GENERALISED NON-ASSOCIATIVE ARITHMETIC

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by

ROGNVALD C. N. HOURSTON

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PREFACE

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R.C.N.H.
CHAPTER ONE

INTRODUCTION
The purpose of this section is to give a summary of those parts of the theory of classes and sets which will be employed in later work, and of the notations which will be used. (A theory of trees which is new and so requires fuller discussion will be dealt with in a later section.) The account is very brief and the reader is referred for fuller discussions to Birkhoff [1].

For the relation of inclusion between relations and between classes the notation '$\subseteq$' will be used. The symbol '$\subset$' will denote proper inclusion. Unions and intersections will be denoted in the usual way by 'curved' symbols, e.g. '$\cup$', while other lattice-operations of join and meet will be denoted by 'angular' symbols, e.g. '$\vee$'. This is a departure from Birkhoff's notation. The term 'class' will be used where, frequently, the term 'set' might seem more natural, but the latter expression will be used to denote a class $C$ together with a relation $R$ whose field is included in $C$. The class associated with a set $S$ will be denoted by '$|S|$'; that is, if $S$ is a set obtained by imposing the relation $R$ on the elements of the class $C$ (where the field of $R$ is included in $C$), then $|S|=C$. We say that a set $S'$ with the relation $R'$ is a subset of a set $S$ with relation $R$, and write '$S' \subseteq S$', if $|S'| \subseteq |S|$ and $R'$ is $R$ with its field restricted to $|S'|$. Thus every subclass of /
of \(|S|\) determines a subset of \(S\). In general we shall use the same symbol for the relation belonging to any subset of \(S\) as we use for that belonging to \(S\), although the two relations are, strictly, distinct.

Wherever a symbol \('\leq'\) or \('\subseteq'\) is employed, we denote by \(''\leq'\) or \(''\subseteq'\) respectively the relations which hold between \(x\) and \(y\) when \(x \leq y\) and \(x \neq y\), or when \(x \subseteq y\) and \(x \neq y\). The sign of equality always denotes identity.

As will be apparent from the above, we shall not always distinguish in notation between a binary connective and the relation derived from it. While this practice is inaccurate to the extent of using the same symbol for different things, there is distinguished precedent for it in the treatment of membership in WHITEHEAD AND RUSSELL [1], *62.

We shall use the expression \('x \in S'\) to mean that \(x \in |S|\).

A set with the relation \(\leq\) is said to be quasi-ordered when
\[
\begin{align*}
&x \leq x \text{ for all } x \text{ in } |S| \quad \text{(Reflexivity in } |S|) \\
&\text{if } x \leq y \text{ and } y \leq z, \text{ then } x \leq z \quad \text{(Transitivity).}
\end{align*}
\]
The set \(S\) is said to be partially ordered when, in addition,
\[
\text{if } x \leq y \text{ and } y \leq x, \text{ then } x = y \quad \text{(Antisymmetry)}.
\]
The relation \(<\) is transitive and irreflexive in a partially ordered set.

The expression \('x \parallel y'\) means that \(x \not\leq y\) and \(y \not\leq x\).

A subset \(S'\) of a partially ordered set \(S\) is said to be convex in \(S\) when, if \(x \in S\) and there are elements \(y\) and \(z\) in \(S'\) such /
such that \( y < x < z \), then \( x \in S' \).

If a partially ordered set has a greatest (least) element we denote it by \( 1 \) (0). A **maximal element** is an element which is not exceeded by any element of the set.

A partially ordered set \( S \) is said to be **directed** if for any pair of elements \( x \) and \( y \) of \( S \) there is an element \( z \) such that \( x \leq z \) and \( y \leq z \); and to be a **chain** if for every pair of elements \( x \) and \( y \) of \( S \) either \( x \leq y \) or \( y \leq x \). A chain is said to be **well-ordered** if every non-null subset has a least element. A **segment**, or **ideal**, of a chain is a subchain consisting of all elements \( x \) such that \( x \leq a \) for a given element \( a \). The complement of a segment is called a **residue**.

The **ascending chain condition** on a partially ordered set \( S \) is the requirement that every non-null subset of \( S \) possess a maximal element. The **descending chain condition** is the dual of this.

We assume the elements of the theory of transfinite ordinal numbers, and make use of transfinite induction. (See SIERPINSKI [1].) The finite ordinals will be denoted by \( 0', 1', 2' \ldots \) (by an adaptation of Birkhoff's notation). The **supremum** \( \text{sup} \{i\} \), of a class \( \{i\} \) of ordinals is the least ordinal \( k \) such that \( i \leq k \) for all \( i \) in \( \{i\} \). The **sequent** \( \text{seq} \{i\} \), of a class \( \{i\} \) of ordinals is the least ordinal \( k \) such that \( i < k \) for all \( i \) in \( \{i\} \). A **limit-ordinal** is an ordinal \( i \) such that there is no ordinal \( j \) for /
for which \( j + 1 \) - 1.

A lattice is a partially ordered set in which every pair of elements \( x \) and \( y \) has a least upper bound, or join, \( x \lor y \), and a greatest lower bound, or meet, \( x \land y \). A sublattice of a lattice is a subset of the lattice which not only is itself a lattice but also has the same joins and meets.

A join-semilattice is a partially ordered set in which every pair of elements \( x \) and \( y \) has a join \( x \lor y \). The dual concept is that of meet-semilattice.

A complete lattice is a partially ordered set \( S \) in which every subclass \( X \) of \(|S|\) has a join \( \lor X \) and a meet \( \land X \). (In this definition we include the case in which \( X \) is the null-class \( \emptyset \); then \( \lor \emptyset = 0 \) and \( \land \emptyset = 1 \). Cf. WHITEHEAD AND RUSSELL [1], *40.2.21 and *41.2.21.) If we define 'complete semilattice' in the obvious way, we have the following result.

**Theorem 1.** A complete join- (or meet-) semilattice \( S \) is a complete lattice.

**Proof.** Let \( X \) be a subclass of \(|S|\). Let \( X' \) be the class of members of \( S \) which are less than every member of \( X \). Then (whether \( X' \) is or is not null) \( \lor X' \) is a greatest lower bound for \( X \).

And dually.

A closure operation on the elements of a partially ordered set /
set is an operation transforming each element \( x \) into an element \( \overline{x} \) in such a way that \( x \leq \overline{x}, \overline{x} = \overline{x} \), and if \( x \leq y \) then \( \overline{x} \leq \overline{y} \). An element \( x \) is said to be closed under the operation if \( x = \overline{x} \).

After this discussion of various kinds of partially ordered sets, we come to the concept of union. If, for all \( i \in \{i\} \), \( S_i \subseteq S \), then the union in \( S \) of the sets \( S_i \) is the subset \( S' \) of \( S \) such that \( |S'| = \bigcup_{S_i} |S_i| \). This rather complex form of the definition is required because two partially ordered sets \( S_1 \) and \( S_2 \) do not, in general, have a unique 'union' because the relations of order which hold between elements of \( S_1 \) and elements of \( S_2 \) are not specified by the partial orderings of \( S_1 \) and of \( S_2 \). Moreover, two partially ordered sets may not have a 'union' at all, for if \( R_1 \) is a partial ordering of \( S_1 \), and \( R_2 \) of \( S_2 \), it may be that there are elements \( x \) and \( y \) in \( |S_1| \cap |S_2| \) such that \( x R_1 y \), \( y R_2 x \) and \( x \neq y \); in which case there can be no partial ordering of \( |S_1| \cup |S_2| \) which includes both \( R_1 \) and \( R_2 \).

**POWERS IN NON-ASSOCIATIVE ALGEBRAS**

In any algebra \( A \) with a binary operation which we call multiplication (and with respect to which \( A \) is in general non-commutative and non-associative) we may define the notion of power as follows:

(i) if \( x \in A \), then \( x \) is a power of \( x \),

(ii)
(ii) if \( y \) and \( z \) are powers of \( x \), then \( y \cdot z \) is also a power of \( x \)

(iii) no entity is a power of \( x \) except in virtue of (i) and (ii).

This definition does not completely accord with the usage often adopted in the case of an algebra whose elements possess reciprocals but do not have finite periods (for then 'negative powers' can arise); but the discussion will deal primarily with the general non-associative case in which such concepts are of less importance. As was pointed out in Etherington [2], the 'shapes' of non-associative products, and hence, in particular, of powers, may be represented by pedigrees. A pedigree is a diagram of the type shown. It has a root-node at the top from

[Diagram]

which there descend either two branches or none; in the former case these terminate in two further nodes. From each of these nodes again there descend either two branches or none; and so on. The diagrams are not the abstract trees of graph-theory because it is necessary to distinguish between nodes on the right and nodes on the left, and so the two pedigrees illustrated are /
are distinct although they would represent the same abstract root-tree. (See Chapter 2 for a discussion of trees.) The pedigrees illustrated represent products of the types
\[(a,b)\cdot(c,d)\cdot(e,f)\] and \[(a,b)\cdot(c,d)\cdot(e,f)\].

To represent the product of two products, we place the pedigree of the factor on the right to the right of the pedigree of the left factor and join their roots to a new root placed above both.

Let us write \(x^1\) to denote the identity function on \(A\) (i.e., the function which assigns to any element \(x\) of \(A\) taken as argument the value \(x\)). The symbol \(1\) placed as an index is just a part of the notation; it certainly does not here denote the number one. We may then go on to write the function whose value for argument \(x\) is \(x\cdot x\) as \(x^1\cdot x\), that whose value is \(x\cdot(x\cdot x)\) as \(x^{1\cdot(1+1)}\), that whose value is \((x\cdot x)\cdot x\) as \(x(1+1)\cdot x\), and so on.

In this way a notation for the powers in the algebra is developed. The "indices" here behave as an (in general non-associative and non-commutative) arithmetic, an arithmetic of functions on \(A\). The properties of this arithmetic depend on the multiplicative properties of powers in the particular algebra \(A\). For if
\[x\cdot(x\cdot x) = (x\cdot x)\cdot(x\cdot x)\]
is an identity in \(A\), then, in the function-arithmetic on \(A\),
\[1 + (1 + 1) = (1 + 1) + (1 + 1)\]; whereas this property does not hold in all algebras. The arithmetic of the power-functions on an algebra \(A\) is isomorphic to the logarithmic (with respect to /
to multiplication) of $A$; the term 'logarithmic' being defined in Etherington [3] and [4]. The logarithmetics of quasi-groups have been studied. (See Popova [1], [2] and [3].)

This is, however, sharply to be contrasted with the use of indices in the case of associative algebras. There the symbol $x^n$, as ordinarily used, denotes a function of two variables, the element $x$ of the algebra and the integer $n$. The integers are logically prior to the application, but their arithmetical has such properties that the index laws

$$x^m \cdot x^n = x^{m+n}$$

$$(x^m)^n = x^{mn}$$

represent properties of all associative algebras. If the expression

$$x^2 = x$$

is an identity in $A$, it does not follow that $2 = 1$.

Let us turn back to the non-associative case. If $m$ and $n$ are two of the power-functions on $A$, and if we define their product $mn$, in the usual way, as the relative product of the two functions, we get the result

$$(x^m)^n = x^{mn}.$$  

Since in an algebra of operators multiplication is necessarily associative (reducing as it does to the relative products of relations) we have the associative law for multiplication of our 'indices'; this law holding for any algebra $A$. It would appear to/
to be a great advantage to have available some form of non-
associative arithmetic whose properties reflect laws which
hold in all non-associative algebras (as does the associative
law of multiplication in the arithmetic). Our index notation
would then represent, as in the associative case, a function
of two variables, one being an element of the algebra in
question and the other an element of the arithmetic. As was
pointed out in ETHERINGTON [2], the pedigrees considered
above give a representation for such an arithmetic; but they
are hardly satisfactory as primitive concepts.

In ROBINSON [1], the required arithmetic, and a number
of associated arithmetics, was axiomatised. It is there
called a 'simple forest' since pedigrees are also called
trees. The axioms for the simple forest $F$ are the following:-

I For all $a$ and $b$ in $F$, there is a unique ele-
ment $c$ in $F$ such that $c = a + b$.

II If $a$, $b$, $c$ and $d$ are in $F$ and $a + b = c + d$, then
$a = c$ and $b = d$.

III If $e_1$ and $e_2$ are elements of $F$ such that there
are no elements $a$ and $b$ of $F$ for which $a + b = e_1$
or $a + b = e_2$, then $e_1 = e_2$.

IV If $F'$ is a non-null subclass of $F$, there is an
element $e'$ in $F'$ such that, if there are elements
$a$ and $b$ in $F$ such that $a + b = e'$, then $a$ and $b$
are in $F - F'$. 

In /
In ETHERINGTON [1], an arithmetic of the kind required was constructed in terms of set-theory. In fact, a much more general programme was undertaken, resulting in the construction of a non-associative arithmetic (or rather two, one being non-commutative, a non-associative analogue for the arithmetic of order-types, and the other a commutative, non-associative analogue of cardinal arithmetic) with counterparts for transfinite numbers. The method adopted for the construction of the 'numbers' of the arithmetic was that of imposing upon an arbitrary chain (or class, in the commutative case) a sequence of progressively finer partitions into convex subsets, these subsets being ultimately unit subsets. Such a partitioned chain is representable by a pedigree in a generalised sense, with possibly infinite numbers of branches and of nodes. In this way the theory of powers in a non-associative algebras was furnished with an arithmetical equipment and placed on a level with that in associative algebras.

The logical process which led to the construction of a non-associative arithmetic is analogous to that which led Cantor to the concept of the arithmetic of ordinal numbers. Cantor required an arithmetic which would serve to provide superscripts for the successive derived sets $E(\mathbb{1}), E(\mathbb{2}), \ldots, E(\omega), \ldots$ of a given point-set $E$. 
We shall here digress to show that there is a simple arithmetical model of the simple forest. This model was suggested by WEYL [1], Appendix A, where a sketch is given of the argument of GÖDEL [1]; this exposition involves the representation of both formulae and formal proofs in terms of pedigrees, and the representation of the pedigrees, in turn, by natural numbers. The nodes of a pedigree may be represented by sequences of zeros and ones as follows:

The root is represented by 1. There descend from it either two branches or none. In the former case, the nodes at the lower ends of these branches are denoted by 10 and 11, that on the left being denoted by the former. From each of these nodes again there descend either two branches or none. At every stage we denote the two nodes below a given node by replacing the initial 1 of its representation by 10 and 11, the former denoting that on the left. This representation seems less natural than Weyl's but gives a more simple representation of addition. The sequences can now be interpreted as integers in binary notation, so that we get, for example, the following figures:

```
1
/ \
10 11
/ \\
100 110 101 111
/ \\
1001 1101
```

Let /
Let $p_1, p_2, p_3, \ldots$ be the sequence of prime numbers 2, 3, 5, ... in their natural order. We represent the whole pedigree by the product of those primes whose subscripts represent nodes of the pedigree. The one shown, for example, has the representation:

$$p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot p_6 \cdot p_7 \cdot p_9 \cdot p_{13} = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 43.$$ 

We can now, by resolving any number into its prime factors, find whether it represents a pedigree and, if so, which. If we denote by " $x'$ " the greatest power of 2 which is less than $x$, and by " $x|y$ " the statement that $x$ is a factor of $y$, the criteria for a number to represent a pedigree can be reduced to the following:

The natural number $n$ represents a pedigree if and only if

1. $n$ is square-free
2. if $p_r | n$, then $p_r - 1 | n$
3. if $p_r | n$ and $p_r + 1 | n$, then $p_r + 2 | n$.

Let $p_{\alpha_1} \cdot p_{\alpha_2} \cdot p_{\alpha_3} \ldots p_{\alpha_j}$ and $p_{\beta_1} \cdot p_{\beta_2} \ldots p_{\beta_k}$ be two numbers which represent pedigrees. It is easily seen that, to get the representation of the pedigree which is the sum of those so represented, we form the product $p_{\gamma_1} \cdot p_{\gamma_2} \ldots p_{\gamma_l}$, where

$$l = j + k + 1$$

$$\gamma_i = 2\alpha_i \quad (1 \leq i \leq j)$$

$$\gamma_i = 2\beta_i + 1 \quad (j + 1 \leq i \leq j + k)$$

$$\gamma_l = 1.$$ 

It is shown in ROBINSON [1] that multiplication in the simple forest /
forest can be defined (recursively) in terms of addition; and, indeed, this is obvious when we consider that the arithmetic arose in the representation of powers with respect to a single operation. Hence it is not necessary to specify independently how to derive from two numbers the number which represents the product of the pedigrees represented by them.

GENERALISATIONS OF CARDINAL AND ORDINAL ARITHMETICS

In his research on the theory of sets, F. Hausdorff found that, by giving a general definition for the product of a chain of order-types, he could derive interesting connections between ordinal numbers and some of the more important order-types. The required definition of the product of a chain of chains was found, however, to result frequently in sets which are not chains, but are only partially ordered. It was in this way that partially ordered sets entered the theory of transfinite arithmetics. The results of this research were published in HAUSDORFF [2] and later, in part, incorporated into HAUSDORFF [1].

Whitehead and Russell, at that time working on "Principia Mathematica", recognised the importance of the results obtained, but
but, owing to their method of approach to the subject, had to generalise the whole theory. Their approach necessitated the consideration of a class of things still wider than that of partially ordered sets, viz. relations in general. To develop arithmetic it was necessary to define the sum and product of any two relations, and hence to introduce a drastic generalisation of Hausdorff's definitions which was not so much interesting in itself (except in Hausdorff's particular case) as demanded by their starting-point. The definitions of addition, multiplication and exponentiation adopted in WHITEHEAD AND RUSSELL [1], *160, *166, and *176 are (approximately) those which later became known as the ordinal operations.

G. Birkhoff, being led from his study of lattices to the consideration of general partially ordered sets, saw that there were applications in this theory not only for the ordinal operations but also for other operations which he called cardinal operations, (suggestions being also derived from the work of O. Ore - see ORE [2]). These new operations were introduced in BIRKHOF [3]. In BIRKHOF [2] these operations were considered more fully and in conjunction with the older ordinal operations. This study was continued in DAY [1] and CAR- RUTH [1]. Parts of this theory are set out in the text and exercises of BIRKHOF [1].

Having given this brief survey of development in this field,
we turn to a consideration of the individual operations of
the arithmetic. Let \( R \) and \( S \) be sets in which relations are
defined, both these relations being denoted by ‘\( \leq \)’. We shall
not here impose any restrictions on the relations. Addition
consists of constructing two disjoint sets \( R' \) and \( S' \) which are
(ordinally) similar to \( R \) and \( S \) respectively, and imposing on
their union a relation. The cardinal sum \( R + S \) is formed by
letting \( \leq \) in this union hold between \( x \) and \( y \) if and only if
\( x \leq y \) in \( R' \) or in \( S' \). When the relations in \( R \) and \( S \) are null
this reduces to the kind of set-addition on which cardinal arith-
metic is based. The ordinal sum \( R \oplus S \) is formed by imposing
on the union of \( R' \) and \( S' \) the relation \( \leq \) where \( x \leq y \) if and only
if \( x \leq y \) in \( R' \) or in \( S' \) or \( x \in R' \) and \( y \in S' \). When \( R \) and \( S \) are
chains and well-ordered sets respectively, this is the opera-
tion on which the addition of order-types and of ordinal num-
 bers is based.

To form the products of the sets \( R \) and \( S \) we construct the
set of ordered pairs \( (r, s) \) where \( r \in R \) and \( s \in S \). The cardinal
product \( RS \) is obtained by imposing the relation on this set
where \( (r_1, s_1) \leq (r_2, s_2) \) if and only if \( r_1 \leq r_2 \) and \( s_1 \leq s_2 \) in \( R \) and \( S \).
Its importance is due to its similarity to the direct product
of algebraic theory, and, in fact, when \( R \) and \( S \) are lattices,
\( RS \) is their direct product. The ordinal product is more diffi-
cult to deal with because it depends on the order of the factors
in the product and there is no standardisation of the manner of writing this order. Cantor, in his ordinal arithmetic, began by writing the multiplier on the left and the multiplicand on the right; but he later changed the order because the index notation (when the index is placed, as is usual, to the right of the base) works out more naturally with this later convention. The second notation is now almost invariable in expositions of the theory of ordinal numbers, and is adopted here wherever ordinals are employed. If, however, we wish to consider products of arbitrary chains of sets, and not just of pairs, the opposite notation becomes convenient because it avoids giving a prominent place to sets whose duals are well-ordered. (Cf. the sections on the multiplication of partitions and of numbers in Chapters 2 and 3.) It is also convenient in this case to write the exponent on the left of the base, when exponentiation is involved. This notation was not, however, employed by Hausdorff himself. In WHITEHEAD AND RUSSELL [1] (*166) where the general standpoint was first adopted, the multiplier is placed on the left. This notation is preserved in BIRKHOFF [1], [2] and [3], DAY [1] and CARRUTH [1]. As we have done in the previous cases, we give Birkhoff’s notation. The ordinal product $R \cdot S$ of $R$ and $S$ is obtained by imposing on the class of ordered pairs the relation $\leq$ where $(r_1, s_1) \leq (r_2, s_2)$ if and only if either $r_1 < r_2$ or both $r_1 = r_2$ and $s_1 \leq s_2$. (The definition of Whitehead and Russell is equivalent /
equivalent to this only when $\leq$ in $R$ and in $S$ is irreflexive. This is a reflection of their use of irreflexivity where Birkhoff employs antisymmetry.)

We shall not in the sequel, be concerned to any extent with exponentiation, but, for the sake of completeness, a discussion of it is included here. By the cardinal power $R^S$ is meant the set of all isotone functions with argument in $S$ and value in $R$, where '$f \leq g$' means that $f(s) \leq g(s)$ in $R$ for all $s$ in $S$. Ordinal exponentiation has a complicated history. The best notation for it (see the preceding paragraph), from BIRKHOF [2], seems to be $S^R$. Here we take the class of all functions with argument in $S$ and value in $R$. It remains to define the relation $\leq$ between such functions. Hausdorff considers only the case in which $R$ and $S$ are both chains. He then imposes the order by "first differences"; i.e. $f < g$ if there is an $s$ in $S$ and that $f(s) < g(s)$ in $R$ while, if $x < s$ in $S$, then $f(x) < g(x)$. The problem is to find a generalisation of this for the case in which $R$ and $S$ are arbitrary. In WHITEHEAD AND RUSSELL [1] (§172, §176) this definition is taken over almost unaltered. An alternative is considered but rejected as being more complicated but with no compensating advantage. It is, roughly: $f < g$ if there is an $s$ in $S$ such that $f(s) < g(s)$ in $R$, and, if $x \leq s$ in $S$, then $f(x) \leq g(x)$. (These definitions have been altered slightly in order to give what amount to equivalent applications when /
when the emphasis is turned from irreflexivity to antisymmetry. The statement \( f \leq g \) is, of course, defined then as meaning that \( f < g \) or \( f = g \). A definition equivalent to this form of the rejected alternative is adopted in Birkhoff [3]. In Birkhoff [2] a new definition is offered, the intention being, apparently, to give the following ordering: \( f \leq g \) means that if \( f(s) \neq g(s) \) in \( R \) then there is an \( s \), such that \( s_1 < s \) in \( S \) and \( f(s_1) < g(s_1) \) in \( R \). But there seems to be a misprint resulting in a reversal of the inequality corresponding to \( s_1 < s \), and so giving an ordering by "last differences". The definition, with the misprint, was adopted in good faith in Day [1] and carried over into Carruth [1]. In Birkhoff [1] a further slight modification of the definition appeared. There, \( f \leq g \) is stated to mean that, if \( f(s) > g(s) \) in \( R \), there is an \( s \), such that \( s_1 < s \) in \( S \) and \( f(s_1) < g(s_1) \) in \( R \). This definition seems to be an error, for \( \mathcal{S} \mathcal{R} \) need not, if it is adopted, be partially ordered even when \( R \) and \( S \) are and \( S \) is finite (as is seen by examining 22). Moreover it does not yield \( \mathcal{S} \mathcal{R} = R \cdot R \). Some theorems from Day [1] are given as exercises without the necessary adjustment required by the adoption of first differences in one case and last differences in the other. (Cf. p 38, Ex.1.(c).) The variety of these definitions is due, in part, to the fact that no useful application has been given for the more troublesome cases and that, consequently, there is little ground for referring one to another.
The aim of the present thesis is to present a generalisation of the arithmetics of the partitioned cardinals and partitioned order-types of Etherington [1] on the lines of the generalised transfinite arithmetics described in the previous section, and to make some study of the resulting arithmetic.

Since the sum of an infinite chain of numbers of finite altitude may be of altitude \( \omega \), and the products of two numbers of altitude \( \omega \) are of altitude \( \omega^2 \), and so on, it is not possible, if the property of closure under the operations of the arithmetic is desired, to consider only numbers of finite altitude. Owing to the rather complex nature of numbers of transfinite altitude, it has been found necessary to go into greater detail on the subject of general partitions than was required in the study of Etherington [1]. While a representation of finite general partitions in terms of the root-trees of graph theory is possible, it is necessary to modify this theory in order to obtain a representation for general partitions of transfinite altitude.

In order to minimise the complications it has been decided to give a preliminary discussion of partitions and their representations in Chapter 2. This enables a simplification of the definitions of the operations of the arithmetic to be made.
for they are now considered in two distinct stages. The first stage is placed in Chapter 2 and consists of giving definitions of sums and products of partitions; and the second, in Chapter 3, consists of applying these definitions to work out the desired arithmetic. Certain laws obeyed by the arithmetic are considered at appropriate stages in the development.

In Chapter 4, a study is made of the subclasses of the arithmetic, "systems" as they are called, which are closed with respect to certain operations. The chief problem which emerges is that of the existence of "bases" (i.e. minimal generating classes). An example shows that a system need not have a basis. The remainder of the argument aims at establishing certain existence and uniqueness theorems for bases. To this end the concept of the "derivative" of a class of numbers with respect to a class of operations is introduced and applied.

An attempt has been made to preserve a reasonably high standard of logical rigour but without presenting the argument in terms of any particular logical system. A few remarks on the way in which the theorems, particularly of Chapter 4, may be affected by different logical systems will not be out of place.

In Chapter 4 the class of all partitioned numbers plays a prominent part; the existence of the closure of a class of partitioned numbers with respect to a class of operations is, for example, made to depend on its existence. Since this class involves the class of all ordinal numbers, we would seem to be dangerously /
dangerously near the antinomy of Burali-Forti. Whitehead and Russell avoid this and other antinomies by means of the theory of types; but this means that we cannot increase classes indefinitely without exhausting a given logical type. The type of any constructed sum or product for two partitioned sets will be different from that of the sets themselves. (In defining the sum of two numbers in Chapter 3, we make use of a class of disjoint sets without proving the existence of such a class. The way in which an attempt to construct such a class results in a change of type can be seen in WHITEHEAD AND RUSSELL [1], *110 and *112.) There may then be no similar partitioned sets contained in the original type; in this case the sum of the numbers of the sets is the null-class as far as their own type is concerned. The arguments proving that systems generated in certain ways are necessarily infinite will thus break down, for the sum of the null-class and any number is the null-class.

In a logical system which avoids the antinomies of set-theory by limiting the possibilities of set-formation, there may be difficulty in establishing the existence of the closure of a class of partitioned numbers with respect to a class $Z$ of operations. It is necessary to show, first, that there is a class which is closed with respect to $Z$ and which includes $A$, and, second, that there must be one such class that is included in any /
any other. In Chapter 4, the first task is performed by citing the class $E_j$ of all partitioned numbers, and the second by taking the intersection of all classes including $A$ and closed with respect to $Z$, which introduces the class of all such classes. It may be that one or other of these methods of procedure is disallowed by the system in question.
CHAPTER TWO

PARTITIONS
TREES

The word 'tree' is used in the theory of graphs to denote a finite, connected graph without circuits (Kreise). A root-tree is a tree in which a particular node \( I \) (chosen arbitrarily) is called the root of the tree. (For the general theory, see KÖNIG [1].) The altitude of a root-tree is the distance from its root to its farthest node. It will be found convenient in later work to have at our disposal root-trees whose altitude is an arbitrary transfinite cardinal. Because of the difficulty of finding within graph-theory a meaning for the expression 'path (Weg) of length \( \omega+1 \)', this generalisation is not readily absorbed into graph theory. The purpose of the present section is to adapt the concept of root-tree for employment in the theory of partially ordered sets and to introduce in that context the desired generalisation and some further concepts which will be found convenient.

Since trees without roots will not be considered, we shall proceed to use the work 'tree' synonymously with 'root-tree'. We define '\( a \leq b \)' where \( a \) and \( b \) are nodes of a tree to mean that \( b \) lies on the (unique) path from \( a \) to \( I \), where \( I \) is the root of the tree. The following theorem is easily proved.

**THEOREM 2.** The nodes of a tree form, under \( \leq \), a directed, partially ordered set in which the upper bounds of any element form a chain.

In fact we can say more than this as the next theorem shows /
shows, but this further result does not generalise in the way required; for infinite partially ordered sets the condition is no longer sufficient.

**Theorem 3.** A necessary and sufficient condition for a finite, partially ordered set in which the upper bounds of any element form a chain to be a join-semilattice is that it be directed.

**Proof.** Since any join-semilattice is directed, the condition is necessary.

Let now $S$ be a partially ordered set of the specified type which is directed. If $a$ and $b$ are two elements of $S$, the class of upper bounds of $a$ and $b$ is finite and not null. Let $a'$ be that member of the class which is least in the chain of upper bounds of $a$, and $b'$ that member which is least in the chain of upper bounds of $b$. Then $a' \leq b'$ since $b'$ occurs in the chain of upper bounds of $a$; and, similarly, $b' \leq a'$. Hence $a' = b' = a \lor b$. Thus every pair of elements of $S$ has a least upper bound and so $S$ is a join-semilattice.

The argument suggests that our purpose will be attained if we define a tree as being a partially ordered set with a greatest element $I$ (its root) such that the upper bounds of any element form a chain. Since we require the altitude of a tree to be an ordinal number we impose the ascending chain condition, and can then weaken the condition of having a greatest /
greatest element to that of being directed, since any directed set with a maximal element has a greatest element. We thus adopt the following definition, and from now on use the word 'tree' in this sense unless the graph-theoretical meaning is explicitly indicated.

A tree is a directed, partially ordered set in which the upper bounds of any element form a chain and in which every non-null subset has a maximal element. Any tree possesses a greatest element \( I \), and this is called the root. Since the ascending chain condition is satisfied, the order-type of the dual of any chain of a tree is an ordinal number. The altitude of a tree is the upper bound of all those ordinals which are the order-types of duals of chains of the tree which do not contain \( I \). (The altitude of a finite tree is the 'length' of the tree as defined e.g. in Birkhoff [1].)

It is frequently convenient to depict our new trees by means of graphs (as is done in the general case of partially ordered sets). When the usual convention of representing greater elements by higher nodes is observed, the graph of a finite tree is a root-tree (in the sense of graph theory) in which the root is uppermost and all the branches descend. This seems rather odd, but it was in this way that Cayley originally represented his trees, and the method is convenient for later applications. Since there is a unique chain from any /
any element of a tree to any other which is greater than it, the Jordan-Dedekind chain condition is trivially satisfied and the nodes of the graph may be "stratified" in such a way that nodes equidistant from the root are placed on the same level. The altitude is then immediately apparent.

A **subtree** of a tree is a subset of the tree which is itself a tree.

**THEOREM 4.** A necessary and sufficient condition for a subset of a tree to be a subtree is that it be directed.

**Proof.** The condition is obviously necessary. It is sufficient because the other defining properties of trees hold for any subset of any set for which they hold.

**COROLLARY.** If a is any element of a tree, the set of elements less than or equal to a is a subtree of the tree.

The subtree consisting of those elements of a tree which are less than or equal to a given element of the tree is called a **principal subtree** (on analogy with the term 'principal ideal' in lattice-theory.)

**LEMMA.** If a partially ordered set S is such that the upper bounds of any element form a chain, then every non-null subset of S has a maximal element if and only if every chain of S has a greatest element.

**Proof.** If every non-null subset of S has a maximal element, then, in particular, every chain of S must have a maximal element;
element; and maximal elements of chains are greatest elements.

Suppose now that every chain in \( S \) has a greatest element, and let \( S' \) be a subset of \( S \). Let \( a \in S' \), and let \( A \) be the chain of upper bounds of \( a \) in \( S \). Let \( A' = A \cap S' \). Then \( A' \) is a chain and has a greatest element \( a' \). Then \( a' \) is maximal in \( S' \), for if \( a' \leq b \) where \( b \in S' \), then \( a \leq b \) and so \( b \in A' \); hence \( b = a' \).

This lemma is stated and proved here because, when \( S \) is a general partially ordered set, the equivalence of the ascending chain condition on \( S \) and the well-ordering condition on the duals of the chains of \( S \) involves an appeal to the Selection Axiom, whereas such an appeal is avoidable in the present case.

**THEOREM 5.** Let \( x \to x' \) be an isotone mapping of a tree \( A \) onto a partially ordered set \( A' \), such that, if \( a' \leq b' \), then for some \( c \), \( c' = b' \) and \( a \leq c \). Then \( A' \) is a tree. Moreover, the altitude of \( A' \) is not greater than that of \( A \).

**Proof.** Since directedness is preserved under all isotone mappings, in order to show that \( A' \) is a tree we must prove (\( \alpha \)) that in the map \( A' \) the upper bounds of any element form a chain and (\( \beta \)) that every subset of \( A' \) contains a maximal element.

(\( \alpha \)) Let \( a' \leq b' \) and \( a' \leq c' \). Then there are elements \( x \) and \( y \) such that \( x' = b' \) and \( y' = c' \), and \( a \leq x \) and \( a \leq y \). Then /
Then, since $A$ is a tree, $x \leq y$ or $y \leq x$, and consequently $b' \leq c'$ or $c' \leq b'$.

In order to prove (1) we prove that if $a$ is any member of $A$, then the chain of upper bounds in $A'$ of $a'$ is an isotonous map of the chain of upper bounds of $a$. Since the map of a chain has a greatest element if the original chain has, the result will follow by the preceding lemma. Now the chain of upper bounds of $a$ is mapped isotonously onto a subchain of the chain of upper bounds of $a'$; it remains to be proved that this subchain is the whole chain; that is, that if $a' \leq b'$, there is a $c$ such that $a \leq c$ and $c' = b'$; but this is the case by hypothesis.

Suppose now that $A'$ contains a chain whose dual has an ordinal greater than that of the dual of any chain of $A$. Then the dual of this chain has a segment whose ordinal is $1+\alpha$ where $\alpha$ is the altitude of $A$. Let $a'$ be the sequent of this segment, and let $a$ be any antecedent in $A$ of $a'$. Then obviously the upper bounds of $a$ cannot be mapped isotonously onto the whole chain of upper bounds of $a'$, and the result proved in the second paragraph of the proof is contradicted. This completes the proof.

**Theorem 6.** Let $x \rightarrow x'$ be an isotonous mapping of a tree $A$ onto a partially ordered set $A'$ such that, if $a' \leq b'$ and $a \parallel b$, then, for some $c$, $c' = b'$, $a \leq c$ and $b \leq c$. Then $A'$ is a tree.

**Proof /**
Proof. This theorem is a particular case of the preceding one. If \( a' = b' \), then \( a \leq b \) or \( b \leq a \). In the first case we may take \( c \) in the preceding theorem to be \( b \), and, in the second, to be \( a \). In the third case, there is, by hypothesis a \( c \) satisfying the conditions of that theorem (and more besides). If \( a' < b' \), then \( a < b \) or \( a \parallel b \), and these two cases are accounted for just as are the second and third above.

The former of the last two theorems does not cover all the mappings of trees which yield trees, but it gives a fairly general, and, at the same time, simple set of sufficient conditions. The second of the two theorems has been stated explicitly because mappings with the properties given in the enunciation of it preserve more of the structure of the tree \( A \) than do the more general ones. Chains of the tree may "shrink" under them, but two chains cannot "coalesce" unless they are both mapped onto the image of a common upper bound. Because of the various structure-preserving properties of these mappings we adopt the following definition:

A \textbf{tree-homomorphism} is an isotone mapping \( x \rightarrow x' \) of a tree \( A \) onto a tree \( A' \) such that if \( a' \leq b' \) then for some \( c \), \( c' = b' \), \( a \leq c \) and \( b \leq c \). Since of the two preceding theorems the latter is a particular of the former, the tree \( A' \) has altitude at most equal to that of \( A \).

(It /
(It may be remarked in passing that if \( x \rightarrow x' \) is an isotone mapping of a lattice \( L \) onto a lattice \( L' \) - or of a join-semilattice \( L \) onto a join-semilattice \( L' \) - then the mapping is a join-homomorphism if and only if whenever \( a' \leq b' \) there is an element \( c \) such that \( c' = b' \), \( a \leq c \) and \( b \leq c \).

For suppose the mapping \( x \rightarrow x' \) satisfies the condition. It is known that \( a' \lor b' \leq (a \lor b)' \). We have therefore to show that \((a \lor b)' \leq a' \lor b' \). If \( c' = a' \lor b' \), there are, by hypothesis, elements \( x \) and \( y \) such that \( x' = y' = c' = a' \lor b' \), \( a \leq x \), \( b \leq y \), \( c \leq x \) and \( c \leq y \). Then there is a further element \( z \) such that \( z' = a' \lor b' \), \( x \leq z \) and \( y \leq z \). Hence \( a \lor b \leq z \), and so \((a \lor b)' \leq z' = a' \lor b' \).

Suppose \( x \rightarrow x' \) is a join-homomorphism. If \( a' \leq b' \), then \((a \lor b)' = a' \lor b' = b' \), so that the condition is satisfied if we take \( c \) to be \( a \lor b \).

The difference between tree-homomorphisms and more general mappings which yield trees will perhaps be illustrated by the pair of diagrams which follow. The same tree is in them mapped onto dissimilar trees. The second case shows a tree-homomorphism while the first shows a mapping of a kind not covered even by the more general of the theorems, as is seen at once on comparison of altitude.
Various further properties of tree-homomorphisms are considered in the following theorems.

THEOREM 7. Let $A_1'$ be a subset of $A'$ and let $A_1$ be the set of all those elements of $A$ whose images under some tree-homomorphisms from $A$ to $A'$ lie in $A_1'$. Then a necessary and sufficient condition for $A_1'$ to be a (principal) subtree of $A'$ is that $A_1$ be a (principal) subtree of $A$.

Proof. If $A_1$ is a subtree of $A$, then $A_1'$ is a subtree of $A'$ since $x_1 \rightarrow x_1'$ is a tree-homomorphism from $A_1$ to $A_1'$. Suppose $A_1$ is a principal subtree of $A$ and suppose $a' < a_1'$, where $a_1' \in A_1'$. Then there is a $b$ such that $b' = a_1'$, $a \leq b$ and $a_1 \leq b$. Hence $b \in A_1$ and consequently $a \in A_1$ and $a' \in A_1'$. This proves the condition sufficient in both cases.

Let now $A_1'$ be a subtree of $A'$. Then the set $A_1$ is a subtree of
Let a and b be two members of \( A \). If it is directed, then \( a' \) and \( b' \) are in \( A' \), and hence have an upper bound \( c' \) in \( A_1' \). As a consequence of the definition of tree-homomorphism, there is an upper bound \( d \) of \( a \) and \( c \) in \( A_1 \), such that \( d' = c' \). Similarly, there is an upper bound \( e \) of \( b \) and \( c \) in \( A_1 \) such that \( e' = c' \). By a repetition of the argument, there is in \( A_1 \) an upper bound \( f \) of \( d \) and \( e \) such that \( f' = d' = e' = c' \), and \( f \) is therefore an upper bound of \( a \) and \( b \).

Finally, let us suppose that \( A_1' \) is a principal subtree of \( A' \). Let \( a \leq a_1 \) where \( a_1 \in A_1 \). Then \( a' \leq a_1' \) and \( a_1' \in A_1' \); consequently \( a' \in A_1' \). Hence \( a \in A_1 \).

(In the above proof, when an element \( x' \) of \( A' \) has been chosen, we mean by "x" an arbitrary antecedent of \( x' \).)

For the truth of the theorem it is necessary that \( A_1 \) contain all the antecedents of each element of \( A_1' \). That the map of a principal subtree is not always a principal subtree and that, when a subtree \( A_1 \) is mapped onto a principal subtree \( A_1' \), the former is not always a principal subtree are both demonstrated by a very simple example.

![Diagram]

The figure represents a tree-homomorphism from a tree of three elements onto a tree of two elements. The subtree consisting...
consisting of the elements represented by the two blackened nodes is not a principal subtree but is mapped onto the whole of the tree on the right; and every tree is a principal subtree of itself. The unblackened node of the diagram on the left represents an element which constitutes a principal subtree, but its image is not a principal subtree of the tree represented on the right.

**Theorem 8.** If $A'_1$ is a convex subtree of $A'$, the set $A'_1$ of the antecedents in $A$ of its elements with respect to a tree-homomorphism from $A$ onto $A'$ is a convex subtree of $A$.

**Proof.** The set $A'_1$ is a subtree of $A'_1$ by Theorem 7. Let now $b \leq c$ where $b \in A'_1$, $c \in A'_1$. Then, for all $x$, if $b \leq x \leq c$, then $b' \leq x' \leq c'$, and so $x' \in A'_1$. Hence $x \in A'_1$.

**Corollary.** If $a' \in A'$, then the set of antecedents of $a$ is a convex subtree of $A$.

**Theorem 9.** The relative product of two tree-homomorphisms is a tree-homomorphism; that is, the relation of being a tree-homomorph is transitive.

**Proof.** Let $x \rightarrow x'$ and $x' \rightarrow x''$ be tree-homomorphisms from a tree $A$ onto a tree $A'$ and from $A'$ onto a third $A''$. Then the mapping $x \rightarrow x''$ is isotone. Suppose $a'' \leq b''$. Then there is an element $c'$ such that $c'' = b''$, and $a' \leq c'$ and $b' \leq c'$. Hence there are elements $d$ and $e$ such that $a \leq d$, $c \leq d$. /
If \( c \leq d \) and \( d' = c' \), and \( b \leq e \), \( c \leq e \) and \( e' = c' \). Since \( A' \) is a tree, \( d' \leq e' \) or \( e' \leq d' \); consequently there is an upper bound \( f \) of \( d \) and \( e \) such that \( f' = d' = e' = e' \). Hence \( f'' = c'' = b'' \). This proves the theorem.

**NOTE.** In BIRKHOFF [1] (p.47), a tree is defined as being a partially ordered set in which the upper bounds of any element form a chain. With this definition, every subset of a tree is itself a tree. The concept of principal subtree appears, in this case, to be unimportant. We may define an equivalence-relation \( \pi \) on the elements of such a set \( S \) by letting \( a \pi b \) if and only if there is an upper bound \( x \) for \( a \) and \( b \) in \( S \). (The relation \( \pi \) is obviously symmetric. It is also transitive, since, if \( a \leq x \), \( b \leq x \), \( b \leq y \) and \( c \leq y \), then \( x \leq y \) or \( y \leq x \) since \( x \) and \( y \) are both upper bounds of \( b \); the greater of \( x \) and \( y \) is an upper bound for \( a \) and \( c \).) The subsets of \( S \) determined by \( \pi \) may be called portions of \( S \). A tree-homomorphism (defined just as for the more restricted trees of the text) then has not only the structure-preserving properties noted previously, but also the property of mapping distinct portions onto distinct portions.

A set-theoretical generalisation of trees without roots is given in SHOLANDER [1].
A simple partition $\pi$ of a non-null class $C$ is an equivalence relation whose field is $C$. It separates $C$ into equivalence-classes, $x$ and $y$ going into the same class if $x \pi y$ and not otherwise. The subclasses of $C$ thus arising are called the $\pi$-subclasses of $C$. The $\pi$-subclasses of $C$ are non-null and disjoint and together include the whole of $C$. Simple partitions may be represented by diagrams showing a rectangle (representing $C$) divided into smaller rectangles (representing the $\pi$-subclasses).

It is natural to proceed to the consideration of the relation-theoretical union and intersection of simple partitions. The intersection $\bigcap_{i} \pi_{i}$ of a class of equivalences $\{\pi_{i}\}$ is again an equivalence; and if, for each $i$, $\pi_{i}$ is a simple partition of a class $C_{i}$, then $\bigcap_{i} \pi_{i}$ is a simple partition of $\bigcap_{i} C_{i}$, provided this class is not null. We proceed now to consider two special cases of union and intersection called respectively the sum and product of simple partitions.

When, for each $i$, $\pi_{i}$ is a simple partition of a class $C$, the intersection $\bigcap_{i} \pi_{i}$ is also a simple partition of $C$ which separates $C$ into all non-null subclasses of the form $\bigcap_{i} C_{i}$ where $C_{i}$ is a $\pi_{i}$-subclass. This simple partition we call the product of the simple partitions $\pi_{i}$. The product of $\pi_{i}$ and $\pi_{j}$ is denoted by $\pi_{i} \cdot \pi_{j}$. If, in the diagrams representing $\pi_{i}$ and $\pi_{j}$, the divisions run vertically /
vertically in the former case and horizontally in the second, then \( \pi_1 \cdot \pi_2 \) is represented by the diagram showing both together.

![Diagram](image)

The relation which holds only between each element of \( C \) and itself is a simple partition \( \pi_0 \) of \( C \) which separates \( C \) into all subclasses which consist each of a single element. It obeys

\[
\pi_0 \cdot \pi = \pi_0
\]

for all simple partitions \( \pi \) of \( C \). For this reason and because of its additive properties it is called the **zero simple partition** of \( C \).

The simple partition \( \pi_e \) which holds between any pair of elements of \( C \) gives rise to only one \( \pi_e \)-subclass, namely \( C \) itself. Since it satisfies

\[
\pi_e \cdot \pi = \pi
\]

for all simple partitions \( \pi \) of \( C \), it is called the **unit simple partition** of \( C \).

The simple partitions \( \pi_e \) and \( \pi_0 \) are respectively the \( 1 \) and \( 0 \) of the lattice of simple partitions of \( C \). (See ORE [1], BIRKHOFF [4].)

Multiplication is an operation defined for simple partitions of the same class \( C \). Addition, on the other hand, is defined only for simple partitions \( \pi_i \) of disjoint classes \( C_i \). The union of /
of all the simple partitions $\pi_i$ is then a simple partition of the union $C = \bigcup_i C_i$ of all the disjoint classes $C_i$. It is called the sum of the simple partitions $\pi_i$ and is denoted by $\sum \pi_i$. This is a particular case of the "strictly disjoint sum" of relations defined in TARSKI [1], p.245. To represent $\pi_i + \pi_j$ diagramatically, it suffices to place the diagrams for $\pi_i$ and for $\pi_j$ side by side.

$\pi_i + \pi_j$:

We see at once that the sum of the zero simple partitions of $C_1$ and $C_2$ (assumed disjoint) is the zero simple partition of the union $C_1 \cup C_2$.

There is no need here to develop in detail the algebra of simple partitions. It may be remarked that multiplication and addition are both commutative and associative, that addition is distributive over multiplication but not vice-versa, that every element is idempotent with respect to multiplication; and that

$$(\pi_i + \pi_j)(\pi_i' + \pi_j') = \pi_i \pi_i' + \pi_i \pi_j'$$

whenever both sides have meaning (see figure).
A subrelation \( \pi' \) of a simple partition \( \pi \) of a class \( C \) is called a portion (Tarski [1], p.245) of \( \pi \) if it is the relation which arises from restricting the field of \( \pi \) to \( C' \), where \( C' \) is a union of \( \pi \)-subclasses of \( C \). Equivalently, \( \pi' \) is a portion of \( \pi \) if \( \pi' \) is a simple partition of a subclass \( C' \) of \( C \), and \( C'' \), the complement of \( C' \) in \( C \), is null or there is a simple partition \( \pi'' \) of \( C'' \) such that \( \pi = \pi' + \pi'' \).

The relation of being a portion is seen to be reflexive, transitive and antisymmetric. It is included in inclusion (\( \subseteq \)) and the portions of a simple partition form a partially ordered set under \( \subseteq \). In addition it is not difficult to prove the following.

**Theorem 10.** The portions of a simple partition form a complete join-semilattice under \( \subseteq \). With the adjunction of the null relation, they form a complete lattice.

**Theorem 11.** If, for all \( i \), \( \pi'_i \subseteq \pi_i \), and \( \sum \pi_i \) exists, then \( \sum_{i} \pi'_i \subseteq \sum \pi_i \). If, for all \( i \), \( \pi'_i \) is a portion of \( \pi_i \), and \( \sum_{i} \pi_i \) exists, then \( \sum_{i} \pi'_i \) is a portion of \( \sum_{i} \pi_i \).

**Proof.** The first part is obvious. Let for each \( i \), \( \pi'_i \) have field \( C_i \) and \( \pi'_i \) have field \( C'_i \). Let \( C_i'' = C_i - C'_i \), and let \( \{ j \} \) be the subset of \( \{ i \} \) for which \( C_i'' \) is not null. If, for each \( j \), there is a simple partition \( \pi''_j \) of \( C_j'' \) such that \( \pi'_j + \pi''_j = \pi_j \), then \( \sum_{i} \pi_i = \sum_{i} \pi'_i + \sum_{i} \pi''_i \). Hence the theorem.
theorem.

**THEOREM 12.** Let a set \( \{\pi_{ij}\} \) of simple partitions be such that, for all \( i \), the field of \( \pi_{ij} \) is included in \( C_i \), where the classes \( C_i \) are mutually disjoint. Then

\[
\bigcup_{i} \bigcap_{j} \pi_{ij} = \bigcap_{j} \bigcup_{i} \pi_{ij}
\]

provided, for each \( i \), the relation \( \bigcup_{j} \pi_{ij} \) is not null.

**Proof.** Since, in the theory of relations, the general distributive law \( \bigcap_{i} \bigcup_{j} \pi_{ij} = \bigcup_{i} \bigcap_{j} \pi_{ij} \) holds, it is necessary only to verify that the equivalences \( \bigcup_{j} \pi_{ij} \), being non-null by hypothesis, are also mutually disjoint. But for each \( i \), \( \bigcap_{j} \pi_{ij} \) is a simple partition of a subclass of \( C_i \), and these classes are mutually disjoint. The theorem follows.

A simple partition \( \pi \) of \( C \) is said to operate on a class \( C' \) when \( C' \) is the field of a partition of \( \pi \); that is, when \( C' \) is a non-null subclass of \( C \) which either wholly includes or totally excludes each \( \pi \)-subclass of \( C \).
GENERAL PARTITIONS.

A general partition of a class C, or more briefly, a partition of C, is an ascending chain $\Pi$ of simple partitions

$$\Pi: \pi_0, \pi_1, \pi_2, \ldots$$

where

1. $\pi_0$ is the unit simple partition of C,
2. $\pi_{\cap i} \subseteq \bigcap \pi_i$ for any set $\{i\}$ of subscripts,
3. the field of $\pi_{\cap i}$ is the field of a portion of $\bigcap \pi_i$,
4. if $x \pi_i y$ is true, and $x \neq y$, then there is a (subsequent) $\pi_j$ such that $x \pi_j y$ is false, but $x \pi_j x$ and $y \pi_j y$ are true.

The ordinal of the chain $\Pi$ when its first term is omitted is called the altitude of the partition.

The meaning of (1) is clear. Condition (ii) implies that $\pi_{i+1}$ is a refinement of $\pi_i$ when the field of the latter is restricted to that of the former. Condition (iii) implies that if $\pi_{i+1}$ operates on a class C', then it operates on some union of $\pi_i$ - subclasses of C which includes C'. Hence, (ii) and (iii) together imply that $\pi_{i+1}$ is a sum of simple partitions of some of the $\pi_i$ - subclasses of C. The general statement of (ii) and (iii) which has been adopted is required in order to cover also the case of limit-ordinals; if i is a limit-ordinal, then $\pi_i$ bears to the intersection of all its predecessors the same relation /
relation as that explained as holding between $\pi_{i+1}$ and $\pi_i$.

Condition (iv) states, loosely speaking, that every element of $C$ is ultimately separated from any other. (The word 'subsequent' is placed in brackets since it may be deduced from (ii) and so need not be stated independently.)

Some of the consequences of these conditions are considered in the following pages, and a method of representing partitions by means of trees is worked out.

It should be noted that the above definition generalises that given in Etherington [1], p. 445. In that paper, (ii) is, in effect, replaced by

$$(ii)' \quad \text{if } \{i\} \text{ is any set of subscripts, and if } \alpha \text{ is a portion of } \bigcup_{i} \pi_{i} \text{ and } \beta \text{ is } \pi_{\text{seq}}\{i\} \text{ restricted to the field of } \alpha, \text{ then } \beta \subset \alpha.$$

Condition (ii)' implies that every $\pi_k$-subclass is a proper subclass of some $\pi_\ell$-subclass whenever $k > \ell$, whereas condition (ii) demands only that it be a subclass. (Cf. the lemma below.) A consequence of this is that the altitude of $\Pi$ may be transfinite even though $C$ is a finite class. For example, if $C$ is a unit class, its unit and zero simple partitions coincide; and so any ascending chain (finite or transfinite) all of whose elements are this simple partition constitutes a partition of $C$. If (ii)' were imposed, however, the only possible partition of $C$ would be the unit chain whose only element is this simple partition. The stronger condition (ii)', while it would /
would simplify some of the subsequent discussion, is not essential to any of the main arguments of the theory.

If we form the set $A$ of all subclasses $X$ of $C$ such that $X$ is a $\pi_i$-subclass of $C$ for some $i$, then $A$ forms a tree under inclusion. (This fact appears below as Theorem 14). The representation of partitions $\mathbb{P}$ by such trees would not, however, be satisfactory. A subclass $X$ of $C$ may appear as a $\pi_i$-subclass and as a $\pi_{i+1}$-subclass, and from the representation of $\mathbb{P}$ by $A$ we should not be able to tell whether or not this were the case. For example, if $C$ is a unit class we have seen that many partitions may be imposed on it; but the tree $A$ representing any one of these would also arise from any other. It would, in fact, consist of the single element $C$ no matter what partition $\mathbb{P}$ we might choose. Hence we resort to the usual device of forming ordered pairs in order to give the subclasses the desired multiplicity in the representation. (When condition (ii) is replaced by (ii)' this difficulty obviously does not arise.)

We construct for each $\pi_i$-subclass $C_i$ the ordered pair $(C_i, i)$, and then form the class $B$ of all such ordered pairs. We define a relation of inequality in $B$ by letting

\[(C_i, i) \leq (C_j, j)\]

mean that $i \geq j$ and $C_i \subseteq C_j$. In this way we get the representation we want. Taking again the case in which /
which $C$ is a unit class, if $\pi$ is the partition of altitude $\alpha$ of $C$, then, in the representation $B$ of $\pi$, $C$ will be paired successively with $0, 1, 2, \ldots$, the ordered pairs forming a descending chain whose dual has ordinal $\omega + \alpha$; this representation is thus seen to be dependent on $\alpha$, whereas the other was not.

**Lemma.** If $k \geq \ell$, every $\pi_k$-subclass is included in one and only one $\pi_\ell$-subclass.

**Proof.** If $k = \ell$ the lemma is obvious. Let now $k > \ell$, and let $\{i\}$ be the set of ordinals preceding $k$. Then $k = \text{seq} \{i\}$. Hence, by (ii),

$$\pi_k \subseteq \bigcap_{i \in \{i\}} \pi_i \subseteq \pi_\ell$$

since $\ell \in \{i\}$. Hence every $\pi_k$-subclass is included in some $\pi_\ell$-subclass. But, if any $\pi_k$-subclass were included in two $\pi_\ell$-subclasses, it would be included also in their intersection, and this is always null. Hence the lemma.

**Theorem 13.** The set $B$ with the relation $\leq$ is a tree whose altitude is the altitude of $\pi$.

**Proof.** The set $B$ is partially ordered since it is a subset of the direct (i.e. cardinal) product of two partially ordered sets. In addition, $B$ is directed since all its elements $X$ satisfy

$$X \leq (C, C)$$

The/
The ascending chain condition holds, for if, $B' \subseteq B$, let $B''$ consist of those members $(C_i, i)$ of $B'$ for which $i$ is least. Then the members of $B''$ are maximal in $B'$ by the preceding lemma.

Let $C_k$ be a particular $\pi_k$-subclass, and let $C_\xi$ and $C_m$ be those $\pi_\xi$- and $\pi_m$-subclasses in which $C_k$ is included, where $k \geq \xi \geq m$. Then $C_\xi$ is included in some $\pi_m$-subclass; let it be $C_m'$. But then $C_k \subseteq C_\xi \subseteq C_m'$ and so $C_k$ is included in both $C_m$ and $C_m'$. Hence $C_m' = C_m$ and $C_\xi \subseteq C_m$. The upper bounds of $(C_k, k)$ therefore form a chain. Thus $B$ is a tree.

The dual of the chain of upper bounds of $(C_k, k)$ contains the root $(C, 0)$ and has ordinal $k + 1$. Since the ordinal of a well-ordered chain is the sequent of the ordinals of its segments, $1 + \alpha$, where $\alpha$ is the altitude of $B$, is the least upper bound of all the ordinals $k$, and thus is the altitude of $\Pi$. This completes the proof.

**Theorem 14.** The set $A$ (defined above) forms a tree under $\subseteq$, and the mapping $(C_i, i) \to C_i$ is a tree-homomorphism from $B$ onto $A$.

**Proof.** Since the first part of the theorem is a consequence of the second, it suffices to prove the second. The mapping in question is isotone. Suppose now $C_i \subseteq C_j$. Let $k$ be the least ordinal such that $C_k = C_j$ for some $\pi_k$-subclass.
\(\pi_k\)-subclass \(C_k\). Then \((C_j, j) \leq (C_k, k)\). But \(C_i \subseteq C_k\), and if, in addition, \(i < k\), we must have \(C_i = C_k\) (since if \(i < k\), then \(C_i\) must include any \(\pi_k\)-subclass which overlaps it, by (ii); and consequently \(k\) is not the least ordinal such that \(C_k = C_j\). Hence \(i \geq k\), and so \((C_i, i) \leq (C_k, k)\). Thus \((C_k, k)\) is an upper bound for \((C_i, i)\) and \((C_j, j)\) such that \(C_k = C_j\). This establishes the theorem.

In virtue of Theorem 13, we now have at our disposal a graphical representation of those partitions of finite classes whose altitude is finite, for these partitions give rise to finite trees \(B\). When the graph of the tree \(B\) is

\[
\begin{array}{c}
\pi_5 \\
\pi_4 \\
\pi_3 \\
\pi_2 \\
\pi_1 \\
\pi_0
\end{array}
\]

stratified, the nodes on a given level represent the \(\pi_i\)-subclasses for some \(i\). In the figure, the root-tree on the left represents a partition \(\Pi\) of a class \(C\) of four elements. The simple partitions \(\pi_0\) and \(\pi_1\) are both the unit simple partition of \(C\). The single \(\pi_1\)-subclass is separated into three \(\pi_2\)-subclasses /
$\pi_2$-subclasses consisting of one, one, and two elements each. The simple partition $\pi_3$ operates on the first and third of these only; it is the sum of the unique simple partition of the former and the zero simple partition of the latter. Finally, $\pi_4$ operates only on the first $\pi_3$-subclass and is the unique simple partition of this unit class. The altitude is 4. On the right is the graph of the tree $A$ derived from the same $\pi$, and the arrows show the tree-homomorphism.

**THEOREM 15.** The tree $B$ representing $\pi$ satisfies:

(v) if $(X, i) \in B$ and $(Y, j) \in B$, then $X \subseteq Y$ or $Y \subseteq X$ or $X \cap Y = \emptyset$, and if $i = j$, then $X = Y$ or $X \cap Y = \emptyset$.

(vi) if $(X, i) \in B$, $x \in X$, $y \in Y$ and $x \neq y$, then there are elements $(Y_1, j)$ and $(Y_2, j)$ of $B$ (for some $j$) such that $Y_1 \cap Y_2 = \emptyset$, $x \in Y_1$ and $y \in Y_2$.

(vii) if $(X, i) \in B$ and $i > j$, then there is an element $(Y, j)$ of $B$ such that $X \subseteq Y$.

**Proof.** The proof presents no difficulties. It depends on (i), (ii) and (iv).

The following theorem states the converse of the preceding one.

**THEOREM 16.** Let $K$ be a tree whose elements are ordered pairs $(X, i)$ where $X$ is a non-null class and $i$ is an ordinal number and ' $(X, i) \leqslant (Y, j) \prime$ means that $X \subseteq Y$ and $i \geq j$. If $K$ satisfies (v) - (vii) (when 'X' is written for 'B'), then there is a class $C$ and a partition $\pi$ of $C$ such that $K$ is
is the tree $B$ representing $\mathcal{T}$; and both $C$ and $\mathcal{T}$ are uniquely determined.

**Proof.** Let $(C, \mathcal{L})$ be the root of the tree. Then $\mathcal{L} = \mathcal{Q}$, by (vii). Let $\rho_2$ be the unit simple partition of $C$.

Suppose that $\rho_i$ has been defined for all ordinals $i$ which precede $k$.

If there exists an element $(X, j)$ of $\mathcal{K}$ where $j \geq k$, then there is an element $(Y, k)$ of $\mathcal{K}$, by (vii). If there is no such element $(X, j)$ of $\mathcal{K}$, we let $\rho_k$ be the null relation. Suppose now that there are elements for which the second member of the pair is $k$. Let $(Y_j, k), j \in \{j\}$ be all such. Then the $Y_j$ are disjoint, by (v), and not null. Hence there is an equivalence relation whose equivalence-classes are the $Y_j$. We define $\rho_k$ to be this equivalence.

In this way $\rho_k$ is defined by a process of transfinite induction for all ordinals $k$. Let $\{i\}$ be the set of ordinals for which $\rho_i$ is not null; then, by (vii), $\{i\}$ is a segment of the chain of all ordinals. For each $i$ of $\{i\}$ we define $\mathcal{T}_i$ to be $\rho_i$. In this way an ascending chain

$$\mathcal{T}: \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \ldots$$

is constructed, all of whose members are equivalences and which is not null, since $\rho_2$ is not null. We have to show that /
that $\pi$ is a partition of $C$.

Now (i) is obviously satisfied.

If $k > j$, then $\pi_k \subseteq \pi_j$ by (vii), and so if $k - \text{seq} \{j\}$, then $\pi_k \subseteq \bigcap_{j \in \mathcal{J}} \pi_j$. Thus (ii) is satisfied.

Let $X$ be a $\pi_j$-subclass, i.e. let $(X, j) \in \mathcal{K}$. Suppose $(Y, k) < (X, j)$. If $Y \subset X$, then there is an $x$ such that $x \in X$ and $x \notin Y$. Let $y \in Y$; then, by (vi), there are elements $(Y_1, \ell)$ and $(Y_2, \ell)$ such that $x \in Y_1$ and $y \in Y_2$ and $Y_1 \cap Y_2 = \varnothing$. If $\ell \geq k$, then there are $\pi_k$-subclasses, one containing $x$ and one containing $y$. That is, for any $j$ and $k$ such that $k > j$, if $x$ is a $\pi_k$-subclass, then, for any element $x$ of $X$, there is a $\pi_j$-subclass containing $x$, or there is, for some $\ell$ intermediate between $k$ and $j$, a $\pi_{\ell}$-subclass which does not contain $x$, but which is included in $X$. Let now $\{\ell\}$ be the class of all predecessors of $k$, $k - \text{seq} \{j\}$, and let $Y_k$ be any $\pi_k$-subclass. Let, for each $j$, $Y_j$ be that $\pi_j$-subclass which includes $Y_k$. Then the $Y_j$ form a chain under $\subseteq$, by (v). Now any element which is contained in each $Y_j$ must be contained in some $\pi_k$-subclass. Hence $\bigcap_{j \in \mathcal{J}} Y_j$ is included in the field of $\pi_k$. But $\bigcap_{j \in \mathcal{J}} Y_j$ is an equivalence-class of $\bigcap_{j \in \mathcal{J}} \pi_j$; and any such equivalence-class is expressible by the intersection of such a chain. Hence the field of $\pi_k$ includes any equivalence-class of $\bigcap_{j \in \mathcal{J}} \pi_j$ which overlaps it. Thus (iii) /
(iii) is verified for \( \mathcal{W} \).

It is seen that (iv) follows from (vi).

It is a consequence of these two theorems that conditions (i) - (iv) are not independent, but that (iii) follows from the other three.

**THEOREM 17.** If \((X, i) \in B, (Y_0, i_0) \in B \) and \( Y_0 \subseteq X \), then there is \( \mathfrak{c} \) (possibly null) class of elements \((Y_r, i_r)\) of \( B \) such that, if \( \{s\} = \{r\} \cup \{0\} \), then \( \bigcup_{i \in \mathfrak{c}} Y_s = X \) and the \( Y_s \) are disjoint.

**Proof.** If \( i \geq i_0 \), then \( Y_0 = X \) and so the class \((Y_r, i_r)\) is null. Suppose \( i_0 > i \). Let now \((Y_r, i_r)\) consist of those \( \pi_{i+1} \)-subclasses (each paired with \( i + 1 \)) which are included in \( X \) but do not include \( Y_0 \), and those \( \pi_{i+2} \)-subclasses (each paired with \( i + 2 \)) which are included in \( X \) but not in any of the selected \( \pi_{i+1} \)-subclasses and do not include \( Y_0 \), and so on by a transfinite inductive process.

Then the \( Y_s \) are obviously disjoint. Suppose that \( x \in X \) but \( x \) is not contained in any of the classes \( Y_s \). Let \( \{j\} \) be the set of ordinals such that \( j > i \) and \( x \) is contained in the field of \( \pi_j \). For each \( j \) of \( \{j\} \), let \( Z_j \) be the \( \pi_j \)-subclass containing \( x \). Then, for all \( j \), \( i_0 > j \). For if \( x \in Z_j \) and \( j \geq i_0 \), then \( x \) must be contained in some \( \pi_{i_0} \)-subclass, and this subclass, or some class including it.
it, must, by the method of construction of \( \{Y_r^l\} \), be contained in \( \{Y_r^l\} \); for it cannot include \( Y_0 \) and must be included in \( X \).

For all \( j \), \( Z_j \) is a class containing \( x \), including \( Y_0 \) and included in \( X \). Let \( \{k\} \) be a subchain of \( \{j\} \). Then, by (iii), \( x \) is contained in some \( \prod_{\text{seq}} \{k\} \)-subclass, and so seq \( \{k\} \in \{j\} \). Hence, by the principle of transfinite induction, every ordinal greater than \( i \) is a member of \( \{j\} \). But this is impossible since \( i_0 \) is an upper bound for \( \{j\} \). Thus there is no such element as \( x \), and the theorem is proved.

The class \( \{(Y_r, i_r)\} \), whose existence is here established may be called a class of **complements** of \( (Y_0, i_0) \) in \( (X, i) \). If \( i = 0 \), we call the class \( (Y_0, i_0) \) a complete class of complements in \( B \). The particular method of construction used in the proof of the theorem leads to what we may call the class of **primary** complements of \( (Y_0, i_0) \) in \( (X, i) \). (This use of the word 'primary' has connections with the term 'primary power' as used in Etherington [2].) That the class of complements of \( (Y_0, i_0) \) in \( (X, i) \) is not in general unique follows from the fact that any member \( (Y_r, i_r) \) of it may be replaced by any inferior element together with a class of complements of it in \( (Y_r, i_r) \). It can be proved that the cardinal of the class of primary complements is at most equal /
equal to that of any other class of complements.

In the figure, the class of primary complements of the element represented by the inked terminal node in that represented by the root-node consists of the elements represented by the three nodes marked in pencil.

This section is concluded by an appendix introducing the concept of "subpartition".

**THEOREM 18.** Let \( \Pi \) be a partition of \( C \) and let \( C' \) be one of the \( \Pi_n \)-subclasses of \( C \). Let a chain

\[ \Pi': \Pi_0', \Pi_1', \Pi_2', \ldots \]

be constructed by restricting to \( C' \) the fields of those members of the chain

\[ \Pi_n, \Pi_{n+1}, \Pi_{n+2}, \ldots \]  

(1)

whose fields overlap \( C' \). Then \( \Pi' \) is a partition of \( C' \) in which the \( \Pi_i' \)-subclasses of \( C' \) are those \( \Pi_n+i \)-subclasses of \( C \) which are included in \( C' \).

**Proof.** The unit simple partition of \( C' \) is obviously \( \Pi_n \) restricted to \( C' \) and is thus \( \Pi_0' \); hence \( \Pi' \) satisfies (i).

For all \( i \), \( \Pi_{n+i} \) restricted to \( C' \) must be a portion of \( \Pi_{n+i} \) or be null, for every \( \Pi_{n+i} \)-subclass of \( C \) which overlaps \( C' \) must, by (ii), be included in \( C' \). Also, if \( i \leq j \) and /
and there is a \( \pi_{n+j} \)-subclass of \( C \) included in \( C' \), there
must be a \( \pi_{n+i} \)-subclass included in \( C' \) and including the
\( \pi_{n+j} \)-subclass, by the lemma above. Hence the subset of
(1) consisting of those members which appear with restricted
fields in \( \Pi' \) is a segment of the chain (1) or the
chain itself. There is, consequently, a unique isotone
correspondence between \( \Pi' \) and the subchain in question,
and so \( \pi'_i \) must, for each subscript-ordinal \( i \), be \( \pi_{n+i} \)
restricted to \( C' \).

Condition (ii) must now be satisfied by \( \Pi' \) because
the relation \( \subseteq \) remains valid when fields are similarly
restricted. Condition (iii) is also satisfied because, if
it were not, there must be an element \( x \) contained for some \( i \),
in that \( \pi'_i \)-subclass which includes some fixed \( \pi'_i \)-\( \pi'_{i+1} \)-subclass,
and not contained in any \( \pi'_i \)-\( \pi'_{i-1} \)-subclass. But
then the same must hold when the dashes are removed and
'\( n + i \)' is substituted for 'i', and so \( \Pi' \) could not satisfy
(iii). Condition (iv) is satisfied also, because, if it
were not, it obviously could not be satisfied by \( \Pi' \) either.
This completes the proof.

In virtue of this theorem we may call \( \Pi' \) a subpartition
of \( \Pi \). Thus each pair \( (C_n, n) \) determines a subpartition of
\( \Pi \). In terms of the representation of partitions by trees, the last
theorem appears as the following.

THEOREM /
THEOREM 19. If $\Pi$ is represented by the tree $B$ and $\Pi'$ by $B'$ and if $\Pi'$ is the subpartition of $\Pi$ determined by the pair $(C_n, n)$, then $B'$ is isomorphic to a principal subtree of $B$ which is mapped onto $B'$ by $(C_{j'}, j) \rightarrow (C_{j'}, k)$ where $k$ is the (unique) ordinal such that $n + k = j$ and the $C_{j'}$ are the $C_j$ which are included in $C_n$.

It follows that when $B$ is represented by a graph, the tree $B'$ is represented by the subgraph consisting of the node representing $(C_n, n)$ and all nodes and edges below it.

ADDITION of PARTITIONS.

Let $\{i\}$ be a non-null class such that to every member $i$ there corresponds a class $C_i$ and a partition

$\{\Pi\} : \{\pi_2, \pi_1, \pi_2, \ldots\}$

of $C_i$, where the classes $C_i$ are disjoint and the altitude of $\{\Pi\}$ is $\alpha_i$. Let $C = \bigcup C_i$; we define an ascending chain

$\Pi : \pi_2, \pi_1, \pi_2, \ldots$

by letting $\pi_2$ be the unit simple partition of $C$, and letting, for each ordinal $k$,

$\pi_{k+1} = \sum_{\{j^{(k)}\}} j^{(k)} \pi_k$

where $\{j^{(k)}\}$ is the subclass of $\{i\}$ consisting of those elements $i$ for
for which \( k < 1 + \alpha_1 \).

Now \( \Pi \) is a partition of \( C \); for (i) is obviously satisfied, (ii) and (iii) hold by virtue of Theorems 11 and 12, and (iv) is readily verified. We say that \( \Pi \) is the sum of the partitions \( i_1 \Pi \) and write
\[
\Pi = \sum_{i_1}^\Pi,
\]
or, in simple cases,
\[
\Pi = i_1 \Pi + i_2 \Pi + \ldots + i_n \Pi.
\]
It follows immediately from this definition that, if \( \alpha \) is the altitude of \( \Pi \), then \( \alpha = 1 + \sup \{ \alpha_i \} \).

**Theorem 20.** If \( \Pi \) is a partition of altitude \( \alpha > 0 \), then there is a unique class \( \{ i_1 \Pi \} \) of subpartitions such that
\[
\Pi = \sum_{i_1}^\Pi \quad (1)
\]

**Proof.** Let \( \{ C_i \} \) be the class of \( i_1 \)-subclasses of \( C \), where \( C \) is the class partitioned by \( \Pi \). If we let \( i_1 \Pi \) be the subpartition of \( \Pi \) determined by \( (C_i, 1) \), then (1) holds. It follows from the definition of addition that if the class \( \{ i_1 \Pi \} \) is replaced by a different class, the resulting sum is different.

**Corollary.** The addition of partitions is not associative.

Since the partitions \( i_1 \Pi \) are the subpartitions of \( \Pi \) determined by the pairs \( (C_i, 1) \), we have, by the last theorem
of the preceding section, the rule that, to construct the graph representing the partition $\Pi$, we place the graphs representing the partitions $^1\Pi$ side by side and join their roots to a new root. For example, in the figure

```
the first graph represents the sum of partitions represented by the other three.

The class $\{i\}$ may have a single member; in that case $\Pi$ is not $^1\Pi$ itself, but the partition derived from it by pre-fixing the unit simple partition of $C_1$ to its chain of simple partitions. If $^1\Pi$ is represented by the last graph in the figure above, then $\Pi$ is represented by the graph:
```

**MULTIPLICATION of PARTITIONS**

The product of two partitions $^1\Pi$ of a class $C_1$ and $^0\Pi$ of a class $C_0$, in that order, is to be a partition $\Pi$ of the product /
product-class \( C \) of \( C_1 \) and \( C_0 \); that is, of the class of all ordered pairs \((x_1, x_0)\) where \( x_1 \in C_1 \) and \( x_0 \in C_0 \). This product-class \( C \) can be regarded as being formed by replacing each \( x_0 \) of \( C_0 \) by the members \((x_1, x_0)\) of the class similar to \( C_1 \) which arises when each \( x_1 \) of \( C_1 \) is paired with the \( x_0 \) under consideration. The partition \( \Pi \) then partitions \( C \) by first imposing the simple partitions induced on it by the successive members of \( \overset{0}{\Pi} \) (which apply to the second term of each pair) and then imposing on each of the classes \( \{(x_1, x_0)\} \) (for a given \( x_0 \)) the successive simple partitions induced by the members of \( \overset{1}{\Pi} \) (except for the first, \( \overset{1}{\Pi}_0 \), which is to be regarded as being absorbed into the intersection of all the preceding simple partitions - cf. the remarks on the definition of a general partition). We proceed to give a precise definition.

For each \( x_1 \) of \( C_1 \) \((i = 0,1)\) let \( K_1(x_1) \) be the segment of \( \overset{i}{\Pi} \) consisting of all those members whose fields contain \( x_1 \). Let \( k_1(x_1) \) be the ordinal of this segment when its first term is omitted. Then

\[
\text{seq } K_1(x_1) = \overset{i}{\Pi}_1 + k_1(x_1)
\]

if the member in question exists. Let a sequence

\[
\Pi: \overset{0}{\Pi}, \overset{1}{\Pi}, \overset{2}{\Pi}, \ldots
\]

whose elements are simple partitions of subclasses of \( C \), be defined as follows:-

\[
\overset{i}{\Pi}/
\]
\( \pi_j \) holds between \((x_1, x_0)\) and \((y_1, y_0)\) if and only if either \( j < \frac{1}{1} + \min\{k_0(x_0), k_0(y_0)\} \) and \( x_0 = y_0 \), or, simultaneously,

\[
\frac{1}{1} + k_0(x_0) < j < \frac{1}{1} + k_0(x_0) + \min\{k_1(x_1), k_1(y_1)\}
\]

\( x_0 = y_0 \)

\[
x_1 = \frac{1}{1} + j \frac{1}{1} y_1,
\]

where \( j^1 \) is the (unique) ordinal such that \( \frac{1}{1} + k_0(x_0) + j^1 = j \).

We denote \( \Pi \) by \( \langle 1, 0 \rangle \times 0 \Pi \); the multiplier is placed on the right and the multiplicand on the left. We call \( \Pi \) the product of \( 1 \Pi \) by \( 0 \Pi \).

**Theorem 21.** The ascending chain \( \Pi = \langle 1, 0 \rangle \times 0 \Pi \) is a partition of \( C \). Its altitude is the sum of those of \( 0 \Pi \) and \( 1 \Pi \), in that order.

**Proof.** Condition (i) is satisfied by \( \Pi \).

Suppose \((x_1, x_0) \leq \Pi \leq (y_1, y_0)\) and let \( j < i \). If \( i < \frac{1}{1} + k_0(x_0) \), then \((x_1, x_0) \leq \Pi_j \leq (y_1, y_0)\) since \( 0 \Pi \) satisfies (ii). If \( \frac{1}{1} + k_0(x_0) \leq j \), then \((x_1, x_0) \leq \Pi_j \leq (y_1, y_0)\) since \( 1 \Pi \) satisfies (ii). Suppose, finally, \( j < \frac{1}{1} + k_0(x_0) \leq i \). Then \( x_0 = y_0 \); but in that case \( 'x_0 \leq 0 \Pi_j \leq y_0' \) holds for all \( j < i \) less than \( \frac{1}{1} + k_0(x_0) \), and so \((x_1, x_0) \leq \Pi_j \leq (y_1, y_0)\). Thus \( \Pi \) satisfies (ii).

Again suppose \((x_1, x_0) \leq \Pi \leq (y_1, y_0)\). If \( x_0 = y_0 \) then \( i < \frac{1}{1} + k_0(x_0) \) and there is a \( j \) such that

\( i / \)
\[ i < j < 1 + k_0(x_0) \]

and \( x_0 \circ \pi_j y_0 \) is false, but \( x_0 \circ \pi_j x_0 \) and \( y_0 \circ \pi_j y_0 \) are true. Hence \( (x_1, x_0) \circ \pi_j (y_1, y_0) \) is false, but \( (x_1, x_0) \circ \pi_j (x_1, x_0) \) and \( (y_1, y_0) \circ \pi_j (y_1, y_0) \) are true. If \( x_0 = y_0 \) and \( x_1 + y_1 \), there is a \( j_1 \) such that \( x_1 \uparrow \pi_{j_1} y_1 \) is false and \( x_1 \downarrow \pi_{j_1} x_1 \) and \( y_1 \downarrow \pi_{j_1} y_1 \) are true. Let \( j_1 \) be the (unique) ordinal such that \( 1 + j_1 = j_1 \), and let, in this case, \( j \) be \( 1 + k_0(x_0) + j_1 \). Then \( (x_1, x_0) \circ \pi_j (y_1, y_0) \) is false but \( (x_1, x_0) \circ \pi_j (x_1, x_0) \) and \( (y_1, y_0) \circ \pi_j (y_1, y_0) \) are true. Hence \( \Pi \) satisfies (iv).

But we have seen that (iii) is a consequence of (ii) and (iv), and so \( \Pi \) is a partition of \( C \).

If \( K((x_1, x_0)) \) is the chain consisting of those members of \( \Pi \) whose fields contain \( (x_1, x_0) \), and \( K(x_1, x_0) \) is the ordinal of this chain when its first term is omitted, then, if \( \alpha, \alpha_0 \) and \( \alpha_1 \) are the altitudes of \( \Pi, ^0 \Pi \) and \( ^1 \Pi \),

\[
\alpha = \sup \{ k((x_1, x_0)) \} \quad \text{(over all pairs } (x_1, x_0) \}\]

\[
= \sup \{ k_0(x_0) + k_1(x_1) \} \quad \text{(by the construction of } \Pi \text{)}
\]

\[
= \sup \{ k_0(x_0) \} + \sup \{ k_1(x_1) \} \quad \text{(since } k_1(x_1) \text{ is independent of } k_0(x_0) \text{)}
\]

\[
= \alpha_0 + \alpha_1
\]

If the graphs representing \( ^0 \Pi \) and \( ^1 \Pi \) are known, that representing \( \Pi = ^1 \Pi \times ^0 \Pi \) is formed by replacing each terminal node /
node of that representing \( 0_{\Pi} \) by a principal subtree like the graph representing \( 1_{\Pi} \). For example, in the figure, the first root-tree represents the product of the partition represented by the second by that represented by the third.

By drawing such diagrams it is easy to see that the multiplication of partitions is not, in general, commutative.

We now give a general definition for the product of any chain of partitions whose dual is well-ordered. Let \( \{ i \} \) be any segment of the chain of ordinal numbers and let there correspond to each \( i \) of \( \{ i \} \) a partition \( i_{\Pi} \) of a class \( C_i \). The product \( \Pi \) over \( \{ i \} \) of the partitions \( i_{\Pi} \) is to be a partition of the class \( C \) whose elements are all the functions which assign to each \( i \) of \( \{ i \} \) an element \( x_i \) of \( C_i \). The altitude of \( \Pi \) will be \( \alpha = \sum_{i=\{i\}} \alpha_i \) where \( \alpha_i \) is the altitude of \( i_{\Pi} \). We retain the notations '\( k_i(x_i) \)' and '\( k_i(x_i) \)'. Let \( x = (\ldots, x_2, x_1, x_0) \) and \( y = (\ldots, y_2, y_1, y_0) \) be two members of \( C \) and let \( j \) be any ordinal.
ordinal number such that \( j < 1 + \sum \alpha_i \). Let \( m \) be the least ordinal such that \( x_m \neq y_m \), or the ordinal of \( \{i\} \) if \( x = y \); and let \( M \) be the segment consisting of all ordinals preceding \( m \). Let \( n \) be the least ordinal such that, if \( N \) is the segment consisting of all ordinals preceding \( 1 + n \), then

\[
j < 1 + \min \left\{ \sum_{i \in N} k_i(x_i), \sum_{i \in N} k_i(y_i) \right\}.
\]

We let \( x \equiv y \) if and only if \( j = 0 \), or \( n < m \), or both \( n = m \)
and \( x_n = y_n \), where

\[
1 + \sum_{i \in M} k_i(x_i) + j^1 = j.
\]

We denote \( \prod \) by \( \mathcal{P} \prod \) and call it the product over \( \{i\} \) of the partitions \( \{i\} \). If, for all \( i \), \( \{i\} \) is the same partition, which we denote now by \( \{i\} \) (this use being distinct from the preceding), we denote \( \mathcal{P} \prod \) by \( \prod_{\{i\}} \), where \( k \)
is the ordinal of \( \{i\} \), and speak of the product of \( \prod \) with exponent \( k \). The proof that \( \mathcal{P} \prod \) is a partition of \( G \) proceeds along lines similar to those in the preceding theorem and is omitted. It is necessary that \( \{i\} \) should be well-ordered, because sums such as \( \sum \alpha_i \) must yield ordinal numbers. The condition that \( \{i\} \) be a segment of the chain of all ordinal numbers is, however, arbitrary, but convenient.

We note that \( \mathcal{P} \prod = (\prod) \prod = (2 \prod) \prod \) and that

\[
\mathcal{P} \prod = \prod \prod \prod. \quad \text{The following theorem states that the associative}
\]
associative law holds to within isomorphism for the multiplication of partitions.

**Theorem 22.** Let \( \{i\} \) be a segment of the chain of all ordinals and let \( \{i\} \) be divided into intervals \( \{i\}_j \) where \( \{j\} \) is another segment of the chain. (We can regard \( \{i\} \) as the ordinal sum of the \( \{i\}_j \) over \( \{j\} \).) If to each \( i \) there corresponds a partition \( i \Pi \) of \( C_i \) and if \( C \) is the product-class of the \( C_i \), the mapping

\[ x \rightarrow (\ldots, x(j), \ldots, x(1), x(0)) \]

where \( x \) is \((\ldots, x_i, \ldots, x_0)\) and \( x(j) \) is the function which assigns to each \( i \) of \( \{i\}_j \) the value \( x_i \) assigned to \( i \) in \( x \), is an isomorphism:

\[ P_{\{i\}_j}^{i \Pi} \cong P_{\{i\}_j}^{x} (P_{\{k\}_j}^{k \Pi}) \]

where \( \{k\}_j \) is the segment of the chain of all ordinals which is similar to \( \{i\}_j \), and where, if \( k \) corresponds to \( i \) under this similarity, \( k \Pi \) is \( i \Pi \).

**Proof.** The detailed proof will not be given; the method is as follows. Let \( ' \Pi \) denote \( P_{\{i\}_j}^{x} i \Pi \), let \( ' \Pi_d \) denote \( P_{\{i\}_j}^{x} (P_{\{k\}_j}^{k \Pi}) \), and let \( m \) be a variable whose values are the subscripts of the chain \( \Pi \). In proving \( \Pi_m \cong \Pi_m \) two possibilities arise: (1) \( m \) is the initial ordinal of some \( \{i\}_j \), and (2) \( m \) occurs in some position other than the first in some \( \{i\}_j \). The two cases have to be /
be considered separately. In both cases the result follows from the definition of multiplication.

**Corollary.** \( \frac{1}{p} \prod x (\frac{1}{q} \prod x 2 \prod) \cong (\frac{1}{p} \prod x \frac{1}{q} \prod) x 2 \prod \)

\cong \( P_{i_1, j_1, q_1}^{i_1, j_1, q_1} \).

**Theorem 23.** \( \prod \sum \prod^{(i)} \cong \sum (\prod x \prod^{(i)}) \). That is, premultiplication is distributive over addition.

**Proof.** The expressions on both sides denote partitions of a class \( C \) whose elements are ordered pairs \((x_1, x_0)\) where, with the usual notational conventions, \( x_1 \in C_1 \) and \( x_0 \in C_0(i) \) for some \( i \). Also, if \( \prod x \prod^{(i)} \) then

\( k_0(x_0^{(i)}) = 1 + k_0^{(i)}(x_0^{(i)}) \).

Let \( \prod = \prod x \prod^{(i)} \). Then \((x_1, x_0)^\prod = \prod x_1 + j(y_1, y_0)\) if and only if either \( j < \min \{k_0(x_0), k_0(y_0)\} \) and \( x_0 \prod x_1 + jy_0 \), or, simultaneously, \( k_0(x_0) \leq j < k_0(x_0) + \min \{k_1(x_1), k_1(y_1)\} \).

The former condition holds if and only if

\[ j < 1 + \min \{k_0(i)(x_0), k_0(i)(y_0)\} \]

and \( x_0 \prod x_1 + j y_0 \) for some \( i \) (because in order that \( x_0 \prod x_1 + j y_0 \) may hold, \( x_0 \) and \( y_0 \) must both belong to the same \( C_0(i) \)). The second condition holds, similarly, if and only if

\[ 1 + k_0(i)(x_0) \leq j < 1 + k_0(i)(x_0) + \min \{k_1(x_1), k_1(y_1)\}, \quad x_0 = y_0 \]

and \( x_1 \prod x_1 + j y_1 \) where

\[ 1 + k_0(i)(x_0) + j = j. \]

But /
But these are just the conditions for $\pi_j(1)$ to hold, where $\pi(1) = 1\pi \times 0\pi(1)$. Hence the theorem.

COROLLARY. If $2\pi = \sum_{i=1}^{2} \pi^{(i)}$ and $j\pi^{(i)} = j\pi$ for $j > 0$, then $\prod_{ij}^* j\pi = \sum_{i=1}^{j} (\prod_{ij}^* j\pi^{(i)})$. 
CHAPTER THREE.

ARITHMETIC OF PARTITIONED NUMBERS.
PARTITIONED SETS

A partitioned set \( T \) is a partially ordered set \( S \) together with a general partition \( \Pi \) of \( |S| \); its altitude is the altitude of \( \Pi \).

In the treatment of partitioned chains ('partitioned series') in Etherington [1] (p. 445) it was stipulated that, for each \( i \), the \( \Pi_i \)-subsets should be convex in \( S \). This condition does not readily generalise to anything very natural when partially ordered sets are considered, and so is omitted; there is imposed no condition on the order-relations which may hold between elements of different \( \Pi_i \)-subsets. (A \( \Pi_i \)-subset is, naturally, a subset \( S_i \) of \( S \) such that \( |S_i| \) is a \( \Pi_i \)-subclass of \( |S| \).)

To form a subset \( T^1 \) of \( T \) we select, for some \( i \), a \( \Pi_i \)-subset \( S^1 \) of \( S \) and impose on \( S^1 \) the subpartition \( \Pi^1 \) of \( \Pi \) which is determined by the pair \( (|S^1|, i) \). We speak of \( T^1 \) as 'the subset determined by \( (|S^1|, i) \)'. There is thus no 'null subset' of a partitioned set. We use the usual notations and terminology when talking about subsets of partitioned sets: "\( T^1 \subseteq T \)", "\( T^1 \) is included in \( T \)", etc. The following theorem is an immediate consequence of the meaning of 'subpartition'; for, if \( \Pi^1 \) is a subpartition of \( \Pi \), then \( \Pi^1 \) must be a partition of a \( \Pi_i \)-subclass of \( |S| \), for some \( i \).

THEOREM 24. The partitioned set \( T^1 \) is a subset of the partitioned /
partitioned set $T$ if and only if $\Pi^1$ is a 'subpartition' of $\Pi$.

Let a set $C$ be formed whose elements are all the ordered pairs of the form $(T_i, i)$ where $S_i$ is some $\Pi_i$ - subset of $S$ and $T_i$ is the subset of $T$ determined by $(|S_i|, i)$. The relation in $C$ is defined by letting 

\[(T_i, i) \preceq (T_j, j)\]

mean 'i $\geq$ j and $T_i \subseteq T_j$'. Then we have the following theorem.

**THEOREM 25.** The set $C$ is a tree and is isomorphic to the tree $B$ which represents $\Pi$.

**Proof.** The statement \[(T_i, i) \preceq (T_j, j)\] is true if and only if $i \geq j$ and $T_i \subseteq T_j$; i.e. if and only if $i \geq j$ and $i |\Pi$ is a subpartition of $j |\Pi$ (where these expressions have their usual meaning). But if $i \geq j$, then $i |\Pi$ is a subpartition of $j |\Pi$ if and only if $S_i \subseteq S_j$. Hence the sets $B$ and $C$ are isomorphic.

**THEOREM 26.** The subsets of a partitioned set $T$ form, under inclusion, a tree $D$. The mapping $(T_i, i) \rightarrow T_i$ is a tree-homomorphism from $C$ onto $D$, and the mapping $T_i \rightarrow S_i$ is a tree-homomorphism from $D$ onto $A$.

**Proof.** The first statement is a consequence of the first half of the second, and both parts of the second are proved by a procedure exactly analogous to that which shows $(S_i, i) \rightarrow \frac{S_i}{S_i}$.
$S_i$ to be a tree-homomorphism from $B$ onto $A$ (Theorem 14).

**Theorem 27.** The subpartitions of $\Pi$ form a tree isomorphic to $D$.

**Proof.** By Theorem 24.

**Theorem 28.** If $T^1 \subseteq T$ and $A^1$, $C^1$ and $D^1$ are defined in the obvious way, then $A^1$, $C^1$ and $D^1$ are principal subtrees of $A$, $C$ and $D$ respectively. Any principal subtrees of $C$ and $D$ are the trees $C^1$ and $D^1$ of subsets $T^1$ of $T$.

The proof of the above theorem presents no difficulties.

**Theorem 29.** If the partitioned set $T$ is of finite altitude, the mapping $(T_i, i) \rightarrow T_i$ is an isomorphism.

**Proof.** Suppose $T_j = T_i$ where $j > i$. Then if $\alpha_i$ and $\alpha_j$ are the altitudes of $T_i$ and $T_j$, we have $\alpha^1 = i + \alpha_i$ and $\alpha^1 = j + \alpha_j$, where $\alpha^1$ is the ordinal of the chain consisting of those members of $\Pi$, other than $\Pi_0$, whose fields overlap $|S_1|$. But $\alpha_i = \alpha_j$, and hence $\alpha^1$ is a transfinite ordinal.

At this point an example may be helpful. Let $S$ be a set of two elements operated on by a partition $\Pi$ of altitude $\omega$, where $\Pi_0$ is the unit simple partition of $S$, $\Pi_1$ separates $S$ into two subsets of one element each, $\Pi_2$ is equal to $\Pi_1$, and the terms of the chain $\Pi_2$, $\Pi_4$, $\Pi_6$, ... (of /
(of ordinal $\omega$) are all the unit simple partition of one of the $\mathcal{T}_1$ - subclasses. In this way a partitioned set $T$ is constructed (conditions (i) - (iv) being readily verified for it). The graphs of the trees $C$, $D$ and $A$ associated with $T$ are shown in the figure.

![Diagram of trees C, D, and A]

The procedure by which the trees are, in practice, constructed is the following. It is more simple to construct the tree $B$ which represents $\mathcal{T}$ and which is isomorphic to $C$ than to think of $C$ directly. The graph of $D$ is then formed by allowing a node $x$ to coalesce with the node $y$ immediately above it if and only if $x$ is the only node immediately below $y$ and the principal subtrees determined by $x$ and $y$ are isomorphic. This means that there can be only one node immediately below $x$, and only one immediately below that, and so on. Hence the subpartition of $\mathcal{T}$ determined by the element represented by the node in question must commence with a subchain /
subchain of ordinal at least \( \omega \) all of whose members are the
unit simple partition of the subclass in question. Thus \( x \)
and \( y \) coalesce if and only if the principal subtree deter-
mined by \( y \) has an upper part consisting only of a chain of
order-type \( \omega^* \). Finally, \( A \) is obtained from \( D \) by allowing
each node \( x \) to be identified with the node \( y \) immediately
above it if \( x \) is the only node immediately below \( y \). (If the
nodes above \( x \) form a chain whose dual has a limit-ordinal
we require two stages in the process; we first allow such
nodes of the chain to coalesce as are permitted by the above
rules to do so, and then, if \( x \) has a node \( y \) immediately
above it after this has been carried out, we apply the rules
again.) The justification of these rules is to be sought in
the preceding theorems.

A partitioned set \( T \) is said to be \textit{finite} when its alti-
tude is finite and the cardinal of \( |S| \) is finite. The par-
titioned set \( T \) constructed in the preceding example is thus
not finite although the corresponding \( S \) has only two elements.

\textbf{PARTITIONED NUMBERS.}

The definition of similarity for partitioned sets is the
obvious one. Let \( T_1 \) and \( T_2 \) be two partitioned sets, derived
from /
from the partially ordered sets $S_1$ and $S_2$ respectively by means of the partitions

$$\begin{align*}
1 \Pi & : \ 1\pi_2, \ 1\pi_1, \ 1\pi_2, \\
2 \Pi & : \ 2\pi_2, \ 2\pi_1, \ 2\pi_2,
\end{align*}$$

they are said to be similar if their altitudes are equal,

there exists a one-one correspondence $R$ between the elements of $S_1$ and $S_2$ such that

$(\alpha)$ if $x_1 R x_2$ and $y_1 R y_2$, then $x_1 \leq y_1$ if and only if $x_2 \leq y_2$

$(\beta)$ if $x_1 R x_2$ and $y_1 R y_2$, then, for all $i$,

$$x_1 \ 1\pi_i \ y_1 \text{ if and only if } x_2 \ 2\pi_i \ y_2.$$ 

The correspondence $R$ is called a correlator of $T_1$ and $T_2$.

We denote the similarity of $T_1$ and $T_2$ by

$$T_1 \sim T_2.$$ 

The relation of similarity is obviously an equivalence.

The class of partitioned sets similar to a given partitioned set $T$ is called the partitioned number belonging to $T$; it is denoted (by an adoption of a notation introduced by Cantor for order-types) by $\overline{T}$. Partitioned numbers will frequently be called numbers in the following. The number belonging to a finite set is said to be a finite number.

If $T_1$ is a subset of $T$, we say that $T_1$ is a subnumber of
of $T$. This we denote by the notation $\bar{T}^1 \leq \bar{T}$.

It must be noted that, just as in the cases of cardinal
and ordinal numbers, $'T^1 \subset T'$ does not, in general, imply
$'T^1 < \bar{T}'$, but only $'T^1 \leq \bar{T}'$. That this is so may be seen by
an example.

Let there correspond to each natural number $n$, an element
$a_n$ and let the set $S$ of these elements have the identity par-
tial ordering. We define a partition

$$\Pi: \pi_2, \pi_3, \pi_4, \ldots$$

of altitude $\omega$ on $S$ as follows:

$\pi_0$ is the unit simple partition
of $|S|$

$\pi_i$ holds, for each $i > 0$, between
every pair of $a_i$, $a_{i+1}$, ..., and
between $a_{i-1}$ and itself.

In this way we have defined a partitioned set $T$. Let $T'$, $T''$
be the subsets of $T$ which are derived from $S - \{a_0\}$,
$S - \{a_0, a_1\}$ by the appropriate subpartitions of $\Pi$. Then

$$T'' \subset T' \subset T \quad \text{and} \quad \bar{T}'' = \bar{T}' = \bar{T}.$$  
(The figure shows the graph of the tree $C$ for this set, and
is the same as that for $D$ and $A$ in this case.)

If, however, the altitude of $T$ is finite and $T^1 \subset T$,
then $\bar{T}^1 < \bar{T}$. This follows from the fact that $(T_i, i) \rightarrow T_i$
is /
is then an isomorphism.

The relation of being a subnumber is antisymmetric for numbers of finite altitude, for if \( T^1 \subset T \) and the altitude of \( T \) is finite, that of \( T^1 \) is a smaller finite number. But for numbers of transfinite altitude, this again breaks down. This we illustrate by an example. Take as the set \( S \) a denumerable set of elements

\[ a_0, a_1, a_2, \ldots \]

with the identity partial ordering. We define \( \pi \) (of altitude \( \omega \)) as follows:

- \( \pi_0 \) is the unit simple partition of \(|S|\)
- \( \pi_{1, 2} \) (for \( i = 0, 1, 2, \ldots \)) holds between \( a_{3i} \)
  and itself and between every pair of the elements \( a_{3i+1}, a_{3i+2}, a_{3i+3}, \ldots \)
- \( \pi_{1, 2} \) (for \( i = 1, 2, 3, \ldots \)) holds between \( a_{3i-2} \) and
  itself, and \( a_{3i-1} \) and itself and between every pair of the elements \( a_{3i}, a_{3i+1}, a_{3i+2}, \ldots \)

In this way we have constructed a partitioned set \( T \).

Let \( T' \) and \( T'' \) be the subsets of \( T \) in which \( S' \) and \( S'' \) respectively consist of the elements \( a_1, a_2, a_3, \ldots \) and

\[ a_3, a_4, a_5, \ldots \]

Then \( T'' \subset T' \subset T \), and \( T" \subset T' \subset T \) and

\( T" = T \). Hence the relation of being a subnumber is a quasi-
quasi-ordering of the class of all numbers, and a partial ordering of the class of all numbers of finite altitude.

**ADDITION**

Let \( \{i\} \) be a chain such that
to every \( i \) there corresponds a unique partitioned set \( T_i \)
\[ T_i = N_i \]

\( T_i \) is derived from the set \( S_i \) by means of the partition
\[ \Pi : \\pi_0, \\pi_1, \\pi_2, \ldots \]
of \( |S_i| \)
all the \( S_i \) are disjoint and there is a set \( S \)
which /
which is the join in itself of all the $S_i$.

Let us now impose on $|S|$ the partition $\overline{N} = \sum_{i \in I} \lambda_i$, so constructing a partitioned set $T$ with number $N$. We call $N$ a sum of the numbers $N_i$ and write

$$N = \sum_{i \in I} N_i.$$

In simple cases we have such notations as

$$N = N_1 + N_2 + N_3.$$

In these notations the asterisks represent the vagueness in the definition which results from the fact that the partial ordering of $S$ is not uniquely determined by the orderings of the $S_i$. An asterisk resembles, in some ways, a variable, and certain types of sums can be characterised by assigning to it "particular values". As explained below, it may sometimes be replaced also by other "variable" symbols such as the obelus.

The ordering within each of the disjoint subsets $S_i$ which together constitute $S$ always remains fixed, but the order-relations which hold between pairs of elements which belong to different subsets $S_i$ and $S_j$ are not specified. If we have $x \parallel y$ for all pairs of elements $(x, y)$ which are not both members of the same set $S_i$, the ordering of $S$ is completely specified. The number $N$ which arises in this case is called the cardinal sum of the $N_i$ and we write

$$N = \sum_{i \in I} c_i N_i,$$

or /
or, in simple cases,

\[ N = N_1 + \circ N_2 + \circ N_3, \]

and so on. If \( x < y \) whenever \( x \in S_i, y \in S_j \) and \( i < j \), the partial ordering of \( S \) is again uniquely determined. In this case the number \( N \) is called the **ordinal sum** of the \( N_i \), and we write

\[ N = \sum_{[i]}^\circ N_i, \]

etc. When the subscript-chain \( \{i\} \) consists of a single element, the sum \( N \) is unique, and the asterisk may be omitted.

But, as was explained in the section ADDITION OF PARTITIONS, the number \( \sum_{[i]}^N N_i \) is not \( N \) itself. If \( N_1 - N \) we denote \( \sum_{[i]}^N N_i \) by '\( \sum N \)'. It is to be noted that no meaning has been attached to

\[ N_1 + \circ N_2 + \circ N_3 \]

and similar expressions.

**THEOREM 30.** Cardinal addition is commutative; i.e. if \( \{i\} \) and \( \{j\} \) are chains such that there is a (not necessarily isotone) one-one correspondence between their elements, such that if \( i \rightarrow j \) then \( N_i = N_j \), then \( \sum_{[i]}^\circ N_i = \sum_{[j]}^\circ N_j \).

**Proof.** The set \( S \) is the same in both cases since the definition of its ordering does not depend on the ordering of the subscript-chain. Also \( \sum_{[i]}^N \) is independent of the ordering of \( \{i\} \); in fact, for strict accordance with the definition of addition of partitions, we should write /
Although the types of sum called cardinal and ordinal sums are defined generally for any chain \( \{i\} \) and any numbers \( N_i \), it is to be noted that the theory of addition cannot be reduced to the consideration of a finite number of such types of sum. In rough language, the number of possible sums increases both with the cardinal of the chain \( \{i\} \) and with the cardinals of the classes \( |S_i| \). If \( |\{i\}| \) has cardinal 2 and \( |S_1| \) and \( |S_2| \) have cardinals \( C_1 \) and \( C_2 \) respectively, the (cardinal) number of pairs \( (S_1, S_2) \) where \( S_1 \in S_1 \) and \( S_2 \in S_2 \) is \( C_1 \times C_2 \); to each of these pairs we assign one of three 'values' given by \( S_1 \leq S_2 \), \( S_1 \ll S_2 \), and \( S_1 \parallel S_2 \). Hence the number of partial orderings is at most \( 3^{C_1 \times C_2} \). The maximum possible number of sums is thus this cardinal. The actual number of distinct sums depends on the orderings of the sets \( S_1 \) and \( S_2 \), and on the way in which \( S_1 \) and \( S_2 \) are related to \( 1^\Pi \) and \( 2^\Pi \). The number of distinct sums \( N \) lies, therefore, between the number of dissimilar sets \( S \) and \( 2^{c_1 \times c_2} \). The number will be the number of different possible orderings of \( S \) whenever \( \Pi \) possesses no automorphism other than identity.

If the number \( N_i \) has, for each \( i \), altitude \( \alpha_i \), the sums \( \sum_{i} \alpha_i \) have altitude \( 1 + \sup \{\alpha_i\} \).

THEOREM
THEOREM 31. Any number \( N \) of altitude \( \alpha > 0 \) can be expressed as a sum \( \sum_{i_1}^x N_i \), and the class \{N_i\} of summands is uniquely determined. (The Selection Axiom is required for the general, infinite case.)

Proof. Let \( T \) have number \( N \). In the usual notation, \( \Pi \) has a unique decomposition \( \sum_{i_1}^x 1_{\Pi} \). We impose now a simple ordering on \{i\} using, for the infinite case, the Selection Axiom. With the obvious meaning for the symbols \( N_i \) we now have

\[
N = \sum_{i_1}^x N_i,
\]

where the class \{N_i\}, being uniquely determined by the class \{i_1\}, is likewise unique.

The numbers \( N_i \) are called the components of \( N \).

COROLLARY. Addition of partitioned numbers is not associative.

Since addition in the present arithmetic is non-associative, addition cannot be reduced to the repeated addition of unit increments. But induction on the altitude of finite numbers is possible. Using this process, the following theorem gives what may be called the Principle of Non-associative Induction.

THEOREM 32. If \( \varphi(x) \) is a propositional matrix such that \( (\alpha)' \varphi(N) \) is true when \( N \) is the number of zero altitude \( (\beta)' \varphi(\sum_{i_1}^x N_i) \) is true whenever \{i\} is a finite chain and \( \varphi(N_i) \) is true for all \( i \)

then /
then \( \varphi(N) \) is true for all finite numbers \( N \).

The Selection Axiom implies that if \((\alpha)\) holds and
\[(\beta)' \quad \varphi\left(\bigcup_{i \in I} N_i\right) \text{ is true when } \varphi(N_1) \text{ is true for all } i \text{ and } \{i\} \text{ is any chain,}\]
then \( \varphi(N) \) is true for all numbers \( N \) of finite altitude (and also for certain members of altitude \( \omega \)).

**Proof.** Let \( N \) be a finite number. Then there is a finite chain \( \{i\} \) to whose elements correspond numbers \( N_i \) such that \( N = \bigcup_{i \in I} N_i \). If \( \alpha \) is the altitude of \( N \) and \( \alpha_i \) of \( N_i \), then \( \alpha_i < \alpha \) for all \( i \) since \( \alpha = 1 + \sup \{\alpha_i\} \) and \( \alpha \) is finite. Thus \((\beta)\) implies that if \( \varphi(N) \) holds for all numbers of altitude less than \( \alpha \), then it holds also for all numbers of altitude \( \alpha \). Hence, by \((\alpha)\) and the Principle of Mathematical Induction we have the first part of the theorem.

The second part is similar. The numbers of altitude \( \omega \) which enter are those whose components are all of finite altitude, together with those whose components are of finite altitude or are numbers whose components are of finite altitude, and so on. (It is evident that a number whose components are all of finite altitude will have altitude \( \omega \) whenever the altitudes of its components are not bounded above by any finite ordinal).

From /
From this theorem the next follows at once.

THEOREM 33. Any finite number either is the number of zero altitude or can be expressed as a suitably bracketed sum all of whose terms are this number.

THEOREM 34. If \( N \) is a finite number of altitude other than \( 0 \) and \( N^1 \) is any subnumber of \( N \), then \( N \) can be expressed as a bracketed sum one of whose terms is \( N^1 \).

If \( T = N \) and \( (S_i, i) \) determines a subset \( T^1 \) of \( T \) such that \( T^1 = N^1 \), then it is seen that there are pairs \( (S_j, j) \) determining subsets whose numbers are the other terms of the bracketed sum and which are complements of \( (S_i, i) \) with respect to \( (3, 0) \). Since, conversely, any such class of complements determines a sum of the desired kind, the choice of primary complements yields an expression having fewer terms than any other. The terms of this expression, other than \( N^1 \) itself, are called the primary complements of \( N^1 \) with respect to \( N \).

In these results there appears the reason why addition was treated in the general manner here adopted. With merely the cardinal and ordinal types (or, indeed, any finite number of types) of addition an arbitrary number \( N \) would not, in general, be decomposable, and little general theory would be
be possible. The reason for adopting an approach so different from that of Birkhoff \cite{2} is that here the splitting of the set $S$ is determined by $\Pi$ and it remains to see how the parts must be combined in order to reconstitute $S$, whereas, in Birkhoff's theory, the possibility of decomposition depends on finding a splitting of $S$ such that order-relations of certain simple kinds may hold between the pieces.

Two sums $\sum_{i \in I} N_i$ and $\sum_{j \in J} N_j$ are said to be similar when there is an ordinal correlator of $\{i\}$ and $\{j\}$ such that, if $i$ corresponds to $j$ and $i^1$ to $j^1$, and $\overline{T_i} = N_i$, etc., then there are correlators $R_i$ from $T_i$ to $T_j$ and $R_{i^1}$ from $T_{i^1}$ to $T_{j^1}$ such that, if $s_i < T_i$, etc., $s_i R_i s_j$ and $s_{i^1} R_{i^1} s_{j^1}$ then $s_i < s_j$ if and only if $s_{i^1} < s_{j^1}$. Thus, in rough language, $\sum_{i \in I}^* N_i$ and $\sum_{j \in J}^* N_j$ are similar if not only $\sum_{i \in I}^* N_i = \sum_{j \in J}^* N_j$ but also the components are 'similarly ordered' by the chains $\{i\}$ and $\{j\}$. When in any sum the summands are rearranged, there is a sum over the rearranged chain which is equal to the original, but it is not in general similar to it. If two sums are similar, the fact may be represented by writing both addition signs with the same symbol attached, i.e. by writing both with asterisks or both with obeli, etc. This usage is consistent /
consistent with the previously defined use of the symbols 
'\Sigma^c' and '\Sigma^0'.

With this convention, two numbers \( N \) and \( N^1 \) which are 
both sums of the two numbers \( N_1 \) and \( N_2 \) can be distinguished 
by writing

\[
N = N_1 +^\leftrightarrow N_2 \quad \quad N^1 = N_1 +^+ N_2.
\]

MULTIPLICATION

In defining the product of numbers we make use of our 
definition of the product of partitions just as in defining 
the sum of numbers we used the definition of the sum of 
partitions. If we are to have \( N = N_1 \times N_0 \), then, adopting 
the usual notations, we saw that \( N \) is a partition of \( |S| \) 
where \( |S| \) is the product-class of \( |S_1| \) and \( |S_0| \). Thus \( S \) 
must be obtained by imposing on this product-class some 
partial ordering. Since, as was pointed out in the consid-
eration of the multiplication of partitions, we consider

\[ N_0 \]
If we place no condition on these relations, but admit all the possible ones as giving rise to products of \( N_1 \) and \( N_0 \), we find that the ordering of \( S_0 \) becomes quite irrelevant to the result. If we replaced \( T_0 \) by any set similarly partitioned, the same class of products would arise. Since this is undesirable and since there are no overwhelming reasons for the inclusion of this general case of multiplication, we define only two special types, the analogues of Birkhoff's cardinal and ordinal multiplication; but, since we regard the second factor as the multiplier, we shall have to reverse the order of the factors. Hence we admit the two products \( N_1 \times^0 N_0 \) and \( N_1 \times^0 N_0 \) which arise when, in the notation of Birkhoff [1] and [2], \( S = S_0 S_1 \) and \( S = S_0 S_1 \) respectively. These we call the cardinal and ordinal products. (The general case of addition, although introducing a certain awkwardness, did not raise such serious difficulties as would the general case of multiplication. The need for its inclusion rested on the need for numbers to be decomposable.)
decomposable; in the case of multiplication, most numbers would be unfactorable no matter how we ordered S.) We proceed to give the full definitions for the case of a chain of numbers whose dual is well-ordered.

Let \{i\} be a segment of the chain of all ordinals, such that to each i of \{i\} there corresponds a number \(N_i\). Let, for each i, \(T_i\) be a partitioned set such that \(\overline{T_i} = N_i\). Let \(|\Sigma|\) be the class of all functions \(\sigma\) which assign to each i of \{i\} a value \(\sigma(i)\) in \(|\Sigma|\). Then \(\Pi = \prod \sigma \times i \Pi_i\) is a partition of \(|\Sigma|\). Let \(\Sigma\) be derived from \(|\Sigma|\) by imposing on it the relation defined by

\[\sigma_1 \leq \sigma_2 \text{ if and only if } \sigma_1(i) \leq \sigma_2(i) \text{ for all } i.\]

Then \(\Sigma\) together with \(\Pi\) constitutes a partitioned set \(\Sigma\); let \(\overline{\Sigma} = N\). We call \(N\) the \textit{cardinal product} over \(|\Sigma|^{\times}\) of the numbers \(N_i\), and write

\[N = \prod_{i \in \Sigma} N_i.\]

Let now \(\Sigma\) be derived from \(|\Sigma|\) by imposing on it the relation defined by

\[\sigma_1 \leq \sigma_2 \text{ if and only if for every } i \text{ such that } \sigma_1(i) \neq \sigma_2(i) \text{ there is an } i^1 \text{ such that } i^1 \leq i \text{ in } \{i\} \text{ and } \sigma_1(i^1) < \sigma_2(i^1).\]

In this case the number \(N\) which arises is called the \textit{ordinal product} over \(|\Sigma|^{\times}\) of the numbers \(N_i\), and we write

\[N = \prod_{i \in \Sigma} N_i.\]

Since /
Since the multiplication of partitions and both cardinal and ordinal multiplication in Birkhoff's arithmetic are known to be associative to within isomorphism, we have the following theorem.

THEOREM 35. Both cardinal and ordinal multiplication are associative. That is, if \{i\}_j is a segment of the chain of ordinals and is split into intervals \{i\}_j, where \(j\) runs through a segment of the chain of ordinals, then

\[
\begin{align*}
\prod_{i}^{c} N_{i} &= \prod_{i}^{c} \left( \prod_{ik}^{c} N_{k} \right) \\
\prod_{i}^{o} N_{i} &= \prod_{i}^{o} \left( \prod_{ik}^{o} N_{k} \right),
\end{align*}
\]

where \(\{k\}_j\) is the segment of the chain of ordinals which is similar to \(\{i\}_j\) and, if \(i\) corresponds to \(k\) under this similarity,

\[N_k = N_i.
\]

The following theorem gives a form of distributivity of multiplication over addition.

THEOREM 36. If \(\sum_{i}^{c} N_{i}\) is a sum of a chain of numbers \(\{N_{i}\}_i\) and \(N\) is any number, there are sums of the forms

\[
\begin{align*}
\sum_{i}^{c} \left( N \times^{c} N_{i} \right) \quad \text{and} \quad \sum_{i}^{c} \left( N \times^{o} N_{i} \right),
\end{align*}
\]

such that

\[
\begin{align*}
N \times^{c} \sum_{i}^{c} N_{i} &= \sum_{i}^{c} \left( N \times^{c} N_{i} \right) \quad (1) \\
N \times^{o} \sum_{i}^{c} N_{i} &= \sum_{i}^{c} \left( N \times^{o} N_{i} \right). \quad (2)
\end{align*}
\]

Proof. It is known that \(\prod_{i}^{c} \prod_{i}^{o} = \sum_{i}^{c} \left( \prod_{i}^{c} \prod_{i}^{o} \right).\) Hence, with the usual notation, it remains to be shown, for the cardinal /
cardinal case that, if the sets $S_i$ are disjoint and $S^1$ denotes any set which is the union in itself of the $S_i$, then, in Birkhoff's notation, the set $SS^1$ can be decomposed into disjoint and exhaustive sets (similar to) $SS_i$. Now $|SS^1|$ consists of pairs $(x, x^1)$ where $x \in S$ and $x^1 \in S_i$ for some $i$, so that the classes of pairs $(x, x^1)$ where $x \in S$ and $x^1 \in S_i$ are disjoint and together make up $|SS^1|$. But the partial ordering induced on these classes by that of $SS^1$ is just that of $SS_i$. Hence the theorem for the cardinal case. The cardinal case is similar.

It is not true that for every sum of the form

$$
\sum_{i_1}^{+} (N \times^0 N_1) \quad \text{or} \quad \sum_{i_1}^{+} (N \times^0 N_1)
$$

there is a sum $\sum_{i_1}^{+} N_1$ such that (1) or (2) holds. This is shown by the following example. Let $T$, $T'$ and $T''$ be partitioned sets whose partially ordered sets and partitions are as illustrated, and let the numbers /
numbers of these sets be \( N, N' \) and \( N'' \) respectively. Then

\[
N = N' +^0 N'
\]

\[
= (N' \times^c N'') +^0 (N' \times^c N'')
\]

Now if there were a sum \( N'' +^0 N'' \) such that

\[
N = N' \times^c (N'' +^c N'')
\]

the set \( S \) would be expressible as a cardinal product similar to \( S'S_1 \) where \( S_1 \) is some two-element partially ordered set. Thus \( S_1 \) must either be similar to \( S' \) or have the identity partial ordering. Neither of these possibilities, however, yields

\[
S \simeq S'S_1,
\]

and so the nonexistence of the required sum is demonstrated.

**Theorem 27.** For any numbers \( N_1 \) and any number \( N \),

\[
N \times^c \sum_{i \in I} N_1 = \sum_{i \in I} (N \times^c N_1)
\]

\[
N \times^0 \sum_{i \in I} N_1 = \sum_{i \in I} (N \times^0 N_1)
\]

\[
N \times^0 \sum_{i \in I} N_1 = \sum_{i \in I} (N \times^0 N_1)
\]

**Proof.** These identities result from the corresponding distributive laws in Birkhoff's arithmetic. See BIRKHOFF [2], (26), (35) and (34).
CHAPTER FOUR

CLOSURE
In this chapter we go on to consider subclasses of the class of all partitioned numbers which are closed under certain operations. These operations are to be "suboperations" of the operations of addition and multiplication defined previously.

The operations of addition and multiplication may be considered as relations holding between chains of numbers \( \mathbb{N}_1 \) and the members of "value-classes". Addition relates to any chain \( \mathbb{N}_1 \), the members of the class of numbers of the form \( \sum_{i=1}^{\infty} N_i \), and the two operations of cardinal and ordinal multiplication each relate to a given chain whose dual is well-ordered a unique number, the only member of the value-class for this chain. A suboperation of an operation is a non-null subrelation of the operation considered, in this way, as a relation.

An operation is said to be uniform in a class \( A \) if its domain contains with any chain of type \( k \) all chains of type \( k \) whose elements are in \( A \). Thus in any field the singulary operation of taking inverses is not uniform since the zero has no inverse.

Since the value-classes of both operations of multiplication /
...cation are unit classes, any proper suboperation can arise only through restriction of the domain of the operation. We may restrict the domain to chains of certain specified types, and so produce the operations of multiplication of index 2 or of index \( \omega^* \), etc. In general, an operation of multiplication with its domain restricted to chains of order-type \( k \) will be called a multiplication of index \( k \). Any other kind of restriction renders the operation non-uniform, and such operations will not be considered.

In the case of addition we may similarly restrict the domain to chains of a given type \( k \). The resulting operation is called addition of index \( k \). We may now further restrict the operation by allowing it, for the chosen \( k \) and for a given chain \( \{N_i\} \) of type \( k \), to take only a subclass of the values of \( \Sigma_{i=1}^{\infty} N_i \) (considered as a function of the "variable" \( \star \)). (If this subclass should consist of a single element we are entitled to speak of the sum of the chain \( \{N_i\} \)). For example, we might wish to consider cardinal addition of index 2. Any operation arising from such a double restriction of \( \Sigma^* \) will be called a restricted addition of index \( k \). Any union of such suboperations of \( \Sigma^* \) will be called a restricted addition. If the class of indices contains only finite ordinals, the operation is said to be finitary.

The /
The operations to be considered are the uniform restricted multiplications and the restricted additions. Multiplication of index 1 will not be considered since it is the identity operation and is therefore trivial.

An ordered pair \((H, Z)\) where \(H\) is a class of numbers and \(Z\) is a non-null class of operations is called a system of numbers if \(H\) is closed with respect to the members of \(Z\). A system \((H, Z)\) is called an arithmetic if \(Z\) contains a restricted addition and a restricted multiplication, and a family if it contains only a restricted addition. If \(Z\) contains more than one restricted addition it may be replaced by the class \(Z'\) formed from \(Z\) by replacing these restricted additions by their union, and analogously for multiplication; the system is, for practical purposes, unaltered. Hence systems in which \(Z\) contains such multiplicities need not be considered.

We write '\((H_1, Z_1) \leq (H_2, Z_2)\)' if \(Z_1 = Z_2\) and \(H_1 \subseteq H_2\).

Consideration of altitude shows at once that if \((H, Z)\) is a system and \(H\) contains a number of finite altitude, then \(H\) is not a finite class. If \((H, Z)\) is an arithmetic, then \(H\) is not a finite class. For, let \(N \in H\) and let the altitude of \(N\) be \(\alpha\). Then, if \(N_i = N\) for all \(i\) and \(\{i\}\) has ordinal \(k\), then \(P_i^{-\alpha} N_i\) and \(P_i^{-\alpha} N_i\) both have altitude \(\alpha k\). But the case \(k = 1\) is excluded, and for all other ordinals \(k, \alpha k > \alpha\).
A family of numbers of transfinite altitude may be finite. In fact, if $S$ consists of a single element and if a partition $\Pi$ of $|S|$ (of altitude $\omega$) is defined, having all its terms equal to the unique simple partition of $|S|$, then a partitioned set $T$ arises whose number $\mathbb{N}$ is such that $(\{N\}, Z)$ is a family, where $Z$ is the class whose only member is the operation of addition of index 1. Thus a family of numbers of transfinite altitude may consist of a single number.

Another example is the following. Let $S$ consist of all irrational members of the interval $[0, 1]$ in their natural order. Let $\Pi$ (of altitude $\omega$) be defined by letting $\Pi_i$ ($i = 0, 1, 2, \ldots$) hold between two members $x$ and $y$ of $S$ if and only if there is an integer $r$ such that $x$ and $y$ both belong to the subinterval $\left[\frac{r}{2^i}, \frac{r+1}{2^i}\right]$. Let the number of the partitioned set $T$ so formed be $N$. Then $N \circ N = N$, and hence $(\{N\}, Z)$ is a family if $Z$ consists of ordinal addition of index 2.

There is, however, an essential triviality about all finite families. For, let a finite family $(H, Z)$ have $n$ members, and let $Z$ contain a restricted addition of (finite) index $r$. Then, since numbers are distinct if their classes of components are distinct, there must be at least as many members /
members of \( H \) as there are ways of taking \( r \) not necessarily distinct members out of the class \( H \). But the number of ways of taking \( r \) objects out of a class of \( n \) objects, where repetitions are allowed, is always greater than \( n \) unless \( n \) or \( r \) is 1.

We sum up these results in a theorem.

**THEOREM 38.** If \((H, Z)\) is a system of numbers, then \( H \) is not a finite class unless \((H, Z)\) is a family and either \( H \) is a unit class or \( Z \) consists of addition of index 1.

---

**GENERATING CLASSES AND BASES.**

This section is devoted to a consideration of generating classes for systems of numbers. All the theory developed here, though expressed in terms of partitioned numbers, goes over to the general case of any class of elements and class \( Z \) of operations (not necessarily one-valued or uniform) whose domain is included in it.

Firstly we approach the question from a point of view which regards as fundamental the class of classes of numbers.
The methods employed in this enquiry are familiar and are to be found in MOORE [1], and BIRKHOFF [1], p.49 et seq.

It is only when a second approach is adopted that the numbers themselves form the fundamental class.

A class $A$ of numbers is said to generate a system $(H, Z)$ when $A \subseteq H$ and, if $(H', Z)$ is a system such that $A \subseteq H'$, then $H \subseteq H'$.

A class $B$ of numbers is called a basis for a system $(H, Z)$ if $B$ generates $(H, Z)$ and no proper subclass of $B$ does so. It is a corollary of Theorem 38 that $H$ is itself the unique basis for the system $(H, Z)$ if it is a finite class.

The following theorem states that the property possessed by $H$ when $(H, Z)$ is a system is extensionally attainable in the sense of MOORE [1].

**THEOREM 39.** If $(H_i, Z)$ is a system for every $i$ in $\{i\}$ (where $Z$ does not depend on $i$), then $(\bigcap_{i \in \{i\}} H_i, Z)$ is a system.

**Proof.** All admissible sums and products of chains of members of $H_i$ belong to all the $H_i$ and hence to $\bigcap_{i \in \{i\}} H_i$.

Theorems 40-44 state results whose analogues are known to hold for any class of classes in which an extensionally attainable property is defined.
THEOREM 40. If A generates a system \((H, Z)\), then \(H\) is the intersection of the class of classes \(G\) such that \((G, Z)\) is a system and \(G \supseteq A\).

THEOREM 41. Every class \(A\) of numbers generates a unique system \((H, Z)\) for a given class \(Z\) of operations.

Proof. Let the class of all numbers be \(H\). Then \((H, Z)\) is a system such that \(H \supseteq A\). Hence the class of systems \((G, Z)\) such that \(G \supseteq A\) is not null. Let \(H\) be the intersection of the class of all such classes \(G\); then \(A\) generates \((H, Z)\), by Theorem 39, and \((H, Z)\) is, by Theorem 40, the unique system generated by \(A\).

If \((H, Z)\) is the system generated by \(A\), we denote \(H\) (which depends on \(A\) and \(Z\)) by \([A, Z]\)', and call \([A, Z]\) the closure of \(A\) with respect to \(Z\).

THEOREM 42. The systems \((H, Z)\), for given \(Z\), for \(m\) under \(\leq\), a complete lattice with \((\mathbb{Q}, Z)\) as \(0\) and \((H, Z)\) as \(1\), where \(H\) is the class of all partitioned numbers; and

\[
\bigwedge_{i} (H, Z) = \left( \bigcap_{i} H, Z \right)
\]

\[
\bigvee_{i} (H, Z) = \left( \bigcup_{i} H, Z \right)
\]

THEOREM 43. If \(A_1 \subseteq A_2\), then \([A_1, Z] \subseteq [A_2, Z]\).

THEOREM 44. For all \(A\) and \(Z\), \([A, Z], Z] = [A, Z]\).
THEOREM 45. If \( A \) generates the system \((H, Z)\) and \( B \) is a basis for \((G, Z)\), then \( A \cap B \) is a basis for \(([A \cap B, Z], Z)\).

**Proof.** Let \( C \subseteq A \cap B \) and suppose \( C \) generates \(([A \cap B, Z], Z)\). Then \([C, Z] = [A \cap B, Z] \supseteq A \cap B\). Hence

\[
\{B - (A \cap B)\} \cap C, Z \supseteq \{B - (A \cap B)\} \cup [C, Z]
\]

\[
\supseteq \{B - (A \cap B)\} \cup (A \cap B)
\]

\[
= B
\]

\[
\{B - (A \cap B)\} \cap C, Z \supseteq [B, Z]. \tag{1}
\]

But \( \{B - (A \cap B)\} \cap C = B - \{(A \cap B) - C\} \subseteq B. \tag{2} \)

Hence, by (1)

\[
[B - (A \cap B)] \cap C, Z = [B, Z]; \tag{3}
\]

and so, by (2) and (3), since \( B \) is a basis for \(([B, Z], Z)\)

\[
B = \{B - (A \cap B)\} \cup C, \text{ whence } C = A \cap B.
\]

THEOREM 46. If \( A \) is a finite class, then there is a basis for \(([A, Z], Z)\) which is included in \( A \).

**Proof.** Let \( A \) have \( n \) members. There are then \( 2^n - 1 \) non-null subclasses of \( A \). We select from this class of subclasses all those members \( A' \) such that \([A', Z] \supseteq A\). Let \( B \) be one of those subclasses \( A' \) which contain fewer, or at most, just as many, elements as any other. Then \( B \) is a basis for \(([A, Z], Z)\), since by construction, it generates this system and has no proper subclass which does so.

It is to be noted that if \( B \) is a basis, the cardinal of...
B is not necessarily minimal, although the converse of this is true. (This can be seen in the case of groups by considering $C_6$ which has both a one-element and a two-element basis). An example is given in the following section to illustrate this point.

We now adopt the second point of view, and set out to specify which numbers do actually belong to $[A, Z]$ for a given class $A$. They are, in rough terms:

- the members of $A$
- the members of $A'$ where $A'$ consists of the members of $A$ together with the numbers of the form $f(\{N_i\})$ where $f \in Z$ and $\{N_i\}$ is a chain of members of $A$,
- the members of $A''$ where $A''$ consists of the members of $A'$ together with numbers of the form $f(\{N_i\})$ where $f \in Z$ and $\{N_i\}$ is a chain of members of $A'$,

etc.

But, in this rough description, the word "etc." must be interpreted so as to cover a transfinite recursive process, since the functions $f$ need not be of finite index. If we allow ourselves the use of transfinite recursive definitions, this description can be made precise. An alternative method which does not require recursive processes is, however, available; the process being the analogue of that which is used for singling /
singling out the finite cardinals from among all the cardinal numbers, and which is due to Frege ("Die Grundlagen der Arithmetik", 1884). The precise description of the members of \([A, Z]\) concludes the present section.

The terminology used in the process is derived from that of Higman [1], but certain modifications are necessary.

We say that the relation \(r\) is, with respect to the class \(Z\) of operations, a **divisibility quasi-ordering** of the class \(\mathbb{H}_1\) of all partitioned numbers if

- \(r\) is reflexive in \(\mathbb{H}_1\) and transitive
- \(\mathbb{H}_1 \cap \mathbb{N}\) whenever \(\mathbb{N} = f(\{\mathbb{N}_i\})\) and \(f \in Z\).

For strict accordance with the definition of Higman [1], we should also add the condition:

- \(\mathbb{N}' \cap \mathbb{N}\) whenever \(\mathbb{N} = f(\{\mathbb{N}_i\}), \mathbb{N}' = f(\{\mathbb{N}_i'\})\)
  + and \(\mathbb{H}_1' \cap \mathbb{H}_1\) for each \(i\);

but reflection on the meaning of this shows that it is not a very "natural" condition when the functions \(f\) are allowed to be many-valued.

We say that the relation \(r\) is, with respect to \(Z\), a **quasi-ordering of divisibility** with respect to the class \(A\), or a **quasi-ordering** \((A, Z)\), if

- (a) \(r\) is reflexive in \(\mathbb{H}_1\) and transitive
- (b) \(\mathbb{N}' \cap \mathbb{N}\) whenever there is a chain \(\{\mathbb{N}_i\}\) of numbers such /
such that \( N' \in \{N_i\} \), \( f(\{N_i\}) = N \) and to each \( N_i \) there corresponds a number \( N'_i \) in \( A \) for which \( N'_i \prec N_i \).

Once again, a closer correspondence with the definition of Higman and with the general notion of an ordered algebra is obtained if a further condition is imposed:

(c) \( N' \prec N \) whenever \( N = f(\{N_i\}) \), \( N' = f(\{N'_i\}) \) and \( N'_i \prec N_i \) for each \( i \), and to each \( N'_i \) there corresponds a number \( N''_i \) in \( A \) for which \( N''_i \prec N'_i \).

This condition is open to the same objection as was the corresponding one in the previous definition; and, in fact, it turns out that, for the subsequent theory, it is immaterial whether or not it is imposed. We shall be concerned only with the class of successors with respect to \( r \) of the members of \( A \). It can be shown that if \( r \) is a relation which satisfies (a) and (b), then there is a relation \( 'r' \) which includes \( r \) and satisfies (a), (b) and (c) and for which the class in question is the same as for \( r \) itself. We shall not impose the condition (c).

It is clear that, if \( A_1 \subseteq A_2 \), then every quasi-ordering \((A_2, Z)\) is also a quasi-ordering \((A_1, Z)\), and that a divisibility quasi-ordering with respect to \( Z \) is a quasi-ordering of /
of divisibility with respect to \( H_1 \) and hence to any other class of numbers.

**THEOREM 47.** The property of being a quasi-ordering \((A, Z)\) (for given \(A\) and \(Z\)) is an extensionally attainable property of relations.

**Proof.** Suppose that, for all \( j \) in \([J]\), \( r_j \) is a quasi-ordering \((A, Z)\). Then obviously \( \bigcup_{j} r_j \) satisfies (a). If there is a chain \( \{N_i\} \) such that \( f(\{N_i\}) = N \) (for some \( f \) in \( Z \)) and for each \( N_i \) there is a number \( N_i' \) such that \( N_i' r_j N_i \) for all \( j \), then \( N' r_j N \) for all \( j \) whenever \( N' \in \{N_i\} \), by hypothesis. Hence in this case \( \bigcup_{j} r_j \) satisfies (b). If there is no such chain \( \{N_i\} \), the condition is trivially satisfied by \( \bigcup_{j} r_j \) (since a conditional whose antecedent is false is true). Hence the theorem is true.

We use the expression \( 'N_1 \prec N_2 \) \((A, Z)\)' to signify that \( N_1 \prec N_2 \) for every relation \( r \) which is a quasi-ordering \((A, Z)\).

By the preceding theorem, \( \prec \) \((A, Z)\) is, itself, such a quasi-ordering.

**THEOREM 48.** The class of all numbers \( N \) such that there is an \( N' \) for which \( N' \in A \) and \( N' \prec N \) \((A, Z)\) is \([A, Z]\).

**Proof.** Let \( A' \) be the class of numbers \( N \) such that there is
is a number $N'$ in $A$ for which $N' \preceq N$, where $\preceq$ is a given quasi-ordering $(A, Z)$. Then, by (b), the class $A'$ is closed with respect to $Z$, and, by (a), $A'$ includes $A$: $A \subseteq A' = [A', Z]$.

Conversely, let $A'$ be a class of numbers such that $A \subseteq A' = [A', Z]$. Let the relation $\preceq$ hold between $N_1$ and $N_2$ whenever $N_1 = N_2$ or $N_1$ and $N_2$ are both in $A'$, and only in these cases. Then the class of numbers $N$ for which there is a number $N'$ in $A$ such that $N' \preceq N$ is $A'$. But $\preceq$ satisfies (a), by definition, and (b), since $A'$ is closed. Hence $\preceq$ is a quasi-ordering $(A, Z)$.

Hence the theorem.

(It may be remarked that, in the second paragraph, it is easily seen that $\preceq$ also satisfies condition (c); this justifies the remark above that this condition is irrelevant to the result.)

---

We now proceed to apply to the particular case of partitioned numbers the general theory set out in the preceding section.
section. The following theorem is readily verified from the
definition of "divisibility quasi-ordering".

**Theorem 49.** The relation $r_0$ which holds between two num-
bers $N_1$ and $N_2$ of altitudes $\alpha_1$ and $\alpha_2$ if and only if
$N_1 = N_2$ or $1 + \alpha_1 \leq \alpha_2$ is with respect to any class $Z$ of
operations, a divisibility quasi-ordering of $H_1$.

Let $A$ be any class of numbers. We denote by $A(\alpha)$,
where $\alpha$ is an ordinal number, the subclass of $A$ which con-
sists of those members whose altitude is less than $\alpha$. Thus
$A(0) = \emptyset$. If $\{\alpha_i\}$ is the class of ordinals which are alti-
tudes of members of $A$, then $A(\alpha) = A$ whenever $\alpha \geq \text{seq}\{\alpha_i\}$;
and, if $\alpha_1 \leq \alpha_2$, then $A(\alpha_1) \subseteq A(\alpha_2)$.

**Theorem 50.** $[A, Z] (\alpha) \subseteq [A(\alpha), Z]$.

*Proof.* Let $r$ be any quasi-ordering $(A(\alpha), Z)$. Then $r_0 \cap r$
is also such a quasi-ordering. We now define the relation
$r'$ by letting $N_1 r' N_2$ if $N_1 (r_0 \cap r) N_2$ or if both $N_1 r_0 N_2$
and $N_2$ has altitude at least $\alpha$. Then $r_0 \cap r \subseteq r'$, and since
the former is reflexive so is the latter. Also $r'$ is trans-
itive. For suppose $N_1 r' N_2$ and $N_2 r' N_3$; then the alti-
tudes of $N_1$, $N_2$ and $N_3$ in that order form an increasing
sequence. If $N_3$ has altitude less than $\alpha$, the result fol-
loows from the transitivity of $r_0 \cap r$, and if $N_3$ has alti-
tude at least $\alpha$, then $N_1 r' N_3$ by definition. In order to /
to prove that $r'$ is a quasi-ordering $(A, Z)$, it remains to prove that it satisfies (b). Let $\{N_i\}$ be a chain of numbers, for each $N_i$ in which there is a number $N_i'$ in $A$ such that $N_i' \succ N_i$. If each $N_i$ has altitude less than $\alpha$, then $N_i' (\tau_0 \circ r) N_1$ and $N_i' \in A(\alpha)$; hence $N' \prec r' f(\{N_i\})$ whenever $N' \in \{N_i\}$, since $\tau_0 \circ r$ is a quasi-ordering $(A(\alpha), Z)$. If some $N_i$ has altitude at least $\alpha$, then $f(\{N_i\})$ has altitude at least $\alpha$, and so, again, (b) is verified. Moreover, $r'$ has the property that those successors with respect to $r'$ of members of $A$, whose altitudes are less than $\alpha$, are all successors of members of $A(\alpha)$. The property of satisfying this condition is easily seen to be extensionally attainable.

Thus to any relation $r$ which is a quasi-ordering $(A(\alpha), Z)$ there corresponds a relation $r'$ which is a quasi-ordering $(A, Z)$, which, restricted to numbers of altitude less than $\alpha$, is included in $r$, and which is such that those successors with respect to $r'$ of members of $A$, which have altitude less than $\alpha$, are successors of members of $A(\alpha)$. This holds, in particular, when $r \prec \prec (A(\alpha), Z)$, and the theorem follows.

The foregoing theorem is, of course, intuitively obvious, and it might be supposed that a more direct proof would be possible. It is, in fact, not difficult to construct one, if
a process of recursive definition followed by transfinite induction is allowed. The above argument shows, however, how such arguments can be avoided. The theorem is required subsequently as a lemma (Theorem 57, etc.).

**Theorem 51.** 
\[ [(A, Z)(\alpha), Z] = [A(\alpha), Z]. \]

*Proof.* Since \( A(\alpha) \subseteq [A, Z](\alpha) \), \([A(\alpha), Z] \subseteq [[A, Z](\alpha), Z]\). Since \([A, Z](\alpha) \subseteq [A(\alpha), Z]\), \([[A, Z](\alpha), Z] \subseteq [[A(\alpha), Z], Z]\]

**Theorem 52.** If \( B \) is a basis for \((H, Z)\), then \( B(\alpha) \) is a basis for \(([H(\alpha), Z], Z)\).

*Proof.* By hypothesis, \( B \) is a basis for \((H, Z)\) and \( H(\alpha) \) generates \(([H(\alpha), Z], Z)\). Hence \( B \cap H(\alpha) \) is a basis for \(([B \cap H(\alpha)], Z)\) by Theorem 45. But \([B \cap H(\alpha), Z]\]

**Theorem 53.** If \( A \) generates \((H, Z)\) and \( Z \) does not contain a multiplication of any index which is the dual of a limit ordinal, then every member of \( H \) possesses a subnumber in \( A \).

*Proof.* Let \( G \) be the class of all numbers which possess a subnumber in \( A \). Then \( A \subseteq G \). Also \((G, Z)\) is a system. For any subnumber of any \( N_1 \) is a subnumber of \( \Sigma_{i_1}^* N_1 \), and any subnumber of \( N_k \) is a subnumber of \( \prod_{i_1}^* N_1 \) and of \( \prod_{i_1}^* N_1 \), where \( k = \sup \{i\} \). Hence \( H \subseteq G \) since \( A \) generates \((H, Z)\), and \(/
and the theorem follows.

We can now show that not every system of numbers has a basis.

Let \( \mathcal{S}_0 \) be the set of irrational numbers of the interval \([0, 1]\) with the identity partial ordering. Let \( \mathcal{O}_\pi \) of altitude \( \omega \), be defined by letting (for \( i = 0, 1, 2, \ldots \)) \( \mathcal{O}_\pi \) hold between \( x \) and \( y \) in \( \mathcal{S}_0 \) if and only if there is an integer \( r \) such that \( x \) and \( y \) both belong to the interval \([r/(i+1)!], (r+1)/(i+1)!]\). Thus \( \mathcal{O}_\pi \) separates \( \mathcal{S}_0 \) into two subclasses, \( \mathcal{O}_\pi^2 \) separates each \( \mathcal{O}_\pi \)-subclass into three subclasses, \( \mathcal{O}_\pi^3 \) separates each \( \mathcal{O}_\pi^2 \)-subclass into four subclasses, etc. We thus arrive at a partitioned set \( \mathcal{T}_0 \). Let \( (i = 0, 1, 2, \ldots) \) \( \mathcal{T}_1 \) be the partitioned subset of \( \mathcal{T}_0 \) determined by the pair \((\mathcal{S}_0, \omega [0, 1/(i+1)!], i)\). Let now \( \mathcal{T}_1 \) be \( \mathcal{T}_1 \) (i = 0, 1, 2, \ldots). Then

\[
\begin{align*}
N_0 &= N_1 + cN_1 \\
N_1 &= N_2 + cN_2 + cN_2 \\
N_k &= N_k + cN_k + cN_k + cN_k \\
&\quad \text{etc.}
\end{align*}
\]

Let \( A = \{N_0, N_1, N_2, \ldots\} \) and \( H = [A, Z] \) where \( Z \) consists of cardinal addition for finite indices. Then the family \( (H, Z) \) has no basis.

The proof of this assertion is divided into sections.

(\( \alpha \)) If \( N \in A \) and \( N' \leq N \), then \( N' \in A \). Because, for each
each \( i \), the partitioned subsets \( T_1^{(1)}, T_1^{(2)} \) of \( T_0 \) determined by pairs \( (\bar{3}_3, [r_1/(i+1)!], (r_1+1)/(i+1)!], 1) \) and \( (\bar{3}_3, [r_2/(i+1)!], (r_2+1)/(i+1)!], 1) \) are similar under the mapping \( x \rightarrow x+(r_2-r_1)/(i+1)! \), and so \( T_1^{(1)} = T_1^{(2)} = N_1 \). Hence all subnumbers of \( N_0 \) are members of \( A \). But any member of \( A \) is a subnumber of \( N_0 \), and so, consequently, are all its subnumbers.

(\( \beta \)) No finite subset of \( A \) generates \( (H, Z) \). For, suppose there is a finite generating set \( A' \). Then every member of \( A \) has some member of \( A' \) as a subnumber, by the preceding theorem. Let now \( N_k \) be a member of \( A \) such that the subscript \( k \) exceeds the subscript of each member of \( A' \). Every subnumber of \( N_k \) is the number of a partitioned set \( T \) for which there are at least \( k \) \( T_1 \) subclasses of \( \bar{3}_3 \). Hence no member of \( A' \) can be a subnumber of \( N_k \). This shows that there is no such class as \( A' \).

(\( \gamma \)) Any infinite subclass of \( A \) generates \( (H, Z) \). For, from the statements (1) it follows that the closure \( [N_k, Z] \) of the class whose only member is \( N_k \) contains all \( N_i \) for which \( i < k \). Hence the closure \( [A', Z] \) of any subclass \( A' \) of \( A \) contains all numbers \( N_i \) such that \( i \) precedes the subscript of some member of \( A' \). Thus, if \( A' \) is infinite, \( A \subset [A', Z] \), and so \( [A', Z] = H \).

From (\( \beta \)) and (\( \gamma \)) it follows that there is no basis for \( (H, Z) \).
(H, Z) which is included in A. The proof is now completed
by (\delta).

(\delta) If A' generates (H, Z), then A' \cap A generates
(H, Z). Every member N of H must have some member of A' as
a subnumber, this being true, in particular, when N \in A.

But if N' \leq N and N \in A, then N' \in A, by (\alpha). It was proved
in (\beta) that if A' \cap A is finite there are members of A which
do not possess subnumbers in A' \cap A; hence A' \cap A must be in-
finite. But in this case, by (\gamma), A' \cap A generates (H, Z).

A second example shows that a system may have two dis-

joint bases.

Let S be a denumerable set of elements \{a_1, a_2, a_3, ....\}
with the identity partial ordering. Let a partition \pi of
altitude \omega be imposed on |S| as follows:-

\pi_0 is the unit simple partition of |S|
\pi_1 holds between every pair of a_2, a_3, .... and between
a_1 and itself
\pi_i (i = 1, 2, 3, ...) holds between every pair of
a_{i,2} + 1, a_{i,2} + 2, ...., and between a_{i,2} and it-

self
\pi_{i,2} + 1 (i = 1, 2, 3, ...) holds between every pair of
a_{i,2} + 2, a_{i,2} + 3, ...., and between a_{i,2} and it-

self and a_{i,2} + 1 and itself.

Thus we have defined a partitioned set T; let its number be \mathbb{N}.

Let /
Let the components of $N$ be $N_1$ and $N_2$, where $N_1$ is of altitude $\omega$ and $N_2$ is of altitude $\emptyset$. Then $N_1$ has two components, one of altitude $1$ which we call $N_3$ and one of altitude $\omega$ which is $\mathbb{N}$.

\[
N_1 + ^c N_2 = N
\]
\[
N_2 + ^c N = N_2
\]

From these two equations we see that the family $(\mathbb{N}, Z)$, where $Z$ consists of cardinal addition of index $\mathbb{Z}$, which is generated by $\{N, N_1, N_2, N_3\}$ has the bases $\{N, N_2, N_3\}$ and $\{N_1, N_2, N_3\}$.

The following example again presents a family with two distinct bases, and these bases possess different numbers of elements.

Let $S$ be the set of irrational numbers of the interval $[0, 1]$ in their natural order. We define $\Pi$ (of altitude $\omega$) by letting $\Pi_{i,2} (i = 0, 1, 2, \ldots)$ hold between $x$ and $y$ in $S$ if and only if there is an integer $r$ such that $x$ and $y$ both belong to $[r/3^i, (r+1)/3^i]$, and letting $\Pi_{i,2} + 1$ hold between /
between \( x \) and \( y \) in \( S \) if and only if there is an integer \( r \) such that \( x \) and \( y \) both belong to \([r/3^i, (3r + 1)/3^i + 1]\) or to \([(3r + 1)/3^i + 1, (r + 1)/3^i]\). We thus have defined a partitioned set \( T \) with number \( N \). (The tree \( B \) representing

\[
\begin{array}{c}
\begin{array}{c}
[0, 1] \\
[0, \frac{1}{3}] & [\frac{1}{3}, 1]
\end{array} \\
\begin{array}{c}
[0, \frac{2}{3}] \\
[\frac{2}{3}, \frac{1}{3}] & [\frac{1}{3}, 1]
\end{array} \\
\begin{array}{c}
[0, \frac{1}{5}] \\
[\frac{1}{5}, \frac{1}{3}] & [\frac{1}{3}, \frac{1}{5}]
\end{array}
\end{array}
\]

etc.

\( T \) is shown in the figure, the element \((S, [\frac{1}{3}, \frac{1}{3}], S)\) of \( B \), for example, being briefly denoted by \(' [\frac{1}{3}, \frac{1}{3}]'\). The number \( N \) has two components, \( N_1 \) which has a unique component and \( N_2 \) which has two components. The components of \( N_1 \) and of \( N_2 \) are, all three, \( N \) itself. Hence

\[
\begin{align*}
N_1 + 0 N_2 &= N \\
\Sigma N &= N_1 \\
N + 0 N &= N_2
\end{align*}
\]

Hence /
Hence the family \( \{\{N, N_1, N_2\}, Z\} \) where \( Z \) consists of ordinal addition of indices \( 1 \) and \( 2 \) is generated by \( \{N\} \) and by \( \{N_1, N_2\} \). Now \( \{N\} \), being a unit class, is certainly a basis for the family. So also is \( \{N_1, N_2\} \). For it is readily proved that any member \( N_0 \) of \( \{\{N_1\}, Z\} \) must be the number belonging to a partitioned set \( T_0 \) such that the tree \( B_0 \) representing its partition \( 0\Pi \) contains a complete class of complements \( \{B_0^{(r)}, i^{(r)}\} \), all the ordinals \( i^{(r)} \) being finite, such that the subset of \( T \) determined by these pairs \( \{B_0^{(r)}, i^{(r)}\} \) all have the number \( N_1 \). Now \( N_2 \) does not satisfy this condition. Similarly \( N_1 \in \{\{N_2\}, Z\} \).

**THE DERIVATIVE**

We denote by \( \mathcal{D}_\alpha (A, Z) \) the class \( A (\alpha + 1) - [A(\alpha), Z] \). Thus \( \mathcal{D}_\alpha (A, Z) \) consists of those members of \( A \) whose altitude is \( \alpha \) and which are not contained in the system (with respect to \( Z \)) generated by the class of those members of \( A \) whose altitude is less than \( \alpha \).

**THEOREM 54.** \( A(\alpha) \subset \bigcup_{i \in i^*} \mathcal{D}_i (A, Z), Z \), where \( \{i\}^* \) is the class /
class of ordinals less than $\alpha$.

**Proof.** Since $A(0) = 0$, the theorem is true when $\alpha = 0$.

Suppose the theorem true for all ordinals less than $\alpha$.

Now, by definition, $D_1(A, Z) = A(1 + 1) - \{A(1), Z\}$.

Hence

$$A(1 + 1) \subseteq D_1(A, Z) \cup \{A(1), Z\}$$

$$\subseteq D_1(A, Z) \cup \bigcup_{\iota \in \kappa} D_j(A, Z), Z$$

(if $\iota \prec \alpha$ and $\{\iota\}$ is the class of ordinals less than $\iota$)

$$\subseteq \left[\bigcup_{\iota \in \kappa} D_j(A, Z), Z\right]$$

Hence if there is an ordinal $\iota$ such that $\alpha = i + 1$, the theorem holds also for $\alpha$. If there is no such ordinal $\iota$, let $\{\iota\}$ be the class of ordinals less than $\alpha$. Then

$$A(\alpha) = \bigcup_{\iota \in \kappa} A(\iota)$$

$$\subseteq \bigcup_{\iota \in \kappa} \left[\bigcup_{\iota \in \kappa} D_j(A, Z), Z\right] \text{ (by hypothesis).}$$

$$\subseteq \left[\bigcup_{\iota \in \kappa} \bigcup_{\iota \in \kappa} D_j(A, Z), Z\right]$$

$$= \left[\bigcup_{\iota \in \kappa} D_1(A, Z), Z\right].$$

The theorem follows by transfinite induction.

**Theorem 55.** $[A(\alpha), Z] = \left[\bigcup_{\iota \in \kappa} D_1(A, Z), Z\right]$. 

**Proof.** $\bigcup_{\iota \in \kappa} D_1(A, Z) \subseteq A(\alpha) \subseteq \left[\bigcup_{\iota \in \kappa} D_1(A, Z), Z\right]$. 

$\therefore \left[\bigcup_{\iota \in \kappa} D_1(A, Z), Z\right] \subseteq [A(\alpha), Z] \subseteq \left[\bigcup_{\iota \in \kappa} D_1(A, Z), Z\right]$. 

**Corollary**
COROLLARY. If \( \{\alpha\} \) is the class of all ordinals, then
\[
\]

We denote the class \( \bigcup_{\alpha} D_\alpha(A, Z) \) by 'D(A, Z)' and call it the derivative of \( A \) with respect to \( Z \). By the above corollary, it is a refinement of \( A \) which likewise generates \( ([A, Z], Z) \). (This use of the word "derivative" has no connection with that in the differential calculus or that in point-set theory).

**THEOREM 56.** If \( B \) is a basis for \( \langle A, Z \rangle \), then
\[
D(B, Z) = B.
\]

**THEOREM 57.** The mapping \( A \to D(A, Z) \) of the generating classes of a system onto their derivatives is isotone. That is, if \( A_1 \subseteq A_2 \subseteq [A_1, Z] \), then \( D(A_1, Z) \subseteq D(A_2, Z) \)

*Proof.* Let \( \varphi(\alpha) \) be the statement:
\[
D_\alpha(A_1, Z) \subseteq D_\alpha(A_2, Z) \subseteq [A_1 (\alpha + 1), Z].
\]

We prove \( \varphi(\alpha) \) for all \( \alpha \) by transfinite induction.

Firstly, \( \varphi(0) \) is true. For \( [A_1 (1), Z] \) either contains the unique number of zero altitude or is null. In the former case the number of zero altitude belongs to \( A_1 \) and hence to \( A_2 \), and so to \( D_0 (A_1, Z) \) and \( D_0 (A_2, Z) \). In the second case both \( D_0 (A_1, Z) \) and \( D_0 (A_2, Z) \) are null.

Suppose now that \( \varphi(i) \) holds for all \( i \) less than \( \alpha \).

Now /
Now $D_{\alpha}(A_1, Z) = A_1(\alpha + 1) - [A_1(\alpha), Z]$ and $D_{\alpha}(A_2, Z) = A_2(\alpha + 1) - [A_2(\alpha), Z]$. But $A_1(\alpha + 1) \subseteq A_2(\alpha + 1)$ since $A_1 \subseteq A_2$. We have thus, to obtain the first inequality, to show that $[A_2(\alpha), Z] \subseteq [A_1(\alpha), Z]$. Now

$[A_2(\alpha), Z] = [\bigcup_{i \in \omega} D_1(A_2, Z), Z]$ (Theorem 55)

where $\omega$ consists of all ordinals less than $\alpha$. By hypothesis, $D_1(A_2, Z) \subseteq [A_1(i + 1), Z]$.

Hence $\bigcup_{i \in \omega} D_1(A_2, Z) \subseteq \bigcup_{i \in \omega} [A_1(i + 1), Z]

\subseteq [\bigcup_{i \in \omega} A_1(i + 1), Z]

= [A_1(\alpha), Z].$

And so $[A_2(\alpha), Z] \subseteq [A_1(\alpha), Z]$.

Thus the first inequality of $\phi(\alpha)$ is established. Now

$D_{\alpha}(A_2, Z) \subseteq A_2(\alpha + 1)$

$\subseteq [A_1, Z](\alpha + 1)$ since $A_2 \subseteq [A_1, Z]

\subseteq [A_1(\alpha + 1), Z]$, by Theorem 50.

Thus the second inequality is also established.

The proof is now completed by taking the union over all $\alpha$ of both sides of the first inequality of $\phi(\alpha)$.

COROLLARY. If $B$ is a basis for $(H, Z)$, then $B \subseteq D(H, Z)$.


Proof. $D_{\alpha}(D(A, Z), Z) = D(A, Z)(\alpha + 1) - [D(A, Z)(\alpha), Z]

= [D_{\alpha}(A, Z) \cup \bigcup_{i \in \omega} D_1(A, Z)] - [\bigcup_{i \in \omega} D_1(A, Z), Z]

(where \{i\} is the class of ordinals less than $\alpha$)

= $D_{\alpha}(A, Z) - [\bigcup_{i \in \omega} D_1(A, Z), Z]

= \{A /
This theorem and the preceding one together with the fact that $D(A, Z) \subseteq A$ state that the operation of forming derivatives with respect to a fixed class $Z$ is a closure operation on the generating classes of a system, with respect to the relation of including $(Z)$.

**Theorem 59.** If $A' \subseteq D(A, Z)$, then $A' = D(A', Z)$.

**Proof.** Let, for each ordinal $\alpha$, $A'_\alpha$ be the class of those members of $A'$ which have altitude $\alpha$. We have to show that $A'_\alpha = D_{\alpha}(A', Z)$ for all $\alpha$.

$$D_{\alpha}(A', Z) = A'(\alpha + 1) - [A'(\alpha), Z]$$

$$= A'_\alpha - [A'(\alpha), Z].$$

By hypothesis, $A'_\alpha \subseteq D_{\alpha}(A, Z)$; and $D_{\alpha}(A, Z)$ and $[A(\alpha), Z]$ are disjoint. But $A' \subseteq D(A, Z) \subseteq A$, and so $A'(\alpha) \subseteq A(\alpha)$. Thus $A'_\alpha$ and $[A'(\alpha), Z]$ are disjoint, and $A'_\alpha - [A'(\alpha), Z] = A'_\alpha$. Hence $D_{\alpha}(A', Z) = A'_\alpha$.

Theorem 59 is required in the proof of Theorem 63.

**Theorem 60.** If $[A_1, Z] = [A_2, Z]$, then

$D(A_1(\omega), Z) = D(A_2(\omega), Z)$.

**Proof.** If $A$ is any class of numbers, then $D(A(\omega), Z) = \bigcup D_i(A, Z)$ where $\{i\}$ is the class of finite ordinals.
We have to prove that $D_i (A_1, Z) = D_i (A_2, Z)$ for each $i$ in $\{i\}_i$; this we do by induction on $i$.

Firstly, $D_0 (A_1, Z) = D_0 (A_2, Z)$; for $[A_1, Z](1) = [A_2, Z](1)$, and $[A, Z](1) = D_0 (A, Z)$ for all $A$.

We now suppose that $D_j (A_1, Z) = D_j (A_2, Z)$ for all $j$ less than a finite $i$; that is, for all $j$ in $\{j\}_i$. Then

\[ [A_1(i), Z] = [\bigcup_{i \leq j} D_j (A_1, Z), Z] = [\bigcup_{i \leq j} D_j (A_2, Z), Z] = [A_2(i), Z]. \]

Now

\[ D_1 (A_1, Z) = A_1(i + 1) - [A_1(i), Z], \]
\[ D_1 (A_2, Z) = A_2(i + 1) - [A_2(i), Z]. \]

If $N \in A_1 (i + 1)$, then $N' \subseteq N (A_2 (i + 1), Z)$ for some $N'$ in $A_2 (i + 1)$, by Theorem 50. Hence $N' \subseteq N$. But if $N' \notin [A_2(i), Z]$, then we must have $N' = N$ since $i + 1 > i$ for finite $i$. Thus, by (1) and (2),

\[ D_1 (A_1, Z) \supseteq D_1 (A_2, Z). \]

Similarly, $D_1 (A_1, Z) \subseteq D_1 (A_2, Z)$.

**Theorem 61.** Every system $(H, Z)$ generated by a class $A$ of numbers of finite altitude has a unique basis $B$.

**Proof.** Since the members of $A$ are of finite altitude, $A = A(\omega)$. Let $B = D(A, Z)$; then $B$ is a basis for $(H, Z)$.

For suppose there is a class $A_1$ such that $A_1 \subseteq B$ and

\[ [A_1, Z] \subseteq H; \]

by the preceding theorem $D(A_1, Z) = D(A, Z) = B$, which is impossible since $D(A_1, Z) \subseteq A_1 \subseteq B$.

If /
If $A'$ generates $(H, Z)$, then $B \subseteq A'$; for $B = D(A, Z) = D(A'(\omega), Z) \subseteq A'$. Hence $B$ is unique.

**Theorem 62.** If $B_1$ and $B_2$ are both bases for $(H, Z)$, then $B_1(\omega) = B_2(\omega)$.

**Proof.** $D(B_1(\omega), Z) = D(B_2(\omega), Z)$. But $B_1(\omega)$ and $B_2(\omega)$ are both bases for $([H(\omega), Z], Z)$ (Theorem 52). Hence $B_1(\omega) = D(B_1(\omega), Z) = D(B_2(\omega), Z) = B_2(\omega)$.

**Theorem 63.** If, for every transfinite ordinal $\alpha$, $D_\alpha(A, Z)$ is a finite class, then the system $([A, Z], Z)$ has a basis. (The Selection Axiom is required for the general case.)

**Proof.** For each transfinite ordinal $\alpha$, the system $([D_\alpha(A, Z), Z], Z)$ has a basis $B_\alpha$ included in $D_\alpha(A, Z)$. Since $D_\alpha(A, Z)$ is finite. Let $B = D(A(\omega), Z) \cup \bigcup_{\alpha \in \omega} B_\alpha$, where $\omega$ is the class of transfinite ordinals. The Selection Axiom is required to establish the existence of $B$ in the case in which for a transfinite number of ordinals $\alpha$, $([D_\alpha(A, Z), Z], Z)$ has more than one basis which is included in $D_\alpha(A, Z)$. We show that $B$ is a basis for $([A, Z], Z)$.


But $[B, Z] = [D(A(\omega), Z) \cup \bigcup_{\alpha \in \omega} B_\alpha, Z]$

$\supseteq [D(A(\omega), Z), Z] \cup \bigcup_{\alpha \in \omega} [B_\alpha, Z]$

$= [A(\omega), Z] \cup \bigcup_{\alpha \in \omega} [D_\alpha(A, Z), Z]$.

$\therefore [B, Z] \supseteq [A(\omega) \cup \bigcup_{\alpha \in \omega} D_\alpha(A, Z), Z]$


Hence /
Hence $B$ generates $([A, Z], Z)$.

Suppose now that $A' \subset B$. By definition $B \subset D(A, Z)$. Hence $A' \subset B = D(B, Z)$, and so $A' = D(A', Z)$ (Theorem 59). Hence $D_\alpha(A', Z) \subset D_\alpha(B, Z) = B_\alpha$ for all transfinite ordinals $\alpha$, and $A'(\omega) \subset B(\omega) = D(A(\omega), Z)$. Since $A' \neq B$, we must have either $A'(\omega) \subset D(A(\omega), Z)$ or $D_\alpha(A', Z) \subset B_\alpha$ for some transfinite ordinal $\alpha$.

If $A'(\omega) \subset D(A(\omega), Z)$, then $[A'(\omega), Z] \subset [D(A(\omega), Z), Z]$, since $D(A(\omega), Z)$ is a basis for $([D(A(\omega), Z), Z], Z)$; and hence

$$[A', Z](\omega) = [A'(\omega), Z](\omega)$$

(by Theorem 50)

$$\subset [D(A(\omega), Z), Z](\omega)$$

$$= [B(\omega), Z](\omega)$$

$$= [A(\omega), Z](\omega)$$

$$= [A, Z](\omega)$$

(by Theorem 50).

Hence $A'$ cannot generate $([A, Z], Z)$; since, if it did so, we should have $[A', Z] = [A', Z]$ and, hence $[A', Z](\omega) = [A, Z](\omega)$.

If $D_\alpha(A', Z) \subset B_\alpha$, then $[D_\alpha(A', Z), Z] \subset [B_\alpha, Z]$ since $B_\alpha$ is a basis for $([B_\alpha, Z], Z)$. By an argument similar to the above we conclude that $[A', Z](\alpha + 1) \subset [A, Z](\alpha + 1)$ and, hence, that $A'$ does not generate $([A, Z], Z)$.

Thus, in neither case can $A'$ generate $([A, Z], Z)$; we conclude that $B$ is a basis for this system.
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$\subseteq, \subset, \cup, \vee, |S|$  

</
$, \parallel$

$\sup \{1\}, \sqcup \{1\}$

$\lor, \land, \mathcal{O}$

$R \circ S, R \circ S, S \circ R, R \circ S$

$R^S, S_R$

$\pi$ - subclass, $\pi_1 \cdot \pi_2$

$\Sigma \pi_1$

$\Delta_1 \Pi$

$\Theta_1^1 \Pi$

$\Pi^* \times \Theta_0 \Pi$

$\mathcal{P} \{1\} \circ \Theta_1 \Pi, \Pi^{k*}$

$T_1 \sim T_2$

$T$

$T^1 \leq T$

$\Sigma^* N_1, \Sigma_0^0 N_1, \Sigma_1^0 N_1, \Sigma^0 N$

$N_1 \times^0 N_0, N_1 \times^0 N_0$

$\mathcal{P}^{0}_{\{1\}} \ast N, \mathcal{P}^{0}_{\{1\}} \ast N$

$[A, Z]$

$N_1 < N_2 (A Z)$

$\Gamma_0, \chi (A)$

$D_{\chi}(A, Z)$

$D(A, Z)$