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Explicit Numerical Schemes of SDEs
Driven by Lévy Noise with Super-linear Coefficients and Their Application to Delay Equations

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Doctor of Philosophy
University of Edinburgh
2015
Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. Further, I declare that this work has not been submitted for any other degree or professional qualification.

Chaman Kumar
Edinburgh, July 2015
Abstract

We investigate an explicit tamed Euler scheme of stochastic differential equation with random coefficients driven by Lévy noise, which has super-linear drift coefficient. The strong convergence property of the tamed Euler scheme is proved when drift coefficient satisfies one-sided local Lipschitz condition whereas diffusion and jump coefficients satisfy local Lipschitz conditions. A rate of convergence for the tamed Euler scheme is recovered when local Lipschitz conditions are replaced by global Lipschitz conditions and drift satisfies polynomial Lipschitz condition. These findings are consistent with those of the classical Euler scheme. New methodologies are developed to overcome challenges arising due to the jumps and the randomness of the coefficients. Moreover, as an application of these findings, a tamed Euler scheme is proposed for the stochastic delay differential equation driven by Lévy noise with drift coefficient that grows super-linearly in both delay and non-delay variables. The strong convergence property of the tamed Euler scheme for such SDDE driven by Lévy noise is studied and rate of convergence is shown to be consistent with that of the classical Euler scheme. Finally, an explicit tamed Milstein scheme with rate of convergence arbitrarily close to one is developed to approximate the stochastic differential equation driven by Lévy noise (without random coefficients) that has super-linearly growing drift coefficient.

Key words. Stochastic differential equation with random coefficient driven by Lévy noise, delay equation, tamed Euler scheme, tamed Milstein scheme, super-linear coefficients.

AMS subject classifications (MSC2010). 60G51, 60G55, 60H05, 60H10, 60H30, 65C30, 65H35, 68U20
Lay Summary

Stochastic differential equations are popular methods of modelling real world phenomena. But they seldom possess explicit solutions, which necessitates development of numerical schemes to obtain their approximate solutions. Euler scheme is the simplest explicit scheme and has a strong order of convergence equal to 0.5, when coefficients of stochastic differential equations grow linearly. However, it is well known that the efficient Euler scheme does not converge at finite time in the case of super-linearly growing coefficients. Recently, several tamed Euler schemes have been developed which are explicit in nature and allow the coefficients to grow super-linearly. In this thesis, we develop explicit tamed Euler scheme for stochastic differential equation with random coefficients driven by Lévy noise which has super-linear drift coefficient. As an application of this finding, we also propose an explicit tamed Euler scheme for stochastic delay differential equation driven by Lévy noise with drift coefficient that grows super-linearly in both delay and non-delay variables. Further, a tamed Milstein scheme for stochastic differential equation driven by Lévy noise with super-linear drift coefficient has been developed. Finally, we conclude that the methodologies developed herein can be extended to other higher order schemes. The results contained in this thesis should appeal to mathematicians and practitioners alike, due to increasing popularity of stochastic (delay) differential equations driven by Lévy noise.
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I dedicate this thesis to maa, papa
&
my beloved wife Neelima.
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Chapter 1

Introduction

1.1 Motivation and Literature Review

Since the seminal work of [8], stochastic differential equations (SDEs) have been widely used to model future uncertainties encountered in real life situations. The models based on SDEs have numerous applications in finance, economics, actuarial science, engineering, physics, chemistry, biology and ecology. One can refer to [34, 36, 43, 57] and references therein for a comprehensive discussion on SDEs and their applications. Often, SDEs do not have explicit solutions, which necessitates the development of numerical schemes to obtain their approximate solutions. Over the years, several explicit and implicit numerical schemes have been developed. The interested reader may refer to [36] and references therein for a comprehensive discussion on strong and weak convergence of explicit and implicit schemes of SDEs. We also mention a paper by [35] on path-wise approximations and its relation to strong convergence. The classical Euler scheme is considered to be the simplest explicit scheme for approximating SDEs. The strong and weak convergence rate of the Euler scheme are known to be $0.5$ and $1.0$ respectively. Recently, it has been proved that the efficient Euler scheme diverges in strong and weak sense at finite time when coefficients of SDE do not satisfy linearity assumption and are allowed to have super-linear growth. For details on strong and weak divergence of the Euler scheme for SDEs with super-linearly growing coefficients, one can refer to [29]. Furthermore, there are implicit schemes which can be used to obtain numerical solutions of SDEs with super-linearly growing coefficients, one can refer to [36]. However, explicit schemes require less computing time and effort than implicit schemes when implemented on computer. Thus, it is desirable to develop explicit numerical schemes of SDEs when their coefficients have super-linear growth. The first publication in this direction is [30], where the authors have proposed a tamed Euler scheme for approximating SDE with super-linearly growing drift coefficient. Motivated by the work of [30], several other tamed schemes of SDEs with super-linear coefficients have been developed in the literature, for example [28, 59, 60, 64, 69].

The event-driven phenomena are frequently observed in real life situations. For example, announcements made by central banks, changes in credit ratings, operational failures, credit defaults, market crashes and booms, etc. are some events, which are random in nature and have sudden and significant effects on the stock price movements. The models based on SDEs are not suitable to capture such event-driven uncertainties. However, SDEs driven by Lévy noise are considered to be more appropriate method of modelling these phenomena. Over the past few years, SDEs driven by Lévy noise have gained much popularity in finance, economics, actuarial science and many other
branches of sciences. For details on the Lévy process and their applications, one can refer to [1, 7, 15, 52, 61] and references therein. As before, SDEs driven by Lévy noise rarely have explicit solutions and hence one often resorts to numerical schemes to obtain their approximate solutions. A detailed discussion on explicit and implicit schemes of SDEs driven by Lévy noise can be found in [9, 10, 54, 56, 62] and references therein. Moreover, SDEs are special cases of SDEs driven by Lévy noise. Thus, due to the work of [29], one concludes that the simple and efficient Euler scheme also diverges in strong and weak sense at finite time when coefficients of SDE driven by Lévy noise have super-linear growth. Hence, one requires to develop explicit tamed Euler schemes to obtain their approximate solutions. To the best of our knowledge, there has been no development on the numerical schemes of SDEs driven by Lévy noise when their coefficients are allowed to grow super-linearly. Motivated by [59], in joint work with Dareiotis and Sabanis [16], we propose an explicit tamed Euler scheme of SDE with random coefficients driven by Lévy noise where drift coefficient is allowed to grow super-linearly. We study strong convergence of the tamed Euler scheme and obtain a rate of convergence which is same as that of the classical Euler scheme. Based on our findings, we conclude that optimal rate of convergence is achieved for mean square convergence which is consistent with findings of [5]. Furthermore, in the absence of jumps, a rate of one-half can be obtained for convergence in \( L^q \) for any \( q \geq 2 \), which is same as that of [28, 59, 60, 64, 69]. At this stage, a natural question to ask is, if such taming techniques can further be extended to Milstein scheme and other higher order schemes? Recently, some work has been done on Milstein scheme of SDE with super-linear coefficients, for example [64, 66, 69]. To the best of our knowledge, Milstein scheme and other higher order schemes of SDE driven by Lévy noise have not been developed so far when the coefficients grow super-linearly. In joint the work with Sabanis [40], we adopt the methodologies developed in [16, 59] and propose an explicit tamed Milstein scheme for SDE driven by Lévy noise which can have super-linear growth in its drift coefficient. The rate of mean square convergence of the tamed Milstein scheme is shown to be arbitrarily close to one. Furthermore, as a special case, we also prove that the tamed Milstein scheme of SDE with super-linear drift coefficient achieves the rate one-half for convergence in \( L^q \) for any \( q \geq 2 \), which is same as that of [64, 66, 69]. Finally, the techniques developed in [40] can be extended to develop other higher order schemes of SDEs with super-linear coefficients.

The SDEs discussed above are assumed to have markovian structure and are popular methods of modelling physical system when the future state of the system depends only on its present state. However, there are real life situations, where the future state of the system not only depends on its present state but also on its past state(s). For example, the effect of change in regulations on the economy is observed after a lap of some time. The stochastic delay differential equations (SDDEs) are more appropriate methods of capturing these causal effects than SDEs. Recently, SDDEs have found applications in physics, biology, ecology, economics and finance, some references are [2, 17, 18, 19, 22, 27, 43, 46, 47, 53, 63, 68] and references therein. Furthermore, SDDEs rarely have explicit solutions and thus one needs to develop numerical schemes to obtain their approximate solutions. The strong and weak numerical schemes for SDDEs have been developed in recent past, for example, [3, 4, 11, 12, 13, 25, 26, 37, 38, 41, 44, 45, 50, 51] and references therein. In joint work with Sabanis [39], we adopt the approach of [25] and prove strong convergence of the Euler scheme by considering SDDEs as a special case of SDEs with random coefficients. With this approach, strong convergence is proved under more relaxed conditions than those existing in the literature, for example, [3, 37, 38, 44]. More
precisely, in [45], strong convergence has been shown by assuming linear growth and local Lipschitz conditions in both variables corresponding to delay and non-delay arguments. Whereas, in [39], authors have relaxed these conditions by assuming polynomial growth and continuity in the argument corresponding to delays. Moreover, no smoothness condition on the initial data is assumed. In addition, it is also shown that the rate of convergence is one-fourth when the drift coefficient satisfies one-sided Lipschitz and polynomial Lipschitz in the non-delay and delay arguments respectively. These findings are in agreement with the findings of [23]. Furthermore, numerical schemes of SDDEs driven by Lévy noise have also been studied in the literature, one can refer to [5, 31, 32, 33, 42, 49, 58, 65, 67] and references therein. However, no result exists on the explicit numerical schemes of SDDEs that have coefficients which grow super-linearly in non-delay variables. In joint work with Dareiotis and Sabanis [16], we adopt the approach of [25] and propose a tamed Euler scheme for SDDE driven by Lévy noise which has drift coefficient that grow super-linearly in both delay and non-delay variables. The proposed scheme is shown to converge in $L^q$ with an optimal rate arbitrarily close to one-half for mean square convergence. A big motivation to study strong convergence of the tamed schemes discussed above comes from the fact that such convergence is needed for the efficient implementation of accelerated Monte Carlo schemes. One can refer to the seminal work of [20, 21] and references therein for details.

We conclude this section with a note that the results contained in this report are based on my joint works [16, 39, 40].

1.2 Brief Summary of Results

In this report, we are concerned with the strong approximation of explicit numerical schemes of SDE and SDDE driven by Lévy noise. In other words, the aim is to construct numerical schemes $(x^n_t)_{t \in [0,T]}$ in order to obtain an approximation of the true solution $(x_t)_{t \in [0,T]}$ of SDE and SDDE driven by Lévy noise such that

$$\left( E \sup_{0 \leq t \leq T} |x_t - x^n_t|^q \right)^{1/q} \leq Kn^{-r}$$

where $r$ is the rate of convergence of the scheme and $1/n$ its step size for any $n \in \mathbb{N}$. The explicit schemes that are discussed in this report, have been recently developed in my joint works [16, 39, 40]. This report contains seven chapters including this introductory chapter. We now give chapter-wise summary of the results included in this report.

In Chapter 2, we discuss the basic framework of the SDE with random coefficients driven by Lévy noise. This framework is used throughout this report. We also prove the boundedness of the solution of SDE driven by Lévy noise in $L^p$ sense for a fixed $p \geq 2$. In Lemma 2.1, moment bounds are shown when drift, diffusion and jump coefficients grow linearly. In Lemma 2.2, moment bounds are proved by relaxing the condition on the drift coefficient so that it grows super-linearly. These proofs are included for the sake of completeness and for the justification of finiteness of the right hand side when applying the Gronwall’s lemma, something that is missing from the existing literature.

In Chapter 3, we introduce two explicit numerical schemes for SDE with random coefficients driven by Lévy noise. In Section 3.1, the classical Euler scheme is discussed which is essentially a generalization of the results obtained in joint work with Sabanis [39] in order to include jumps. The strong convergence of the Euler scheme is shown in Theorem 3.1 of this report under the assumptions that - (a) drift satisfies linear growth and local one-sided Lipschitz condition whereas (b) diffusion and jump coefficients
satisfy linear growth and local Lipschitz conditions. The methodologies developed in [39] allow one to overcome challenges arising due to randomness of the coefficients. The classical rate of convergence of the Euler scheme is recovered in Theorem 3.2 when local Lipschitz conditions are replaced by global Lipschitz conditions. As mentioned before, the Euler scheme fails to converge in strong and weak sense when coefficients of SDE grow super-linearly, refer to [29] for details. Hence, in Section 3.2, we introduce an explicit tamed Euler scheme for approximating SDE with random coefficients driven by Lévy noise in order to allow the drift coefficient to grow super-linearly. These results on the tamed Euler scheme are taken from my joint work with Dareiotis and Sabanis [16]. The taming assumption in [16] is in the same spirit as used in [59]. We establish moment bounds of the tamed Euler scheme in Lemma 3.4. For this purpose, new techniques are developed to overcome challenges arising due to jumps. Further, strong convergence of the tamed Euler scheme is also shown under local one-sided Lipschitz condition on drift and local Lipschitz condition on diffusion and jump coefficients. Finally, rate of strong convergence is shown to be equal to that of the Euler scheme when local Lipschitz conditions are replaced by global Lipschitz conditions. We notice that the optimal rates of convergence of both Euler scheme and tamed Euler scheme of SDE with random coefficients driven by Lévy noise are arbitrarily close to 0.5. These optimal rates are achieved for mean square convergence.

An important application of the results discussed in Chapter 3 is to find numerical schemes for approximating SDDE driven by Lévy noise. This is included in Chapter 4 where we use the approach of [25] and consider SDDE driven by Lévy noise as a particular case of SDE with random coefficients driven by Lévy noise. By virtue of this approach, in Theorem 4.1, we prove existence and uniqueness of the solution of SDDE driven by Lévy noise under more relaxed conditions than those existing in the literature. In Theorem 4.2, strong convergence result of Euler scheme for SDDE driven by Lévy noise is proved under the assumptions that - (a) drift, diffusion and jump coefficients have linear growth in non-delay variable and have super-linear growth in delay variable, (b) local one-sided Lipschitz condition on drift and local Lipschitz condition on diffusion and jump coefficients in non-delay variables, and (c) drift, diffusion and jump coefficients satisfy continuity in delay variables. Further, no smoothness condition on the initial data is required. The rate of convergence of the Euler scheme of SDDE driven by Lévy noise is obtained in Theorem 4.3, when local Lipschitz conditions in non-delay argument are replaced by the corresponding global Lipschitz conditions and continuity assumptions in delay variables are replaced by polynomial Lipschitz conditions. These findings are obtained under more relaxed conditions than those of existing results, see for example, [3, 5, 35, 37, 44, 45]. These results are also in agreement with the findings of [5]. By adopting the similar approach as used in Section 4.2 for Euler scheme, we propose a tamed Euler scheme of SDDE driven by Lévy noise in Section 4.3. Thus, we find an explicit tamed Euler scheme of SDDE driven by Lévy noise which has the same rate as that of the Euler scheme and allows the drift coefficient to grow super-linearly in both delay and non-delay variables. Finally, we conclude that the optimal rates of convergence of both Euler scheme and tamed Euler scheme are arbitrarily close to 0.5. These optimal rates are achieved for mean square convergence.

In Chapter 5, we discuss the results obtained in my joint work with Sabanis [40]. In this chapter, we drop the settings of randomness of coefficients since the application of Itô’s formula on the random coefficients leads to anticipative integrals. This makes the analysis difficult for higher order numerical schemes of SDE with random coefficients. Thus, we work with SDE driven by Lévy noise and propose a tamed Milstein scheme to approximate its solution when the drift coefficient is allowed to grow super-linearly.
The mean square convergence is shown under one-sided Lipschitz condition on drift and Lipschitz condition on diffusion and jump coefficients. The methodologies developed here can be extended to obtain other higher order schemes for SDEs driven by Lévy noise.

In Chapter 6, we demonstrate theoretical findings with the help of some simulation results. Finally, the report concludes with explanation of some future projects in Chapter 7.

1.3 Notations

In this section, we introduce notations that shall be used throughout this report. The symbol $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}, P)$ denotes a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ of its $\sigma$-algebra $\mathcal{F}$ that satisfies the usual conditions, i.e. $\mathcal{F}_0$ contains all $P$-null sets (complete) and the filtration is right continuous. $w$ stands for $\mathbb{R}^m$-valued standard Wiener process. Also, $(Z, \mathcal{Z}, \nu)$ represents a $\sigma$-finite measure space and $N(dt, dz)$ is a Poisson random measure defined on $(Z, \mathcal{Z}, \nu)$ with intensity $\nu$. The compensated Poisson random measure is denoted by $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$. We work under the above mentioned setting throughout this report.

For a vector $x \in \mathbb{R}^d$, $|x|$ denotes its Euclidean norm and $x^i$ its $i$-th element for $i = 1, \ldots, d$. For a matrix $\sigma \in \mathbb{R}^{d \times m}$, $|\sigma|$ stands for its Hilbert-Schmidt norm and $\sigma^{(i,j)}$ for its $(i,j)$-th component for $i = 1, \ldots, d$ and $j = 1, \ldots, m$. The transpose of $\sigma$ is denoted by $\sigma^*$. The indicator function of a set $A$ is denoted by $I_A$ whereas $[a]$ stands for the integral part of a real number $a$. Again, $\mathcal{P}$ refers to predictable sigma-algebra on $\Omega \times \mathbb{R}_+$ and $\mathcal{B}(V)$ stands for the sigma-algebra of Borel sets of a topological space $V$. Also, for a fixed $T > 0$, $L^p$ denotes the set of non-negative measurable functions $g$ on $[0, T]$ such that

$$\int_0^T |g_t|^p dt < \infty.$$ 

For a random variable $X$, the notation $X \in L^p$ means $E|X|^p < \infty$. By $\mathcal{A}$, we denote the class of non-negative predictable processes $L := (L_t)_{t \in [0, T]}$ such that

$$\int_0^T L_t dt < \infty$$

for almost every $\omega \in \Omega$. The notation $x^n \overset{P}{\rightharpoonup} x$ means that the sequence $(x_n)_{n \in \mathbb{N}}$ of random variables converges in probability to a random variable $x$. Finally, $K$ stands for a generic constant which does not depend on $n$ and changes values from place to place.

1.4 Some Useful Results

We recall some well-known inequalities which are frequently used in this report.

Lemma 1.1 (Young’s Inequality). Let $a \geq 0$, $b \geq 0$ and $\epsilon > 0$ be real numbers, then

$$ab \leq \frac{\epsilon}{p} a^p + \frac{1}{q^p \epsilon^{q/p}} b^q$$

where $p, q > 1$ satisfy $1/p + 1/q = 1$. 

5
Lemma 1.2 (Hölder’s Inequality). Let \((X, \Sigma, \mu)\) be a measure space. Then, for all measurable functions \(f, g : X \to \mathbb{R}\)
\[
\int_X |f(x)g(x)|\mu(dx) \leq \left( \int_X |f(x)|^p \mu(dx) \right)^{1/p} \left( \int_X |g(x)|^q \mu(dx) \right)^{1/q}
\]
where \(p, q \in [1, \infty] \) satisfy \(1/p + 1/q = 1\).

Lemma 1.3 (Gronwall’s lemma). Let \(T > 0\) and \(c \geq 0\) be fixed constants. Suppose \(u : [0, T] \to \mathbb{R}_+\) is a bounded and Borel measurable function, and \(v : [0, T] \to \mathbb{R}_+\) is an integrable function. If
\[
u(t) \leq c + \int_0^t v(s)u(s)ds
\]
holds for all \(0 \leq t \leq T\), then
\[
u(t) \leq c \exp \left( \int_0^t v(s)ds \right)
\]
for all \(0 \leq t \leq T\).

The following inequality on the bounds of a continuous martingale can be found in [14].

Lemma 1.4 (Burkholder-Davis-Gundy Inequality). Let \(M := (M_t)_{t \geq 0}\) be a continuous martingale on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) such that \(M_0 = 0\) almost surely and \([M]\) denotes the quadratic variation of \(M\). Then, for any \(0 < r < \infty\), there exist constants \(0 < c_r, C_r < \infty\) (depend only on \(r\)) such that
\[
c_r E([M]_t)^{\frac{r}{2}} \leq E \sup_{0 \leq s \leq t} |M_s|^r \leq C_r E([M]_t)^{\frac{r}{2}}
\]
for all \(t \geq 0\).

The proof of the following lemma can be found in [48].

Lemma 1.5. Let \(r \geq 2\) and \(T > 0\). There exists a constant \(K\), depending only on \(r\), such that for every real-valued, \(\mathcal{P} \otimes \mathcal{Z}\)–measurable function \(g\) satisfying
\[
\int_0^T \int_Z |g_t(z)|^2 \nu(dz)dt < \infty
\]
almost surely, then the following estimate holds,
\[
E \sup_{0 \leq t \leq T} \left| \int_0^t \int_Z g_s(z)\bar{N}(ds,dz) \right|^r \leq KE \left( \int_0^T \int_Z |g_t(z)|^2 \nu(dz)dt \right)^{r/2} + KE \int_0^T \int_Z |g_t(z)|^r \nu(dz)dt.
\]
(1.1)

It is known that if \(1 \leq r \leq 2\), then the second term in (1.1) can be dropped.

The proof of the following Domination-based inequality can be found in [57] (see Proposition IV.4.7 and Exercise IV.4.31/1).
Lemma 1.6. Let $X$ be a positive, adapted right continuous process and $A$ be a continuous increasing process such that

$$E[X_\tau|\mathcal{F}_0] \leq E[A_\tau|\mathcal{F}_0]$$

for any bounded stopping time $\tau$. Then for any $\zeta \in (0,1)$,

$$E[(X^*_\infty)^\zeta] \leq \frac{2-\zeta}{1-\zeta} E[(A_\infty)^\zeta].$$

where $X^*_\infty = \sup_{t \geq 0} X_t$.

Finally, we give Itô’s formula for semi-martingales (with jumps). The proof can be found in [55].

Lemma 1.7 (Itô’s Formula). Let $x := (x_t)_{t \geq 0}$ be a $d$-dimensional semi-martingale issuing from $x_0$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a twice continuously differentiable function. Then, $f(x)$ is a semi-martingale and

$$f(x_t) = f(x_0) + \sum_{i=1}^d \int_{0}^{t} f_{x_i}(x_{s-})dx^i_s + \frac{1}{2} \sum_{i,j=1}^d \int_{0}^{t} f_{x_ix_j}(x_{s-})d[x^i_s,x^j_s]_s$$

$$+ \sum_{0<s \leq t} \left( f(x_s) - f(x_{s-}) - \sum_{i=1}^d f_{x_i}(x_{s-}) \Delta x^i_s \right)$$

where $f_{x_i}$ is the first order partial derivative w.r.t. $x_i$ and $f_{x_ix_j}$ is the second order partial derivative of $f$ w.r.t. $x_i$ and $x_j$ for all $i,j = 1,\ldots,d$. 

7
Chapter 2

SDE with Random Coefficients Driven by Lévy Noise

In this chapter, we introduce set up for SDE with random coefficients driven by Lévy noise, which is used subsequently in Chapters 3, 4.

Let \( b_t(x) \) and \( \sigma_t(x) \) be \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable functions which take values in \( \mathbb{R}^d \) and \( \mathbb{R}^d \times \mathbb{R}^m \) respectively. Further, assume that \( \gamma_t(x,z) \) is \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Z} \)-measurable function which takes values in \( \mathbb{R}^d \). Also, let \( t_0 \) and \( t_1 \) be fixed constants satisfying \( 0 \leq t_0 < t_1 \leq T \).

We consider the following SDE on \((\Omega, \mathcal{F}, \mathbb{P})\) defined by,

\[
 dx_t = b_t(x_t)dt + \sigma_t(x_t)dw_t + \int_Z \gamma_t(x_t, z) \tilde{N}(dt, dz)
\]

almost surely for any \( t \in [t_0, t_1] \) with initial value \( x_{t_0} \) which is an \( \mathcal{F}_{t_0} \)-measurable random variable in \( \mathbb{R}^d \).

Remark 2.1. For notational convenience, we write \( x_t \) instead of \( x_{t-} \) on the right hand side of the above equation. This does not cause any problem since the compensators of the martingales driving the equation are continuous. This notational convention shall be adopted throughout this report.

2.1 Existence and Uniqueness

For the purpose of this section, the set of assumptions are as follows.

A-1. There exists an \( \mathcal{M} \in \mathcal{A} \) such that

\[
 xb_t(x) + |\sigma_t(x)|^2 + \int_Z |\gamma_t(x, z)|^2 \nu(dz) \leq \mathcal{M}_t(1 + |x|^2)
\]

almost everywhere on \( \Omega \times [t_0, t_1] \) for any \( x \in \mathbb{R}^d \).

A-2. For every \( R > 0 \), there exists an \( \mathcal{M}(R) \in \mathcal{A} \) such that,

\[
 (x - \bar{x})(b_t(x) - b_t(\bar{x})) + |\sigma_t(x) - \sigma_t(\bar{x})|^2 + \int_Z |\gamma_t(x, z) - \gamma_t(\bar{x}, z)|^2 \nu(dz) \leq \mathcal{M}_t(R)|x - \bar{x}|^2
\]

almost everywhere on \( \Omega \times [t_0, t_1] \) whenever \( |x|, |\bar{x}| \leq R \).

A-3. Almost everywhere on \( \Omega \times [t_0, t_1] \), the function \( b_t(x) \) is continuous in \( x \in \mathbb{R}^d \).
Proof of the following theorem can be found in [24].

**Theorem 2.1.** Let Assumptions A-1 to A-3 be satisfied. Then, there exists a unique càdlàg process satisfying SDE (2.1).

### 2.2 Moment Bounds

In this section, the moment bounds of the solution of SDE (2.1) have been established. In Lemma 2.1, the coefficients of SDE (2.1) are assumed to have linear growth, whereas in Lemma 2.2, this condition is relaxed to allow super-linear growth in the drift coefficient.

**A-4.** For a fixed \( p \geq 2 \), \( E|x_{t_0}|^p < \infty \).

**A-5.** There exist a constant \( L > 0 \) and a non-negative random variable \( M \) satisfying \( EM^2 < \infty \) such that

\[
|b_t(x)|^2 \vee |\sigma_t(x)|^2 \vee \int_Z |\gamma_t(x, z)|^2 \nu(dz) \leq L(M + |x|^2)
\]

almost surely for any \( t \in [t_0, t_1] \) and \( x \in \mathbb{R}^d \).

**A-6.** There exist a constant \( L > 0 \) and a non-negative random variable \( M' \) satisfying \( EM' < \infty \) such that

\[
\int_Z |\gamma_t(x, z)|^p \nu(dz) \leq L(M' + |x|^p)
\]

almost surely for any \( t \in [t_0, t_1] \) and \( x \in \mathbb{R}^d \).

The following is probably well-known. However, the proof is provided for the sake of completeness and for the justification of finiteness of the right hand side when applying Gronwall’s lemma, something that is missing from the existing literature.

**Lemma 2.1.** Let Assumptions A-2 to A-6 hold. Then, there exists a unique solution \((x_t)_{t \in [t_0, t_1]}\) of SDE (2.1), and the following estimate holds

\[
E \sup_{t_0 \leq t \leq t_1} |x_t|^p \leq K,
\]

where \( K := K(t_0, t_1, L, p, E|x_{t_0}|^p, EM^2, EM') \).

**Proof.** The existence and uniqueness of solution of SDE (2.1) follows immediately from Theorem 2.1 by noting that due to Assumption A-5, Assumption A-1 is satisfied.

As jump can occur at the terminal time \( t_1 \), we consider SDE (2.1) on the extended interval \([t_0, t_1']\), by re-defining \( b_t(\cdot), \sigma_t(\cdot) \) and \( \gamma_t(\cdot) \) as \( b_t(\cdot)I_{t_0 \leq t \leq t_1}, \sigma_t(\cdot)I_{t_0 \leq t \leq t_1} \) and \( \gamma_t(\cdot)I_{t_0 \leq t \leq t_1} \) for \( t \in [t_0, t_1'] \), for some \( t_1' > t_1 \). Now we consider the stopping time \( \pi_R := \inf \{t \geq t_0 : |x_t| > R\} \wedge t_1' \) for some \( R > 0 \), then clearly \(|x_t| \leq R\) for \( t_0 \leq t < \pi_R \).

By equation (2.1), one has

\[
|x_t|^p \leq 4^{p-1}|x_{t_0}|^p + 4^{p-1} \left| \int_{t_0}^t b_s(x_s)ds \right|^p + 4^{p-1} \left| \int_{t_0}^t \sigma_s(x_s)dw_s \right|^p + 4^{p-1} \left| \int_{t_0}^t \int_Z \gamma_s(x_s, z)\tilde{N}(ds, dz) \right|^p.
\]
Thus, on taking suprema over \([t_0, u \wedge \pi_R]\) for a \(u \in [t_0, t'_1]\) and expectations, one obtains

\[
E \sup_{t_0 \leq t < u \wedge \pi_R} |x_t|^p \leq 4^{p-1}E|x_{t_0}|^p + 4^{p-1}E \sup_{t_0 \leq t < u \wedge \pi_R} \left| \int_{t_0}^t b_s(x_s)ds \right|^p
\]

\[
+ 4^{p-1}E \sup_{t_0 \leq t < u \wedge \pi_R} \left| \int_{t_0}^t \sigma_s(x_s)dw_s \right|^p
\]

\[
+ 4^{p-1}E \sup_{t_0 \leq t < u \wedge \pi_R} \left| \int_{t_0}^t \int_Z \gamma_s(x_s, z)\tilde{N}(ds, dz) \right|^p
\]

\[
=: A_1 + A_2 + A_3 + A_4. \quad (2.2)
\]

Notice that \(A_1 := 4^{p-1}E|x_{t_0}|^p\). By using Hölder’s inequality, \(A_2\) can be estimated by

\[
A_2 := 4^{p-1}E \sup_{t_0 \leq t < u \wedge \pi_R} \left| \int_{t_0}^t b_s(x_s)ds \right|^p \leq KE \int_{t_0}^u I_{t_0 \leq s < \pi_R} |b_s(x_s)|^p ds
\]

which on using Assumption A-5 gives

\[
A_2 \leq KE \int_{t_0}^u I_{t_0 \leq s < \pi_R} (M + |x_s|^2)\tilde{N}(ds, dz) \leq KEM\tilde{N} + KE \int_{t_0}^u I_{t_0 \leq s < \pi_R} |x_s|^p ds.
\]

Notice that for a cádlág process \(y_t\) and a stopping time \(\tau\),

\[
\sup_{t < \tau} y_t = \sup_{t \leq \tau} y_t
\]

and therefore expressions in the above relation are measurable. Since \(x_t\) has countably many discontinuities, almost surely

\[
I_{t_0 \leq s < \pi_R} |x_s|^p \leq I_{t_0 \leq s < \pi_R} \sup_{t_0 \leq t < s} |x_t|^p \leq \sup_{t_0 \leq t < s \wedge \pi_R} |x_t|^p,
\]

for almost every \(s \in [t_0, t'_1]\).

Thus, on using (2.3), one obtains

\[
A_2 \leq KEM\tilde{N} + K \int_{t_0}^u E \sup_{t_0 \leq t < s \wedge \pi_R} |x_t|^p ds < \infty. \quad (2.4)
\]

Further, by using Burkholder-Davis-Gundy inequality, \(A_3\) is estimated by

\[
A_3 := 4^{p-1}E \sup_{t_0 \leq t < u \wedge \pi_R} \left| \int_{t_0}^t \sigma_s(x_s)dw_s \right|^p \leq KE \left( \int_{t_0}^u I_{t_0 \leq s < \pi_R} |\sigma_s(x_s)|^2 ds \right)^\frac{p}{2}
\]

which on the application of Assumption A-5 along with Hölder’s inequality gives

\[
A_3 \leq KE \int_{t_0}^u I_{t_0 \leq s < \pi_R} (M + |x_s|^2)\tilde{N} ds.
\]

Hence by using (2.3), one obtains the following estimate of \(A_3\),

\[
A_3 \leq K \left( EM\tilde{N} + \int_{t_0}^u E \sup_{t_0 \leq t < s \wedge \pi_R} |x_t|^p ds \right) < \infty. \quad (2.5)
\]

We now proceed to obtain the estimate of \(A_4\).
For estimating $A_4$, one uses Lemma 1.5 to get the following,

$$A_4 := 4^{p-1}E \sup_{t_0 \leq t < u \wedge \pi_R} \left| \int_{t_0}^{u} \int_{Z} \gamma_s(x_s, z) \tilde{N}(ds, dz) \right|^p$$

$$\leq KE \left( \int_{t_0}^{u} \int_{Z} I_{t_0 \leq s < \pi_R} |\gamma_s(x_s, z)|^2 \nu(ds) \right)^{\frac{p}{2}}$$

$$+ KE \int_{t_0}^{u} \int_{Z} I_{t_0 \leq s < \pi_R} |\gamma_s(x_s, z)|^p \nu(ds) ds$$

which due to Assumptions A-5, A-6 and Hölder’s inequality gives

$$A_4 \leq KE \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} (M + |x_s|^2) \frac{p}{2} ds + KE \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} (M' + |x_s|^p) ds$$

$$\leq KE \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} (M \frac{p}{2} + |x_s|^p) ds + KE \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} (M' + |x_s|^p) ds$$

$$\leq KEM \frac{p}{2} + KEM' + KE \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} |x_s|^p ds.$$ 

As before, by using (2.3), one obtains

$$A_4 \leq K \left( EM \frac{p}{2} + EM' + \int_{t_0}^{u} E \sup_{r < s \wedge \pi_R} |x_r|^p ds \right) < \infty. \quad (2.6)$$

On substituting the estimates from (2.4), (2.5) and (2.6) in (2.2), the following estimate is obtained,

$$E \sup_{t_0 \leq t < u \wedge \pi_R} |x_t|^p \leq K \left( E|x_{t_0}|^p + EM \frac{p}{2} + EM' + \int_{t_0}^{u} E \sup_{r < s \wedge \pi_R} |x_r|^p ds \right) < \infty$$

for any $u \in [t_0, t_1')$. Consequently, by using Gronwall’s lemma,

$$E \sup_{t_0 \leq t < t_1' \wedge \pi_R} |x_t|^p \leq K( E|x_{t_0}|^p + EM \frac{p}{2} + KEM').$$

where the constant $K > 0$ depends on $t_0, t_1, p, L$ and does not depend on $R$. Notice also that when $R \to \infty$, then $t_1' \wedge \pi_R \to t_1'$. Thus due to Fatou’s lemma,

$$E \sup_{t_0 \leq t < t_1'} |x_t|^p \leq K( E|x_{t_0}|^p + EM \frac{p}{2} + KEM').$$

The result follows by noticing that

$$E \sup_{t_0 \leq t \leq t_1} |x_t|^p \leq E \sup_{t_0 \leq t < t_1'} |x_t|^p \leq K( E|x_{t_0}|^p + EM \frac{p}{2} + KEM')$$

which completes the proof. \qed

The moment bounds in the above lemma are obtained under the condition that coefficients of SDE (2.1) are assumed to grow linearly. In the following lemma, moment bounds are obtained by relaxing this condition to allow super-linear growth in the drift coefficient. For this purpose, Assumption A-5 is replaced by the following assumption.

**A-7.** There exist a constant $L > 0$ and a non-negative random variable $M$ satisfying
\[ EM^\frac{p}{2} < \infty \text{ such that} \]
\[ \|x_t\| + |\sigma_t(x)|^2 + \int_Z |\gamma_t(x,z)|^2 \nu(dz) \leq L(M + |x|^2) \]

almost surely for any \( t \in [t_0, t_1] \) and \( x \in \mathbb{R}^d \).

In Examples [1, 2] of Chapter 6, we consider equations which satisfy Assumption A-7.

**Lemma 2.2.** Let Assumptions A-2 to A-4, A-6 and A-7 be satisfied. Then, there exists a unique solution \((x_t)_{t \in [t_0, t_1]}\) of SDE (2.1) and the following estimate holds
\[ E \sup_{t_0 \leq t \leq t_1} |x_t|^p \leq K, \]
where \( K := K(t_0, t_1, L, p, E|x_{t_0}|^p, EM^{\frac{p}{2}}, EM') \).

**Proof.** The existence and uniqueness of solution to SDE (2.1) follows immediately from Theorem 2.1 by noting that due to Assumption A-7, Assumption A-1 is satisfied.

As before, to include the case when jump can occur at terminal time \( t_1 \), we consider SDE (2.1) on the extended interval \([t_0, t'_1]\), by re-defining \( b_t(.)\), \( \sigma_t(.)\) and \( \gamma_t(.)\) as \( b_t(.)I_{t_0 \leq t \leq t_1}, \sigma_t(.)I_{t_1 \leq t \leq t_1} \) and \( \gamma_t(.)I_{t_0 \leq t \leq t_1} \) for \( t \in [t_0, t'_1] \), for some \( t'_1 > t_1 \). Also we again consider the stopping time \( \pi_R := \inf \{ t \geq t_0 : |x_t| > R \} \wedge t'_1 \) for some \( R > 0 \) and also observe that \( |x_t| \leq R \) for \( t_0 \leq t < \pi_R \).

By Itô formula (recall that \( p \geq 2 \)),
\[ |x_t|^p = |x_{t_0}|^p + p \int_{t_0}^t |x_s|^{p-2} x_s b_s(x_s) ds + p \int_{t_0}^t |x_s|^{p-2} x_s \sigma_s(x_s) dw_s \]
\[ + \frac{p(p-2)}{2} \int_{t_0}^t |x_s|^{p-4} |\sigma_s(x_s)|^2 ds + \frac{p}{2} \int_{t_0}^t |x_s|^{p-2} |\sigma_s(x_s)|^2 ds \]
\[ + p \int_{t_0}^t \int_Z |x_s|^{p-2} x_s \gamma_s(x_s, z) N(ds, dz) \]
\[ + \int_{t_0}^t \int_Z \{|x_s + \gamma_s(x_s, z)|^p - |x_s|^p - p|x_s|^{p-2} x_s \gamma_s(x_s, z)\} N(ds, dz) \quad (2.7) \]

almost surely for any \( t \in [t_0, t_1] \). By virtue of Assumption A-7, the second term of equation (2.7) is estimated by
\[ p \int_{t_0}^t |x_s|^{p-2} x_s b_s(x_s) ds \leq p \int_{t_0}^t |x_s|^{p-2}(M + |x_s|^2) ds \]
which on the application of Young’s inequality gives
\[ p \int_{t_0}^t |x_s|^{p-2} x_s b_s(x_s) ds \leq K M^{\frac{p}{2}} + K \int_{t_0}^t |x_s|^p ds. \quad (2.8) \]

Similarly, the fourth and fifth terms of equation (2.7) can be estimated together. By
using Schwarz inequality, Young’s inequality and Assumption A-7, one obtains,
\[
\frac{p(p - 2)}{2} \int_{t_0}^{t} |x_s|^{p-4} |\sigma^*_s(x_s)x_s|^2 ds + \frac{p}{2} \int_{t_0}^{t} |x_s|^{p-2} |\sigma_s(x_s)|^2 ds
\]
\[
\leq \frac{p(p - 1)}{2} \int_{t_0}^{t} |x_s|^{p-2} |\sigma_s(x_s)|^2 ds \leq K \int_{t_0}^{t} |x_s|^p ds + K \int_{t_0}^{t} |\sigma_s(x_s)|^p ds
\]
\[
\leq K \int_{t_0}^{t} |x_s|^p ds + K \int_{t_0}^{t} (M^{\frac{2}{p}} + |x_s|^p) ds \leq KM^{\frac{2}{p}} + K \int_{t_0}^{t} |x_s|^p ds. \tag{2.9}
\]

Further, since the map \( y \to |y|^p \) is of class \( C^2 \), by the formula for the remainder, for any \( y_1, y_2 \in \mathbb{R}^d \), one gets
\[
|y_1 + y_2|^p - |y_1|^p - p|y_1|^{p-2} y_1 y_2 \leq K \int_{t_0}^{1} |y_1 + qy_2|^{p-2} |y_2|^2 dq
\]
\[
\leq K(|y_1|^{p-2}|y_2|^2 + |y_2|^p). \tag{2.10}
\]

The above inequality (2.10) plays an important role and is used frequently in this report. Here, using this fact, one estimates seventh term of equation (2.7) by

\[
\int_{t_0}^{t} \int_{Z} (|x_s + \gamma_s(x_s, z)|^p - |x_s|^p - p|x_s|^{p-2} x_s \gamma_s(x_s, z)) N(ds, dz)
\]
\[
\leq K \int_{t_0}^{t} \int_{Z} (|x_s|^{p-2} |\gamma_s(x_s, z)|^2 + |\gamma_s(x_s, z)|^p) N(ds, dz). \tag{2.11}
\]

Thus on substituting the estimates from (2.8), (2.9) and (2.11) in (2.7), one obtains
\[
|x_t|^p \leq |x_{t_0}|^p + K M^{\frac{p}{2}} + K \int_{t_0}^{t} |x_s|^p ds + p \int_{t_0}^{t} |x_s|^{p-2} x_s \sigma_s(x_s) dw_s
\]
\[
+ p \int_{t_0}^{t} \int_{Z} |x_s|^{p-2} x_s \gamma_s(x_s, z) \tilde{N}(ds, dz)
\]
\[
+ K \int_{t_0}^{t} \int_{Z} \{|x_s|^{p-2} |\gamma_s(x_s, z)|^2 + |\gamma_s(x_s, z)|^p\} N(ds, dz).
\]

By taking suprema over \([t_0, u \wedge \pi_R]\) for a \( u \in [t_0, t'] \) and expectations, one has
\[
E \sup_{t_0 \leq t < u \wedge \pi_R} |x_t|^p \leq E|x_{t_0}|^p + KEM^{\frac{p}{2}} + KE \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} |x_s|^p ds
\]
\[
+ pE \sup_{t_0 \leq t < u \wedge \pi_R} \left| \int_{t_0}^{t} |x_s|^{p-2} x_s \sigma_s(x_s) dw_s \right|
\]
\[
+ pE \sup_{t_0 \leq t < u \wedge \pi_R} \left| \int_{t_0}^{t} \int_{Z} |x_s|^{p-2} x_s \gamma_s(x_s, z) \tilde{N}(ds, dz) \right|
\]
\[
+ KE \int_{t_0}^{u} \int_{Z} I_{t_0 \leq s < \pi_R} \left\{|x_s|^{p-2} |\gamma_s(x_s, z)|^2 + |\gamma_s(x_s, z)|^p\right\} N(ds, dz)
\]
\[
=: C_1 + C_2 + C_3 + C_4 + C_5. \tag{2.12}
\]

Notice that \( C_1 := E|x_{t_0}|^p + KEM^{\frac{p}{2}} \). For the term \( C_2 \), one uses similar arguments as
adopted in the inequality (2.3) to obtain the following,

\[
C_2 := KE \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} |x_s|^p ds \leq K \int_{t_0}^{u} E \sup_{t_0 \leq r < s \leq \pi_R} |x_r|^p ds < \infty. \tag{2.13}
\]

Further, by the Burkholder-Davis-Gundy inequality, \(C_3\) can be estimated as

\[
C_3 := pE \sup_{t_0 \leq t < u \land \pi_R} \left| \int_{t_0}^{t} |x_s|^{p-2} x_s \sigma_s(x_s) dw_s \right|
\]

\[
\leq KE \left( \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} |x_s|^{2p-2} |\sigma_s(x_s)|^2 ds \right)^{1/2}
\]

\[
\leq KE \sup_{t_0 \leq t < u \land \pi_R} |x_t|^{p-1} \left( \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} |\sigma_s(x_s)|^2 ds \right)^{1/2}
\]

which on the application of Young’s inequality gives

\[
C_3 \leq \frac{1}{4} E \sup_{t_0 \leq t < u \land \pi_R} |x_t|^p + KE \left( \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} |\sigma_s(x_s)|^2 ds \right)^{p/2}
\]

and then due to Hölder’s inequality and Assumption A-7, one has

\[
C_3 \leq \frac{1}{4} E \sup_{t_0 \leq t < u \land \pi_R} |x_t|^p + KE \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} |\sigma_s(x_s)|^p ds
\]

\[
\leq \frac{1}{4} E \sup_{t_0 \leq t < u \land \pi_R} |x_t|^p + KE \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} (M^\nu + |x_s|^p) ds.
\]

Also, by using (2.3), one obtains

\[
C_3 \leq \frac{1}{4} E \sup_{t_0 \leq t < u \land \pi_R} |x_t|^p + KEM^\nu + K \int_{t_0}^{u} E \sup_{t_0 \leq r < s \leq \pi_R} |x_r|^p ds < \infty. \tag{2.14}
\]

To estimate \(C_4\), one uses Lemma 1.5 to write

\[
C_4 := pE \sup_{t_0 \leq t < u \land \pi_R} \left| \int_{t_0}^{t} \int_{Z} |x_s|^{p-2} x_s \gamma_s(x_s, z) \tilde{N}(ds, dz) \right|
\]

\[
\leq KE \left( \int_{t_0}^{u} \int_{Z} I_{t_0 \leq s < \pi_R} |x_s|^{2p-2} |\gamma_s(x_s, z)|^2 \nu(dz) ds \right)^{1/2}
\]

\[
\leq KE \sup_{t_0 \leq t < u \land \pi_R} |x_t|^{p-1} \left( \int_{t_0}^{u} \int_{Z} I_{t_0 \leq s < \pi_R} |\gamma_s(x_s, z)|^2 \nu(dz) ds \right)^{1/2}
\]

which due to Young’s inequality, Assumption A-7 and Hölder’s inequality implies

\[
C_4 \leq \frac{1}{4} E \sup_{t_0 \leq t < u \land \pi_R} |x_t|^p + KE \left( \int_{t_0}^{u} \int_{Z} I_{t_0 \leq s < \pi_R} |\gamma_s(x_s, z)|^2 \nu(dz) ds \right)^{\frac{p}{2}}
\]

\[
\leq \frac{1}{4} E \sup_{t_0 \leq t < u \land \pi_R} |x_t|^p + KE \left( \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} (M + |x_s|^2) ds \right)^{p/2}
\]

\[
\leq \frac{1}{4} E \sup_{t_0 \leq t < u \land \pi_R} |x_t|^p + KEM^{\frac{p}{2}} + KE \int_{t_0}^{u} I_{t_0 \leq s < \pi_R} |x_s|^p ds.
\]
Thus, on using (2.3), one obtains
\[ C_4 \leq \frac{1}{4} E \sup_{t_0 \leq t < u \wedge \pi_R} |x_t|^p + KEM^\frac{p}{2} + K \int_{t_0}^u E \sup_{t_0 \leq r < s \wedge \pi_R} |x_r|^p ds < \infty. \tag{2.15} \]

For \( C_5 \), by Assumptions A-6, A-7 and Young’s inequality, one has
\[
C_5 := KE \int_{t_0}^u \int_{I_{t_0} \leq s < \pi_R} \left( |x_s|^{p-2} |\gamma_s(x_s, z)|^2 + |\gamma_s(x_s, z)|^p \right) \nu(dz) ds
\leq KE \int_{t_0}^u I_{t_0} \leq s < \pi_R (|x_s|^{p-2} (M + |x_s|^2) + M' + |x_s|^p) ds
\]
which due to Young’s inequality gives
\[ C_5 \leq KE \int_{t_0}^u I_{t_0} \leq s < \pi_R (M^\frac{p}{2} + M' + |x_s|^p) ds \]
and again using (2.3), one obtains
\[ C_5 \leq KEM^\frac{p}{2} + KEM' + K \int_{t_0}^u E \sup_{t_0 \leq r < s \wedge \pi_R} |x_r|^p ds < \infty. \tag{2.16} \]

By substituting the estimates from (2.13)-(2.16) in (2.12), one has
\[
E \sup_{t_0 \leq t < u \wedge \pi_R} |x_t|^p \leq \frac{1}{2} E \sup_{t_0 \leq t < u \wedge \pi_R} |x_t|^p + E|x_{t_0}|^p + KEM^\frac{p}{2} + KEM'
+ K \int_{t_0}^u E \sup_{t_0 \leq r < s \wedge \pi_R} |x_r|^p ds,
\]
which implies
\[ E \sup_{t_0 \leq t < u \wedge \pi_R} |x_t|^p \leq K \left( E|x_{t_0}|^p + EM^\frac{p}{2} + EM' + \int_{t_0}^u E \sup_{t_0 \leq r < s \wedge \pi_R} |x_r|^p ds \right) < \infty. \]
for any \( u \in [t_0, t_1) \). Finally, the application of Gronwall’s lemma gives
\[ E \sup_{t_0 \leq t < t' \wedge \pi_R} |x_t|^p \leq K \left( E|x_{t_0}|^p + EM^\frac{p}{2} + EM' \right) \]
where \( K \) depends only on \( t_0, t_1, p, L \) and does not depend on \( R \). Notice that when \( R \to \infty \), then \( t'_1 \wedge \pi_R \to t'_1 \). Therefore on taking \( R \to \infty \) and applying Fatou’s lemma, one obtains
\[ E \sup_{t_0 \leq t < t_1'} |x_t|^p \leq K \left( E|x_{t_0}|^p + EM^\frac{p}{2} + EM' \right). \]
The result follows by observing that
\[ E \sup_{t_0 \leq t \leq t_1} |x_t|^p \leq E \sup_{t_0 \leq t < t_1'} |x_t|^p \leq K \left( E|x_{t_0}|^p + EM^\frac{p}{2} + EM' \right) \]
which completes the proof. \( \Box \)
Chapter 3

Numerical Schemes of SDE with Random Coefficients Driven by Lévy Noise

In this chapter, we discuss two explicit numerical schemes of SDE (2.1). The first scheme is Euler scheme and the second scheme is tamed Euler scheme.

Section 3.1 on the Euler scheme is a generalization of the results obtained in my joint work with Sabanis [39] in order to include jumps. The strong convergence of the Euler scheme is proved under the conditions that - (a) drift, diffusion and jump coefficients satisfy linear growth condition, and (b) drift coefficient satisfies one-sided local Lipschitz condition whereas diffusion and jump coefficients satisfy local Lipschitz condition. New methodologies have been implemented for strong convergence to overcome challenges arising due to randomness of the coefficients. Furthermore, a rate of convergence is obtained when local Lipschitz conditions are replaced by global Lipschitz conditions and under polynomial Lipschitz condition on the drift coefficient.

Section 3.2 is based on the results obtained in my joint work with Dareiotis and Sabanis [16]. We propose an explicit tamed Euler scheme for SDE (2.1) with super-linearly growing drift coefficient. New techniques have been developed to prove moment bounds of the tamed Euler scheme. Moreover, strong convergence of the tamed Euler scheme is shown under assumptions (a) and (b) as stated above. Also, a rate convergence of the tamed Euler scheme is shown to be consistent with the one obtained for the Euler scheme.

Finally, we conclude that for both Euler and tamed Euler schemes, the optimal rate of convergence is arbitrarily close to 0.5, which is attained for mean square convergence. A strong motivation to study explicit 0.5 numerical schemes for SDE with random coefficients driven by Lévy noise is a recently developed methodology in [25], where SDDE is regarded as a special case of SDE with random coefficients. By adopting the approach of [25], we are able to recover results similar to the one obtained in this chapter for the case of SDDE driven by Lévy noise, which are discussed in Chapter 4.

We now introduce a generic notation for the numerical schemes of SDE (2.1) in the sense that these notations are used to represent both the Euler scheme and the tamed Euler scheme.

For every $n \in \mathbb{N}$, let $b^n_t(x)$ and $\sigma^n_t(x)$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions which take values in $\mathbb{R}^d$ and $\mathbb{R}^{d \times m}$ respectively. Also, for every $n \in \mathbb{N}$, let $\gamma^n_t(x, z)$ be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Z}$-measurable function which takes values in $\mathbb{R}^d$. For every $n \in \mathbb{N}$, we consider
the scheme of SDE (2.1) defined by,
\[ dx^n_t = b^n_t(x^n_{\kappa(n,t)})dt + \sigma^n_t(x^n_{\kappa(n,t)})dw_t + \int_Z \gamma^n_t(x^n_{\kappa(n,t)},z)\tilde{N}(dt,dz), \]
almost surely for any \( t \in [t_0, t_1] \) where the initial value \( x^n_{t_0} \) is an \( \mathcal{F}_{t_0} \)-measurable random variable which takes values in \( \mathbb{R}^d \) and the function \( \kappa \) is defined by
\[ \kappa(n, t) := \frac{[n(t - t_0)]}{n} + t_0 \]
for any \( t \in [t_0, t_1] \).

**Remark 3.1.** The scheme defined in (3.1) can be easily implemented for autonomous SDEs. Some examples are discussed in Chapter 6. For non-autonomous SDEs, one still has to discretize the time-dependent coefficients. This can be done, for example, using integral averaging over the time grid but the exact details are beyond the scope of this thesis.

### 3.1 Euler Scheme

The Euler scheme to be discussed in this section is represented by equation (3.1). We remark that the equation (3.1) is considered as the classical Euler scheme, when Assumption B-2 as mentioned below is imposed on its coefficients.

#### 3.1.1 Moment Bounds

The moment bound of the Euler scheme scheme (3.1) is obtained under the following conditions. Recall that \( p \geq 2 \) from Assumption A-4.

**B-1.** \( \sup_{n \in \mathbb{N}} E|x^n_{t_0}|^p < \infty \).

**B-2.** There exist a constant \( L > 0 \) and a sequence \((M_n)_{n \in \mathbb{N}}\) of non-negative random variables satisfying \( \sup_{n \in \mathbb{N}} E M_n^p < \infty \) such that
\[ |b^n_t(x)|^2 \vee |\sigma^n_t(x)|^2 \vee \int_Z |\gamma^n_t(x,z)|^2 \nu(dz) \leq L(M_n + |x|^2) \]
almost surely for any \( t \in [t_0, t_1] \), \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^d \).

**B-3.** There exist a constant \( L > 0 \) and a sequence \((M'_n)_{n \in \mathbb{N}}\) of non-negative random variables satisfying \( \sup_{n \in \mathbb{N}} E M'_n < \infty \) such that
\[ \int_Z |\gamma'^n_t(x,z)|^p \nu(dz) \leq L(M'_n + |x|^p) \]
almost surely for any \( t \in [t_0, t_1] \), \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^d \).

**Lemma 3.1.** Let Assumptions B-1 to B-3 hold, then
\[ \sup_{n \in \mathbb{N}} E \sup_{t_0 \leq t \leq t_1} |x^n_t|^p \leq K, \]
where \( K := K(t_0, t_1, p, L, \sup_{n \in \mathbb{N}} E|x^n_{t_0}|^p, \sup_{n \in \mathbb{N}} E M_n^p, \sup_{n \in \mathbb{N}} E M'_n). \)

**Proof.** The proof follows by adopting the similar arguments as used in Lemma 2.1. \( \square \)
3.1.2 Convergence in $\mathcal{L}^q$

The assumptions for $\mathcal{L}^q$-convergence of the Euler scheme (3.1) are given below. For the purpose of stating these assumptions, one considers, for every $R > 0$, an $\mathcal{F}_{t_0}$-measurable random variable $C_R$ which satisfies,

$$\lim_{R \to \infty} P(C_R > f(R)) = 0$$  \hspace{1cm} (3.3)

for a non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$. This family of random variables is introduced to overcome the challenges arising due to randomness of the coefficients of SDE (2.1) and scheme (3.1). Such representation helps in developing the corresponding theory for the stochastic delay differential equations driven by Lévy noise which is discussed in Chapter 4. The above notation for the family of random variables satisfying (3.3) is used throughout this report.

In order to establish $\mathcal{L}^q$-convergence of the Euler scheme (3.1), one replaces Assumption A-2 on the coefficients of SDE (2.1) by the following assumptions.

**A-8.** For every $R > 0$ and $t \in [t_0, t_1]$,

$$(x - \bar{x})(b_t(x) - b_t(\bar{x})) \lor |\sigma_t(x) - \sigma_t(\bar{x})|^2 \lor \int_Z |\gamma_t(x, z) - \gamma_t(\bar{x}, z)|^2 \nu(dz) \leq C_R |x - \bar{x}|^2$$

almost surely whenever $|x|, |\bar{x}| \leq R$.

**A-9.** For every $R > 0$ and $t \in [t_0, t_1]$,

$$\sup_{|x| \leq R} |b_t(x)| \leq C_R,$$

almost surely.

Further, one also requires below mentioned convergence among coefficients of (2.1) and (3.1).

**B-4.** For every $R > 0$ and $B(R) := \{\omega \in \Omega : C_R \leq f(R)\}$,

$$\lim_{n \to \infty} E \int_{t_0}^{t_1} I_{B(R)}(t) \sup_{|x| \leq R} |b^n_t(x) - b_t(x)|^2 dt = 0$$

$$\lim_{n \to \infty} E \int_{t_0}^{t_1} I_{B(R)}(t) \sup_{|x| \leq R} |\sigma^n_t(x) - \sigma_t(x)|^2 dt = 0$$

$$\lim_{n \to \infty} E \int_{t_0}^{t_1} I_{B(R)}(t) \sup_{|x| \leq R} \int_Z |\gamma^n_t(x, z) - \gamma_t(x, z)|^2 \nu(dz) dt = 0.$$

Finally, one also requires convergence in probability among initial values of (2.1) and (3.1).

**B-5.** For every $n \in \mathbb{N}$, the initial values of SDE (2.1) and the scheme (3.1) satisfy

$$|x^n_{t_0} - \bar{x}^n_{t_0}| \overset{P}{\to} 0$$

as $n \to \infty$. 

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Let us introduce families of stopping times that are used frequently in this report. For every $R > 0$ and $n \in \mathbb{N}$, let
\[
\pi_R := \inf\{t \geq t_0 : |x_t| \geq R\}, \quad \pi_n R := \inf\{t \geq t_0 : |x^n_t| \geq R\}, \quad \sigma_{n,R} := \pi_R \wedge \pi_{nR}
\] (3.4)
for any $\omega \in \Omega$. It is easy to observe that they are stopping times.

**Remark 3.2.** By virtue of Assumptions B-2 and B-3, there exist a constant $L > 0$ and a sequence $(M'_n)_{n \in \mathbb{N}}$ of non-negative random variables satisfying $\sup_{n \in \mathbb{N}} EM'_n < \infty$ such that
\[
\int Z |\gamma^n_t(x,z)|^r \nu(dz) \leq L(M'_n + |x|^r)
\] almost surely for any $2 \leq r \leq p$, $t \in [t_0, t_1]$, $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$.

We now proceed to obtain rate of one-step error of the Euler scheme (3.1).

**Lemma 3.2.** Let Assumptions B-1 to B-3 be satisfied, then one-step error of the Euler scheme (3.1) is given by
\[
\sup_{t_0 \leq t \leq t_1} \mathbb{E}|x^n_t - x^n_{\kappa(n,t)}|^r \leq K n^{-1}
\] for every $2 \leq r \leq p$, where $K > 0$ does not depend on $n$.

**Proof.** From the definition of the scheme (3.1), one writes,
\[
\mathbb{E}|x^n_t - x^n_{\kappa(n,t)}|^r \leq 3^{r-1} \mathbb{E}\left| \int_{\kappa(n,t)}^t b^n_s(x^n_{\kappa(n,s)})ds \right|^r + 3^{r-1} \mathbb{E}\left| \int_{\kappa(n,t)}^t \sigma^n_s(x^n_{\kappa(n,s)})dw_s \right|^r
\]
\[
+ 3^{r-1} \mathbb{E}\left| \int_{\kappa(n,t)}^t \int Z \gamma^n_s(x^n_{\kappa(n,s)}, z) \tilde{N}(ds, dz) \right|^r
\]
\[
=: D_1 + D_2 + D_3.
\] (3.5)
The term $D_1$ can be estimated by the application of Hölder’s inequality as
\[
D_1 := 3^{r-1} \mathbb{E}\left( \int_{\kappa(n,t)}^t b^n_s(x^n_{\kappa(n,s)})ds \right)^r \leq K n^{-(r-1)} \mathbb{E}\left( \int_{\kappa(n,t)}^t |b^n_s(x^n_{\kappa(n,s)})|^r ds \right)
\] which on using Assumption B-2 and Lemma 3.1 gives
\[
D_1 \leq K n^{-(r-1)} \mathbb{E}\left( \int_{\kappa(n,t)}^t (M_n + |x^n_{\kappa(n,s)}|^2)^{\frac{r}{2}} ds \right) \leq K n^{-r} \left( EM_n^{\frac{r}{2}} + \mathbb{E}|x^n_{\kappa(n,t)}|^r \right) \leq K n^{-r}.
\] (3.6)
Also, by an elementary stochastic inequality, one has
\[
D_2 := 3^{r-1} \mathbb{E}\left( \int_{\kappa(n,t)}^t \sigma^n_s(x^n_{\kappa(n,s)})dw_s \right)^r \leq K \mathbb{E}\left( \int_{\kappa(n,t)}^t |\sigma^n_s(x^n_{\kappa(n,s)})|^2 ds \right)^{\frac{r}{2}}
\] which on the application of Hölder’s inequality and Assumption B-2 gives
\[
D_2 \leq K n^{-\left(\frac{r}{2}-1\right)} \mathbb{E}\left( \int_{\kappa(n,t)}^t |\sigma^n_s(x^n_{\kappa(n,s)})|^r ds \right) \leq K n^{-\left(\frac{r}{2}-1\right)} \mathbb{E}\left( \int_{\kappa(n,t)}^t (M_n + |x^n_{\kappa(n,s)}|^2)^{\frac{r}{2}} ds \right)
\]
and then by using Lemma 3.1, one obtains

\[ D_2 \leq Kn^{-\frac{5}{2}}(EM_n^2 + E|x_{n(t)}^n|') \leq Kn^{-\frac{5}{2}}. \]  

Finally, \( D_3 \) can be estimated by using Lemma 1.5 as

\[
D_3 := 3^{r-1}E\left| \int_{\kappa(n,t)}^{t} \int Z \gamma_n^s(x_{n(s),s}, z)N(ds, dz) \right|^{r'} \leq KE\left( \int_{\kappa(n,t)}^{t} \int Z |\gamma_n^s(x_{n(s),s}, z)|^2 \nu(dz)ds \right)^{\frac{r}{2}} + KE\int_{\kappa(n,t)}^{t} \int Z |\gamma_n^s(x_{n(s),s}, z)|^r \nu(dz)ds \]

which on using Assumption B-2 and Remark 3.2 implies

\[
D_3 \leq KE\left( \int_{\kappa(n,t)}^{t} (M_n + |x_{n(s),s}|^2) ds \right)^{\frac{r}{2}} + KE\int_{\kappa(n,t)}^{t} (M_n' + |x_{n(s),s}|') ds \]

and then on the application of Lemma 3.1, one obtains

\[
D_3 \leq Kn^{-\frac{5}{2}}(EM_n^2 + E|x_{n(t)}^n|') + Kn^{-1}(EM_n + E|x_{n(t)}^n|') \leq Kn^{-\frac{5}{2}} + Kn^{-1}. \]  

By combining estimates from (3.6), (3.7) and (3.8) in (3.5), one completes the proof. \( \square \)

We now prove strong convergence of the Euler scheme (3.1) in the following theorem.

**Theorem 3.1.** Let Assumptions A-3 to A-6, A-8 and A-9 be satisfied. Also assume that Assumptions B-1 to B-5 hold. Then, the Euler scheme (3.1) converges to the solution of SDE (2.1) in \( \mathcal{L}^q \) i.e.,

\[ \lim_{n \to \infty} E \sup_{t_0 \leq t \leq t_1} |x_t - x_t^n|^q = 0 \]

for all \( q < p \).

**Proof.** Let us define the error as \( e_t^n := x_t - x_t^n \) for any \( t \in [t_0, t_1] \). In order to simplify the notation, let us define,

\[
\tilde{b}_t^n := b_t(x_{t^n}) - b_t^n(x_{n(t^n)}), \bar{\sigma}_t^n := \sigma_t(x_{t^n}) - \sigma_t^n(x_{n(t^n)}), \tilde{\gamma}_t^n(z) := \gamma_t(x_t, z) - \gamma_t^n(x_{n(t^n)}, z) \]

almost surely for any \( t \in [t_0, t_1] \). In this simplified notation, \( e_t^n \) can be written as

\[
e_t^n = e_{t_0}^n + \int_{t_0}^{t} \{b_s(x_s) - b_s^n(x_{n(s),s})\} ds + \int_{t_0}^{t} \{\sigma_s(x_s) - \sigma_s^n(x_{n(s),s})\} dw_s + \int_{t_0}^{t} \int Z \{\gamma_s(x_s) - \gamma_s^n(x_{n(s),s})\} \tilde{N}(ds, dz) \leq e_{t_0}^n + \int_{t_0}^{t} \tilde{b}_s^n ds + \int_{t_0}^{t} \bar{\sigma}_s^n dw_s + \int_{t_0}^{t} \int Z \tilde{\gamma}_s^n(z) \tilde{N}(ds, dz) \]

almost surely for any \( t \in [t_0, t_1] \). Further, by using the stopping times defined in equation (3.4) and random variables defined in (3.3), let us partition the sample space
\( \Omega \) with the help of \( \Omega_1 \) and \( \Omega_2 \) defined below,

\[
\Omega_1 = \{ \omega \in \Omega : \pi_R \leq t_1 \text{ or } \pi_{nR} \leq t_1 \text{ or } C_R > f(R) \}
\]

\[
= \{ \omega \in \Omega : \pi_R \leq t_1 \} \cup \{ \omega \in \Omega : \pi_{nR} \leq t_1 \} \cup \{ \omega \in \Omega : C_R > f(R) \}
\]

\[
\Omega_2 = \Omega \setminus \Omega_1 = \{ \omega \in \Omega : \omega \notin \Omega_1 \}
\]

\[
= \{ \omega \in \Omega : \pi_R > t_1 \} \cap \{ \omega \in \Omega : \pi_{nR} > t_1 \} \cap \{ \omega \in \Omega : C_R <= f(R) \}
\]

\[
= \{ \omega \in \Omega : \pi_R > t_1 \} \cap \{ \omega \in \Omega : \pi_{nR} > t_1 \} \cap B(R)
\]

where \( B(R) := \{ \omega \in \Omega : C_R \leq f(R) \} \) as defined in Assumption B-4. Also note that \( I_{\Omega} = I_{\Omega_1 \cup \Omega_2} \leq I_{\Omega_1} + I_{\Omega_2} \). By using this fact, for any \( q < p \), one could write the following,

\[
E \sup_{t_0 \leq t \leq t_1} |e_t^n|^q = E \sup_{t_0 \leq t \leq t_1} |e_t^n|^q I_\Omega = E \sup_{t_0 \leq t \leq t_1} |e_t^n|^q I_{\Omega_1} + E \sup_{t_0 \leq t \leq t_1} |e_t^n|^q I_{\Omega_2}
\]

\[
=: E_1 + E_2. \quad (3.11)
\]

The first term \( E_1 \) of equation (3.11) can be estimated as follows. By the application of Hölder’s inequality, one could write,

\[
E_1 := E \sup_{t_0 \leq t \leq t_1} |e_t^n|^q I_{\Omega_1} \leq \left( E \sup_{t_0 \leq t \leq t_1} |e_t^n|^q I_{\Omega_1} \right)^{\frac{q}{p}} \left( E I_{\Omega_1} \right)^{\frac{p-q}{p}}
\]

\[
\leq K \left( E \sup_{t_0 \leq t \leq t_1} |x_t|^p + E \sup_{t_0 \leq t \leq t_1} |x_t^n|^p \right)^{\frac{q}{p}} \left( E I_{\Omega_1} \right)^{\frac{p-q}{p}}.
\]

Then, on using Lemma 2.1 and Lemma 3.1 along with the above mentioned definition of \( \Omega_1 \), one obtains

\[
E_1 \leq K \left\{ P(I_{\Omega_1}) \right\}^{\frac{p-q}{p}} \leq K \left\{ P(\{ \omega \in \Omega : \pi_R \leq t_1 \} \cup \{ \omega \in \Omega : \pi_{nR} \leq t_1 \} \cup \{ \omega \in \Omega : C_R > f(R) \}) \right\}^{\frac{p-q}{p}} \leq K \left\{ P(\{ \omega \in \Omega : \pi_R \leq t_1 \}) + P(\{ \omega \in \Omega : \pi_{nR} \leq t_1 \}) + P(\{ \omega \in \Omega : C_R > f(R) \}) \right\}^{\frac{p-q}{p}}. \quad (3.12)
\]

Now, by using Lemma 2.1, one has

\[
P(\{ \omega \in \Omega : \pi_R \leq t_1 \}) = EI_{\{ \omega \in \Omega : \pi_R \leq t_1 \}} \leq EI_{\{ \omega \in \Omega : \pi_{nR} \leq t_1 \}} \frac{|x_{\pi_R}|^p}{R^p} \leq \frac{1}{R^p} E \sup_{t_0 \leq t \leq t_1} |x_t|^p \leq \frac{K}{R^p}. \quad (3.13)
\]

Similarly, by using Lemma 3.1, one has

\[
P(\{ \omega \in \Omega : \pi_{nR} \leq t_1 \}) = EI_{\{ \omega \in \Omega : \pi_{nR} \leq t_1 \}} \leq EI_{\{ \omega \in \Omega : \pi_{nR} \leq t_1 \}} \frac{|x_{\pi_{nR}}|^p}{R^p} \leq \frac{1}{R^p} E \sup_{t_0 \leq t \leq t_1} |x_t^n|^p \leq \frac{K}{R^p}. \quad (3.14)
\]

Thus on substituting estimates from (3.13) and (3.14) in (3.12), one obtains,

\[
E_1 \leq K \left( \frac{1}{R^p} + P(\{ \omega \in \Omega : C_R > f(R) \}) \right)^{\frac{p-q}{p}}. \quad (3.15)
\]

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where constant $K > 0$ does not depend on $n$. Having obtained estimates for $E_1$, we now proceed to obtain estimates for $E_2$. For this, one recalls equation (3.10) and uses Itô’s formula to obtain the following,

$$
|e^n_t|^2 = |e^n_{t_0}|^2 + 2 \int_{t_0}^{t} e^n_s b^n_s ds + 2 \int_{t_0}^{t} e^n_s \sigma^n_s dw_s + \int_{t_0}^{t} |\sigma^n_s|^2 ds
$$

$$
+ 2 \int_{t_0}^{t} \int_Z e^n_s \gamma^n_s(z) \tilde{N}(ds, dz) + \int_{t_0}^{t} \int_Z |\gamma^n_s(z)|^2 N(ds, dz)
$$

almost surely for any $t \in [t_0, t_1]$. Also, to estimate the second term of equation (3.16), one uses the following splitting,

$$
e^n_s b^n_s = (x_s - x^n_s)(b_s(x_s) - b^n_s(x^n_{\kappa(n,s)}))
$$

almost surely for any $s \in [t_0, t_1]$. Notice that $E_2$ is non-zero only on $\Omega_2$, thus one can henceforth restrict all the calculations in the estimation of $E_2$ on the interval $[t_0, t_1 \wedge \tau_{nR}]$ which also means that $\|x_t \vee x^n_t\| < R$ for any $t \in [t_0, t_1 \wedge \tau_{nR}]$. As a consequence, on the application of Assumption A-8 and Schwarz inequality, one obtains

$$
e^n_s b^n_s \leq C_R |x_s - x^n_{\kappa(n,s)}|^2 + |x_s - x^n_{\kappa(n,s)}||b_s(x^n_{\kappa(n,s)}) - b^n_s(x^n_{\kappa(n,s)})|
$$

almost surely for any $s \in [t_0, t_1 \wedge \tau_{nR}]$. Further, one can write $x_s - x^n_{\kappa(n,s)} = x_s - x^n_s + x^n_s - x^n_{\kappa(n,s)}$ and then use Young’s inequality to get,

$$
e^n_s b^n_s \leq 2C_R |x_s - x^n_s|^2 + 2C_R |x^n_s - x^n_{\kappa(n,s)}|^2 + \frac{1}{2} |x_s - x^n_{\kappa(n,s)}|^2 + \frac{1}{2} |x_s - x^n_{\kappa(n,s)}|^2
$$

almost surely for any $s \in [t_0, t_1 \wedge \tau_{nR}]$. By using Assumption A-9, one also obtains

$$
e^n_s b^n_s \leq (2C_R + 1)|x_s - x^n_s|^2 + (2C_R + 3) |x^n_s - x^n_{\kappa(n,s)}|^2 + |b_s(x^n_{\kappa(n,s)}) - b^n_s(x^n_{\kappa(n,s)})|^2
$$

almost surely for any $s \in [t_0, t_1 \wedge \tau_{nR}]$. Thus on substituting estimates from (3.18) in equation (3.16), one obtains

$$
|e^n_t|^2 \leq |e^n_{t_0}|^2 + (2C_R + 1) \int_{t_0}^{t} |e^n_s|^2 ds + (2C_R + \frac{3}{2}) \int_{t_0}^{t} |x^n_s - x^n_{\kappa(n,s)}|^2 ds
$$

$$
+ \int_{t_0}^{t} |b_s(x^n_{\kappa(n,s)}) - b^n_s(x^n_{\kappa(n,s)})|^2 ds + 2C_R \int_{t_0}^{t} |x^n_s - x^n_{\kappa(n,s)}| ds + 2 \int_{t_0}^{t} e^n_s \sigma^n_s dw_s
$$

$$
+ \int_{t_0}^{t} |\sigma^n_s|^2 ds + 2 \int_{t_0}^{t} \int_Z e^n_s \gamma^n_s(z) \tilde{N}(ds, dz) + \int_{t_0}^{t} \int_Z |\gamma^n_s(z)|^2 N(ds, dz)
$$

(3.19)
almost surely for any \( t \in [t_0, t_1 \wedge \tau_nR] \). Now, by using the definition of \( \Omega_2 \) and of \( \tau_nR \) in equation (3.4), one has
\[
E_2 := E \sup_{t_0 \leq t \leq t_1} |e^n_t| q I_{\Omega_2} \\
= E \sup_{t_0 \leq t \leq t_1} |e^n_t| q I_{\{\omega \in \Omega; \pi R > t_1\} \cap \{\omega \in \Omega; \pi R > t_1\} \cap B(R)} \\
= E \sup_{t_0 \leq t \leq t_1} |e^n_t| q I_{\{\omega \in \Omega; \pi R > t_1\} \cap \{\omega \in \Omega; \pi R > t_1\} I_{B(R)} \\
\leq E \sup_{t_0 \leq t \leq t_1} |e^n_{t \wedge \tau_nR}| q I_{B(R)}. \tag{3.20}
\]
Thus using the estimate obtained in (3.19), one obtains
\[
E \sup_{t_0 \leq t \leq u} |e^n_{t \wedge \tau_nR}|^2 I_{B(R)} \leq E|e^n_{t_0}|^2 + E(2C_R + 1)I_{B(R)} \int_{t_0}^{u \wedge \tau_nR} |e^n_s|^2 ds \\
+ E(2C_R + \frac{3}{2})I_{B(R)} \int_{t_0}^{u \wedge \tau_nR} |x^n_s - x^n_{\kappa(n,s)}|^2 ds \\
+ 2EC_R I_{B(R)} \int_{t_0}^{u \wedge \tau_nR} |x^n_s - x^n_{\kappa(n,s)}| ds \\
+ E \int_{t_0}^{u \wedge \tau_nR} I_{B(R)} |b_s(x^n_{\kappa(n,s)}) - b_s(x^n_{\kappa(n,s)})|^2 ds \\
+ 2E \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^{u \wedge \tau_nR} I_{B(R)} e^n_s \sigma^n_x dw_s \right| + E \int_{t_0}^{u \wedge \tau_nR} I_{B(R)} |\sigma^n_x|^2 ds \\
+ 2E \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^{u \wedge \tau_nR} \int_{\mathbb{R}} I_{B(R)} e^n_s \tilde{\gamma}^n_s(z) N(ds, dz) \right| \\
+ E \sup_{t_0 \leq t \leq u} \int_{t_0}^{u \wedge \tau_nR} \int_{\mathbb{R}} I_{B(R)} |\tilde{\gamma}^n_s(z)|^2 N(ds, dz) =: F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7 + F_8 + F_9 \tag{3.21}
\]
for every \( R > 0 \) and \( u \in [t_0, t_1 \wedge \tau_nR] \). Here \( F_1 := E|e^n_{t_0}|^2 \). \( F_2 \) is estimated by
\[
F_2 := E(2C_R + 1)I_{B(R)} \int_{t_0}^{u \wedge \tau_nR} |e^n_s|^2 ds \\
\leq (2f(R) + 1) \int_{t_0}^{u} E \sup_{t_0 \leq t \leq s} |e^n_{t \wedge \tau_nR}|^2 I_{B(R)} ds \tag{3.22}
\]
for every \( R > 0 \) and \( u \in [t_0, t_1 \wedge \tau_nR] \). Also \( F_3 \) is estimated as
\[
F_3 := E(2C_R + \frac{3}{2})I_{B(R)} \int_{t_0}^{u \wedge \tau_nR} |x^n_s - x^n_{\kappa(n,s)}|^2 ds \\
\leq (2f(R) + \frac{3}{2}) \sup_{t_0 \leq s \leq t_1} E|\bar{x}^n_s - x^n_{\kappa(n,s)}|^2 \tag{3.23}
\]
Similarly, \( F_4 \) can be estimated by
\[
F_4 := 2EC_R I_{B(R)} \int_{t_0}^{u \wedge \tau_nR} |x^n_s - x^n_{\kappa(n,s)}| ds \leq 2f(R) \sup_{t_0 \leq s \leq t_1} E|\bar{x}^n_s - x^n_{\kappa(n,s)}| \tag{3.24}
\]
for every $R > 0$. Again, $F_5$ has following estimate,

$$F_5 := E \int_{t_0}^{\tau_{\alpha R}} I_{B(R)} |b_s(x_{\kappa(n,s)}^n) - b_s^n(x_{\kappa(n,s)}^n)|^2 ds$$

$$= E \int_{t_0}^{u} I_{\{t_0 \leq s < \tau_{\alpha R}\}} I_{B(R)} |b_s(x_{\kappa(n,s)}^n) - b_s^n(x_{\kappa(n,s)}^n)|^2 ds. \quad (3.25)$$

To estimate $F_6$, one uses Burkholder-Davis-Gundy inequality to write

$$F_6 := 2E \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^{t \wedge \tau_{\alpha R}} I_{B(R)} e_s^n \bar{\sigma}_s^n ds \right|$$

$$\leq KE \left( \int_{t_0}^{u} I_{B(R)} |e_s^n| |\bar{\sigma}_s^n|^2 |ds \right)^{\frac{1}{2}}$$

$$\leq KE \left( \int_{t_0}^{u} I_{\{t_0 \leq s < \tau_{\alpha R}\}} I_{B(R)} |e_s^n|^2 |\bar{\sigma}_s^n|^2 |ds \right)^{\frac{1}{2}}$$

which further implies

$$F_6 \leq KE \sup_{t_0 \leq s \leq u} |e_s^n| I_{\{t_0 \leq s < \tau_{\alpha R}\}} I_{B(R)} \left( \int_{t_0}^{u} I_{\{t_0 \leq s < \tau_{\alpha R}\}} I_{B(R)} |\bar{\sigma}_s^n|^2 |ds \right)^{\frac{1}{2}}$$

$$\leq KE \sup_{t_0 \leq s \leq u} |e_s^n| I_{\{t_0 \leq s \wedge \tau_{\alpha R}\}} I_{B(R)} \left( \int_{t_0}^{u} I_{\{t_0 \leq s < \tau_{\alpha R}\}} I_{B(R)} |\bar{\sigma}_s^n|^2 |ds \right)^{\frac{1}{2}}$$

and then on the application of Young’s inequality, one obtains the following estimates of $F_6$,

$$F_6 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |e_s^n| |\bar{\sigma}_s^n|^2 I_{B(R)} + KE \int_{t_0}^{u} I_{\{t_0 \leq s < \tau_{\alpha R}\}} I_{B(R)} |\bar{\sigma}_s^n|^2 ds. \quad (3.26)$$

One observes that the second term of the above inequality is same as $F_7$ of (3.21). Thus their estimates can be obtained together by writing

$$F_6 + F_7 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |e_s^n| |\bar{\sigma}_s^n|^2 I_{B(R)} + KE \int_{t_0}^{u} I_{\{t_0 \leq s < \tau_{\alpha R}\}} I_{B(R)} |\bar{\sigma}_s^n|^2 ds \quad (3.26)$$

for every $R > 0$ and $u \in [t_0, t_1]$ where constant $K > 0$ does not depend on $R$ and $n$. In order to estimate the second term of the above inequality, one uses the following splitting of $\bar{\sigma}_s^n$,

$$\bar{\sigma}_s^n = \sigma_s(x_s) - \sigma_s^n(x_{\kappa(n,s)}^n)$$

$$= (\sigma_s(x_s) - \sigma_s^n(x_{\kappa(n,s)}^n)) + (\sigma_s(x^n_s) - \sigma_s(x_{\kappa(n,s)})) + (\sigma_s(x_{\kappa(n,s)}) - \sigma_s^n(x_{\kappa(n,s)})) \quad (3.27)$$

which implies

$$|\bar{\sigma}_s^n|^2 \leq 3|\sigma_s(x_s) - \sigma_s^n(x_{\kappa(n,s)}^n)|^2 + 3|\sigma_s(x^n_s) - \sigma_s(x_{\kappa(n,s)})|^2 + 3|\sigma_s(x_{\kappa(n,s)}) - \sigma_s^n(x_{\kappa(n,s)})|^2$$

almost surely for any $s \in [t_0, t_1]$. As before, one again notices that $|x_s| \leq R$ and $|x_s^n| \leq R$ whenever $s \in [t_0, t_1 \wedge \tau_{\alpha R})$. Thus, on the application of Assumption A-8, one obtains

$$|\bar{\sigma}_s^n|^2 \leq 3C_R |e_s^n|^2 + 3C_R |x_s^n - x_{\kappa(n,s)}|^2 + 3|\sigma_s(x^n_s) - \sigma_s(x_{\kappa(n,s)})|^2$$

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almost surely \( s \in [t_0, t_1 \wedge \tau_{nR}) \). Hence substituting this estimate in (3.26) gives

\[
F_6 + F_7 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |e_{s \wedge \tau_{nR}}^n|^2 I_B(R) \\
+ K E C_R \int_{t_0}^{u} I_{\{t_0 \leq s < \tau_{nR}\}} I_B(R) |e_s^n|^2 ds \\
+ K E C_R \int_{t_0}^{u} I_{\{t_0 \leq s < \tau_{nR}\}} I_B(R) |x_s^n - x_{\kappa(n,s)}|^2 ds \\
+ K E \int_{t_0}^{u} I_{\{t_0 \leq s < \tau_{nR}\}} I_B(R) |\sigma_s(x_s^n) - \sigma_s^n(x_{\kappa(n,s)})|^2 ds
\]

which further implies

\[
F_6 + F_7 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |e_{s \wedge \tau_{nR}}^n|^2 I_B(R) \\
+ K f(R) \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |e_{r \wedge \tau_{nR}}^n|^2 I_B(R) ds \\
+ K f(R) \sup_{t_0 \leq s \leq t_1} E|a_s^n - x_{\kappa(n,s)}|^2 \\
+ K E \int_{t_0}^{u} I_{\{t_0 \leq s < \tau_{nR}\}} I_B(R) |\sigma_s(x_s^n) - \sigma_s^n(x_{\kappa(n,s)})|^2 ds
\]

(3.28)

for any \( u \in [t_0, t_1] \). To estimate \( F_8 \), one writes

\[
F_8 := 2E \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^{t \wedge \tau_{nR}} \int_Z I_B(R)e_s^n(z)\tilde{N}(ds,dz) \right| \\
= 2E \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^{t} \int_Z I_{\{t_0 \leq s < \tau_{nR}\}} I_B(R)e_s^n(z)\tilde{N}(ds,dz) \right|
\]

which on the application Lemma 1.5 gives

\[
F_8 \leq KE \left( \int_{t_0}^{u} \int_Z I_{\{t_0 \leq s \leq \tau_{nR}\}} I_B(R)|e_s^n|^2|\tilde{\gamma}_s^n(z)|^2 \nu(dz) ds \right)^{\frac{1}{2}} \\
\leq KE \sup_{t_0 \leq s \leq u} |e_{s \wedge \tau_{nR}}^n| \left( \int_{t_0}^{u} \int_Z I_{\{t_0 \leq s < \tau_{nR}\}} I_B(R)|\tilde{\gamma}_s^n(z)|^2 \nu(dz) ds \right)^{\frac{1}{2}}
\]

and then by using Young’s inequality, one obtains

\[
F_8 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |e_{s \wedge \tau_{nR}}^n|^2 I_B(R) + KE \int_{t_0}^{u} \int_Z I_{\{t_0 \leq s < \tau_{nR}\}} I_B(R)|\tilde{\gamma}_s^n(z)|^2 \nu(dz) ds
\]

(3.29)

for any \( u \in [t_0, t_1] \). Further also observe that \( F_9 \) can be written as

\[
F_9 := E \sup_{t_0 \leq t \leq u} \int_{t_0}^{t \wedge \tau_{nR}} \int_Z I_B(R)|\tilde{\gamma}_s^n(z)|^2 N(ds,dz) \\
= E \int_{t_0}^{u \wedge \tau_{nR}} \int_Z I_B(R)|\tilde{\gamma}_s^n(z)|^2 N(ds,dz) \\
= E \int_{t_0}^{u} \int_Z I_{\{t_0 \leq s < \tau_{nR}\}} I_B(R)|\tilde{\gamma}_s^n(z)|^2 \nu(dz) ds
\]

(3.30)
for any \( u \in [t_0, t_1] \). Thus one could combine the estimates from (3.29) and (3.30) to write

\[
F_8 + F_9 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |e^n_{s \wedge \tau_R}|^2 I_B(R) + KE \int_{t_0}^u \int_Z I_{\{t_0 \leq s < \tau_R\}} I_B(R) |\hat{\gamma}^n_s(z)|^2 \nu(dz) ds
\]

(3.31)

for any \( u \in [t_0, t_1] \). In order to estimate the second term of the above inequality, one uses the following splitting,

\[
\hat{\gamma}^n_s(z) = \gamma(x_s, z) - \gamma^n_s(x^n_{\kappa(n,s)}, z)
\]

\[
= (\gamma(x_s, z) - \gamma_s(x^n_s, z)) + (\gamma_s(x^n_s, z) - \gamma_s(x^n_{\kappa(n,s)}, z))
\]

\[
+ (\gamma_s(x^n_{\kappa(n,s)}, z) - \gamma^n_s(x^n_{\kappa(n,s)}, z))
\]

(3.32)

which implies

\[
|\hat{\gamma}^n_s(z)|^2 \leq 3|\gamma(x_s, z) - \gamma_s(x^n_s, z)|^2 + 3|\gamma_s(x^n_s, z) - \gamma_s(x^n_{\kappa(n,s)}, z)|^2
\]

\[
+ 3|\gamma_s(x^n_{\kappa(n,s)}, z) - \gamma^n_s(x^n_{\kappa(n,s)}, z)|^2
\]

almost surely for any \( s \in [t_0, t_1] \) and \( z \in Z \). Recall that \( |x_s| \leq R \) and \( |x^n_s| \leq R \) for any \( s \in [t_0, t_1 \wedge \tau_R] \). Thus, by using Assumption A-8, one has

\[
\int Z |\hat{\gamma}^n_s(z)|^2 \nu(dz) \leq 3 \int Z |\gamma(x_s, z) - \gamma_s(x^n_s, z)|^2 \nu(dz) + 3 \int Z |\gamma_s(x^n_s, z) - \gamma_s(x^n_{\kappa(n,s)}, z)|^2 \nu(dz)
\]

\[
+ 3 \int Z |\gamma_s(x^n_{\kappa(n,s)}, z) - \gamma^n_s(x^n_{\kappa(n,s)}, z)|^2 \nu(dz)
\]

\[
\leq 3C_R |e^n_s|^2 + 3C_R |x^n_s - x^n_{\kappa(n,s)}|^2 + 3 \int Z |\gamma_s(x^n_{\kappa(n,s)}, z) - \gamma^n_s(x^n_{\kappa(n,s)}, z)|^2 \nu(dz)
\]

for any \( s \in [t_0, t_1 \wedge \tau_R] \). Hence on substituting the above estimates in (3.31), one obtains

\[
F_8 + F_9 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |e^n_{s \wedge \tau_R}|^2 I_B(R) + KE C_R \int_{t_0}^u \int_I_{\{t_0 \leq s < \tau_R\}} I_B(R) |e^n_{s \wedge \tau_R}|^2 ds
\]

\[
+ KE C_R \int_{t_0}^u \int_{\{t_0 \leq s < \tau_R\}} I_B(R) |x^n_s - x^n_{\kappa(n,s)}|^2 ds
\]

\[
+ KE \int_{t_0}^u \int Z I_{\{t_0 \leq s < \tau_R\}} I_B(R) |\gamma_s(x^n_{\kappa(n,s)}, z) - \gamma^n_s(x^n_{\kappa(n,s)}, z)|^2 \nu(dz) ds
\]

(3.33)

which further implies the following,

\[
F_8 + F_9 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |e^n_{s \wedge \tau_R}|^2 I_B(R) + Kf(R) E \int_{t_0}^u E \sup_{t_0 \leq s \leq t_1} |e^n_{s \wedge \tau_R}|^2 I_B(R) ds
\]

\[
+ Kf(R) \sup_{t_0 \leq s \leq t_1} E |x^n_s - x^n_{\kappa(n,s)}|^2
\]

\[
+ KE \int_{t_0}^u \int Z I_{\{t_0 \leq s < \tau_R\}} I_B(R) |\gamma_s(x^n_{\kappa(n,s)}, z) - \gamma^n_s(x^n_{\kappa(n,s)}, z)|^2 \nu(dz) ds
\]

(3.33)

for any \( u \in [t_0, t_1] \). On combining estimates obtained from (3.22), (3.23), (3.24), (3.25),
which implies from (3.20) that

\[ q < p \]

of random variables is uniformly integrable for any \( R > 0 \). Consequently \( \sup_{t_0 \leq t \leq t_1} |e_{t \wedge \tau_{n, R}}^n I_{B(R)}| \to 0 \) in probability, as \( n \to \infty \).

By Lemma 2.1 and Lemma 3.1, we have that the sequence, \( \sup_{t_0 \leq t \leq t_1} |e_{t \wedge \tau_{n, R}}^n I_{B(R)}|_{n \in \mathbb{N}} \) of random variables is uniformly integrable for any \( q < p \). Hence, for each \( R > 0 \) we have

\[ E \sup_{t_0 \leq t \leq t_1} |e_{t \wedge \tau_{n, R}}^n I_{B(R)}| \to 0, \text{ as } n \to \infty \]

which implies from (3.20) that \( E_2 \to 0 \) as \( n \to \infty \) for every \( R > 0 \). Also, when \( R \to \infty \), \( E_1 \to 0 \) due to (3.15) and (3.3). This completes the proof.

\[ \Box \]

### 3.1.3 Rate of Convergence

We now proceed to obtain a rate of convergence of the scheme (3.1). For this, one replaces Assumption A-8 by the following assumption.
**Remark 3.3.** Due to (3.34) and Assumption A-9, one immediately obtains

\[ |b_t(x)|^2 \leq K (1 + |x|^{\chi+2}) \]  

(3.35)

almost surely for any \( t \in [t_0, t_1] \) and \( x \in \mathbb{R}^d \).

**Remark 3.4.** Due to (3.34), \( b_t(x) \) is a continuous function in \( x \in \mathbb{R}^d \).

Furthermore, one replaces Assumption B-4 by the following assumption.

**B-6.** There exists a constant \( C > 0 \) such that

\[
E \int_{t_0}^{t_1} |b_t^n(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)})|^q dt \leq C n^{-\frac{q}{1+\delta}}
\]

\[
E \int_{t_0}^{t_1} |\sigma_t^n(x^n_{\kappa(n,t)}) - \sigma_t(x^n_{\kappa(n,t)})|^q dt \leq C n^{-\frac{q}{1+\delta}}
\]

\[
E \int_{t_0}^{t_1} \left( \int_{\mathbb{R}^d} |\gamma_t^n(x^n_{\kappa(n,t)}, z) - \gamma_t(x^n_{\kappa(n,t)}, z)|^q \nu(dz) \right)^{\frac{2}{q}} dt \leq C n^{-\frac{q}{1+\delta}}
\]

for \( \zeta = 2, q \).

Finally, Assumption B-5 is replaced by the following assumption.

**B-7.** There exists a constant \( C > 0 \) such that initial values satisfy

\[ E|x_{t_0} - x^n_{t_0}|^q \leq C n^{-\frac{q}{1+\delta}}. \]

Notice that convergence rate in \( \mathcal{L}^q \) is achieved for any \( q < p \) satisfying \( \max \{(\chi + 2)q, \frac{q\chi q + \delta}{2}\} \leq p \). If Assumption A-4 and Assumption B-1 hold for every \( p \geq 2 \) (for example, \( x_{t_0} = x^n_{t_0} = c \), where \( c \) is constant), then \( \delta \) can be taken as small as possible. Hence, one obtains an optimal rate of convergence equal to 0.5— as a consequence of the following theorem. This is demonstrated in Chapter 6 with the help of Examples [1, 2].

**Theorem 3.2.** Let Assumptions A-4 to A-6, A-9 and A-10 be satisfied. Also, suppose that Assumptions B-1 to B-3, B-6 and B-7 hold. Then, the Euler scheme (3.1) converges to the solution of SDE (2.1) in \( \mathcal{L}^q \) with rate arbitrarily close to 1/q, i.e.

\[ E \sup_{t_0 \leq t \leq t_1} |x_t - x^n_t|^q \leq K n^{-\frac{q}{1+\delta}} \]

where constant \( K > 0 \) does not depend on \( n \).
Proof. First of all, let us recall the notations used in the proof of Theorem 3.1. As before, define \( e_t^n : = x_t - x_t^n \) and thus write

\[
e_t^n = e_{t_0}^n + \int_{t_0}^t \tilde{b}_s^n \, ds + \int_{t_0}^t \tilde{\sigma}_s^n \, dw_s + \int_{t_0}^t \int_Z \gamma_s^n(z) \tilde{N}(ds, dz)\]

where \( \tilde{b}_t^n := b_t(x_t) - b_t^n(x_t^n, \sigma_t, n) \), \( \tilde{\sigma}_t^n := \sigma_t(x_t) - \sigma_t^n(x_t^n, \sigma_t, n) \) and \( \gamma_t^n(z) := \gamma_t(x_t, z) - \gamma_t^n(x_t^n, \sigma_t, n, z) \) almost surely for any \( t \in [t_0, t_1] \).

By the application of Itô’s formula, one obtains

\[
|e_t^n|^q = |e_{t_0}^n|^q + q \int_{t_0}^t |e_s^n|^{q-2} e_s^n \tilde{b}_s^n \, ds + q \int_{t_0}^t |e_s^n|^{q-2} e_s^n \tilde{\sigma}_s^n \, dw_s + \frac{q(q - 2)}{2} \int_{t_0}^t |e_s^n|^{q-4} |\tilde{\sigma}_s^n| \, ds + q \int_{t_0}^t |e_s^n|^{q-2} |\tilde{\sigma}_s^n|^2 \, ds + q \int_{t_0}^t \int_Z |e_s^n|^{q-2} e_s^n \gamma_s^n(z) \tilde{N}(ds, dz)
\]

almost surely for any \( t \in [t_0, t_1] \). In Theorem 3.1, one uses splitting given in (3.17) but to obtain a rate of convergence of scheme (3.1), one uses the following splitting,

\[
e_s^n \tilde{b}_s^n = (x_s - x_s^n)(b_s(x_s) - b_s^n(x_s^n)) + (x_s - x_s^n)(b_s(x_s^n) - b_s^n(x_s^n, \sigma_t, n)) + (x_s - x_s^n)(b_s^n(x_s^n, \sigma_t, n) - b_s^n(x_s^n, \sigma_t, n))
\]

which on the application of Assumption A-10 and Schwarz inequality gives

\[
e_s^n \tilde{b}_s^n \leq C|\sigma_s^n|^q + |\tilde{\sigma}_s^n|^{q-1} |b_s^n(x_s^n) - b_s^n(x_s^n, \sigma_t, n)| + |\tilde{\sigma}_s^n|^{q-1} |b_s^n(x_s^n, \sigma_t, n) - b_s^n(x_s^n, \sigma_t, n)|
\]

almost surely for any \( s \in [t_0, t_1] \). This can further be written as

\[
|e_s^n|^{q-2} e_s^n \tilde{b}_s^n \leq C|e_s^n|^q + |e_s^n|^{q-1} |b_s^n(x_s^n) - b_s^n(x_s^n, \sigma_t, n)| + |e_s^n|^{q-1} |b_s^n(x_s^n, \sigma_t, n) - b_s^n(x_s^n, \sigma_t, n)|
\]

and then on using Young’s inequality and Assumption A-10, one obtains

\[
|e_s^n|^{q-2} e_s^n \tilde{b}_s^n \leq K|e_s^n|^q + K|b_s^n(x_s^n) - b_s^n(x_s^n, \sigma_t, n)|^{q} + K|b_s^n(x_s^n, \sigma_t, n) - b_s^n(x_s^n, \sigma_t, n)|^{q}
\]

almost surely for any \( s \in [t_0, t_1] \). Thus, by substituting estimates from (3.8) and (2.10) in equation (3.36), one obtains

\[
|e_t^n|^q \leq |e_{t_0}^n|^q + K \int_{t_0}^t |e_s^n|^q \, ds + K \int_{t_0}^t |b_s^n(x_s^n, \sigma_t, n) - b_s^n(x_s^n, \sigma_t, n)|^q \, ds
\]

\[
+ K \int_{t_0}^t \int_Z |e_s^n|^{q-2} e_s^n \gamma_s^n(z) \tilde{N}(ds, dz) + K \int_{t_0}^t \int_Z |e_s^n|^{q-2} |\tilde{\sigma}_s^n|^2 \, ds + q \int_{t_0}^t |e_s^n|^{q-2} |\tilde{\sigma}_s^n|^2 \, ds
\]

almost surely for any \( t \in [t_0, t_1] \). Therefore by taking suprema over \( [t_0, u] \) for a \( u \in
Theorem 3.2. For any $u \in [t_0, t_1]$ and expectations,

\[
E \sup_{t_0 \leq t \leq u} |e_t^n|^q \leq E|e_{t_0}^n|^q + KE \int_{t_0}^u |e_s^n|^q ds + KE \int_{t_0}^u |b_s(x^n_s) - b_s(x^n_{\kappa(n,s)})|^q ds
\]

+ $KE \int_{t_0}^u |b_s(x^n_{\kappa(n,s)}) - b_s^+(x^n_{\kappa(n,s)})|^q ds + qE \sup_{t_0 \leq t \leq u} \int_{t_0}^t |e_s^n|^{q-2} e_s^n \tilde{\sigma}_s^n dw_s$

+ $\frac{q}{2} E \int_{t_0}^u |e_s^n|^{q-2} |\tilde{\sigma}_s^n|^2 ds$

+ $qE \sup_{t_0 \leq t \leq u} \int_{t_0}^t \int Z |e_s^n|^{q-2} e_s^n \tilde{\gamma}_s^n(z) \tilde{N}(ds, dz)$

+ $KE \sup_{t_0 \leq t \leq u} \int_{t_0}^t \int Z (|e_s^n|^{q-2} |\tilde{\gamma}_s^n(z)|^2 + |\tilde{\gamma}_s^n(z)|^q) N(ds, dz)$

\[
= G_1 + G_2 + G_3 + G_4 + G_5 + G_6 + G_7 + G_8 + G_9
\]

(3.39) for any $u \in [t_0, t_1]$. Here $G_1 := E|e_{t_0}^n|^q$ and $G_2$ can be estimated by

\[
G_2 := KE \int_{t_0}^u |e_s^n|^q ds \leq K \int_{t_0}^u E \sup_{t_0 \leq t \leq s} |e_t^n|^q ds
\]

(3.40) for any $u \in [t_0, t_1]$. By the application of Assumption A-10, $G_3$ can be estimated by

\[
G_3 := KE \int_{t_0}^u |b_s(x^n_s) - b_s(x^n_{\kappa(n,s)})|^q ds
\]

\[
\leq KE \int_{t_0}^u (1 + |x^n_s|^\chi + |x^n_{\kappa(n,s)}|^\chi)^\frac{q}{2} |x^n_s - x^n_{\kappa(n,s)}|^q ds
\]

\[
\leq KE \int_{t_0}^u (1 + |x^n_s|^\chi^\frac{q}{2} + |x^n_{\kappa(n,s)}|^\chi^\frac{q}{2}) |x^n_s - x^n_{\kappa(n,s)}|^q ds
\]

which due to Hölder’s inequality gives

\[
G_3 \leq K \int_{t_0}^u \left( E(1 + |x^n_s|^\chi^\frac{q}{2} + |x^n_{\kappa(n,s)}|^\chi^\frac{q}{2}) \right)^\frac{q}{q + \delta} \left( E|x^n_s - x^n_{\kappa(n,s)}|^q + \delta \right)^\frac{q}{q + \delta} ds
\]

\[
\leq K \int_{t_0}^u \left( 1 + E|x^n_s|^\chi^\frac{q + \delta}{2} + E|x^n_{\kappa(n,s)}|^\chi^\frac{q + \delta}{2} \right)^\frac{q}{q + \delta} \left( E|x^n_s - x^n_{\kappa(n,s)}|^q + \delta \right)^\frac{q}{q + \delta} ds
\]

for any $u \in [t_0, t_1]$. Also, notice from Assumption A-10 that $\chi^\frac{q + \delta}{2} \leq p$. Therefore, the result of Lemma 3.1 can be used and hence

\[
G_3 \leq K \int_{t_0}^{t_1} \left( E|x^n_s - x^n_{\kappa(n,s)}|^q + \delta \right)^\frac{q}{q + \delta} ds.
\]

(3.41)

Moreover, $G_4$ can be estimated by

\[
G_4 := KE \int_{t_0}^u |b_s(x^n_{\kappa(n,s)}) - b_s^+(x^n_{\kappa(n,s)})|^q ds
\]

\[
\leq KE \int_{t_0}^{t_1} |b_s(x^n_{\kappa(n,s)}) - b_s^+(x^n_{\kappa(n,s)})|^q ds.
\]

(3.42)
By the application of Burkholder-Davis-Gundy inequality, one obtains

\[
G_5 := qE \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^{t} e_s^n |q-2e_s^n \sigma_s^n dw_s \right|
\]

\[
\leq KE \left( \int_{t_0}^{u} |e_s^n|^{2q-2} |\sigma_s^n|^2 ds \right)^{\frac{1}{2}}
\]

\[
\leq KE \sup_{t_0 \leq s \leq u} \left| e_s^n \right|^{q-1} \left( \int_{t_0}^{u} \left| \sigma_s^n \right|^2 ds \right)^{\frac{1}{2}}
\]

which due to Young’s inequality and Hölder’s inequality gives

\[
G_5 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} \left| e_s^n \right|^q + KE \int_{t_0}^{u} \left| \sigma_s^n \right|^q ds \tag{3.43}
\]

for any \( u \in [t_0, t_1] \). Further, due to Schwarz inequality, \( G_6 \) and \( G_7 \) can be estimated together by

\[
G_6 + G_7 := \frac{q(q-2)}{2} E \int_{t_0}^{u} \left| e_s^n \right|^{q-4} |\sigma_s^n| e_s^n |e_s^n|^{2q-2} |\sigma_s^n| ds + \frac{q}{2} E \int_{t_0}^{u} \left| e_s^n \right|^{q-2} |\sigma_s^n|^2 ds
\]

\[
\leq \frac{q(q-1)}{2} E \int_{t_0}^{u} \left| e_s^n \right|^{q-2} |\sigma_s^n|^2 ds
\]

which on the application of Young’s inequality gives

\[
G_6 + G_7 \leq KE \sup_{t_0 \leq r \leq s} \left| e_r^n \right|^q ds + KE \int_{t_0}^{u} \left| \sigma_s^n \right|^q ds \tag{3.44}
\]

for any \( u \in [t_0, t_1] \). On combining estimates from (3.43) and (3.44),

\[
G_5 + G_6 + G_7 \leq \frac{1}{4} \sup_{t_0 \leq s \leq u} \left| e_s^n \right|^q + K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} \left| e_r^n \right|^q ds + KE \int_{t_0}^{u} \left| \sigma_s^n \right|^q ds \tag{3.45}
\]

for any \( u \in [t_0, t_1] \). Now, one uses the splitting of \( \sigma_s^n \) given in (3.27) to write

\[
\left| \sigma_s^n \right|^q \leq K|\sigma_s(x_s) - \sigma_s(x^n_s)|^q + K|\sigma_s(x_s) - \sigma_s(x^n_{\kappa(n,s)})|^q + K|\sigma_s(x^n_{\kappa(n,s)}) - \sigma_s(x^n_{\kappa(n,s)})|^q
\]

which on using Assumption A-10 gives

\[
\left| \sigma_s^n \right|^q \leq K|x_s - x^n_s|^q + K|x^n_{\kappa(n,s)} - x^n_{\kappa(n,s)}|^q + K|\sigma_s(x^n_{\kappa(n,s)}) - \sigma_s(x^n_{\kappa(n,s)})|^q
\]

for any \( s \in [t_0, t_1] \). On substituting this estimate in (3.45), one obtains

\[
G_5 + G_6 + G_7 \leq \frac{1}{4} \sup_{t_0 \leq s \leq u} \left| e_s^n \right|^q + K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} \left| e_r^n \right|^q ds
\]

\[
+ K \int_{t_0}^{t_1} E|x_s^n - x^n_{\kappa(n,s)}|^q ds
\]

\[
+ KE \int_{t_0}^{t_1} |\sigma_s(x^n_{\kappa(n,s)}) - \sigma_s(x^n_{\kappa(n,s)})|^q ds \tag{3.46}
\]

for any \( u \in [t_0, t_1] \).
Now, for estimating $G_8$, one uses the splitting of $\gamma^n_t(z)$ given in (3.32) to write

\[
G_8 \leq E \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^t \int_Z \left| e^n_s \right|^{q-2} e^n_s \left( \gamma_s(x_s, z) - \gamma_s(x^n_s, z) \right) \tilde{N}(ds, dz) \right| \\
+ E \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^t \int_Z \left| e^n_s \right|^{q-2} e^n_s \left( \gamma_s(x^n_s, z) - \gamma_s(x^n_{\kappa(n,s)}, z) \right) \tilde{N}(ds, dz) \right| \\
+ E \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^t \int_Z \left| e^n_s \right|^{q-2} e^n_s \left( \gamma_s(x^n_{\kappa(n,s)}, z) - \gamma_s^n(x^n_{\kappa(n,s)}, z) \right) \tilde{N}(ds, dz) \right|
\]

which due to Lemma 1.5 gives

\[
G_8 \leq KE \left( \int_{t_0}^u \int_Z \left| e^n_s \right|^{2q-2} \left| \gamma_s(x_s, z) - \gamma_s(x^n_s, z) \right|^2 \nu(dz)ds \right)^{\frac{1}{2}} \\
+ KE \left( \int_{t_0}^u \int_Z \left| e^n_s \right|^{2q-2} \left| \gamma_s(x^n_s, z) - \gamma_s(x^n_{\kappa(n,s)}, z) \right|^2 \nu(dz)ds \right)^{\frac{1}{2}} \\
+ KE \left( \int_{t_0}^u \int_Z \left| e^n_s \right|^{2q-2} \left| \gamma_s(x^n_{\kappa(n,s)}, z) - \gamma_s^n(x^n_{\kappa(n,s)}, z) \right|^2 \nu(dz)ds \right)^{\frac{1}{2}}
\]

for any $u \in [t_0, t_1]$. This can further be estimated as

\[
G_8 \leq KE \sup_{t_0 \leq s \leq u} \left| e^n_s \right|^{q-1} \left( \int_{t_0}^u \int_Z \left| \gamma_s(x_s, z) - \gamma_s(x^n_s, z) \right|^2 \nu(dz)ds \right)^{\frac{1}{2}} \\
+ KE \sup_{t_0 \leq s \leq u} \left| e^n_s \right|^{q-1} \left( \int_{t_0}^u \int_Z \left| \gamma_s(x^n_s, z) - \gamma_s(x^n_{\kappa(n,s)}, z) \right|^2 \nu(dz)ds \right)^{\frac{1}{2}} \\
+ KE \sup_{t_0 \leq s \leq u} \left| e^n_s \right|^{q-1} \left( \int_{t_0}^u \int_Z \left| \gamma_s(x^n_{\kappa(n,s)}, z) - \gamma_s^n(x^n_{\kappa(n,s)}, z) \right|^2 \nu(dz)ds \right)^{\frac{1}{2}}
\]

and then on the application of Young’s inequality, one obtains,

\[
G_8 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} \left| e^n_s \right|^q \\
+ KE \left( \int_{t_0}^u \int_Z \left| \gamma_s(x_s, z) - \gamma_s(x^n_s, z) \right|^2 \nu(dz)ds \right)^{\frac{3}{2}} \\
+ KE \left( \int_{t_0}^u \int_Z \left| \gamma_s(x^n_s, z) - \gamma_s(x^n_{\kappa(n,s)}, z) \right|^2 \nu(dz)ds \right)^{\frac{3}{2}} \\
+ KE \left( \int_{t_0}^u \int_Z \left| \gamma_s(x^n_{\kappa(n,s)}, z) - \gamma_s^n(x^n_{\kappa(n,s)}, z) \right|^2 \nu(dz)ds \right)^{\frac{3}{2}}
\]

which due to Hölder’s inequality further implies

\[
G_8 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} \left| e^n_s \right|^q \\
+ KE \int_{t_0}^u \left( \int_Z \left| \gamma_s(x_s, z) - \gamma_s(x^n_s, z) \right|^2 \nu(dz) \right)^{\frac{3}{2}} ds \\
+ KE \int_{t_0}^u \left( \int_Z \left| \gamma_s(x^n_s, z) - \gamma_s(x^n_{\kappa(n,s)}, z) \right|^2 \nu(dz) \right)^{\frac{3}{2}} ds \\
+ KE \int_{t_0}^u \left( \int_Z \left| \gamma_s(x^n_{\kappa(n,s)}, z) - \gamma_s^n(x^n_{\kappa(n,s)}, z) \right|^2 \nu(dz) \right)^{\frac{3}{2}} ds.
\]
Thus, by using Assumption A-10, one has
\[
G_8 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |e^n_s|^q + K \int_{t_0}^u E \sup_{t_0 \leq r \leq s} |e^n_r|^q ds \\
+ K \int_{t_0}^{t_1} E |x^n_s - x^n_{\kappa(n,s)}|^q ds \\
+ KE \int_{t_0}^{t_1} \left( \int_Z |\gamma_s(x^n_{\kappa(n,s)}, z) - \gamma^n_s(x^n_{\kappa(n,s)}, z)|^2 \nu(dz) \right)^{\frac{q}{2}} ds
\]
for any \( u \in [t_0, t_1] \).

Finally, for estimating \( G_9 \), one writes it as sum of of two terms as given below,
\[
G_9 := KE \sup_{t_0 \leq t \leq u} \int_{t_0}^t \int_Z (|e^n_s|^q - \bar{\gamma}_s^n(z))^2 \nu(dz) ds \\
+ KE \int_{t_0}^u \int_Z |\bar{\gamma}_s^n(z)|^q \nu(dz) ds
\]
for any \( u \in [t_0, t_1] \). In order to estimate \( H_1 \), one recalls the splitting of \( \gamma^n_s(z) \) given in (3.32) to get,
\[
H_1 := KE \int_{t_0}^u \int_Z |e^n_s|^q - \bar{\gamma}_s^n(z))^2 \nu(dz) ds \\
\leq KE \int_{t_0}^u \int_Z |e^n_s|^q - \bar{\gamma}_s(x_s, z) - \gamma^n_s(x^n_{\kappa(n,s)}, z)|^2 \nu(dz) ds \\
+ KE \int_{t_0}^u \int_Z |e^n_s|^q - \bar{\gamma}_s(x^n_{\kappa(n,s)}, z) - \gamma^n_s(x^n_{\kappa(n,s)}, z)|^2 \nu(dz) ds
\]
which on using Assumption A-10 gives
\[
H_1 \leq KE \int_{t_0}^u |e^n_s|^q ds + KE \int_{t_0}^u |e^n_s|^q - |x^n_s - x^n_{\kappa(n,s)}|^2 ds \\
+ KE \int_{t_0}^u \int_Z |e^n_s|^q - |\gamma_s(x^n_{\kappa(n,s)}, z) - \gamma^n_s(x^n_{\kappa(n,s)}, z)|^2 \nu(dz) ds
\]
for any \( u \in [t_0, t_1] \). By the application of Young’s inequality and Hölder’s inequality, one obtains
\[
H_1 \leq K \int_{t_0}^u E \sup_{t_0 \leq r \leq s} |e^n_r|^q ds \\
+ K \int_{t_0}^{t_1} E |x^n_s - x^n_{\kappa(n,s)}|^q ds \\
+ KE \int_{t_0}^{t_1} \left( \int_Z |\gamma_s(x^n_{\kappa(n,s)}, z) - \gamma^n_s(x^n_{\kappa(n,s)}, z)|^2 \nu(dz) \right)^{\frac{q}{2}} ds
\]
for any $u \in [t_0, t_1]$. For $H_2$, one again uses the splitting of $\tilde{\gamma}_n^s(z)$ given in equation (3.32) to get,

$$H_2 := KE \int_{t_0}^{u} \int_{Z} |\tilde{\gamma}_n^s(z)|^q \nu(dz) ds$$

$$\leq KE \int_{t_0}^{u} \int_{Z} |\gamma_s(x_s, z) - \gamma_s(x_n^s, z)|^q \nu(dz) ds$$

$$+ KE \int_{t_0}^{u} \int_{Z} |\gamma_s(x_n^s, z) - \gamma_s(x_{nK(n,s)}^s, z)|^q \nu(dz) ds$$

$$+ KE \int_{t_0}^{u} \int_{Z} |\gamma_s(x_{nK(n,s)}^s, z) - \gamma_n^s(x_{nK(n,s)}^s, z)|^q \nu(dz) ds$$

which on using Assumption A-10 gives

$$H_2 \leq K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |\epsilon_r^n|^q ds + K \int_{t_0}^{t_1} E|x_n^s - x_{nK(n,s)}^s|^q ds$$

$$+ KE \int_{t_0}^{t_1} \left( \int_{Z} |\gamma_s(x_{nK(n,s)}^s, z) - \gamma_n^s(x_{nK(n,s)}^s, z)|^2 \nu(dz) \right)^{\frac{q}{2}} ds$$

$$+ KE \int_{t_0}^{t_1} \int_{Z} |\gamma_s(x_{nK(n,s)}^s, z) - \gamma_n^s(x_{nK(n,s)}^s, z)|^q \nu(dz) ds$$

(3.50)

for any $u \in [t_0, t_1]$. Hence, on combining estimates obtained from (3.49) and (3.50) in (3.48), one obtains

$$G_9 \leq K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |\epsilon_r^n|^q ds + K \int_{t_0}^{t_1} E|x_n^s - x_{nK(n,s)}^s|^q ds$$

$$+ KE \int_{t_0}^{t_1} \left( \int_{Z} |\gamma_s(x_{nK(n,s)}^s, z) - \gamma_n^s(x_{nK(n,s)}^s, z)|^2 \nu(dz) \right)^{\frac{q}{2}} ds$$

$$+ KE \int_{t_0}^{t_1} \int_{Z} |\gamma_s(x_{nK(n,s)}^s, z) - \gamma_n^s(x_{nK(n,s)}^s, z)|^q \nu(dz) ds$$

(3.51)

for any $u \in [t_0, t_1]$. Thus, one can substitute estimates from (3.40), (3.41), (3.42), (3.46), (3.47) and (3.51) in (3.39) to obtain the final estimate as

$$E \sup_{t_0 \leq r \leq u} |\epsilon_r^n|^q \leq E|\epsilon_{t_0}^n|^q + \frac{1}{2} E \sup_{t_0 \leq s \leq u} |\epsilon_s^n|^q + K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |\epsilon_r^n|^q ds$$

$$+ K \int_{t_0}^{t_1} \left( E|x_n^s - x_{nK(n,s)}^s|^q \right)^{\frac{q}{q+1}} ds$$

$$+ K \int_{t_0}^{t_1} E|x_n^s - x_{nK(n,s)}^s|^q ds$$

$$+ KE \int_{t_0}^{t_1} |b_s(x_{nK(n,s)}^s) - b_n^s(x_{nK(n,s)}^s)|^q ds$$

$$+ KE \int_{t_0}^{t_1} |\sigma_s(x_{nK(n,s)}^s) - \sigma_n^s(x_{nK(n,s)}^s)|^q ds$$

$$+ KE \int_{t_0}^{t_1} \left( \int_{Z} |\gamma_s(x_{nK(n,s)}^s, z) - \gamma_n^s(x_{nK(n,s)}^s, z)|^2 \nu(dz) \right)^{\frac{q}{2}} ds$$

$$+ KE \int_{t_0}^{t_1} \int_{Z} |\gamma_s(x_{nK(n,s)}^s, z) - \gamma_n^s(x_{nK(n,s)}^s, z)|^q \nu(dz) ds$$

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for any \( u \in [t_0, t_1] \). Then, the application of Gronwall’s inequality gives,

\[
E \sup_{t_0 \leq t \leq t_1} |e_t^n|^q \leq E|e_{t_0}^n|^q + K \int_{t_0}^{t_1} \left( E|x_s^n - x_{n(s)}^n|^q + \frac{q}{q+\delta} \right) ds
\]

\[
+ K \int_{t_0}^{t_1} E|x_s^n - x_{n(s)}^n|^q ds + KE \int_{t_0}^{t_1} |b_s(x_{n(s)}^n) - b_{n(s)}^n(x_{n(s)}^n)|^q ds
\]

\[
+ KE \int_{t_0}^{t_1} |\sigma_s(x_{n(s)}^n) - \sigma_{s(s)}^{n(s)}|^q ds
\]

\[
+ KE \int_{t_0}^{t_1} \left( \int Z |\gamma_s(x_{n(s)}^n, z) - \gamma_{s(s)}^{n(s)}(x_{n(s)}^n, z)|^2 \nu(dz) \right)^{\frac{q}{2}} ds
\]

\[
+ KE \int_{t_0}^{t_1} \int Z |\gamma_s(x_{n(s)}^n, z) - \gamma_{s(s)}^{n(s)}(x_{n(s)}^n, z)|^q \nu(dz) ds.
\]

Finally, by using Assumptions B-6, B-7 and Lemma 3.2, one obtains,

\[
E \sup_{t_0 \leq t \leq t_1} |e_t^n|^q \leq Kn^{-\frac{q}{q+\delta}}
\]

which completes the proof. \( \square \)

### 3.2 Tamed Euler Scheme

In this section, we introduce tamed Euler scheme of SDE (2.1) with super-linearly growing drift coefficient. As mentioned before, we use equation (3.1) to denote our tamed Euler scheme to be discussed in this section. The equation (3.1) represents tamed Euler scheme when Assumption B-2 on its coefficients is replaced by Assumptions B-8 and B-9, which are mentioned below. In fact, Assumption B-9 is our taming assumption on the drift coefficient of the scheme (3.1). This approach of taming drift coefficient of scheme (3.1) enables one to assume super-linear growth on drift coefficient of SDE (2.1). Thus, Assumption A-5 is replaced with Assumption A-7. At the end of this section, we present a simple example of tamed Euler scheme of SDE driven by Lévy noise (i.e. when coefficients are non-random). Throughout this section, above mentioned modifications are followed.

#### 3.2.1 Moment Bounds

The following assumptions are made for the moment bounds of the scheme (3.1).

**B-8.** There exist a constant \( L > 0 \) and a sequence \( (M_n)_{n \in \mathbb{N}} \) of non-negative random variables satisfying \( \sup_{n \in \mathbb{N}} EM_n^2 < \infty \) such that

\[
xb_t^n(x) \vee |\sigma_t^n(x)|^2 \vee \int Z |\gamma_t^n(x, z)|^2 \nu(dz) \leq L(M_n + |x|^2)
\]

almost surely for any \( t \in [t_0, t_1] \), \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^d \).

The following taming assumption for scheme (3.1) is in the same spirit as adopted in [59].

**B-9.** For any \( t \in [t_0, t_1] \), \( x \in \mathbb{R}^d \),

\[
|b_t^n(x)| \leq n^\theta
\]
almost surely with \( \theta \in (0, \frac{1}{2}] \) for every \( n \in \mathbb{N} \).

Note that due to B-9, for each \( n \geq 1 \), the norm of \( b^n \) is a bounded function of \( t \) and \( x \) which along with B-1 and B-8 guarantee the existence of a unique solution to tamed Euler scheme (3.1). Moreover, they also guarantee that for each \( n \geq 1 \),

\[
E \sup_{0 \leq t \leq T} |x^n_t|^p < \infty. \tag{3.52}
\]

Clearly, one cannot claim at this point that this bound is independent of \( n \). Nevertheless, as a result of this observation, one needs not to apply stopping time arguments, similar to the one used in the proofs of Lemmas [2.1, 2.2], in the proofs of Lemma 3.3 and Lemma 3.4 mentioned below.

**Lemma 3.3.** Let Assumptions B-3, B-8 and B-9 hold. Then,

\[
\int_{t_0}^{u} E|x^n_t - x^n_{\kappa(n,t)}|^p dt \leq Kn^{-1} + Kn^{-1} \int_{t_0}^{u} E|x^n_{\kappa(n,t)}|^p dt
\]

for any \( u \in [t_0, t_1] \) where \( K := K(t_0, t_1, L, p, \sup_{n \in \mathbb{N}} E M_n^2, \sup_{n \in \mathbb{N}} E M'_n) \).

**Proof.** From the definition of the scheme (3.1), one writes,

\[
E|x^n_t - x^n_{\kappa(n,t)}|^p \leq KE \int_{\kappa(n,t)}^{t} b^\sigma_s(x^n_{\kappa(n,s)}) ds |b^\sigma_s(x^n_{\kappa(n,s)})|^p + KE \int_{\kappa(n,t)}^{t} \sigma^\eta_s(x^n_{\kappa(n,s)}) ds |\sigma^\eta_s(x^n_{\kappa(n,s)})|^p + KE \int_{\kappa(n,t)}^{t} \gamma^\zeta_s(x^n_{\kappa(n,s)}, z) ds \zeta (ds, dz).
\]

On the application of Hölder’s inequality and an elementary inequality of stochastic integrals along with Lemma 1.5, one obtains

\[
E|x^n_t - x^n_{\kappa(n,t)}|^p \leq Kn^{-(p-1)} E \int_{\kappa(n,t)}^{t} |b^\sigma_s(x^n_{\kappa(n,s)})|^p ds + KE \left( \int_{\kappa(n,t)}^{t} |\sigma^\eta_s(x^n_{\kappa(n,s)})|^p ds \right)^{\frac{p}{2}} + KE \left( \int_{\kappa(n,t)}^{t} \gamma^\zeta_s(x^n_{\kappa(n,s)}, z) |\zeta|^p \nu(dz) ds \right)
\]

which on using Assumption B-9 and Hölder’s inequality gives,

\[
E|x^n_t - x^n_{\kappa(n,t)}|^p \leq Kn^{-p(1-\theta)} + Kn^{-\left(\frac{p}{2}-1\right)} E \int_{\kappa(n,t)}^{t} |\sigma^\eta_s(x^n_{\kappa(n,s)})|^p ds + Kn^{-\left(\frac{p}{2}-1\right)} E \left( \int_{\kappa(n,t)}^{t} \gamma^\zeta_s(x^n_{\kappa(n,s)}, z) |\zeta|^p \nu(dz) ds \right)^{\frac{p}{2}} + KE \left( \int_{\kappa(n,t)}^{t} \gamma^\zeta_s(x^n_{\kappa(n,s)}, z) |\zeta|^p \nu(dz) ds \right)
\]

Further, by using Assumptions B-3 and B-8,

\[
E|x^n_t - x^n_{\kappa(n,t)}|^p \leq Kn^{-p(1-\theta)} + Kn^{-\frac{p}{2}} (E|\zeta|^p + E|x^n_{\kappa(n,t)}|^p) + Kn^{-\frac{p}{2}} (E|\zeta|^p + E|x^n_{\kappa(n,t)}|^p) + Kn^{-1} (E|\zeta|^p + E|x^n_{\kappa(n,t)}|^p)
\]

which completes the proof by observing that \( \theta \in (0, \frac{1}{2}] \).
Lemma 3.4. Let Assumptions B-1, B-3, B-8 and B-9 be satisfied. Then,\[
\sup_{n \in \mathbb{N}} \mathbb{E} \sup_{t_0 \leq t \leq t_1} |x_t^n|^p \leq K,
\]
where \(K := K(t_0, t_1, L, p, \sup_{n \in \mathbb{N}} \mathbb{E}|x_{t_0}^n|^p, \sup_{n \in \mathbb{N}} EM_n^2, \sup_{n \in \mathbb{N}} EM'_n)\).

Proof. By the application of Itô’s formula, one gets
\[
|x_t^n|^p = |x_{t_0}^n|^p + p \int_{t_0}^t |x_s^n|^{p-2} x_s^n b_s^n(x_{\kappa(n,s)})ds
+ p \int_{t_0}^t |x_s^n|^{p-2} x_s^n \sigma_s^n(x_{\kappa(n,s)})dw_s
+ \frac{p(p-2)}{2} \int_{t_0}^t |x_s^n|^{p-4} |\sigma_s^n(x_{\kappa(n,s)})|^2 ds
+ \frac{p}{2} \int_{t_0}^t |x_s^n|^{p-2} |\sigma_s^n(x_{\kappa(n,s)})|^2 ds
\]
\[
+ p \int_{t_0}^t \int_{\mathbb{R}} \{ |x_s^n + \gamma_s^n(x_{\kappa(n,s)}, z)|^p - |x_s^n|^p - p|x_s^n|^{p-2} \gamma_s^n(x_{\kappa(n,s)}, z) \} N(ds, dz)
\]
for any \(t \in [t_0, t_1]\). In order to estimate first term of (3.53), one writes
\[
x_s^n b_s^n(x_{\kappa(n,s)}) = (x_s^n - x_{\kappa(n,s)}^n)b_s^n(x_{\kappa(n,s)}) + x_{\kappa(n,s)}^n b_s^n(x_{\kappa(n,s)})
\]
which due to Assumption B-8 gives
\[
x_s^n b_s^n(x_{\kappa(n,s)}) = b_s^n(x_{\kappa(n,s)}) \int_{\kappa(n,s)}^s b_r^n(x_{\kappa(n,r)})dr
+ b_s^n(x_{\kappa(n,s)}) \int_{\kappa(n,s)}^s \sigma_r^n(x_{\kappa(n,r)}) dw_r
+ b_s^n(x_{\kappa(n,s)}) \int_{\kappa(n,s)}^s \int_{\mathbb{R}} \gamma_r^n(x_{\kappa(n,r)}, z) \tilde{N}(dr, dz)
+ L(M_n + |x_{\kappa(n,s)}|^2)
\]
\[
\leq |b_s^n(x_{\kappa(n,s)})| \int_{\kappa(n,s)}^s b_r^n(x_{\kappa(n,r)})dr + |b_s^n(x_{\kappa(n,s)})| \int_{\kappa(n,s)}^s \sigma_r^n(x_{\kappa(n,r)}) dw_r
+ |b_s^n(x_{\kappa(n,s)})| \int_{\kappa(n,s)}^s \int_{\mathbb{R}} \gamma_r^n(x_{\kappa(n,r)}, z) \tilde{N}(dr, dz)
+ L(M_n + |x_{\kappa(n,s)}|^2).
\]
Further, Assumption B-9 implies
\[
x_s^n b_s^n(x_{\kappa(n,s)}) \leq n^{2\theta-1} + n^{\theta} \int_{\kappa(n,s)}^s \sigma_r^n(x_{\kappa(n,r)}) dw_r
+ n^{\theta} \int_{\kappa(n,s)}^s \int_{\mathbb{R}} \gamma_r^n(x_{\kappa(n,r)}, z) \tilde{N}(dr, dz) + K(M_n + |x_{\kappa(n,s)}|^2)
\]
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Thus, on using estimates from (3.54) and (2.10) in equation (3.53), one obtains

\[ |x_s^n|^{p-2}x_s^n b_s^n(x_{\kappa(n,s)}) \leq |x_{s_0}^n|^{p} \int_{\kappa(n,s)} \sigma^p_r(x_{\kappa(n,r)}) dw_r + n^\theta |x_s^n|^{p-2}\int_{\kappa(n,s)} \int_Z \gamma^n_r(x_{\kappa(n,r)}, z) \tilde{N}(dr, dz) \]

\[ + K|x_s^n|^{p-2}(M_n + |x_{\kappa(n,s)}|^2) \]

which due to Young’s inequality gives

\[ |x_s^n|^{p-2}x_s^n b_s^n(x_{\kappa(n,s)}) \leq 1 + K|x_s^n|^p + Kn^\theta \int_{\kappa(n,s)} \sigma^p_r(x_{\kappa(n,r)}) dw_r \mid \gamma^n_r(x_{\kappa(n,r)}, z) \tilde{N}(dr, dz) \mid \]

\[ + Kn^\theta |x_s^n|^{p-2} \int_{\kappa(n,s)} \int_Z \gamma^n_r(x_{\kappa(n,r)}, z) \tilde{N}(dr, dz) \mid |x_{\kappa(n,s)}|^p. \]

Also, on using \( |x_s^n|^{p-2} = |x_{s_0}^n - x_{\kappa(n,s)}^n + x_{\kappa(n,s)}^n|^{p-2} \leq 2p^{-3}|x_{s_0}^n - x_{\kappa(n,s)}^n|^{p-2} + 2p^{-3}|x_{\kappa(n,s)}^n|^{p-2} \), one obtains

\[ |x_s^n|^{p-2}x_s^n b_s^n(x_{\kappa(n,s)}) \leq 1 + K|x_s^n|^p + Kn^\theta \int_{\kappa(n,s)} \sigma^p_r(x_{\kappa(n,r)}) dw_r \mid \gamma^n_r(x_{\kappa(n,r)}, z) \tilde{N}(dr, dz) \mid \]

\[ + Kn^\theta |x_s^n - x_{\kappa(n,s)}^n|^{p-2} \int_{\kappa(n,s)} \int_Z \gamma^n_r(x_{\kappa(n,r)}, z) \tilde{N}(dr, dz) \mid + K(|M_n|^\frac{p}{2} + |x_{\kappa(n,s)}|^p). \]

Thus, on using estimates from (3.54) and (2.10) in equation (3.53), one obtains

\[ |x_t^n|^p \leq |x_{t_0}^n|^p + K + K \int_{t_0}^t |x_s^n|^p ds \]

\[ + Kn^\theta \int_{t_0}^t \int_{\kappa(n,s)} \sigma^p_r(x_{\kappa(n,r)}) dw_r \mid ds \]

\[ + Kn^\theta \int_{t_0}^t |x_{\kappa(n,s)}|^p \int_{\kappa(n,s)} \int_Z \gamma^n_r(x_{\kappa(n,r)}, z) \tilde{N}(dr, dz) \mid ds \]

\[ + Kn^\theta \int_{t_0}^t |x_{s}^n - x_{\kappa(n,s)}^n|^p \int_{\kappa(n,s)} \int_Z \gamma^n_r(x_{\kappa(n,r)}, z) \tilde{N}(dr, dz) \mid ds \]

\[ + K \int_{t_0}^t (|M_n|^\frac{p}{2} + |x_{\kappa(n,s)}|^p) ds + p \int_{t_0}^t |x_s^n|^{p-2}x_s^n \sigma^p_r(x_{\kappa(n,s)}) dw_s \]

\[ + \frac{p(p-1)}{2} \int_{t_0}^t |x_s^n|^{p-2}|\sigma^p_r(x_{\kappa(n,s)})|^2 ds \]

\[ + p \int_{t_0}^t \int_Z |x_s^n|^{p-2}x_s^n \gamma^n_r(x_{\kappa(n,s)}, z) \tilde{N}(ds, dz) \]

\[ + K \int_{t_0}^t \int_Z (|x_s^n|^{p-2}|\gamma^n_r(x_{\kappa(n,s)}, z)|^p + |\gamma^n_r(x_{\kappa(n,s)}, z)|^p) N(ds, dz) \]

\[ . \]
which on taking suprema over \( t \in [t_0, u] \) for a \( u \in [t_0, t_1] \) and expectation gives

\[
E \sup_{t_0 \leq t \leq u} |x_t^n|^p \leq E|x_{t_0}^n|^p + K + KE \int_{t_0}^{u} |x_s^n|^p ds + Kn^{\theta \varphi} E \int_{t_0}^{u} | \int_{\kappa(n,s)}^{s} \sigma_r^n(x_{\kappa(n,r)}) dw_r |^2 ds
\]

\[
+ Kn^{\theta \varphi} \int_{t_0}^{u} \int_{\kappa(n,s)}^{s} \int_{Z} |x_{\kappa(n,s)}^n| |x_{\kappa(n,r)}^n|^{-2} \gamma_r^n(x_{\kappa(n,r)}, z) \tilde{N}(dr, dz) | ds
\]

\[
+ Kn^{\theta \varphi} \int_{t_0}^{u} |x_s^n - x_{\kappa(n,s)}^n|^p | \int_{\kappa(n,s)}^{s} \gamma_r^n(x_{\kappa(n,r)}, z) \tilde{N}(dr, dz) | ds
\]

\[
+ KE \int_{t_0}^{u} (|M_n|^p + |x_{\kappa(n,s)}^n|)^p ds + pE \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^{t} |x_s^n|^p x_s^n \sigma_s^n(x_{\kappa(n,s)}) dw_s \right|
\]

\[
+ KE \int_{t_0}^{u} |x_s^n|^{p-2} |x_s^n \sigma_s^n(x_{\kappa(n,s)})|^2 ds + pE \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^{t} \int_{\kappa(n,s)}^{s} |x_s^n|^p x_s^n \gamma_s^n(x_{\kappa(n,s)}, z) \tilde{N}(dr, dz) \right|
\]

\[
+ KE \int_{t_0}^{u} \int_{\kappa(n,s)}^{s} \int_{Z} (|x_{\kappa(n,s)}^n|^p - |x_s^n|^p)^2 ds + \left| \gamma_s^n(x_{\kappa(n,s)}, z)^p \right| N(ds, dz)
\]

\[= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9 + J_{10}. \]  

(3.55)

One notes that \( J_1 := E|x_{t_0}^n|^p + K \). Also, one estimates \( J_2 \) by

\[
J_2 := KE \int_{t_0}^{u} |x_s^n|^p ds \leq K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |x_r^n|^p ds. \]  

(3.56)

For \( J_3 \), one uses an elementary inequality of stochastic integrals to obtain

\[
J_3 := Kn^{\theta \varphi} E \int_{t_0}^{u} \left| \int_{\kappa(n,s)}^{s} \sigma_r^n(x_{\kappa(n,r)}) dw_r \right|^2 ds
\]

\[
\leq Kn^{\theta \varphi} \int_{t_0}^{u} E \left( \int_{\kappa(n,s)}^{s} |\sigma_r^n(x_{\kappa(n,r)})|^2 dr \right) \frac{\varphi}{2} ds
\]

which due to Assumption B-8 gives

\[
J_3 \leq Kn^{\theta \varphi - \varphi} \int_{t_0}^{u} E(M_n + |x_{\kappa(n,s)}^n|^2)^{\frac{\varphi}{2}} ds \leq Kn^{\theta \varphi(2\varphi - 1)} \int_{t_0}^{u} (1 + E(M_n + |x_{\kappa(n,s)}^n|^2)^{\varphi}) ds
\]

\[
\leq K + KEM_n^{\varphi} + K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |x_r^n|^p ds \]  

(3.57)

for any \( u \in [t_0, t_1] \). Further to estimate \( J_4 \), one uses Lemma 1.5 to write,

\[
J_4 := Kn^{\theta} \int_{t_0}^{u} \left| \int_{\kappa(n,s)}^{s} \int_{Z} |x_{\kappa(n,s)}^n|^p \gamma_r^n(x_{\kappa(n,r)}, z) \tilde{N}(dr, dz) \right| ds
\]

\[
\leq Kn^{\theta} \int_{t_0}^{u} E \left( \int_{\kappa(n,s)}^{s} \int_{Z} |x_{\kappa(n,s)}^n|^2 \gamma^4_r(x_{\kappa(n,r)}, z)^2 \nu(dz) dr \right)^{\frac{1}{2}} ds
\]

and then by applying Assumption B-8, one obtains

\[
J_4 \leq Kn^{\theta - \frac{1}{2}} \int_{t_0}^{u} E|x_{\kappa(n,s)}^n|^{p-2} (M_n + |x_{\kappa(n,s)}^n|^2)^{\frac{1}{2}} ds
\]

\[
\leq K \int_{t_0}^{u} E|x_{\kappa(n,s)}^n|^{p-2} (1 + M_n + |x_{\kappa(n,s)}^n|^2) ds
\]
which further simplifies to the following estimates,

$$J_4 \leq K + KEM_n^p + K\int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |x_r^n|^p ds \tag{3.58}$$

Further, to estimate $J_5$, one uses Young’s inequality to obtain

$$J_5 := Kn^\theta E \int_{t_0}^{u} |x_s^n - x_n^{\kappa(n,s)}|^p dr dz \bigg\|_{ds}^{s} \leq Kn^\theta \int_{t_0}^{u} E |x_s^n - x_n^{\kappa(n,s)}|^p dr dz + Kn^{\theta - 1} \int_{t_0}^{u} E \bigg\|_{ds}^{s} \leq Kn^{\theta - 1} \int_{t_0}^{u} E |x_s^n - x_n^{\kappa(n,s)}|^p dr dz$$

which on the application of Lemma 3.3 implies,

$$J_5 \leq Kn^\theta - 1 + Kn^{\theta - 1} \int_{t_0}^{u} E |x_s^n - x_n^{\kappa(n,s)}|^p dr dz + Kn^{\theta - 1} \int_{t_0}^{u} E (M + |x_s^n - x_n^{\kappa(n,s)}|^2) \nu(dz) dr ds$$

and hence on noticing that $\theta \in (0, \frac{1}{2}]$, one obtains due to Lemma 1.5,

$$J_5 \leq K + K \int_{t_0}^{u} E |x_s^n - x_n^{\kappa(n,s)}|^p dr dz + Kn^{\theta - 1} \int_{t_0}^{u} E (M + |x_s^n - x_n^{\kappa(n,s)}|^2) \nu(dz) dr ds$$

Thus, on using Assumptions B-3 and B-8,

$$J_5 \leq K + K \int_{t_0}^{u} E |x_s^n - x_n^{\kappa(n,s)}|^p dr dz + Kn^{\theta - 1} \int_{t_0}^{u} E (M + |x_s^n - x_n^{\kappa(n,s)}|^2) \nu(dz) dr ds$$

which again due to $\theta \in (0, \frac{1}{2}]$ gives

$$J_5 \leq K \left(1 + \int_{t_0}^{u} E |x_s^n - x_n^{\kappa(n,s)}|^p dr dz + \int_{t_0}^{u} E (M + |x_s^n - x_n^{\kappa(n,s)}|^2) \nu(dz) dr ds + \int_{t_0}^{u} E (M + |x_s^n - x_n^{\kappa(n,s)}|^2) \nu(dz) dr ds \right)$$

which further implies

$$J_5 \leq K \left(1 + EM_n^p + EM'_n + \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |x_r^n|^p dr ds \right) \tag{3.59}$$

for any $u \in [t_0, t_1]$.

It is easy to observe that $J_6$ can be estimated by

$$J_6 := KE \int_{t_0}^{u} (M_n^p + |x_s^n - x_n^{\kappa(n,s)}|^p) dr dz \leq KEM_n^p + K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |x_r^n|^p dr ds. \tag{3.60}$$
Moreover, due to Burkholder-Davis-Gundy inequality, one obtains

\[ J_7 := pE \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^{t} |x_s^n|^{p-2} x_s^n \sigma_s^n(x_{\kappa(n,s)})dw_s \right| \]
\[ \leq KE \left( \int_{t_0}^{u} |x_s^n|^{2p-2} |\sigma_s^n(x_{\kappa(n,s)})|^2 ds \right)^{\frac{1}{2}} \]
\[ \leq KE \sup_{t_0 \leq s \leq u} |x_s^n|^{p-1} \left( \int_{t_0}^{u} |\sigma_s^n(x_{\kappa(n,s)})|^2 ds \right)^{\frac{1}{2}} \]

which on using Young’s inequality gives

\[ J_7 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |x_s^n|^p + KE \left( \int_{t_0}^{u} |\sigma_s^n(x_{\kappa(n,s)})|^2 ds \right)^{\frac{3}{2}} \]

and then by applying Hölder’s inequality with Assumption B-8, one obtains

\[ J_7 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |x_s^n|^p + KE \int_{t_0}^{u} |\sigma_s^n(x_{\kappa(n,s)})|^p ds \]
\[ \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |x_s^n|^p + KEM_n^p + KE \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |x_r^n|^p ds. \quad (3.61) \]

For \( J_8 \), due to Young’s inequality,

\[ J_8 := KE \int_{t_0}^{u} |x_s^n|^{p-2} |\sigma_s^n(x_{\kappa(n,s)})|^2 ds \]
\[ \leq KE \int_{t_0}^{u} |x_s^n|^p ds + KE \int_{t_0}^{u} |\sigma_s^n(x_{\kappa(n,s)})|^p ds \]

and again on using Assumption B-8, one obtains

\[ J_8 \leq KEM_n^p + KE \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |x_r^n|^p ds. \quad (3.62) \]

Also, one uses Lemma 1.5 to estimate \( J_9 \) by,

\[ J_9 := pE \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^{t} \int_{Z} |x_s^n|^{p-2} x_s^n \gamma_s^n(x_{\kappa(n,s)}, z) \tilde{N}(ds, dz) \right| \]
\[ \leq KE \left( \int_{t_0}^{u} \int_{Z} |x_s^n|^{2p-2} |\gamma_s^n(x_{\kappa(n,s)}, z)|^2 \nu(dz) ds \right)^{\frac{1}{2}} \]
\[ \leq KE \sup_{t_0 \leq s \leq u} |x_s^n|^{p-1} \left( \int_{t_0}^{u} \int_{Z} |\gamma_s^n(x_{\kappa(n,s)}, z)|^2 \nu(dz) ds \right)^{\frac{1}{2}} \]

which due to Young’s inequality and Hölder’s inequality gives

\[ J_9 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |x_s^n|^p + KE \left( \int_{t_0}^{u} \int_{Z} |\gamma_s^n(x_{\kappa(n,s)}, z)|^2 \nu(dz) ds \right)^{\frac{3}{2}}. \]

Thus, by using Assumption B-8, one obtains

\[ J_9 \leq \frac{1}{4} E \sup_{t_0 \leq s \leq u} |x_s^n|^p + KEM_n^p + K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |x_r^n|^p ds. \quad (3.63) \]
Finally, $J_{10}$ can be estimated as follow,

$$J_{10} := KE \int_{t_0}^{u} \int_{Z} |x_s^n - x_{\kappa(n,s)}|^2 ds dz + K E \sum_{t_0}^{u} |x_s^n - x_{\kappa(n,s)}|^2 ds dz$$

$$= KE \int_{t_0}^{u} \int_{Z} |x_s^n|^2 \gamma_s^n(x_{\kappa(n,s)}, z) \nu(ds, dz)$$

$$\leq KE \sum_{t_0}^{u} |x_s^n|^2 \nu(ds, dz) + K E \sum_{t_0}^{u} \gamma_s^n(x_{\kappa(n,s)}, z) \nu(ds, dz)$$

which on the application of Assumptions B-3 and B-8 gives

$$J_{10} \leq KE \sum_{t_0}^{u} |x_s^n|^2 (M_n + |x_{\kappa(n,s)}|^2) ds + K E \sum_{t_0}^{u} (M_n' + |x_{\kappa(n,s)}|^p) ds$$

and then Young’s inequality further gives

$$J_{10} \leq KEM_n^p + KEM_n' + K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |x_r^n|^p ds$$

for any $u \in [t_0, t_1]$. By substituting estimates from (3.56)-(3.64) in (3.55), one obtains due to equation (3.52),

$$E \sup_{t_0 \leq t \leq u} |x_t^n|^p \leq \frac{1}{2} E \sup_{t_0 \leq t \leq u} |x_t^n|^p + K KEM_n^p + KEM_n' + K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |x_r^n|^p ds < \infty.$$  

The application of Gronwall’s lemma completes the proof. \hfill \Box

### 3.2.2 Convergence in $L^q$

The one-step error of the tamed Euler scheme (3.1) is given in the following lemma.

**Lemma 3.5.** Let Assumptions B-1, B-3, B-8 and B-9 be satisfied. Then,

$$\sup_{t_0 \leq t \leq t_1} E |x_t^n - x_{\kappa(n,t)}|^r \leq Kn^{-1}$$

for any $2 \leq r \leq p$, where $K := K(t_0, t_1, L, p, \sup_{n \in \mathbb{N}} E |x_t^n|^p, \sup_{n \in \mathbb{N}} EM_n^p, \sup_{n \in \mathbb{N}} EM_n')$.

**Proof.** The lemma follows immediately from Lemma 3.3 and Lemma 3.4. \hfill \Box

The below mentioned theorem gives the convergence in $L^q$ of the tamed Euler scheme (3.1), introduced in this section, to the true solution of SDE (2.1) with super-linear drift coefficient.

**Theorem 3.3.** Let Assumptions A-3, A-4, A-6 to A-9 be satisfied. Also, suppose that Assumptions B-1, B-3 to B-5, B-8 and B-9 hold. Then, the tamed Euler scheme (3.1) converges to the solution of SDE (2.1) in $L^q$ sense, i.e.

$$\lim_{n \to \infty} E \sup_{t_0 \leq t \leq t_1} |x_t^n|^q = 0$$

for all $q < p$.

**Proof.** The proof is similar to the proof of the Theorem 3.1 with a note that one uses Lemmas [2.2, 3.4, 3.5] \hfill \Box
3.2.3 Rate of Convergence

The following theorem provides the rate of $L^q$-convergence of the tamed Euler scheme (3.1), introduced in this section, to the true solution of SDE (2.1) with super-linear drift coefficient. One observes that the rate of convergence is consistent with that of Euler scheme. Further, one notices that the mean square convergence gives optimal convergence rate, which is arbitrarily close to $0.5$. Moreover, one can also conclude that in the absence of jump, the tamed Euler scheme gives rate equal to $0.5$ for convergence in $L^2$, which is also the rate for Euler scheme.

**Theorem 3.4.** Let Assumptions A-4, A-6, A-7, A-9 and A-10 be satisfied. Also, suppose that Assumptions B-1, B-3, B-6 to B-9 hold. Then, the tamed Euler scheme (3.1) converges to the solution of SDE (2.1) in $L^q$ with rate arbitrarily close to $1/q$, i.e.

$$
E \sup_{t_0 \leq t \leq t_1} |x_t - x^n_t|^q \leq K n^{-\frac{q}{q+3}}
$$

where $K > 0$ does not depend on $n$.

**Proof.** The proof is similar to the proof of the Theorem 3.2 with a note that one uses Lemmas [2.2, 3.4, 3.5].

3.2.4 A Simple Example

We now introduce a tamed Euler scheme of SDEs driven by Lévy noise which have coefficients that are not random. For this purpose, we only highlight the modifications needed in the settings of our previous discussion. In SDE (2.1), $b_t(x)$ and $\sigma_t(x)$ are $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions with values in $\mathbb{R}^d$ and $\mathbb{R}^{d \times m}$ respectively. Also $\gamma_t(x, z)$ is a $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$-measurable function with values in $\mathbb{R}^d$. Moreover, one modifies Assumptions A-6 and A-7 by assigning $M = M' = 1$. Further, for every $n \in \mathbb{N}$, the scheme (3.1) is given by defining

$$
b^n_t(x) = \frac{b_t(x)}{1 + n^{-\theta} |b_t(x)|}, \sigma^n_t(x) = \sigma_t(x) \text{ and } \gamma^n_t(x, z) = \gamma_t(x, z) \quad (3.65)
$$

with $\theta \in (0, \frac{1}{2}]$ for any $t \in [t_0, t_1]$, $x \in \mathbb{R}^d$ and $z \in \mathbb{Z}$. Then, it is easy to observe that Assumptions B-3, B-8 and B-9 hold since $M_n = M'_n = 1$ and $\theta \in (0, \frac{1}{2}]$. Hence Lemmas [2.2, 3.3, 3.4, 3.5] follow immediately. Finally, $\mathcal{F}_{t_0}$ measurable random variable $C_R$ in Assumptions A-8 and A-9 is a constant for every $R$. In this new settings, one obtains the following corollaries for SDE (2.1) and scheme (3.1) with coefficients given by (3.65).

**Corollary 1.** Let Assumptions A-3, A-4, A-6 to A-9 be satisfied by the coefficients of SDE given immediately above. Also suppose that Assumptions B-1 and B-5 hold. Then, the numerical scheme (3.1) with coefficients given by (3.65) converges to the solution of SDE (2.1) in $L^q$ sense i.e.

$$
\lim_{n \to \infty} E \sup_{t_0 \leq t \leq t_1} |x_t - x^n_t|^q = 0
$$

for all $q < p$.

**Proof.** Assumption A-8 and A-9 are satisfied on taking $f(R) = C_R$ in equation (3.3).
For Assumption B-4, one observes due to (3.65) and Assumption A-9,

\[ E \int_{t_0}^{t_1} I_{B(R)} \sup_{|x| \leq R} |b^n_t(x) - b_t(x)|^2 dt \leq E \int_{t_0}^{t_1} I_{B(R)} \sup_{|x| \leq R} |b_t(x)|^4 n^{-2\theta} dt \]
\[ \leq K f(R)^4 n^{-2\theta} \to 0 \]
as \( n \to \infty \) for every \( R \). Also for diffusion and jump coefficients, Assumption B-4 holds trivially. Thus, Theorem 3.1 completes the proof.

For rate of convergence of scheme (3.1), one takes \( \theta = \frac{1}{2} \) in equation (3.65).

**Corollary 2.** Let Assumptions A-3, A-4, A-6, A-7, A-9 and A-10 be satisfied by the coefficients of SDE given immediately above. Also suppose that Assumptions B-1 and B-7 hold. Then, the numerical scheme (3.1) with coefficients given by (3.65) achieves the classical rate (of Euler scheme) in \( L^q \) sense i.e.

\[ E \sup_{t_0 \leq t \leq t_1} |x_t - x^n_t|^q \leq Kn^{-\frac{q}{\sigma+\delta}} \]

where constant \( K > 0 \) does not depend on \( n \).

**Proof.** By using equation (3.65) and Remark 3.3, one obtains

\[ E \int_{t_0}^{t_1} |b^n_t(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)})|^q dt \leq n^{-2\theta} E \int_{t_0}^{t_1} |b_t(x^n_{\kappa(n,t)})|^{2q} dt \]
\[ \leq Kn^{-\frac{q}{2}} (1 + E \sup_{t_0 \leq t \leq t_1} |x^n_t|^q \chi + 2) \leq Kn^{-\frac{q}{2}} \]
since \( \theta = \frac{1}{2} \). Hence, Assumption B-6 for drift coefficients follows due to Lemma 3.4. For diffusion and jump coefficients, Assumption B-6 holds trivially. The proof is completed by Theorem 3.2.
Chapter 4

Application to Delay Equations

This chapter is based on my joint works [16, 39]. We demonstrate an application of the results discussed in previous Chapters [2, 3] to the case of delay equations with the help of the approach introduced in [25]. By adopting the approach of [25], SDDE driven by Lévy noise can be regarded as a special case of SDE with random coefficients driven by Lévy noise. Thus, in Section 4.1, we prove existence and uniqueness of the solution to SDDE driven by Lévy noise under more relaxed conditions than those present in the literature. In Section 4.2, we construct an Euler scheme of SDDE driven by Lévy noise by using the methods of [25]. The strong convergence of the Euler scheme is proved under the assumptions that - (a) drift, diffusion and jump coefficients grow linearly in non-delay variable whereas they are allowed to grow super-linearly in delay variable, (b) drift coefficient satisfies one-sided local Lipschitz condition while jump and diffusion coefficients satisfy local Lipschitz conditions in non-delay variable, and (c) drift, diffusion and jump coefficients are continuous functions in delay variable. Further, a rate of convergence is obtained when local Lipschitz conditions are replaced by global Lipschitz conditions in addition to polynomial Lipschitz conditions on the coefficients. In Section 4.3, we introduce tamed Euler scheme for SDDE driven by Lévy noise which allows the drift coefficient of the Lévy driven SDDE to have super-linear growth in both delay and non-delay variables. The results on strong convergence and rate of convergence of the tamed scheme are also proved, which are in agreement with those of the Euler scheme. Finally, one concludes that mean square convergence gives the optimal convergence rate for both schemes, which is arbitrarily close to 0.5.

4.1 SDDE Driven by Lévy Noise

In this section, we introduce the set up of the SDDE driven by Lévy noise, which is used throughout this chapter. Let βt(y1, . . . , yk, x) and αt(y1, . . . , yk, x) be $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^{d\times k}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions which take values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$ respectively. Also, let λt(y1, . . . , yk, x, z) be $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^{d\times k}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$-measurable function which takes values in $\mathbb{R}^d$. For fixed $T, H > 0$, we consider a $d$-dimensional SDDE driven by Lévy noise on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}, P)$ defined by,

$$dx_t = \beta_t(y_t, x_t)dt + \alpha_t(y_t, x_t)dw_t + \int_Z \lambda_t(y_t, x_t, z)\tilde{N}(dt, dz), \quad t \in [0, T],$$

$$x_t = \xi_t, \quad t \in [-H, 0], \quad (4.1)$$
where $\xi: [-H,0] \times \Omega \to \mathbb{R}^d$ and $y_t := (x_{\delta_1(t)}, \ldots, x_{\delta_k(t)})$. The delay parameters $\delta_1(t), \ldots, \delta_k(t)$ are increasing functions of $t$ and satisfy $-H \leq \delta_j(t) \leq \lfloor t/h \rfloor h$ for some $h > 0$ and $j = 1, \ldots, k$.

**Remark 4.1.** In the following, we assume without loss of generality that $T$ is a multiple of $h$. If not, then SDDE (4.1) can be defined for $T' > T$ so that $T' = N'h$, where $N'$ is a positive integer. The results proved in this article are then recovered for the original SDDE (4.1) by choosing parameters as $\beta I_t \leq T$, $\alpha I_t \leq T$ and $\lambda I_t \leq T$.

**Remark 4.2.** We remark that two popular cases of delay viz. $\delta_i(t) = t - h$ and $\delta_i(t) = \lfloor t/h \rfloor h$ can be addressed by our findings which have been widely used in literature, for example, [18, 19, 47, 53] and references therein.

### 4.1.1 Existence and Uniqueness

The existence and uniqueness of the solution to SDDE (4.1) are proved under the assumptions given below.

**C-1.** For every $R > 0$, there exists an $M(R) \in L^1$ such that

$$x\beta_t(y,x) + |\alpha_t(y,x)|^2 + \int_\mathbb{Z} |\lambda_t(y,x,z)|^2 \nu(dz) \leq M_t(R)(1 + |x|^2)$$

for any $t \in [0,T]$ whenever $|y| \leq R$ and $x \in \mathbb{R}^d$.

**C-2.** For every $R > 0$, there exists an $M(R) \in L^1$ such that

$$(x - \bar{x})(\beta_t(y,x) - \beta_t(y,\bar{x})) + |\alpha_t(y,x) - \alpha_t(y,\bar{x})|^2 + \int_\mathbb{Z} |\lambda_t(y,x,z) - \lambda_t(y,\bar{x},z)|^2 \nu(dz)$$

$$\leq M_t(R)|x - \bar{x}|^2$$

for any $t \in [0,T]$ whenever $|x|, |\bar{x}|, |y| \leq R$.

**C-3.** The function $\beta_t(y,x)$ is continuous in $x \in \mathbb{R}^d$ for any $t \in [0,T]$ and $y \in \mathbb{R}^{d \times k}$.

**Theorem 4.1.** Let Assumptions C-1 to C-3 be satisfied. Then, there exists a unique solution of SDDE (4.1).

**Proof.** We adopt the approach of [25] and consider SDDE (4.1) as a special case of SDE (2.1) with coefficients given by,

$$b_t(x) = \beta_t(y_t, x), \sigma_t(x) = \alpha_t(y_t, x), \gamma_t(x, z) = \lambda_t(y_t, x, z)$$

(4.2)

almost surely for any $t \in [0,T]$. Then, the proof is a straightforward generalization of Theorem 2.1 of [25] and follows by using Theorem 2.1. \qed

### 4.1.2 Moment Bounds

For establishing the moment bounds of solution to SDDE (4.1), one replaces Assumption C-1 with the following assumptions.

**C-4.** For a fixed $p \geq 2$, $E \sup_{-H \leq t \leq 0} |\xi_t|^p < \infty$. 

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There exist constants $L > 0$ and $\chi \geq 2$ such that
\[ |\beta_t(y, x)|^2 \vee |\alpha_t(y, x)|^2 \vee \int_Z |\lambda_t(y, x, z)|^2 \nu(dz) \leq L(1 + |y|^\chi + |x|^2) \]
for any $t \in [0, T]$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{d \times k}$.

There exist constants $L > 0$ and $\chi \geq 2$ such that
\[ \int_Z |\lambda_t(y, x, z)|^p \nu(dz) \leq L(1 + |y|^\chi^p + |x|^p) \]
for any $t \in [0, T]$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{d \times k}$.

Let us define,
\[ p_i = \left(\frac{2}{\chi}\right)^i p \]  \hspace{1cm} (4.3)
for $i = 1, \ldots, N'$, where $\chi$ and $p$ satisfy $p/2 \geq (\chi/2)^{N'}$. Further, let us define
\[ p^* := \min_{i \in \mathbb{N}} p_i = \left(\frac{2}{\chi}\right)^{N'} p. \]  \hspace{1cm} (4.4)
We remark that throughout this chapter, we restrict to the case when $p^* \geq 2$.

Remark 4.3. By virtue of Assumptions C-5 and C-6,
\[ \int_Z |\lambda_t(y, x, z)|^r \nu(dz) \leq L(1 + |y|^\chi^r + |x|^r) \]
for any $2 \leq r \leq p$, $t \in [0, T]$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{d \times k}$.

Lemma 4.1. Let Assumptions C-2 to C-6 be satisfied. Then, there exists a unique solution to SDDE (4.1) and moreover,
\[ E \sup_{0 \leq t \leq T} |x_t|^{p^*} \leq K. \]

Proof. The lemma is proved by adopting the approach of [25], i.e. by using Lemma 2.1 inductively. In other words, one requires to prove
\[ E \sup_{(i-1)h \leq t \leq ih} |x_t|^{p_i} \leq K \]  \hspace{1cm} (4.5)
for every $i = 1, \ldots, N'$. For this, one verifies Assumptions A-4 to A-6 inductively.

Case $t \in [0, h]$. Notice that SDDE (4.1) can be regarded as a special case of SDE (2.1) with $t_0 = 0$, $t_1 = h$, $x_{t_0} = \xi_0$ and coefficients given by
\[ b_t(x) = \beta_t(y_t, x), \sigma_t(x) = \alpha_t(y_t, x), \gamma_t(x, z) = \lambda_t(y_t, x, z) \]  \hspace{1cm} (4.6)
almost surely for any $t \in [0, h]$, $x \in \mathbb{R}^d$ and $z \in Z$.

Verify A-4. Assumption A-4 holds due to Assumption C-4.
Verify A-5. Assumption A-5 follows due to Assumptions C-4 and C-5 since
\[ |b_t(x)|^2 \vee |\sigma_t(x)|^2 \vee \int_Z |\gamma_t(x, z)|^2 \nu(dz) \]
\[ = |\beta_t(y_t, x)|^2 \vee |\alpha_t(y_t, x)|^2 \vee \int_Z |\lambda_t(y_t, x, z)|^2 \nu(dz) \]
\[ \leq L(1 + |y_t|^2 + |x|^2) \leq L(1 + \Psi + |x|^2) \]
\[ = L(M + |x|^2) \]
almost surely for any \( t \in [0, h] \) and \( x \in \mathbb{R}^d \), where
\[ \Psi := \sup_{0 \leq t \leq h} |y_t| = \sup_{0 \leq t \leq h} |(\xi_{\delta_1(t)}, \ldots, \xi_{\delta_k(t)})|. \tag{4.7} \]
Thus, one takes \( M = 1 + \Psi \) and observes that
\[ EM^\frac{p}{2} \leq K(1 + E\Psi^\frac{p}{2}) = K(1 + E\Psi) < \infty \]
due to Assumption C-4.

Verify A-6. Assumption A-6 follows from Assumption C-6 since
\[ \int_Z |\gamma_t(x, z)|^p \nu(dz) = \int_Z |\lambda_t(y_t, x, z)|^p \nu(dz) \]
\[ \leq L(1 + |y_t|^2 \vee |x|^p) \leq L(1 + \Psi^2 \vee |x|^p) \]
\[ = L(M' + |x|^p) \]
almost surely for any \( t \in [0, h] \) and \( x \in \mathbb{R}^d \). Further, one takes \( M' = 1 + \Psi^2 = 1 + \Psi \)
and observes that \( EM' < \infty \) due to Assumption C-4. Thus, (4.5) holds for \( i = 1 \) due to Lemma 2.1.

For inductive arguments, one assumes that (4.5) holds for \( i = r \) i.e.
\[ E \sup_{(r-1)h \leq t \leq rh} |x_t|^p \leq K. \tag{4.8} \]

Case \( t \in [rh, (r + 1)h] \). Again notice that SDDE (4.1) can be regarded as a special case of SDE (2.1) with \( t_0 = rh, t_1 = (r + 1)h, x_{t_0} = x_{rh} \) and coefficients given by
\[ b_t(x) = \beta_t(y_t, x), \sigma_t(x) = \alpha_t(y_t, x), \gamma_t(x, z) = \lambda_t(y_t, x, z) \tag{4.9} \]
amost surely for any \( t \in [rh, (r + 1)h], x \in \mathbb{R}^d \) and \( z \in Z \).

Verify A-4. Assumption A-4 holds due to (4.8).

Verify A-5. On using Assumption C-5,
\[ |b_t(x)|^2 \vee |\sigma_t(x)|^2 \vee \int_Z |\gamma_t(x, z)|^2 \nu(dz) \]
\[ = |\beta_t(y_t, x)|^2 \vee |\alpha_t(y_t, x)|^2 \vee \int_Z |\lambda_t(y_t, x, z)|^2 \nu(dz) \]
\[ \leq L(1 + |y_t|^2 + |x|^2) \leq L(1 + \sup_{rh \leq t \leq (r+1)h} |y_t|^2 + |x|^2) = L(M + |x|^2) \]
amost surely for any \( t \in [rh, (r + 1)h] \) and \( x \in \mathbb{R}^d \). Thus, one takes
\[ M = 1 + \sup_{rh \leq t \leq (r+1)h} |y_t|^2 \]

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which due to (4.8) gives
\[ EM^{\frac{p_{r+1}}{2}} \leq K \left( 1 + E \sup_{rh \leq t \leq (r+1)h} |y_t|^{\frac{p_{r+1}}{2}} \right) \]
\[ = K \left( 1 + E \sup_{rh \leq t \leq (r+1)h} |y_t|^p \right) < \infty. \]

Verify A-6. On using Remark 4.3, one obtains,
\[ \int_Z |\gamma_t(x,z)|^{p_{r+1}} \nu(dz) = \int_Z |\lambda_t(y_t, x, z)|^{p_{r+1}} \nu(dz) \]
\[ \leq L(1 + |y_t|^{\frac{p_{r+1}}{2}} + |x|^{p_{r+1}}) \]
\[ \leq L(1 + \sup_{rh \leq t \leq (r+1)h} |y_t|^{\frac{p_{r+1}}{2}} + |x|^{p_{r+1}}) \]
\[ = L(M' + |x|^{p_{r+1}}) \]
almost surely for any \( t \in [r h, (r + 1)h] \) and \( x \in \mathbb{R}^d \). Thus, one takes
\[ M' = 1 + \sup_{rh \leq t \leq (r+1)h} |y_t|^{\frac{p_{r+1}}{2}} = 1 + \sup_{rh \leq t \leq (r+1)h} |y_t|^p \]
which due to (4.8) gives \( EM' < \infty \). Thus, (4.5) holds for \( i = r + 1 \). This completes the proof.

Now, we replace Assumption C-5 with the following assumption.

C-7. There exist constants \( L > 0 \) and \( \chi \geq 2 \) such that
\[ x \beta_t(y, x) \vee |\alpha_t(y, x)|^2 \vee \int_Z |\lambda_t(y, x, z)|^2 \nu(dz) \leq L(1 + |y|^\chi + |x|^2) \]
for any \( t \in [0, T] \), \( x \in \mathbb{R}^d \) and \( y \in \mathbb{R}^{d \times k} \).

**Lemma 4.2.** Let Assumptions C-2 to C-4, C-6 and C-7 be satisfied. Then, there exists a unique solution to SDDE (4.1) and moreover,
\[ E \sup_{0 \leq t \leq T} |x_t|^{p'} \leq K. \]

**Proof.** The proof is similar to the proof of Lemma 4.1 and we only highlight the differences. As before, one adopts the approach of [25] and inductively proves (4.5) for every \( i = 1, \ldots, N' \) with the help of Lemma 2.2. Notice that Assumptions A-4 and A-6 have been verified during the proof of Lemma 4.1. Thus, one only requires to verify Assumption A-7 inductively.

**Case** \( t \in [0, h] \). As before, SDDE (4.1) can be regarded as a special case of SDE (2.1) with \( t_0 = 0, t_1 = h, x_{t_0} = \xi_0 \) and coefficients given by (4.6). Then Assumption A-7 holds due to Assumptions C-7 and C-4. Thus, (4.5) holds for \( i = 1 \) by using Lemma 2.2.

**Case** \( t \in [rh, (r+1)h] \). Again, SDDE (4.1) can be regarded as a special case of SDE (2.1) with \( t_0 = rh, t_1 = (r+1)h, x_{t_0} = x_{rh} \) and coefficients given by (4.9). Also, Assumption A-7 holds due to Assumptions C-7 and inductive assumption. Hence, (4.5) holds for \( i = r + 1 \) on using Lemma 2.2. This completes the proof. \( \square \)
4.2 Euler Scheme

In this section, we discuss Euler scheme of SDDE (4.1) and its strong convergence properties. For every \( n \in \mathbb{N} \), define Euler scheme of SDDE (4.1) as

\[
dx_t^n = \beta_t(y_t^n, x_{\kappa(n,t)})dt + \alpha_t(y_t^n, x_{\kappa(n,t)})dw_t + \int_Z \lambda_t(y_t^n, x_{\kappa(n,t)}, z)\tilde{N}(dt, dz), \quad t \in [0, T],
\]

\[x_t^n = \xi_t, \quad t \in [-H, 0],\]

(4.10)

where \( y_t^n := (x_{\delta_1(t)}^n, \ldots, x_{\delta_k(t)}^n) \) and \( \kappa \) is defined by (3.2) with \( t_0 = 0 \) i.e. \( \kappa(n, t) = [nt]/n \).

4.2.1 Moment Bounds

In the following lemma, we establish moment bounds for the scheme (4.10).

Lemma 4.3. Let Assumptions C-4 to C-6 be satisfied. Then,

\[
\sup_{n \in \mathbb{N}} E \sup_{0 \leq t \leq T} |x_t^n|^{p'} \leq K.
\]

Proof. The lemma is proved by adopting the approach of [25], i.e. one uses Lemma 3.1 inductively to show

\[
\sup_{n \in \mathbb{N}} E \sup_{i=1, \ldots, N_h} |x_{i h}^n|^{p_i} \leq K \tag{4.11}
\]

for every \( i = 1, \ldots, N_h \).

Case \( t \in [0, h] \). The scheme (4.10) can be regarded as a special case of scheme (3.1) with \( t_0 = 0, t_1 = h, x_{t_0}^n = \xi_0 \) and coefficients given by

\[
b_t^n(x) = \beta_t(y_t^n, x), \sigma_t^n(x) = \alpha_t(y_t^n, x), \gamma_t^n(x, z) = \lambda_t(y_t^n, x, z) \tag{4.12}
\]

almost surely for any \( t \in [0, h], x \in \mathbb{R}^d \) and \( z \in Z \).

Verify B-1. Assumption B-1 holds due to Assumption C-4.

Verify B-2. Assumption B-2 follows due to Assumptions C-4 and C-5 since

\[
|b_t^n(x)|^2 \vee |\sigma_t^n(x)|^2 \vee \int_Z |\gamma_t^n(x, z)|^2 \nu(dz)
\]

\[= |\beta_t(y_t^n, x)|^2 \vee |\alpha_t(y_t^n, x)|^2 \vee \int_Z |\lambda_t(y_t^n, x, z)|^2 \nu(dz)
\]

\[\leq L(1 + |y_t^n|^2 + |x|^2) \leq L(1 + \Psi |x|^2 + |x|^2) = L(M_n + |x|^2)
\]

almost surely for any \( t \in [0, h] \) and \( x \in \mathbb{R}^d \), where \( \Psi \) is defined in equation (4.7). Thus, one can take \( M_n = 1 + \Psi |x|^2 \) and observes that \( EM_n^{\Psi x} \leq K(1 + E \Psi |x|^2) = K(1 + E \Psi |x|^2) < \infty \) due to Assumption C-4.

Verify B-3. Assumption B-3 follows from Assumption C-6 since

\[
\int_Z |\gamma_t^n(x, z)|^p \nu(dz) = \int_Z |\lambda_t(y_t^n, x, z)|^p \nu(dz) \leq L(1 + |y_t^n|^\frac{p}{2} + |x|^p)
\]

\[\leq L(1 + \Psi |x|^\frac{p}{2} + |x|^p) = L(M_n' + |x|^p)
\]

almost surely for any \( t \in [0, h] \) and \( x \in \mathbb{R}^d \). Further, one can take \( M_n' = 1 + \Psi |x|^\frac{p}{2} = 1 + \Psi |x|^\frac{p}{2} \) and observes that \( EM_n' < \infty \) due to Assumption C-4. Thus (4.11) holds for
Lemma 3.1. This completes the proof.

For inductive arguments, one assumes that (4.11) holds for \( i = r \) i.e.

\[
\sup_{n \in \mathbb{N}} E \sup_{(r-1)h \leq t \leq rh} |x_t^n|^p \leq K. \tag{4.13}
\]

**Case** \( t \in [rh, (r+1)h] \). Again notice that scheme (4.10) can be regarded as a special case of scheme (3.1) with \( t_0 = rh, t_1 = (r+1)h, x_t^n = x^n_{rh} \) and coefficients given by

\[
b^n_t(x) = b_t(y^n_t, x), \sigma_t^n(x) = \sigma_t(y^n_t, x), \gamma^n_t(x, z) = \gamma_t(y^n_t, x, z) \tag{4.14}
\]

almost surely for any \( t \in [rh, (r+1)h] \), \( x \in \mathbb{R}^d \) and \( z \in Z \).

**Verify B-1.** Assumption B-1 holds due to (4.13).

**Verify B-2.** Due to equation (4.14) and Assumption C-5, one has

\[
|b^n_t(x)|^2 \vee |\sigma^n_t(x)|^2 \vee \int_Z |\gamma^n_t(x, z)|^2 \nu(dz)
= |b_t(y^n_t, x)|^2 \vee |\sigma_t(y^n_t, x)|^2 \vee \int_Z |\lambda_t(y^n_t, x, z)|^2 \nu(dz)
\leq L(1 + |y^n_t| + |x|^2) \leq L(1 + \sup_{rh \leq t \leq (r+1)h} |y^n_t| + |x|^2)
= L(M_n + |x|^2)
\]

almost surely for any \( t \in [rh, (r+1)h] \) and \( x \in \mathbb{R}^d \). Thus, one can take

\[
M_n = 1 + \sup_{rh \leq t \leq (r+1)h} |y^n_t|
\]

and observes that

\[
EM_n^{p_{r+1}} \leq K \left( 1 + E \sup_{rh \leq t \leq (r+1)h} |y^n_t|^{\frac{p_{r+1}}{2}} \right) = K \left( 1 + E \sup_{rh \leq t \leq (r+1)h} |y^n_t|^{p_{r+1}} \right) < \infty
\]

due to (4.13).

**Verify B-3.** By using (4.14) and Remark 4.3, one writes

\[
\int_Z |\gamma^n_t(x, z)|^{p_{r+1}} \nu(dz) = \int_Z |\lambda_t(y^n_t, x, z)|^{p_{r+1}} \nu(dz) \leq L(1 + |y^n_t|^{\frac{p_{r+1}}{2}} + |x|^{p_{r+1}})
\leq L(1 + \sup_{rh \leq t \leq (r+1)h} |y^n_t|^{\frac{p_{r+1}}{2}} + |x|^{p_{r+1}}) = L(M_n' + |x|^{p_{r+1}})
\]

almost surely for any \( t \in [rh, (r+1)h] \) and \( x \in \mathbb{R}^d \). Further one can take,

\[
M_n' = 1 + \sup_{rh \leq t \leq (r+1)h} |y^n_t|^{\frac{p_{r+1}}{2}} = 1 + \sup_{rh \leq t \leq (r+1)h} |y^n_t|^{p_{r}}
\]

and observes that \( EM' < \infty \) due to (4.13). Thus, (4.11) holds for \( i = r + 1 \) due to Lemma 3.1. This completes the proof.

\[\square\]

### 4.2.2 Convergence in \( L^q \)

For convergence in \( L^q \) of the Euler scheme (4.10), one needs following additional assumptions.
C-8. For every $R > 0$, there exists a constant $L_R > 0$ such that

$$(x - \bar{x})(\beta_t(y, x) - \beta_t(y, \bar{x})) \lor |\alpha_t(y, x) - \alpha_t(y, \bar{x})|^2 \lor \int Z |\lambda_t(y, x, z) - \lambda_t(y, \bar{x}, z)|^2 \nu(dz)$$

$$\leq L_R|x - \bar{x}|^2$$

for any $t \in [0, T]$ whenever $|x|, |y|, |\bar{x}| < R$.

C-9. For every $R > 0$, there exists a constant $L_R > 0$ such that

$$\sup_{|x|, |y| \leq R} |\beta_t(y, x)|^2 \leq L_R$$

for any $t \in [0, T]$.

C-10. For every $R > 0$ and $t \in [0, T]$,

$$\sup_{|x| \leq R} \left\{ \left| \beta_t(y, x) - \beta_t(y', x) \right|^2 + |\alpha_t(y, x) - \alpha_t(y', x)|^2 + \int Z |\lambda_t(y, x, z) - \lambda_t(y', x, z)|^2 \nu(dz) \right\} \to 0$$

when $y' \to y$.

Lemma 4.4. Let Assumptions C-4 to C-6 be satisfied, then one-step error of the Euler scheme (4.10) of SDDE (4.1) is given by

$$\sup_{0 \leq t \leq T} E |x_t^n - x_{\kappa(n, t)}^n|^r \leq Kn^{-1}$$

for any $2 \leq r \leq p^*$.

Proof. The proof follows by using Lemma 3.2 inductively. Notice that Assumptions B-1 to B-3 are already verified during the proof of Lemma 4.3. Hence, the proof follows immediately.

Theorem 4.2. Let Assumptions C-3 to C-6 and C-8 to C-10 be satisfied. Then, the Euler scheme (4.10) of SDDE (4.1) converges in $\mathcal{L}^q$, i.e.

$$\lim_{n \to \infty} E \sup_{0 \leq t \leq T} |x_t - x_t^n|^q = 0$$

for any $q < p^*$.

Proof. The theorem is proved by adopting the approach of [25], i.e. one has to prove

$$\lim_{n \to \infty} E \sup_{(i-1)h \leq t \leq ih} |x_t - x_t^n|^q = 0 \quad (4.15)$$

inductively for any $q < p_i$ and for every $i = 1, \ldots, N'$ with the help of Theorem 3.1. One observes that Assumptions A-3 to A-6 have been verified during the proof of Lemma 4.1 whereas Assumptions B-1 to B-3 have been verified during the proof of Lemma 4.3. Thus, one only requires to verify Assumptions A-8, A-9, B-4 and B-5. Also notice that moment bounds of Lemmas [4.1, 4.3] are achieved for $p^*$ defined by equation (4.4) and hence the desired convergence of scheme (4.10) is obtained for any $q < p^*$.

Case $t \in [0, h]$. As before, one considers SDDE (4.1) as a special case of SDE (2.1)
with \( t_0 = 0, t_1 = h, x_{t_0} = \xi_0 \) and coefficients given by equation (4.6). Similarly, scheme (4.10) can be regarded as a special case of scheme (3.1) with \( t_0 = 0, t_1 = h, x_{t_0} = \xi_0 \) and coefficients given by equation (4.12).

Verify A-8. For every \( R > 0 \) and \( t \in [0, h] \), Assumption C-8 implies

\[
(x - \bar{x})(b_t(x) - b_t(\bar{x})) \vee |\sigma_t(x) - \sigma_t(\bar{x})|^2 \vee \int_{\mathbb{Z}} |\gamma_t(x, z) - \gamma_t(\bar{x}, z)|^2 \nu(dz) \\
= (x - \bar{x})(\beta_t(y_t, x) - \beta_t(y_t, \bar{x})) \vee |\alpha_t(y_t, x) - \alpha_t(y_t, \bar{x})|^2 \\
\vee \int_{\mathbb{Z}} |\lambda_t(y_t, x, z) - \lambda_t(y_t, \bar{x}, z)|^2 \nu(dz) \leq C_R |x - \bar{x}|^2
\]

almost surely whenever \(|x|, |\bar{x}| \leq R\), where \( \mathcal{F}_0 \)-measurable random variable \( C_R \) is given by

\[
C_R := L_R I_{\Omega_R} + \sum_{j=R}^{\infty} L_{j+1}I_{\Omega_{j+1} \setminus \Omega_j}
\] (4.16)

where \( \Omega_j := \{\omega \in \Omega : \Psi \leq j\} \) and \( \Psi \) is defined in (4.7). Further, one takes \( f(R) := L_R \) and uses Markov’s inequality along with Assumption C-4 to obtain

\[
P(C_R > f(R)) = P(C_R > L_R) \leq 1 - P(\Omega_R) = P(\Psi > R) \leq \frac{E \Psi^p}{R^p} \leq \frac{K}{R^p} \to 0
\] (4.17)
as \( R \to \infty \).

Verify A-9. As before, for every \( R > 0 \) and \( t \in [0, h] \), Assumption C-9 implies

\[
\sup_{|x| \leq R} |b_t(x)|^2 = \sup_{|x| \leq R} |\beta_t(y_t, x)|^2 \leq C_R
\]
amost surely, where \( C_R \) is given in (4.16) and satisfies (4.17).

Verify B-4. Notice that initial data of SDDE (4.1) and scheme (4.10) satisfy

\[
y_t = y_t^n = (\xi_{\delta_1(t)}, \ldots, \xi_{\delta_n(t)})
\] (4.18)
for any \( t \in [0, h] \). Consequently, one obtains

\[
b_t^n(x) - b_t(x) = \beta_t(y_t^n, x) - \beta_t(y_t, x) \equiv 0 \\
\sigma_t^n(x) - \sigma_t(x) = \alpha_t(y_t^n, x) - \alpha_t(y_t, x) \equiv 0 \\
\gamma_t^n(x, z) - \gamma_t(x, z) = \lambda_t(y_t^n, x, z) - \lambda_t(y_t, x, z) \equiv 0
\]
amost surely for any \( t \in [0, h] \), \( x \in \mathbb{R}^d \) and \( z \in \mathbb{Z} \). Thus Assumption B-4 holds.

Verify B-5. This follows immediately due to (4.18).

Thus, when \( i = 1 \), (4.15) holds for any \( q < p_1 \) due to Theorem 3.1 and also by noticing that Lemmas [4.1, 4.3] are true for \( p_1 \)-th moment bounds. For inductive arguments, one assumes,

\[
\lim_{n \to \infty} E \sup_{(r-1)h \leq t \leq rh} |x_t - x_t^n|^q = 0
\] (4.19)
for any \( q < p_r \).

Case \( t \in [rh, (r + 1)h] \). As before, one considers SDDE (4.1) as a special case of SDE
(2.1) with \( t_0 = rh \), \( t_1 = (r + 1)h \), \( x_{t_0} = x_{rh} \) and coefficients given by equation (4.9). Similarly, scheme (4.10) can be regarded as a special case of scheme (3.1) with \( t_0 = rh \), \( t_1 = (r + 1)h \), \( x_{t_0}^n = x_{rh}^n \) and coefficients given by equation (4.14).

Verify A-8. For every \( R > 0 \) and \( t \in [rh, (r + 1)h] \), Assumption C-8 implies

\[
(x - \bar{x})(b_t(x) - b_t(\bar{x})) \vee |\sigma_t(x) - \sigma_t(\bar{x})|^2 \vee \int_Z |\gamma_t(x, z) - \gamma_t(\bar{x}, z)|^2 \nu(dz)
\]

\[
= (x - \bar{x})(\beta_t(y_t, x) - \beta_t(\bar{y}_t, \bar{x})) \vee |\alpha_t(y_t, x) - \alpha_t(\bar{y}_t, \bar{x})|^2
\]

\[
\vee \int_Z |\lambda_t(y_t, x, z) - \lambda_t(\bar{y}_t, \bar{x}, z)|^2 \nu(dz)
\]

\[
\leq C_R |x - \bar{x}|^2
\]

almost surely whenever \(|x|, |\bar{x}| \leq R\) with \( \mathcal{F}_{rh} \)-measurable random variable \( C_R \) given by

\[
C_R := L_RI_{\Omega_R} + \sum_{j=R}^{\infty} L_{j+1}I_{\Omega_{j+1}\setminus \Omega_j}
\]

(4.20)

where \( \Omega_j := \{ \omega \in \Omega : \sup_{t \in [rh, (r+1)h]} |y_t| \leq j \} \). Further take \( f(R) := L_R \) and then on the application of Markov’s inequality and Lemma 4.1, one obtains

\[
P(C_R > f(R)) = P(C_R > L_R) = 1 - P(C_R \leq L_R) \leq 1 - P(\Omega_R)
\]

\[
= P \left( \sup_{rh \leq t < (r+1)h} |y_t| > R \right)
\]

\[
\leq \frac{1}{R^{rp}} E \sup_{rh \leq t < (r+1)h} |y_t|^p_r
\]

\[
\leq \frac{K}{R^{rp}} \rightarrow 0 \text{ as } R \rightarrow \infty. \quad (4.21)
\]

Verify A-9. Again, for every \( R > 0 \) and \( t \in [rh, (r + 1)h] \), due to Assumption C-9,

\[
\sup_{|x| \leq R} |b_t(x)|^2 = \sup_{|x| \leq R} |\beta_t(y_t, x)|^2 \leq C_R
\]

almost surely, where \( \mathcal{F}_{rh} \)-measurable random variable \( C_R \) is defined in (4.20) and satisfies \( P(C_R > f(R)) \rightarrow 0 \) as \( R \rightarrow \infty \) due to (4.21).

Verify B-4. The inductive assumption (4.19) gives that \(|y^n_t - y_t| \rightarrow 0\) in probability as \( n \rightarrow \infty \). Thus, Assumption C-10 implies \( \sup_{|x| \leq R} |\beta_t(y^n_t, x) - \beta_t(y_t, x)| \rightarrow 0 \) in probability as \( n \rightarrow \infty \) for every \( R > 0 \) and \( t \in [rh, (r+1)h] \). Furthermore, the sequence

\[
\{ I_{B(R)} \sup_{|x| \leq R} |\beta_t(y^n_t, x) - \beta_t(y_t, x)|^2 \}_{n \in \mathbb{N}} \quad (4.22)
\]

is uniformly integrable since it has bounded \( p_r \)-th moments for every \( R > 0 \). This can be seen as follows. Due to Assumption C-5, Lemmas [4.1, 4.3], one has

\[
EI_{B(R)} \sup_{|x| \leq R} |\beta_t(y^n_t, x) - \beta_t(y_t, x)|^{pr+1} \leq KE \sup_{|x| \leq R} \left\{ |\beta_t(y^n_t, x)|^{pr+1} + |\beta_t(y_t, x)|^{pr+1} \right\}
\]

\[
\leq KE \sup_{|x| \leq R} \left\{ 1 + |y^n_t|^{pr} + |y_t|^{pr} + |x|^{pr} \right\}
\]

\[
\leq K + |R|^{pr+1}
\]

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for every $R > 0$ and $t \in [rh, (r + 1)h]$. Therefore,

$$E \int_{rh}^{(r+1)h} I_{B(R)} \sup_{|x| \leq R} |b_t^n(x) - b_t(x)|^2 dt$$

$$= E \int_{rh}^{(r+1)h} I_{B(R)} \sup_{|x| \leq R} |\beta_t^n(y_t^n, x) - \beta_t(y_t, x)|^2 dt \to 0 \text{ as } n \to \infty$$

for every $R > 0$. By adopting similar arguments, one obtains the corresponding convergence for diffusion and jump coefficients.

Verify B-5. This follows immediately due to (4.19).

Thus, (4.15) holds for any $q < p_{r+1}$ on using Theorem 3.1 and by noticing that Lemmas $[4.1, 4.3]$ are true for $p_{r+1}$-th moment moment bounds. This completes the proof. \(\square\)

### 4.2.3 Rate of Convergence

We now proceed to obtain the rate of convergence of the Euler scheme (4.10). For this purpose, one replaces Assumptions C-8 and C-10 by the following assumptions.

**C-11.** There exist constants $C > 0$, $q \geq 2$ and $\chi > 0$ such that,

$$(x - \bar{x})(\beta_t(x) - \beta_t(\bar{x})) \vee |\alpha_t(x) - \alpha_t(\bar{x})|^2$$

$$\vee \int_Z |\lambda_t(x, z) - \lambda_t(\bar{x}, z)|^2 \nu(dz) \leq C|x - \bar{x}|^2$$

$$\int_Z |\lambda_t(x, z) - \lambda_t(\bar{x}, z)|^q \nu(dz) \leq C|x - \bar{x}|^q$$

$$|\beta_t(x) - \beta_t(\bar{x})|^2 \leq C(1 + |x|^\chi + |\bar{x}|^\chi)|x - \bar{x}|^2$$

for any $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$, $y \in \mathbb{R}^{d \times k}$ and a $\delta \in (0, 1)$ such that $\max\{(\chi+2)q, \frac{2\chi}{q} + \frac{\delta}{q}\} \leq p^*$.  

**C-12.** There exist constants $C > 0$, $q \geq 2$ and $\chi > 0$ such that,

$$|\beta_t(y) - \beta_t(\bar{y})|^2 + |\alpha_t(y) - \alpha_t(\bar{y})|^2 \leq C(1 + |y|^\chi + |\bar{y}|^\chi)|y - \bar{y}|^2$$

$$\left(\int_Z |\lambda_t(y, z) - \lambda_t(\bar{y}, z)|^q \nu(dz)\right)^{\frac{2}{q}} \leq C(1 + |y|^\chi + |\bar{y}|^\chi)|y - \bar{y}|^q$$

where $\zeta = 2q$, for any $t \in [0, T]$, $x \in \mathbb{R}^d$ and $y, \bar{y} \in \mathbb{R}^{d \times k}$.

**Remark 4.4.** Due to Assumptions C-9, C-11 and C-12, there exists a constant $C > 0$ such that

$$|\beta_t(y)|^2 \leq C(1 + |y|^{\chi+2} + |x|^{\chi+2})$$

for any $t \in [0, T]$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{d \times k}$.

The following theorem gives the rate of convergence of the scheme (4.10).

**Theorem 4.3.** Let Assumptions C-4 to C-6, C-9, C-11 and C-12 be satisfied. Then, the Euler scheme (4.10) converges to the solution of SDDE (4.1) in $\mathcal{L}^q$ with rate arbitrarily close to $1/q$, i.e.

$$E \sup_{0 \leq t \leq T} |x_t - y^n_t|^q \leq Kn^{-\frac{q}{q+\chiq}}$$
for any $q < p^*$.

Proof. As before, one uses the approach of [25] to prove

$$E \sup_{(i-1)h \leq t \leq ih} |x_t^n - x_t^\ast|^q \leq Kn^{-\frac{q}{q+\delta}}$$

(4.23)

inductively for any $q < p_1$ and for every $i = 1, \ldots, N'$ with the help of Theorem 3.2. Notice that Assumptions A-3 to A-6, A-9, B-1 to B-3 have already been verified during the proofs of Lemmas [4.1, 4.3] and Theorem 4.2. Thus, one only needs to verify Assumptions A-10, B-6 and B-7.

Case $t \in [0, h]$. As before, one considers SDDE (4.1) and scheme (4.10) as a special case of SDE (2.1) and scheme (3.1) with $t_0 = 0$, $t_1 = h$, $x_{t_0} = \xi_0$ and coefficients given by equations (4.6) and (4.12) respectively.

Verify A-10. This follows immediately due to Assumption C-11.

Verify B-6. Since the initial data of SDDE (4.1) and scheme (4.10) are equal, thus Assumption B-6 follows trivially.

Verify B-7. This follows again due to same initial data of SDDE (4.1) and scheme (4.10).

Thus, equation (4.23) holds for $i = 1$ by using Theorem 3.2. For inductive arguments, one assumes that equation (4.23) holds for $i = r$ i.e.

$$E \sup_{(r-1)h \leq t \leq rh} |x_t^n - x_t^\ast|^q \leq Kn^{-\frac{q}{q+\delta}}$$

(4.24)

for an $q < p_r$.

Case $t \in [rh, (r+1)h]$. As before, one considers SDDE (4.1) and scheme (4.10) as a special case of SDE (2.1) and scheme (3.1) with $t_0 = rh$, $t_1 = (r+1)h$, $x_{t_0} = x_{rh}$ and coefficients given by equations (4.9) and (4.14) respectively.

Verify A-10. This follows immediately by Assumption C-11.

Verify B-6. Due to Assumption C-12, one has

$$E \int_{rh}^{(r+1)h} |b^n_t(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)})|^q dt = E \int_{rh}^{(r+1)h} |\beta_t(y^n_t, x^n_{\kappa(n,t)}) - \beta_t(y_t, x^n_{\kappa(n,t)})|^q dt$$

$$\leq KE \int_{rh}^{(r+1)h} (1 + |y^n_t|^\frac{q^2}{2} + |y_t|^\frac{q^2}{2}) |y_t - y^n_t|^q dt$$

which on the application of Hölder’s inequality, Lemma 4.1 and Lemma 4.3 implies

$$E \int_{rh}^{(r+1)h} |b^n_t(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)})|^q dt \leq K \int_{rh}^{(r+1)h} (E|y_t - y^n_t|^{q+\delta})^\frac{q}{q+\delta} dt$$

and the inductive assumption (4.19) gives

$$E \int_{rh}^{(r+1)h} |b^n_t(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)})|^q dt \leq Kn^{-\frac{q}{q+\delta(r+1)\delta}}.$$ 

One can similarly verify Assumption B-6 for the diffusion and jump coefficients.

Verify B-7. This follows due to (4.24).

Thus, (4.24) holds due to Theorem 3.2. This completes the proof. 

\[\square\]
4.3 Tamed Euler Scheme

In this section, we replace the Euler scheme (4.10) by the following tamed Euler scheme which allows the drift coefficient of SDDE (4.1) to grow super-linearly in non-delay variable.

For every $n \in \mathbb{N}$, the tamed Euler scheme of SDDE (4.1) is given by

$$dx^n_t = \beta^n_t(y^n_t, x^n_{\kappa(n,t)})dt + \alpha_t(y^n_t, x^n_{\kappa(n,t)})dw_t + \int_Z \lambda_t(y^n_t, x^n_{\kappa(n,t)}, z)\tilde{N}(dt, dz), \quad t \in [0, T],$$

$$x^n_t = \xi_t, \quad t \in [-H, 0],$$

(4.25)

where $y^n_t := (x^n_{\delta_1(t)}, \ldots, x^n_{\delta_k(t)})$ and $\kappa$ is defined by (3.2) with $t_0 = 0$ i.e. $\kappa(n,t) = [nt]/n$. Further, for every $n \in \mathbb{N}$, drift coefficient in the above equation is assumed to have the following form,

$$\beta^n_t(y, x) := \frac{1}{1 + n^{-\theta} |\beta_t(y, x)|} \beta_t(y, x)$$

for any $t \in [0, T], \ x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{d\times k}$, where $\theta \in (0, \frac{1}{2}]$.

**Remark 4.5.** One can easily verify that

$$|\beta^n_t(y, x)| \leq \min(n^\theta, |\beta_t(y, x)|)$$

for any $t \in [0, T], \ x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{d\times k}$, where $\theta \in (0, \frac{1}{2}]$. Also notice that Assumption B-9 is satisfied.

4.3.1 Moment Bounds

In the following lemma, we establish moment bounds of scheme (4.25).

**Lemma 4.5.** Let Assumptions C-4, C-6 and C-7 be satisfied. Then,

$$\sup_{n \in \mathbb{N}} E \sup_{0 \leq t \leq T} |x^n_t|^p \leq K.$$  

**Proof.** The proof is similar to the proof of Lemma 4.3, so we only highlight the differences. As before, one adopts the approach of [25] to prove (4.11) for every $i = 1, \ldots, N'$. For this, one can use Lemma 3.4 inductively. Further, Assumptions B-1 and B-3 have already been verified in the proof of Lemma 4.3. Also notice that Assumption B-9 holds from Remark 4.5. Thus, one only requires to verify Assumption B-8 inductively. Moreover, for diffusion and jump coefficients, Assumption B-8 is already verified during the proof of Lemma 4.3.

**Case $t \in [0, h]$.** As before, scheme (4.25) can be regarded as a special case of scheme (3.1) with $t_0 = 0, t_1 = h, x_{t_0} = \xi_0$ and drift coefficient given by

$$b^n_t(x) = \frac{\beta_t(y^n_t, x)}{1 + n^{-\theta} |\beta_t(y^n_t, x)|}$$

(4.26)

almost surely for any $t \in [0, h]$ and $x \in \mathbb{R}^d$ where $\theta \in (0, \frac{1}{2}]$. Note that $y^n_t = (\xi_{\delta_1(t)}, \ldots, \xi_{\delta_k(t)})$ whenever $t \in [0, h]$. Due to Assumption C-7, one has

$$xb^n_t(x) = \frac{x\beta_t(y^n_t, x)}{1 + n^{-\theta} |\beta_t(y^n_t, x)|} \leq x\beta_t(y^n_t, x) \leq L(M_n + |x|^2)$$
where $M_n = 1 + \Psi^i$ with $\Psi$ given by equation (4.7). Moreover, $EM_n^{\frac{p^*}{2}} < \infty$ as before. Thus (4.11) holds for $i = 1$ due to Lemma 3.4. For the inductive arguments, let us assume that (4.11) holds for $i = r$, then one proves that (4.11) also holds for $i = r + 1$. **Case $t \in [rh, (r + 1)h]$.** Again, one considers scheme (4.25) as a special case of scheme (3.1) with $t_0 = rh$, $t_1 = (r + 1)h$, $x_{t_0} = x_{rh}$ and drift coefficient given by

$$b_t^n(x) = \frac{\beta_t(y_t^n, x)}{1 + n^{-\theta}|\beta_t(y_t^n, x)|} \tag{4.27}$$

almost surely for any $t \in [rh, (r + 1)h]$ and $x \in \mathbb{R}^d$ where $\theta \in (0, \frac{1}{2}]$. Note that $y_t^n = (x_{\delta_1(t)}, \ldots, x_{\delta_k(t)})$ whenever $t \in [0, h]$. Further, Assumption C-7 gives

$$xb_t^n(x) = \frac{x\beta_t(y_t^n, x)}{1 + n^{-\theta}|\beta_t(y_t^n, x)|} \leq x\beta_t(y_t^n, x) \leq L(M_n + |x|^2)$$

where $M_n = 1 + |y_t^n|$ which satisfy $EM_n^{\frac{p^*}{2} + 1} < \infty$ due to inductive assumption. Thus (4.11) holds for $i = r + 1$. This completes the proof. \hfill \Box

### 4.3.2 Convergence in $\mathcal{L}^q$

We now proceed to prove the strong convergence of the tamed Euler scheme (4.25).

**Theorem 4.4.** Let Assumptions C-3, C-4, C-6 to C-10 be satisfied. Also suppose that Remark 4.4 holds. Then, the tamed Euler scheme (4.25) converges to the solution of SDDE (4.1) in $\mathcal{L}^q$, i.e.

$$\lim_{n \to \infty} E \sup_{0 \leq t \leq T} |x_t^n - x_t|^q = 0$$

for any $q < p^*$.

**Proof.** The proof of the theorem adopts similar line of arguments as adopted in the proof of Theorem 4.2. Hence, we only outline the differences in their proofs. Notice that Assumptions A-4, A-6 to A-9, B-1, B-3, B-5, B-8 and B-9 are already verified during the proofs of Lemmas [4.2, 4.5] and Theorem 4.2. Thus, one verifies Assumption B-4 only for the drift coefficient.

**Case $t \in [0, h]$.** As before, SDDE (4.1) can be regarded as a special case of SDE (2.1) with $t_0 = 0$, $t_1 = h$, $x_{t_0} = \xi_0$ and coefficients given by equation (4.6). Similarly, scheme (4.25) can be regarded as a special case of scheme (3.1) with $t_0 = 0$, $t_1 = h$, $x_{t_0} = \xi_0$ and diffusion and jump coefficients given by (4.12) whereas drift coefficient by (4.26). By noticing that $y_t = y_t^n =: \Phi_t$ for $t \in [0, h]$, so for every $R > 0$,

$$E \int_0^h I_{B(R)} \sup_{|x| \leq R} |b_t^n(x) - b_t(x)|^2 dt$$

$$= E \int_0^h I_{B(R)} \sup_{|x| \leq R} \left| \frac{\beta_t(y_t^n, x)}{1 + n^{-\theta}|\beta_t(y_t^n, x)|} - \beta_t(y_t, x) \right|^2 dt$$

$$= E \int_0^h I_{B(R)} \sup_{|x| \leq R} \left| \frac{\beta_t(\Phi_t, x)}{1 + n^{-\theta}|\beta_t(\Phi_t, x)|} - \beta_t(\Phi_t, x) \right|^2 dt$$

$$\leq n^{-2\theta} E \int_0^h I_{B(R)} \sup_{|x| \leq R} \left| \beta_t(\Phi_t, x) \right|^4 dt$$
and due to Remark 4.4 and Assumption C-4, one obtains
\[ E \int_{0}^{h} I_{B(R)} \sup_{|x| \leq R} |b^n_t(x) - b_t(x)|^2 dt \leq Kn^{-2\theta}E \int_{0}^{h} I_{B(R)} (1 + |\Psi_t|^{2\chi} + R^4) dt \]
\[ \leq Kn^{-2\theta}(1 + E|\Psi_t|^{2\chi} + R^4) \to 0 \text{ as } n \to \infty \]
for every $R > 0$. Therefore, by the application of Theorem 3.3, one obtains (4.15) for $i = 1$. For inductive arguments, let (4.15) holds for $i = r$, then one needs to prove (4.15) for $i = r + 1$.

Case $t \in [rh, (r+1)h]$. As before, SDDE (4.1) can be regarded as a special case of SDE (2.1) with $t_0 = rh$, $t_1 = (r+1)h$, $x_{t_0} = x_{rh}$ and coefficients given by equation (4.9). Similarly, scheme (4.25) can be regarded as a special case of scheme (3.1) with $t_0 = rh$, $t_1 = (r+1)h$, $x_{t_0} = x_{rh}$ and diffusion and jump coefficients given by (4.14) whereas drift coefficient by (4.27). Further,
\[ E \int_{rh}^{(r+1)h} I_{B(R)} \sup_{|x| \leq R} |b^n_t(x) - b_t(x)|^2 dt \]
\[ \leq 2E \int_{0}^{h} I_{B(R)} \sup_{|x| \leq R} \frac{\beta_t(y^n_t, x)}{1 + n^{-\theta}|\beta_t(y^n_t, x)|} - \beta_t(y^n_t, x) dt \]
\[ + 2E \int_{0}^{h} I_{B(R)} \sup_{|x| \leq R} |\beta_t(y^n_t, x) - \beta_t(y_t, x)|^2 dt \]
\[ =: T_1 + T_2 \]
for every $R > 0$. For $T_1$, one uses Remark 4.4 and Lemma 4.5 to obtain,
\[ T_1 := 2E \int_{0}^{h} I_{B(R)} \sup_{|x| \leq R} \frac{\beta_t(y^n_t, x)}{1 + n^{-\theta}|\beta_t(y^n_t, x)|} - \beta_t(y^n_t, x) dt \]
\[ \leq n^{-2\theta} K \left( 1 + E \sup_{rh \leq t \leq (r+1)h} |y^n_t|^{2\chi} + R^4 \right) \to 0 \text{ as } n \to \infty \]
for every $R > 0$. For $T_2$, one observes $y^n_t \to y_t$ in probability as $n \to \infty$ due to inductive assumption. Also, Assumption C-10 implies $\beta_t(y^n_t, x) \to \beta_t(y_t, x)$ in probability as $n \to \infty$. Moreover, one observes that due to Remark 4.4, Lemmas [4.2, 4.5], the sequence
\[ \left\{ \int_{rh}^{(r+1)h} I_{B(R)} \sup_{|x| \leq R} |\beta_t(y^n_t, x) - \beta_t(y_t, x)|^2 dt \right\}_{n \in \mathbb{N}} \]
is uniformly integrable which further implies
\[ T_2 := E \int_{rh}^{(r+1)h} I_{B(R)} \sup_{|x| \leq R} |\beta_t(y^n_t, x) - \beta_t(y_t, x)|^2 dt \to 0 \text{ as } n \to \infty \]
for every $R > 0$. Therefore, by the application of Theorem 3.3, one obtains (4.15) for $i = r + 1$ which completes the proof.

4.3.3 Rate of Convergence

For the rate of convergence of tamed Euler scheme (4.25), one can take $\theta = \frac{1}{2}$.

**Theorem 4.5.** Let Assumptions C-4, C-6, C-7, C-9, C-11 and C-12 be satisfied. Then,
the tamed Euler scheme (4.25) of SDDE (4.1) converges in $\mathcal{L}^q$ with rate arbitrarily close to $1/q$, i.e.

$$E \sup_{0 \leq t \leq T} |x_t^n - x_t^n|^q \leq Kn^{-\frac{q}{q+N_2}}$$

for any $q < p^\ast$.

Proof. The theorem can be proved by adopting the same arguments as used in the proof of Theorem 4.3. Thus we outline only the differences between them. Here one can use Theorem 3.4 inductively to obtain the desired rate given by (4.23). Further, one only requires to verify Assumption B-6 for the drift coefficient.

Case $t \in [0,h]$. As before, one considers SDDE (4.1) as a special case of SDE (2.1) with $t_0 = 0$, $t_1 = h$, $x_{t_0} = \xi_0$ and coefficients given by equation (4.6). Also, scheme (4.25) can be considered as a special case of scheme (3.1) with $t_0 = 0$, $t_1 = h$, $x_{t_0} = \xi_0$ and diffusion and jump coefficients given by (4.12) whereas drift coefficient by (4.26). Again notice that $y_t = y_t^n =: \Phi_t$ for $t \in [0,h]$ which implies

$$E \int_0^h |b_t^n(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)})|^q dt$$

$$= E \int_0^h \frac{\beta_t(\Phi_t, x^n_{\kappa(n,t)})}{1 + n^{-\theta}|\beta_t(\Phi_t, x^n_{\kappa(n,t)})|} - \beta_t(\Phi_t, x^n_{\kappa(n,t)})|^q dt$$

$$\leq n^{-\theta q}E \int_0^h \beta_t(\Phi_t, x^n_{\kappa(n,t)})|^{2q} dt$$

which on using Remark 4.4, Assumption C-4 and Lemma 4.5 gives

$$E \int_0^h |b_t^n(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)})|^q dt$$

$$\leq n^{-\theta q} K(1 + E\psi(x+2)^q + E \sup_{0 \leq t \leq h} |x^n_{\kappa(n,t)}|^{(x+2)^q}) \leq Kn^{-\frac{q}{2}}$$

for any $q < p_1$ because $\theta = \frac{1}{2}$. Thus, by Theorem 3.4, one obtains that equation (4.23) holds for $i = 1$. For inductive arguments, one assumes that equation (4.23) holds for $i = r$ and then verifies it for $i = 1 + r$.

Case $t \in [r h, (r + 1)h]$. Again, consider SDDE (4.1) as a special case of SDE (2.1) with $t_0 = rh$, $t_1 = (r+1)h$, $x_{t_0} = x_{rh}$ and coefficients given by equation (4.9). Similarly, consider scheme (4.25) as a special case of scheme (3.1) with $t_0 = rh$, $t_1 = (r+1)h$, $x_{t_0} = x_{rh}$, diffusion and jump coefficients given by (4.14) whereas drift coefficient by (4.26). Now,

$$E \int_0^h |b_t^n(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)})|^q dt$$

$$\leq KE \int_0^h \frac{\beta_t(y_t^n, x^n_{\kappa(n,t)})}{1 + n^{-\theta}|\beta_t(y_t^n, x^n_{\kappa(n,t)})|} - \beta_t(y_t^n, x^n_{\kappa(n,t)})|^q dt$$

$$+ KE \int_0^h \beta_t(y^n_t, x^n_{\kappa(n,t)}) - \beta_t(y_t, x^n_{\kappa(n,t)})|^q dt$$

$$\leq Kn^{-\theta q}E \int_0^h |\beta_t(y^n_t, x^n_{\kappa(n,t)})|^{2q} dt + KE \int_0^h |\beta_t(y_t^n, x^n_{\kappa(n,t)}) - \beta_t(y_t, x^n_{\kappa(n,t)})|^q dt$$

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which on the application of Remark 4.4 and Assumption C-12 gives

\[ \mathbb{E} \int_0^h \left| b^n_t(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)}) \right|^q dt \leq K n^{-q\theta} \mathbb{E} \int_0^h (1 + |y_t^n|^{(x+2)q} + |x^n_{\kappa(n,t)}|^{(x+2)q}) dt \]

\[ + K \mathbb{E} \int_0^h (1 + |y_t^n|^{\frac{q}{2}} + |y_t^n|^{\frac{q}{2}}) |y_t - y_t^n|^q dt \]

and then on the application of Hölder’s inequality and Lemmas [4.2, 4.5], one obtains

\[ \mathbb{E} \int_0^h \left| b^n_t(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)}) \right|^q dt \leq K n^{-q\theta} + K \mathbb{E} \int_0^h (E|y_t - y_t^n|^{q+\delta})^{\frac{q}{q+\delta}} dt. \]

Finally, on using the inductive assumption and \( \theta = \frac{1}{2} \),

\[ \mathbb{E} \int_0^h \left| b^n_t(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)}) \right|^q dt \leq K n^{-\frac{q}{2}} + K n^{-\frac{q}{2(\gamma+1)}} \]

and hence (4.23) holds for \( i = r + 1 \). This completes the proof. \( \square \)
Chapter 5

Tamed Milstein Scheme of SDE Driven by Lévy Noise

In this chapter, we introduce a tamed Milstein scheme for SDE driven by Lévy noise with super-linearly growing drift coefficient. It is important to note that the application of Itô’s formula on the random coefficients can lead to anticipative integrals, which brings complexity in the analysis of numerical schemes of rate higher than 0.5. One can refer to [26] for delay equations, which is a special case of SDE with random coefficients. Thus, we restrict ourselves to the case, when the coefficients of SDE driven by Lévy noise are non-random in order to discuss its tamed Milstein scheme. This chapter is based on the results obtained in [40] and the methodologies developed herein can be extended to higher order schemes. However, for simplicity, we only consider tamed Milstein scheme. To the best of authors’ knowledge, this is the first result on 1.0 order scheme for SDE driven by Lévy noise with super-linear drift coefficient. Throughout this chapter, intensity measure is assumed to be finite i.e., $\nu(Z) < \infty$.

5.1 SDE Driven by Lévy Noise

Let $b(x)$ and $\sigma(x)$ be $\mathcal{B}(\mathbb{R}^d)$-measurable functions with values in $\mathbb{R}^d$ and $\mathbb{R}^{d \times m}$ respectively. Also, let $\gamma(x, z)$ be a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{I}$-measurable function with values in $\mathbb{R}^d$. Moreover, $b(x)$, $\sigma(x)$ and $\gamma(x, z)$ are assumed to be twice differentiable functions in $x \in \mathbb{R}^d$. We consider the following $d$-dimensional SDE on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}, P)$,

$$
x_t = \xi + \int_0^t b(x_s)ds + \int_0^t \sigma(x_s)dw_s + \int_0^t \int_Z \gamma(x_s, z)\tilde{N}(ds, dz),
$$

(5.1)

almost surely for any $t \in [0, T]$, where $\xi$ is an $\mathcal{F}_0$-measurable random variable in $\mathbb{R}^d$.

Remark 5.1. As before, on the right hand side of the above equation (5.1), we write $x_t$ instead of $x_{t-}$ in order to ease the notation. This does not cause any problem, since the compensator of the martingales driving the equation are continuous. This notational convenience shall be followed throughout this chapter. One also notices that SDE (5.1) can be regarded as a special case of SDE (2.1).

5.1.1 Existence and Uniqueness

The existence and uniqueness for the solution to SDE (5.1) follows under the following assumptions.
D-1. There exists a constant $L > 0$ such that

$$xb(x) + |\sigma(x)|^2 + \int_Z |\gamma(x, z)|^2 \nu(dz) \leq L(1 + |x|^2)$$

for any $x \in \mathbb{R}^d$.

D-2. For every $R > 0$, there exists a constant $L_R > 0$ such that

$$(x - \bar{x})(b(x) - b(\bar{x})) + |\sigma(x) - \sigma(\bar{x})|^2 + \int_Z |\gamma(x, z) - \gamma(\bar{x}, z)|^2 \nu(dz) \leq L_R|x - \bar{x}|^2$$

for any $|x|, |\bar{x}| \leq R$.

D-3. Let $b(x)$ be a continuous function in $x \in \mathbb{R}^d$.

The proof of the following theorem can be found in [24]

**Theorem 5.1.** Let Assumptions D-1 to D-3 be satisfied. Then, there exists a unique solution for SDE (5.1).

### 5.1.2 Moment Bounds

For moment bounds of solution of SDE (5.1), one replaces Assumption D-1 with the following assumptions.

D-4. For a fixed $p \geq 2$, $E|\xi|^p < \infty$.

D-5. There exists a constant $L > 0$ such that

$$xb(x) \vee |\sigma(x)|^2 \vee \int_Z |\gamma(x, z)|^2 \nu(dz) \leq L(1 + |x|^2)$$

for any $x \in \mathbb{R}^d$.

D-6. There exits a constant $L > 0$ such that

$$\int_Z |\gamma(x, z)|^p \nu(dz) \leq L(1 + |x|^p)$$

for any $x \in \mathbb{R}^d$.

The following well-known result on moment bounds also follows from Lemma 2.2.

**Lemma 5.1.** Let Assumptions D-2 to D-6 be satisfied. Then, there exists a unique solution to SDE (5.1) and moreover

$$E \sup_{0 \leq t \leq T} |x_t|^p < K$$

where $K := K(L, T, p, m, d, E|\xi|^p)$.

### 5.2 Tamed Milstein Scheme

For every $n \in \mathbb{N}$, we propose the following form for taming the super-linearly growing drift coefficient of SDE (5.1),

$$\tilde{b}^n(x) = \frac{b(x)}{1 + n^{-\theta} |b(x)|^{2g(1+\chi)}} \quad (5.2)$$
for any $\theta \geq \frac{1}{2}$, and $x \in \mathbb{R}^d$ where $\lambda > 0$ comes from Assumption D-9 (stated below). One notes that $\tilde{b}^n$ is an $\mathbb{R}^d$-valued function and its $i$-th element is denoted by $\tilde{b}^{n,i}$ for $i = 1, \ldots, d$.

**Remark 5.2.** One observes that when $\theta = \frac{1}{2}$, then we obtain tamed Euler schemes similar to those discussed in [16, 59]. In this article, we discuss a tamed Milstein scheme of SDE (5.1) by taking $\theta = 1$. It is important to note that by assigning different values to $\theta$ and appropriately including multiple stochastic integrals in the scheme, one could write a tamed scheme and then perform calculations similar to the methodology developed in this article to achieve an order $\theta$. For the purpose of this article, we only discuss the case when $\theta = 1$ i.e. the tamed Milstein scheme of Lévy driven SDE with super-linear drift coefficient. From now onward, throughout this article, we take $\theta = 1$.

**Remark 5.3.** For every $n \in \mathbb{N}$, equation (5.2) implies

$$|\tilde{b}^n(x)| \leq \min(n^{1/2}, |b(x)|)$$

for any $x \in \mathbb{R}^d$ which is in the same spirit as in [16, 59].

For every $n \in \mathbb{N}$, we propose the following tamed Milstein scheme,

$$x^n_t = x^n_0 + \int_0^t \tilde{b}^n(x^n_{\kappa(n,s)})ds + \int_0^t \tilde{\sigma}^n(x^n_{\kappa(n,s)})dw_s + \int_0^t \int_z \tilde{\gamma}(x^n_{\kappa(n,s)}, z_2)N(ds, dz_2)$$

(5.3)

almost surely for any $t \in [0,T]$ where $\kappa(n,t) := [nt]/n$. The drift coefficient $\tilde{b}^n$ of scheme (5.3) is defined in equation (5.2) whereas diffusion $\tilde{\sigma}$ and jump $\tilde{\gamma}$ coefficients of the scheme (5.3) are defined below. For every $n \in \mathbb{N}$ and $s \in [0,T]$, the diffusion coefficient $\tilde{\sigma}(x^n_{\kappa(n,s)})$ of scheme (5.3) is given by

$$\tilde{\sigma}(x^n_{\kappa(n,s)}) := \sigma(x^n_{\kappa(n,s)}) + \sigma_1(x^n_{\kappa(n,s)}) + \sigma_2(x^n_{\kappa(n,s)}) + \sigma_3(x^n_{\kappa(n,s)})$$

(5.4)

where $\sigma_1(x^n_{\kappa(n,s)})$, $\sigma_2(x^n_{\kappa(n,s)})$ and $\sigma_3(x^n_{\kappa(n,s)})$ are $d \times m$ matrices with their $(i,k)$-th elements given by

$$\sigma_1^{(i,k)}(x^n_{\kappa(n,s)}) := \sum_{j=1}^m \sum_{u=1}^d \int_{\kappa(n,s)} \sigma^{(i,j)}(x^n_{\kappa(n,r)}) \frac{\partial \sigma^{(i,k)}(x^n_{\kappa(n,r)})}{\partial x^u} dw_r$$

(5.5)

$$\sigma_2^{(i,k)}(x^n_{\kappa(n,s)}) := \sum_{u=1}^d \int_{\kappa(n,s)} \int_Z \frac{\partial \sigma^{(i,k)}(x^n_{\kappa(n,r)})}{\partial x^u} \gamma^u(x^n_{\kappa(n,r)}, z_1) N(dr, dz_1)$$

$$\sigma_3^{(i,k)}(x^n_{\kappa(n,s)}) := \int_{\kappa(n,s)} \int_Z \left( \sigma^{(i,k)}(x^n_{\kappa(n,r)}) + \gamma(x^n_{\kappa(n,r)}, z_1) \right) N(dr, dz_1)$$

- \sum_{u=1}^d \frac{\partial \sigma^{(i,k)}(x^n_{\kappa(n,r)})}{\partial x^u} \gamma^u(x^n_{\kappa(n,r)}, z_1) N(dr, dz_1)

respectively, for every $i = 1, \ldots, d$, $k = 1, \ldots, m$. Similarly, for every $n \in \mathbb{N}$, $s \in [0,T]$ and $z_2 \in Z$, the jump coefficient $\tilde{\gamma}(x^n_{\kappa(n,s)}, z_2)$ of scheme (5.3) is given by

$$\tilde{\gamma}(x^n_{\kappa(n,s)}, z_2) := \gamma(x^n_{\kappa(n,s)}, z_2) + \gamma_1(x^n_{\kappa(n,s)}, z_2) + \gamma_2(x^n_{\kappa(n,s)}, z_2) + \gamma_3(x^n_{\kappa(n,s)}, z_2)$$

(5.6)

where $\gamma_1(x^n_{\kappa(n,s)}, z_2)$, $\gamma_2(x^n_{\kappa(n,s)}, z_2)$ and $\gamma_3(x^n_{\kappa(n,s)}, z_2)$ are $d$-dimensional vectors with
their $i$-th elements given by,

\[
\gamma_i^i(x^n_{\kappa(s)}, z_2) := \sum_{j=1}^{m} \sum_{u=1}^{d} \int_{\kappa(s)}^{s} \frac{\partial \gamma_i^i(x^n_{\kappa(r)}, z_2)}{\partial x^u} \sigma^{(u,j)}(x^n_{\kappa(r)}) d\omega^j \\
\gamma_2^i(x^n_{\kappa(s)}, z_2) := \sum_{u=1}^{d} \int_{\kappa(s)}^{s} \int_{Z} \frac{\partial \gamma_i^i(x^n_{\kappa(r)}, z_2)}{\partial x^u} \gamma_i^u(x^n_{\kappa(r)}, z_1) \tilde{N}(dr, dz) \\
\gamma_3^i(x^n_{\kappa(s)}, z_2) := \int_{\kappa(s)}^{s} \int_{Z} \left( \gamma_i^i(x^n_{\kappa(r), z_1}), z_2) - \gamma_i^i(x^n_{\kappa(r), z_2}) - \sum_{u=1}^{d} \frac{\partial \gamma_i^i(x^n_{\kappa(r)}, z_2)}{\partial x^u} \gamma_i^u(x^n_{\kappa(r), z_1}) \right) N(dr, dz)
\]

respectively, for every $i = 1, \ldots, d$.

### 5.2.1 Moment Bounds

For the moment bounds of the scheme (5.3), one replaces Assumption D-5 with the following assumptions.

**D-7.** There exists a constant $L > 0$ such that

\[
(x - \bar{x})(b(x) - b(\bar{x})) \vee |\sigma(x) - \sigma(\bar{x})|^2 \vee \int_{Z} |\gamma(x, z) - \gamma(\bar{x}, z)|^2 \nu(dz) \leq L|x - \bar{x}|^2
\]

for any $x, \bar{x} \in \mathbb{R}^d$.

**D-8.** There exists a constant $L > 0$ such that

\[
\left| \frac{\partial \sigma^{(i,j)}(x)}{\partial x^k} - \frac{\partial \sigma^{(i,j)}(\bar{x})}{\partial x^k} \right|^2 \vee \int_{Z} \left| \frac{\partial \gamma_i^i(x, z)}{\partial x^k} - \frac{\partial \gamma_i^i(\bar{x}, z)}{\partial x^k} \right|^2 \nu(dz) \leq L|x - \bar{x}|^2
\]

for any $x, \bar{x} \in \mathbb{R}^d$ and for all $i, k = 1, \ldots, d$ and $j = 1, \ldots, m$.

**Remark 5.4.** Assumption D-5 holds due to Assumption D-7.

**Remark 5.5.** Due to Assumptions D-7 and D-8, there exists a constant $L > 0$ such that

\[
\left| \frac{\partial \sigma^{(i,j)}(x)}{\partial x^k} \right| \vee \left| \frac{\partial^2 \gamma_i^i(x, z)}{\partial x^k \partial x^u} \right| \vee \int_{Z} \left| \frac{\partial \gamma_i^i(x, z)}{\partial x^k} \right| \nu(dz) \vee \int_{Z} \left| \frac{\partial^2 \gamma_i^i(x, z)}{\partial x^k \partial x^u} \right| \nu(dz) \leq L
\]

for any $x \in \mathbb{R}^d$, $i, k, u = 1, \ldots, d$ and $j = 1, \ldots, m$.

One observes that due to Remark 5.3, for a fixed $n \in \mathbb{N}$, the drift coefficient of the scheme (5.3) is bounded. Also, the diffusion and jump coefficients grow linearly due to Remark 5.4. Hence, one can refer to [54] to conclude that Assumptions D-4, D-6 and D-7 along with equation (5.2) imply that, for a fixed $n \in \mathbb{N}$,

\[
E \sup_{0 \leq t \leq T} |x^n_t|^p < \infty. \tag{5.7}
\]

Clearly, one can not claim at this stage that the bound is independent of $n$. However, it guarantees that all local martingales appearing henceforth are in fact true martingales and thus the use of stopping time arguments is avoided. Before proving the moment bounds of the scheme (5.3) in Lemma 5.9, one requires to show the following lemmas.
Lemma 5.2. Let Assumption D-7 holds, then

\[ E|\sigma_1(x_{n,r})|^p \leq Kn^{-\frac{p}{2}}(1 + E|x_{n,r}|^p), \]

for every \( n \in \mathbb{N} \) and \( s \in [0, T] \), where \( K := K(L, p, m, d) \).

Proof. First one writes,

\[
E|\sigma_1(x_{n,r})|^p = E\left( \sum_{k=1}^{m} \sum_{i=1}^{d} |\sigma_1^{(i,k)}(x_{n,r})|^2 \right)^{\frac{p}{2}} \\
\leq K \sum_{k,j=1}^{m} \sum_{i,u=1}^{d} E \int_{s}^{n} |\sigma_{(u,j)}(x_{n,r})| \frac{\partial(\sigma^{(i,k)}(x_{n,r}))}{\partial x_i} dw_j^p \\
\]

which on the application of an elementary inequality for stochastic integrals and H"older’s inequality yields

\[
E|\sigma_1(x_{n,r})|^p \leq K \sum_{k,j=1}^{m} \sum_{i,u=1}^{d} E \left( \int_{s}^{n} |\sigma_{(u,j)}(x_{n,r})| \frac{\partial(\sigma^{(i,k)}(x_{n,r}))}{\partial x_i} dr \right)^{\frac{p}{2}} \\
\leq Kn^{-\left(\frac{p}{2}-1\right)} \sum_{k,j=1}^{m} \sum_{i,u=1}^{d} E \int_{s}^{n} |\sigma_{(u,j)}(x_{n,r})|^p \left| \frac{\partial(\sigma^{(i,k)}(x_{n,r}))}{\partial x_i} \right| dr \\
\]

and then due to Remarks [5.4, 5.5], one obtains

\[
E|\sigma_1(x_{n,r})|^p \leq Kn^{-\left(\frac{p}{2}-1\right)} E \int_{s}^{n} (1 + |x_{n,r}|^p) dr = Kn^{-\frac{p}{2}}(1 + E|x_{n,r}|^p) \\
\]

for every \( n \in \mathbb{N} \) and \( s \in [0, T] \). This completes the proof. \( \square \)

Lemma 5.3. Let Assumptions D-6 and D-7 hold, then

\[ E|\sigma_2(x_{n,r})|^p \leq Kn^{-1}(1 + E|x_{n,r}|^p), \]

for every \( n \in \mathbb{N} \) and \( s \in [0, T] \), where \( K := K(L, p, m, d) \).

Proof. One notes that,

\[
E|\sigma_2(x_{n,r})|^p = E\left( \sum_{i=1}^{m} \sum_{k=1}^{d} |\sigma_2^{(i,k)}(x_{n,r})|^2 \right)^{\frac{p}{2}} \leq K \sum_{i=1}^{m} \sum_{k=1}^{d} E|\sigma_2^{(i,k)}(x_{n,r})|^p \\
\leq K \sum_{i,u=1}^{d} \sum_{k=1}^{m} E \left| \int_{s}^{n} \int_{Z} \frac{\partial(\sigma^{(i,k)}(x_{n,r}))}{\partial x_i} \gamma^u(x_{n,r}, z_1) \tilde{N}(dr, dz_1) \right|^p \\
\]

which due to Lemma 1.5 yields

\[
E|\sigma_2(x_{n,r})|^p \leq K \sum_{i,u=1}^{d} \sum_{k=1}^{m} E \left( \int_{s}^{n} \left| \int_{Z} \frac{\partial(\sigma^{(i,k)}(x_{n,r}))}{\partial x_i} \gamma^u(x_{n,r}, z_1) \right|^2 \nu(dz_1) dr \right)^{\frac{p}{2}} \\
+ K \sum_{i,u=1}^{d} \sum_{k=1}^{m} E \left( \int_{s}^{n} \left| \int_{Z} \frac{\partial(\sigma^{(i,k)}(x_{n,r}))}{\partial x_i} \gamma^u(x_{n,r}, z_1) \right|^p \nu(dz_1) dr \right) \\
\]

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and then on using Remarks [5.4, 5.5] and Assumption D-6, one obtains
\[
E|\sigma_2(x_{\kappa(n,s)}^n)|^p \leq Kn^{-\frac{p}{2}} (1 + E|x_{\kappa(n,s)}^n|^p) + Kn^{-1} (1 + E|x_{\kappa(n,s)}^n|^p)
\]
for every \( n \in \mathbb{N} \) and \( s \in [0, T] \). This finishes the proof.

**Lemma 5.4.** Let Assumptions D-6 to D-8 hold, then
\[
E|\sigma_3(x_{\kappa(n,s)}^n)|^p \leq Kn^{-1} (1 + E|x_{\kappa(n,s)}^n|^p),
\]
for every \( n \in \mathbb{N} \) and \( s \in [0, T] \), where \( K := K(L, p, m, d) \).

**Proof.** By using Assumption D-7 and Remark 5.5, one obtains
\[
E|\sigma_3(x_{\kappa(n,s)}^n)|^p = E\left( \sum_{i=1}^{d} \sum_{k=1}^{m} |\sigma_3^{(i,k)}(x_{\kappa(n,s)}^n)|^2 \right)^{\frac{p}{2}}
\]
\[
\leq KE \sum_{i=1}^{d} \sum_{k=1}^{m} \left[ \int_{\kappa(n,s)}^{s} \int_{Z} \left| \sigma_3^{(i,k)}(x_{\kappa(n,r)}^n + \gamma(x_{\kappa(n,r)}^n, z_1)) - \sigma_3^{(i,k)}(x_{\kappa(n,r)}^n) \right|
\right. \\
+ \left. \sum_{u=1}^{d} \left| \frac{\partial \sigma_3^{(i,k)}(x_{\kappa(n,r)}^n)}{\partial x_u} \right| \left| \gamma(x_{\kappa(n,r)}^n, z_1) \right| \right] N(dr, dz_1)^p
\]
\[
\leq KE \left( \int_{\kappa(n,s)}^{s} \int_{Z} |\gamma(x_{\kappa(n,r)}^n, z_1)| N(dr, dz_1)^p \right)^{\frac{p}{2}}
\]
which due to Lemma 1.5 gives
\[
E|\sigma_3(x_{\kappa(n,s)}^n)|^p \leq KE \left( \int_{\kappa(n,s)}^{s} \int_{Z} |\gamma(x_{\kappa(n,r)}^n, z_1)| \tilde{\nu}(dr, dz_1)^p \right)^{\frac{p}{2}}
\]
\[
+ KE \left( \int_{\kappa(n,s)}^{s} \int_{Z} |\gamma(x_{\kappa(n,r)}^n, z_1)| \nu(dz_1)dr \right)^{\frac{p}{2}}
\]
and this further implies the following
\[
E|\sigma_3(x_{\kappa(n,s)}^n)|^p \leq KE \left( \int_{\kappa(n,s)}^{s} \int_{Z} |\gamma(x_{\kappa(n,r)}^n, z_1)|^2 \nu(dz_1)dr \right)^{\frac{p}{2}}
\]
\[
+ KE \int_{\kappa(n,s)}^{s} \int_{Z} |\gamma(x_{\kappa(n,r)}^n, z_1)|^p \nu(dz_1)dr
\]
\[
+ KE \left( \int_{\kappa(n,s)}^{s} \int_{Z} |\gamma(x_{\kappa(n,r)}^n, z_1)| \nu(dz_1)dr \right)^{\frac{p}{2}}
\]
and then on applying Remark 5.4 and Assumption D-6, one obtains
\[
E|\sigma_3(x_{\kappa(n,s)}^n)|^p \leq Kn^{-\frac{p}{2}} (1 + E|x_{\kappa(n,s)}^n|^p) + Kn^{-1} (1 + E|x_{\kappa(n,s)}^n|^p)
\]
\[
+ Kn^{-p} (1 + E|x_{\kappa(n,s)}^n|^p)
\]
for every \( n \in \mathbb{N} \) and \( s \in [0, T] \). This completes the proof.

The following corollary is a consequence of Lemmas [5.2, 5.3, 5.4].

**Corollary 3.** Let Assumptions D-6 to D-8 hold, then
\[
E|\tilde{\sigma}(x_{\kappa(n,s)}^n)|^p \leq K (1 + E|x_{\kappa(n,s)}^n|^p),
\]
for every \( n \in \mathbb{N} \) and \( s \in [0,T] \), where \( K := K(L, p, m, d) \).

**Lemma 5.5.** Let Assumptions D-6 to D-8 hold, then

\[
E \int_{Z} |\gamma_1(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) \leq Kn^{-\frac{p}{2}}(1 + E|x_{\kappa(n,s)}^n|^p),
\]

for every \( n \in \mathbb{N} \) and \( s \in [0,T] \), where \( K := K(L, p, m, d) \).

**Proof.** One observes that

\[
E \int_{Z} |\gamma_1(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) = E \int_{Z} \left( \sum_{i=1}^{d} |\gamma_i^1(x_{\kappa(n,s)}^n, z_2)|^2 \right)^{\frac{p}{2}} \nu(dz_2)
\]

\[
\leq K \sum_{i=1}^{d} \int_{Z} \sum_{j=1}^{m} \sum_{u=1}^{d} E \left| \int_{\kappa(n,s)}^{s} \frac{\partial \gamma_i^1(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \sigma^{(u,j)}(x_{\kappa(n,r)}^n) dw_j^p \right|^p \nu(dz_2)
\]

which on using an elementary inequality for stochastic integrals and Hölder’s inequality implies

\[
E \int_{Z} |\gamma_1(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2)
\]

\[
\leq K \sum_{i=1}^{d} \sum_{j=1}^{m} \int_{Z} E \left( \int_{\kappa(n,s)}^{s} \left| \frac{\partial \gamma_i^1(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \sigma^{(u,j)}(x_{\kappa(n,r)}^n) \right|^2 dw_j^p \right)^{\frac{p}{2}} \nu(dz_2)
\]

\[
\leq Kn^{-\frac{p}{2}} \sum_{i=1}^{d} \sum_{j=1}^{m} \int_{Z} E \left| \int_{\kappa(n,s)}^{s} \frac{\partial \gamma_i^1(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \sigma^{(u,j)}(x_{\kappa(n,r)}^n) \right|^p dw_j^p \nu(dz_2)
\]

and then due to Remarks [5.4, 5.5], one obtains

\[
E \int_{Z} |\gamma_1(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) \leq Kn^{-\frac{p}{2}}(1 + E|x_{\kappa(n,s)}^n|^p)
\]

for every \( n \in \mathbb{N} \) and \( s \in [0,T] \). Hence the proof follows. \hfill \( \Box \)

**Lemma 5.6.** Let Assumptions D-6 to D-8 hold, then

\[
E \int_{Z} |\gamma_2(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) \leq Kn^{-1}(1 + E|x_{\kappa(n,s)}^n|^p),
\]

for every \( n \in \mathbb{N} \) and \( s \in [0,T] \), where \( K := K(L, p, m, d) \).

**Proof.** One observes that

\[
E \int_{Z} |\gamma_2(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) = E \int_{Z} \left( \sum_{i=1}^{d} |\gamma_i^2(x_{\kappa(n,s)}^n, z_2)|^2 \right)^{\frac{p}{2}} \nu(dz_2)
\]

\[
\leq K \int_{Z} \sum_{i=1}^{d} E \left| \int_{\kappa(n,s)}^{s} \frac{\partial \gamma_i^1(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \gamma^{(u)}(x_{\kappa(n,r)}^n, z_1) \tilde{N}(dr, dz_1) \right|^p \nu(dz_2)
\]
which due to Lemma 1.5 yields

\[ E \int_Z |\gamma_2(x^n_{(n,s)}, z_2)|^p \nu(dz_2) \leq K \int_Z \sum_{i,u=1}^d E \left( \int_{\kappa(n,s)}^s \int_Z \left| \frac{\partial \gamma^i(x^n_{(n,r)}, z_2)}{\partial x^u} \gamma^n(x^n_{(n,r)}, z_1) \right|^2 \nu(dz_1)dr \right)^{\frac{p}{2}} \nu(dz_2) + K \int_Z \sum_{i,u=1}^d E \left( \int_{\kappa(n,s)}^s \int_Z \left| \frac{\partial \gamma^i(x^n_{(n,r)}, z_2)}{\partial x^u} \gamma^n(x^n_{(n,r)}, z_1) \right|^p \nu(dz_1)dr \nu(dz_2) \right) \]

and then on using Remarks [5.4, 5.5] and Assumption D-6, one obtains,

\[ E \int_Z |\gamma_2(x^n_{(n,s)}, z_2)|^p \nu(dz_2) \leq Kn^{-\frac{p}{2}} (1 + E|x^n_{(n,s)}|^p) + Kn^{-1} (1 + E|x^n_{(n,s)}|^p) \]

for every \( n \in \mathbb{N} \) and \( s \in [0, T] \). Thus the proof finishes.

**Lemma 5.7.** Let Assumptions D-6 to D-8 hold, then

\[ E \int_Z |\gamma_3(x^n_{(n,s)}, z_2)|^p \nu(dz_2) \leq K n^{-1} (1 + E|x^n_{(n,s)}|^p), \]

for every \( n \in \mathbb{N} \) and \( s \in [0, T] \), where \( K := K(L, p, m, d) \).

**Proof.** On using Assumption D-7 and Remark 5.5, one writes

\[ E \int_Z |\gamma_3(x^n_{(n,s)}, z_2)|^p \nu(dz_2) \leq KE \left( \sum_{i=1}^d |\gamma^i(x^n_{(n,s)}, z_2)|^2 \right)^{\frac{p}{2}} \nu(dz_2) \]

\[ = KE \sum_{i=1}^d \int_{\kappa(n,s)}^s \int_Z \left( \gamma^i(x^n_{(n,r)} + \gamma(x^n_{(n,r)}, z_1), z_2) - \gamma^i(x^n_{(n,r)}, z_2) \right) \frac{\partial \gamma^i(x^n_{(n,r)}, z_2)}{\partial x^u} \nu(dz_1) \nu(dz_2) \]

\[ \leq KE \sum_{i=1}^d \left( \int_{\kappa(n,s)}^s \int_Z \left| \gamma^i(x^n_{(n,r)} + \gamma(x^n_{(n,r)}, z_1), z_2) - \gamma^i(x^n_{(n,r)}, z_2) \right| \right)^{\frac{p}{2}} \nu(dz_2) \]

which due to Assumption D-6, Lemma 1.5 and Remark 5.4 gives,

\[ E \int_Z |\gamma_3(x^n_{(n,s)}, z_2)|^p \nu(dz_2) \leq K \int_Z E \left( \int_{\kappa(n,s)}^s \int_Z |\gamma(x^n_{(n,r)}, z_1)| N(dr, dz_1) \right)^{\frac{p}{2}} \nu(dz_2) \]

\[ + K \int_Z E \left( \int_{\kappa(n,s)}^s \int_Z |\gamma(x^n_{(n,r)}, z_1)| \nu(dz_1)dr \nu(dz_2) \right) \]

\[ \leq Kn^{-\frac{p}{2}} (1 + E|x^n_{(n,s)}|^p) + Kn^{-1} (1 + E|x^n_{(n,s)}|^p) + Kn^{-p} (1 + E|x^n_{(n,s)}|^p) \]

for every \( n \in \mathbb{N} \) and \( s \in [0, T] \). This completes the proof. \( \square \)
The following corollary is a consequence of Lemmas [5.5, 5.6, 5.7].

**Corollary 4.** Let Assumptions D-6 to D-8 hold, then

\[ E \int _{Z} | \gamma (x_{\kappa (n,s)}^{n}, z_{2}) |^{p} \nu (dz_{2}) \leq K (1 + E | x_{\kappa (n,s)}^{n} |^{p}), \]

for every \( n \in \mathbb{N} \) and \( s \in [0, T] \), where \( K := K(L, p, m, d) \).

**Lemma 5.8.** Let Assumptions D-6 to D-8 hold, then

\[ \int _{0}^{u} E | x_{s}^{n} - x_{\kappa (n,s)}^{n} |^{p} ds \leq Kn^{-1} (1 + \int _{0}^{u} E | x_{\kappa (n,s)}^{n} |^{p} ds), \]

for any \( u \in [0, T] \), \( n \in \mathbb{N} \), where \( K := K(L, p, m, d) \).

**Proof.** Due to Hölder's inequality, Lemma 1.5 and Remark 5.3, one observes that

\[
E | x_{s}^{n} - x_{\kappa (n,s)}^{n} |^{p} = E \left| \int _{\kappa (n,s)}^{s} \tilde{b}^{n}(x_{\kappa (n,r)}^{n}) dr + \int _{\kappa (n,s)}^{s} \sigma (x_{\kappa (n,r)}^{n}) dw_{r} \right| ^{p} \\
+ \int _{\kappa (n,s)}^{s} \int _{Z} \gamma (x_{\kappa (n,r)}^{n}, z_{2}) \tilde{N} (dr, dz_{2}) \right| ^{p} \\
\leq KE \left| \int _{\kappa (n,s)}^{s} \tilde{b}^{n}(x_{\kappa (n,r)}^{n}) dr \right| ^{p} + KE \left| \int _{\kappa (n,s)}^{s} \sigma (x_{\kappa (n,r)}^{n}) dw_{r} \right| ^{p} \\
+ KE \left| \int _{\kappa (n,s)}^{s} \int _{Z} \gamma (x_{\kappa (n,r)}^{n}, z_{2}) \tilde{N} (dr, dz_{2}) \right| ^{p} \\
\leq K | s - \kappa(n,s) |^{p-1} \int _{\kappa(n,s)}^{s} | \tilde{b}^{n}(x_{\kappa(n,r)}^{n}) |^{p} dr \\
+ K | s - \kappa(n,s) |^{p-1} \int _{\kappa(n,s)}^{s} E | \tilde{\sigma}(x_{\kappa(n,r)}^{n}) |^{p} dr \\
+ K | s - \kappa(n,s) |^{p-1} \int _{\kappa(n,s)}^{s} E \left( \int _{Z} | \gamma (x_{\kappa(n,r)}^{n}, z_{2}) |^{2} \nu (dz_{2}) \right)^{\frac{p}{2}} dr \\
+ K \int _{\kappa(n,s)}^{s} E \int _{Z} | \gamma (x_{\kappa(n,r)}^{n}, z_{2}) |^{p} \nu (dz_{2}) dr
\]

and then on using Corollaries [3, 4], one obtains

\[
E | x_{s}^{n} - x_{\kappa (n,s)}^{n} |^{p} \leq Kn^{-\frac{p}{2}} + Kn^{-\frac{p}{2}} (1 + E | x_{\kappa (n,s)}^{n} |^{p}) \\
+ K n^{-\frac{p}{2}} (1 + E | x_{\kappa (n,s)}^{n} |^{p}) \\
+ Kn^{-1} (1 + E | x_{\kappa (n,s)}^{n} |^{p})
\]

for any \( s \in [0, T] \), \( n \in \mathbb{N} \). This completes the proof. \( \square \)

**Lemma 5.9.** Let Assumptions D-4 and D-6 to D-8 hold, then

\[
\sup _{n \in \mathbb{N}} E \sup _{0 \leq t \leq T} | x_{t}^{n} |^{p} \leq K,
\]

where \( K := K(L, T, p, m, d, E | \xi |^{p}) \).
Proof. By Itô’s formula,

\[ |x_t^n|^p = |\xi|^p + p \int_0^t |x_s^n|^{p-2} x_s^n \tilde{b}_s^n(x^n_{\kappa(s),n})ds + p \int_0^t |x_s^n|^{p-2} x_s^n \tilde{\sigma}(x^n_{\kappa(s),n})dw_s \]
\[ + \frac{p(p-2)}{2} \int_0^t |x_s^n|^{p-4} |\tilde{\sigma}(x^n_{\kappa(s),n})x_s^n|^2 ds \]
\[ + \frac{p}{2} \int_{t_0}^t \left( |x_s^n|^{p-2} |\tilde{\sigma}(x^n_{\kappa(s),n})|^2 ds \right) \]
\[ + p \int_0^t \int_Z |x_s^n|^{p-2} x_s^n \tilde{\gamma}(x^n_{\kappa(s),n}, z_2) \tilde{N}(ds, dz_2) \]
\[ + \int_0^t \int_Z \{|x_s^n + \tilde{\gamma}(x^n_{\kappa(s),n}, z_2)|^p - |x_s^n|^p - p|x_s^n|^{p-2} x_s^n \tilde{\gamma}(x^n_{\kappa(s),n}, z_2)|N(ds, dz_2) \right) \]  

(5.8)

almost surely for any \( t \in [0, T] \) and \( n \in \mathbb{N} \). For estimating the second term on the right hand side of the above equation, one writes

\[ x_s^n \tilde{b}_n(x^n_{\kappa(s),n}) = (x_s^n - x^n_{\kappa(s),n}) \tilde{b}_n(x^n_{\kappa(s),n}) + x^n_{\kappa(s),n} \tilde{b}_n(x^n_{\kappa(s),n}) \]

which on using Remark 5.4 gives

\[ x_s^n \tilde{b}_n(x^n_{\kappa(s),n}) \leq \left\{ \int_{\kappa(s)} \tilde{b}_n(x^n_{\kappa(s),r})dr + \int_{\kappa(s)} \tilde{\sigma}(x^n_{\kappa(s),r})dw_r \right\} \]
\[ + \int_{\kappa(s)} \int_Z \tilde{\gamma}(x^n_{\kappa(s),r}, z_2) \tilde{N}(dr, dz_2) \tilde{b}_n(x^n_{\kappa(s),n}) \]
\[ + L(1 + |x^n_{\kappa(s),n}|^2) \]

and then by using Remark 5.3 and Schwartz inequality, one obtains

\[ x_s^n \tilde{b}_n(x^n_{\kappa(s),n}) \leq |\tilde{b}_n(x^n_{\kappa(s),n})| \left| \int_{\kappa(s)} \tilde{b}_n(x^n_{\kappa(s),r})dr + |\tilde{b}_n(x^n_{\kappa(s),n})| \left| \int_{\kappa(s)} \tilde{\sigma}(x^n_{\kappa(s),r})dw_r \right| \right| \]
\[ + |\tilde{b}_n(x^n_{\kappa(s),n})| \left| \int_{\kappa(s)} \int_Z \tilde{\gamma}(x^n_{\kappa(s),r}, z_2) \tilde{N}(dr, dz_2) \right| \]
\[ + L(1 + |x^n_{\kappa(s),n}|^2) \]
\[ \leq 1 + n^{\frac{1}{2}} \left| \int_{\kappa(s)} \tilde{\sigma}(x^n_{\kappa(s),r})dw_r \right| + L(1 + |x^n_{\kappa(s),n}|^2) \]
\[ + n^{\frac{1}{2}} \left| \int_{\kappa(s)} \int_Z \tilde{\gamma}(x^n_{\kappa(s),r}, z_2) \tilde{N}(dr, dz_2) \right| \]

almost surely for any \( s \in [0, T] \) and \( n \in \mathbb{N} \). Further, due to Young’s inequality,

\[ |x_s^n|^{p-2} x_s^n \tilde{b}_n(x^n_{\kappa(s),n}) \leq 1 + K|x_s^n|^p + K|x^n_{\kappa(s),n}|^p \]
\[ + n^{\frac{1}{2}} \left| \int_{\kappa(s)} \tilde{\sigma}(x^n_{\kappa(s),r})dw_r \right|^{\frac{p}{2}} \]
\[ + n^{\frac{1}{2}} |x_s^n|^{p-2} \left| \int_{\kappa(s)} \int_Z \tilde{\gamma}(x^n_{\kappa(s),r}, z_2) \tilde{N}(dr, dz_2) \right| \]  

(5.9)

almost surely for any \( s \in [0, T] \) and \( n \in \mathbb{N} \). Hence, one first substitutes the estimates from (5.9) in (5.8) and uses (2.10) to estimate the last term of (5.8) which on taking
suprema and expectation gives

$$E \sup_{0 \leq t \leq u} |x_t^n|^p \leq E|\xi|^p + K + K \int_0^u E \sup_{0 \leq r \leq s} |x_r^n|^p \, ds$$

$$+ KE \int_0^u |x_s^n|^{p-2} \tilde{\sigma}(x_{n(n,s)})^2 \, ds$$

$$+ Kn^2 \int_0^u \left( \int_{\kappa(n,s)} E \tilde{\sigma}(x_{n(n,r)}) \, dw_r \right)^{\frac{p}{2}} \, ds$$

$$+ pE \sup_{0 \leq t \leq u} \left| \int_0^t |x_s^n|^{p-2} x_s^n \tilde{\sigma}(x_{n(n,s)}) \, dw_s \right|$$

$$+ pE \sup_{0 \leq t \leq u} \left| \int_0^t \int_{\kappa(n,s)} E \tilde{\sigma}(x_{n(n,r)}) \int_{\kappa(n,s)} |\gamma(x_{n(n,r)}, z_2) \tilde{N}(ds, dz_2)| \, ds \right|$$

$$+ E \int_0^u \int_{\kappa(n,s)} \left\{ |x_s^n|^{p-2} |\gamma(x_{n(n,s)}, z_2)|^2 + |\gamma(x_{n(n,s)}, z_2)|^p \right\} N(ds, dz_2)$$

$$+ Kn^2 \int_0^u \left( \int_{\kappa(n,s)} E |x_s^n|^{p-2} \int_{\kappa(n,s)} |\gamma(x_{n(n,r)}, z_2) \tilde{N}(dr, dz_2)| \, ds \right)$$

$$=: C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7. \quad (5.10)$$

Here, $C_1$ is given by

$$C_1 := E|\xi|^p + K \int_0^u E \sup_{0 \leq r \leq s} |x_r^n|^p \, ds$$

for any $u \in [0, T]$. Due to Young’s inequality and Corollary 3, $C_2$ can be estimated by

$$C_2 := KE \int_0^u |x_s^n|^{p-2} \tilde{\sigma}(x_{n(n,s)})^2 \, ds$$

$$\leq K \int_0^u E|x_s^n|^p \, ds + K \int_0^u E|\tilde{\sigma}(x_{n(n,s)})|^p \, ds$$

$$\leq K + K \int_0^u E|x_{n(n,s)}|^p \, ds. \quad (5.11)$$

For $C_3$, one uses an elementary stochastic inequality and Corollary 3 to obtain

$$C_3 := Kn^2 \int_0^u \left( \int_{\kappa(n,s)} E \tilde{\sigma}(x_{n(n,r)}) \, dw_r \right)^{\frac{p}{2}} \, ds$$

$$\leq Kn^2 \int_0^u E \left( \int_{\kappa(n,s)} E \tilde{\sigma}(x_{n(n,r)})^2 \, dr \right)^{\frac{p}{2}} \, ds$$

$$\leq 1 + Kn^2 \int_0^u E \left( \int_{\kappa(n,s)} |\tilde{\sigma}(x_{n(n,r)})|^2 \, dr \right)^{\frac{p}{2}} \, ds$$

$$\leq Kn \int_0^u \int_{\kappa(n,s)} E|\tilde{\sigma}(x_{n(n,r)})|^p \, dr \, ds$$

$$\leq K + K \int_0^u E|x_{n(n,s)}|^p \, ds. \quad (5.12)$$

Also, by using Burkholder-Gundy-Davis, Young’s and Hölder’s inequalities, $C_4$ can be
estimated as

\[ C_4 := pE \sup_{0 \leq t \leq u} \left| \int_0^t |x_s^n|^{p-2} x_s^n \hat{\sigma}(x_{n,s}^n) dw_s \right| \]
\[ \leq KE \left( \int_0^u |x_s^n|^{2p-2} |\hat{\sigma}(x_{n,s}^n)|^2 ds \right)^{\frac{1}{2}} \]
\[ \leq KE \sup_{0 \leq t \leq u} |x_t^n|^{p-1} \left( \int_0^u |\hat{\sigma}(x_{n,s}^n)|^2 ds \right)^{\frac{1}{2}} \]
\[ \leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + KE \left( \int_0^u |\hat{\sigma}(x_{n,s}^n)|^2 ds \right)^{\frac{p}{2}} \]
\[ \leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K \int_0^u E|\hat{\sigma}(x_{n,s}^n)|^p ds \]

which due to Corollary 3 gives

\[ C_4 \leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K + K \int_0^u E|x_{n,s}^n|^p ds. \] (5.13)

Further, due to Lemma 1.5, Young’s and Hölder’s inequalities, one obtains

\[ C_5 := E \sup_{0 \leq t \leq u} \int_0^t \int_Z |x_s^n|^{p-2} x_s^n \hat{\gamma}(x_{n,s}^n, z_2) |\hat{N}(ds, dz_2) \]
\[ \leq KE \int_0^u \left( \int_Z |x_s^n|^{2p-2} |\hat{\gamma}(x_{n,s}^n, z_2)|^2 \nu(dz_2) \right)^{\frac{1}{2}} ds \]
\[ \leq KE \sup_{0 \leq t \leq u} |x_t^n|^{p-1} \int_0^u \left( \int_Z |\hat{\gamma}(x_{n,s}^n, z_2)|^2 \nu(dz_2) \right)^{\frac{1}{2}} ds \]
\[ \leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K \int_0^u \int_Z |\hat{\gamma}(x_{n,s}^n, z_2)|^p \nu(dz_2) ds \]

which due to Corollary 4 gives

\[ C_5 \leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K + K \int_0^u E|x_{n,s}^n|^p ds. \] (5.14)

Moreover, \( C_6 \) can be estimated as

\[ C_6 := E \int_0^u \int_Z \left\{ |x_s^n|^{p-2} |\hat{\gamma}(x_{n,s}^n, z_2)|^2 + |\hat{\gamma}(x_{n,s}^n, z_2)|^p \right\} \hat{N}(ds, dz_2) \]
\[ \leq E \int_0^u \int_Z \left\{ |x_s^n|^{p-2} |\hat{\gamma}(x_{n,s}^n, z_2)|^2 + |\hat{\gamma}(x_{n,s}^n, z_2)|^p \right\} \nu(dz_2) ds \]
\[ \leq KE \int_0^u \int_Z |x_s^n|^{2p-2} |\hat{\gamma}(x_{n,s}^n, z_2)|^2 \nu(dz_2) ds \]
\[ + KE \int_0^u \int_Z |\hat{\gamma}(x_{n,s}^n, z_2)|^p \nu(dz_2) ds \]
\[ \leq KE \sup_{0 \leq t \leq u} |x_t^n|^{p-2} \int_0^u \int_Z |\hat{\gamma}(x_{n,s}^n, z_2)|^2 \nu(dz_2) ds \]
\[ + KE \int_0^u \int_Z |\hat{\gamma}(x_{n,s}^n, z_2)|^p \nu(dz_2) ds \]
which on the application of Young’s inequality, Hölder’s inequality and Corollary 4 implies

\[
C_6 \leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x^n_t|^p + K \int_0^u E \int_Z |\gamma(x^n_{\kappa(n,s)}, z_2)|^p \nu(dz_2) ds \\
\leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x^n_t|^p + K + K \int_0^u E|x^n_{\kappa(n,s)}|^p ds.
\]  

(5.15)

Finally, for estimating \( C_7 \), one uses the inequality \(|x^n_{t} - x^n_{\kappa(n,s)}|^{p-2} \leq 2^{p-3}|x^n_{\kappa(n,s)}|^{p-2} + 2^{p-3}|x^n_{t} - x^n_{\kappa(n,s)}|^{p-2} \) to obtain,

\[
C_7 := Kn^\frac{1}{2} \int_0^u E\left| \int_{\kappa(n,s)} |x^n_{\kappa(n,s)}|^{p-2} \int_Z |\gamma(x^n_{\kappa(n,r)}, z_2)| \tilde{N}(dr, dz_2) \right| ds \\
\leq Kn^\frac{1}{2} \int_0^u E\left| \int_{\kappa(n,s)} |x^n_{\kappa(n,s)}|^{p-2} \int_Z |\gamma(x^n_{\kappa(n,r)}, z_2)| \tilde{N}(dr, dz_2) \right| ds \\
+ Kn^\frac{1}{2} \int_0^u E|\gamma(x^n_{t} - x^n_{\kappa(n,s)}, z_2)|^{p-2} \int_Z |\gamma(x^n_{\kappa(n,r)}, z_2)| \tilde{N}(dr, dz_2) \right| ds \\
=: C_{7a} + C_{7b}.
\]  

(5.16)

To estimate \( C_{7a} \), one uses Lemma 1.5 to write

\[
C_{7a} := Kn^\frac{1}{2} \int_0^u E\left( \int_{\kappa(n,s)} |x^n_{\kappa(n,s)}|^{p-2} \int_Z |\gamma(x^n_{\kappa(n,r)}, z_2)|^2 \nu(dz_2)| dr \right)^{\frac{1}{2}} ds
\]

which on using Young’s and Hölder’s inequalities gives

\[
C_{7a} \leq KE \sup_{0 \leq t \leq u} |x^n_t|^{p-2} n^\frac{1}{2} \int_0^u \left( \int_{\kappa(n,s)} \int_Z |\gamma(x^n_{\kappa(n,r)}, z_2)|^2 \nu(dz_2) dr \right)^{\frac{1}{2}} ds \\
\leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x^n_t|^p + Kn^\frac{3}{2} E \int_0^u \left( \int_{\kappa(n,s)} \int_Z |\gamma(x^n_{\kappa(n,r)}, z_2)|^2 \nu(dz_2) dr ds \right)^{\frac{3}{2}} \\
\leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x^n_t|^p + 1 + Kn^\frac{3}{2} E \int_0^u \left( \int_{\kappa(n,s)} \int_Z |\gamma(x^n_{\kappa(n,r)}, z_2)|^2 \nu(dz_2) dr ds \right)^{\frac{3}{2}} \\
\leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x^n_t|^p + 1 + Kn \int_0^u \int_{\kappa(n,s)} E \int_Z |\gamma(x^n_{\kappa(n,r)}, z_2)|^p \nu(dz_2) dr ds
\]

and then using Corollary 4,

\[
C_{7a} \leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x^n_t|^p + K + K \int_0^u E|x^n_{\kappa(n,s)}|^p ds.
\]  

(5.17)

Further, to estimate \( C_{7b} \), one observes that due to Young’s inequality,

\[
C_{7b} := Kn^\frac{1}{2} \int_0^u E|\gamma(x^n_{t} - x^n_{\kappa(n,s)}, z_2)|^p ds + Kn^\frac{1}{2} \int_0^u E\left| \int_{\kappa(n,s)} |\gamma(x^n_{\kappa(n,r)}, z_2)| \tilde{N}(dr, dz_2) \right|^{\frac{p}{2}} ds
\]

\[
\leq Kn^\frac{1}{2} \int_0^u E|\gamma(x^n_{t} - x^n_{\kappa(n,s)}, z_2)|^p ds + Kn^\frac{1}{2} \int_0^u E\left| \int_{\kappa(n,s)} |\gamma(x^n_{\kappa(n,r)}, z_2)| \tilde{N}(dr, dz_2) \right|^{\frac{p}{2}} ds
\]  

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which also implies the following estimates,

\[ C_7 \leq Kn^{\frac{1}{2}} \int_0^u E |x^n_s - x^{n,s}_n|^p ds + 1 + Kn \int_0^u E \int_{\kappa(n,s)}^{s} \int_{Z} \tilde{\gamma}(x^{n}_{\kappa(n,r)}, z_2) \tilde{N}(dr, dz_2)^p ds \]

and then Lemmas [1.5, 5.8] give

\[ C_7 \leq Kn^{\frac{1}{2}} \left( 1 + \int_0^u E |x^{n}_{\kappa(n,s)}|^p ds \right) + 1 \]

\[ + Kn \int_0^u E \left( \int_{\kappa(n,s)}^{s} \int_{Z} |\tilde{\gamma}(x^{n}_{\kappa(n,r)}, z_2)|^p \nu(dz_2) dr \right)^{\frac{p}{q}} ds \]

\[ + Kn \int_0^u E \int_{\kappa(n,s)}^{s} \int_{Z} |\tilde{\gamma}(x^{n}_{\kappa(n,r)}, z_2)|^p \nu(dz_2) dr ds. \]

Also, with the help of Hölder's inequality, one gets the following,

\[ C_7 \leq K + K \int_0^u E |x^{n}_{\kappa(n,s)}|^p ds \]

\[ + Kn^{\frac{2-p}{2}} \int_0^u \int_{\kappa(n,s)}^{s} E \int_{Z} |\tilde{\gamma}(x^{n}_{\kappa(n,r)}, z_2)|^p \nu(dz_2) dr ds \]

\[ + Kn \int_0^u E \int_{\kappa(n,s)}^{s} \int_{Z} |\tilde{\gamma}(x^{n}_{\kappa(n,r)}, z_2)|^p \nu(dz_2) dr ds. \]

Finally, due to Corollary 4, one obtains

\[ C_7 \leq K + K \int_0^u E |x^{n}_{\kappa(n,s)}|^p ds. \]  

(5.18)

Thus, on using estimates from (5.17) and (5.18) in (5.16), one has

\[ C_7 \leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x^n_t|^p + K \int_0^u E |x^{n}_{\kappa(n,s)}|^p ds. \]  

(5.19)

Hence, on substituting estimates from (5.11), (5.12), (5.13), (5.14), (5.15) and (5.19) in (5.10), one obtains due to (5.7),

\[ E \sup_{0 \leq t \leq u} |x^n_t|^p \leq \frac{1}{2} E \sup_{0 \leq t \leq u} |x^n_t|^p + E|\xi|^p + K + K \int_0^u E \sup_{0 \leq r \leq s} |x^n_r|^p ds < \infty \]

which on the application of Gronwall’s lemma completes the proof. □

5.2.2 Rate of Convergence

In order to obtain the rate of convergence of the scheme (5.3), an additional assumption is needed which is listed below.

D-9. There exist constants \( L > 0 \) and \( \chi > 0 \) such that

\[ \left| \frac{\partial b^i(x)}{\partial x^k} - \frac{\partial b^i(\bar{x})}{\partial x^k} \right| \leq L(1 + |x|^{\chi} + |\bar{x}|^{\chi})|x - \bar{x}| \]

for any \( x, \bar{x} \in \mathbb{R}^d, i, k = 1, \ldots, d. \)
Proof. The proof follows due to Lemmas [5.8, 5.9] and Remark 5.7.

Remark 5.6. Due to Assumption D-9, there exist constants $L > 0$ and $\chi > 0$ such that
\[
\left| \frac{\partial b^j(x)}{\partial x^k} \right| \leq L(1 + |x|^\chi + 1) \quad \text{and} \quad \left| \frac{\partial^2 b(x)}{\partial x^j \partial x^k} \right| \leq L(1 + |x|)\
\]
for any $x \in \mathbb{R}^d$ and $i, j, k = 1, \ldots, d$.

Remark 5.7. Due to Remark 5.6, there exist constants $L > 0$ and $\chi > 0$ such that
\[
|b(x) - b(\bar{x})| \leq L(1 + |x|^\chi + |\bar{x}|^\chi) |x - \bar{x}|
\]
which further implies that
\[
|b(x)| \leq L(1 + |x|^{\chi + 2})
\]
for any $x, \bar{x} \in \mathbb{R}^d$.

In what follows, we assume $6(\chi + 2) \leq p$ and $\delta \in (4/(p - 2), 1)$.

Lemma 5.10. Let Assumptions D-4 and D-6 to D-9 hold, then
\[
\sup_{0 \leq t \leq T} E|\varepsilon_t^n - \varepsilon_{n(n,t)}|^r \leq Kn^{-1}
\]
for any $2 \leq r \leq p$.

Proof. The proof follows due to Lemmas [5.8, 5.9] and Remark 5.7. \qed

Lemma 5.11. Let Assumptions D-4 and D-6 to D-9 hold, then
\[
\sup_{0 \leq t \leq T} E|\sigma(\varepsilon_{n(n,t)})|^2|\varepsilon_t^n - \varepsilon_{n(n,t)}|^2 \leq Kn^{-1}.
\]

Proof. By Remark 5.4 and Hölder’s inequality,
\[
E|\sigma(\varepsilon_{n(n,t)})|^2|\varepsilon_t^n - \varepsilon_{n(n,t)}|^2 \leq KE(1 + |\varepsilon_{n(n,t)}|^2)|\varepsilon_t^n - \varepsilon_{n(n,t)}|^2
\]
\[
\leq KE \int_{n(n,t)}^T (1 + |\varepsilon_{n(n,t)}|^2)\tilde{b}^n(\varepsilon_{n(n,r)})dr
\]
\[
+ KE \int_{n(n,t)}^T (1 + |\varepsilon_{n(n,t)}|^2)\tilde{\sigma}(\varepsilon_{n(n,r)})dw_r
\]
\[
+ KE \int_{n(n,t)}^T \int_{Z} (1 + |\varepsilon_{n(n,t)}|^2)\tilde{\gamma}(\varepsilon_{n(n,r)}, z_2)\tilde{N}(dr, dz_2)
\]
\[
\leq Kn^{-1} E \int_{n(n,t)}^T (1 + |\varepsilon_{n(n,t)}|^2)\tilde{b}^n(\varepsilon_{n(n,r)})|dr
\]
\[
+ KE \int_{n(n,t)}^T (1 + |\varepsilon_{n(n,t)}|^2)\tilde{\sigma}(\varepsilon_{n(n,r)})|dr
\]
\[
+ KE \int_{n(n,t)}^T \int_{Z} (1 + |\varepsilon_{n(n,t)}|^2)\tilde{\gamma}(\varepsilon_{n(n,r)}, z_2)^2\nu(dz_2)dr.
\]
Then, on further application of Hölder’s inequality along with Remark 5.7 and Corollaries [3, 4], one completes the proof. \qed

Lemma 5.12. Let Assumptions D-4 and D-6 to D-9 hold, then
\[
\sup_{0 \leq t \leq T} E \int_{Z} |\gamma(\varepsilon_{n(n,t)}, z_2)|^2|\varepsilon_t^n - \varepsilon_{n(n,t)}|^2\nu(dz_2) \leq Kn^{-1}.
\]
Proof. By Remark 5.4, one obtains,
\[
\int_{Z} |\gamma(x_{\kappa(n,t)}^{n}, z_2) - \hat{\gamma}(x_{\kappa(n,t)}^{n}, z_2)|^2 |x_t^n - x_{\kappa(n,t)}^{n}|^2 \nu(dz_2) \leq KE(1 + |x_{\kappa(n,t)}^{n}|^2) |x_t^n - x_{\kappa(n,t)}^{n}|^2
\]
and then one uses the same arguments as used in the proof of Lemma 5.11 to complete the proof. \qed

Lemma 5.13. Consider equation (5.2) and let Assumptions D-4 and D-6 to D-9 be satisfied, then
\[
\sup_{0 \leq t \leq T} E|b(x_t^n) - \hat{b}(x_t^n)|^2 \leq Kn^{-2}.
\]
Proof. The proof immediately follows due to equation (5.2), Remarks [5.3, 5.7] and Lemma 5.9. \qed

Lemma 5.14. Let Assumptions D-4 and D-6 to D-9 hold, then
\[
\sup_{0 \leq t \leq T} E \int_{Z} |\gamma(x_t^n, z_2) - \hat{\gamma}(x_{\kappa(n,t)}^{n}, z_2)|^2 \nu(dz_2) \leq Kn^{-2}.
\]
Proof. By Itô’s formula,
\[
\gamma^j(x_t^n, z_2) = \gamma^j(x_{\kappa(n,t)}^{n}, z_2) + \sum_{u=1}^{d} \int_{\kappa(n,t)}^{t} \frac{\partial \gamma^j(x_t^n, z_2)}{\partial x^u} \hat{b}(x_t^n, z_2) \, dz
\]
\[
+ \frac{1}{2} \sum_{u_1, u_2=1}^{d} \sum_{l_1=1}^{m} \int_{\kappa(n,t)}^{t} \frac{\partial^2 \gamma^j(x_t^n, z_2)}{\partial x^{u_1} \partial x^{u_2}} \hat{\sigma}(x_t^n, z_2) \gamma^{(u_1, l_1)}(x_{\kappa(n,t)}^{n}, z_2) \, dz
\]
\[
+ \sum_{u=1}^{d} \sum_{j=1}^{m} \int_{\kappa(n,t)}^{t} \frac{\partial \gamma^j(x_t^n, z_2)}{\partial x^u} \hat{\sigma}(x_t^n, z_2) \, dz
\]
\[
+ \sum_{u=1}^{d} \int_{\kappa(n,t)}^{t} \int_{Z} \frac{\partial \gamma^j(x_t^n, z_2)}{\partial x^u} \gamma^u(x_{\kappa(n,t)}^{n}, z_1) \, N(dr, dz_1)
\]
\[
+ \int_{\kappa(n,t)}^{t} \int_{Z} \left( \gamma^j(x_t^n, z_2) - \gamma^j(x_{\kappa(n,t)}^{n}, z_1) \right) \, N(dr, dz_1).
\]
Also, one recalls equation (5.6),
\[
\hat{\gamma}^j(x_{\kappa(n,t)}^{n}, z_2) = \gamma^j(x_{\kappa(n,t)}^{n}, z_2) + \sum_{j=1}^{m} \sum_{u=1}^{d} \int_{\kappa(n,t)}^{t} \frac{\partial \gamma^j(x_{\kappa(n,t)}^{n}, z_2)}{\partial x^u} \hat{\sigma}(x_{\kappa(n,t)}^{n}, z_2) \, dz
\]
\[
+ \sum_{u=1}^{d} \int_{\kappa(n,t)}^{t} \int_{Z} \frac{\partial \gamma^j(x_{\kappa(n,t)}^{n}, z_2)}{\partial x^u} \gamma^u(x_{\kappa(n,t)}^{n}, z_1) \, N(dr, dz_1)
\]
\[
+ \int_{\kappa(n,t)}^{t} \int_{Z} \left( \gamma^j(x_{\kappa(n,t)}^{n}, z_2) - \gamma^j(x_{\kappa(n,t)}^{n}, z_1) \right) \, N(dr, dz_1).
\]
Then, one takes expectation of squared difference of (5.20) and (5.21) to get,

\[
E \int_Z |\gamma_i^n(x_r^n, z_2) - \tilde{\gamma}_i^n(x_r^n, z_2)|^2 \nu(dz_2)
\]

\[
\leq K \int_Z E \left[ \sum_{u=1}^d \int_{\kappa(n,t)}^t \frac{\partial \gamma_i^n(x_r^n, z_2)}{\partial x_u} \sigma^{(u,i)}(x_r^n, z_2) \lambda_u(x_r^n, z_2) dz_2 \right] \nu(dz_2)
\]

\[
+ K \int_Z E \left[ \sum_{u=1}^d \sum_{u_2=1}^d \sum_{l_1=1}^d \int_{\kappa(n,t)}^t \frac{\partial^2 \gamma_i^n(x_r^n, z_2)}{\partial x_u \partial x_{u_2}} \tau_{u_2,l_1}(x_r^n, z_2) \lambda_u(x_r^n, z_2) \right] \nu(dz_2)
\]

\[
+ K \int_Z E \left[ \sum_{u=1}^d \sum_{u_2=1}^d \sum_{l_1=1}^d \int_{\kappa(n,t)}^t \left( \frac{\partial \gamma_i^n(x_r^n, z_2)}{\partial x_u} - \frac{\partial \gamma_i^n(x_r^n, z_2)}{\partial x_{u_2}} \right) \sigma^{(u,i)}(x_r^n, z_2) \lambda_u(x_r^n, z_2) \right] \nu(dz_2)
\]

\[
+ K \int_Z E \left[ \sum_{u=1}^d \int_{\kappa(n,t)}^t \left( \frac{\partial \gamma_i^n(x_r^n, z_2)}{\partial x_u} - \frac{\partial \gamma_i^n(x_r^n, z_2)}{\partial x_{u_2}} \right) \sigma^{(u,i)}(x_r^n, z_2) \lambda_u(x_r^n, z_2) \right] \nu(dz_2)
\]

\[
+ K \int_Z E \left[ \sum_{u=1}^d \int_{\kappa(n,t)}^t \left( \frac{\partial \gamma_i^n(x_r^n, z_2)}{\partial x_u} - \frac{\partial \gamma_i^n(x_r^n, z_2)}{\partial x_{n,r}} \right) \gamma^{(u)}(x_r^n, z_2) \lambda_u(x_r^n, z_2) \right] \nu(dz_2)
\]

\[
+ K \int_Z \left[ \sum_{u=1}^d \int_{\kappa(n,t)}^t \left( \frac{\partial \gamma_i^n(x_r^n, z_2)}{\partial x_u} - \frac{\partial \gamma_i^n(x_r^n, z_2)}{\partial x_{n,r}} \right) \gamma^{(u)}(x_r^n, z_2) \lambda_u(x_r^n, z_2) \right] \nu(dz_2)
\]

\[
+ K \int Z [E \left[ \sum_{u=1}^d \int_{\kappa(n,t)}^t \left( \gamma_i^n(x_r^n, z_2) - \gamma_i^n(x_r^n, z_2) \right) \right] \right] \nu(dz_2)
\]

\[
+ K \int Z \left[ \sum_{u=1}^d \left| \frac{\partial \gamma_i^n(x_r^n, z_2)}{\partial x_u} \right| \left| \gamma_i^n(x_r^n, z_2) \right| + \sum_{u=1}^d \left| \frac{\partial \gamma_i^n(x_r^n, z_2)}{\partial x_u} \right| \right] \nu(dz_2)
\]

\[
+ K \int Z \left[ \sum_{u=1}^d \left| \frac{\partial \gamma_i^n(x_r^n, z_2)}{\partial x_u} \right| \left| \gamma_i^n(x_r^n, z_2) \right| \right] \nu(dz_2)
\]
which due to Hölder’s inequality, Fubini theorem and Lemma 1.5 gives,

\[ E \int_Z |\gamma^i(x^n, z_2) - \tilde{\gamma}^i(x^n_{\kappa(n,r)}, z_2)|^2 \nu(dz_2) \]

\[ \leq K n^{-1} E \sum_{u=1}^{d} \sum_{t} \int_{\kappa(n,t)} \left( \int_{Z} \left| \frac{\partial \gamma^i(x^n_{\kappa(n,t)}, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right)^{1/2} \nu(dz_2) \]

\[ + K n^{-1} E \sum_{u=1}^{d} \sum_{r} \int_{\kappa(n,r)} \left( \int_{Z} \left| \frac{\partial \gamma^i(x^n_{\kappa(n,r)}, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right)^{1/2} \nu(dz_2) \]

\[ + K E \sum_{u=1}^{d} \sum_{j=1}^{m} \int_{\kappa(n,t)} \left( \int_{Z} \left| \frac{\partial \gamma^i(x^n_{\kappa(n,t)}, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right)^{1/2} \sigma_1^{(u,j)}(x^n_{\kappa(n,r)}) \right)^{1/2} \nu(dz_2) \]

\[ + K E \sum_{u=1}^{d} \sum_{j=1}^{m} \int_{\kappa(n,t)} \left( \int_{Z} \left| \frac{\partial \gamma^i(x^n_{\kappa(n,t)}, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right)^{1/2} \sigma_2^{(u,j)}(x^n_{\kappa(n,r)}) \right)^{1/2} \nu(dz_2) \]

\[ + K E \sum_{u=1}^{d} \sum_{j=1}^{m} \int_{\kappa(n,t)} \left( \int_{Z} \left| \frac{\partial \gamma^i(x^n_{\kappa(n,t)}, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right)^{1/2} \sigma_3^{(u,j)}(x^n_{\kappa(n,r)}) \right)^{1/2} \nu(dz_2) \]

\[ + K E \sum_{u=1}^{d} \int_{\kappa(n,t)} \left( \int_{Z} \left| \frac{\partial \gamma^i(x^n_{\kappa(n,t)}, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right)^{1/2} \gamma^i(x^n_{\kappa(n,r)}, z_1) \right)^{1/2} \nu(dz_1) \]

\[ + K E \sum_{u=1}^{d} \int_{\kappa(n,t)} \left( \int_{Z} \left| \frac{\partial \gamma^i(x^n_{\kappa(n,t)}, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right)^{1/2} \gamma_1^i(x^n_{\kappa(n,r)}, z_1) \right)^{1/2} \nu(dz_1) \]

\[ + K E \sum_{u=1}^{d} \int_{\kappa(n,t)} \left( \int_{Z} \left| \frac{\partial \gamma^i(x^n_{\kappa(n,t)}, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right)^{1/2} \gamma_2^i(x^n_{\kappa(n,r)}, z_1) \right)^{1/2} \nu(dz_1) \]

\[ + K E \sum_{u=1}^{d} \int_{\kappa(n,t)} \left( \int_{Z} \left| \frac{\partial \gamma^i(x^n_{\kappa(n,t)}, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right)^{1/2} \gamma_3^i(x^n_{\kappa(n,r)}, z_1) \right)^{1/2} \nu(dz_1) \]

\[ + K E \int_{\kappa(n,t)} \left( \int_{Z} \left| \gamma^i(x^n + \tilde{\gamma}(x^n_{\kappa(n,r)}, z_1), z_2) - \gamma^i(x^n_{\kappa(n,r)}, z_1), z_2) \right|^2 \nu(dz_2) \right)^{1/2} \nu(dz_1) \]

\[ + \int_{\kappa(n,t)} \left( \int_{Z} \left| \gamma^i(x^n, z_2) - \gamma^i(x^n_{\kappa(n,r)}, z_2) \right|^2 \nu(dz_2) \right)^{1/2} \nu(dz_1) \]

\[ + \sum_{u=1}^{d} \int_{Z} \left( \left| \frac{\partial \gamma^i(x^n_{\kappa(n,r)}, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right)^{1/2} \gamma_1^i(x^n_{\kappa(n,r)}, z_1) \right)^{1/2} \nu(dz_1) \]

\[ + \sum_{u=1}^{d} \int_{Z} \left( \left| \frac{\partial \gamma^i(x^n_{\kappa(n,r),}, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right)^{1/2} \gamma_2^i(x^n_{\kappa(n,r),}, z_1) \right)^{1/2} \nu(dz_1) \]

\[ + \sum_{u=1}^{d} \int_{Z} \left( \left| \frac{\partial \gamma^i(x^n_{\kappa(n,r),}, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right)^{1/2} \gamma_3^i(x^n_{\kappa(n,r),}, z_1) \right)^{1/2} \nu(dz_1) \]
Thus, due to Assumption D-8 and Remark 5.5,

$$E \int_Z |\gamma(x^n_t, z_2) - \tilde{\gamma}(x^n_{\kappa(n,t)}, z_2)|^2 \nu(dz_2) \leq Kn^{-1} E \int_{\kappa(n,t)} |\tilde{\gamma}^n(x^n_{\kappa(n,r)})|^2 dr$$

$$+ Kn^{-1} E \int_{\kappa(n,t)} |\tilde{\gamma}^n(x^n_{\kappa(n,r)})|^2 dr + KE \int_{\kappa(n,t)} |\sigma(x^n_{\kappa(n,r)})|^2 |x^n_r - x^n_{\kappa(n,r)}|^2 dr$$

$$+ KE \int_{\kappa(n,t)} (|\sigma_1(x^n_{\kappa(n,r)})|^2 + |\sigma_2(x^n_{\kappa(n,r)})|^2 + |\sigma_3(x^n_{\kappa(n,r)})|^2) dr$$

$$+ K \int_{\kappa(n,t)} \int_Z |x^n_r - x^n_{\kappa(n,r)}|^2 |\gamma(x^n_{\kappa(n,r)}, z_2)|^2 \nu(dz_2) dr$$

$$+ KE \int_{\kappa(n,t)} \int_Z |\gamma_1(x^n_{\kappa(n,r)}, z_2)|^2 \nu(dz_2) dr$$

$$+ KE \int_{\kappa(n,t)} \int_Z |\gamma_2(x^n_{\kappa(n,r)}, z_2)|^2 \nu(dz_2) dr$$

$$+ KE \int_{\kappa(n,t)} \int_Z |\gamma_3(x^n_{\kappa(n,r)}, z_2)|^2 \nu(dz_2) dr$$

and finally Remarks [5.3, 5.4, 5.7], Lemmas [5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.9, 5.10, 5.11, 5.12] along with Corollaries [3, 4] give

$$E \int_Z |\gamma(x^n_t, z_2) - \tilde{\gamma}(x^n_{\kappa(n,t)}, z_2)|^2 \nu(dz_2) \leq Kn^{-2}.$$ 

This completes the proof.

\[\square\]

**Lemma 5.15.** Let Assumptions D-4 and D-6 to D-9 hold, then

$$\sup_{0 \leq t \leq T} E|\sigma(x^n_t) - \tilde{\sigma}(x^n_{\kappa(n,t)})|^2 \leq Kn^{-2}.$$

**Proof.** For any \(k = 1, \ldots, d\) and \(v = 1, \ldots, m\), by Itô’s formula,

$$\sigma^{(k,v)}(x^n_t) = \sigma^{(k,v)}(x^n_{\kappa(n,t)}) + \sum_{u=1}^d \int_{\kappa(n,t)}^{t} \frac{\partial \sigma^{(k,v)}(x^n_{r})}{\partial x^u} \overline{b}^{u,n}_r (x^n_{\kappa(n,r)}) dr$$

$$+ \frac{1}{2} \sum_{u_1, u_2=1}^d \sum_{l_1=1}^m \int_{\kappa(n,t)}^{t} \frac{\partial^2 \sigma^{(k,v)}(x^n_{r})}{\partial x^{u_1} \partial x^{u_2}} \overline{\sigma}^{(u_1,l_1)}(x^n_{\kappa(n,r)}) \overline{\sigma}^{(u_2,l_1)}(x^n_{\kappa(n,r)}) dr$$

$$+ \sum_{u=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^{t} \frac{\partial \sigma^{(k,v)}(x^n_{r})}{\partial x^u} \overline{\sigma}^{(u,j)}(x^n_{\kappa(n,r)}) dw^j_r$$

$$+ \sum_{u=1}^d \int_{\kappa(n,t)}^{t} \int_Z \frac{\partial \sigma^{(k,v)}(x^n_{r})}{\partial x^u} \overline{\sigma}^{(u,j)}(x^n_{\kappa(n,r)}, z_2) \tilde{N}(dr, dz_2)$$

$$+ \int_{\kappa(n,t)}^{t} \int_Z \left( \sigma^{(k,v)}(x^n_{r} + \tilde{\gamma}(x^n_{\kappa(n,r)}, z_2)) - \sigma^{(k,v)}(x^n_{r}) - \sum_{u=1}^d \frac{\partial \sigma^{(k,v)}(x^n_{r})}{\partial x^u} \tilde{\sigma}^{(u,j)}(x^n_{\kappa(n,r)}, z_2) \right) N(dr, dz_2).$$

\[\tag{5.22}\]
Also, one recalls equation (5.4),

\[ \tilde{\sigma}^{(k,v)}(x_{n,t}^{n}) = \sigma^{(k,v)}(x_{n,t}^{n}) + \sum_{j=1}^{m} \sum_{u=1}^{d} \int_{\kappa(n,t)}^{t} \sigma^{(u,j)}(x_{n,r}^{n}) \frac{\partial \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u} dw^j_r \]

\[ + \sum_{u=1}^{d} \int_{\kappa(n,t)}^{t} \sum_{Z} \frac{\partial \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u} \gamma^u(x_{n,r}^{n}, z_2) \tilde{N}(dr, dz) \]

\[ + \int_{\kappa(n,t)}^{t} \sum_{Z} (\sigma^{(k,v)}(x_{n,r}^{n}) + \gamma(x_{n,r}^{n}, z)) - \sigma^{(k,v)}(x_{n,r}^{n}) \]

\[ - \sum_{u=1}^{d} \frac{\partial \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u} \gamma^u(x_{n,r}^{n}, z_2) \tilde{N}(dr, dz). \]  

(5.23)

Thus, on taking expectation of the square of the difference between (5.22) and (5.23), one obtains

\[ E[\sigma^{(k,v)}(x_{n,t}^{n}) - \tilde{\sigma}^{(k,v)}(x_{n,t}^{n})]^2 \leq K \sum_{u=1}^{d} E \left| \int_{\kappa(n,t)}^{t} \frac{\partial \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u} b_n,w(x_{n,r}^{n}) dr \right|^2 \]

\[ + K \sum_{u_1, u_2 = 1}^{d} \sum_{l_1 = 1}^{m} E \left| \int_{\kappa(n,t)}^{t} \frac{\partial^2 \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u \partial x^{u_2}} \tilde{\sigma}^{(u_1,l_1)}(x_{n,r}^{n}) \tilde{\sigma}^{(u_2,l_1)}(x_{n,r}^{n}) dr \right|^2 \]

\[ + K \sum_{u=1}^{d} \sum_{j=1}^{m} E \left| \int_{\kappa(n,t)}^{t} \sigma^{(u,j)}(x_{n,r}^{n}) \left( \frac{\partial \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u} - \frac{\partial \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u} \right) dw^j_r \right|^2 \]

\[ + K \sum_{u=1}^{d} \sum_{j=1}^{m} E \left| \int_{\kappa(n,t)}^{t} \frac{\partial \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u} \sigma^{(u,j)}(x_{n,r}^{n}) dw^j_r \right|^2 \]

\[ + K \sum_{u=1}^{d} \sum_{j=1}^{m} E \left| \int_{\kappa(n,t)}^{t} \frac{\partial \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u} \sigma^{(u,j)}(x_{n,r}^{n}) dw^j_r \right|^2 \]

\[ + K \sum_{u=1}^{d} \sum_{j=1}^{m} E \left| \int_{\kappa(n,t)}^{t} \frac{\partial \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u} \sigma^{(u,j)}(x_{n,r}^{n}) dw^j_r \right|^2 \]

\[ + K \sum_{u=1}^{d} \left| \int_{\kappa(n,t)}^{t} \sum_{Z} \gamma^u(x_{n,r}^{n}, z_2) \left( \frac{\partial \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u} - \frac{\partial \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u} \right) \tilde{N}(dr, dz) \right|^2 \]

\[ + K \sum_{u=1}^{d} \left| \int_{\kappa(n,t)}^{t} \sum_{Z} \frac{\partial \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u} \gamma^u(x_{n,r}^{n}, z_2) \tilde{N}(dr, dz) \right|^2 \]

\[ + K \sum_{u=1}^{d} \left| \int_{\kappa(n,t)}^{t} \sum_{Z} \frac{\partial \sigma^{(k,v)}(x_{n,r}^{n})}{\partial x^u} \gamma^u(x_{n,r}^{n}, z_2) \tilde{N}(dr, dz) \right|^2 \]

\[ + KE \left| \int_{\kappa(n,t)}^{t} \sum_{Z} (\sigma^{(k,v)}(x_{n,r}^{n} + \gamma(x_{n,r}^{n}, z)) - \sigma^{(k,v)}(x_{n,r}^{n} + \gamma(x_{n,r}^{n}, z))) N(dr, dz) \right|^2 \]

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which on the application of Hölder’s inequality and Lemma 1.5 yields

\[E|\sigma(x^n_t) - \hat{\sigma}(x^n_t)|^2\]

\[\leq Kn^{-1} \sum_{u_1,u_2=1}^d \int_{\kappa(n,t)}^t E \left[ \frac{\partial \sigma(x^n_r)}{\partial x^u} \right]^2 |\hat{\kappa}^n_u(x^n_r)|^2 dr\]

\[+ K \sum_{u=1}^d E \left[ \frac{\partial \sigma(x^n_r)}{\partial x^u} \right]^2 |\hat{\kappa}_u^n(x^n_r)|^2 dr\]

\[+ \sum_{u=1}^d \sum_{j=1}^m E \left[ \frac{\partial \sigma(x^n_r)}{\partial x^u} \right]^2 |\hat{\sigma}^{(u,j)}(x^n_r)|^2 dr\]

\[+ \sum_{u=1}^d \sum_{j=1}^m E \left[ \frac{\partial \sigma(x^n_r)}{\partial x^u} \right]^2 |\hat{\sigma}_2^{(u,j)}(x^n_r)|^2 dr\]

\[+ \sum_{u=1}^d \sum_{j=1}^m E \left[ \frac{\partial \sigma(x^n_r)}{\partial x^u} \right]^2 |\hat{\sigma}_3^{(u,j)}(x^n_r)|^2 dr\]

\[+ \sum_{u=1}^d \int_{\kappa(n,t)}^t \left[ \frac{\partial \sigma(x^n_r)}{\partial x^u} \right]^2 |\hat{\gamma}^n_u(x^n_r, z_2)|^2 \nu(dz_2) dr\]

\[+ \sum_{u=1}^d \int_{\kappa(n,t)}^t \left[ \frac{\partial \sigma(x^n_r)}{\partial x^u} \right]^2 |\hat{\gamma}_1^u(x^n_r, z_2)|^2 \nu(dz_2) dr\]

\[+ \sum_{u=1}^d \int_{\kappa(n,t)}^t \left[ \frac{\partial \sigma(x^n_r)}{\partial x^u} \right]^2 |\hat{\gamma}_2^u(x^n_r, z_2)|^2 \nu(dz_2) dr\]

\[+ \sum_{u=1}^d \int_{\kappa(n,t)}^t \left[ \frac{\partial \sigma(x^n_r)}{\partial x^u} \right]^2 |\hat{\gamma}_3^u(x^n_r, z_2)|^2 \nu(dz_2) dr\]

\[+ KE \int_{\kappa(n,t)}^t Z \left[ \sigma(x^n_r + \hat{\gamma}(x^n_r, z_2)) - \sigma(x^n_r) \right]^2 \nu(dz_2) dr\]

\[+ KE \int_{\kappa(n,t)}^t Z \left[ \sigma(x^n_r) - \sigma(x^n_r) \right]^2 \nu(dz_2) dr\]

\[+ K \sum_{u=1}^d \int_{\kappa(n,t)}^t \left[ \frac{\partial \sigma(x^n_r)}{\partial x^u} \right]^2 |\hat{\gamma}^n_u(x^n_r, z_2)|^2 \nu(dz_2) dr\]
Almost surely for any \( t \) and then due to Assumptions D-7, D-8 and Remark 5.5, 

Let Assumptions D-4 and D-6 to D-9 hold, then for any Lemma 5.16.

This completes the proof.

and finally Remarks [5.3, 5.4, 5.7], Lemmas [5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.9, 5.10, 5.11, 5.12] along with Corollaries [3, 4] give

\[ E|\sigma(x^n_t) - \tilde{\sigma}(x^n_{\kappa(n,t)})|^2 \leq Kn^{-2} \]

This completes the proof. \( \square \)

Before, proceeding further, let us define \( e^n_t := x_t - x^n_t \) i.e.

\[
e^n_t = \int_0^t (b(x_s) - \tilde{b}(x^n_{\kappa(n,s)}))ds + \int_0^t (\sigma(x_s) - \tilde{\sigma}(x^n_{\kappa(n,s)}))dw_s
\]

\[
+ \int_0^t \int_Z (\gamma(x_s, z_2) - \tilde{\gamma}(x^n_{\kappa(n,s)}, z_2))\tilde{N}(ds, dz_2)
\]

almost surely for any \( t \in [0, T] \).

**Lemma 5.16.** Let Assumptions D-4 and D-6 to D-9 hold, then for any \( t \in [0, T] \)

\[ E \int_0^t e^n_s (b(x^n_s) - b(x^n_{\kappa(n,s)}))ds \leq K \int_0^t \sup_{0 \leq r \leq s} E|e^n_r|^2ds + Kn^{-\frac{2}{2+\varepsilon} - 1}. \]
Proof. By Itô’s formula,

\[ b^k(x^n_s) - b^k(x^n_{\kappa(n,s)}) = \sum_{i=1}^{d} \int_{\kappa(n,s)}^{s} \frac{\partial b^k(x^n_r)}{\partial x^i} \hat{b}^{n,i}(x^n_{\kappa(n,r)}) dr \]

\[ + \frac{1}{2} \sum_{i,j=1}^{d} \int_{\kappa(n,s)}^{s} \frac{\partial^2 b^k(x^n_r)}{\partial x^i \partial x^j} \sum_{l=1}^{m} \hat{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) \hat{\sigma}^{(j,l)}(x^n_{\kappa(n,r)}) dr \]

\[ + \sum_{i=1}^{d} \int_{\kappa(n,s)}^{s} \frac{\partial b^k(x^n_r)}{\partial x^i} \sum_{l=1}^{m} \sigma^{(i,l)}(x^n_{\kappa(n,r)}) dw^l_r \]

\[ + \int_{\kappa(n,s)}^{s} \int_{Z} \left( b^k(x^n_r + \hat{\gamma}(x^n_{\kappa(n,r)}, z_2)) - b^k(x^n_r) \right. \]

\[ \left. \quad - \frac{\partial b^k(x^n_r)}{\partial x^i} \hat{\gamma}^i(x^n_{\kappa(n,r)}, z_2) \right) N(dr, dz_2) \]

(5.24)

and thus one could write

\[ E \int_{0}^{t} e_s^n(b(x^n_s) - b(x^n_{\kappa(n,s)})) ds \]

\[ = E \sum_{k=1}^{d} \int_{0}^{t} e_s^{n,k} \left\{ b^k(x^n_s) - b^k(x^n_{\kappa(n,s)}) \right\} ds \]

\[ = E \sum_{k=1}^{d} \int_{0}^{t} e_s^{n,k} \int_{\kappa(n,s)}^{s} \frac{\partial b^k(x^n_r)}{\partial x^i} \hat{b}^{n,i}(x^n_{\kappa(n,r)}) dr ds \]

\[ + \frac{1}{2} \sum_{k,i,j=1}^{d} E \int_{0}^{t} e_s^{n,k} \int_{\kappa(n,s)}^{s} \frac{\partial^2 b^k(x^n_r)}{\partial x^i \partial x^j} \sum_{l=1}^{m} \hat{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) \hat{\sigma}^{(j,l)}(x^n_{\kappa(n,r)}) dr ds \]

\[ + \sum_{k,i,j=1}^{d} E \int_{0}^{t} e_s^{n,k} \int_{\kappa(n,s)}^{s} \frac{\partial b^k(x^n_r)}{\partial x^i} \sum_{l=1}^{m} \sigma^{(i,l)}(x^n_{\kappa(n,r)}) dw^l_r ds \]

\[ + \sum_{k,i,j=1}^{d} E \int_{0}^{t} e_s^{n,k} \int_{\kappa(n,s)}^{s} \int_{Z} \frac{\partial b^k(x^n_r)}{\partial x^i} \gamma^i(x^n_{\kappa(n,r)}, z_2) \tilde{N}(dr, dz_2) ds \]

\[ + \sum_{k=1}^{d} E \int_{0}^{t} e_s^{n,k} \int_{\kappa(n,s)}^{s} \int_{Z} \left( b^k(x^n_r + \hat{\gamma}(x^n_{\kappa(n,r)}, z_2)) - b^k(x^n_r) \right. \]

\[ \left. \quad - \frac{\partial b^k(x^n_r)}{\partial x^i} \hat{\gamma}^i(x^n_{\kappa(n,r)}, z_2) \right) N(dr, dz_2) ds \]

\[ =: F_1 + F_2 + F_3 + F_4 + F_5. \]

(5.25)
By using Young’s and Hölder’s inequalities, $F_1$ can be estimated as

$$F_1 := \sum_{k,i=1}^{d} E \int_0^t e^{n,k}_s \int_{\kappa(n,s)}^{s} \frac{\partial b^k(x^n_s)}{\partial x^i} \tilde{b}^{n,i}(x^n_{\kappa(n,r)})dr \, ds$$

$$\leq KE \int_0^t |e^{n}_s|^2 ds + Kn^{-1} \sum_{k,i=1}^{d} E \int_0^t \int_{\kappa(n,s)}^{s} \left| \frac{\partial b^k(x^n_s)}{\partial x^i} \right|^2 |\tilde{b}^{n,i}(x^n_{\kappa(n,r)})|^2 dr \, ds$$

and then due to Remarks [5.3, 5.6, 5.7] and Lemma 5.9, one obtains

$$F_1 \leq K \int_0^t \sup_{0 \leq r \leq s} E |e^{n}_r|^2 dr + Kn^{-2}. \quad (5.26)$$

Similarly, $F_2$ can be estimated by using Young’s and Hölder’s inequalities along with Remarks [5.4, 5.6], Corollary 3 and Lemma 5.9 as,

$$F_2 := \frac{1}{2} \sum_{k,i,j=1}^{d} E \int_0^t e^{n,k}_s \int_{\kappa(n,s)}^{s} \frac{\partial^2 b^k(x^n_s)}{\partial x^i \partial x^j} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) \tilde{\sigma}^{(j,l)}(x^n_{\kappa(n,r)}) dr \, ds$$

$$\leq K \int_0^t \sup_{0 \leq r \leq s} E |e^{n}_r|^2 dr + Kn^{-2}. \quad (5.27)$$

For the estimation of $F_3$, one uses

$$e^{n,k}_s = e^{n,k}_{\kappa(n,s)} + \int_{\kappa(n,s)}^{s} (b^k(x_r) - \tilde{b}^{n,k}(x^n_{\kappa(n,r)})) dr$$

$$+ \sum_{v=1}^{m} \int_{\kappa(n,s)}^{s} (\sigma^{(k,v)}(x_r) - \tilde{\sigma}^{(k,v)}(x^n_{\kappa(n,r)})) dw^v_r$$

$$+ \int_{\kappa(n,s)}^{s} \int_{Z} (\gamma^k(x_r, z_2) - \tilde{\gamma}^k(x^n_{\kappa(n,r)}, z_2)) \tilde{N}(dr, dz_2)$$

to obtain the following,

$$F_3 := K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_0^t e^{n,k}_s \int_{\kappa(n,s)}^{s} \frac{\partial b^k(x^n_s)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) dw^l_r \, ds$$

$$\leq K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_0^t \int_{\kappa(n,s)}^{s} \frac{\partial b^k(x^n_s)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) dw^l_r \, ds$$

$$+ K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_0^t \int_{\kappa(n,s)}^{s} (b^k(x_r) - \tilde{b}^{n,k}(x^n_{\kappa(n,r)})) dr$$

$$\times \int_{\kappa(n,s)}^{s} \frac{\partial b^k(x^n_s)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) dw^l_r \, ds$$

$$+ K \sum_{k,i=1}^{d} \sum_{l,v=1}^{m} E \int_0^t \int_{\kappa(n,s)}^{s} (\sigma^{(k,v)}(x_r) - \tilde{\sigma}^{(k,v)}(x^n_{\kappa(n,r)})) dw^v_r$$

$$\int_{\kappa(n,s)}^{s} \frac{\partial b^k(x^n_s)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) dw^l_r \, ds$$
\[ + K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_0^t \int_{\kappa(n,s)} Z \left( \gamma^k(x_r, z_2) - \tilde{\gamma}^k(x_r^{n,(i,l)}, z_2) \right) \tilde{N}(dr, dz_2) \int_{\kappa(n,s)} \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_r^{n,(i,l)}) dw_r^i ds \]
\[ =: F_{31} + F_{32} + F_{33} + F_{34}. \tag{5.28} \]

Here, \( F_{31} \) is given by
\[
F_{31} := K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_0^t \int_{\kappa(n,s)} \left| e_{n,k}^{n,(i,l),s} \right|^2 ds \\
+ K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_0^t \int_{\kappa(n,s)} \left| \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_1^{(i,l)}(x_r^{n,(i,l)}) \right|^2 dw_r^i ds \\
+ K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_0^t \int_{\kappa(n,s)} \left| \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_2^{(i,l)}(x_r^{n,(i,l)}) \right|^2 dw_r^i ds \\
+ K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_0^t \int_{\kappa(n,s)} \left| \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_3^{(i,l)}(x_r^{n,(i,l)}) \right|^2 dw_r^i ds
\]

which on the application of Young’s inequality gives,
\[
F_{31} \leq KE \int_0^t \left| e_{n,k}^{n,(i,l),s} \right|^2 ds \\
+ K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_0^t \int_{\kappa(n,s)} \left| \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_1^{(i,l)}(x_r^{n,(i,l)}) \right|^2 dw_r^i ds \\
+ K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_0^t \int_{\kappa(n,s)} \left| \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_2^{(i,l)}(x_r^{n,(i,l)}) \right|^2 dw_r^i ds \\
+ K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_0^t \int_{\kappa(n,s)} \left| \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_3^{(i,l)}(x_r^{n,(i,l)}) \right|^2 dw_r^i ds
\]

and then on using an elementary inequality of stochastic integrals along with Remark 5.6, one obtains
\[
F_{31} \leq KE \int_0^t \left| e_{n,k}^{n,(i,l),s} \right|^2 ds \\
+ KE \int_0^t \int_{\kappa(n,s)} \left( 1 + |x_r^n|^2 \right) \left| \sigma_1(x_r^{n,(i,l)}) \right|^2 dr ds \\
+ KE \int_0^t \int_{\kappa(n,s)} \left( 1 + |x_r^n|^2 \right) \left| \sigma_2(x_r^{n,(i,l)}) \right|^2 dr ds \\
+ KE \int_0^t \int_{\kappa(n,s)} \left( 1 + |x_r^n|^2 \right) \left| \sigma_3(x_r^{n,(i,l)}) \right|^2 dr ds.
\]
Also, by using Hölder’s inequality and Lemmas [5.2, 5.3, 5.4, 5.9],

$$F_{31} \leq KE \int_0^t |e_{\kappa(n,s)}|^2 ds$$

$$+ K \int_0^t \int_{\kappa(n,s)} \left( E(1 + |x_r^n|^\chi) \frac{2(2+\delta)}{4} \right) \frac{\delta}{s} \left( E|\sigma_1(x_{\kappa(n,r)}^n)|^{2+\delta} \right) \frac{2}{s} dr ds$$

$$+ K \int_0^t \int_{\kappa(n,s)} \left( E(1 + |x_r^n|^\chi) \frac{2(2+\delta)}{4} \right) \frac{\delta}{s} \left( E|\sigma_2(x_{\kappa(n,r)}^n)|^{2+\delta} \right) \frac{2}{s} dr ds$$

$$+ K \int_0^t \int_{\kappa(n,s)} \left( E(1 + |x_r^n|^\chi) \frac{2(2+\delta)}{4} \right) \frac{\delta}{s} \left( E|\sigma_3(x_{\kappa(n,r)}^n)|^{2+\delta} \right) \frac{2}{s} dr ds$$

$$\leq K \int_0^t \sup_{0 \leq r \leq s} E|e_r^n|^2 ds + Kn^{-\frac{2}{1+\delta}}. \quad (5.29)$$

Further, due to Remark 5.7,

$$|b^k(x_r) - \tilde{b}^{n,k}(x_{\kappa(n,r)}^n)| \leq L(1 + |x_r|^\chi + |x_r^n|^\chi)|x_r - x_r^n|$$

$$+ L(1 + |x_r^n|^\chi + |x_{\kappa(n,r)}^n|^\chi)|x_r^n - x_{\kappa(n,r)}^n|$$

$$+ |b^k(x_{\kappa(n,r)}^n) - \tilde{b}^{n,k}(x_{\kappa(n,r)}^n)| \quad (5.30)$$

for any $r \in [0, T]$, $k = 1, \ldots, d$, which on substituting in $F_{32}$ gives

$$F_{32} := K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \int_{\kappa(n,s)} (b^k(x_r) - \tilde{b}^{n,k}(x_{\kappa(n,r)}^n)) dr$$

$$\times \int_{\kappa(n,s)} \frac{\partial b^k(x_r^n)}{\partial x^l} \sigma^{(i,l)}(x_{\kappa(n,r)}^n) dw^l_r ds$$

$$\leq K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \int_{\kappa(n,s)} (1 + |x_r|^\chi + |x_r^n|^\chi)|x_r - x_r^n| dr$$

$$\times \left| \int_{\kappa(n,s)} \frac{\partial b^k(x_r^n)}{\partial x^l} \sigma^{(i,l)}(x_{\kappa(n,r)}^n) dw^l_r \right| ds$$

$$+ K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \int_{\kappa(n,s)} (1 + |x_r^n|^\chi + |x_{\kappa(n,r)}^n|^\chi)|x_r^n - x_{\kappa(n,r)}^n| dr$$

$$\times \left| \int_{\kappa(n,s)} \frac{\partial b^k(x_r^n)}{\partial x^l} \sigma^{(i,l)}(x_{\kappa(n,r)}^n) dw^l_r \right| ds$$

$$+ K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \int_{\kappa(n,s)} |b^k(x_{\kappa(n,r)}^n) - \tilde{b}^{n,k}(x_{\kappa(n,r)}^n)| dr$$

$$\times \left| \int_{\kappa(n,s)} \frac{\partial b^k(x_r^n)}{\partial x^l} \sigma^{(i,l)}(x_{\kappa(n,r)}^n) dw^l_r \right| ds.$$

Now, one uses Hölder’s inequality with exponent $\sqrt{2}$ and $2 + \sqrt{2}$ to obtain,

$$F_{32} \leq K \sum_{k,i=1}^d \sum_{l=1}^m \int_0^t \left( n^{-\sqrt{2}+1} E \int_{\kappa(n,s)} (1 + |x_r|^\chi + |x_r^n|^\chi)^{\sqrt{2}} |x_r - x_r^n| dr \right)^{\frac{1}{2}}$$

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Thus, due to Young’s inequality, one obtains

\[ F \]

Again, for any \( K \)

\[ K \]

\[ K \]

\[ K \]

\[ K \]

\[ K \]

\[ K \]

which on the application of Hölder’s inequality along with Remark 5.6, Lemmas [5.1, 5.9, 5.13] and Corollary 3 yields the following,

\[
F_{32} \leq Kn^{-\frac{1}{2}} \int_{0}^{t} \left( n^{-\frac{1}{2}} + \int_{\kappa(s)}^{s} (E(1 + |x_{r}| + |x_{r}^{n}|)|x_{r}^{n}|)\sqrt{2} ds \right) \leq \frac{1}{2} \] 

\[
\times \left( \frac{1}{2} \frac{1}{2} \right) \frac{1}{2} ds 
\]

\[
+ Kn^{-\frac{1}{2}} \int_{0}^{t} \left( n^{-\frac{1}{2}} + \int_{\kappa(s)}^{s} (E(1 + |x_{r}| + |x_{r}^{n}|)|x_{r}^{n}|)\sqrt{2} ds \right) \leq \frac{1}{2} \] 

\[
\times \left( \frac{1}{2} \frac{1}{2} \right) \frac{1}{2} ds + Kn^{-\frac{1}{2}} 
\]

\[
\leq Kn^{-\frac{1}{2}} \int_{0}^{t} \left( \sup_{0 \leq r \leq s} E|e_{r}^{n}|^{2} \right) ds + Kn^{-\frac{1}{2}} + Kn^{-\frac{1}{2}}. 
\]

Thus, due to Young’s inequality, one obtains

\[
F_{32} \leq K \int_{0}^{t} \sup_{0 \leq r \leq s} E|e_{r}^{n}|^{2} ds + Kn^{-\frac{1}{2}}. \quad (5.31) 
\]

Again, for any \( r \in [0, T], \ k = 1, \ldots, d \) and \( v = 1, \ldots, m \), one uses

\[
\sigma^{(k,v)}(x_{r}) - \sigma^{(k,v)}(x_{\kappa(n,r)}) = (\sigma^{(k,v)}(x_{r}) - \sigma^{(k,v)}(x_{r}^{n})) 
\]

\[
+ (\sigma^{(k,v)}(x_{r}^{n}) - \sigma^{(k,v)}(x_{\kappa(n,r)})) 
\]

\[
(5.32) 
\]

to express \( F_{33} \) as below,

\[
F_{33} := K \sum_{k,i=1}^{d} \sum_{l,v=1}^{m} E \int_{0}^{t} \int_{\kappa(s)}^{s} (\sigma^{(k,v)}(x_{r}) - \sigma^{(k,v)}(x_{\kappa(n,r)}))dw_{r}^{v} 
\]

\[
\times \left( \frac{1}{2} \frac{1}{2} \right) \frac{1}{2} ds 
\]

\[
= K \sum_{k,i=1}^{d} \sum_{l,v=1}^{m} E \int_{0}^{t} \int_{\kappa(s)}^{s} (\sigma^{(k,v)}(x_{r}) - \sigma^{(k,v)}(x_{r}^{n}))dw_{r}^{v} 
\]

\[
= K \sum_{k,i=1}^{d} \sum_{l,v=1}^{m} E \int_{0}^{t} \int_{\kappa(s)}^{s} (\sigma^{(k,v)}(x_{r}) - \sigma^{(k,v)}(x_{r}^{n}))dw_{r}^{v} 
\]

\[
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\]
and then Assumption D-7 and Lemma 5.15 give

Finally, for any $F_r$ the following,

which again due to Hölder’s inequality, Remark 5.6, Lemma 5.9 and Corollary 3 yields

Further, Hölder’s inequality implies

which again due to Hölder’s inequality, Remark 5.6, Lemma 5.9 and Corollary 3 yields the following,

and then Assumption D-7 and Lemma 5.15 give

Thus, on the application of Young’s inequality, one obtains

Finally, for any $r \in [0, T]$ $k = 1, \ldots, d$ and $z_2 \in Z$, one uses

$$
\gamma^k(x_r, z_2) - \gamma^k(x_{r, n}^{(\kappa(n, r)), z_2}) = (\gamma^k(x_r, z_2) - \gamma^k(x_{r}^{n}, z_2)) + (\gamma^k(x_r^{n}, z_2) - \gamma^k(x_{\kappa(n, r), z_2}^{n}))
$$
to express $F_{34}$ as the following

$$
F_{34} := K \sum_{k,i=1}^{d} \sum_{l,v=1}^{m} E \int_{0}^{t} \int_{\kappa(n,s)}^{s} \left( \gamma^k(x_r, z_2) - \gamma^k(x_{\kappa(n, r), z_2}^{n}) \right) \tilde{N}(dr, dz_2)
$$

$$
\times \int_{\kappa(n,s)}^{s} \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n, r), z_2}^{n}) dw_r^i ds
$$

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By adopting the same approach as followed in the estimation of $F$, obtains Thus substituting the estimates from (5.29), (5.31), (5.33) and (5.34) in (5.28), one obtains

Thus, on the application of Young’s inequality, one obtains

which due to Hölder’s inequality gives,

Further, the application of Assumption D-7, Remark 5.6, Lemmas [1.5, 5.14] and Corollary 3 gives

Further, the application of Young’s inequality, one obtains

Thus substituting the estimates from (5.29), (5.31), (5.33) and (5.34) in (5.28), one obtains

By adopting the same approach as followed in the estimation of $F_3$, one could estimate
Proof. \[ \delta \text{ with } (5.25) \] completes the proof.

Finally, combining estimates from (5.26), (5.27), (5.35), (5.36) and (5.37) in equation (5.25) completes the proof. 

\[ \square \]

**Theorem 5.2.** Let Assumptions D-4 and D-6 to D-9 hold with \( p \geq 6(\chi + 2) \). Then, the tamed Milstein scheme (5.3) converges to the solution of SDE (5.1) in \( L^2 \) with rate of convergence arbitrarily close to 1, i.e.,

\[ \sup_{0 \leq t \leq T} E|\epsilon^n_t - x^n_t|^2 \leq Kn^{-2\delta^{-1}} \]

with \( \delta \in (4/(p - 2), 1) \).

**Proof.** By Itô’s formula,

\[
|\epsilon^n_t|^2 = 2 \int_0^t e^n_s (b(x_s) - \tilde{b}^n(x^n_{\kappa(n,s)})) ds + 2 \int_0^t e^n_s (\sigma(x_s) - \tilde{\sigma}^n(x^n_{\kappa(n,s)})) dw_s \\
+ \int_0^t \left| \sigma(x_s) - \tilde{\sigma}(x^n_{\kappa(n,s)}) \right|^2 ds + 2 \int_0^t \int_{\mathbb{R}} e^n_s (\gamma(x_s, z_2) - \tilde{\gamma}(x^n_{\kappa(n,s)}, z_2)) \tilde{N}(ds, dz_2) \\
+ \int_0^t \int_{\mathbb{R}} (|e^n_s + \gamma(x_s, z_2) - \tilde{\gamma}(x^n_{\kappa(n,s)}, z_2)|^2 - |e^n_s|^2 - 2e^n_s (\gamma(x_s, z_2) - \tilde{\gamma}(x^n_{\kappa(n,s)}, z_2))) N(ds, dz_2)
\]

almost surely for any \( t \in [0, T] \), which on taking expectation implies

\[
E|\epsilon^n_t|^2 = 2E \int_0^t e^n_s (b(x_s) - \tilde{b}^n(x^n_{\kappa(n,s)})) ds + E \int_0^t \left| \sigma(x_s) - \tilde{\sigma}(x^n_{\kappa(n,s)}) \right|^2 ds \\
+ E \int_0^t \int_{\mathbb{R}} (|e^n_s + \gamma(x_s, z_2) - \tilde{\gamma}(x^n_{\kappa(n,s)}, z_2)|^2 - |e^n_s|^2 \\
- 2e^n_s (\gamma(x_s, z_2) - \tilde{\gamma}(x^n_{\kappa(n,s)}, z_2))) \nu(dz_2) ds
\]

\[ =: W_1 + W_2 + W_3. \] (5.38)

for any \( t \in [0, T] \). Now, for estimating \( W_1 \), one uses the following,

\[
e^n_s (b(x_s) - \tilde{b}^n(x^n_{\kappa(n,s)})) = e^n_s (b(x_s) - b(x^n_s)) + e^n_s (b(x^n_s) - b(x^n_{\kappa(n,s)})) \\
+ e^n_s (b(x^n_{\kappa(n,s)}) - \tilde{b}^n(x^n_{\kappa(n,s)}))
\]

which on using Assumption D-7 and Young’s inequality gives

\[
e^n_s (b(x_s) - \tilde{b}^n(x^n_{\kappa(n,s)})) \leq K|e^n_s|^2 + e^n_s (b(x^n_s) - b(x^n_{\kappa(n,s)})) \\
+ K|b(x^n_{\kappa(n,s)}) - \tilde{b}^n(x^n_{\kappa(n,s)})|^2
\] (5.39)
for any $s \in [0, T]$. Thus, $W_1$ can be estimated by

$$W_1 := 2E \int_0^t e_s^n(b(x_s) - \tilde{b}^n(x_{\kappa(n,s)}))ds$$

$$\leq KE \int_0^t |e_s^n|^2 ds + 2E \int_0^t e_s^n(b(x_s^n) - b(x_{\kappa(n,s)}))ds$$

$$+ KE \int_0^t |b(x_{\kappa(n,s)}) - \tilde{b}^n(x_{\kappa(n,s)})|^2 ds$$

which on the application of Lemma [5.13, 5.16] implies

$$W_1 \leq K \int_0^t \sup_{0 \leq r \leq s} E|e_r^n|^2 ds + Kn^{-\frac{1}{2}\pi^{-\frac{3}{2}}}.$$  \hspace{1cm} (5.40)$$

Further, for estimating $W_2$, one writes,

$$\sigma(x_s) - \sigma(x_{\kappa(n,s)}) = \sigma(x_s) - \sigma(x_s^n) + \sigma(x_s^n) - \sigma(x_{\kappa(n,s)})$$

which on using Assumption D-7 yields,

$$|\sigma(x_s) - \sigma(x_{\kappa(n,s)})|^2 \leq K|e_s^n|^2 + 2|\sigma(x_s^n) - \sigma(x_{\kappa(n,s)})|^2$$ \hspace{1cm} (5.41)$$

for any $s \in [0, T]$. Thus, due to Lemma 5.15, one obtains

$$W_2 := E \int_0^t |\sigma(x_s) - \sigma(x_{\kappa(n,s)})|^2 ds \leq K \int_0^t \sup_{0 \leq r \leq s} E|e_r^n|^2 ds + Kn^{-\frac{2}{2}\pi^{-\frac{1}{2}}}. \hspace{1cm} (5.42)$$

Finally, for estimating $W_3$, one uses equation (2.10) to write

$$W_3 := E \int_0^t \int_Z \left(|e_s^n + \gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}, z_2)|^2 - |e_s^n|^2ight.$$  

$$\left. - 2e_s^n(\gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}, z_2)))\nu(dz_2)ds \right.$$

$$\leq KE \int_0^t \int_Z \left|\gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}, z_2)|^2\nu(dz_2)ds \right.$$

and then applying the following splitting

$$\gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}, z_2) = \gamma(x_s, z_2) - \gamma(x_s^n, z_2) + \gamma(x_s^n, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}, z_2),$$

one could write

$$W_3 \leq KE \int_0^t \int_Z \left|\gamma(x_s, z_2) - \gamma(x_s^n, z_2)|^2\nu(dz_2)ds \right.$$

$$+ KE \int_0^t \int_Z \left|\gamma(x_s^n, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}, z_2)|^2\nu(dz_2)ds \right.$$

which due to Assumption D-7 and Lemma 5.14 yields

$$W_3 \leq K \int_0^t \sup_{0 \leq r \leq s} E|e_r^n|^2 ds + Kn^{-\frac{2}{2}\pi^{-\frac{1}{2}}}.$$ \hspace{1cm} (5.43)$$
Thus by substituting estimates from (5.40), (5.42) and (5.43) in (5.38), one obtains,

$$\sup_{0 \leq t \leq u} E|e_t^n|^2 \leq K \int_0^u \sup_{0 \leq r \leq s} E|e_r^n|^2 ds + Kn^{-\frac{1}{3} - \frac{3}{2}}$$

for any $u \in [0, T]$. The Gronwall’s lemma completes the proof.

5.3 Continuous Case

In this section, the case when $\nu \equiv 0$ or $\gamma \equiv 0$ in SDE (5.1) is discussed, i.e. one considers the following SDE,

$$x_t = \xi + \int_0^t b(x_s)ds + \int_0^t \sigma(x_s)dw_s$$ (5.44)

almost surely for any $t \in [0, T]$, where $\xi$ is an $\mathcal{F}_0$-measurable random variable in $\mathbb{R}^d$. Further, the scheme (5.3) is replaced by the following,

$$x_t^n = \xi + \int_0^t \tilde{b}_n(x_{n,s}^n)ds + \int_0^t \tilde{\sigma}(x_{n,s}^n)dw_s,$$ (5.45)

almost surely for any $t \in [0, T]$. The drift coefficient $\tilde{b}_n$ in scheme (5.45) is given by equation (5.2) and the diffusion coefficient $\tilde{\sigma}$ is defined as

$$\tilde{\sigma}(x_{\kappa(n,s)}^n) := \sigma(x_{\kappa(n,s)}^n) + \sigma_1(x_{\kappa(n,s)}^n)$$ (5.46)

where $\sigma_1$ is given by (5.5). Also, Assumptions D-6 to D-8 hold since $\gamma \equiv 0$. This implies that Lemma 5.1 holds under Assumptions D-3, D-4, and D-7. Further, Lemma 5.2 and Corollary 3 hold under Assumptions D-7 and D-8 while Lemma 5.9 holds under Assumptions D-4, D-7 and D-8. One also notes that Lemma 5.3 to Lemma 5.7 and Corollary 4 are not required for this section since $\gamma \equiv 0$.

We now proceed for the derivation of the rate of convergence of scheme (5.45) and prove that this is same as that obtained by [66]. In what follows, rate of convergence is achieved for any $q \geq 2$ satisfying $q \leq \frac{p}{3(\chi+1)}$.

**Lemma 5.17.** Consider Remark 5.3 and let Assumptions D-4, D-7 and D-8 (with $\gamma \equiv 0$) hold, then

$$\sup_{0 \leq t \leq T} E|x_t^n - x_{\kappa(n,t)}^n|^q \leq Kn^{-\frac{q}{2}}.$$

**Proof.** This follows due to Lemmas [5.8, 5.9].

**Lemma 5.18.** Consider equation (5.2) and let Assumptions D-4, D-7 and D-8 (with $\gamma \equiv 0$) be satisfied, then

$$\sup_{0 \leq t \leq T} E|b(x_t^n) - \tilde{b}_n(x_t^n)|^q dt \leq Kn^{-q}.$$

**Proof.** This is an immediate consequence of equation (5.2), Remark 5.4 and Lemma 5.9. 

**Lemma 5.19.** Let Assumptions D-4, D-7 and D-8 (with $\gamma \equiv 0$) hold, then

$$\sup_{0 \leq t \leq T} E|\sigma(x_t^n) - \tilde{\sigma}(x_{\kappa(n,t)}^n)|^q \leq Kn^{-q}.$$
Proof. Recall equations (5.22) with \( \gamma = 0 \) and (5.46) to write,

\[
E |\sigma(x^n_t) - \tilde{\sigma}(x^n_{\kappa(n,t)})|^q = E \left( \sum_{k=1}^{d} \sum_{v=1}^{m} |\sigma^{(k,v)}(x^n_t) - \tilde{\sigma}^{(k,v)}(x^n_{\kappa(n,t)})|^2 \right)^{\frac{q}{2}} \\
\leq K \sum_{u,k=1}^{d} \sum_{v=1}^{m} E \left| \int_{\kappa(n,t)}^{t} \frac{\partial \sigma^{(k,v)}(x^n_r)}{\partial x^u} \tilde{\gamma}^{n,u}(x^n_{\kappa(n,r)}) dr \right|^q \\
+ K \sum_{u,k,u_1,u_2=1}^{d} \sum_{v_1,v_2=1}^{m} E \left| \int_{\kappa(n,t)}^{t} \frac{\partial^2 \sigma^{(k,v)}(x^n_r)}{\partial x^{u_1} \partial x^{u_2}} \tilde{\sigma}^{(u_1,l_1)}(x^n_{\kappa(n,r)}) \tilde{\sigma}^{(u_2,l_1)}(x^n_{\kappa(n,r)}) dr \right|^q \\
+ K \sum_{u,k=1}^{d} \sum_{v,j=1}^{m} E \left| \int_{\kappa(n,t)}^{t} \sigma^{(u,j)}(x^n_{\kappa(n,r)}) \left( \frac{\partial \sigma^{(k,v)}(x^n_r)}{\partial x^u} - \frac{\partial \tilde{\sigma}^{(k,v)}(x^n_{\kappa(n,r)})}{\partial x^u} \right) dr \right|^q \\
+ K \sum_{u,k=1}^{d} \sum_{v,j=1}^{m} E \left| \int_{\kappa(n,t)}^{t} \frac{\partial \sigma^{(k,v)}(x^n_r)}{\partial x^u} \sigma^{(u,j)}(x^n_{\kappa(n,r)}) dw^j_r \right|^q \\
\]  

and the application of Remarks [5.4, 5.5, 5.3, 5.7], Assumption D-8, Lemmas [5.2, 5.17] and Corollary 3 completes the proof.

Lemma 5.20. Let Assumptions D-3, D-4 and D-7 to D-9 (with \( \gamma = 0 \)) hold, then

\[
E \int_0^t |e^n_s|^q - 2e^n_s(b(x^n_s) - b(x^n_{\kappa(n,s)})) ds \leq K \int_0^t \sup_{0 \leq r \leq s} E |e^n_r|^q ds + Kn^{-q} 
\]

for any \( t \in [0,T] \).

Proof. As before, one uses Ito’s formula to write \( b^k(x^n_s) - b^k(x^n_{\kappa(n,s)}) \) and then obtains the following,

\[
E \int_0^t |e^n_s|^q - 2e^n_s(b(x^n_s) - b(x^n_{\kappa(n,s)})) ds = \sum_{k=1}^{d} E \int_0^t |e^n_s|^q - 2e^n_s,b^k(x^n_s) - b^k(x^n_{\kappa(n,s)}) ds \\
= \sum_{k,i=1}^{d} E \int_0^t |e^n_s|^q - 2e^n_s,b^k \int_{\kappa(n,s)}^s \frac{\partial b^k(x^n_r)}{\partial x^i} \tilde{\gamma}^{n,i}(x^n_{\kappa(n,r)}) dr ds \\
+ \frac{1}{2} \sum_{k,i,j=1}^{d} E \int_0^t |e^n_s|^q - 2e^n_s,b^k \int_{\kappa(n,s)}^s \frac{\partial^2 b^k(x^n_r)}{\partial x^i \partial x^j} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) \tilde{\sigma}^{(j,l)}(x^n_{\kappa(n,r)}) dr ds \\
+ \sum_{k,i=1}^{d} E \int_0^t |e^n_s|^q - 2e^n_s,b^k \int_{\kappa(n,s)}^s \frac{\partial b^k(x^n_r)}{\partial x^i} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) dw^l_s ds \\
=: T_1 + T_2 + T_3. 
\]  

(5.47)

By using Schwarz and Young’s inequalities, \( T_1 \) can be estimated as

\[
T_1 := \sum_{k,i=1}^{d} E \int_0^t |e^n_s|^q - 2e^n_s,b^k \int_{\kappa(n,s)}^s \frac{\partial b^k(x^n_r)}{\partial x^i} \tilde{\gamma}^{n,i}(x^n_{\kappa(n,r)}) dr ds \\
\leq KE \int_0^t |e^n_s|^q ds + K \sum_{k,i=1}^{d} E \int_0^t \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x^n_r)}{\partial x^i} \tilde{\gamma}^{n,i}(x^n_{\kappa(n,r)}) dr \right|^q ds 
\]

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which on the application of Remarks [5.3, 5.6, 5.7] and Lemma 5.9 implies

$$T_1 \leq K \int_0^t \sup_{0 \leq r \leq s} E|e_n^r|^q ds + Kn^{-q}. \quad (5.48)$$

By using Schwarz, Young’s and Hölder’s inequalities, Remark 5.6, Corollary 3 and Lemma 5.9, $T_2$ can be estimated as,

$$T_2 := \frac{1}{2} \sum_{k,i,j=1}^d E \int_0^t |e_s^n|^{q-2} e_s^{n,k} \int_{\kappa(n,s)} \partial^2 b^k(x^n_s) \sum_{l=1}^m \sigma^{(i,l)}(x^n_{\kappa(n,r)}) \hat{\sigma}^{(j,l)}(x^n_{\kappa(n,r)}) dr ds$$

$$\leq K \int_0^t \sup_{0 \leq r \leq s} E|e_r^n|^q ds + Kn^{-q}. \quad (5.49)$$

For the last term of equation (5.47), one observes that

$$de_s^{n,k} = (b^k(x_s) - \hat{b}^n(x^n_{\kappa(n,s)})) ds + \sum_{v=1}^m (\sigma^{(k,v)}(x_s) - \hat{\sigma}^{(k,v)}(x^n_{\kappa(n,s)})) dw_s^v$$

and also uses Itô’s formula to obtain the following,

$$d|e_s^n|^{q-2} = (q - 2)|e_s^n|^{q-4} e_s^n (b(x_s) - \hat{b}^n(x^n_{\kappa(n,s)})) ds$$

$$+ (q - 2)|e_s^n|^{q-4} e_s^n (\sigma(x_s) - \hat{\sigma}(x^n_{\kappa(n,s)})) dw_s$$

$$+ \frac{(q - 2)(q - 4)}{2} |e_s^n|^{q-6} |(\sigma(x_s) - \hat{\sigma}(x^n_{\kappa(n,s)}))|^2 e_s^n ds$$

$$+ \frac{q - 2}{2} |e_s^n|^{q-4} |\sigma(x_s) - \hat{\sigma}(x^n_{\kappa(n,s)})|^2 ds$$

which on multiplying gives

$$d|e_s^n|^{q-2} de_s^{n,k} = \left( \sum_{v=1}^m (\sigma^{(k,v)}(x_s) - \hat{\sigma}^{(k,v)}(x^n_{\kappa(n,s)})) dw_s^v \right)$$

$$\times \left( (q - 2)|e_s^n|^{q-4} \sum_{u=1}^d \sum_{v=1}^m e_s^{n,u} \sum_{v_1=1}^m (\sigma^{(u,v)}(x_s) - \hat{\sigma}^{(u,v)}(x^n_{\kappa(n,s)})) dw_s^{v_1} \right)$$

$$= (q - 2) \sum_{v=1}^m (\sigma^{(k,v)}(x_s) - \hat{\sigma}^{(k,v)}(x^n_{\kappa(n,s)}))|e_s^n|^{q-4}$$

$$\times \sum_{u=1}^d e_s^{n,u} (\sigma^{(u,v)}(x_s) - \hat{\sigma}^{(u,v)}(x^n_{\kappa(n,s)})) ds.$$

Thus, one uses product rule to obtain the following

$$d(|e_s^n|^{q-2} e_s^{n,k}) = |e_s^n|^{q-2} de_s^{n,k} + e_s^{n,k} d|e_s^n|^{q-2} + d|e_s^n|^{q-2} de_s^{n,k}$$

$$= |e_s^n|^{q-2} (b^k(x_s) - \hat{b}^n(x^n_{\kappa(n,s)})) ds$$

$$+ |e_s^n|^{q-4} \sum_{v=1}^m (\sigma^{(k,v)}(x_s) - \hat{\sigma}^{(k,v)}(x^n_{\kappa(n,s)})) dw_s^v$$

$$+ (q - 2) e_s^{n,k} |e_s^n|^{q-4} e_s^n (b(x_s) - \hat{b}^n(x^n_{\kappa(n,s)})) ds.$$
for any $s \in [0,T]$, which on integrating over the interval $[s, \kappa(n,s)]$ implies

$$\left|e^n_s\right|^q - e^{n,k}_s = \left|e^n_{\kappa(n,s)}\right|^q - e^{n,k}_{\kappa(n,s)} + \int_{\kappa(n,s)}^{s} \left|e^n_r\right|^q - 2 \sum_{v=1}^{m} \left(e^{(k,v)}_r(x_r) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)})\right)dw^v_r \right)dr \\
+ (q-2) \int_{\kappa(n,s)}^{s} e^{n,k}_r \left|e^n_r\right|^{q-4} e^n_r (b(x_r) - \tilde{b}^n(x_{\kappa(n,r)}))dr \\
+ (q-2) \int_{\kappa(n,s)}^{s} e^{n,k}_r \left|e^n_r\right|^{q-4} \left(e^n_r (\sigma(x_r) - \tilde{\sigma}(x_{\kappa(n,r)}))\right)dw_r \\
+ \frac{(q-2)(q-4)}{2} \int_{\kappa(n,s)}^{s} \left(e^{n,k}_r \left|e^n_r\right|^{q-6} (\sigma(x_r) - \tilde{\sigma}(x_{\kappa(n,r)}))\right)^2 e^n_r dr \\
+ \frac{q-2}{2} \int_{\kappa(n,s)}^{s} e^{n,k}_r \left|e^n_r\right|^{q-4} \left(\sigma(x_r) - \tilde{\sigma}(x_{\kappa(n,r)})\right)^2 dr \\
+ (q-2) \int_{\kappa(n,s)}^{s} \sum_{v=1}^{m} \left(e^{(k,v)}_r(x_r) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)})\right)|e^n_r|^{q-4} \\
\times \sum_{u=1}^{d} e^{n,u}_r (\sigma^{(u,v)}(x_r) - \tilde{\sigma}^{(u,v)}(x_{\kappa(n,r)}))dr \right)$$

(5.50)

almost surely for any $s \in [0,T]$. Hence, on substituting the values from equation (5.50) in $T_3$ of equation (5.47) gives

$$T_3 := \sum_{k,i=1}^{d} E \int_0^t \left|e^n_s\right|^q - 2 e^{n,k}_s \int_{\kappa(n,s)}^{s} \frac{\partial b^k(x^n_r)}{\partial x^i} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)})dw^l_r dr \right)
\\
= \sum_{k,i=1}^{d} E \int_0^t \left|e^n_{\kappa(n,s)}\right|^q - 2 e^{n,k}_s \int_{\kappa(n,s)}^{s} \frac{\partial b^k(x^n_r)}{\partial x^i} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)})dw^l_r dr \right)
\\
+ \sum_{k,i=1}^{d} E \int_0^t \left|e^n_s\right|^q - 2 \int_{\kappa(n,s)}^{s} (b^k(x_r) - \tilde{b}^n(x_{\kappa(n,r)}))dr \\
\times \int_{\kappa(n,s)}^{s} \frac{\partial b^k(x^n_r)}{\partial x^i} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)})dw^l_r dr \right)$$

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\[
+ \sum_{k,i=1}^{d} E \int_{0}^{t} \int_{\kappa(n,s)} |e_{r}^{n,|2} e_{r}^{n,k}(x_{r}^{n}) - \tilde{\sigma}(k,v)(x_{\kappa(n,r)})| dw_{r}^{v} \\
\times \int_{\kappa(n,s)} \frac{\partial b_{k}^{n}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}) dw_{l}^{i} ds \\
+ K \sum_{k,i=1}^{d} E \int_{0}^{t} \int_{\kappa(n,s)} e_{r}^{n,k} |e_{r}^{n,|4} e_{r}^{n}(b(x_{r}) - \tilde{b}(x_{\kappa(n,r)})) dr \\
\times \int_{\kappa(n,s)} \frac{\partial b_{k}^{n}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}) dw_{l}^{i} ds \\
+ K \sum_{k,i=1}^{d} E \int_{0}^{t} \int_{\kappa(n,s)} e_{r}^{n,k} |e_{r}^{n,|4} e_{r}^{n}(\sigma(x_{r}) - \tilde{\sigma}(x_{\kappa(n,r)})) dw_{r} \\
\times \int_{\kappa(n,s)} \frac{\partial b_{k}^{n}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}) dw_{l}^{i} ds \\
+ K \sum_{k,i=1}^{d} E \int_{0}^{t} \int_{\kappa(n,s)} e_{r}^{n,k} |e_{r}^{n,|6} |(\sigma(x_{r}) - \tilde{\sigma}(x_{\kappa(n,r)}))^{2} dw_{r} \\
\times \int_{\kappa(n,s)} \frac{\partial b_{k}^{n}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}) dw_{l}^{i} ds \\
+ K \sum_{k,i=1}^{d} E \int_{0}^{t} \int_{\kappa(n,s)} \sum_{v=1}^{m} (\sigma^{(k,v)}(x_{r}) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)})) e_{r}^{n,q-4} \\
\times \sum_{u=1}^{d} e_{r}^{n,u}(\sigma^{(u,v)}(x_{r}) - \tilde{\sigma}^{(u,v)}(x_{\kappa(n,r)})) dr \\
\times \int_{\kappa(n,s)} \frac{\partial b_{k}^{n}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}) dw_{l}^{i} ds \\
\]  

which on the application of Schwarz inequality gives 

\[
T_{3} = \sum_{k,i=1}^{d} E \int_{0}^{t} |e_{\kappa(n,s)}^{n,|2} e_{\kappa(n,s)}^{n,k}(x_{\kappa(n,r)})| dw_{r}^{v} \\
\times \int_{\kappa(n,s)} \frac{\partial b_{k}^{n}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}) dw_{l}^{i} ds \\
+ \sum_{k,i=1}^{d} E \int_{0}^{t} \int_{\kappa(n,s)} |e_{r}^{n,|2}(b(x_{r}) - \tilde{b}(x_{\kappa(n,r)}))| dr \\
\times \int_{\kappa(n,s)} \frac{\partial b_{k}^{n}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}) dw_{l}^{i} ds \\
\]

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\[ + \sum_{k,i=1}^{d} E \int_{0}^{t} \int_{\kappa(n,s)}^{s} |e_{r}^{n}|^{q-2} \sum_{v=1}^{m} (\sigma(k,v)(x_{r}) - \tilde{\sigma}(k,v)(x_{\kappa(n,\cdot)}))dw_{r}^{v} \]
\[ \times \int_{\kappa(n,s)}^{s} \frac{\partial b_{k}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \sigma^{(i,l)}(x_{\kappa(n,\cdot)})dw_{r}^{l}ds \]
\[ + K \sum_{k,i=1}^{d} E \int_{0}^{t} \int_{\kappa(n,s)}^{s} e_{r}^{n,k} |e_{r}^{n}|^{q-2} e_{r}^{n}(\sigma(x_{r}) - \tilde{\sigma}(x_{\kappa(n,\cdot)}))dw_{r} \]
\[ \times \int_{\kappa(n,s)}^{s} \frac{\partial b_{k}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \sigma^{(i,l)}(x_{\kappa(n,\cdot)})dw_{r}^{l}ds \]
\[ + K \sum_{k,i=1}^{d} E \int_{0}^{t} \int_{\kappa(n,s)}^{s} |e_{r}^{n}|^{q-3}|\sigma(x_{r}) - \tilde{\sigma}(x_{\kappa(n,\cdot)})|^{2}dr \]
\[ \times \left| \int_{\kappa(n,s)}^{s} \frac{\partial b_{k}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \sigma^{(i,l)}(x_{\kappa(n,\cdot)})dw_{r}^{l} \right|ds \]
\[ =: T_{31} + T_{32} + T_{33} + T_{34} + T_{35}. \quad (5.51) \]

Furthermore, for estimating \( T_{31} \), one writes,
\[ T_{31} := K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_{0}^{t} |e_{\kappa(n,s)}^{n}|^{q-2} e_{\kappa(n,s)}^{n,k} \int_{\kappa(n,s)}^{s} \frac{\partial b_{k}(x_{r}^{n})}{\partial x^{i}} \sigma^{(i,l)}(x_{\kappa(n,\cdot)})dw_{r}^{l}ds \]
\[ = K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_{0}^{t} |e_{\kappa(n,s)}^{n}|^{q-2} e_{\kappa(n,s)}^{n,k} \int_{\kappa(n,s)}^{s} \frac{\partial b_{k}(x_{r}^{n})}{\partial x^{i}} \sigma^{(i,l)}(x_{\kappa(n,\cdot)})dw_{r}^{l}ds \]
\[ + K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_{0}^{t} |e_{\kappa(n,s)}^{n}|^{q-2} e_{\kappa(n,s)}^{n,k} \int_{\kappa(n,s)}^{s} \frac{\partial b_{k}(x_{r}^{n})}{\partial x^{i}} \sigma_{1}^{(i,l)}(x_{\kappa(n,\cdot)})dw_{r}^{l}ds \]

which on using the Young’s inequality, Hölder’s inequality and an elementary inequality of stochastic integrals gives
\[ T_{31} \leq K \int_{0}^{t} \sup_{0 \leq r \leq s} E|e_{r}^{n}|^{q}ds + K \sum_{k,i=1}^{d} \sum_{l=1}^{m} E \int_{0}^{t} \left| \int_{\kappa(n,s)}^{s} \frac{\partial b_{k}(x_{r}^{n})}{\partial x^{i}} \sigma^{(i,l)}(x_{\kappa(n,\cdot)})dw_{r}^{l} \right|^{q}ds \]
\[ \leq K \int_{0}^{t} \sup_{0 \leq r \leq s} E|e_{r}^{n}|^{q}ds + Kn^{-\frac{q}{2}+1}E \int_{0}^{t} \int_{\kappa(n,s)}^{s} (1 + |x_{r}^{n}|)^{q} |\sigma_{1}(x_{\kappa(n,\cdot)})|^{q}drds \]
\[ \leq K \int_{0}^{t} \sup_{0 \leq r \leq s} E|e_{r}^{n}|^{q}ds \]
\[ + Kn^{-\frac{q}{2}+1}E \int_{0}^{t} \int_{\kappa(n,s)}^{s} (E(1 + |x_{r}^{n}|)^{q} \frac{\sigma_{1}(x_{\kappa(n,\cdot)})}{|\sigma_{1}(x_{\kappa(n,\cdot)})|^{p}})^{\frac{q}{p}} \frac{q}{p}drds \]

and then application of Lemmas [5.2, 5.9] gives,
\[ T_{31} \leq K \int_{0}^{t} \sup_{0 \leq r \leq s} E|e_{r}^{n}|^{q}ds + Kn^{-q}. \quad (5.52) \]
Also, to estimate $T_{32}$, one uses the splitting (5.30) to obtain,

$$T_{32} := \sum_{k,i=1}^{d} E \int_{0}^{t} \int_{k(n,s)} |e_{r}^{n}|^{q-2} |b(x_{r}) - \hat{b}^{n}(x_{k(n,r)})| dr$$

$$\times \left| \int_{k(n,s)}^{s} \frac{\partial b^{k}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{k(n,r)}) dw_{r}^{l} \right| ds$$

$$\leq K \sum_{k,i=1}^{d} E \int_{0}^{t} \int_{k(n,s)} |e_{r}^{n}|^{q-1} (1 + |x_{r}| + |x_{k(n,r)}|) dr$$

$$\times \left| \int_{k(n,s)}^{s} \frac{\partial b^{k}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{k(n,r)}) dw_{r}^{l} \right| ds$$

$$+ K \sum_{k,i=1}^{d} E \int_{0}^{t} \int_{k(n,s)} |e_{r}^{n}|^{q-2} (1 + |x_{r}^{n}| + |x_{k(n,r)}|) |x_{r}^{n} - x_{k(n,r)}| dr$$

$$\times \left| \int_{k(n,s)}^{s} \frac{\partial b^{k}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{k(n,r)}) dw_{r}^{l} \right| ds$$

which on the application of Hölder’s inequality with exponents $(q - 0.5)/(q - 1)$ and $(q - 0.5)/0.5$ on first term and with exponents $(q - 1.5)/(q - 2)$ and $(q - 1.5)/0.5$ on the second and third terms yields,

$$T_{32} \leq K \sum_{k,i=1}^{d} \int_{0}^{t} \left( E \left[ \int_{k(n,s)}^{s} |e_{r}^{n}|^{q-1} (1 + |x_{r}^{n}| + |x_{k(n,r)}|) dr \right] \right)^{\frac{q-0.5}{q-1.5}} \left( E \left[ \int_{k(n,s)}^{s} \frac{\partial b^{k}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{k(n,r)}) dw_{r}^{l} \right] \right)^{\frac{q-0.5}{q-0.5}} ds$$

$$+ K \sum_{k,i=1}^{d} \int_{0}^{t} \left( E \left[ \int_{k(n,s)}^{s} |e_{r}^{n}|^{q-2} (1 + |x_{r}^{n}| + |x_{k(n,r)}|) |x_{r}^{n} - x_{k(n,r)}| dr \right] \right)^{\frac{q-1.5}{q-2}} \left( E \left[ \int_{k(n,s)}^{s} \frac{\partial b^{k}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{k(n,r)}) dw_{r}^{l} \right] \right)^{\frac{q-1.5}{q-1.5}} ds$$

$$+ K n^{-1} \sum_{k,i=1}^{d} \int_{0}^{t} \left( E \left[ \int_{k(n,s)}^{s} |e_{r}^{n}|^{q-2} |b(x_{k(n,r)})|^{2} dr \right] \right)^{\frac{q-2}{q-1.5}} \left( E \left[ \int_{k(n,s)}^{s} \frac{\partial b^{k}(x_{r}^{n})}{\partial x^{i}} \sum_{l=1}^{m} \tilde{\sigma}^{(i,l)}(x_{k(n,r)}) dw_{r}^{l} \right] \right)^{\frac{q-2}{q-0.5}} ds.$$
one obtains

\[ T_{32} \leq K \sum_{k,i=1}^{d} \int_0^t \left( n^{\frac{q-0.5}{q-1}} E \int_{\kappa(n,s)} |e_{r}^{n,q-0.5}(1 + |x_{r}|^{q} + |x_{r}^{n}|^{q})^{\frac{q-0.5}{q-1}} dr \right)^{\frac{q-1}{q-0.5}} \times \left( \sum_{l=1}^{m} E \left( \int_{\kappa(n,s)} \left| \frac{\partial h^{k}(x_{r}^{n})}{\partial x^{l}} \right|^2 \bar{\sigma}(i,l)(x_{r}^{n})^{2} dr \right)^{q-0.5} \right)^{\frac{0.5}{q-0.5}} ds \]

\[ + K \sum_{k,i=1}^{d} \int_0^t \left( n^{\frac{q-0.5}{q-2}+1} E \int_{\kappa(n,s)} |e_{r}^{n,q-1.5}(1 + |x_{r}|^{q} + |x_{r}^{n}|^{q}) \times |x_{r}^{n} - x_{r}^{n}(n,r)|^{\frac{q-1.5}{q-2}} ds \right)^{\frac{q-2}{q-1.5}} \times \left( \sum_{l=1}^{m} E \left( \int_{\kappa(n,s)} \left| \frac{\partial h^{k}(x_{r}^{n})}{\partial x^{l}} \right|^2 \bar{\sigma}(i,l)(x_{r}^{n})^{2} dr \right)^{q-1.5} \right)^{\frac{0.5}{q-1.5}} ds \]

\[ + Kn^{-1} \sum_{k,i=1}^{d} \int_0^t \left( n^{\frac{q-0.5}{q-2}+1} E \int_{\kappa(n,s)} |e_{r}^{n,q-1.5}|b(x_{r}^{n}(n,r)\right)^{2(q-0.5)}| \bar{\sigma}(x_{r}^{n})^{2(q-0.5)} ds \right)^{\frac{q-2}{q-1.5}} \times \left( \sum_{l=1}^{m} E \left( \int_{\kappa(n,s)} \left| \frac{\partial h^{k}(x_{r}^{n})}{\partial x^{l}} \right|^2 \bar{\sigma}(i,l)(x_{r}^{n})^{2} dr \right)^{q-1.5} \right)^{\frac{0.5}{q-1.5}} ds \]

which on using Hölder’s inequality along with Lemmas [5.1, 5.9] and Remarks [5.6, 5.7] gives

\[ T_{32} \leq K \int_0^t \left( n^{\frac{q-0.5}{q-1}} \int_{\kappa(n,s)} (E|e_{r}^{n,q}|^{q-0.5})^{\frac{q-1}{q-0.5}} dr \right)^{\frac{q-1}{q-0.5}} \times \left( n^{q-1.5} E \int_{\kappa(n,s)} (1 + |x_{r}^{n}|^{q+1})^{2(q-0.5)}| \bar{\sigma}(x_{r}^{n})^{2(q-0.5)} ds \right)^{\frac{q-1}{q-0.5}} ds \]

\[ + K \int_0^t \left( n^{\frac{q-1.5}{q-2}+1} \int_{\kappa(n,s)} (E|e_{r}^{n,q}|^{q-1.5})^{\frac{q-1.5}{q}} dr \right)^{\frac{q-2}{q-1.5}} \times \left[ E \left( (1 + |x_{r}^{n}|^{q} + |x_{r}^{n}(n,r)|^{q})|x_{r}^{n} - x_{r}^{n}(n,r)| \right)^{\frac{q-1.5}{q} \bar{\sigma}(x_{r}^{n})^{2(q-1.5)} ds \right)^{\frac{q-1.5}{q}} ds \]

\[ + Kn^{-1} \int_0^t \left( n^{\frac{q-1.5}{q-2}+1} \int_{\kappa(n,s)} (E|e_{r}^{n,q}|^{q-1.5})^{\frac{q-1.5}{q}} dr \right)^{\frac{q-2}{q-1.5}} \times \left[ E \left( (1 + |x_{r}^{n}|^{q} + |x_{r}^{n}(n,r)|^{q})|x_{r}^{n} - x_{r}^{n}(n,r)| \right)^{\frac{q-1.5}{q} \bar{\sigma}(x_{r}^{n})^{2(q-1.5)} ds \right)^{\frac{q-1.5}{q}} ds \]

Moreover, due to Hölder’s inequality, Lemma 5.9 and Corollary 3,

\[ T_{32} \leq K \int_0^t n^{-\frac{3}{2}} \left( \sup_{0 \leq r \leq s} E|e_{r}^{n,q}|^{q} \right)^{\frac{q-1}{q}} ds \]

\[ + Kn^{-\frac{1}{2}} \int_0^t \left( n^{\frac{q-1.5}{q-2}+1} \int_{\kappa(n,s)} (E|e_{r}^{n,q}|^{q-1.5})^{\frac{q-1.5}{q}} \left[ (E|x_{r}^{n} - x_{r}^{n}(n,r)|^{q})^{\frac{q-1.5}{q} \bar{\sigma}(x_{r}^{n})^{2(q-1.5)} ds \right)^{\frac{q-1.5}{q}} ds \right)^{\frac{q-2}{q-1.5}} ds \]

\[ + Kn^{-\frac{5}{2}} \int_0^t \left( \sup_{0 \leq r \leq s} E|e_{r}^{n,q}|^{q} \right)^{\frac{q-2}{q}} ds \]

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and then using Young’s inequality along with Lemma 5.17, one obtains

\[
T_{32} \leq K \int_0^t \sup_{0 \leq r \leq s} E|e^n_r|^q ds + Kn^{-\frac{3q}{4}}
\]

\[
+ Kn^{-\frac{3}{2}} \int_0^t \left( \sup_{0 \leq r \leq s} E|e^n_r|^q \right)^{\frac{q-2}{q}} ds + Kn^{-\frac{3q}{4}}
\]

which by Young’s inequality gives,

\[
T_{32} \leq K \int_0^t \sup_{0 \leq r \leq s} E|e^n_r|^q ds + Kn^{-\frac{3q}{4}}.
\] (5.53)

For estimating \( T_{33} \), one uses the splitting (5.32) to write

\[
T_{33} := \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)} |e^n_r|^{q-2} \sum_{v=1}^m \left( \sigma^{(k,v)}(x_r) - \tilde{\sigma}^{(k,v)}(x^n_{\kappa(n,r)}) \right) dw^v_r
\]

\[
\times \int_{\kappa(n,s)} \frac{\partial h^k(x^n_r)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) dw^l_r ds
\]

\[
= \sum_{k,i=1}^d \sum_{v=1}^m E \int_0^t \int_{\kappa(n,s)} |e^n_r|^{q-2} \left( \sigma^{(k,v)}(x_r) - \sigma^{(k,v)}(x^n_{\kappa(n,r)}) \right) dw^v_r
\]

\[
\times \int_{\kappa(n,s)} \frac{\partial h^k(x^n_r)}{\partial x^i} \tilde{\sigma}^{(i,v)}(x^n_{\kappa(n,r)}) dw^v_r ds
\]

\[
+ \sum_{k,i=1}^d \sum_{v=1}^m E \int_0^t \int_{\kappa(n,s)} |e^n_r|^{q-2} \left( \sigma^{(k,v)}(x_r) - \tilde{\sigma}^{(k,v)}(x^n_{\kappa(n,r)}) \right) dw^v_r
\]

\[
\times \int_{\kappa(n,s)} \frac{\partial h^k(x^n_r)}{\partial x^i} \tilde{\sigma}^{(i,v)}(x^n_{\kappa(n,r)}) dw^v_r ds
\]

\[
= \sum_{k,i=1}^d \sum_{v=1}^m E \int_0^t \int_{\kappa(n,s)} |e^n_r|^{q-2} \left( \sigma^{(k,v)}(x_r) - \sigma^{(k,v)}(x^n_{\kappa(n,r)}) \right)
\]

\[
\times \int_{\kappa(n,s)} \frac{\partial h^k(x^n_r)}{\partial x^i} \tilde{\sigma}^{(i,v)}(x^n_{\kappa(n,r)}) dr ds
\]

\[
+ \sum_{k,i=1}^d \sum_{v=1}^m E \int_0^t \int_{\kappa(n,s)} |e^n_r|^{q-2} \left( \sigma^{(k,v)}(x_r) - \tilde{\sigma}^{(k,v)}(x^n_{\kappa(n,r)}) \right)
\]

\[
\times \int_{\kappa(n,s)} \frac{\partial h^k(x^n_r)}{\partial x^i} \tilde{\sigma}^{(i,v)}(x^n_{\kappa(n,r)}) dr ds
\]

and then Assumption D-7 and Remark 5.7 give

\[
T_{33} \leq KE \int_0^t \int_{\kappa(n,s)} |e^n_r|^{q-2} \left( 1 + |x^n_r|^\xi^+ \right) |\tilde{\sigma}(x^n_{\kappa(n,r)})| dr ds
\]

\[
+ KE \int_0^t \int_{\kappa(n,s)} |e^n_r|^{q-2} |\sigma(x^n_r) - \tilde{\sigma}(x^n_{\kappa(n,r)})| \left( 1 + |x^n_r|^\xi^+ \right) |\tilde{\sigma}(x^n_{\kappa(n,r)})| dr ds
\] (5.54)
which due to Hölder’s inequality becomes

\[
T_{33} \leq K \int_0^t \int_{\kappa(n,s)} \left( E|e_r^n|^q \right)^{\frac{q-1}{q}} \left( E(1 + |x_r^n|^\gamma + 1)^q |\tilde{\sigma}(x^n_{\kappa(n,r)})|^{\gamma} \right)^{\frac{1}{q}} dr ds \\
+ K \int_0^t \int_{\kappa(n,s)} \left( E|e_r^n|^q \right)^{\frac{q-2}{q}} \left( E|\sigma(x^n_r) - \tilde{\sigma}(x^n_{\kappa(n,r)})|^{\gamma} \right)^{\frac{1}{q}} dr ds \\
\times \left( 1 + |x_r^n|^\gamma \right)^{\frac{2}{q}} |\tilde{\sigma}(x^n_{\kappa(n,r)})|^{\frac{2}{q}} dr ds \\
\leq K \int_0^t \sup_{0 \leq r \leq s} E|e_r^n|^q ds \\
+ K \int_0^t \int_{\kappa(n,s)} \left( E(1 + |x_r^n|^\gamma + 1)^q |\tilde{\sigma}(x^n_{\kappa(n,r)})|^{\gamma} \right)^{\frac{1}{q}} dr ds \\
\times \left( 1 + |x_r^n|^\gamma \right)^{\frac{2}{q}} |\tilde{\sigma}(x^n_{\kappa(n,r)})|^{\frac{2}{q}} dr ds.
\]

Also, due to Young’s inequality and Hölder’s inequality, one gets

\[
T_{33} \leq K \int_0^t E|e_r^n|^q ds + Kn^{-q+1} \int_0^t \int_{\kappa(n,s)} E(1 + |x_r^n|^\gamma + 1)^q |\tilde{\sigma}(x^n_{\kappa(n,r)})|^{\gamma} dr ds \\
+ K n^{-\frac{q}{2} + 1} \int_0^t \int_{\kappa(n,s)} E|\sigma(x^n_r) - \tilde{\sigma}(x^n_{\kappa(n,r)})|^{\gamma} \left( 1 + |x_r^n|^\gamma \right)^{\frac{2}{q}} |\tilde{\sigma}(x^n_{\kappa(n,r)})|^{\frac{2}{q}} dr ds \\
\leq K \int_0^t \sup_{0 \leq r \leq s} E|e_r^n|^q ds \\
+ K n^{-q+1} \int_0^t \int_{\kappa(n,s)} \left( E(1 + |x_r^n|^\gamma + 1)^{2q} E|\tilde{\sigma}(x^n_{\kappa(n,r)})|^{2q} \right)^{\frac{1}{2}} dr ds \\
+ K n^{-\frac{q}{2} + 1} \int_0^t \int_{\kappa(n,s)} \left( E|\sigma(x^n_r) - \tilde{\sigma}(x^n_{\kappa(n,r)})|^{\gamma} E(1 + |x_r^n|^\gamma + 1)^{\frac{q}{2}} |\tilde{\sigma}(x^n_{\kappa(n,r)})|^{\frac{q}{2}} \right)^{\frac{1}{2}} dr ds.
\]

which on the application of Lemmas [5.9, 5.19] and Corollary 3 yields

\[
T_{33} \leq K \int_0^t \sup_{0 \leq r \leq s} E|e_r^n|^q ds + Kn^{-q}.
\]

To estimate \( T_{34} \), one observes that

\[
T_{34} := K \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)} e_r^n |e_r^n|^{q-4} e_r^n (\sigma(x^n_r) - \tilde{\sigma}(x^n_{\kappa(n,r)})) dw_r \\
\times \int_{\kappa(n,s)} \frac{\partial \sigma^{(i,j)}(x^n_{\kappa(n,r)})}{\partial x^i} du_r^i ds \\
= K \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)} e_r^n |e_r^n|^{q-4} \sum_{u=1}^d \sum_{v=1}^m e_r^n (\sigma^{(u,v)}(x^n_r) - \tilde{\sigma}^{(u,v)}(x^n_{\kappa(n,r)})) dw_r^v \\
\times \int_{\kappa(n,s)} \frac{\partial \sigma^{(i,j)}(x^n_{\kappa(n,r)})}{\partial x^i} du_r^i ds.
\]
\[ T_{34} \leq KE \int_0^t \int_{\kappa(n,s)} |e^n_r|^q |1 + |x^n_r|^\chi+1| |\tilde{\sigma}(x^n_{\kappa(n,r)})| drds \]
\[ + KE \int_0^t \int_{\kappa(n,s)} |e^n_r|^{q-2} |\sigma(x^n_r) - \tilde{\sigma}(x^n_{\kappa(n,r)})| (1 + |x^n_r|^\chi+1) |\tilde{\sigma}(x^n_{\kappa(n,r)})| drds \]

which is exactly the same as \( T_{33} \) given in equation (5.54). Thus, by proceeding exactly the same, one obtains,

\[ T_{34} \leq K \int_0^t \sup_{0 \leq r \leq s} E|e^n_r|^q ds + Kn^{-q}. \quad (5.56) \]

Moreover, for estimating \( T_{35} \), one again uses the splitting (5.32) to obtain,

\[ T_{35} := K \sum_{k,i=1}^d \int_0^t \int_{\kappa(n,s)} |e^n_r|^{q-3} |\sigma(x_r) - \tilde{\sigma}(x^n_{\kappa(n,r)})|^2 dr \]
\[ \times \left| \int_{\kappa(n,s)} \frac{\partial b^k(x^n_r)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) dw^l_r \right| ds \]
\[ \leq K \sum_{k,i=1}^d \int_0^t \int_{\kappa(n,s)} |e^n_r|^{q-1} dr \left( \int_{\kappa(n,s)} \frac{\partial b^k(x^n_r)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) dw^l_r \right) ds \]
\[ + K \sum_{k,i=1}^d \int_0^t \int_{\kappa(n,s)} |e^n_r|^{q-3} |\sigma(x^n_r) - \tilde{\sigma}(x^n_{\kappa(n,r)})|^2 dr \]
\[ \times \left| \int_{\kappa(n,s)} \frac{\partial b^k(x^n_r)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) dw^l_r \right| ds \]

which due to Hölder’s inequality implies

\[ T_{35} \leq K \sum_{k,i=1}^d \int_0^t \left( E\left[ \int_{\kappa(n,s)} |e^n_r|^{q-1} dr \right] ^{\frac{q}{q-1}} \right) ^{\frac{q-1}{q}} \]
\[ \times \left( \int_{\kappa(n,s)} \left| \int_{\kappa(n,s)} \frac{\partial b^k(x^n_r)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x^n_{\kappa(n,r)}) dw^l_r \right| ^q \right) ^{\frac{1}{q}} ds \]
Let us take 

\[ \text{Proof.} \]

(5.45)

Then, the tamed Milstein scheme completes the proof.

Finally, on combining the estimates from (5.48), (5.49) and (5.58) in (5.47), one obtains

\[
T_{35} \leq Kn^{-\frac{2}{q}} \int_0^t \left( n^{-\frac{q-1}{q}} + E \int_{\kappa(n,s)} |e_{r}^{n}|^q dr \right)^{\frac{q-1}{q}} ds
\]

\[
+ Kn^{-\frac{2}{q}} \int_0^t \left( n^{-\frac{q-1}{q}} + E \int_{\kappa(n,s)} |e_{r}^{n}|^{(q-3)q} |\sigma(x_{r}^{n}) - \tilde{\sigma}(x_{\kappa(n,s)}^{n})|^{q/4} dr \right)^{\frac{q-1}{q}} ds
\]

\[
\leq Kn^{-\frac{2}{q}} \int_0^t \left( \sup_{0 \leq r \leq s} E|e_{r}^{n}|^q \right)^{\frac{q-1}{q}} ds + Kn^{-\frac{2}{q}} \int_0^t \left( \sup_{0 \leq r \leq s} E|e_{r}^{n}|^q \right)^{\frac{q-1}{q}} ds + Kn^{-\frac{2}{q}} \int_0^t \left( \sup_{0 \leq r \leq s} E|e_{r}^{n}|^q \right)^{\frac{q-3}{q}} ds.
\]

Thus, by Young’s inequality,

\[
T_{35} \leq K \int_0^t \sup_{0 \leq r \leq s} E|e_{r}^{n}|^q ds + Kn^{-\frac{2q}{4}} + Kn^{-\frac{2q}{4}}.
\]  

(5.57)

Hence, on substituting the estimates from (5.52), (5.53), (5.55), (5.56) and (5.57) in (5.51), one obtains

\[
T_3 \leq K \int_0^t \sup_{0 \leq r \leq s} E|e_{r}^{n}|^q ds + Kn^{-q}.
\]  

(5.58)

Finally, on combining the estimates from (5.48), (5.49) and (5.58) in (5.47), one completes the proof.

\[
\square
\]

**Theorem 5.3.** Let Assumptions D-4 and D-7 to D-9 (for \( \gamma \equiv 0 \)) with \( p \geq 3\chi + 6 \) hold. Then, the tamed Milstein scheme (5.45) converges to the solution of SDE (5.44) in \( L^q \) with rate 1 i.e.,

\[
\sup_{0 \leq t \leq T} E|x_t - x_t^n|^q \leq K n^{-q}
\]

where \( 0 < q \leq p/(3\chi + 6) \).

**Proof.** Let us take \( e_t^n := x_t - x_t^n \) i.e.,

\[
e_t^n = \int_0^t (b(x_s) - \tilde{b}(x_{\kappa(n,s)}))ds + \int_0^t (\sigma(x_s) - \tilde{\sigma}(x_{\kappa(n,s)}))dw_s
\]
for any $t \in [0, T]$. By Itô’s formula, for any $q \geq 2$,

$$
|e^n_t|^q = q \int_0^t |e^n_s| q^{-2} e^n_s (b(x_s) - b^*(x^n_{\kappa(n,s)})) ds \\
+ q \int_0^t |e^n_s| q^{-2} e^n_s (\sigma(x_s) - \tilde{\sigma}(x^n_{\kappa(n,s)})) dw_s \\
+ \frac{q(q - 2)}{2} \int_0^t |e^n_s| q^{-4} |(\sigma(x_s) - \tilde{\sigma}(x^n_{\kappa(n,s)})) e^n_s|^2 ds \\
+ \frac{q}{2} \int_0^t |e^n_s| q^{-2} |\sigma(x_s) - \tilde{\sigma}(x^n_{\kappa(n,s)})|^2 ds
$$

almost surely for any $t \in [0, T]$, which on taking expectation and using Schwarz inequality implies

$$
E|e^n_t|^q \leq qE \int_0^t |e^n_s| q^{-2} e^n_s (b(x_s) - b^*(x^n_{\kappa(n,s)})) ds \\
+ K \int_0^t |e^n_s| q^{-2} |\sigma(x_s) - \tilde{\sigma}(x^n_{\kappa(n,s)})|^2 ds.
$$

(5.59)

Thus, by using the estimates from (5.39) and (5.41), one obtains,

$$
E|e^n_t|^q \leq KE \int_0^t |e^n_s|^q ds + KE \int_0^t |e^n_s| q^{-2} e^n_s (b(x^n_s) - b^n(x^n_{\kappa(n,s)})) ds \\
+ KE \int_0^t |b(x^n_{\kappa(n,s)}) - b^n(x^n_{\kappa(n,s)})|^q ds \\
+ KE \int_0^t |\sigma(x^n_s) - \tilde{\sigma}(x^n_{\kappa(n,s)})|^q ds
$$

for any $t \in [0, T]$. Further, the application of Lemmas [5.18, 5.19, 5.20] gives

$$
\sup_{0 \leq t \leq u} E|e^n_t|^q \leq K \int_0^u \sup_{0 \leq r \leq s} E|e^n_r|^q ds + Kn^{-q}
$$

for any $u \in [0, T]$ and hence Gronwall’s lemma completes the proof. \(\square\)

**Corollary 5.** Let Assumptions D-4 and D-7 to D-9 (for $\gamma \equiv 0$) with $p \geq 3\chi + 6$ hold. Then, the tamed Milstein scheme (5.45) converges to the solution of SDE (5.44) in $L^q$ with rate 1 i.e.,

$$
\sup_{0 \leq t \leq T} E|x_t - x^n_t|^q \leq Kn^{-q}
$$

where $0 < q < p/(3\chi + 6)$.

**Proof.** This follows due to Theorem 5.3 and Lemma 1.6. \(\square\)
Chapter 6

Numerical Simulation

In this chapter, we present some numerical simulations to illustrate theoretical results obtained in this report. Let us first introduce the notations for this chapter. SDEs and their schemes considered in this chapter are defined on the interval \([0, 1]\), which is divided into \(2^n\) sub-intervals of equal length i.e. the step-size is \(h = 2^{-n}\) for some \(n \in \mathbb{N}\). Further, \(x_{lh}^h\) stands for the approximation obtained by the scheme under consideration at \(lh\)-th grid for \(l = 0, \ldots, 2^n - 1\). Also, \(\Delta w_{lh}^j := w_{(l+1)h}^j - w_{lh}^j\) denotes the increment of \(w^j\) in the interval \([lh, (l+1)h]\) for \(j = 1, \ldots, m\) and \(l = 0, \ldots, 2^n - 1\). Moreover, \(N(t, \mathbb{R})\) is a Poisson process with intensity \(\lambda\) and \(z_i\) denotes \(i\)-th mark at jump-time \(\tau_i\) for any \(i = N(lh, \mathbb{R}) + 1, \ldots, N((l+1)h, \mathbb{R})\) and \(l = 0, \ldots, 2^n - 1\). Finally, \(I_{\{j=k\}}\) is equal to 1 if \(j = k\) and 0 otherwise.

C codes of tamed Euler scheme are included in Appendix A and that of Milstein scheme are given in Appendix B. All the codes are implemented in MPI parallel system.

6.1 Tamed Euler Scheme

In this section, we shall discuss several SDEs and SDDEs driven by Lévy noise with non-linear drift coefficients. The simulation findings that are presented below confirm our theoretical results.

**Example 1.** Consider the following SDE,

\[
\begin{align*}
\frac{dx_t}{dt} &= x_t - 0.10x_t^3dt + x_t dw_t + \int_{\mathbb{R}} x_t z \tilde{N}(dt, dz) \\
&= x_t(1 - 0.10x_t^3 + \int_{\mathbb{R}} x_t z \tilde{N}(dt, dz))
\end{align*}
\]

(6.1)

for any \(t \in [0, 1]\) with \(x_0 = 1\). The tamed Euler scheme of SDE (6.1) at \((l+1)h\)-th grid point is given by

\[
x_{(l+1)h}^h = x_{lh}^h + \frac{x_{lh}^h - 0.10(x_{lh}^h)^3}{1 + \sqrt{h}|x_{lh}^h - 0.10(x_{lh}^h)^3|}h + x_{lh}^h \Delta w_{lh} + x_{lh}^h \sum_{i=N(lh,\mathbb{R})}^{N((l+1)h,\mathbb{R})} z_i
\]

(6.2)

with \(x_0^h = x_0 = 1\) for \(l = 0, \ldots, 2^n - 1\), where step-size \(h = 2^{-n}\) for \(n \in \mathbb{N}\). Since SDE (6.1) has no explicit solution, thus tamed Euler scheme (6.2) with step-size \(h = 2^{-25}\) is taken as the true solution of SDE (6.1). In Table 6.1 and Figure 6.1, the number of paths is taken to be 24000. Table 6.1a and their corresponding Figures [6.1a, 6.1b] are constructed under the assumption that the mark distribution is normal with mean 0 and variance 0.125. Similarly, mark random variables are assumed to follow uniform on \([-1/4, 1/4]\) in Table 6.1b and their corresponding Figures [6.1c, 6.1d]. One can observe...
from the simulation outputs in Table 6.1 and Figure 6.1 that the rate of convergence depends on average number of jumps and mark distribution.

\[
h = 2^{-n} \quad \sqrt{E|\mathbf{x}_T - \mathbf{x}_h|^2}
\]

<table>
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<tr>
<th>$h = 2^{-n}$</th>
<th>$\lambda = 3.0$</th>
<th>$\lambda = 5.0$</th>
<th>$h = 2^{-n}$</th>
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<th>$\lambda = 5.0$</th>
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<tr>
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<td>$2^{-10}$</td>
<td>0.15211174</td>
<td>0.17555676</td>
</tr>
</tbody>
</table>

(a) Mark is normal with mean 0 and variance 0.125. (b) Mark is uniform on $[-1/4, 1/4]$.

<table>
<thead>
<tr>
<th>$\lambda = 3.0$</th>
<th>$\lambda = 5.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-24}$</td>
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<tr>
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<tr>
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<tr>
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<td>0.13868676</td>
</tr>
<tr>
<td>$2^{-11}$</td>
<td>0.20299659</td>
</tr>
<tr>
<td>$2^{-10}$</td>
<td>0.30699265</td>
</tr>
</tbody>
</table>

(c) Mark is normal with mean 0 and variance 0.125. (d) Mark is uniform on $[-1/4, 1/4]$.

The rates of convergence in Figures 6.1a, 6.1b, 6.1c and 6.1d are respectively
0.59456, 0.62021, 0.59003 and 0.60298. These rates are close to the theoretical rate 0.5 of convergence. The reasons of deviation of these rates from the theoretical ones are sampling error, inherent bias in pseudo random number generation and rounding error. However, the sampling error is usually the most significant among these three. One can also notice that the mark distribution significantly affects the experimental rates of convergence since sampling error primarily depends on the mark distribution. In the following examples, similar behaviour of the experimental rates of convergence vis-a-vis the corresponding theoretical rates of convergence are observed.

Example 2. Consider the following SDDE,

\[ dx_t = (x_t - x_t^3 - x_t^{2.5})dt + (x_t - x_t^3)dw_t + \int_{\mathbb{R}} (x_t + x_t^{2.5})z\tilde{N}(dt, dz) \] (6.3)

for \( t \in [0, 1] \) with initial data \( \xi_t = 1 + t \) for \( t \in [-0.5, 0] \). The tamed Euler scheme of SDE (6.3) is given by

\[ x_{lh+h}^h = x_{lh}^h + \frac{x_{lh}^h - (x_{lh}^h)^3 - (x_{lh+h-0.5}^h)^3}{1 + \sqrt{h}} \left( x_{lh}^h - (x_{lh}^h)^3 - (x_{lh+h-0.5}^h)^3 \right) + (x_{lh}^h + (x_{lh+h-0.5}^h)^3)\Delta w_t \]

for \( \ell = 0, \ldots, 2^n - 1 \) with the same initial data as given for SDDE (6.3). In Table 6.2 and Figure 6.2, the tamed Euler scheme (6.4) with \( h = 2^{-25} \) is taken as true solution of SDDE (6.3).

Table 6.2a, Figure 6.2a and Figure 6.2b are based on 60000 paths with mark distribution as normal with mean 0 and variance 0.125 where two cases \( \lambda = 1 \) and \( \lambda = 3 \) are considered.

Table 6.2b, Figure 6.2c and Figure 6.2d are based on 60000 paths with mark distribution as uniform on \([-1/4, 1/4]\) where two cases \( \lambda = 1 \) and \( \lambda = 3 \) are considered.

\[
\begin{array}{cccc} 
| h = 2^{-n} | & | \sqrt{E[x_T - x_T^h]^2} | & \lambda = 1.0 & \lambda = 3.0 \\
2^{-24} & 0.00045939 & 0.00125835 & \\
2^{-23} & 0.00100674 & 0.00301770 & \\
2^{-22} & 0.00175679 & 0.00556465 & \\
2^{-21} & 0.00277471 & 0.00937316 & \\
2^{-20} & 0.00430431 & 0.01512999 & \\
2^{-19} & 0.00633029 & 0.02370223 & \\
2^{-18} & 0.00967029 & 0.03593179 & \\
2^{-17} & 0.01440737 & 0.05305247 & \\
2^{-16} & 0.02055054 & 0.07896327 & \\
2^{-15} & 0.02962555 & 0.13558553 & \\
2^{-14} & 0.04377431 & 0.20160320 & \\
2^{-13} & 0.07131762 & 0.30480352 & \\
2^{-12} & 0.14524791 & 0.58792957 & \\
2^{-11} & 0.28893682 & 1.25012444 & \\
2^{-10} & 0.50634323 & 1.75807788 & \\
\end{array}
\]

\[
\begin{array}{cccc} 
| h = 2^{-n} | & | \sqrt{E[x_T - x_T^h]^2} | & \lambda = 1.0 & \lambda = 3.0 \\
2^{-24} & 0.00027530 & 0.00030324 & \\
2^{-23} & 0.00053211 & 0.00058797 & \\
2^{-22} & 0.00088093 & 0.00096236 & \\
2^{-21} & 0.00136662 & 0.00149429 & \\
2^{-20} & 0.00204834 & 0.00224583 & \\
2^{-19} & 0.00302734 & 0.00334352 & \\
2^{-18} & 0.00438932 & 0.00483870 & \\
2^{-17} & 0.00634224 & 0.00704321 & \\
2^{-16} & 0.00921768 & 0.01027825 & \\
2^{-15} & 0.01322013 & 0.01507885 & \\
2^{-14} & 0.01901493 & 0.02245324 & \\
2^{-13} & 0.02752557 & 0.03535582 & \\
2^{-12} & 0.04060207 & 0.06105942 & \\
2^{-11} & 0.06250819 & 0.10558130 & \\
2^{-10} & 0.09849757 & 0.18046795 & \\
\end{array}
\]

(a) Mark is normal with mean 0 and variance 0.125. (b) Mark is uniform on \([-1/4, 1/4]\).

Table 6.2: Tamed Euler scheme (6.4) of SDDE (6.3).
(a) $\lambda = 1$ and mark is normal with mean 0 and variance 0.125

(b) $\lambda = 3$ and mark is normal with mean 0 and variance 0.125

(c) $\lambda = 1$ and mark is uniform on $[-1/4, 1/4]$  

(d) $\lambda = 3$ and mark is uniform on $[-1/4, 1/4]$

Figure 6.2: $L^2$-convergence rate of tamed Euler scheme (6.4) of SDDE (6.3).

In Figures 6.2a, 6.2b, 6.2c and 6.2d, the rates of convergence are respectively $0.66382$, $0.72915$, $0.57345$ and $0.61814$ which are close to the theoretical rate 0.5. In addition to the factors as explained in Example 1, the presence of delay also affects the sampling error and hence the experimental rates of convergence.

### 6.2 Tamed Milstein Scheme

The tamed Milstein scheme introduced in Chapter 5 are not implementable on computer due to the presence of multiple stochastic integrals. However, appropriate commutative conditions (as mentioned below) on the coefficients of SDEs (5.1) and (5.44) allow implementation of corresponding schemes (5.3) and (5.45) on computer. One can refer to [36, 54] for details.

First, let us consider the continuous case. The commutative condition on the diffusion coefficient of SDE (5.44) is as follows,

$$\sum_{u=1}^{d} \sigma^{(u,j)}(x) \frac{\partial \sigma^{(i,k)}(x)}{\partial x^u} = \sum_{u=1}^{d} \sigma^{(u,k)}(x) \frac{\partial \sigma^{(i,j)}(x)}{\partial x^u}$$

(6.5)

for any $k, j = 1, \ldots, m, i = 1, \ldots, d$ and $x \in \mathbb{R}^d$. This commutative property is useful
since one can use the following relationship
\[
\int_{lh}^{lh+h} \int_{lh}^{s} dw_r^j dw_s^k + \int_{lh}^{lh+h} \int_{lh}^{s} dw_r^k dw_s^j = \frac{1}{2} (\Delta w_r^j \Delta w_s^k - h I_{j=k})
\]
and hence can write the scheme (5.45) at \((lh + h)\)-th grid as given below,
\[
x_{lh+h}^h = x_{lh}^h + \hat{b}^n(x_{lh}^h)h + \sum_{k=1}^{m} \sigma^{(k)}(x_{lh}^h) \Delta w_{lh}^k
\]
\[
+ \frac{1}{2} \sum_{j,k=1}^m \sum_{u=1}^d \sigma^{(u,j)}(x_{lh}^h) \frac{\partial \sigma^{(k)}(x_{lh}^h)}{\partial x^u} \{\Delta w_{lh}^j \Delta w_{lh}^k - h I_{j=k}\}
\] (6.6)
where \(\sigma^{(k)}\) denotes the \(k\)-th column of \(\sigma\) for \(k = 1, \ldots, m\). The scheme defined in (6.6) can be easily simulated. Below, we consider some SDEs which satisfy the commutative condition (6.5) and verify that the tamed Milstein scheme (6.6) converges in \(L^q\)-sense to the true solution of SDE (5.44) with rate 1.0.

**Example 3.** Consider the following one-dimensional SDE,
\[
dx_t = (x_t - x_t^3) dt + x_t dw_t
\] (6.7)
for any \(t \in [0, 1]\) with \(x_0 = 1\). One can easily check that the commutative property (6.5) is satisfied by the diffusion coefficient \(\sigma(x) = x\). The tamed Milstein scheme of SDE (6.7) at \((lh + h)\)-th grid is given by
\[
x_{lh+h}^h = x_{lh}^h + \frac{x_{lh}^h - (x_{lh}^h)^3}{1 + h(x_{lh}^h - (x_{lh}^h)^3)^2} h + x_{lh}^h \Delta w_{lh} + \frac{1}{2} x_{lh}^h ((\Delta w_{lh})^2 - h)
\] (6.8)
for \(l = 0, \ldots, 2^n - 1\) where \(h = 2^{-n}\) for some \(n \in \mathbb{N}\). The tamed Milstein scheme (6.8) with \(h = 2^{-25}\) is taken as true solution. Table 6.3 and Figure 6.3 are based on 10000 paths and show that the tamed Milstein scheme (6.8) converges to the true solution of SDE (6.7) in \(L^q\)-sense. Moreover, the rates of convergence are 0.98615, 1.0229 and 1.0386 for \(L^1\), \(L^2\) and \(L^3\) respectively. One can notice that these rates of convergence become very close to the predicted rate 1.0 of convergence with less number of paths than the number of paths required when trajectories have jumps. Similar findings are observed in the following examples.

| \(h = 2^{-n}\) | \(E|T - x_T^h|\) | \(\sqrt{E|T - x_T^h|^2}\) | \(\frac{3}{2} \sqrt{E|T - x_T^h|^3}\) |
|---|---|---|---|
| 2^{-21} | 0.000013 | 0.000013 | 0.000013 |
| 2^{-20} | 0.000027 | 0.000027 | 0.000027 |
| 2^{-19} | 0.000055 | 0.000055 | 0.000055 |
| 2^{-18} | 0.000112 | 0.000112 | 0.000112 |
| 2^{-17} | 0.000226 | 0.000226 | 0.000226 |
| 2^{-16} | 0.000452 | 0.000452 | 0.000452 |
| 2^{-15} | 0.000908 | 0.000908 | 0.000908 |
| 2^{-14} | 0.001839 | 0.001839 | 0.001839 |
| 2^{-13} | 0.003755 | 0.003755 | 0.003755 |
| 2^{-12} | 0.007828 | 0.007828 | 0.007828 |
| 2^{-11} | 0.017083 | 0.017083 | 0.017083 |
| 2^{-10} | 0.040887 | 0.040887 | 0.040887 |

Table 6.3: Tamed Milstein scheme (6.8) of SDE (6.7)
Example 4. Let us consider another one-dimensional SDE defined by,
\[ dx_t = (x_t - x_t^5)dt + x_t dw_t \]  
(6.9)
for any \( t \in [0,1] \) with \( x_0 = 1 \). The tamed Milstein scheme of SDE (6.9) at \((lh + h)\)-th grid is given by
\[ x_{lh}^{h} + \sqrt{h} x_{lh}^{(x_{lh}^5)} h + x_{lh}^{h} \Delta w_{lh} + \frac{1}{2} x_{lh}^{h} \{ (\Delta w_{lh})^2 - h \} \]  
(6.10)
for \( l = 0, \ldots, 2^n - 1 \) with \( h = 2^{-n} \). The scheme with \( h = 2^{-21} \) is taken as true solution of SDE (6.9). From Table 6.4 and Figure 6.4, one can notice that \( L^q \) convergence rates of scheme (6.10) are approximately 1.0 for all \( q = 1, \ldots, 5 \). The number of paths considered is 60000.  

| \( h = 2^{-n} \) | \( E|x_T - x_T^h| \) | \( \sqrt{E}(x_T - x_T^h)^2 \) | \( \sqrt{E}(x_T - x_T^h)^3 \) | \( \sqrt{E}(x_T - x_T^h)^4 \) | \( \sqrt{E}(x_T - x_T^h)^5 \) |
|---|---|---|---|---|---|
| \( 2^{-20} \) | 0.0000007223 | 0.0000072739 | 0.0000071338 | 0.0000133872 | 0.0000000505 |
| \( 2^{-19} \) | 0.00000020488 | 0.0000081546 | 0.0000037173 | 0.0000144860 | 0.0000000579 |
| \( 2^{-18} \) | 0.00000046829 | 0.0000081546 | 0.0000037173 | 0.0000144860 | 0.0000000579 |
| \( 2^{-17} \) | 0.00000095256 | 0.0000081546 | 0.0000037173 | 0.0000144860 | 0.0000000579 |
| \( 2^{-16} \) | 0.0000205589 | 0.0000081546 | 0.0000037173 | 0.0000144860 | 0.0000000579 |
| \( 2^{-15} \) | 0.0000417394 | 0.0000081546 | 0.0000037173 | 0.0000144860 | 0.0000000579 |
| \( 2^{-14} \) | 0.0000839482 | 0.0000081546 | 0.0000037173 | 0.0000144860 | 0.0000000579 |
| \( 2^{-13} \) | 0.0001710052 | 0.0000081546 | 0.0000037173 | 0.0000144860 | 0.0000000579 |
| \( 2^{-12} \) | 0.0003470798 | 0.0000081546 | 0.0000037173 | 0.0000144860 | 0.0000000579 |
| \( 2^{-11} \) | 0.0007231189 | 0.0000081546 | 0.0000037173 | 0.0000144860 | 0.0000000579 |

Table 6.4: Tamed Milstein scheme (6.10) of SDE (6.9)

We now proceed to SDE (5.1) and its tamed Milstein scheme (5.3). As mentioned before, one requires commutative assumptions in order to make the scheme (5.3) implementable. In addition to the commutative property (6.5) on diffusion coefficient, one also needs commutative property on jump coefficient which allows to obtain explicit values of multiple stochastic integrals appearing in scheme (5.3). For the purpose of
simplicity, let us assume that
\[ \int_Z \gamma(x, z) \nu(dz) = 0 \] (6.11)
for any \( x \in \mathbb{R}^d \). Thus, SDE (5.1) becomes
\[ x_t = \xi + \int_0^t b(x_s) ds + \int_0^t \sigma(x_s) dw_s + \int_0^t \int_Z \gamma(x_s, z) N(ds, dz) \] (6.12)
for any \( t \in [0, T] \). As a consequence, the tamed Milstein scheme (5.3) at the \((lh + h)\)-th grid can be written as,
\[
x_{lh+h}^h = x_{lh}^h + \tilde{b}^n(x_{lh}^h)h + \sum_{k=1}^m \sigma^{(k)}(x_{lh}^h) \Delta w_{lh}^k + \int_{lh}^{lh+h} \int_Z \gamma(x_{lh}^h, z_2) N(ds, dz_2)
\]
\[ + \sum_{j=1}^m \sum_{k=1}^d \int_{lh}^{lh+h} \int_{lh}^s \sigma^{(u,j)}(x_{lh}^h) \frac{\partial \sigma^{(k)}(x_{lh}^h)}{\partial x^u} d w_i^j d w^k_s
\]
\[ + \sum_{j=1}^m \int_{lh}^{lh+h} \int_{lh}^s \left\{ \sigma^{(j)}(x_{lh}^h + \gamma(x_{lh}, z_1)) - \sigma^{(j)}(x_{lh}^h) \right\} N(dr, dz_1) d w^j_s
\]
\[ + \sum_{j=1}^m \sum_{k=1}^d \int_{lh}^{lh+h} \int_{lh}^s \int_{lh}^Z \frac{\partial \gamma(x_{lh}^h, z_2)}{\partial x^u} \sigma^{(u,j)}(x_{lh}^h) d w_i^j N(ds, dz_2)
\]
\[ + \int_{lh}^{lh+h} \int_{lh}^s \int_{lh}^Z \left\{ \gamma(x_{lh}^h + \gamma(x_{lh}, z_1), z_2) - \gamma(x_{lh}^h, z_2) \right\} N(dr, dz_1) N(ds, dz_2) \] (6.13)
for any \( l = 0, \ldots, 2^n - 1 \), where \( h = 2^{-n} \) for \( n \in \mathbb{N} \). The drift coefficient \( \tilde{b}^n \) of scheme (6.13) is defined in equation (5.2). In this section, one-dimensional case of scheme (6.13) is considered which due to commutative condition (6.5) gives the following scheme
\[
x_{lh+h}^h = x_{lh}^h + \tilde{b}^n(x_{lh}^h)h + \sigma(x_{lh}^h) \Delta w_{lh} + \sum_{i=\mathbb{N}(lh,Z)+1}^{N(lh+h,Z)} \gamma(x_{lh}^h, z_i)
\]
\[ + \frac{1}{2} \sigma(x_{lh}^h) \frac{\partial \sigma(x_{lh}^h)}{\partial x^u} \left\{ (\Delta w_{lh})^2 - h \right\}
\]
\[ + \sum_{i=\mathbb{N}(lh,Z)+1}^{N(lh+h,Z)} \left\{ \sigma(x_{lh}^h + \gamma(x_{lh}, z_i)) - \sigma(x_{lh}^h) \right\} (w_{lh+h} - w_{lh})
\]
\[ + \sigma(x_{lh}^h) \sum_{i=\mathbb{N}(lh,Z)+1}^{N(lh+h,Z)} \frac{\partial \gamma(x_{lh}^h, z_i)}{\partial x^u} (w_{lh} - w_{lh})
\]
\[ + \sum_{j=\mathbb{N}(lh,Z)+1}^{N(lh+h,Z)} \sum_{i=\mathbb{N}(lh,Z)+1}^{N(lj,Z)} \left\{ \gamma(x_{lh}^h + \gamma(x_{lh}, z_i), z_j) - \gamma(x_{lh}^h, z_j) \right\} \] (6.14)
for \( l = 0, \ldots, 2^n - 1 \). This one dimensional scheme is easy to implement and several examples are presented below to illustrate that it has rate of convergence equal to 1.0. The multi-dimensional case of scheme (6.13) with mark-dependent jump coefficient is
computationally difficult to simulate. However, the mark-independent case i.e. when \( \gamma(x,z) = \gamma(x) \) for any \( x \in \mathbb{R}^d \) and \( z \in \mathbb{Z} \), can be simplified by imposing the jump commutative condition as given below,

\[
\sigma^{(k,j)}(x + \gamma(x)) - \sigma^{(k,j)}(x) = \sum_{u=1}^{d} \frac{\partial \gamma^{(k)}}{\partial x^{u}} \sigma^{(u,j)}(x) \tag{6.15}
\]

for any \( x \in \mathbb{R}^d \), \( k = 1, \ldots, d \) and \( j = 1, \ldots, m \). Thus, the tamed Milstein scheme \( (6.14) \) of SDE \( (6.12) \) with diffusion and jump coefficients satisfying commutative conditions \( (6.5) \) and \( (6.15) \), can be written as

\[
x_h^{lh} = x_h^{0} + b^h(x_h^{0})h + \sum_{k=1}^{m} \sigma^{(k)}(x_h^{0}) \Delta w_h^k + \gamma(x_h^{0}) \Delta N(lh, Z) \\
+ \frac{1}{2} \sum_{j,h}^{m} \sum_{u=1}^{d} \sigma^{(u,j)}(x_h^{0}) \frac{\partial \sigma^{(k)}}{\partial x^{u}} \{ \Delta w_h^j \Delta w_h^k - hI_{(j=k)} \} \\
+ \sum_{j=1}^{m} \{ \sigma^{(j)}(x_h^{0}) + \gamma(x_h^{0}) \} \Delta N(lh, Z) \Delta w_h^j \\
+ \frac{1}{2} \{ \gamma(x_h^{0}) \} \{ (\Delta N(lh, Z))^2 - \Delta N(lh, Z) \} \tag{6.16}
\]

where \( \Delta N(lh, Z) = N(lh + h, Z) - N(lh, Z) \) (i.e. the number of jumps in \([lh, lh + h]\)) for any \( l = 0, \ldots, 2^n - 1 \).

Notice that when assumption \( (6.11) \) does not hold, one can implement the scheme \( (6.13) \) by replacing the drift coefficient \( b^0(x) \) with \( b^0(x) - \int_{Z} \gamma(x,z) \nu(dz) \).

Let us now consider some examples of non-linear one dimensional SDE and implement tamed Milstein scheme \( (6.14) \) to verify our theoretical findings.

**Example 5.** Consider the following one-dimensional SDE,

\[
dx_t = -0.10x^2_t dt + x_t dw_t + \int_{\mathbb{R}} x_t \tilde{N}(dt, dz) \tag{6.17}
\]

for any \( t \in [0, 1] \) with \( x_0 = 1 \). The examples below satisfy equation \( (6.11) \) and thus tamed Milstein scheme \( (6.14) \) can be used. More specially, we consider the following tamed Milstein scheme of SDE \( (6.17) \),

\[
x_h^{lh} = x_h^{0} + \frac{-0.10(x_h^{0})^3}{1 + h|0.10x_h^{0}|^6} h + x_h^{0} \Delta w_h + x_h^{0} \sum_{i=N(lh, \mathbb{R})}^{N(lh+h, \mathbb{R})} z_i + \frac{1}{2} x_h^{0} \{ (\Delta w_h)^2 - h \} \\
+ x_h^{0} \sum_{i=N(lh, \mathbb{R})}^{N(lh+h, \mathbb{R})} z_i(w_{l+h} - w_{l}) + x_h^{0} \sum_{i=N(lh, \mathbb{R})}^{N(lh+h, \mathbb{R})} z_i(w_{l} - w_{l+h}) \\
+ x_h^{0} \sum_{j=N(lh, \mathbb{R})}^{N(lh+h, \mathbb{R})} \sum_{i=N(lh, \mathbb{R})}^{N(lh+h, \mathbb{R})} z_i z_j \tag{6.18}
\]

for \( l = 0, \ldots, 2^n - 1 \).

The case \( \lambda = 3 \) in Table 6.5a and its corresponding plot in Figure 6.5a are based on 60,000 trajectories while case \( \lambda = 5 \) of Table 6.5a and its corresponding plot in
Figure 6.5b are based on 360,000 paths. In both cases, the mark random variables $z_i$s are assumed to follow normal distribution with mean 0 and variance 0.125.

Similarly, cases $\lambda = 3$ and $\lambda = 5$ in Table 6.5b and their corresponding plots in Figure 6.5c and Figure 6.5d are based on 240,000 simulations. Here, the mark random variables $z_i$s are assumed to follow uniform distribution on $[-1/4, 1/4]$.

\[
h = 2^{-n} \quad \sqrt{E[x_T - x_T^n]^2} \quad \lambda = 3.0 \quad \lambda = 5.0
\]

<table>
<thead>
<tr>
<th>$h = 2^{-n}$</th>
<th>$\lambda = 3.0$</th>
<th>$\lambda = 5.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-20}$</td>
<td>0.00067484</td>
<td>0.00621555</td>
</tr>
<tr>
<td>$2^{-19}$</td>
<td>0.00204889</td>
<td>0.02203719</td>
</tr>
<tr>
<td>$2^{-18}$</td>
<td>0.00515874</td>
<td>0.06003098</td>
</tr>
<tr>
<td>$2^{-17}$</td>
<td>0.01321011</td>
<td>0.27309100</td>
</tr>
<tr>
<td>$2^{-16}$</td>
<td>0.03146860</td>
<td>0.45542612</td>
</tr>
<tr>
<td>$2^{-15}$</td>
<td>0.06349005</td>
<td>0.66561201</td>
</tr>
<tr>
<td>$2^{-14}$</td>
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<td>0.85082102</td>
</tr>
<tr>
<td>$2^{-13}$</td>
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<td>1.55620505</td>
</tr>
<tr>
<td>$2^{-12}$</td>
<td>0.40437133</td>
<td>2.07503830</td>
</tr>
<tr>
<td>$2^{-11}$</td>
<td>0.57251083</td>
<td>2.41922833</td>
</tr>
</tbody>
</table>

(a) Mark is normal with mean 0 and variance 0.125.

<table>
<thead>
<tr>
<th>$h = 2^{-n}$</th>
<th>$\sqrt{E[x_T - x_T^n]^2}$</th>
<th>$\lambda = 3.0$</th>
<th>$\lambda = 5.0$</th>
</tr>
</thead>
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<td>0.00005260</td>
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<td>0.00013058</td>
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<td>0.00028420</td>
</tr>
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<td>$2^{-17}$</td>
<td></td>
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<td>0.00059787</td>
</tr>
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<td>$2^{-16}$</td>
<td></td>
<td>0.00112643</td>
<td>0.00125615</td>
</tr>
<tr>
<td>$2^{-15}$</td>
<td></td>
<td>0.00242178</td>
<td>0.00272857</td>
</tr>
<tr>
<td>$2^{-14}$</td>
<td></td>
<td>0.00563126</td>
<td>0.00673629</td>
</tr>
<tr>
<td>$2^{-13}$</td>
<td></td>
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<td>0.01747566</td>
</tr>
<tr>
<td>$2^{-12}$</td>
<td></td>
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</tr>
<tr>
<td>$2^{-11}$</td>
<td></td>
<td>0.07302734</td>
<td>0.07900926</td>
</tr>
</tbody>
</table>

(b) Mark is uniform on $[-1/4, 1/4]$.

Table 6.5: Tamed Milstein scheme (6.18) of SDE (6.17).

It is evident from plots in Figure 6.5 that rate of convergence of the tamed Milstein scheme of SDE driven by Lévy noise depends significantly on the distribution of mark random variable and the jump intensity. These simulations indicate that the tamed Milstein scheme (6.18) achieves a rate close 1.0 and hence confirm the theoretical findings of Chapter 5. □
(a) $\lambda = 3$ and mark is normal with mean 0 and variance 0.125

(b) $\lambda = 5$ and mark is normal with mean 0 and variance 0.125

(c) $\lambda = 3$ and mark is uniform on $[-1/4, 1/4]$

(d) $\lambda = 5$ and mark is uniform on $[-1/4, 1/4]$

Figure 6.5: $\mathbb{L}^2$-convergence rate of tamed Milstein scheme (6.18) of SDE (6.17).
Chapter 7

Future Aims

In this report, we discussed explicit tamed Euler schemes of stochastic differential equation driven by Lévy noise and stochastic delay differential equation driven by Lévy noise, which have drift coefficients that can grow super-linearly. A potential future project can be to extend the methodologies developed here to the case, when diffusion coefficient is also allowed to grow super-linearly. Further, we are also able to extend our techniques to tamed Milstein scheme of stochastic differential equation driven by Lévy noise. Then, one can explore the possibility of deriving tamed Milstein and higher order schemes for stochastic delay differential equations driven by Lévy noise. Moreover, the results on the tamed Milstein scheme are obtained under the finiteness of jump intensity, which is a restrictive assumption and one would also like to extend our results when this finite intensity assumption is not satisfied.
Appendices

A  C codes for Tamed Euler Scheme

/*
========================================================================
Program for Euler/Tamed Euler scheme of one -dim SDEs with jumps
========================================================================
Author : Chaman Kumar
E-mail : rkchaman@gmail.com
Institution : School of Mathematics, University of Edinburgh
Date : 13 April 2015
========================================================================
Notes:
(1) Modify "numer" and "denom" of c function "b_n" (drift coefficient)
    to switch between Euler and Tamed Euler schemes
(2) Modify "numer" and "denom" of c function "sig_n" (diffusion
    coefficient) to switch between Euler and Tamed Euler schemes
(3) Modify "numer" and "denom" of c function "gam_n" (jump coefficient)
    to switch between Euler and Tamed Euler schemes.
========================================================================
*/
#include<mpi.h>
#include<stdio.h>
#include<stdlib.h>
#include<math.h>
#include<time.h>
#define NSTEP 10
#define NPATHs 100

double snv_bm();
void jump_timesize(double T, double LAMBDA, int *njT, double *jtime,
    double *jsize);
void path_info(long N, double dt, int njT, double *jtime, double *w,
    int *cnjump);
double jump_inc(double dt, double xn, int cnjumpsta, int cnjumpend,
    double *jsize);
double b_n(double x, double stepsize);
double sig_n(double x, double stepsize);
double gam_n(double x, double sizej, double stepsize);

int main(int argc, char** argv)
{
    MPI_Init(&argc,&argv);
    int iproc, nproc;
    MPI_Comm_rank(MPI_COMM_WORLD,&iproc);
    MPI_Comm_size(MPI_COMM_WORLD,&nproc);
    return 0;
}
int njumT, R, p, ipath, MAX_NJUMPs=2000;
long N=pow(2,21), k, L;
double LAMBDA=1.0, T=1.0, x0=1.0, dt= (1.0*T)/(1.0*N);
double xT, xnT, Dt, errortemp;
double xerror2[NPATHs][NSTEP], sum2[NSTEP], ssum2[NSTEP];
double mean2[NSTEP], ErrorL2[NSTEP];
double *jtime=(double *)malloc(MAX_NJUMPs*sizeof(double));
double *jsize =(double *) malloc(MAX_NJUMPs*sizeof(double));
int *njT=(int *) malloc(sizeof(int));
int *cnjump = (int *) malloc( ((int)N+1)*sizeof(int) );
double *w=(double *) malloc( ((int)N+1)*sizeof(double) );

//----- seed -------
int myseed;
myseed=((int)time(NULL))*((int)getpid());
srand(myseed);

for (ipath=1;ipath<=(long)NPATHs;ipath++)
{
    jump_timesize(T, LAMBDA, njT, jtime, jsize);
    njumT=*njT;
path_info(N, dt, njumT, jtime, w, cnjump);
    //------------ True Solution ------------------
xT=x0;
    for(k=1;k<=N;k++)
    {
        xT=xT+b_n(xT, dt)*dt+sig_n(xT, dt)*(w[k]-w[k-1])
            +jump_inc(dt, xT, cnjump[k-1], cnjump[k], jsize);
    }
    //--------- Euler/Tamed Euler Scheme ----------
    for(p=1;p<=(int)NSTEP;p++)
    {
        R=(unsigned)pow(2, p); Dt=(double )R*dt; L=N/R;
xnT=x0;
        for(k=1;k<=L;k++)
        {
            xnT=xnT+b_n(xnT, Dt)*Dt+sig_n(xnT, Dt)*(w[R*k]-w[R*(k-1)])
                +jump_inc(Dt, xnT, cnjump[R*(k-1)], cnjump[R*k], jsize);
        }
        errortemp=fabs(xT-xnT);
xerror2[ipath-1][p-1]=pow(errortemp,2.0);
    }
}
for(p=1; p<=(int)NSTEP; p++)
{
  sum2[p-1]=0.0;
  for (ipath=1; ipath<=(long)NPATHs; ipath++)
  {
    sum2[p-1]=sum2[p-1]+xerror2[ipath-1][p-1];
  }
}
MPI_Reduce(&sum2, &ssum2, (int)NSTEP, MPI_DOUBLE, MPI_SUM, 0,
           MPI_COMM_WORLD);
for(p=1; p<=(int)NSTEP; p++)
{
  mean2[p-1]=ssum2[p-1]/(1.0*(double)NPATHs*nproc);
  ErrorL2[p-1]=sqrt(mean2[p-1]);
}
if (iproc==0)
{
  printf("No of Paths=%d, N=%lu, Lambda=%lf, X0=%lf \n",
         (1*nproc*(int)NPATHs), N, LAMBDA, x0);
  printf("------------\n");
  printf("L2 Error \n");
  printf("------------\n");
  for(p=1; p<=(int)NSTEP; p++)
  {
    printf("%.8lf ", ErrorL2[p-1]);
  }
  printf("\n");
}
MPI_Finalize();
free(njT);
free(jtime);
free(jsize);
free(w);
free(cnjump);
return 0;

// Function : Drift Coefficient
double b_n(double x, double stepsize)
{
  double val;
  double numer, denom;
  numer=(x-pow(x,5.0))*0.10;
  denom=1.0+sqrt(stepsize)*fabs(numer);
  val=numer/denom;
  return val;
}
// Function : Diffusion Coefficient
double sig_n(double x, double stepsize)
{
    double val;
    double numer, denom;
    numer=x;
    denom=1.0;
    val=numer/denom;
    return val;
}

// Function : Jump Coefficient
double gam_n(double x, double sizej, double stepsize)
{
    double val;
    double numer, denom;
    numer=sizej*x;
    denom=1.0;
    val=numer/denom;
    return val;
}

// Function : Jump time and Jump-Size
void jump_timesize(double T, double LAMBDA, int *njT, double *jtime, double *jsize)
{
    unsigned j=0;
    double u, tau;
    u_lab1:
    u=(double)rand()*(1.0/RAND_MAX);
    if(u==0.0||u==1.0)
        goto u_lab1;
    tau=-(double)log(u)/LAMBDA;
    while(tau<=(unsigned)T)
    {
        jtime[j]=tau;
        u_labjs:
        u=(double)rand()*(1.0/RAND_MAX);
        if(u==0.0||u==1.0)
            goto u_labjs;
        jsize[j]=-0.25+u*0.5; //--- Jump-Size : U(-1/4,1/4)
        j=j+1;
        u_lab2:
        u=(double)rand()*(1.0/RAND_MAX);
        if(u==0.0||u==1.0)
            goto u_lab2;
        tau=tau-(double)log(u)/LAMBDA;
    }
    *njT=j;
    return;
}
// Function: Wiener Process and Cumulative number of Jumps
void path_info(long N, double dt, int njT, double *jtime, double *w, int *cnjump) {
    long k;
    int jtemp=0;
    w[0]=0;
    cnjump[0]=0;
    for(k=1;k<=N;k++) {
        if( (jtime[jtemp]<k*dt) && (jtemp<njT) ) {
            w[k]=w[k-1]+snv_bm()*sqrt(jtime[jtemp]-(k-1)*dt);
            cnjump[k]=cnjump[k-1]+1;
            jtemp=jtemp+1;
            while( (jtime[jtemp]<k*dt) && (jtemp<njT) ) {
                w[k]=w[k]+snv_bm()*sqrt(jtime[jtemp]-jtime[jtemp-1]);
                cnjump[k]=cnjump[k]+1;
                jtemp=jtemp+1;
            }
            w[k]=w[k]+snv_bm()*sqrt(k*dt-jtime[jtemp-1]);
        } else {
            w[k]=w[k-1]+snv_bm()*sqrt(dt);
            cnjump[k]=cnjump[k-1];
        }
    }
    return;
}

// Function: Standard Normal Variate
double snv_bm() {
    double u1,u2, z1, z2, z;
    double two_pi=2.0*3.14159265358979323846;

    u1_lab:
    u1=(double)rand()*(1.0/RAND_MAX);
    if(u1==0.0||u1==1.0) goto u1_lab;
    u2_lab:
    u2=(double)rand()*(1.0/RAND_MAX);
    if(u2==0.0||u2==1.0) goto u2_lab;
    z1=((double)sqrt(-2.0*log(u1)))*((double)cos(two_pi*u2));
    z2=((double)sqrt(-2.0*log(u1)))*((double)sin(two_pi*u2));
    z=(z1+z2)/((double)sqrt(2));
    return z;
}
/ Function : Jump Increment
double jump_inc(double dt, double xn, int cnjumpsta, int cnjumpend,
                     double *jsize)
{
    int jtemp=cnjumpsta;
    double val=0;
    while(jtemp<cnjumpend)
    {
        val=val+gam_n(xn, jsize[jtemp], dt);
        jtemp=jtemp+1;
    }
    return val;
}
B C codes for Tamed Milstein Scheme

/*
========================================================================
Program for Milstein/Tamed Milstein scheme of one -dim SDEs with jumps
========================================================================
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Date : 21 April 2015
========================================================================
Notes:
(1) Modify "numer" and "denom" of c function "b_n" (drift coefficient)
to switch between Milstein and Tamed Milstein schemes
(2) Modify "numer" and "denom" of c function "sig_n" (diffusion
coefficient) to switch between Milstein and Tamed Milstein schemes
(3) Modify "numer" and "denom" of c function "gam_n" (jump coefficient)
to switch between Milstein and Tamed Milstein schemes
(4) Modify c function "sig_nprime" (derivative of diffusion coefficient)
(5) Modify c function "gam_nprime" (derivative of diffusion coefficient)
========================================================================*/
#include<mpi.h>
#include<stdio.h>
#include<stdlib.h>
#include<math.h>
#include<time.h>
#define NSTEP 10
#define NPATHs 1000
double snv_bm();
void jump_timesize(double T, double LAMBDA, int *njT, double *jtime,
                   double *jsize);
void path_info(long N, double dt, int njT, double *jtime, double *w,
               double *wjtime, int *cnjump);
void jump_terms(double dt, double xn, long ksta, long kend, int *cnjump,
                double *w, double *wjtime, double *jsize, double *jterm);
double b_n(double x, double stepsize);
double sig_n(double x, double stepsize);
double gam_n(double x, double sizej, double stepsize);
double sig_nprime(double x, double stepsize);
double gam_nprime(double x, double sizej, double stepsize);
int main(int argc, char** argv)
{
  MPI_Init(&argc,&argv);
  int iproc, nproc;
  MPI_Comm_rank(MPI_COMM_WORLD,&iproc);
  MPI_Comm_size(MPI_COMM_WORLD,&nproc);
int njumT, R, p, ipath, MAX_NJUMPs=1000;
long N=pow(2,21), k, L;
double LAMBDA=1.0, T=1.0, x0=1.0, dt= (1.0*T)/(1.0*N);
double xT, xnT, Dt, errortemp;
double sum2[NSTEP], ssum2[NSTEP], mean2[NSTEP];
double xerror2[NPATHs][NSTEP], ErrorL2[NSTEP];
double *jtime=(double *) malloc(MAX_NJUMPs*sizeof(double));
double *jsize =(double *) malloc(MAX_NJUMPs*sizeof(double));
int *njT=(int *) malloc(sizeof(int));
int *cnjump = (int *) malloc((N+1)*sizeof(int));
double *w=(double *) malloc((N+1)*sizeof(double));
double *wjtime=(double *) malloc(MAX_NJUMPs*sizeof(double));
double *jterm=(double *) malloc(4*sizeof(double));
//-seed-
int myseed;
myseed=((int)time(NULL))*((int)getpid());
srand(myseed);
for (ipath=1;ipath<=(long)NPATHs;ipath++)
{
    jump_timesize(T, LAMBDA, njT, jtime, jsize);
njumT=*njT;
    path_info(N, dt, njumT, jtime, w, wjtime, cnjump);
    //-------------- True Solution -------------------
    xT=x0;
    for(k=1;k<=N;k++)
    {
        jump_terms(dt, xT, k-1, k, cnjump, w, wjtime, jsize, jterm);
        xT=xT+b_n(xT, dt)*dt + sig_n(xT, dt)*(w[k]-w[k-1])
+0.5*sig_n(xT,dt)*sig_nprime(xT,dt)*(pow(w[k]-w[k-1], 2.0)-dt)
    }
    //--------- Milstein/Tamed Milstein Scheme ------
    for(p=1;p<=(int)NSTEP;p++)
    {
        R=(long)pow(2, p); Dt=(double)R*(double)dt; L=N/R;
        xnT=x0;
        for(k=1;k<=L;k++)
        {
            jump_terms(Dt, xnT, R*(k-1), R*k, cnjump, w, wjtime, jsize, jterm);
            xnT=xnT+b_n(xnT, Dt)*Dt + sig_n(xnT, Dt)*(w[R*k]-w[R*(k-1)])
+0.5*sig_n(xnT, Dt)*sig_nprime(xnT, Dt)*(pow(w[R*k]-w[R*(k-1)], 2.0)-Dt)
        }
        errortemp=fabs(xT-xnT);
        xerror2[ipath-1][p-1]=pow(errortemp,2.0);
    }
}
for(p=1;p<=(int)NSTEP;p++)
{
    sum2[p-1]=0.0;
    for (ipath=1;ipath<=(long)NPATHs;ipath++)
    {
        sum2[p-1]=sum2[p-1]+xerror2[ipath-1][p-1];
    }
}
MPI_Reduce(&sum2, &ssum2, (int)NSTEP, MPI_DOUBLE, MPI_SUM, 
0, MPI_COMM_WORLD);
for(p=1;p<=(int)NSTEP;p++)
{
    mean2[p-1]=ssum2[p-1]/((double)NPATHs*(1.0*nproc));
    ErrorL2[p-1]=sqrt(mean2[p-1]);
}
if(iproc==0)
{
    printf("No of Paths=%ld, N=%ld, lambda=%lf, x0=%lf\n",
            (nproc*(long)NPATHs), N, LAMBDA, x0);

    printf("\n----------------\n");
    printf("L2 Error \n");
    printf("----------------\n");
    for(p=1;p<=(int)NSTEP;p++)
    {
        printf("%.8lf ", ErrorL2[p-1]);
    }
    printf("\n");
}
MPI_Finalize();
free(njT);
free(jT);
free(jsize);
free(w);
free(wjtime);
free(cnjump);
return 0;
}

// Function : Drift Coefficient
double b_n(double x, double stepsize)
{
    double val;
    double numer, denom;
    numer= -pow(x,3.0)*0.10;
    denom=1.0+stepsize*pow(fabs(numer), 2.0);
    val=numer/denom;
    return val;
}
// Function : Diffusion Coefficient
double sig_n(double x, double stepsize)
{
    double val;
    double numer, denom;
    numer=x;
    denom=1.0;
    val=numer/denom;
    return val;
}

// Function : Derivative of Diffusion Coefficient
double sig_nprime(double x, double stepsize)
{
    double val;
    double numer, denom;
    numer=1.0;
    denom=1.0;
    val=numer/denom;
    return val;
}

// Function : Jump Coefficient
double gam_n(double x, double sizej, double stepsize)
{
    double val;
    double numer, denom;
    numer=x*sizej;
    denom=1.0;
    val=numer/denom;
    return val;
}

// Function : Derivative of Jump Coefficient
double gam_nprime(double x, double sizej, double stepsize)
{
    double val;
    double numer, denom;
    numer=sizej;
    denom=1.0;
    val=numer/denom;
    return val;
}
// Function : Jump Time and Jump Size
void jump_timesize(double T, double LAMBDA, int *njT, double *jtime,
                    double *jsize)
{
    unsigned j=0;
    double u, tau;
    u_lab1:
    u=(double)rand()*(1.0/RAND_MAX);
    if(u==0.0||u==1.0)
        goto u_lab1;
    tau=-(double)log(u)/LAMBDA;
    while(tau<=(unsigned)T)
    {
        jtime[j]=tau;
        u_labjs:
        u=(double)rand()*(1.0/RAND_MAX);
        if(u==0.0||u==1.0)
            goto u_labjs;
        jsize[j]= -0.25+u*0.5; //--- Jump Size : U(-1/4,1/4)
        j=j+1;
        u_lab2:
        u=(double)rand()*(1.0/RAND_MAX);
        if(u==0.0||u==1.0)
            goto u_lab2;
        tau=tau-(double)log(u)/LAMBDA;
    }
    *njT=j;
    return;
}

// Function : Standard Normal Variate
double snv_bm()
{
    double u1,u2, z1, z2, z;
    double two_pi=2.0*3.14159265358979323846;

    u1_lab:
    u1=(double)rand()*(1.0/RAND_MAX);
    if(u1==0.0||u1==1.0)
        goto u1_lab;
    u2_lab:
    u2=(double)rand()*(1.0/RAND_MAX);
    if(u2==0.0||u2==1.0)
        goto u2_lab;
    z1=((double)sqrt(-2.0*log(u1)))*((double)cos(two_pi*u2));
    z2=((double)sqrt(-2.0*log(u1)))*((double)sin(two_pi*u2));
    z=(z1+z2)/((double)sqrt(2));
    return z;
}
// Function : Jump Terms of the Scheme
void jump_terms(double dt, double xn, long ksta, long kend, int *cnjump,
        double *w, double *wjtime, double *jsize, double *jterm)
{
    int jtemp=cnjump[ksta];
    int jtempnew;
    jterm[0]=0.0;
    jterm[1]=0.0;
    jterm[2]=0.0;
    jterm[3]=0.0;
    while(jtemp<cnjump[kend])
    {
        jterm[0]=jterm[0]+gam_n(xn, jsize[jtemp], dt);
        jterm[1]=jterm[1]
            +gam_nprime(xn, jsize[jtemp], dt)*(wjtime[jtemp]-w[ksta]);
        jterm[2]=jterm[2]+( sig_n(xn+gam_n(xn, jsize[jtemp], dt), dt)
            -sig_n(xn, dt) )*(w[kend]-wjtime[jtemp]);
        jtempnew=cnjump[ksta];
        while(jtempnew<=jtemp)
        {
                +gam_n(xn+gam_n(xn, jsize[jtempnew], dt), jsize[jtemp], dt)
                -gam_n(xn, jsize[jtemp], dt);
            jtempnew=jtempnew+1;
        }
        jtemp=jtemp+1;
    }
    return;
}
// Function : Wiener Process and Cumulative Number of Jumps
void path_info(long N, double dt, int njT, double *jtime, double *w,
    double *wjtime, int *cnjump)
{
    long k; int jtemp=0;
    w[0]=0; cnjump[0]=0;
    for(k=1;k<=N;k++)
    {
        if( (jtime[jtemp]<k*dt) && (jtemp<njT) )
        {
            w[k]=w[k-1]+snv_bm()*sqrt(jtime[jtemp]-(k-1)*dt);
            wjtime[jtemp]=w[k];
            cnjump[k]=cnjump[k-1]+1;
            jtemp=jtemp+1;
            while( (jtime[jtemp]<k*dt) && (jtemp<njT) )
            {
                w[k]=w[k]+snv_bm()*sqrt(jtime[jtemp]-jtime[jtemp-1]);
                wjtime[jtemp]=w[k];
                cnjump[k]=cnjump[k]+1;
                jtemp=jtemp+1;
            }
            w[k]=w[k]+snv_bm()*sqrt(k*dt-jtime[jtemp-1]);
        }
        else
        {
            w[k]=w[k-1]+snv_bm()*sqrt(dt);
            cnjump[k]=cnjump[k-1];
        }
    }
    return;
}
Bibliography


