\perp \) such that \( x \in J(R^\perp) \). Then if \( a \in R \):

\[
(x \triangleleft x) T_a = (0) \quad \text{and therefore} \quad (x \triangleleft x) T_a = 0.
\]

Now there is some \( b \) in \( \mathcal{O} \) such that \( x \triangleleft x = T_b \) and \( T_b = 0 \); since the representation is faithful, \( ba = 0 \) for all \( a \) in \( R \) and thus \( beR_1^\perp \).

(iii) \( \Rightarrow \) (i): Let \( x \) be a given element of \( \mathcal{L} \); let \( S \) be \( x^\perp \) and let \( R = J(S^\perp) \). Consider \( R \); this is non-zero since \( R \) is proper, and so it contains a minimal left ideal which will be \( \{ a \in \mathcal{O} : T_a = u \triangleleft v \} \) for some fixed \( v \) in \( \mathcal{L} \) and \( u \) varying in \( \mathcal{L}^\perp \). We will show that \( v \in [x] \). Let \( h \in S \), then \( (u \triangleleft v) h = 0 \), and thus \( (h, v) u = 0 \) for all \( u \) in \( \mathcal{L} \); it follows that \( (h, v) = 0 \); but \( h \) was an arbitrary element of \( S \) and so \( v \in S^\perp = [x] \). Thus \( \lambda x \in \mathcal{L}^\omega \) for some non-zero \( \lambda \) in \( C \), and therefore \( x \in \mathcal{L}^\omega \).

**Theorem 3.5.3.** If \( \mathcal{O} \) is a primitive Banach algebra with a continuous right complementor \( p \), the following are equivalent:

(i) \( \mathcal{O} \) is a left annihilator algebra;

(ii) \( \mathcal{O} \) is a right annihilator algebra;

(iii) \( \mathcal{O} \) is an annihilator algebra;

(iv) \( \mathcal{O} \) is dual.
PROOF. (i)⇒(iii): Let \( a^* \) be the representation of Theorem 3.3.13, where \( L \) is chosen to be a minimal left ideal that contains a p-projection. Then, by Proposition 3.5.2, the image of \( \sigma \) contains all operators of finite rank on \( L \). Also \( L \), being a Hilbert space, is reflexive. Now, since the socle of \( \sigma \) is dense in \( \sigma \), and the image of the socle in the representation is the set of all operators of finite rank on \( L \), Theorem 2.8.23 in [8] completes the proof.

(iii)⇒(i): By Lemma 1.3, \( a \in \sigma \) for any \( a \) in \( \sigma \). Now, by the proof of Theorem 2.8.27 in [8], we see that \( \sigma \) is dual.

(ii)⇒(iii): We first show that no proper closed subspace \( S \) of \( L^\sigma \) can be dense in \( L \). Let \( L = \{ a \in \sigma : T_a \in \sigma \} \); then \( L \) is a closed left ideal of \( \sigma \) and, if \( x \notin S \) but \( x \in L^\sigma \), then there is an \( a \) in \( \sigma \) such that \( T_a = x \), and clearly \( a \notin L \) so that \( L \) is proper. Therefore there is a non-zero closed right ideal \( R \) that annihilates \( L \). Suppose that \( S \) is dense in \( L \). Then, for all \( h, h' \) in \( L \), \( (LRh, h') = 0 \) and so, since for \( x \) in \( L^\sigma \) and some \( \beta \) in \( C \| x \| \leq \beta \| x \|_\sigma \) (see proof of Proposition 3.5.2), it can be seen that \( (Rh, S) = 0 \); then, by the density in \( L \) of \( S \), \( Rh = 0 \). Therefore \( R = (0) \) which is a contradiction. Thus no proper closed subspace of \( L^\sigma \) is dense in \( L \).

Now consider the identity map \( I \) from the normed linear space \( (L^\sigma, \| \cdot \|) \) onto the normed linear space \( (L, \| \cdot \|) \). Denote these spaces by \( X_1, X_2 \) respectively. Let \( S \) be a maximal closed subspace of \( X_2 \); then \( I(S) \) is certainly closed in \( X_1 \).

If it is not maximal closed then it is contained in a maximal closed subspace \( M; I(M) \) contains \( S \) and, by the maximality of \( S \), is dense in \( X_2 \); but this is a contradiction since
M is proper. Thus \( \mathcal{I}^1(S) \) is maximal closed. Now let \( Q \) be a maximal closed subspace of \( \mathcal{K}_1 \); then \( \mathcal{I}(Q) \) is not dense in \( \mathcal{K}_2 \) and so there is a maximal closed subspace \( N \) of \( \mathcal{K}_2 \) that contains \( \mathcal{I}(Q) \). Then \( \mathcal{I}^1(N) \) is a maximal closed subspace of \( \mathcal{K}_1 \) that contains \( Q \); therefore it must be equal to \( Q \) and so \( N=\mathcal{I}(Q) \). Thus \( \mathcal{I}(Q) \) is maximal closed. Now by Lemma B in [5] \( I \) is a homeomorphism. It is then clear that \( \mathcal{L} \) is complete with respect to \( \| \cdot \| \) and thus that \( \mathcal{L}^* \). Thus by Proposition 3.5.2 \( \mathcal{A} \) is a left annihilator algebra.

The remaining implications are trivial. The Theorem is thus proved if the dimension of \( \mathcal{L} \) is not 1, or 2, but these cases are clear. This completes the proof.

We have shown that, if the algebra \( \mathcal{A} \) is an annihilator or dual algebra, then the representation Theorem 3.3.13 can be strengthened; that is, we can now say that the image of the socle is the set of all operators of finite rank on \( \mathcal{L} \).

This result generalises Theorem 7 in [12]. We now give a converse.

**THEOREM 3.5.4.** Let \( \mathcal{A} \) be an algebra of operators on a Hilbert space \( H \) that is the closure with respect to a norm that majorises the operator norm of the set of all operators of finite rank on \( H \). If \( \mathcal{A} \) also satisfies \( E \mathcal{A} C \mathcal{M} \) for every orthogonal projection \( E \) on \( H \), then it is a dual algebra with a continuous right complementor.

**PROOF.** Immediate from Theorems 3.3.14 and 3.5.3.

In Section 6 we shall show by a counter-example that a simple Banach algebra with a continuous right complementor need not be an annihilator algebra.
We now extend some of these results to the semi-simple case.

**THEOREM 3.5.5.** Let \( \mathfrak{A} \) be a semi-simple Banach algebra with a continuous right complementor \( p \); then the following are equivalent:

(i): \( \mathfrak{A} \) is a left annihilator algebra;

(ii): \( \mathfrak{A} \) is a right annihilator algebra;

(iii): \( \mathfrak{A} \) is an annihilator algebra;

(iv): every closed right ideal of \( \mathfrak{A} \) is an annihilator ideal.

**PROOF.** (i) \( \Rightarrow \) (iii): Let \( \{ \mathfrak{J}_\lambda \}_{\lambda \in \Lambda} \) be the family of all minimal closed two-sided ideals of \( \mathfrak{A} \). Then, from Theorem 3.4.1 and the proof of Theorem 8 in \([1]\), each \( \mathfrak{J}_\lambda \) is a simple left annihilator algebra with a continuous right complementor \( p_\lambda \). Therefore, by Theorem 3.5.3, it is an annihilator algebra. Then, by Theorem 2.8.29 in \([8]\), \( \mathfrak{A} \) is an annihilator algebra.

(ii) \( \Rightarrow \) (iii): is exactly the same with right annihilator substituted throughout for left annihilator.

(i) \( \Rightarrow \) (iv): as above each \( \mathfrak{J}_\lambda \) is a simple left annihilator algebra with a continuous right complementor. Therefore by Theorem 3.5.3 \( \mathfrak{J}_\lambda \) is dual. Also by Lemma 1.3, \( \mathfrak{A} \) is complemented for every \( a \) in \( \mathfrak{A} \); then by the proof of Theorem 2.8.29 in \([8]\) we see that every closed right ideal of \( \mathfrak{A} \) is an annihilator ideal.

The remaining implications are clear.

As a corollary we obtain a generalisation of Theorem 9 of \([12]\).

**COROLLARY 3.5.6.** If \( \mathfrak{A} \) satisfies the conditions of the Theorem and also a left complementor can be defined on \( \mathfrak{A} \), then \( \mathfrak{A} \) is dual.

**PROOF.** Immediate from the analogue of Lemma 1.3 for a left complemented algebra and Theorem 2.8.29 in \([8]\).
We now return to the simple case. Let $\mathfrak{A}$ be a simple Banach algebra with a continuous right complementor $p$. Suppose also that $\mathfrak{A}$ has no left ideals of dimension less than three. Then $\mathfrak{A}$ has a continuous representation $a \mapsto T_a$ in the algebra $K(H)$ of all compact operators on a Hilbert space $H$. The involution in $K(H)$ induces a partial involution (i.e. an involution that is defined on a subspace of $\mathfrak{A}$) in $\mathfrak{A}$. If $\mathfrak{A}$ is also known to be an annihilator algebra then its socle will be self-adjoint with respect to the involution. Again, if $\mathfrak{A}$ is an annihilator (and hence dual) algebra, then there is a natural mapping $L \mapsto L^q$ defined on the set of all closed left ideals of $\mathfrak{A}$ by $L^q = L_r^p$. It is easy to see that this satisfies the analogues for left complementors of $C(i), C(ii)'$, $C(iii)$, and $C(iv)$.

There remain two unanswered questions:

(i) When is the image of $\mathfrak{A}$ self-adjoint in $K(H)$? Equivalently, when is the partial involution an involution?

(ii) When is $q$ a left complementor?

We give partial answers to these questions in the next two Theorems.

THEOREM 3.5.7. Let $\mathfrak{A}$ be a simple annihilator algebra with a continuous right complementor $p$ and no minimal left ideals of dimension less than three. Then there is a constant $K$ such that $\|x^*\| \leq K\|x\|$ for all $x$ in $\mathfrak{A}$ whose image $T_x$ in the representation is of rank one.
PROOF. Let $e_1$ be any $p$-projection in $\mathfrak{A}$. In $L$ the norm $\|\cdot\|$ and the inner product norm $\langle\cdot,\cdot\rangle$ are equivalent (Theorem 3.3.11), so let $a, b$ satisfy $\beta|y| \leq \|y\| \leq \alpha|y|$ ($y \in L$).

Now $T_{e_1} = e_1 a e_1 \overline{e_1}$, and thus

$$\|e_1\| = \|T_{e_1}\| = \|e_1 a e_1 \overline{e_1}\| \leq \alpha \|e_1 a e_1\| \leq \alpha \|e_1\|^2 = \alpha \|e_1\|^2 / (e_1 e_1) = \alpha \lambda.$$ 

($\lambda$ is the constant of Proposition 3.5.2).

Now suppose that $K$ cannot be found: there exists a sequence $\{a_n\}$ such that $a_n$ is a minimal right ideal and $\|a_n\| \leq 1/n$ and $\|a_n^*\| = 1$. Let $e_n$ be the $p$-projection contained in $a_n$ ($e_n$ exists since all maximal ideals of $\mathfrak{A}$ are modular).

Then $\|a_n^*\| = \|T_{a_n}\| = \|x_n \otimes y_n\| \leq \alpha \|x_n\| \|y_n\| \leq \lambda \alpha^2 \|x_n\| \|y_n\| = \lambda \alpha^2 \|x_n \otimes y_n\|

= \lambda \alpha^2 \|y_n \otimes x_n\| \leq (\lambda \alpha^2 / \beta) \|y_n \otimes x_n\| = (\lambda \alpha^2 / \beta) \|T_{a_n}\|

= (\lambda \alpha^2 / \beta) \|a_n\| \leq (\lambda \alpha^2 / \beta n)$, which is the required contradiction.

**THEOREM 3.5.8.** Let $\mathfrak{A}$ be a simple annihilator algebra with a continuous right complementor $p$ and with no minimal left ideals whose dimension is less than three. Then, of the following, (i), (ii), (iii) are equivalent and (iv) is a consequence of the rest:

(i) the partial involution induced in $\mathfrak{A}$ by the representation $a \mapsto T_a$ is continuous;

(ii) $\mathfrak{A}$ is self-adjoint with respect to this involution;

(iii) $R^\mathfrak{A} = R_1^\mathfrak{A}$ for all closed right ideals $R$ of $\mathfrak{A}$;

(iv) $q$ is a left complementor.

**PROOF.** (i) $\Rightarrow$ (ii): The socle is dense and self-adjoint with respect to the involution, and so the result is clear.
(ii) ⇒ (i): If $\mathcal{A}$ is a Banach*-algebra then it is an $A^*$-algebra where $|a| = |T_a|$ is the auxiliary norm. Therefore Theorem 4.1.15 in [8] gives the result.

(ii) ⇒ (iii): Let $R$ be any closed right ideal of $\mathcal{A}$ and let $S = \mathcal{D}(R)$. Now, using Theorem 3.3.13, we have

\[ R^P = \left\{ a: (T_a^* H, T_b H) = 0 \ b \in R \right\} = \left\{ a: (H, T_a^* b H) = 0 \ b \in R \right\} = \{ a: T_a b = 0 \ b \in R \} := R_1^* . \]

(iii) ⇒ (ii): Suppose $a \in S, a^* \notin S$; let $R = (a^a)^P$; then by hypothesis $R^P = R_1^*$; however, by Lemma 1.3, $R^P$ contains $a$ and thus $R_1$ contains $a^*$, which is a contradiction.

(ii) ⇒ (iv): Let $L$ be any closed left ideal of $\mathcal{A}$. Then let $E$ be the orthogonal projection of $H$ onto $\mathcal{D}(R)$ ($R = L_1$). Then

\[ L_r = \left\{ a: T_a = T_b E \right\}; \text{ now } L_r^P = L_{r_1} = L^*; \text{ thus } L^d = L_r^P = L_{r_1}^* = L_r^* = \left\{ a: T_a = T_b(1-E) \right\} . \]

Also, since $\mathcal{A}$ is self-adjoint with respect to the involution induced by the natural involution in $K(H)$, we have for any $a$ there are elements $c, d$ in $\mathcal{A}$ such that $T_c = T_c^* = (E T_a)^* = (T_a E)$ and $T_d = T_d^* = ((1-E) T_a^*) = (T_a^* (1-E))$. It is now clear that $\delta L = L + L^d$ and thus that $\delta$ is a left complementor.
6. SOME COUNTER-EXAMPLES.

EXAMPLE 3.6.1. A Banach algebra with a continuous right complementor that is not an annihilator algebra.

Let \( H \) be any infinite dimensional Hilbert space; let \( T \) be a lower semi-bounded self-adjoint transformation on \( H \) (i.e. \( (x,x) \preceq (Tx,x) \) for all \( x \) in \( H \)) that is not bounded\(^t\). Let the domain of \( T \) be \( H_{\infty} \), and for \( x,y \) in \( H_{\infty} \) define \((x,y)'\) to be \((Tx,y)\). Then \( H_{\infty} \) with the inner product \((,)'\) is a pre-Hilbert space; then, if \( H \) is the Hilbert space completion of \( H_{\infty} \) and the inner product is extended by continuity to \( H \), we have \((x,y)'=(Tx,y)\) \((y \in H_{\infty}, x \in H_{\infty})\)
(see results on page 335 in [91]).

Consider the algebra \( \mathfrak{a} \) of operators on \( H \) generated by the set of all operators of the form \((x \otimes y)\) where \( x \in H, y \in H_{\infty} \).

Define \( \sigma \) on \( \mathfrak{a} \) by \( \sigma(a) = \sup \{ ||aTh|| : h \in H_{\infty}, ||h|| = 1 \} \), where \( || ||_{\infty} \) denotes the inner product norm in \( H_{\infty} \). Then it is clear that \( \sigma \) is a norm on \( \mathfrak{a} \). Also \( \sigma(a) \geq ||a|| \) where \( || || \) is the operator norm in \( \mathfrak{a} \). Using this, it can be verified that \( \sigma(ab) \leq \sigma(a) \sigma(b) \) for all \( a, b \) in \( \mathfrak{a} \). Let \( \mathfrak{a} \) be its \( \sigma \)-completion.

Now \( \{ y \in H_{\infty} : x \otimes y \text{ fixed and } y \text{ varying through } H \} \) is clearly a minimal left ideal \( L \) of \( \mathfrak{a} \) and \( \{ x \in H_{\infty} : x \otimes y \text{ fixed and } y \text{ varying through } H \} \) is a minimal right ideal \( \mathfrak{r} \) of \( \mathfrak{a} \). Also \( H \) is topologically equivalent with \( L \) and \( H_{\infty} \) with \( R \).

However, if \( \mathfrak{a} \) is an annihilator algebra, \( H_{\infty} = H \) and \( \sigma(x \otimes y) \leq ||x|| \cdot ||y|| \) for all \( x, y \) in \( H \). Thus \( ||x \otimes y|| \leq ||x|| \cdot ||y|| \), and it can be concluded that \( T \) is bounded.

We show that \( \mathfrak{a} \) is right complemented; since \( H \) is infinite
dimensional the right complementor will then be continuous. For any closed right ideal $R$ of $\mathcal{J}$ define $R^\circ$ to be
$$\{a \in \mathcal{J} : (ah, RH) = 0 \text{ for all } h \in H\}.$$ We show that, if $\mathcal{O}$ denotes the completion of $\mathcal{J}$ in the norm $\| \cdot \|_{\mathcal{O}}$ (i.e. $\mathcal{O} = K(H)$), then $\mathcal{O}^\circ = \mathcal{O}$ (i.e. $\mathcal{O} = \mathcal{O}$). It will then follow that $\mathcal{O}$ satisfies the hypothesis of Theorem 3.3.2 (ii).

Suppose that $b \in \mathcal{O}$ and $a$ is an element of finite rank of $\mathcal{O}$. Then $ba \in \mathcal{O}$ and $\mathcal{O}(ba) \leq \|b\| \|\cdot\|(a)$. The result follows, since elements of finite rank are dense in $\mathcal{O}$.

Notice that this example shows that an algebra of operators satisfying the stronger relation of Theorem 3.3.2 (i.e. (ii)) need not have either of the one-sided annihilator properties.

In Example 2 of [10] Saworotnow gives an example of a bi-complemented algebra that is also a Hilbert space, and in which $R^\circ = R^\perp$ for all closed right ideals $R$ and a similar relationship holds for closed left ideals, but which is not an annihilator algebra.
BIBLIOGRAPHY


(2). J. DIXMIER, 'Les $C^*$-algèbres et leurs représentations', Cahiers Scientifiques, Fasc XXIX.


NOTATION

adjoint p-derived map, \( P' \), \( (42) \)
continuous complementor (B*-case) \( (9) \)
continuous right complementor \( (65) \)
initial point \( (16) \)
projection by \( (S,T) \) \( (67) \)
p-convergence (B*-case) \( (47) \)
\( \text{general case} \) \( (65) \)
p-derived map, \( P \), \( (9) \)
p-projection \( (4) \)
p-representing operator, \( T \), \( (19) \)

\( ', \sim \) projection \( --(9) \); \( \preccurlyeq \), \( \prec \), \( \| \), \( \| \) \( --(11) \)
\( e_x \) \( --(16) \); \( f_x \) \( --(20) \); \( [x] \) \( --(11) \);
\( J, J (B^*-\text{case}) \) \( --(12) \); \( J, J (\text{general case}) \) \( --(69) \)
\( J_x \) \( --(70) \); \( J_x \) \( --(71) \)
\( R_\lambda \) \( --(58) \); \( E_\lambda \) \( --(78) \); \( E_R \) \( --(35,78) \)
\( P \) \( --(12) \).