SOME ASPECTS OF THE PROBLEM OF VIBRATION OF TURBINE BLADES

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NOTATION

A - Area of cross section
E - Elastic modulus or Young's Modulus
f - Frequency
g - Acceleration due to gravity
I - Second moment of area of cross section
l - Length of beam
M - Bending moment
p - Circular frequency
Q - Shear force
t - Time
V,v - Displacement in Y and y directions respectively
W,w - Displacement in Z and z directions respectively
x - Distance along beam from datum axes
y - Displacement in y direction
XX & ZZ - Principal axes of beam at clamped i.e. datum
yy & zz - Rotating principal axes for pre-twisted beam
\rho - Density of material per unit volume
\phi - Angle of twist per unit length
\theta - Total angle of twist for beam length l
\; \text{i.e. } \theta = \phi \cdot l
CHAPTER 1

1.1 INTRODUCTION.

When the basic design of a steam turbine or a gas turbine has been decided on it is necessary to carry out a vibration analysis of the system to determine all the exciting forces which are likely to be present. The frequency of these forces and the natural frequencies of all the turbine blades are required to make sure that no blade is being forced to vibrate at its natural frequency.

There are a number of possible sources of excitation in steam or gas turbines mostly associated with wakes from supporting parts of the structure or preceding rows of blades. Fluctuations in pressure due to partial admission or supporting webs in the inlet or intake can induce large amplitudes of vibration. In the case of turbines driving through reduction gearing, incorrectly meshed gear teeth can cause vibrations but these are seldom of sufficient magnitude to cause serious damage to blades.

Once all the possible sources of excitation have been ascertained and the respective frequencies in relation to the speed of rotation of the turbine have also been estimated, it only remains to calculate the natural frequencies of vibration of all the blades to determine which, if any, will have natural frequencies coinciding /
coinciding with those of exciting forces. In order to be able to design the turbine blades to fulfil their thermodynamic function efficiently and yet avoid dangerous frequencies it is necessary to be able to calculate accurately the natural frequencies of each design.

The calculation of the natural frequencies of a uniform beam with known end conditions can be performed easily by using the formula obtained from the solution of the differential equation of motion. The turbine blades used in both steam turbines and gas turbines however tend to be anything but uniform in cross-sectional area along their length. Not only can they taper from root to tip but the actual shape of the cross-section can vary and in some cases the blade has a slight twist along its length to improve the efficiency. In the case of many steam turbines and gas turbines the running speed is constant and so the effects of centrifugal force remain constant but in some, for example marine turbines and in particular in aero gas turbines the speed of rotation can be varied over a considerable range. This makes the task of the designer or vibration engineer more difficult since the effects of rotation vary with the speed.

Other design features which can alter the natural frequencies of vibration of the turbine blades are ones which
which are introduced to add extra stiffening to rather long blades or to blades which have to stand up to unusually high pressure. Such features are shroud bands or lacing wires which are attached at the tip or at a section along the beam. Yet another factor which can alter the natural frequency is the temperature at which the blade operates.

So in calculating the natural frequencies of vibration of turbine blades several factors have to be taken into consideration some being constant for a particular blade while others vary according to the speed and power required. In this particular instance the effect of pre-twist on the natural frequencies of vibration is the subject of investigation.

To limit the number of variables the investigation will be carried out using uniform beams of rectangular cross-section. In this way it is hoped to determine the effect of pre-twist on cantilever beams and on clamped-pinned beams. A cantilever beam simulates the case of a turbine blade attached at the root only to a disc or casing and a clamped-pinned beam is assumed to be equivalent to a turbine blade attached at the root to a disc or casing and to a shroud at the tip.

The chapter on methods of calculating the natural frequencies of vibration describes several methods which could
could be employed to determine the frequencies of turbine blades and includes one or two particular cases using some of these methods. The cases of a cantilever beam and a clamped-pinned beam with uniform pre-twist are mentioned in detail.

In a later chapter the experimental results are compared with those predicted in theory and possible explanations are given for any differences which occur.
CHAPTER 2

2.1. Methods of Calculating Natural Frequencies

For the calculation of the natural frequencies and normal modes of vibration of elastic bodies, there are three main avenues of approach which give rise to various methods of exact or approximate solution. These methods are derived

i) from the differential equation for the system
ii) from the integral equation (with flexibility coefficients)
iii) from the consideration of energy

The displacement for these cases can be approximated by specifying it at a number of definite points or by specifying a continuous displacement function or linear combination of such functions. Thus in approximate methods, the actual system which has an infinite number of degrees of freedom is usually replaced by one with a finite number of degrees of freedom. Further approximations are usually made in particular cases. For instance, the effects of shear and rotatory inertia may be neglected in flexure, as in the following sub-sections where only the simpler cases of flexure are considered.

A similar account of the approximate methods available is made in a comprehensive report by Minhinnick (1956) with the emphasis on the application to the problem of aircraft flutter. Both this paper and a survey by Barr (1957) have been found particularly useful in preparing this section.
2.2 The Differential Equation

In many cases the differential equation of the system can be set up and solved analytically giving frequency equations from which the natural frequencies or eigen values may be obtained along with the corresponding mode shapes or eigen functions. In most cases the frequency equation will be of a transcendental nature and graphical or numerical methods have to be employed to obtain solutions. Some examples of this direct type of solution are given in Section 2.2.2.

2.2.1. Differential Equation of Lateral Vibration of a Beam

The differential equation for the deflection curve of a beam in equilibrium under an applied bending moment, $M$, is known from Strength of Materials to be

$$EI \frac{d^4y}{dx^4} = -M$$

in which $E I$ is the flexural rigidity and $M$ is the bending moment at any cross-section. The direction of the axis and the positive directions of bending moments and shearing forces $Q$ are shown in Figure I.
Differentiating equation (1) twice with respect to \( x \) we have

\[
\frac{d}{dx} \left[ EI \frac{d^2 y}{dx^2} \right] = -\frac{dW}{dx} = -\phi \quad (2)
\]

\[
\frac{d^2}{dx^2} \left[ EI \frac{d^2 y}{dx^2} \right] = -\frac{d\phi}{dx} = W \quad (3)
\]

This last equation represents the differential equation of a bar subjected to a uniformly distributed load \( w \) and can be used to obtain the equation of lateral vibration. The load on the beam is caused by alternating inertia forces which vary along the length of the bar, and the intensity is given by

\[
\frac{d^2 u}{dx^2} = \frac{\rho A}{g} \frac{d^2 y}{dx^2} \quad (4)
\]

where \( \rho \) is the weight per unit volume and \( A \) is the cross-sectional area.

Substituting (4) for \( W \) in (3) the general equation for the lateral vibration of a beam becomes

\[
\frac{d^2}{dx^2} \left( EI \frac{d^2 y}{dx^2} \right) = -\frac{\rho A}{g} \frac{d^2 y}{dx^2} \quad (5)
\]

2.2.2. **Case of a Uniform Beam**

In this particular case the flexural rigidity \( EI \) remains /
remains constant along the length of the beam and equation (3.2) becomes

$$\frac{EI}{\beta} \frac{d^2y}{dx^2} + \frac{\rho a}{\beta} \frac{d^2y}{dt^2} = 0 \quad (6)$$

let $C = \sqrt{\frac{EI}{\beta \rho a}}$ the above equation can then be re-written as

$$C^2 \frac{d^2y}{dx^2} + \frac{d^2y}{dt^2} = 0 \quad (7)$$

Let $y = X(x) T(t)$. On substitution (7) becomes

$$C^2 \frac{d^4X}{dx^4} + X \frac{d^2T}{dt^2} = 0$$

$$C^2 \frac{d^4X}{dx^4} = - \frac{1}{T} \frac{d^2T}{dt^2}$$

Now since $X$ and $T$ are independent of one another the above expressions must be constant. Let this constant be $p^2$ and equating the above expressions to this quantity, two equations, one in $T$ and the other in $X$ are obtained

$$\begin{align*}
\frac{d^2T}{dt^2} &= -p^2 T \\
8 \frac{d^4X}{dx^4} &= \frac{p^2}{C^2} X
\end{align*} \quad (10)$$

From /
From a knowledge of simple differential equations it can be shown that

$$T = (A \cos pt + B \sin pt)$$

$$X = (C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx)$$

When the deflection of a beam varies harmonically with time it is said to be performing a normal mode of vibration. $X$ is a function of the co-ordinate $x$ defining the shape of the normal mode of vibration under consideration and $X = (C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx)$ is known as the normal function.

The constants $C_1$, $C_2$, $C_3$ and $C_4$ are determined from the boundary conditions, i.e. according to the type of support at the ends of the beam. In addition to evaluating these constants, an equation relating $\sin kx$, $\cos kx$, $\sinh kx$ and $\cosh kx$ is found, from which consecutive roots may be obtained for '$k_1l$' where '$l$' is the length of the beam.

From (10) it is known that $\nu_n = \frac{ck_n^2}{l}$ where $\nu_n$ is the angular frequency for the $n^{th}$ root $k_n l$ and

$$\nu_n = (k_n l)^2 \frac{\sqrt{k_n^2}}{\nu_n}$$

thus the corresponding natural frequency is given by

$$f_n = \frac{k_n}{2\pi} \sqrt{\frac{\nu_n}{\nu_n}}$$
2.2.2.1. Cantilever Beam

In this case the beam is fixed at one end and left free at the other. Thus the end conditions which have to be satisfied are, at the fixed end, that the deflection and slope are zero and at the free end that the bending moment and shear force are zero. In terms of the normal function these conditions are

1) \( X = 0 \) at \( x = 0 \) 
2) \( \frac{dX}{dx} = 0 \) at \( x = 0 \) 
3) \( \frac{d^2X}{dx^2} = 0 \) at \( x = c \) 
4) \( \frac{d^3X}{dx^3} = 0 \) at \( x = c \)

Referring to equation (12) conditions 1) and 2) require

\[ C_2 + C_4 = 0 \] and \[ C_1 + C_3 = 0 \] and the remaining conditions 3) and 4) require

\[ C_1 = C_4 \frac{\sinh kl - \sin kl}{\cos kl + \cosh kl} \]

The frequency equation \( \cos kl \cosh kl + 1 = 0 \) is obtained from the remaining conditions (3) and the following are the first five consecutive roots

\[ k_1, k_2, k_3, k_4, k_5 \]

1.875 4.694 7.855 10.996 14.137

Thus the fundamental mode of vibration will have a frequency

\[ f_1 = \frac{(1.875)^2 \sqrt{\frac{E \ell}{\rho A}}}{2 \pi c^2} = \frac{3.515}{2 \pi c^2} \sqrt{\frac{E \ell}{\rho A}} \]
2.2.2.2. Fixed-Pinned Beam

In this case the end conditions are that the deflection and slope are zero at the fixed end and that the deflection and bending moment are zero at the pinned end. Therefore

1) \( X = 0 \) at \( x = 0 \)

2) \( \frac{dX}{dx} = 0 \) at \( x = 0 \)

3) \( X = 0 \) at \( x = L \)

4) \( \frac{d^2X}{dx^2} = 0 \) at \( x = L \).

In satisfying these end conditions the frequency equation is found to be

\[ \tan kl = \tan kl' \]

which has roots

\[
\begin{align*}
  k_1 & \quad k_2 \quad k_3 \quad k_4 \quad k_5 \\
  3.927 & \quad 7.069 & \quad 10.210 & \quad 13.351 & \quad 16.493
\end{align*}
\]

The fundamental frequency of a fixed-pinned beam is therefore

\[
f_1 = \frac{(3.927)^2 \sqrt{EI}}{2EI} \]

2.2.3. The Differential Equation of Motion of Non-uniform Beams

The term non-uniformity in a beam means a variation in the mass and stiffness along the length of the beam or blade with the result that the frequencies and modes of vibration /
vibration are different from those of the uniform beam previously considered.

According to the simple beam theory, the differential equation of motion in flexure of a non-uniform beam is

$$\frac{d}{dx^2} \left( \frac{d^2 y}{dx^2} \right) + \rho A(x) \frac{d^2 y}{dx^2} = 0 \quad (15)$$

In most cases the variation of the second moment of area, $I$, and the cross-sectional area, $A$, can be represented by suitable functions of the distance along the beam. If this is the case it is possible to obtain exact analytical solutions to the equations, usually in terms of power series or Bessel functions.

Todhunter and Pearson in their History of the Theory of Elasticity discuss the work of Kirchoff on such a solution when the co-ordinates of the boundary of the cross-section were taken in the form

$$Y = Ax^m, \quad Z = Bx^n \quad (16)$$

The particular cases of this form for a sharp wedge, when $m = 1, n = 0$, and for a pointed cone, when $m = 1, n = 1$, were considered since the solution could be expressed in terms of Bessel functions. It was found that the ratio of the fundamental frequency to that of a uniform rectangular cross-section cantilever in the case of a sharp wedge was 5.315 to 3.516 and for the case of the pointed cone was 8.718 to 3.516.
There are many similar works on beams with restricted types of taper which contain results obtained by solving the differential equation of motion. Although these works are of interest, the cases considered do not always bear much resemblance to turbine blades and therefore are only included in the bibliography.

One paper which is of particular interest is by Cranch and Adler (1954) and deals with many beams which have definite relationships between breadth and length, and depth and length. Some of these could be applied to certain types of turbine blade with advantage and are quoted below.

<table>
<thead>
<tr>
<th>$\delta / \delta_0$</th>
<th>$h / h_0$</th>
<th>FREQUENCY EQUATION</th>
<th>Roots of FREQUENCY EQUATION</th>
<th>BEAM SHAPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$\cosh\cosh=1$</td>
<td>1.875, 4.694, 7.855</td>
<td>b, a, b</td>
</tr>
<tr>
<td>$\chi / \chi_0$</td>
<td>1</td>
<td>$I_1(\chi)I_2(\chi)=0$</td>
<td>4.61, 7.80, 11.0</td>
<td>b, a, b</td>
</tr>
<tr>
<td>$\chi / \chi_0$</td>
<td>1</td>
<td>$I_2(\chi)I_1(\chi)=0$</td>
<td>18.94, 96.81, 255.4</td>
<td>b, a, b</td>
</tr>
</tbody>
</table>

Cranch and Adler obtained these solutions by modifying the form of the differential equation of motion slightly and then solving the equation by means of Bessel Functions.

* $I_1(\chi), I_2(\chi), I_3(\chi) \& I_4(\chi)$ are Bessel Functions.
2.2.4. The Differential Equation of Motion of a Pre-twisted Beam

In order to derive the differential equation of motion of a pre-twisted beam, it is necessary to make assumptions regarding the angle of pre-twist and the cross-section. These assumptions are that the pre-twist is uniform along the length of the beam and that the cross-section is symmetrical about both principal axes, as well as uniform along the length. The symmetry of the cross-section means that there is no coupling between the flexural and torsional modes of vibration, as the centre of flexure and the centroid are coincident.

The principal axes at the fixed end are taken as datum, the beam is of length $L$ and any section at distance $x$ from datum is taken. The angle of the principal axes at any section, distance $x$ from datum, with respect to datum is $\phi x$ radians, where $\phi$ is the rate of twist per unit length, i.e. $\phi L$ radians is total angle of twist.

![Diagram](image-url)

*Fig. II*

$YZ$ are principal axes at datum

$Y_2Z_2$ are principal axes at any section distance $x$ from datum.
Absolute displacements in $y$ and $z$ directions consist of components of $v$ and $w$ as follows:

$$V = v \cos \phi x - w \sin \phi x \quad (17)$$

$$W = v \sin \phi x + w \cos \phi x \quad (18)$$

and absolute bending moments are:

$$M_y = M_y \cos \phi x - M_z \sin \phi x \quad (19)$$

$$M_z = M_y \sin \phi x + M_z \cos \phi x \quad (20)$$

Successive differentiation of the expressions for the displacements gives:

$$\frac{d^2 V}{dx^2} = \left( \frac{d^2 v}{dx^2} - 2 \phi \frac{d^2 w}{dx^2} + \phi^2 v \right) \cos \phi x - \left( \frac{d^2 w}{dx^2} + 2 \phi \frac{d^2 w}{dx^2} - \phi^2 w \right) \sin \phi x \quad (21)$$

and

$$\frac{d^2 W}{dx^2} = \left( \frac{d^2 w}{dx^2} - 2 \phi \frac{d^2 w}{dx^2} + \phi^2 v \right) \sin \phi x + \left( \frac{d^2 w}{dx^2} + 2 \phi \frac{d^2 w}{dx^2} - \phi^2 w \right) \cos \phi x \quad (22)$$

2.2.4.1. Moments of Inertia

From Appendix 'A' the moment of inertia has components:

$$I_y = I_z \sin^2 \phi x + I_y \cos^2 \phi x \quad - - (23)$$

$$I_z = I_z \cos^2 \phi x + I_y \sin^2 \phi x \quad - - (24)$$

$$I_{yz} = (I_z - I_y) \sin \phi x \cos \phi x \quad - - (25)$$

From simple beam theory the expressions for the bending moments of a beam deflecting in two directions simultaneously are:

$$M_y = - \frac{EI_y}{I_y} \left( \frac{d^2 y}{dx^2} \right) - \frac{EI_y}{I_z} \left( \frac{d^2 y}{dx^2} \right) - \frac{EI_y}{I_z} \left( \frac{d^2 y}{dx^2} \right) \quad (26)$$

$$M_z = - \frac{EI_z}{I_z} \left( \frac{d^2 z}{dx^2} \right) - \frac{EI_z}{I_y} \left( \frac{d^2 z}{dx^2} \right) - \frac{EI_z}{I_y} \left( \frac{d^2 z}{dx^2} \right) \quad (27)$$
Substitution of these expressions for in equations (26) and (27) gives

\[ M_y \cos \phi x - M_y \sin \phi x = E I \left[ \frac{3}{2} \left( \frac{\partial^2 \phi}{\partial x^2} - 2 \frac{\partial^2 \phi}{\partial y^2} - \phi^2 \right) \sin \phi x \right] 
+ E I \left( \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial^2 \phi}{\partial y^2} - \phi^2 \right) \cos \phi x \]  
\]

and

\[ M_y \sin \phi x + M_y \cos \phi x = E I \left[ \frac{3}{2} \left( \frac{\partial^2 \phi}{\partial x^2} - 2 \frac{\partial^2 \phi}{\partial y^2} - \phi^2 \right) \cos \phi x \right] 
+ E I \left( \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial^2 \phi}{\partial y^2} - \phi^2 \right) \sin \phi x \]  
\]

Solving for \( M_y \) and \( M_z \)

\[ M_y = -E I \left[ \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial^2 \phi}{\partial y^2} - \phi^2 \right] \]  
\[ M_z = E I \left[ \frac{\partial^2 \phi}{\partial x^2} - 2 \frac{\partial^2 \phi}{\partial y^2} - \phi^2 \right] \]  

where \( M_y \) and \( M_z \) are the bending moments in the principal directions at any section of the beam.

Now for a beam in a state of sustained vibration at a natural frequency, the load on it is caused by alternating inertia forces. Thus the loads are

\[ \frac{d^2}{dx^2}(M_y) = -\frac{\rho A}{2} \frac{d^4 \phi}{dt^4} \text{ and } \frac{d^2}{dx^2}(M_z) = \frac{\rho A}{2} \frac{d^4 \phi}{dt^4} \]  

Substitution of the expressions obtained for \( M_y, M_z, \phi \) and \( \phi_z \) gives

\[ \frac{d^2}{dx^2} \left( E I \left[ \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial^2 \phi}{\partial y^2} - \phi^2 \right] \sin \phi x \right) 
+ E I \left( \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial^2 \phi}{\partial y^2} - \phi^2 \right) \cos \phi x = \frac{\rho A}{2} \left[ \frac{\partial^2 \phi}{\partial t^2} \sin \phi x + \frac{\partial^2 \phi}{\partial x^2} \cos \phi x \right] \]  
\]

and

\[ \frac{d^2}{dx^2} \left( E I \left[ \frac{\partial^2 \phi}{\partial x^2} - 2 \frac{\partial^2 \phi}{\partial y^2} - \phi^2 \right] \cos \phi x \right) 
- E I \left( \frac{\partial^2 \phi}{\partial x^2} - 2 \frac{\partial^2 \phi}{\partial y^2} - \phi^2 \right) \sin \phi x = \frac{\rho A}{2} \left[ \frac{\partial^2 \phi}{\partial t^2} \cos \phi x - \frac{\partial^2 \phi}{\partial x^2} \sin \phi x \right] \]  
\]
differentiating and collecting like terms gives from equation (33)

\[
\begin{aligned}
& [I_y \frac{\partial^2 \phi}{\partial x^2} + 2\phi (I_y + I_z) \frac{\partial^3 \phi}{\partial x^3} - 2\phi^2 (I_y + I_z) \frac{\partial^2 \phi}{\partial x^2} - 2\phi^3 (I_y + I_z) \frac{\partial \phi}{\partial x} + \phi \frac{\partial^3 \phi}{\partial x^3} \cos \phi x

+ [I_y \frac{\partial^2 \phi}{\partial x^2} - 2\phi (I_y + I_z) \frac{\partial^3 \phi}{\partial x^3} - 2\phi^2 (I_y^2 + 2I_y) \frac{\partial^2 \phi}{\partial x^2} + 2\phi^3 (I_y + I_z) \frac{\partial \phi}{\partial x} + \phi \frac{\partial^3 \phi}{\partial x^3} \sin \phi x

= \frac{\rho A}{\frac{\partial^2 \phi}{\partial t^2}} \left\{ \frac{\partial^3 \phi}{\partial x^3} \cos \phi x - \frac{\partial \phi}{\partial x} \sin \phi x \right\} \quad (36)
\end{aligned}
\]

Similarly from equation (34) comes the expression

\[
\begin{aligned}
& [I_z \frac{\partial^2 \phi}{\partial x^2} + 2\phi (I_y + I_z) \frac{\partial^3 \phi}{\partial x^3} - 2\phi^2 (I_y + I_z) \frac{\partial^2 \phi}{\partial x^2} - 2\phi^3 (I_y + I_z) \frac{\partial \phi}{\partial x} + \phi \frac{\partial^3 \phi}{\partial x^3} \cos \phi x

+ [I_z \frac{\partial^2 \phi}{\partial x^2} + 2\phi (I_y + I_z) \frac{\partial^3 \phi}{\partial x^3} - 2\phi^2 (I_y + I_z) \frac{\partial^2 \phi}{\partial x^2} - 2\phi^3 (I_y + I_z) \frac{\partial \phi}{\partial x} + \phi \frac{\partial^3 \phi}{\partial x^3} \sin \phi x

= \frac{\rho A}{\frac{\partial^2 \phi}{\partial t^2}} \left\{ \frac{\partial^3 \phi}{\partial x^3} \cos \phi x - \frac{\partial \phi}{\partial x} \sin \phi x \right\} \quad (36)
\end{aligned}
\]

2.2.2. Equation of Motion

Hence we obtain the two simultaneous partial differential equations of motion

\[
\begin{aligned}
& \frac{d^2 \phi}{dx^2} + 2\phi (I_y + I_z) \frac{\partial^3 \phi}{\partial x^3} - 2\phi^2 (I_y + I_z) \frac{\partial^2 \phi}{\partial x^2} - 2\phi^3 (I_y + I_z) \frac{\partial \phi}{\partial x} + \phi \frac{\partial^3 \phi}{\partial x^3} \cos \phi x

+ \frac{\rho A}{\frac{\partial^2 \phi}{\partial t^2}} \left\{ \frac{\partial^3 \phi}{\partial x^3} \cos \phi x - \frac{\partial \phi}{\partial x} \sin \phi x \right\} \quad (37)
\end{aligned}
\]

\[
\begin{aligned}
& \frac{d^2 \phi}{dx^2} - 2\phi (I_y + I_z) \frac{\partial^3 \phi}{\partial x^3} - 2\phi^2 (I_y + I_z) \frac{\partial^2 \phi}{\partial x^2} + 2\phi^3 (I_y + I_z) \frac{\partial \phi}{\partial x} + \phi \frac{\partial^3 \phi}{\partial x^3} \sin \phi x

+ \frac{\rho A}{\frac{\partial^2 \phi}{\partial t^2}} \left\{ \frac{\partial^3 \phi}{\partial x^3} \cos \phi x - \frac{\partial \phi}{\partial x} \sin \phi x \right\} \quad (38)
\end{aligned}
\]

Since the motion of a vibrating body, which is vibrating at its natural frequency, is simple harmonic, \( w \) and \( v \) may be expressed thus

\[
w = W \cos \omega t \quad \text{and} \quad v = V \cos \omega t \quad (39)
\]
Substituting in the equations of motion above gives

\[ I \dot{W}'' + 2 \dot{\phi}(I_y + I_z) \dot{W}' - 2 \phi^2 (I_y + I_z) W' - \phi^3 (I_y + I_z) V' \\
+ (\phi^4 (I_y + I_z^2) \dot{V} = 0 \tag{40} \]

\[ I \ddot{V} - 2 \phi^2 (I_y + I_z) \ddot{W} - 2 \phi^3 (I_y + I_z) V' + 2 \phi^4 (I_y + I_z) W' \\
+ (\phi^5 (I_y + I_z^2) \ddot{V} = 0 \tag{41} \]

By satisfying the boundary conditions the characteristic equation may be obtained if it is assumed that the solution is of the type

\[ W = A \exp(\lambda \phi) \quad \text{and} \quad V = B \exp(\lambda \phi) \tag{42} \]

2.2.4.3. Boundary Conditions

It is useful considering the boundary conditions which must be satisfied when solving the differential equations of motion.

Cantilever Beam

At the clamped end the deflection and slope must be zero, i.e. at \( x = 0 \) and \( \theta = 0 \)
\[ W = 0 \quad \text{and} \quad V = 0 \]
also \( W' = 0 \) and \( V' = 0 \)

at the free end the bending moment and shear force should be zero, i.e. at \( x = L \) and \( \theta = \phi L \)
\[ W = 0 \quad \text{and} \quad V = 0 \]

\[ EI_y [W'' + 2 \phi V' - \phi^2 W] = 0 \quad \text{and} \]
\[ EI_z [V'' - 2 \phi W' - \phi^2 V] = 0 \]

Primes denote differentiation with respect to \( x \)
Clamped-Pinned Beam

The conditions at the clamped end are the same as above, i.e. at \( x = 0 \) and \( \theta = 0 \):
- \( W = 0 \) and \( V = 0 \)
- \( W' = 0 \) and \( V' = 0 \)

at the pinned end the deflection and the bending moment are zero, i.e. at \( x = L \) and \( \theta = \phi L \):
- \( W = 0 \) and \( V = 0 \)

and

\[
\int_0^L [W'' + 2\phi V' - \phi^2 W] = 0
\]
\[
\int_0^L [V'' - 2\phi W' - \phi^2 V] = 0
\]

In either case an eighth order characteristic equation would be obtained and the solution could be obtained by equating an eight by eight determinant to zero. The calculation involved would be most conveniently carried out by a computer which was not readily available.

The solution of these differential equations for the case of a cantilever beam with an infinitely thin cross-section has been computed by Troesch, Anliker and Ziegler and their results will be mentioned in a later chapter. Anliker also solved, using a computer, the equations for a beam clamped at one end and pinned at the other having different breadth to depth ratios. The latter set of results are most useful for /
for comparison purposes with the results obtained in this investigation. Several other investigators have solved approximately the case of a pre-twisted cantilever beam basing their calculations on the differential equation of motion derived in this section.

2.2.5. The Myklestad Method:

The Myklestad Method is basically the same as the Holzer method applied to the determination of the uncoupled bending modes and frequencies of a beam. Because of the tedious calculations required for the application of Holzer's Method to the bending problem this method has seldom been used for determination of bending frequencies. Myklestad's contribution to this problem lies in the development of a tabular method which greatly reduces the work required in calculating the bending frequencies and modes. Furthermore, the Myklestad (like the Holzer) Method may be used to determine higher order modes without the necessity of determining the lower order modes first. Also the accuracy of the higher order modes is not at all dependent on the accuracy of determination of the lower modes and frequencies.

In this method it is assumed that the mass of the beam is concentrated at discrete points along its axis, and that each section of the beam between two such points or 'stations' is massless but possesses the elastic properties of the real beam. The number of stations chosen depends on the /
the number of frequencies and the accuracy required. As a rule, at least twice as many concentrated masses should be used as the number of frequencies to be obtained, but if the relative amplitudes curve also is required, the number of concentrated masses should not be less than six.

For a cantilever beam the points where the masses are concentrated are numbered from the free end of the beam. Let \( m_n \) = mass at point \( n \).

\[ \begin{align*}
    W_n & = \text{vertical deflection at point } n. \\
    \gamma_n & = \text{slope or angular deflection at point } n. \\
    S_n & = \text{shear force at point } n. \\
    M_n & = \text{bending moment at point } n. \\
    l_n & = \text{distance between points } n \text{ and } n+1. \\
    x_n & = \text{distance of point } n \text{ from the "base" of the beam.}
\end{align*} \]

Elastic coefficients or flexibility coefficients, which must be calculated for each section of the beam, are defined by considering deflection and slope of the length \( l_n \) clamped at point \( (n+1) \) and with unit bending moment or force applied at the point \( n \).

Thus \( d_{Fn} \) and \( \gamma_{Fn} \) are respectively deflection and slope at \( n \) for unit force there.

\( d_{Mn} \) and \( \gamma_{Mn} \) are respectively deflection and slope at \( n \) for unit moment there.

The above symbols for deflection are those used by Myklestad in /
in his original publication of this method. In the latest edition of his book an improved method of derivation replaces \( d_{n} \) and \( d_{Mn} \) by \( u_{n} \) and \( u_{Mn} \).

For a uniform beam the elastic coefficients become

\[
\frac{V_{n}}{V_{Mn}} = \frac{u_{n}}{u_{Mn}}, \quad \frac{V_{Mn}}{V_{Mn}} = \frac{Eln}{EIn} \quad \text{and} \quad \frac{u_{n}}{u_{Mn}} = \frac{Eh}{6EIn}
\]

If the beam is forced to vibrate with harmonic motion the slope of the beam at the \( n \)th station will be \( \gamma_{n} \cos \pi t \) and the deflection \( W_{n} \cos \pi t \). Then the maximum inertia forces on each of the masses will be \( m \pi^{2} \gamma \) and the maximum deflection of the beam during vibration will be the same as if it were statically loaded with these inertia forces.

The deflection curve due to these static loads is easily found as follows:

Consider the \( n \)th section of the beam and assume that the slope \( \gamma_{n} \) and the linear deflection \( W_{n} \) are both known. The shear force immediately to the left of the \( n \)th station is \( S_{n} \cos \pi t \) and the bending moment at station \( n \) is \( M_{n} \cos \pi t \).

The following relationships are obtained from the figure:

\[
\begin{align*}
\gamma_{n+1} &= \gamma_{n} - S_{n} \gamma_{n} + M_{n} \gamma_{Mn} \\
W_{n+1} &= W_{n} - E_{n} \gamma_{n} + S_{n} \gamma_{n} - M_{n} \gamma_{Mn} \\
S_{n+1} &= S_{n} + m_{n+1} \pi^{2} W_{n+1} \\
M_{n+1} &= M_{n} - S_{n} E_{n}
\end{align*}
\]
Let the free end of the beam vibrate with a known deflection but an unknown slope $\phi$

then we have $\varphi_1 = \phi$ (47)

and $w_1 = l$ (48)

By starting from the free end of the beam and applying equations (43) to (46) repeatedly it can be shown that all the quantities involved can be expressed as linear functions of the unknown slope $\phi$.

That is

$S_n = G\phi_n + G_n$ (49)

$M_n = H\phi_n - H_n$ (50)

$\varphi_n = H\phi_n - H_n$ (51)

$w_n = G\phi_n + g_n$ (52)

where $G\phi_n$, $G_n$, $H\phi_n$, $H_n$, $H\phi_n$, $H_n$, $g\phi_n$ and $g_n$ are known as the amplitude coefficients. The negative signs in equations (49) to (52) are introduced merely to make the amplitude coefficients come out positive in most practical problems, which makes calculation easier.

On substituting equations (49) to (52) in equations (43) to (46) and collecting terms containing $\phi$ and those not containing $\phi$ the expressions obtained are of the form $a + b\phi = 0$. Since for any value of $\phi$ the expressions must be satisfied, the quantities $a$ and $b$ must both be zero.

The following equations result:

$g\phi_{n+1} = g\phi_n + M_nB^2g\phi_{n+1}$ (53)

$g\phi_{n+1} = g\phi_n + m\phi_n B^2g\phi_{n+1}$ (54)

$H\phi_{n+1} = H\phi_n + E_n g\phi_n$ (55)

$H\phi_{n+1} = H\phi_n + E_n g\phi_n$ (56)

$H\phi_{n+1} = H\phi_n + 2E_n g\phi_n + \frac{1}{2}E_n^2 H\phi_n$ (57)

$H\phi_{n+1} = H\phi_n + 2E_n g\phi_n + \frac{1}{2}E_n^2 H\phi_n$ (58)

$H\phi_{n+1} = H\phi_n + E_n g\phi_n + H_n H\phi_n$ (59)

$H\phi_{n+1} = H\phi_n + E_n g\phi_n + H_n H\phi_n$ (60)
From the free end it is now possible to proceed to the base by means of equations (53) to (60) and to obtain all the amplitude coefficients for this part of the beam.

2.2.5.1. Initial Conditions

The initial conditions are essential as the equations (53) to (60) cannot be applied successively without them.

Beam with a free end

Such is the case of a cantilever beam or a free-free beam; it is always assumed that the deflection $W$ is unity and the slope $\psi$ is $\beta$. Also the shear force $S_1$ is $m_1p^2$ and the bending moment $M_1 = 0$. The initial conditions for the amplitude coefficients become

$$G_0 = 0, \quad H_0 = 0, \quad \phi_0 = 1, \quad \lambda_0 = 0$$

$$g_0 = m_1p^2, \quad h_0 = 0, \quad \lambda_0 = 0, \quad \gamma_0 = 0$$

Beam with hinged end

For a beam with a hinged end at station number 1 it is assumed that the deflection is equal to unity, which gives

$$G_0 = 0, \quad H_0 = 0, \quad \phi_0 = 1, \quad \lambda_0 = 0$$

$$g_0 = 1, \quad h_0 = 0, \quad \lambda_0 = 0, \quad \gamma_0 = 0$$

Conditions at the base of the beam

The last station of the beam is called the base of the beam, and at a natural frequency two additional boundary conditions must be satisfied at this point. One of these two conditions /
conditions can be satisfied by using the appropriate value for the slope of $\phi$. The other is satisfied only at the natural frequency $p$, and by plotting this boundary condition as a function of $p$ the natural frequency will be obtained.

**Cantilever Beam**

Since the base is clamped, the slope and deflection should be zero i.e.

$$\phi_b - \phi_s = 0 \quad \text{or} \quad \phi = \phi_s$$

The value of $\gamma_b = L_\phi \phi - L_b$ should then be plotted as a function of $p$ and every time it becomes zero there is a natural frequency.

**Simply Supported Beam**

In this case the base is hinged and the deflection and bending moment are zero. The value of $\phi$ is obtained from $\phi_0 = \phi S \phi = 0$ and the other condition gives $M_b = H_\phi \phi - H_b$ which should be plotted as a function of $p$.

This method of finding the natural frequencies and normal modes of vibration may be applied to uniform beams and also to non-uniform beams, it being particularly suitable for the latter. It was originally applied by Myklestad (1944) to the uncoupled bending vibrations of aeroplane wings and beams but later it was applied with the aid of additional expressions for twisting about the axis of the beam to the coupled bending torsion vibration of beams, again by Myklestad.
Prohl (1945) developed a method for determining the critical speed of flexible rotors which is fundamentally the same as the method described above.

2.8.1. The Stodola Method

The Stodola method, sometimes known as the Stodola-Vianello method, was originally devised as a method of determining the lateral vibrations of turbine rotors, but it can easily be extended to obtain the bending or torsional natural frequencies and modes of any non-uniform beam. In this method the differential equation is solved by a process of iterative integration. The differential equation for pure flexure has already been shown in section (2.2.1) to be

\[ \frac{d^2}{dx^2} \left\{ \varepsilon I(x) \frac{d^2 y}{dx^2} \right\} = \omega^2 \mu(x) y(x) \]  

(61)

and that this may be written as a set of equations

\[ \frac{dS}{dx} = -\omega^2 \mu(x) y(x) \]  

(62)

\[ \frac{dM}{dx} = -S(x) \]  

(63)

\[ \varepsilon I(x) \frac{d^2 y}{dx^2} = N y(x) \]  

(64)

\[ \frac{dy}{dx} = \nu(x) \]  

(65)

The /
The following equations may be derived from the above and constitute one stage of the integration.

\[ S_r(x) = \int_x^1 S_r(u) \, du \quad \text{(66)} \]
\[ M_r(x) = \int_x^1 M_r(u) \, du \quad \text{(67)} \]
\[ V_r(x) = \int_x^1 V_r(u) \, du \quad \text{(68)} \]
\[ Y_r(x) = \int_x^1 Y_r(u) \, du \quad \text{(69)} \]

and

\[ \omega^2 V_r(1) = Y_{r-1}(1) \quad \text{(70)} \]

where \( S_r(x), M_r(x) \), etc. are the approximations to \( S(x) \), etc. obtained during the \( r \)th iteration. The cantilever is clamped at the point \( x = 0 \) and free end is at \( x = 1 \). Since both the mode shapes \( \gamma(x) \) and the natural frequency \( \omega \) are unknown, it is necessary to assume an arbitrary value for \( \omega \), say 1, and also to assume a deflection shape \( \gamma_1(x) \), which is something like the actual deflection curve. On substituting these values in the first integral and carrying out the integrations, values are obtained for \( S_r, M_r, V_r, \) and \( Y_r \).

This new expression for the deflection will only be the exact mode shape if first the value for \( \omega \) chosen was exactly the normal mode shape. The ratio of the ordinates \( \gamma_1(x) \) to \( \gamma_2(x) \) gives the first approximation to the natural frequency as is shown in equation. (70)

With a reasonable assumption for the deflection curve, the accuracy of this procedure is very good. If greater accuracy is required, this procedure may be repeated with \( \gamma_2(x) \) as the original assumption thus obtaining a third curve \( \gamma_3(x) \). In fact it is seldom necessary to repeat the procedure /
procedure more than twice since the convergence is so rapid (Den Hartog p. 162), although theoretically the procedure should be repeated until the desired accuracy is obtained.

For the second and higher modes the Stodola process is not convergent unless certain modifications to the process are made, the most important of these being the purification of the deflection curve from its first mode content. In order to do this the shape of the first mode must be known with sufficient accuracy (Den Hartog p. 162.3). For the third or higher modes the procedure is similar, but the assumed curve for the third harmonic has to be purified from the first as well as from the second harmonic. Thus the Stodola process cannot be applied to a higher mode of vibration until all the lower modes have been obtained with sufficient accuracy.

2.3. Energy Methods:

2.3.1. The Rayleigh and Rayleigh-Ritz Method

The Rayleigh Method, a method suggested by Lord Rayleigh in the nineteenth century, is one of great practical utility and is particularly useful in obtaining an estimate of the fundamental natural frequency of any system. The value obtained by this method for the fundamental frequency will always be either equal to or greater than the exact frequency, so that an upper limit to the frequency is given by this method.
Both the Rayleigh and the Rayleigh-Ritz Methods are derived from Hamilton's Principle for Dynamical Systems. Briefly Hamilton's principle applied to systems under consideration at present, states that if \( T \) represents the total kinetic energy of a dynamical system (e.g. a vibrating body) at any time and \( V \) represents the potential energy at the same time, then the actual motion of the system between arbitrary times \( t_0 \) and \( t_1 \) is such that the integral

\[
H = \int_{t_0}^{t_1} (T - V) \, dt
\]  

(71)

has a stationary value compared with the values it has for any other type of motion, provided these neighbouring types of motion are the same as the actual motion at times \( t_0 \) and \( t_1 \). In the language of the calculus of variations, the variation of the integral in (71) is zero for the actual motion, or

\[
S \int_{t_0}^{t_1} (T - V) \, dt = 0
\]  

(72)

Now in problems where the natural frequencies of vibration are sought, the time variation of the quantities \( T \) and \( V \) is known since the motion is simple harmonic with time. Thus /
Thus for a vibrating beam say, the displacement may be expressed as

$$w(x, t) = w(x) \sin pt$$

while

$$V = \frac{1}{2} \int \left[ p \left( \frac{\partial w}{\partial x} \right)^2 \right] dx$$

and

$$T = \frac{1}{2} \int \left[ \rho A(x) \left( \frac{\partial w}{\partial x} \right)^2 \right] dx$$

where $l$ is the length of the beam

$A$ is the cross-sectional area

$p$ is the mass density

and $F$ is a homogeneous quadratic function in $\frac{\partial w}{\partial x}$ and $\frac{\partial^2 w}{\partial x^2}$

Thus

$$T - V = p^2 K \cos^2 pt - u \sin^2 pt$$

(73)

where

$$u = \int \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx$$

and

$$K = \frac{1}{2} \int \left( \frac{\partial A(x)}{\partial x} \right)^2 W^2 dx$$

primes denote differentiation with respect to $x$.

The time integral of (72) can now be evaluated using (73), in accordance with Hamilton's principle the times $t_0$ and $t_1$ have to be chosen so that the variations in $w$ are zero, so take $t_0 = 0$ and $t_1 = T_1 / \rho$. Thus (73) gives rise to the variational equation

$$\delta (p^2 K - u) = 0$$

(74)

where $u$ and $K$ have the values already obtained from (73).

Thus /
Thus it represents the maximum potential energy appearing at moments of the largest deflection and $p^2K$ is the maximum kinetic energy appearing when the system passes through the position of rest.

Considering the fundamental frequency $p_1$; if this were exactly known along with the corresponding eigen function or shape function $\Phi_1(x)$, $K_1$ and $p_1^2$ would be evaluated from (74) thus $p_1^2 = \frac{\Phi_1}{K_1}$

If, however, $\Phi$ and $K$ are the values corresponding to some assumed shape function then by (74)

$$\frac{\Phi}{K} = \frac{\Phi_1}{K_1} (75)$$

that is, if a shape function is chosen which satisfies the geometrical boundary conditions of the problem and the maximum kinetic and potential energies are evaluated, their quotient gives an upper limit to the first natural frequency.

To obtain a value close to the fundamental in practical cases, it is usually necessary to satisfy the dynamical as well as the geometrical boundary conditions. A comprehensive study of the theory and applications of Rayleigh's method is given by Temple and Bickley (1933). Biesanzo and Grammel (1954) study the theory and compare the degree of accuracy it is possible to achieve by improving the approximation to the eigen function.
The method used by Ritz (1900) in the study of the flexural vibration of plates is a development of Rayleigh's Method which obtains simultaneously approximations to the first \( n \) frequencies. The assumed shape function for substitution in (74) is taken in the form of a series such as

\[
W(x) = a_1 \xi_1(x) + a_2 \xi_2(x) + \cdots + a_n \xi_n(x)
\]

where the \( n \) linearly independent functions \( \xi_i(x) \) are each chosen to satisfy the geometric boundary conditions. The constants \( a_i \) are considered as parameters which provide the variations indicated in (74) and as Ritz proposed to make the quantity \( (p^2 K - \lambda) \) a minimum or in other words so that

\[
\frac{d}{da_i} (p^2 K - \lambda) = 0
\]

Carrying out the indicated operations yields a system of \( n \) equations homogeneous and linear in \( a_i \); equating the determinant of these equations to zero will yield the frequency equation the first \( n \) roots of which will correspond to the first \( n \) natural frequencies. Timoshenko (1955) employs this method to obtain the natural frequencies of beams with varying cross-sections and obtains results for a wedge and a conical bar. Also the case of a cantilever beam in which the cross-sectional area and the second moment of area are proportional to a power of the distance \( x \) along the beam is used to illustrate this procedure, generally known as the Rayleigh-Ritz Method.
2.3.2. Galerkin's Method

The Galerkin method proceeds from the variational equation (4) which for a simple beam problem may be written thus

\[ \int_0^L \left( \frac{\partial^2}{\partial x^2} N(x) W^2 - f(x), W', W'' \right) dx = 0 \]  

(72)

If the variations are performed according to theory of the calculus of variations equation (7) becomes

\[ \int_0^L \left( \frac{\partial}{\partial W} \right)^2 \left[ \frac{\partial}{\partial W} \right]^2 \left[ \frac{\partial}{\partial W} \right]^2 \left[ \frac{\partial}{\partial W} \right]^2 \right] dx = 0 \]  

(78)

For this equation to be true for any value of \( \delta W \), a first condition is that the terms under the integral sign must vanish. This leads therefore to the differential equation of the eigen value problem, i.e.

\[ (N - \beta^2 W) \phi = \frac{d}{dx} \left( \frac{\partial}{\partial W} \right) \left[ \frac{\partial}{\partial W} \right]^2 \left[ \frac{\partial}{\partial W} \right]^2 \left[ \frac{\partial}{\partial W} \right]^2 \right] \phi = 0 \]  

(79)

A second condition is that the expressions in the curly brackets in equation (78) must also vanish, thus the dynamical boundary conditions are obtained in advance.

At /
At this stage Galerkin's method, like the Rayleigh-Ritz procedure, requires the assumption that the displacement function can be represented by a function of the type

\[ w(x) = a_1 \xi_1(x) + a_2 \xi_2(x) + \cdots + a_n \xi_n(x) \]  \hspace{1cm} (80)

where the co-ordinate functions \( \xi_i(x) \) must satisfy the geometrical and dynamical boundary conditions. Only these variations \( \delta w \) are admitted that can be represented as variations of the assumed function (80) and thus have the form

\[ \delta w = \xi_1(x) \delta a_1 + \xi_2(x) \delta a_2 + \cdots + \xi_n(x) \delta a_n \]  \hspace{1cm} (81)

where \( \delta a_i \) are arbitrary. Since all possible tentative functions are now no longer admitted we shall in general no longer obtain the exact solution \( \Phi^* \) but only an upper limit \( \beta^2 \leq \Phi^* \) hence an approximation to the lowest value.

The terms in the curly brackets in the variational equation (78) now vanish because of (81) and on substituting the assumed displacement function in equation (78) the following equation is obtained

\[ \int L \left( \frac{d^2 w}{dx^2} \right) \delta w \, dx = 0 \]  \hspace{1cm} (82)

Because the \( \delta a_i \) quantities can be chosen arbitrarily and introducing /
introducing the differential \( L(W) \) as defined in equation (79) the variational equation (82) becomes

\[
\sum_{\alpha=1}^{n} \left[ \frac{\partial^2}{\partial x^2} \left( \sum_{\beta=1}^{n} \alpha \phi_{\beta}(x) \right) - \beta^2 \phi_{\alpha}(x) \right] dx = 0 \quad (\alpha = 1, 2, 3, \ldots, n) \tag{83}
\]

Alternatively equation (83) may be written as

\[
\sum_{\alpha=1}^{n} \phi_{\alpha}(x) \left[ \frac{\partial^2}{\partial x^2} \sum_{\beta=1}^{n} \phi_{\beta}(x) \right] dx = 0 \quad (\alpha = 1, 2, 3, \ldots, n) \tag{84}
\]

and these equations are known as Galerkin's equations. They are \( n \) linear homogeneous equations in \( a_1 \). Since the determinant formed by the coefficients of these equations must vanish, the equation obtained on satisfying this condition is the frequency equation from which are obtained \( n \) values for \( p^2 \). The smallest of these values gives an upper limit to the fundamental frequency and the other values are approximations to the higher natural frequencies.

Galerkin's method is particularly suitable when the differential equation

\[
L(W) - \beta^2 W = 0.
\]

of the problem is already known. Then by (83) an assumed shape function of the correct type is introduced into the differential /
differential equation which is multiplied in turn by each of the co-ordinate functions \( f_k \) and after integration of the entire expression over the length of the beam, is equated to zero. It should be noted, however, that equation (83) or (84) will not have their right hand sides equal to zero if the boundary conditions are time dependent, (i.e. contain the unknown eigen value \( p^2 \)). Such conditions arise from an elastic support or a free end with a spring mass system suspended there. Biezeno and Grammel (1954) give an example of the procedure necessary in such a case.

2.3.3. The Galerkin-Grammel or Complementary Energy Method

This method introduced by Grammel (1939) is developed in a very similar way to the Galerkin method and is derived from equation (82). However, instead of proceeding from the differential equation of the problem implicit in equation (82) the fact that the differential equation is equivalent to the integral equation for the beam in terms of Green's functions is employed.

Once again an approximating function of the type

\[
\psi(x) = \sum_{i=1}^{n} a_i \phi_i(x)
\]

is assumed but on this occasion the functions need only satisfy the geometrical boundary conditions. Thus equation (83) and (84) may now be written as

\[
\int \left[ \sum_{i=1}^{n} a_i \phi_i(x) - \frac{1}{2} \int \phi_i(x, \xi) \phi_i(x, \xi) dx \right] dx = 0 \quad (k = 1, 2, \ldots, n)
\]
where \( g(x, \xi) \) is Green's Function corresponding to the eigenvalue problem.

These are again \( n \) linear homogeneous equations in the \( a_j \) and the determinant formed by the coefficients of these equations must again be equated to zero, with the result that \( n \) values are obtained for the natural frequencies. These values are again upper limits to the respective frequencies.

This method is sometimes preferable to the Galerkin method in that the assumed functions \( g_i \) need only satisfy the geometrical boundary conditions. Furthermore, it can be proved (Grammel, 1939) that the approximations resulting from this method always supply a lower, hence more accurate, upper limit for \( p_1^2 \) than Galerkin's approximations, starting from the same co-ordinate functions \( g_i \). Further explanations of this method are to be found in Biezeno and Grammel (1954 and 1955).

2.3.4. Particular Cases

Although any of the methods described in the preceding sections could be employed to obtain the natural frequencies of vibration of the beams mentioned in the following sections, only the Rayleigh and the Rayleigh-Ritz methods are actually considered. Thus only the frequency of the fundamental mode of vibration will be found.
The case of the uniform beam is more accurately solved by using the differential equation of motion but these examples will show the degree of accuracy obtainable.

a) Cantilever Beam:

Consider the beam to be acted on by its own mass only, then the bending moment is

$$-EI \frac{d^{2} \omega}{dx^{2}} = - \frac{P \omega}{2l} (l-x)^{2} \quad (85)$$

the slope is obtained by integrating this to give

$$-EI \frac{d \omega}{dx} = - \frac{P \omega}{2l} \left( \frac{1}{2}x^{2} - \frac{1}{6}x^{3} + \frac{1}{3}x \right) + C \quad (86)$$

and the deflection is obtained by integrating once more giving

$$-EI \omega = - \frac{P \omega}{2l} \left( \frac{1}{12}x^{3} - \frac{1}{3}x^{4} + \frac{1}{2}x \right) + Cx + D. \quad (87)$$

Boundary Conditions

at the fixed end, i.e. \( x = 0 \)

the deflection is zero, i.e. \( \omega = 0 \)

and the slope is zero, i.e. \( \frac{d \omega}{dx} = 0 \)

\( C = D = 0. \)

at the free end, i.e. \( x = l \)

the bending moment is zero, i.e. \( -EI \frac{d^{2} \omega}{dx^{2}} = 0. \)

and the shear force is zero, i.e. \( -EI \frac{d \omega}{dx} = 0. \)

Both these conditions have already been satisfied.

Thus /
Thus the expressions required are

a) the deflection which is

\[ \omega = -\frac{EI}{2y} \left( 5x \frac{C1}{x^2} - \frac{C2}{x^3} \right) \]  

and b) the bending moment which is

\[ -\frac{EI}{y} \frac{d^2y}{dx^2} = -\frac{PA}{2y} \left( e-x \right)^2 \]  

Now the Kinetic Energy is known to be

\[ \frac{1}{2} \int \frac{e}{y} \left( C1 \right)^2 \left( 6x^2 - 4x + x^3 \right) dx \]

\[ = \frac{2^2 (PA)^3 C^4}{9^3 (21)^2} \left\{ \frac{13}{90 \times 72} \right\} \]

and the Strain Energy is

\[ \frac{1}{2} \int \frac{M^2}{E1} dx \]

i.e.

\[ \frac{(PA)^2 C^5}{90 E1 Y} \]

Thus from equation (5) the expression for \( \omega^2 \) is

\[ \omega^2 = \frac{\frac{1}{2} \int \frac{e}{y} \left( C1 \right)^2 dx}{\frac{1}{2} \int \frac{e}{y} \left( C1 \right)^2 dx} \]

Substituting the values already obtained for the integrals gives

\[ \omega^2 = \frac{E1 Y}{PA^2} \left\{ \frac{90 \times 72}{13 \times 40} \right\} \]

and therefore the frequency is

\[ f = \frac{3.53}{2\pi^2} \sqrt{\frac{E1 Y}{PA}} \]
b) **Clamped-Pinned Beams**

The expression for the bending moment in this case is

\[ -EI \frac{d^2w}{dx^2} = -\frac{P}{8} \left[ 3x^2 - 5x + 4x^2 \right] \]  \hspace{1cm} (95)

on integrating the slope becomes

\[ -EI \frac{dw}{dx} = -\frac{P}{8} \left[ \frac{x^3}{3} - \frac{5}{2} x^2 + \frac{4}{3} x^3 \right] + C. \]  \hspace{1cm} (96)

and integrating once more to obtain the deflection

\[ -EI w = -\frac{P}{8} \left[ \frac{x^3}{3} - \frac{5}{2} x^2 + \frac{4}{3} x^3 \right] + Cx + D. \]  \hspace{1cm} (97)

**Boundary Conditions**

At the clamped end i.e. \( x = 0 \)

the deflection is zero, i.e. \( w = 0 \)

and the slope is zero, i.e. \( \frac{dw}{dx} = 0. \)

Thus \( C = D = 0 \)

and the deflection becomes

\[ w = \frac{P}{8I} \left[ \frac{3x^2}{2} - 5x^2 + 4x^2 \right] \]  \hspace{1cm} (98)

At the pinned end, i.e. \( x = L \)

the deflection is zero, i.e. \( w = 0 \)

and the bending moment is zero, i.e. \( -EI \frac{d^2w}{dx^2} = 0. \)

Both these conditions are satisfied.
Then the Kinetic Energy is

\[
\frac{(μA)^3 L^2 c^2}{55x56x128}  \tag{99}
\]

and the Strain Energy is

\[
\frac{(μA)^2 L^6}{5x128} \tag{100}
\]

Equating the two energies gives

\[
\beta^2 = \frac{EIG}{μAE^2} \left\{ \frac{81x56}{19} \right\} \tag{101}
\]

and the frequency is

\[
f = \frac{15.45}{2\pi^2 E^2} \sqrt{\frac{EIG}{μA}} \tag{102}
\]

2.3.4.2. Non-Uniform Beam

Here the Rayleigh–Ritz Method will be employed to determine the natural frequency of the fundamental mode of vibration of a wedge-shaped cantilever beam. As in previous examples, a deflection shape of the form

\[
y = X\cos(βt) \tag{103}
\]

is assumed for the vibrating beam.

The maximum potential energy and the maximum Kinetic energy are obtained as; the Potential Energy

\[
V = \frac{1}{2} \int_0^L E I \left( \frac{d^2X}{dx^2} \right) dx \tag{104}
\]

and the Kinetic Energy

\[
T = \frac{1}{2} \int_0^L \frac{μA}{2} \beta^2 X^2 dx. \tag{105}
\]

Now /
Now if the expression for $X$ be taken in the form

$$X = a_0 f_0(x) + a_1 f_1(x) + \cdots + a_n f_n(x)$$  \hfill (106)

then, according to Ritz, the quantity $(T - V)$ should be a minimum or

$$\frac{d}{dx} (T - V) = 0.$$  \hfill (107)

### Case of a Wedge

The expressions for the Area and Inertia of the wedge in terms of distance along are

- Area $A = \frac{1}{2} \ell b d$  
- Inertia $I = \frac{1}{12} (2d \ell)^2$

where $\ell$ is the length of the beam, $b$ is the width of the beam, and $2d$ is the depth at the fixed end.

### End Conditions

At the free end the bending moment and shear force are zero, i.e. $x = 0$.

$$EI \frac{d^2X}{dx^2} = 0 \text{ and } \frac{d}{dx} \left( EI \frac{d^2X}{dx^2} \right) = 0.$$  \hfill (108)

At the fixed end the deflection and slope are zero, i.e. $x = \ell$.

$$X = a_0 (1 - \ell^2) + a_2 \ell (1 - \ell)^2$$  \hfill (109)
as the series expression for the deflection and substituting
in the expressions for the potential and kinetic energies
\((T-V)\) becomes

\[
\frac{26d^2}{\varepsilon g} \frac{a_0^2}{30} + \frac{a_0^2 a_1}{105} + \frac{a_0^2 b_2}{280} - \frac{2b0}{363} \left\{ \frac{(a_1-a_0)^2}{3} + \frac{24}{5} a_0 (a_1-2a_0) + 6b_0^2 \right\}
\]

From equation (107) the conditions for this expression to be a minimum, are that

\[
\frac{2}{\partial a_0} (T-V) = 0 \text{ and } \frac{2}{\partial a_2} (T-V) = 0.
\]

which yields the following two linear equations

\[
\left( \frac{b_0^2}{30} - \frac{256}{5} \frac{a_0^2}{364} \right) a_1 + \left( \frac{b_0^2}{105} - \frac{2}{5} \frac{56}{364} a_0^2 \right) a_2 = 0. \tag{111}
\]

and

\[
\left( \frac{b_0^2}{105} - \frac{2}{5} \frac{56}{364} a_0^2 \right) a_1 + \left( \frac{b_0^2}{280} - \frac{2}{5} \frac{56}{364} a_0^2 \right) a_2 = 0. \tag{112}
\]

Equating the determinant of these equations to zero gives

\[
\left( \frac{b_0^2}{30} - \frac{256}{5} \frac{a_0^2}{364} \right) \left( \frac{b_0^2}{280} - \frac{2}{5} \frac{56}{364} a_0^2 \right) - \left( \frac{b_0^2}{105} - \frac{2}{5} \frac{56}{364} a_0^2 \right)^2 = 0 \tag{113}
\]

\(b_0^2\) can be calculated from this equation.

The smallest of the roots given

\[
f = \frac{5.319}{2\pi} \frac{a_0}{e^2 \sqrt{\frac{56}{364}}}
\]

\(\text{(114)}\)
2.3.4.3. **Pre-twisted Beams**

It is reasonable to expect that the Rayleigh and the Rayleigh-Ritz or any of the other methods based on consideration of the energies of vibration, can be used to obtain the natural frequencies. This is true provided the expressions for the bending moments and the deflections are modified to take into consideration the twist of the beam and thus the coupling of motion in the two principal directions.

a) **Cantilever Beam**

Recently two papers have been written by Carnegie (1958 and 1959) using the Rayleigh Energy Method. In the first paper, Carnegie obtained expressions for the static deflections due to the uniformly distributed load of its own weight as follows:

\[
\begin{align*}
\dot{y} &= \frac{3}{2} \left( \frac{I}{I_y} I_x + I_x I_y \right) \left[ \frac{3}{2} I_2 \sin \left( \frac{2 \pi x}{L} \right) + \frac{3}{2} \frac{I_y}{I_x} \sin \left( \frac{2 \pi y}{L} \right) + \frac{2}{2 \pi x} \cos \left( \frac{2 \pi x}{L} \right) \right] \\
\dot{z} &= \frac{3}{2} \left( \frac{I}{I_y} I_x + I_x I_y \right) \left[ \frac{3}{2} I_2 \cos \left( \frac{2 \pi x}{L} \right) - \frac{2}{2 \pi y} \sin \left( \frac{2 \pi y}{L} \right) \right]
\end{align*}
\]

(115)

and

\[
\begin{align*}
\dot{y} &= \frac{3}{2} \left( \frac{I}{I_y} I_x + I_x I_y \right) \left[ \frac{3}{2} I_2 \sin \left( \frac{2 \pi y}{L} \right) + \frac{3}{2} \frac{I_y}{I_x} \sin \left( \frac{2 \pi x}{L} \right) + \frac{2}{2 \pi y} \cos \left( \frac{2 \pi y}{L} \right) \right] \\
\dot{z} &= \frac{3}{2} \left( \frac{I}{I_y} I_x + I_x I_y \right) \left[ \frac{3}{2} I_2 \cos \left( \frac{2 \pi y}{L} \right) - \frac{2}{2 \pi x} \sin \left( \frac{2 \pi x}{L} \right) \right]
\end{align*}
\]

(116)

The /
The modified expressions for the Potential Energy and Kinetic Energy are

\[ V = \int \frac{\varepsilon}{2} \left( \frac{\partial \dot{y}}{\partial x} \right)^2 + \frac{\varepsilon}{2} \frac{\partial^2 y}{\partial x^2} \left( \frac{\partial \dot{y}}{\partial x} \right)^2 + \frac{\varepsilon}{2} \frac{\partial^2 y}{\partial x^2} \left( \frac{\partial \dot{y}}{\partial x} \right)^2 \, dx \]

and Kinetic Energy

\[ T = \int \frac{\dot{y}^2}{2} \left( \frac{\partial \dot{y}}{\partial x} \right)^2 \, dx. \]

Therefore the expression for the square of the angular natural frequency is

\[ \omega^2 = \frac{\int \frac{\varepsilon}{2} \left( \frac{\partial \dot{y}}{\partial x} \right)^2 + \frac{\varepsilon}{2} \frac{\partial^2 y}{\partial x^2} \left( \frac{\partial \dot{y}}{\partial x} \right)^2 + \frac{\varepsilon}{2} \frac{\partial^2 y}{\partial x^2} \left( \frac{\partial \dot{y}}{\partial x} \right)^2 \, dx}{\int \frac{\dot{y}^2}{2} \left( \frac{\partial \dot{y}}{\partial x} \right)^2 \, dx} \]

Substitution of the appropriate values of all the terms involving \( \dot{y} \) and \( \ddot{y} \) and then suitably arranging all the terms yields an expression for the fundamental frequency as follows:

\[ f = \frac{3.63 K}{2 \pi c^2} \sqrt{\frac{\varepsilon}{\pi}} \]

where the factor \( K \) is given by the equation

\[ K = \left\{ \frac{2\left[ 1 + \frac{a^2}{m^2} \right] + 15\left[ 1 - \frac{a^2}{m^2} \right] A}{\left[ 1 + \frac{a^2}{m^2} \right]^2 + \left[ 1 - \frac{a^2}{m^2} \right]^2 B + \left[ 1 - \frac{a^2}{m^2} \right]^2 C} \right\} \]

This expression applies if the width, \( m \), is equal to or greater than the depth, \( n \).

The quantities \( A \), \( B \) and \( C \) are all functions of the pre-twist angle \( \alpha \) as follows.

\[ A = \left\{ \frac{\sin 2\alpha - 2 \alpha + \frac{3}{2} \alpha^3}{2 \alpha^5} \right\} \]

\[ B = \left\{ \frac{15.5 \sin 2\alpha + 15 \alpha \cos 2\alpha + 3/5 \alpha^3 - 3/5 \alpha^3}{1/16 \alpha^4} \right\} \]

\[ C = \left\{ \frac{-2/5 \sin 2\alpha + 3 \alpha \sin 2\alpha + 63 \alpha \cos 2\alpha - 3/5 \alpha^3 - 3/5 \alpha^3}{1/16 \alpha^4} \right\} \]
Thus a value for $K$ can be obtained for any values of $m$ and $n$ and for any angle of pre-twist. There are two particular cases which give unity for $K$ i.e. when $m = n$ and when $\alpha_2 = 0$.

Di Prima and Handelman (1954-5) in a paper on the vibration of Twisted Beams after deriving the differential equation of motion go on to obtain an approximate solution using Rayleigh's Method. Their results are given for three angles of pre-twist for three different breadth to depth ratios.

<table>
<thead>
<tr>
<th>$\alpha_2$</th>
<th>$(b/d)^2$</th>
<th>$f/f_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/6$</td>
<td>48/64/144</td>
<td>1.01/1.01/1.03</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>48/64/144</td>
<td>1.04/1.05/1.10</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>48/64/144</td>
<td>1.07/1.04/1.17</td>
</tr>
</tbody>
</table>

b) **Clamped-Pinned Beam**

In this case it is proposed to follow the same line of approach as Carnegie uses for the cantilever beam, using the appropriate boundary conditions.

**Boundary Conditions**

At the clamped end i.e. $x = 0$ the deflections are zero, i.e. $\nu = 0$ and $w = 0$ and the slopes are zero, i.e. $\frac{d\nu}{dx} = 0$ and $\frac{dw}{dx} = 0$.

where $\nu$ and $w$ are the deflections in the $y$ and $z$ directions.
At the pinned end, i.e. $x = L$
the deflections are zero, i.e. $v_2 = 0$ and $\omega_2 = 0$
and the bending moments are zero, i.e. $M_y = 0$ and $M_z = 0$.

**Bending Moment Expressions**

It was shown in the section on the differential equation of motion for pre-twisted beams that the bending moment expressions were

$$M_y = -EJ_y \left[ \frac{3v}{2} + 2x \frac{dv}{dx} - x^2 \right]$$

and

$$M_z = -EJ_z \left[ \frac{2v}{2} + x \frac{dv}{dx} - x^2 \right]$$

Now considering a clamped-pinned beam of length $L$, breadth $b$, and depth $d$, deflecting under its own weight, $\rho A$ pounds per unit length.
The bending moments are

\[ M_y = -\frac{PA}{2}(e-x)^2 \cos \phi + P(l-x) \]  

(127)

and

\[ M_z = -\frac{PA}{2}(e-x)^2 \sin \phi + q(l-x)^2 \]  

(128)

where, \( P \) and \( q \) are components in the \( y \) and \( z \) directions of the reaction at the pin end.

Equating the two expressions for the bending moments given in (125) and (126) to the above, gives

\[ \frac{d^2 \phi}{dx^2} + 2 \phi \frac{d \phi}{dx} - \phi^2 = \frac{1}{2} PA \{(e-x)^2 \cos \phi + q(l-x)^2 \} \]  

(129)

and

\[ \frac{d^2 \phi}{dx^2} - 2 \phi \frac{d \phi}{dx} - \phi^2 = -\frac{1}{2} PA \{(e-x)^2 \sin \phi + q(l-x)^2 \} \]  

(130)

by differentiating and substituting for terms in \( \phi \) an equation for \( \phi \) is obtained.

\[ \frac{d^4 \phi}{dx^4} + 2 \phi^2 \frac{d^2 \phi}{dx^2} + \phi \frac{d \phi}{dx} = \frac{1}{2} PA \left[ 2 \cos \phi + \phi^2 \{(e-x) - 2 \phi \frac{e-x}{2} \} \right] 
\[ - \left( 1 - \frac{e-x}{2} \right) \left( \phi^2 (e-x)^2 \cos \phi x - \phi \left( e-x \right) \sin \phi x \right) \]  

(131)

In the same way it is possible to obtain a similar differential equation in \( \phi \), i.e.

\[ \frac{d^4 \phi}{dx^4} + 2 \phi^2 \frac{d^2 \phi}{dx^2} + \phi \frac{d \phi}{dx} = \frac{1}{2} PA \left[ 2 \cos \phi + \phi^2 \{(e-x) - 2 \phi \frac{e-x}{2} \} \right] 
\[ + 4 \phi (e-x) \cos \phi x + 2 \frac{e-x}{2} \sin \phi x - \phi \left( e-x \right) \]  

(132)\]
It is possible to solve these two differential equations to obtain expressions for the deflections \( \nu \) and \( \omega \), using the static values for the reactions \( P \) and \( Q \) for a first approximation. Thus the kinetic energy of vibration could be evaluated and equated to the Potential Energy of vibration, giving the fundamental angular frequency in terms of the ratios of the inertias and angles of pre-twist. It is the author's intention to follow up this approach and find suitable functions for \( \nu \) and \( \omega \).
3.1 INTRODUCTION

In this chapter the apparatus used to excite and measure the natural frequencies of vibration of rectangular cross-section beams is described in detail. Two types of beams were used in the investigations a) a cantilever beam and b) a clamped-pinned beam. It was assumed that type (a) represented a turbine blade secured to a disc or an outer casing at one end and free at the other and (b) represented a blade attached to a disc or an outer casing at one end and to a shroud band at the other end.

The first beam used was a cantilever beam, 10 inches long, 1.5 inches wide and 0.5 inch deep, machined integral with its base from a solid piece of metal. Later beams were made out of 1 inch by 0.5 inch rolled steel bar and the length could be adjusted as required. These bars could be clamped at one end and left free at the other or clamped at one end and pinned at the other.

To assist the theoretical analysis a short investigation was carried out to see if the helical shape of the beam fibres has any effect since the theory used assumes they are straight lines.

The sections on the method of carrying out the investigation, including the section on the accuracy of the experiments /
experiments show the care taken to ensure that the variations on the natural frequencies measured were in fact due to pre-twist.

3.2. Cantilever Beams

3.2.1. Cantilever Beam with breadth, depth ratio 3/1.

The cantilever beam first used in these experiments had already been manufactured from a solid piece of metal when the author started the investigation into the effects of pre-twist. The reason for machining the beam from a large piece of metal was to make certain the clamped end condition was definitely obtained. The dimensions of the beam were length 10 inches, width 1½ inches and depth ½ inch. (See Figures IV).

The mass of metal providing the fixed end of the beam was bolted to an old lathe bed which also supported the moving coil exciter attached to the beam. (Figure VI). Each time the beam was due to be twisted a few degrees more the whole assembly /
assembly had to be dismantled and the beam mounted in a torsion machine. Once the beam had been given a pre-twist it was impossible to return to the previous value of twist if there was any doubt about the frequencies obtained.

3.2.2. Cantilever Beam with Breadth/Depth Ratio 8/1.

It was because of difficulty about returning to a previous condition that a different type of clamping device was decided upon and designed. As the changes in natural frequency due to pre-twist were very small, in some cases, it was decided to change the breadth to depth ratio from 3/1 to 8/1 in the hope that the variations due to pre-twist would be increased. This new clamping device was designed so that a piece of rolled mild steel bar, 1 inch wide by ½ inch thick could be securely held between the two blocks which were clamped together by fourteen bolts.

A complete set of beams with pre-twists of degrees 0, 5, 10, 20, 25, 30, 45 and 0 degrees approximately were made up by twisting each length of bar by hand with one end held in a vice before it was cut exactly to length. In this way the portion at the free end which, due to being held in the clamps used for twisting, was untwisted, could be cut off and thus leave a cantilever beam with an approximately uniform twist over its whole length. Any one of these eight beams could be placed in the clamping block which was tightened by using a torque spanner on each of the fourteen bolts in a set order.
3.3. **Clamped-Pinned Beams**

The other type of end condition likely to be encountered in actual steam turbines is clamped at one end and pinned at the other. To investigate the effect of pre-twist on a beam with these end conditions the same eight beams described in section (3.2.2.) were made with a $\frac{1}{2}$ inch ball bearing soldered to the centre of the end section. This ball bearing was held in a hemispherical seating attached to a pedestal which could be slid into position and bolted to the same base plates as the large clamping block at the fixed end. *(Figure IV).* As in the case of the cantilever beams with the same breadth to depth ratio of 8/1 any beam of the set could be frequency checked. The actual method of obtaining the natural frequencies is described in full in section (3.6.).

3.4. **Method of Excitation**

There were three possible methods of exciting the vibration of the beams and to determine the best method for these experiments each method was tried in turn. The three methods which were considered were:

1) Moving coil excitation or electro-mechanical excitation.
2) Electro-magnetic excitation.
3) Crystal excitation.

3.4.1. **Moving Coil Excitation**

This type of exciter gives the best results when actually attached to the beam and this means that a small mass /
mass is added to the beam. In the case of a light weight beam this could have a considerable effect on the natural frequencies and could thus be a great disadvantage in an investigation of this type.

A characteristic of this type of vibration generator is the increase in the impedance of the moving coil as the frequency is increased. The frequency range experienced in the investigations on the cantilever beam with a breadth to depth of 3/1 was from 150 cycles per second up to 35 kilocycles per second. The upper limit of this range is rather high for this type of exciter and also for the type of amplifier being used to drive it but, by monitoring the current and the waveform it was possible to obtain the frequency of these higher resonances accurately and quite easily.

3.4.2. Electro-Magnetic Excitation

As all the beams were made from mild steel it was possible to induce vibration by means of an electro-magnet. The electro-magnet used was a relatively small one and was at the time the only one available, so the results obtained and the conclusions drawn from them cannot be claimed to be true for all electro-magnetic exciters. The method used was to pass an alternating current through the windings of the magnet which was placed close to the beam. Unfortunately in this method the frequency of vibration induced in the beam was/
The first two difficulties were the deciding factors in the choice of the type of exciter as the cost of each crystal made experimenting with different positions and different methods of attaching the crystals out of the question.

Finally, the amplitudes of vibration obtained using this method of excitation were very small and thus identifying each mode shape was at times rather difficult and always required very careful comparison of the relative amplitudes at many different points on the beam.

3.5. Method of Picking Up Vibration

In these frequency measurements only true resonances of the beam were required and in order to satisfy such requirements the power of the impact to the exciter had to be kept at a minimum to avoid exciting harmonics of the frequency being examined. The net result of this was that the amplitudes obtained at resonances were small which made it difficult to detect any vibration and to identify the mode.

At first a Phillips moving coil pick up was used both for detection and identification of resonances but it was found that as the frequency increased the signal from the pick up became very small and it was very difficult to identify the mode. When identifying the nodal pattern of a particular mode of vibration it was necessary to move the pick up over the surfaces of the beam which was not always convenient in the case of a small beam as the pick up assembly /
assembly was quite large. To ensure that the size of pick-up
although the moving parts in this case were light, did not
affect in any way the natural frequencies of the beam, it
was decided to design a small light weight pick-up. Two
possible ways of overcoming the problem of the weight of
the pick-up were considered, 1) to make use of crystals and
2) to employ a capacitance pick-up.

3.5.1. Crystal Pick-up

The first design of the crystal pick-up was in the
form of a barium titanate gauge attached to a perspex former
in such a way that it was suspended as a fixed-fixed beam.
A short probe was then attached to the centre of the crystal
and it was found that when the probe was laid against the
beam the crystal produced a signal which could easily be
displayed on an oscilloscope. This pick-up was very
sensitive indeed and could pick up a large number of
resonances over a wide range of frequencies. As it was
very small and light it could be moved easily and did not
affect the resonances to any great extent. There was,
however, the disadvantage that the crystal was extremely
fragile and had to be handled very carefully indeed.

In an attempt to improve the strength of this pick-up,
a slight modification was made to the design. Instead of
using the crystal as the beam a strip of spring steel was
used and the crystal attached to it. This slight change
did /
did not reduce the sensitivity too much but it did help to prolong the life of a crystal which was a major consideration in view of the cost of one crystal.

3.5.2. Capacitance Pick-up

The advantage of the capacitance type of pick-up was that no extra weight needed to be added to the beam since the beam acted as part of the condenser and the movement of the beam caused a change in the capacitance thus producing a signal which could be displayed on an oscilloscope or actually measured on a galvanometer. Initially this pick-up required very careful setting up but once properly adjusted it was very sensitive. Unfortunately it was difficult to use this pick-up to determine nodal patterns and also the output tended to decrease when higher modes of vibration were being excited. It was for these reasons that it was decided to use the crystal pick-up in preference to the capacitance pick-up.

3.6. Frequency Measurements

The main object of all the experiments was to obtain the natural frequencies of the beams, both untwisted and twisted, as accurately as possible and therefore some convenient means of measuring the frequency of each resonance had to be found. The most common method is to compare the frequency of the pick-up signal with that from an accurate decade oscillator, Muirhead Decade Oscillator, type 650 B, using Lissajous' figures.
3.7. **Accuracy of Experiments**

Since the effect of twist on some of the natural frequencies of both the cantilever beam and the fixed-pinned beam was slight, it was decided to examine the effects of certain variables. These variables were associated with apparatus and the laboratory and therefore could not be completely controlled. The five main factors which caused concern were:

1) the tightness of the bolts
2) the ambient temperature
3) the position of the pick-up
4) the position of the exciter
5) the accuracy of the decade oscillator

3.7.1. **Tightness of the Bolts**

A certain amount of control could be exercised in this instance by using a torque spanner to tighten the clamping bolts. To evaluate the magnitude of any error that might arise from variations in tightness, extreme cases were considered, e.g. all bolts slack and then all tight and in both cases a value for a particular natural frequency was obtained. Intermediate cases of one or two bolts tight and the others slack were also considered, and the number of tight bolts was increased each time until they were all tight. It was considered advisable to have a definite order for tightening the bolts to make sure that the clamping plates were pulled together correctly.
3.7.2. **Ambient Temperature**

It was difficult to control the ambient temperature so it was decided to take temperature readings and obtain the natural frequencies corresponding to these temperatures. These measurements had to be made over a considerable period of time, since it could sometimes take a few days for the temperature to change sufficiently. Although the temperature of the beam could have been controlled it would have required rather elaborate apparatus which was not considered to be justified.

3.7.3. **Position of the Pick-up**

The position of even the lightweight pick-up was found to influence the natural frequencies of the beam to a certain extent and to find out more exactly the magnitude of this effect a short investigation was carried out. Since the effect of the position of the pick-up could be different for each mode of vibration each mode of vibration was investigated. In each case the pick-up was placed at a set number of positions along the beam in turn and the frequency determined accurately for each position. In this way the positions which had the greatest and the least effects were found and the variation in frequency was found to be in the region of half a per cent.

3.7.4. **Position of Exciter**

By employing the same technique as described in the previous /
previous section on the position of the pick-up, the effect of the position of the exciter was determined and the best position for each mode obtained. It was found that by placing the exciter very close to a node, although not actually on it, the effect of the mass of the moving coil was kept at a minimum. This method was employed in all the frequency measurements made subsequent to these investigations.

3.7.5. Accuracy of the Decade Oscillator

The question of the accuracy of the Decade Oscillator was particularly important when the cantilever beam with a breadth to depth ratio of 3 to 1 was being used. It was found that up to a frequency of 10 kilocycles per second the oscillator was within the limits specified by the manufacturers. For frequencies higher than 10 kilocycles per second it was found necessary to use the frequency trimmer control every 1,000 cycles. To find the correct position for this control the output frequency of the oscillator was compared with the signal from a Standard 1,000 cycles per second valve maintained tuning fork manufactured by Muirhead correct to 10 parts per million. It was possible using this tuning fork to check the accuracy of the Decade Oscillator at any multiple or sub-multiple of 1,000 cycles.

3.8. Method of Twisting

It was decided that it would be unpractical to machine /
machine twisted beams and so the pre-twist was produced by twisting the beams in a torsion machine. The effect of this cold twisting on the natural frequencies was neglected. Each beam was mounted in a torsion machine which had been adapted for the purpose, and the twist produced by loading it beyond its yield point, then twisting the required amount to produce a permanent twist. Experience showed just how much to allow for the elasticity of the material but it was still very difficult to obtain exactly the value of pre-twist required. It was therefore necessary to measure each specimen after twisting to determine the true angle of twist. No heat treatment was employed to relieve the stresses produced during the process of twisting.

3.9. Experiment on Square Bar

In connection with the theoretical investigations it was decided to use a square bar to determine if the fibre inclination, which causes a slight reduction in stiffness, was of any importance in changing the natural frequencies of the beam. Since a square section bar has the same stiffness in both planes there could be no coupling effect due to twist between the two stiffnesses.

A short length of square section bar was suspended on rubber at each end to produce the conditions of a free beam and the natural frequencies obtained in the same manner.
manner as described below. This bar was then twisted a known amount and the natural frequencies obtained once more. This procedure was repeated, the angle of twist being increased a little each time until an angle of approximately $270^\circ$ twist was obtained.

3.10. **Experimental Procedure**

To obtain the natural frequencies of vibration of any of the beams mentioned in the preceding sections it was necessary to mount the clamping blocks on the lathe bed which was used as a base, and position the exciter underneath the beam. An Advance oscillator covering the frequency range 5 cycles/second to 50 kilocycles/second was used to drive the exciter through an amplifier, as shown in Figure V. The response of the beam was detected by means of the lightweight pick-up already described and displayed on a cathode ray oscilloscope, which was used in conjunction with a Decade oscillator to obtain a Lissajous' figure and thus the frequency of the response. A second cathode ray oscilloscope was used to monitor the input to the exciter to ensure that the input was reasonably sinusoidal. The normal procedure was to place the pick-up at a particular point on the beam and gradually increase the frequency of the signal from the Advance signal generator until the pick-up detected a resonance.
A resonance causes the signal from the pick-up to increase very rapidly from a relatively small amplitude, if not zero, to a much larger amplitude and then reduce again for a very small change in frequency. The frequency was adjusted at each resonance until the pick-up signal was a maximum and then a quick check made on the mode shape to ascertain the best positions for both the pick-up and exciter. Once the positions of the pick-up and exciter had been obtained, the frequency was adjusted to give the maximum amplitude and the frequency again measured. The next stage was to examine carefully the relative amplitudes and directions of motion of points along the beam to ascertain the number of nodes and their position for each resonant frequency.

The procedure was the same for all three types of beam described in earlier sections of this chapter and for all values of pre-twist. As the angle of pre-twist increased it became increasingly difficult to identify the mode shape properly since, as was to be expected, the pre-twist influenced the shape as well as the frequency. Another difficulty encountered was that as the pre-twist increased the number of resonant frequencies appeared to increase and it could take a long time to examine each resonance to decide which one was in fact the true resonance of the beam. In many instances there were more than one true resonance but some /
some could, from their shape, be said to be a combination of a flexural and an edgewise vibration.

The lightweight pick-up described earlier was most useful when these different modes of vibration were being classified since none of the conventional pick-ups were small enough to be manoeuvred into all the necessary positions. This was particularly noticeable in the case of the clamped-pinned beam when there were supports at both ends of the beam. The sensitivity of the pick-up was improved still further by using a small transistor amplifier.
Results of the experimental and theoretical investigations

Experimental Results

4.1.1. Cantilever Beam with Breadth/Depth Ratio of 3/1

This was the first beam to be used in the investigation, and the frequencies of the flexural, torsional and edgewise modes of vibration were found to be as follows:

(1) Flexural modes

<table>
<thead>
<tr>
<th>Angle of Twist</th>
<th>No. of Nodes</th>
<th>0°</th>
<th>7°6'</th>
<th>16°35'</th>
<th>24°10'</th>
<th>29°25'</th>
<th>32°45'</th>
<th>34°55'</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>157</td>
<td>157</td>
<td>157</td>
<td>157</td>
<td>158</td>
<td>159.4</td>
<td>158</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>992</td>
<td>952</td>
<td>961</td>
<td>976</td>
<td>982</td>
<td>1,017</td>
<td>977</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2,818</td>
<td>2,740</td>
<td>2,546</td>
<td>2,970</td>
<td>2,943</td>
<td>2,942</td>
<td>2,856</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>5,221</td>
<td>5,160</td>
<td>5,102</td>
<td>5,032</td>
<td>5,071</td>
<td>5,079</td>
<td>5,030</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>8,720</td>
<td>8,500</td>
<td>8,444</td>
<td>8,520</td>
<td>8,500</td>
<td>8,461</td>
<td>8,390</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>12,290</td>
<td>11,920</td>
<td>11,800</td>
<td>11,860</td>
<td>11,800</td>
<td>11,680</td>
<td>11,700</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>16,100</td>
<td>15,740</td>
<td>16,090</td>
<td>16,040</td>
<td>15,880</td>
<td>16,100</td>
<td>15,700</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>20,190</td>
<td>19,050</td>
<td>18,550</td>
<td>18,990</td>
<td>-</td>
<td>19,130</td>
<td>19,230</td>
</tr>
</tbody>
</table>
### (2) Torsional Modes

<table>
<thead>
<tr>
<th>Angle of Twist</th>
<th>Frequencies</th>
<th>Cycles/second</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0°</td>
<td>7°6'</td>
</tr>
<tr>
<td>No. of Nodes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1,811</td>
<td>1,834</td>
</tr>
<tr>
<td>1</td>
<td>5,460</td>
<td>5,458</td>
</tr>
<tr>
<td>2</td>
<td>9,153</td>
<td>9,154</td>
</tr>
<tr>
<td>3</td>
<td>13,100</td>
<td>13,090</td>
</tr>
<tr>
<td>4</td>
<td>17,160</td>
<td>16,940</td>
</tr>
<tr>
<td>5</td>
<td>21,220</td>
<td>21,330</td>
</tr>
<tr>
<td>6</td>
<td>25,640</td>
<td>25,650</td>
</tr>
<tr>
<td>7</td>
<td>-</td>
<td>30,125</td>
</tr>
</tbody>
</table>

### (3) Edgewise Modes

<table>
<thead>
<tr>
<th>Angle of Twist</th>
<th>Frequencies</th>
<th>Cycles/second</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0°</td>
<td>7°6'</td>
</tr>
<tr>
<td>No. of Nodes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>449</td>
<td>446</td>
</tr>
<tr>
<td>1</td>
<td>2,630</td>
<td>2,792</td>
</tr>
<tr>
<td>2</td>
<td>6,138</td>
<td>6,564</td>
</tr>
<tr>
<td>3</td>
<td>10,836</td>
<td>11,030</td>
</tr>
<tr>
<td>4</td>
<td>16,790</td>
<td>17,000</td>
</tr>
<tr>
<td>5</td>
<td>22,820</td>
<td>21,600</td>
</tr>
<tr>
<td>6</td>
<td>29,690</td>
<td>28,820</td>
</tr>
<tr>
<td>7</td>
<td>34,590</td>
<td>34,640</td>
</tr>
</tbody>
</table>
These results were felt to be rather unsatisfactory due to the difficulties encountered in trying to identify the mode shapes. Also, since the one beam was used all the time, the angle of twist being increased a little each time, it was impossible to refer back to a previous case if there was any doubt about a particular natural frequency.

4.1.2. Cantilever Beam with breadth/depth ratio of 8/1

This time a set of beams was used, each beam having a different angle of pre-twist. The frequencies of some of the flexural modes of vibration of these cantilevers were found to be as follows:

<table>
<thead>
<tr>
<th>Angle of Twist</th>
<th>0°</th>
<th>5° 10'</th>
<th>12° 25'</th>
<th>21° 15'</th>
<th>25° 45'</th>
<th>30° 5'</th>
<th>45° 50'</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Nodes</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>39.8</td>
<td>39.7</td>
<td>39.8</td>
<td>39.8</td>
<td>39.9</td>
<td>39.9</td>
<td>40.3</td>
</tr>
<tr>
<td>1</td>
<td>248.7</td>
<td>247.2</td>
<td>236</td>
<td>222.3</td>
<td>214.3</td>
<td>204.5</td>
<td>181.9</td>
</tr>
<tr>
<td>2</td>
<td>695.6</td>
<td>696.2</td>
<td>691.3</td>
<td>684</td>
<td>678.7</td>
<td>674.6</td>
<td>646</td>
</tr>
<tr>
<td>3</td>
<td>1361.4</td>
<td>1359.2</td>
<td>1356.5</td>
<td>1345.8</td>
<td>1357.1</td>
<td>1348.2</td>
<td>1324.8</td>
</tr>
<tr>
<td>4</td>
<td>2270.3</td>
<td>2237.3</td>
<td>2236.8</td>
<td>2227.7</td>
<td>2247.5</td>
<td>2263.2</td>
<td>2233</td>
</tr>
</tbody>
</table>

Only the frequencies of the flexural modes of vibration of these beams were measured since it was felt that it would be easier to examine one type of vibration at a time, after the difficulties experienced when using the other cantilever beam. The same beams, however, could be used...
used at a later date to investigate the effect of pre-twist on the frequencies of the torsional and edgewise modes of vibration.

### 4.1.3. Clamped-Pinned Beam

Only one type of beam was used with these end conditions and that was one with a breadth to depth ratio of 8 to 1.

The same set of beams was used as in the cantilever case but the free end was supported by a ball and socket. It was found that this support did provide the end condition of a pin joint. The ball and socket was used instead of a hinge to avoid the difficulties which would occur when beams with different angles of pre-twist were used.

<table>
<thead>
<tr>
<th>No. of Nodes</th>
<th>Angle of Twist</th>
<th>Frequency cycles/second</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0°</td>
<td>5°10' 12°25' 21°15' 25°45' 28°5° 45°50'</td>
</tr>
<tr>
<td>0</td>
<td>175.8</td>
<td>179.6 183.3 203.5 213.8 236.3 300</td>
</tr>
<tr>
<td>1</td>
<td>566.7</td>
<td>562.2 551.3 553.7 538 530.7 528</td>
</tr>
<tr>
<td>2</td>
<td>1179.4</td>
<td>1162 1158.2 1133.7 1153.7 1172.1 1155</td>
</tr>
<tr>
<td>3</td>
<td>1930.6</td>
<td>1983.7 1974 1943.6 1992.7 2016.9 2011.2</td>
</tr>
<tr>
<td>4</td>
<td>3068.6</td>
<td>2918.3 3037.5 2946.6 3023.7 3039.3 3048</td>
</tr>
</tbody>
</table>
4.2. Calculated Natural Frequencies

For the untwisted beams whether the cantilever beam or the clamped-pinned beam, the natural frequencies of the flexural or edgewise modes are readily obtainable using the formula already given in the previous Chapter. The pre-twisted beams, however, are not so easy to cope with when calculating the natural frequencies of these same modes of vibration since a certain amount of coupling between the two modes takes place.

4.2.1. Natural Frequencies of Cantilever Beams

The formula in this case is

\[ f_n = \left( \frac{ksn_e}{2\pi L} \right)^2 \sqrt{\frac{EI}{\rho A}} \]

where the values of \((knI)\) are given in Chapter 2, section 2.2.2.1.

For the cantilever beam with breadth/depth ratio 3/1 the natural frequencies are:

<table>
<thead>
<tr>
<th>No. of Nodes</th>
<th>Flexure Frequency (\text{c/s})</th>
<th>Edgewise Frequency (\text{c/s})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>161</td>
<td>483</td>
</tr>
<tr>
<td>1</td>
<td>1007</td>
<td>3021</td>
</tr>
<tr>
<td>2</td>
<td>2225</td>
<td>8475</td>
</tr>
<tr>
<td>3</td>
<td>5550</td>
<td>16650</td>
</tr>
<tr>
<td>4</td>
<td>9160</td>
<td>27580</td>
</tr>
<tr>
<td>5</td>
<td>13620</td>
<td>40860</td>
</tr>
<tr>
<td>6</td>
<td>19010</td>
<td>57030</td>
</tr>
<tr>
<td>7</td>
<td>25300</td>
<td>75900</td>
</tr>
</tbody>
</table>

The Cantilever Beam with breadth to depth ratio of 3 to 1
The Cantilever Beam with breadth to depth ratio of 8 to 1

The formula for the natural frequencies is the same as that used above; the only difference is in the value of the second moment of area.

<table>
<thead>
<tr>
<th>No. of Nodes</th>
<th>Frequency c/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>40.3</td>
</tr>
<tr>
<td>1</td>
<td>252.5</td>
</tr>
<tr>
<td>2</td>
<td>707.1</td>
</tr>
<tr>
<td>3</td>
<td>1385.1</td>
</tr>
<tr>
<td>4</td>
<td>2290.1</td>
</tr>
</tbody>
</table>

These results are all plotted on a graph and are compared with the experimental results.

4.2.2. Natural Frequencies of the Clamped-Finned Beams

The same formula applies in this case also but the values of \((k \cdot L)\) are different owing to the different boundary conditions.

<table>
<thead>
<tr>
<th>No. of Nodes</th>
<th>Frequency c/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>176.7</td>
</tr>
<tr>
<td>1</td>
<td>572.7</td>
</tr>
<tr>
<td>2</td>
<td>1194</td>
</tr>
<tr>
<td>3</td>
<td>2043</td>
</tr>
<tr>
<td>4</td>
<td>3117</td>
</tr>
</tbody>
</table>

Once again these values for the natural frequencies are plotted on the same graph as the experimental results for comparison purposes.
4.2.3. Calculation of the Natural Frequencies of Pre-twisted Beams

Although it had been intended to calculate the natural frequencies of the pre-twisted beams using the formula suggested by Carnegie (1959) for cantilever beams and the formula which would have been obtained from section 2.3.4.3. for clamped-pinned beams, it has only been possible to calculate the frequencies for the cantilever beams. The theoretical results used in the case of the clamped-pinned beam were obtained by interpolation from figures given by Anliger (1955). If it had been possible to obtain the solution of the differential equation of motion given in section 2.2.4. using a digital computer then there would have been more theoretical results available.

4.3.1. Experiment on Square Section Bar

The natural frequencies of a square section free-free beam for different angles of pre-twist obtained experimentally. Only the frequencies of flexural modes of vibration were considered in this case.

<table>
<thead>
<tr>
<th>Angle of Twist</th>
<th>0</th>
<th>31</th>
<th>60</th>
<th>78</th>
<th>125</th>
<th>270</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode of Vibration</td>
<td>1st Flexural</td>
<td>706</td>
<td>706</td>
<td>706</td>
<td>698</td>
<td>702</td>
</tr>
<tr>
<td></td>
<td>2nd Flexural</td>
<td>1,194</td>
<td>1,917</td>
<td>1,915</td>
<td>1,912</td>
<td>1,913</td>
</tr>
<tr>
<td></td>
<td>3rd Flexural</td>
<td>3,714</td>
<td>3,708</td>
<td>3,705</td>
<td>3,694</td>
<td>3,703</td>
</tr>
<tr>
<td></td>
<td>4th Flexural</td>
<td>5,910</td>
<td>5,894</td>
<td>5,887</td>
<td>5,885</td>
<td>5,894</td>
</tr>
<tr>
<td></td>
<td>5th Flexural</td>
<td>8,894</td>
<td>8,884</td>
<td>8,867</td>
<td>8,855</td>
<td>8,843</td>
</tr>
<tr>
<td></td>
<td>6th Flexural</td>
<td>12,190</td>
<td>12,170</td>
<td>12,150</td>
<td>12,140</td>
<td>12,170</td>
</tr>
</tbody>
</table>
4.3.2. Calculation of Natural Frequencies of Square Bar

The only frequencies calculated in this case were for the untwisted case of the free free beam and they were found to be:

<table>
<thead>
<tr>
<th>Mode of Vibration</th>
<th>Frequency c/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Flexure</td>
<td>726</td>
</tr>
<tr>
<td>2nd Flexure</td>
<td>2,006</td>
</tr>
<tr>
<td>3rd Flexure</td>
<td>3,910</td>
</tr>
<tr>
<td>4th Flexure</td>
<td>6,500</td>
</tr>
<tr>
<td>5th Flexure</td>
<td>9,670</td>
</tr>
<tr>
<td>6th Flexure</td>
<td>13,580</td>
</tr>
</tbody>
</table>

The formula used for this calculation was

\[ n \frac{\pi}{2} \]

where \( n \) could have values 2, 3, 4, .... \( n \).

The values of \( knl \) were taken from tables of beam functions and the values were 4.730, 7.853, 10.996, 14.137, 17.279, \( n \frac{\pi}{2} \).

4.4. Determination of Magnitude of Errors

The results of the investigations to determine the magnitude of the factors which were thought to contribute to the experimental errors were as follows:

1. Tightness of bolts \(+0.5\%\)
2. Ambient temperature \(+0.3\%\)
3. Position of Pick-up \(+0.25\%\)
4. Position of Exciter \(+0.5\%\)
5. Decade Oscillator \(+0.05\%\)
Thus the experimental results can only be said to be accurate within a range of ± 1.6%, and this will be shown on all graphs using the results given in this Chapter.
CHAPTER 5

5.1. INTRODUCTION

5.1.1. In the preceding Chapters methods of calculating the natural frequencies of beams of uniform and non-uniform cross sections with various end conditions have been described in some detail. Another Chapter described the experiments carried out to determine the natural frequencies of uniform cantilever and clamped-pinned beams with various angles of pre-twist. In this the last Chapter an attempt is made to compare, where possible, the results of the theoretical and experimental investigations and also to comment on the findings of these investigations.

Several works on the determination of the natural frequencies of pre-twisted beams are discussed and a comparison of the results made wherever possible. There are many works on the vibration of turbine blades but only those that have some connection with the experimental work carried out are mentioned in this Chapter.

5.2. Cantilever Beams

5.2.1. Comparison of the experimental and theoretical values of the natural frequencies of the untwisted cantilever beams for corresponding number of nodes show that there is good agreement between the results of the two methods. At high frequencies the theoretical results are /
are higher than the experimental results. This difference is due to the effects of rotatory inertia and shearing force (Timoshenko, 1955). In cases where the cross sectional dimensions are small compared with the length, the simple equation used in the earlier Chapter is quite adequate but the effect of the cross section can be considerable at the higher frequencies when a vibrating beam is subdivided by nodal cross sections into short lengths.

5.2.2. For the edgewise modes of vibration the differences between the theoretical and experimental values for the natural frequencies are greater than those for the flexural modes of vibration. While this could be due to the effects of rotatory inertia and shearing force, it is also possible that errors were made in identifying the edgewise modes of vibration since the surface area normal to the direction of motion was smaller and the point of excitation was not changed for each mode of vibration at this stage in the experiments.

5.2.3. The natural frequencies obtained for the flexural, torsional and edgewise modes of vibration for the cantilever with angles of pre-twist up to a maximum of 35 degrees and a breadth to depth ratio of 3 to 1 do show certain trends which will be discussed again in connection with /
with a theory put forward by Rosard (1953). The natural frequencies of the flexural modes of vibration all decreased initially with increase in the angle of pre-twist except for the fundamental frequency which remained unchanged until a pre-twist of 30 degrees was reached. The natural frequencies of the torsional modes of vibration showed very little variation up to the maximum angle of pre-twist considered. The variations in the natural frequencies of the edgewise modes of vibration are more complex and some of them would appear to have been confused with the flexural and torsional modes of vibration although it is possible that these were coupled modes.

5.2.5. It might be expected that the natural frequencies of the cantilever beam with a breadth to depth ratio of 8 to 1, would change more rapidly than the cantilever beam with a breadth to depth ratio of 3 to 1, since the difference between the stiffness in the flexural and edgewise directions is greater and therefore the change in stiffness for the same angle of pre-twist would be greater. The experimental results quoted in the preceding Chapter do not, however, support this theory. In general the changes were smaller for the beam with the breadth to depth ratio of 8 to 1 than for the beam with the ratio of 3 to 1. The difficulty with very small changes /
changes in frequency is that they are too small to be accepted as genuine changes in frequency and can be included in the experimental error.

5.2.5. Let us consider the stress distribution in a cantilever beam when vibrating in the fundamental mode. The maximum stress occurs at or near to the root of the blade. Now owing to the method used to produce the pre-twist, (see section 3.8.) the root is a point where the pre-twisting has had the least effect. At each end of the beam approximately one-twentieth of its total length remained untwisted, and therefore there has been no change in the stiffness. In the higher modes of vibration the position of maximum stress will alter and therefore the effect of pre-twist will vary accordingly.

5.3. Clamped-Pinned Beam

5.3.1. If the argument put forward in the previous section is applied in the case of a clamped-pinned beam, then the position of maximum stress in the fundamental mode of vibration and maximum change of angle of pre-twist will be very close together, if not actually coincident. In this case then the change in the fundamental natural frequency with pre-twist might be expected to be more rapid than in the previous cases.

5.3.2. The results of the experiments carried out on pre-twisted clamped-pinned beams show that when an angle of /
of pre-twist of 45 degrees was reached the fundamental natural frequency had increased by 71%; the second natural frequency had decreased by 72%; and the next three flexural modes had varied by 50%. In the case of the higher modes of vibration this maximum variation was reached before the angle of pre-twist reached 45 degrees.

5.4. **Square Section Bar**

5.4.1. The experiments on the square section bar were carried out to a maximum angle of pre-twist of 270 degrees. Over this range of pre-twist the variation in the frequencies of the flexural modes of vibration were so small that it could be concluded that these frequencies did not change, the variations being due to experimental error. The calculated values for the natural frequencies do not agree so closely as those obtained for the cantilever and clamped-pinned. There are two possible reasons for this difference. First the bar was only roughly finished to the dimension of half an inch square, and second, the mass of the vibrator was attached to the centre of the beam and would reduce the frequency of all modes which had an antinode at this point.

5.4.2. The purpose of this investigation was to check that the fibre inclination, which causes a slight reduction in stiffness, could be ignored in a theoretical analysis of the problem. From the results obtained it would appear that this assumption is reasonable.
5.5. Identification of Modes of Vibration

5.5.1. The subject of the confusion which might occur in identifying, or classifying, the modes of vibration of the cantilever and clamped-pinned beams, is one worthy of careful consideration. Normally it is a relatively simple task to identify the mode of vibration of uniform beams by obtaining the number of stationary points or nodes and the direction of nodal lines on all the faces of the beam. This is particularly true for the lower natural frequencies when the nodes are well spaced and the intervals between the natural frequencies are larger, thus the flexural, edgewise and torsional modes can exist independent of one another. At the higher frequencies, however, the natural frequencies appear to be closer together and the nodes more numerous, thus nearer to each other. This is particularly true of the types of beams used in this investigation which were relatively small in both width and thickness. It sometimes happens that say a torsional mode and an edgewise mode of vibration occur at frequencies which are very close, e.g. the fifth and sixth torsional and edgewise frequencies of the cantilever beam with a breadth to depth ratio of 3 to 1 (see Graph 1), when the beam is given a pre-twist all the frequencies change by different amounts, some increasing and then decreasing, and thus natural frequencies which were originally separate may now coincide or appear to do so. In cases where torsional and edgewise frequencies appear
to coincide there are sometimes as many as four resonances. For instance, when a resonance search is being carried out the first of a group of resonances might turn out to be a torsional mode; the second a coupled torsional and edgewise mode with torsion the more predominant; the third another coupled mode of torsion and edgewise vibration, but this time the edgewise motion the more predominant; and the fourth resonance a pure edgewise resonance. It might happen that these modes which occur so close together all appear to be coupled modes, that is, that no pure torsional or edgewise modes appeared to exist, and it is in cases like this that confusion arises. A solution to this problem is suggested by Geiger (1950), and it is to classify the resonances of the pre-twisted beam by the direction of motion of the end or tip section.

5.6. Discussion of Other Authors' Work

5.6.1. In the preceding pages various authors have been referred to in connection with either the behaviour of pre-twisted beams or the estimation of the natural frequencies of this type of beam. While it would take too long to summarise all of these works at this stage, there are one or two points which have particular application in this Chapter.

5.6.2. /
5.6.2. Geiger was one of the first to carry out investigations into the effect of pre-twist on the natural frequencies of turbine blades and the direction of vibration. These investigations were carried out on cantilever beams with breadth to depth ratios of 2 to 1, 4 to 1, 6 to 1, 10 to 1 and 20 to 1, with angles of pre-twist up to 45 degrees which is the most that might be encountered in practice. The fundamental frequencies of these beams were found to alter very little with pre-twist; the beam with the breadth to depth ratio of 2 to 1 showed a slight increase while the other beams all showed a slight decrease up to 3%. The direction of motion of the tip cross section was found to be a function of the angle of pre-twist. The first overtone frequencies for the beams with breadth to depth ratio of 2 to 1 and 4 to 1 showed an increase in frequency with pre-twist up to an angle of 45 degrees. The beams with the larger ratios of breadth to depth or flatter profiles were found to have the appearance of the superposition of two vibrations, one with a node further from the free end than the corresponding untwisted case and one without a node, the frequencies being lower than the normal first overtone frequency.
Rosard (1953) carried out a theoretical and experimental investigation into the effect of pre-twist on the natural frequencies of beams. The theoretical analysis of the problem was carried out using a method based on Myklestad's Method of calculating the natural frequencies of beams. From this analysis it was possible to determine the motion of a number of sections along the beam.

The results of this investigation were given as the ratio of the natural frequencies to the fundamental frequency of the untwisted bar. This ratio was plotted against the breadth to depth ratio and is reproduced in Graphs 12 and 13. These Graphs give the ratios of the flexural $f_1$, $f_2$, $f_3$, etc. and edgewise $E_1$, $E_2$, $E_3$, etc. natural frequencies to the fundamental flexural frequency and shows the ratios for beams which had been pre-twisted to 40 degrees. The line $T_1$ corresponds to the first frequencies encountered when investigating the beams with 40 degrees of twist, the next line $T_2$ to the next frequencies, up to a breadth to depth ratio of 6.36 the resonances which have been effected were the first edgewise and after that it was the second flexural resonances, and lastly $T_3$ which shows how the second flexural resonances of beams with a breadth to depth ratio /
ratio less than 6.36 to 1 and the first edgewise resonances of beams with a breadth to depth ratio greater than 6.36 to 1 behave. Using this kind of diagram in conjunction with theoretical and experimental results it would be possible to produce a series of curves $T_1, T_2, T_3, \ldots$ for beams of different breadth to depth ratios and angles of pre-twist. This method could also be applied to the clamped-pinned beam (Graph 13) thus giving a useful method of obtaining the effect of pre-twist on beams with any ratio of breadth to depth. It would only be necessary to know the angle of pre-twist, the fundamental frequency and the breadth to depth ratio to obtain the frequency factors for all the flexural and edgewise modes of vibration.

5.6.4. The differential equation of motion was derived in Chapter 2, section 2.2.4., and the method of solution employed by Anliker (1955) mentioned. The results which this solution yielded are used to plot the Graphs 9, 10 and 11. Anliker did not consider the case of the beam with a breadth to depth ratio of 8 to 1, but approximate results were obtained for this case by interpolation of the results given. The results were for beams clamped at one end and simply supported or pinned at the other, but a similar method could be used to obtain results for cantilever beams as only the boundary conditions would have /
have to be altered. In order to solve these equations satisfactorily a digital computer is required since the calculations involved are lengthy.

Although the accuracy of the curves obtained from Anliker's results by interpolation cannot be regarded as very good, they do show agreement with the experimental points. It would appear that this method of estimating the natural frequencies of pre-twisted cantilever and clamped-pinned beams would give results near to what happens in practice. In this age of digital computers this method of calculation could be used to produce the necessary information to draw up the graphs of frequency factor against breadth to depth ratio mentioned in section 5.6.3., and illustrated in Graphs 12 and 13.

5.6.5. In more recent years an approximate solution based on the Rayleigh Energy Method has been suggested by Carnegie (1959) giving an expression for the fundamental frequency of a cantilever beam with pre-twist. This method has one limitation which makes it less useful than Anliker's method, and that is that it can only be applied to the fundamental frequency which for most cantilever beams shows little change with pre-twist. The accuracy of this method is not as good as that to be expected from the differential equation. Di Prima and Handelman (1954-5) gave /
gave an approximate solution for the cantilever beam which is compared with the results of Carnegie's Method and the experimental results in Graph 5.

In Chapter 2, section 2.3.4., 3b., the author describes a possible approach for obtaining an expression for the frequency of a clamped-pinned beam with pre-twist. Unfortunately attempts to obtain suitable expression for the static displacement of such a beam have so far been unsuccessful. The final equation would be very similar to that obtained by Carnegie for the cantilever but the three quantities involving functions of the angle of pre-twist would be different. This method would only yield the fundamental frequency of the beam which, although often the most important, is not always the only frequency which can be excited in sufficient magnitude to cause trouble.

5.7. Comparison of Experimental and Theoretical Results

5.7.1. The results of the theoretical values for the fundamental frequency of the cantilever beam using the methods suggested by Di Prima and Handelman and Carnegie are obtained in this investigation. The approximations given by Di Prima and Handelman increase more rapidly than the experimental values. At 28.6 degrees their method has increased by 1.08% where the experimental values have only increased by 1.001%. Carnegie's method on /
on the other hand tends to under-estimate the changes in the frequency. At 40 degrees of twist Carnegie's method shows an increase of 1.002% while the experimental values show an increase of 1.01%.

The values obtained from Anliker's curves compare very well with the experimental ones as shown in Graph 5 and also in Graphs 9, 10 and 11. The experimental points are also shown on Graphs 12 and 13. The curves on Graph 12 are those obtained by Rosard, but since there are only two points from the experimental investigations, very little can be gained from comparison.

5.6. Final Comments

5.6.1. It may appear that these investigations into the effect of pre-twist on the natural frequencies of vibration are incomplete. Had it been possible to obtain the solution to the differential equation of motion for pre-twisted beams and also to obtain an expression for the fundamental frequency of a clamped-pinned beam using the Rayleigh Energy Method then there would have been more results to compare with experimental results. The frequencies of the cantilever and clamped-pinned beams with a breadth to depth ratio of 8 to 1 are also incomplete since the torsional and edgewise modes of vibration have not been investigated. This was caused by the decision to try to investigate the flexural vibration.
vibration alone but since with pre-twist both flexural and edgewise frequencies change it is not really practical to investigate the changes in one plane only.

5.8.2. If it is ever possible to continue this particular line of investigation the first step should be to develop a technique of identifying modes of vibration by the direction of motion of the end section in addition to obtaining the nodal patterns which may sometimes be complex. Secondly, to obtain the natural frequencies of beams of different breadth to depth ratios with angle of pre-twist theoretical investigation should be carried out to obtain the solution of the differential equation of motion of the pre-twisted beam for the same cases as those investigated experimentally. All the results could then be presented in the form of graphs like the ones suggested by Rosard.
APPENDIX 'A'

Relationships between \( I_{yy}, I_{zz}, I_{yz}, I_{zx} \) and \( I_{xy} \)

The above figure shows the cross section of a beam with principal axes \( YY \) and \( ZZ \).

Considering the small element \( dA \) at point \( (a, b) \) with reference to \( YY \) and \( ZZ \) axes.

Then the moment of area

\[ I_{yy} = \int b^2 dA = \int \int b^2 dydz \]

where \( y \) and \( z \) are co-ordinates with reference to the principal axes.

Thus since \( b = y \sin \phi + z \cos \phi \) it follows that /
that
\[ I_{xy} = \int (y^2 \sin^2 \phi + z^2 \cos^2 \phi + 2yz \sin \phi \cos \phi) \, dy \, dz \]
\[ = I_{zz} \sin^2 \phi + I_{yy} \cos^2 \phi + I_{yz} \sin \phi \cos \phi \]
and as \( I_{yz} \) the product moment of area about the principal axes is zero
\[ I_{xy} = I_{zz} \sin^2 \phi + I_{yy} \cos^2 \phi \]

Similarly the moment of area
\[ I_{zz} = \int a^2 \, dy \, dz \]
where \( a = y \cos \phi + z \sin \phi \)

and the following expression can be deduced for \( I_{zz} \)
\[ i.e. I_{zz} = I_{zz} \cos^2 \phi + I_{ys} \sin^2 \phi \]
The product of moment of area \( I_{yz} \) about the reference axes \( YY \) and \( ZZ \) is given by
\[ I_{yz} = \int a \, b \, dA = \int a \, b \, dy \, dz \]
which on substituting for \( a \) and \( b \) becomes
\[ I_{yz} = \int (y \sin \phi + z \cos \phi)(y \cos \phi - z \sin \phi) \, dy \, dz \]
\[ = \int (y^2 \sin \phi \cos \phi - z^2 \sin \phi \cos \phi + yz \cos^2 \phi - yz \sin^2 \phi) \, dy \, dz \]
\[ = (I_{zz} - I_{yy}) \sin \phi \cos \phi \]

Thus
\[ I_{yz} = \frac{(I_{zz} - I_{yy})}{2} \sin 2\phi \]
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CANTILEVER BEAM

BRETTA TO DEPTH RATIO 3/1

Frequency in Cycles per Second

ANGLE OF PRE- TWIST DEGREES
CANTILEVER BEAM

GRAPH 3

Frequency in Cycles per Second

Angle of Pre-Twist - Degrees
Frequency of Clamped Pinned Beam

Frequency in Cycles per Second

Angle of Pre-Twist Degrees

1. Theoretical Values, taken from Analysis
2. Experimental Values
Square Section Beam

Frequency in Cycles per Second

No of Nodes

Theoretical Values
Experimental Values
First flexural frequency

Experiment values

Angle of pre-twist radians

Graph 9
SECOND FLEXURAL FREQUENCY

\[ \times \] EXPERIMENTAL VALUES

ANGLE OF PRE-TWIST RADIAN
FREQUENCY CONSTANTS

CLAMPED PINNED BEAMS

THIRD FLEXURAL FREQUENCY

X EXPERIMENTAL VALUES

ANGLE OF PRE-TWIST RADIANS

GRAPH II
FREQUENCY RATIOS: CLAMPED PINNED BEAMS

BREADTH TO DEPTH RATIO

GRAPH 13
CANTILEVER BEAM

Solid Base

Solid Base

Breadth to Depth Ratio 3/1
CHAPTELEVER BEAM

SOLID BASE

SOLID BASE

Breadth to depth ratio 3/1

Fig III