SURGERY ON SIMPLY CONNECTED
Poincaré Spaces

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To my mother
The following material is submitted as a thesis in support of an application for the degree of Doctor of Philosophy at the University of Edinburgh, having been submitted for no other degree. Except where acknowledgement is made, the work is original.
I am extremely grateful to Elmer Rees for his support and encouragement over the last few years. I'd especially like to express my thanks for the many interesting problems he provided early on which made research such a pleasure. My thanks are also due to Mike Eggar, whose door was always open, and to Andrew Ranicki for lending me his cat. I was very much encouraged and helped by conversations with Frank Quinn to whom I am in debt for giving me so much of his time and whose ideas are at the foundation of this thesis.

I would have given up long ago if it were not for friends who shared the successes and disappointments of my work. To Robin, Lisa, Polly, Mark, Mary Jo, George, Jennifer and many others – thank you.
A theory of surgery below the middle dimension on simply-connected Poincaré complexes is presented, generalising the classical theory of surgery on manifolds. However, unlike previous approaches to the problem, manifold-theoretic properties of Poincaré complexes are not referred to, but instead solely homotopy-theoretic properties are used.

It is shown that given a simply-connected normal space $X$ of formal dimension $n$ there is a normal space $W$ which is $([n-1]/2) - 1$-Poincaré and a $([n]/2) + 1$-connected normal map $f: W \to X$. This is achieved by considering the homotopy fibre of a certain map from the normal space into a sphere as an approximation to removing cells which disturb duality. A normal structure is defined on the fibre restoring duality in the relevant dimensions. An iteration of this argument removes the excess cells in the fibre.

A counter-example is given to an extension of cofibrations result in [13] which was previously used to remove the cells disturbing duality in [14]. It is shown that the conditions for extension given in [13] are satisfied if the map to be extended is weakly-coprincipal. The counter-example is a weakly-coprincipal map in which appropriate Massey higher products act as an obstruction to extension.
INTRODUCTION

The study of the relationship between manifolds and homotopy theory gave rise to the classical theory of surgery. The central problem in surgery theory may be stated as follows: Let $X$ be a CW–complex. Given a pair $(M, f)$ where $M$ is a manifold of dimension $n$ (PL, topological or smooth) and $f: M \to X$ is a map, can we manipulate $M$ (and $f$) in such a way so as to obtain a homotopy equivalence $f_1: M_1 \to X$, where $M_1$ is again a manifold. The basic construction is surgery on a class $\phi \in \pi_{i+1}(f)$. Suppose $\phi$ can be represented by an embedding $e: S^i \times D^{n-i} \to M$ (with a null–homotopy of $f \cdot e_{|S^i \times \{0\}}$) then we may form the space $W$ given by

$$W = M \times I \cup_{S^i \times D^{n-i}} D^{i+1} \times D^{n-i}$$

Now $W$ is a manifold with boundary $\partial W = M \sqcup M'$ and so extending $f \cdot \pi_1: M \times I \to X$ over the $(i + 1)$–handle $D^{i+1} \times D^{n-i}$ gives a map $f': M' \to X$. We say $(M', f')$ is obtained from $(M, f)$ by surgery on $\phi$.

We may think of a sequence of surgeries beginning with $(M, f)$ and ending with $(M', f')$ as defining a cobordism between $M$ and $M'$, that is a manifold $W$ such that $\partial W = M \sqcup M'$ together with a map $F: W \to X$ such that $F_{|M} = f, F_{|M'} = f'$. Conversely any cobordism gives a sequence of surgeries by considering a Morse decomposition of $W$.

If a homotopy equivalence $f_1: M_1 \to X$ exists then obviously $X$ has the
homotopy properties of a manifold. In particular $X$ satisfies Poincaré duality on some class $[X] \in H_\ast(X)$ (where $H_\ast$ is homology with the appropriate coefficients). According to Spivak [17] this is sufficient to define a $(k - 1)$-spherical fibration $\xi$ over $X$ together with an element $\alpha \in \pi_{n+k}T\xi$ such that the Thom isomorphism $\cap U: \tilde{H}_{n+k}(T\xi) \cong H_n(X)$ takes the image of $\alpha$ under the Hurewicz homomorphism to $[X]$. (The space $T\xi$ is the Thom complex of $\xi$.) Furthermore the existence of $f_i$ implies that the classifying map $j: X \to BO$ of $\xi$ lifts into $BO$ and so we may think of $\xi$ as a vector bundle over $X$.

We therefore restrict ourselves to cobordisms $F: W \to X$, which can be covered by a bundle map $B: \nu_W \to \xi$ where $\nu_W$ is the normal bundle of $W$ and $\xi$ a vector bundle over $X$. Such a cobordism is called a normal cobordism. If $X$ is a Poincaré space with Spivak bundle $\xi$ then given a normal map $(M, f)$ the obstructions to finding a normally cobordant homotopy equivalence are well known. See [18, p. 341 for example.

Normal cobordism defines an equivalence relation on normal maps, that is triples $(M, f, b)$, with $M$ and $f$ as before and $b: \nu_M \to \xi$ a bundle map covering $f$. The set of equivalence classes is a group, denoted $\Omega^X_n$, with addition defined by disjoint union. The Pontrjagin–Thom construction provides the essential link between normal cobordism and homotopy theory. Let $(M, f, b) \in \Omega^X_n$. Suppose the manifold $M$ embeds in $S^{n+k}$ with regular neighbourhood $N$. Then the collapse map $S^{n+k} \to N/\partial N$ gives a class $\eta \in \pi_{n+k}(T\nu_M)$. Given a normal map $(M, f, b)$ define its Thom invariant to be the class $T(b)\eta \in \pi_{n+k}(T\xi)$. Thus
we have the Pontrjagin–Thom isomorphism

\[ \Phi_X : \Omega_n^X \xrightarrow{\cong} \{S^{n+k}, T\xi\} \]

where \( \Phi_X (M, f, b) = \{T(B)\eta\}, \) \( \{\cdot\} \) being stable homotopy classes. For example, if \( X = BO \) we have \( \Omega_*^{BO} \cong \pi_* (MO) \) where \( MO \) is the Thom spectrum associated with \( BO \). We may also define the bordism groups over a space \( Y, \) \( \Omega_*^{BO} (Y) \), and in this case we have an isomorphism \( \Omega_*^{BO} (Y) \cong H_* (Y; MO) \).

Transversality plays an essential role in proving \( \Phi_X \) to be an isomorphism.

It is natural to ask to what extent these theorems can be reproduced in the Poincaré duality category. For example, we may define the Poincaré cobordism groups \( \Omega_*^{PD} \) to be equivalence classes under Poincaré cobordism of Poincaré spaces of formal dimension \( n \). Thus two Poincaré spaces \( P_1, P_2 \) are equivalent if there is a Poincaré space \( W \) such that \( \partial W = P_1 \sqcup P_2 \) with the appropriate relations on the Spivak normal fibrations. (See [9].) Then in the same way we have a homomorphism

\[ \Phi_{PD} : \Omega_*^{PD} (Y) \rightarrow H_* (Y; MSG) \]

The map is not an isomorphism [9]. (The results on transversality of manifolds do not hold in the Poincaré duality category.) However we may fit \( \Phi_{PD} \) into a long exact sequence. First define a normal space \( (X, \zeta, \alpha) \) to be a space \( X \) together with a \((k-1)\)-spherical fibration over \( X \) and a class \( \alpha \in \pi_{n+k} (T\xi) \). Then we have normal cobordism groups \( \Omega_n^N (Y) \) (not to be confused with the above normal cobordism groups) and a Thom–Pontrjagin map \( \Phi_N : \Omega_n^N (Y) \rightarrow \)
$H_n(X; MSG)$. It is easy to show that $\Phi_N$ is an isomorphism and the map $(\Phi_N)^{-1} \cdot \Phi_P : \Omega_P^D(Y) \to \Omega_N^P(Y)$ is just the forgetful map. Moreover this map fits into a long exact sequence

$$\cdots \to \Omega_{n+1}^{P,N}(Y) \to \Omega_n^P(Y) \to \Omega_n^P(Y) \to \Omega_n^{P,N}(Y) \to \cdots$$

where $\Omega_{n+1}^{P,N}(Y)$ is the bordism group of normal spaces with Poincaré boundary.

Now suppose $f : W \to X$ is a normal map where $W$ is Poincaré. Then the mapping cylinder of $f$ is a normal space with Poincaré boundary. It may also be thought of as a normal cobordism. Similarly a normal map $(W, Z) \to (X, Z)$ with $(W, Z)$ and $Z$ Poincaré gives rise to a normal cobordism. Thus we are led to study such normal maps and cobordisms. We would like to prove the following

"NORMAL SURGERY THEOREM" If $X$ is a normal space of formal dimension $n$ there is an obstruction $\sigma(X) \in L_{n-1}(\pi_1 X)$ such that $X$ is normally bordant to a Poincaré complex iff $\sigma(X) = 0$.

A 'sketch proof' is given in [14], the essential step being the removal of cells above the middle dimension which disturb Poincaré duality, by the extension of cofibrations. A special argument is required for the middle dimension. With a relativised version of this theorem the bordism groups $\Omega_n^{P,N}(Y)$ are identified with the Wall groups $L_n(\pi_1 Y)$. Hence we have the long exact sequence

$$(*) \quad \cdots \to L_n(\pi_1 Y) \to \Omega_n^P(Y) \to H_n(Y; MSG) \to L_{n-1}(\pi_1 Y) \to \cdots$$
Detailed proofs of these results have not been published. The extension of cofibrations result is given in \[13\] but the proof is incomplete.

In this thesis we present a counter-example to the extension of a map as a cofibration result in \[13\], (see Chapter 3). We give a necessary condition for extension which appears more in character with earlier extension results (i.e. desuspension) and show that this implies the conditions for extension in \[13\]. The counter-example relies on higher cohomology products as an obstruction to extension as a cofibration. Since the results of \[13\] were essential in the normal surgery theorem in \[14\], a correct proof is still outstanding. This we give, below the middle dimension, in Chapter 2. Although in the context of normal surgery the higher products are zero we do not obtain our results by an extension theorem. Instead we first obtain a normal structure on the fibre (the usual first approximation for an extension) and then show, by an iteration argument, that the same construction can be used to remove the 'error'. Chapter 1 contains some necessary homotopy results.

Several different proofs of the existence of \(\ast\) have in fact been given. See N. Levitt \[8,9\], L. Jones \[7\] and more recently J. Hausmann and P. Vogel \[6\]. The general approach of these papers is to try to pull information in the manifold category into the Poincaré duality category. Then the usual manifold program is pursued to give Poincaré surgery results. The homotopy-theoretic approach adopted here, following Quinn, seems appropriate to the Poincaré duality category.
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§1 HOMOTOPY CONSTRUCTIONS

We begin by recalling some standard homotopy constructions. All spaces will be of the homotopy type of countable CW–complexes. Let \( A, B, C \) be spaces.

**Definition** Given two maps \( f : A \to B, \ g : C \to B \), define the homotopy pullback, \( W_{f, g} \), of \( f \) and \( g \) to be the space

\[
\{(a, \gamma, c) \in A \times PB \times C \mid f(a) = \gamma(0), g(c) = \gamma(1)\}
\]

together with projection maps \( \Pi_A : W_{f, g} \to A, \Pi_C : W_{f, g} \to C \) taking \((a, \gamma, c)\) to \( a \) and \( c \) respectively.

Here \( PB \) is the space of continuous maps \( I \to B \) with the compact–open topology. The map \( H_{f, g} : W_{f, g} \times I \to B \) defined by \( H_{f, g}((a, \gamma, c), t) = \gamma(t) \) is a homotopy from \( f \cdot \Pi_A \) to \( g \cdot \Pi_C \) called the canonical homotopy along the paths.

The homotopy pullback satisfies the ‘universal’ property that given \( \alpha : Z \to A, \beta : Z \to C \) such that \( f \cdot \alpha = g \cdot \beta \), there exists a \( \chi : Z \to W_{f, g} \) such that \( \Pi_A \cdot \chi = \alpha, \Pi_C \cdot \chi = \beta \).

The homotopy pullback is not universal in the categorical sense, since \( \chi \) is
not necessarily unique. In fact a choice of map \( \chi \) making the diagram commute is equivalent to a choice of homotopy \( f \cdot \alpha \simeq g \cdot \beta \). In the following there will always be given a particular homotopy and so \( \chi \) will be well defined.

**Definition** A diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\beta} & C \\
\alpha \downarrow & & \downarrow g \\
A & \xrightarrow{f} & B
\end{array}
\]

together with a given homotopy \( f \cdot \alpha \simeq g \cdot \beta \), is said to be a homotopy pullback square if the induced map

\[
\chi : W \to W_{f,\alpha}
\]

is a homotopy equivalence.

Suppose now that \( B \) is a pointed space, that is, we have a fixed map \( * \to B \), where \( * \) is the one point space. Then the homotopy pullback of \( f : A \to B \), and \( * \to B, W_{f,*} \), is called the homotopy fibre of \( f \), denoted \( F_f \).

By naturality a homotopy commutative diagram of pointed spaces and maps

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\alpha} & A_2 \\
\downarrow f_1 & & \downarrow f_2 \\
B_1 & \xrightarrow{\beta} & B_2
\end{array}
\]

gives rise to a map of homotopy fibres

\[
\phi_{\alpha,\beta} : F_f \to F_{f'}
\]
Again \( \phi_{\alpha, \beta} \) will depend on a choice of homotopy \( \beta \cdot f_1 \simeq f_2 \cdot \alpha \), but once it is given \( \phi_{\alpha, \beta} \) is well defined.

**Lemma 1.1.1** If

\[
\begin{array}{ccc}
W & \xrightarrow{\beta} & C \\
\downarrow \alpha & & \downarrow g \\
A & \xrightarrow{f} & B
\end{array}
\]

is a homotopy pullback square of pointed spaces then \( \phi_{\alpha, g} : F_\beta \to F_f \) and \( \phi_{\beta, f} : F_\alpha \to F_\gamma \) are homotopy equivalences.

**Proof** Given a path \( \gamma \in PB \) for \( t \in [0,1] \) let \( \gamma_t \in PB \), be defined by

\[
\gamma_t(s) = \begin{cases} 
\gamma(s) & s \leq t \\
\gamma(t) & s > t
\end{cases}
\]

and \( \gamma' \) the corresponding path from \( \gamma(t) \) to \( \gamma(1) \).

Now

\[
F_{\Pi A} = \{ (a, \gamma, c, \omega) \in A \times PB \times C \times PA | f(a) = \gamma(0), g(c) = \gamma(1), a = \omega(0), * = \omega(1) \}
\]

and

\[
F_\gamma = \{ (c, \gamma) \in C \times PB | \gamma(0) = g(c), \gamma(1) = * \}
\]

Then \( \phi_{\Pi c, f} : F_{\Pi A} \to F_\gamma \) is given by \( \phi_{\Pi c, f}(a, \gamma, c, \omega) = (c, (-\gamma) + f_*(\omega)) \). There is an inclusion \( i : F_\gamma \to F_{\Pi A} \) where \( i(c, \gamma) = (*, (-\gamma), c, *) \). Then \( \phi_{\Pi c, f} \circ i = 1_{F_\gamma} \). Furthermore the map \( H : F_{\Pi A} \times I \to F_{\Pi A} \) where \( H(a, \gamma, c, \omega, t) = \)
(ω(t), −f_*(ω_k) + γ_1, c, ω') provides a homotopy i • φ_{π_A,f} \cong 1_{F_\alpha} and so φ_{π_C,f} is a homotopy equivalence.

For a general homotopy pullback, \( W \), we have

\[
\begin{array}{ccc}
W & \xrightarrow{\chi} & W_{f,g} \\
\downarrow{\alpha} & & \downarrow{\Pi_A} \\
A & \xrightarrow{f} & B
\end{array}
\]

where \( \Pi_C \cdot \chi = \beta \). Then we have the commutative diagram

\[
\begin{array}{ccc}
F_\alpha & \xrightarrow{\phi_{\chi,1}} & F_{\Pi_A} \\
\downarrow{\phi_{\beta,f}} & & \downarrow{\phi_{\Pi_C,f}} \\
F_\beta & \xrightarrow{f} & F_\beta
\end{array}
\]

Now \( \phi_{\Pi_C,f} \) is a homotopy equivalence by the above argument and since \( \chi \) is a homotopy equivalence by assumption, \( \phi_{\chi,1} \) is also. Hence \( \phi_{\beta,f} \) is a homotopy equivalence, as is \( \phi_{\alpha,g} \) by symmetry.

Let \( B \) be a pointed space. The diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & B \\
\downarrow \cong & & \downarrow \cong \\
X & \xrightarrow{f} & B
\end{array}
\]

gives a map \( e: F_f \to W_{f,1_B} \), on taking homotopy pullbacks. We shall write \( \tilde{X} \) for the space \( W_{f,1_B} \), and \( \tilde{f} \) for the map \( \Pi_B \). Then \( \tilde{f}: \tilde{X} \to B \) is a fibration and \( e \) is the inclusion of the fibre of \( \tilde{f} \) over the basepoint of \( B \). It is called the (Hurewicz) fibration associated with \( f \).
There is an inclusion \( k: X \rightarrow \tilde{X} \), where \( k(x) = (x, \gamma_f(x), f(x)) \), making \( X \) a strong deformation retract of \( \tilde{X} \) with retraction \( \Pi_X \). Here \( \gamma_f(x) \in PB \) satisfies 
\( \gamma_f(x)(t) = f(x) \) for all \( t \). Furthermore

\[
\begin{array}{ccc}
X & \xrightarrow{k} & \tilde{X} \\
\downarrow f & & \downarrow \tilde{f} \\
B & \xrightarrow{=} & B
\end{array}
\]

commutes. In this way we think of every map as being equivalent, up to homotopy, to a fibration.

The homotopy pushout of two maps \( f: B \rightarrow A \), \( g: B \rightarrow C \), denoted \( M_{f,g} \), is the space

\[A \cup (I \times B) \cup C / \{(0,b) \sim f(b), (1,b) \sim g(b)\}\]

together with inclusions \( i_A: A \rightarrow M_{f,g} \), \( i_C: C \rightarrow M_{f,g} \) such that \( i_A(a) = \{a\}_\sim \) and \( i_C(c) = \{c\}_\sim \) respectively.

The homotopy pushout has properties dual to the homotopy pullback. In particular, given maps \( \alpha: A \rightarrow Z \), \( \beta: C \rightarrow Z \) and a homotopy \( \alpha \cdot f \simeq \beta \cdot g \), there is a map \( \chi: M_{f,g} \rightarrow Z \) such that \( \chi \cdot i_A = \alpha \), and \( \chi \cdot i_B = \beta \). We also have general homotopy pushouts, homotopy cofibres (or mapping cones) of a map \( f \), denoted \( C_f \), and cofibrations associated with maps (or mapping cylinders). The definitions will be omitted.

We shall require
LEMMA 1.1.2. If

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\gamma \downarrow & & \downarrow \delta \\
C & \xrightarrow{\beta} & D
\end{array}
\]

is a homotopy commutative diagram then \( C_{\alpha,\beta} \cong C_x \).

NOTATION We shall use the notation \( \phi_{\beta,\alpha} : C_\gamma \rightarrow C_\delta \) to denote the induced map on mapping cones. It will be clear from the context whether \( \phi_{\beta,\alpha} \) maps between fibres or mapping cones.

PROOF Consider the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\gamma \downarrow & & \downarrow i_B \\
C & \xrightarrow{i_C} & M_{\gamma,\alpha} & \xrightarrow{\chi} & D \\
\downarrow i_1 & & \downarrow i_2 & & \downarrow i_D \\
C_\gamma & \xrightarrow{\phi_{i_C,\alpha}} & C_\delta & \xrightarrow{i_3} & M_{r,\chi} \\
\Downarrow & & \Downarrow \omega & & \Downarrow \\
C_\gamma & \xrightarrow{\phi_{\alpha,\beta}} & C_\delta
\end{array}
\]

where \( i_1, i_2 \) and \( i_3 \) are the obvious inclusion maps, and \( \omega : M_{r,\chi} \rightarrow C_\delta \) is defined by

\[
\omega \mid_D : D \hookrightarrow D \cup_\delta CB
\]

\[
\omega \mid_{M_{\gamma,\alpha} \times I} : M_{\gamma,\alpha} \times I \xrightarrow{\pi} M_{\gamma,\alpha} \xrightarrow{\chi} D \hookrightarrow D \cup_\delta CB
\]

\[
\omega \mid_{C_\delta} = \phi_{\chi,1_3}
\]
Clearly the definition of $\omega$ is compatible with the identification $\sim$. Then $\omega \cdot i_\beta = \phi_{X,1_\beta}$ and so $\omega \cdot i_\beta \cdot \phi_{i_C,a} = \phi_{X,1_\beta} \cdot \phi_{i_C,a} = \phi_{a,b}$, and so the lower rectangle commutes and we have a commutative diagram.

Now there is an inclusion $k: C_\beta \to M_{r,x}$ where

$$k(d) = d \in D \subset M_{r,x}$$
$$k(b,t) = (b,2t) \in M_{2,x} \times I \subset M_{r,x} \quad 0 \leq t \leq \frac{1}{2}$$
$$= (b,2t-1) \in C_{i_D} \subset M_{r,x} \quad \frac{1}{2} \leq t \leq 1$$

It is easily seen that $\omega \cdot k \simeq 1_{C_\beta}$, and $k \cdot \omega \simeq 1_{M_{r,x}}$. Thus $\omega$ is a homotopy equivalence.

Since $\omega$, $\phi_{i_C,a}$ and $\phi_{i_D,r}$ are homotopy equivalences we have

$$C_{\phi_{i_D,r}} \cong C_{\phi_{i_C,a}} \cong C_{i_D} \cong C_{i_C} \cong C$$

Hence result. □

REMARK We have a given homotopy equivalence $C_{\phi_{i_D,r}} \cong C$ which is constructed in a natural way.

§2 Thom Complexes and Whitney Sums

Let $\xi$ be a fibration with $p: E_\xi \to X$ the projection of the total space onto the base space. Define the Thom complex of $\xi$, denoted $T_\xi$, to be the mapping
cone of \( p \). If \( f: \xi \to \eta \) is a map of fibrations over \( X \), there is an induced map 
\[ Tf: T\xi \to T\eta. \]

**EXAMPLE** Consider the trivial spherical fibration \( p: X \times S^{k-1} \to X \), denoted by \( X \times S^{k-1} \). Then \( T(X \times S^{k-1}) \cong \Sigma^k(X_+) \), where \( X_+ \) is the disjoint union of \( X \) and a point.

Thus the Thom complex of a fibration can be thought of as a generalised suspension. In fact we have

**THOM ISOMORPHISM THEOREM** Suppose \( X \) is simply-connected and \( \xi \) is a \( S^{k-1} \)-fibration over \( X \), (that is, it has fibre \( S^{k-1} \) up to homotopy). Then there is a class \( U \in \tilde{H}^k(T\xi) \), such that
\[
\cap U: \tilde{H}^{*+k}(T\xi) \to H_*(X)
\]
\[
\cup U: H^*(X) \to \tilde{H}^{*+k}(T\xi)
\]
are isomorphisms, and \((\phi_*j)^*(U) \in \tilde{H}^k(\Sigma S^{k-1}) \) is a generator, where the map \( j \) is the inclusion of the fibre in the total space. The class \( U \) is called the Thom class of \( \xi \).

**PROOF** See [3, Theorem II.2.3.]

Given two fibrations \( \eta, \xi \) over \( X \) with projection maps \( p, q \) respectively,
let $\tilde{W}_{p,q}$ be the topological pullback of $p$ and $q$, together with projection maps

$\Pi_\eta : \tilde{W}_{p,q} \to E\eta$ and $\Pi_\xi : \tilde{W}_{p,q} \to E\xi$. Then there is an induced map

$\chi : M_{\Pi_\eta, \Pi_\xi} \to X$. This is a fibration, called the Whitney join of $\eta$ and $\xi$, and denoted $\eta \oplus \xi$. Its fibre is the join of the fibres of $\eta$ and $\xi$. In particular, if $\eta$ and $\xi$ are spherical fibrations then so is $\eta \oplus \xi$.

**Lemma 1.2.1.** $\eta \oplus (\xi \oplus \omega) = (\eta \oplus \xi) \oplus \omega$

**Proof.** Each total space is homeomorphic over $X$ to the total space of the fibration $\eta \oplus \xi \oplus \omega$ which is formed from the pullback–pushout of the three projection maps. See the Appendix for details.

**Lemma 1.2.2.** If $X \times Y$ is the trivial fibration over $X$ and $\xi$ is any fibration over $X$ then $T(\xi \oplus (X \times Y)) = \Sigma Y \wedge T\xi$.

**Proof.** We have

\[ \begin{array}{ccc}
E\xi \times Y & \xrightarrow{\omega} & p \times 1 \\
\downarrow \Pi & & \Pi \downarrow \Pi_p
\end{array} \]

\[ W_{p,pX} \rightarrow X \times Y \]

\[ E\xi \rightarrow X \]

where $\omega : E\xi \times Y \leftrightarrow \tilde{W}_{p,pX}$ such that $\omega(e,y) = (e, (p(e), y))$, for $e \in E\xi$ and...
y \in Y$. Since $p_x$ is the trivial projection, $E^\xi \times Y \subset \overline{W}_{p.p_x}$ is a retract and so $\omega$ is a homotopy equivalence. Thus $C_{p,p_x} \cong C_{p,p \times 1} \cong \Sigma Y \wedge T\xi$. However, by Lemma 1.1.2., $C_{p,p_x} \cong C_x = T(\xi \oplus (X \times Y))$ and the result follows.

Given $\eta$, $\xi$, fibrations over $X$, we may form the commutative cube

\[
\begin{array}{cccccccc}
\overline{W}_{p,q} & \xrightarrow{\Pi_q} & E^\xi & \xrightarrow{\Pi_p} & q & \xrightarrow{p} & X \\
\downarrow J & \downarrow \Delta_{E\xi} & \downarrow 1 \wedge q & \downarrow \Delta_{E\xi} & \downarrow \Delta \\
E\eta \wedge E^\xi & \xrightarrow{p \wedge 1} & X \wedge E^\xi & \xrightarrow{p} & E\eta \wedge X & \xrightarrow{p \wedge 1} & X \wedge X
\end{array}
\]

where the commutative squares are the upper and lower faces of the cube and the middle square represents the vertical maps. Also $J(a,b) = a \wedge b$ whilst $\Delta_{E\xi} : E^\xi \xrightarrow{\Delta} E^\xi \wedge E^\xi \xrightarrow{p \wedge 1} X \wedge E^\xi$, and similarly for $\Delta_{E\eta}$. Taking mapping cones gives

\[
\begin{array}{cccc}
C_{\Pi_q} & \xrightarrow{\phi_{\Delta_{E\xi}, J}} & C_{1 \wedge q} & \xrightarrow{\cong} & E\eta \wedge T\xi \\
\phi_{p, \Pi_q} & \downarrow & \phi_{p \wedge 1, p \wedge 1} & \downarrow & p \wedge 1 \\
C_q & \xrightarrow{\phi_{\Delta_{E\eta}, \Delta_{E\xi}}} & C_{1 \wedge q} & \xrightarrow{\cong} & X \wedge T\xi
\end{array}
\]

Finally, taking mapping cones of the vertical maps gives a diagonal map

$\Delta_{\eta, \xi} : T(\eta \oplus \xi) \to T\eta \wedge T\xi$

**Lemma 1.2.3.** Let $\xi$, $\eta$, and $\omega$ be fibrations over $X$. Then the following diagram commutes up to homotopy

\[
\begin{array}{ccc}
T(\xi \oplus \eta \oplus \omega) & \xrightarrow{\Delta_{\xi \oplus \eta, \omega}} & T(\xi \oplus \eta) \wedge T\omega \\
\Delta_{\xi, \eta \oplus \omega} & \downarrow & \Delta_{\xi, \eta} \wedge 1 \\
T(\xi) \wedge T(\eta \oplus \omega) & \xrightarrow{1 \wedge \Delta_{\xi, \omega}} & T\xi \wedge T\eta \wedge T\omega
\end{array}
\]
**PROOF**  See Appendix I

**EXAMPLE**  Suppose $\xi$ is a $S^{k-1}$-fibration over $X$ with Thom class $U \in \tilde{H}^k(T\xi)$. Then we have the diagonal

$$\Delta_{X \times S^{i-1}, \xi} : T(X \times S^{i-1} \oplus \xi) \cong \Sigma^i T\xi \to \Sigma^i (X_+) \wedge T\xi$$

and hence maps

$$\cap U : \tilde{H}_{*+k}(T\xi) \to H_*(X)$$

$$\cup U : H^*(X) \to \tilde{H}^*+k(T\xi)$$

These are immediately seen to be the Thom isomorphisms, from the definition.

See [19, p. 350].

**EXAMPLE**  As a 'non-stable' version of the diagonal consider the diagram

$$\begin{array}{cccc}
E\xi & \overset{p}{\longrightarrow} & X & \longrightarrow & C_p \\
\Delta_{Et} & \downarrow & \Delta & \downarrow & \phi_{\Delta, \Delta_{et}} \\
E\xi \wedge X & \overset{p \wedge 1}{\longrightarrow} & X \wedge X & \longrightarrow & C_{p \wedge 1}
\end{array}$$

which defines a diagonal map $\Delta_\xi : T\xi \to T\xi \wedge X$

**REMARK**  The diagram

$$\begin{array}{ccc}
T(\eta \oplus \xi) & \overset{\Delta_\eta, \xi}{\longrightarrow} & T\eta \wedge T\xi \\
\Delta_\eta \wedge \xi & \downarrow & 1 \wedge \Delta_\xi \\
T(\eta \oplus \xi) \wedge X & \overset{\Delta_\eta, \xi \wedge 1}{\longrightarrow} & T\eta \wedge T\xi \wedge X
\end{array}$$
commutes.

**Lemma 1.2.4.** If $\xi$ is a $S^{k-1}$-fibration over $X$ with Thom class $U \in \tilde{H}^k(T\xi)$ and $\eta$ is any fibration over $X$ then

\[ \cap U: \tilde{H}_{*+k}(T(\eta \oplus \xi)) \rightarrow \tilde{H}_*(T\eta) \]
\[ \cup U: \tilde{H}^*(T\eta) \rightarrow \tilde{H}^{*+k}(T(\eta \oplus \xi)) \]

are isomorphisms.

**Proof** We shall show that the cap-product is an isomorphism. The proof for the cup-product is similar. From diagram (\*), p. 10, we obtain a commutative diagram of cap-products $\cap_1$ and $\cap_2$

\[ \tilde{C}_{*+k}(C_{n_x}) \xrightarrow{\cap_1 U} C_*(E\eta) \]
\[ \xrightarrow{p_\ast} \]
\[ \tilde{C}_{*+k}(C_q) \xrightarrow{\cap_2 U} C_*(X) \]

The cap-product $\cap U$ is formed, by definition, by taking mapping cones and so it is an isomorphism on homology if both $\cap_1$ and $\cap_2$ are. Now $\cap_2$ induces the Thom isomorphism and so we need only show that $\cap_1$ induces an isomorphism on homology.

But we have the commutative diagram

\[
\begin{array}{cccccc}
W & \xrightarrow{\Delta_W} & E\eta \wedge W & \xrightarrow{\Pi_\eta} & 1 \wedge \Pi_\eta & \xrightarrow{\Delta_\eta} & E\eta \wedge E\eta \\
J & \downarrow & 1 \wedge \Pi_\eta & \downarrow & 1 \wedge q & \downarrow & 1 \wedge p \\
E\eta \wedge E\xi & \xrightarrow{\Delta_\eta} & E\eta \wedge E\xi & \xrightarrow{\Delta_\eta} & E\eta \wedge X & \xrightarrow{\Delta_\eta} & E\eta \wedge X
\end{array}
\]
and taking mapping cones gives the commutative diagram

\[
\begin{array}{ccc}
C_{n*} & \xrightarrow{\Delta_{\xi^*}} & E\eta \wedge C_{n*} \\
\downarrow \phi_{p,\Delta^*} & & \downarrow 1 \wedge \phi_{p,\Pi^{n*}} \\
E\eta \wedge T\xi & \xrightarrow{} & E\eta \wedge T\xi
\end{array}
\]

where \( \xi^* \) is the induced \( S^{k-1} \)-fibration \( \Pi_{p,q} : W_{p,q} \to E\eta \) with Thom class \( U^* = (\phi_{p,\Pi^{n*}})^*(U) \in H^k(T\xi) \). However, \( \cap U^* : H^{*+k}(C_{n*}) \to H^*(E\eta) \) is an isomorphism and so \( \cap_1 U \) is also.

§3 S–DUALITY AND NORMAL FIBRATIONS

Let \( X \) be a finite CW-complex, then \( X \) is homotopy equivalent to a finite dimensional polyhedron \( K \). Embed \( K \) in a sphere \( S^N, N \) large, and let \( U \subset S^N \) be a regular neighbourhood of \( K \) in \( S^N \), with boundary \( \partial U \). Form the fibration associated with the inclusion \( \partial U \hookrightarrow U \) and pull it back to \( X \). This fibration is called the normal fibration of \( X \), denoted \( \nu(X) \). There is a given homotopy equivalence \( U/\partial U \cong T(\nu^N(X)) \) and so the collapse map \( S^N \to U/\partial U \) gives an element \( \{X\}_N \in \pi_N(T\nu^N(X)) \). If \( N \) is large enough all embeddings of \( K \) in \( S^N \) will be isotopic and so \( T\nu^N(X) \) and \( \{X\}_N \) will be well-defined (up to homotopy for the former).

A map \( \alpha : S^n \to A \wedge B \) is said to be an \( n \)-duality map if \( (\alpha_*[S^n]) \cap : \tilde{H}^{n+*}A \to \tilde{H} B \) is an isomorphism for all \( * \). We write \( A = D^n B \).
Suppose $\alpha: S^n \to A \wedge B$ is an $n$-duality map then we have the map

$$\alpha_{p,q} : S^{n+p+q} \to \Sigma^p A \wedge \Sigma^q B$$

$$t_1 \wedge \cdots \wedge t_n \wedge t'_1 \wedge \cdots \wedge t'_p \wedge t''_1 \wedge \cdots \wedge t''_q$$

$$\mapsto t'_1 \wedge \cdots \wedge t'_p \wedge \alpha_{p,q}(t_1 \wedge \cdots \wedge t_n) \wedge t''_1 \wedge \cdots \wedge t''_q$$

If $\alpha: S^n \to A \wedge A^*$ and $\beta: S^n \to B \wedge B^*$ are $n$-duality maps there is an isomorphism

$$D^n(\alpha, \beta) : \{A, B\} \to \{B^*, A^*\}$$

such that for sufficiently large $r$ if $f : \Sigma A \to \Sigma B$ and $f' : \Sigma B^* \to \Sigma A^*$ then $D^n\{f\} = \{f'\}$ iff the following diagram commutes up to homotopy

$$
\begin{array}{ccc}
\Sigma A \wedge A^* & \xleftarrow{\alpha_{r,0}} & S^{n+r} \xrightarrow{\beta_{0,r}} B \wedge \Sigma B^* \\
\downarrow f \wedge 1 & & \downarrow 1 \wedge f' \\
\Sigma B \wedge A^* & \xrightarrow{T} & B \wedge \Sigma A^*
\end{array}
$$

See [16]

**Lemma 1.3.1.** $D^MD^N A \cong \Sigma^{M-N} A$

**Proof** Immediate

**Theorem 1.3.2.** The map

$$S^N \xrightarrow{(X)_N} T(\nu^N(X)) \xrightarrow{\Delta_{\nu}} (X_+) \wedge T(\nu^N(X))$$

is an $N$-duality map. Hence $D^N(X_+) = T(\nu^N(X))$. 

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PROOF Follows from the Lefschetz duality theorem.

**Theorem 1.3.3.** If $\xi$ is a $S^{k-1}$-fibration over $X$ and $-\xi$ is a $S^{i-1}$-fibration over $X$ such that $\xi \oplus -\xi \cong S^{k+i-1} \times X$, then

$$S^{N+k+l} \xrightarrow{(X)_{N+k+l}} T((\nu^N \oplus -\xi) \oplus \xi) \xrightarrow{\Delta_{\nu^N \oplus -\xi, \xi}} T(\nu \oplus -\xi) \wedge T\xi$$

is a $N + k + l$-duality map. Hence $D^{N+k+l}(T\xi) = T(\nu \oplus -\xi)$.

**Proof** By Lemma 1.2.3. we have the commutative diagram

$$
\begin{array}{ccc}
T(\nu \oplus -\xi \oplus \xi) & \xrightarrow{\Delta_{\nu \oplus -\xi, \xi}} & T(\nu \oplus -\xi) \wedge T\xi \\
\Delta_{\nu \oplus -\xi, \xi} & & \downarrow 1 \wedge \Delta_{\xi} \\
T(\nu \oplus -\xi \oplus \xi) \wedge X & \xrightarrow{\Delta_{\nu \oplus -\xi, \xi} \wedge 1} & T(\nu \oplus -\xi) \wedge T\xi \wedge X
\end{array}
$$

Then

$$
\tilde{H}^*(T(\nu \oplus -\xi)) \xrightarrow{\Sigma^{k+l}(X) \cap} \tilde{H}_{N+k+l-*}(T\xi) \\
\cup U \downarrow \cap U \\
\tilde{H}^{*+k}(T(\nu \oplus -\xi \oplus \xi)) \xrightarrow{(X)_{N+k+l \cap}} H_{N+k+l-*}(X)
$$

commutes, where $U$ is the Thom class of $\xi$, and so $\Sigma^{k+l}(X) \cap$ is an isomorphism.

§4 Geometric Products and Poincaré Spaces

Let $\xi$ be a fibration over $X$ and $f: T\xi \to Y$, then we have the map

$$
T\xi \xrightarrow{\Delta_{\xi}} T\xi \wedge X \xrightarrow{f \wedge 1} Y \wedge X
$$
called the geometric product of \( f \), denoted \( \cap f \). Dually, let \( \xi \) be a \( S^{k-1} \)-fibration over \( X \) and \( g: Y \to T\xi \), then we have the stable map

\[
D^{N+k+1}(X_+) \cong T(\nu \oplus -\xi \oplus \xi) \xrightarrow{\Delta_{\nu \oplus -\xi \oplus \xi}} T(\nu \oplus -\xi) \wedge T\xi \xrightarrow{D^{N+k+1}g \wedge 1} D^{N+k+1}Y \wedge T\xi
\]
called the geometric product of \( g \), denoted \( g \cap \).

**Example** If \( f: T(X \times S^{k-1}) \cong \Sigma^k X_+ \to S^r \) then \( \cap f: \Sigma^k X_+ \to \Sigma^r X_+ \) induces the usual cap-product on homology.

**Definition** A space \( X \) together with a class \([X] \in H_n(X)\) is said to be a Poincaré space of formal dimension \( n \) if

\[
[X]: H^*(X) \to H_{n-*}(X)
\]
is an isomorphism for all \(*\). \([X]\) is called the fundamental class of \( X \).

**Theorem 1.4.1.** Let \( X \) be a simply-connected space. The following are equivalent

1. \( X \) is a Poincaré space of formal dimension \( n \).
2. There is a spherical fibration \( \xi \) over \( X \) with fibre \( S^{k-1} \) and a map

\[
\alpha: S^{n+k} \to T\xi \text{ such that } \alpha \cap: D^{n+2k+1}X_+ \to \Sigma^{k+1}T\xi \text{ is a homotopy equivalence.}
\]
3. \( \nu^{n+k}(X) \) is a spherical fibration with fibre \( S^{k-1} \).
Proof (1 \Rightarrow 3) See [17].

(3 \Rightarrow 2) Take \( \xi \) to be \( \nu^{n+k}(X) \) and \( \alpha \) to be \( \{X\}_{n+k} \).

Then \( \alpha \cap: D^{n+k+1}(X_{+}) \to \Sigma T\nu \) is just the identity map.

(2 \Rightarrow 1) By repeated application of Lemma 1.2.3. we have the commutative diagram

\[
\begin{array}{ccccccccc}
& & T(\nu \oplus -\xi \oplus \xi) \land X & \xrightarrow{\Delta_{\nu \oplus -\xi \oplus \xi}^{1}} & T(\nu \oplus -\xi) \land T\xi \land X & \xrightarrow{1 \lor 1 \land \Delta_{\xi}} & T(\nu \oplus -\xi) \land T\xi \land X \land X \\
& & & \uparrow^{1 \land \Delta_{\xi}} & \downarrow^{1 \land \Delta_{\xi}} & & \\
& & T(\nu \oplus -\xi \oplus \xi) & \xrightarrow{\Delta_{\nu \oplus -\xi \oplus \xi}^{1}} & T(\nu \oplus -\xi) \land T\xi \\
& & & \downarrow^{1 \land \Delta_{\xi}} & & \\
T(\nu \oplus -\xi \oplus \xi) \land X & \xrightarrow{\Delta_{\nu \oplus -\xi \oplus \xi}^{1}} & T(\nu \oplus -\xi) \land T\xi \land X \land X & \xrightarrow{1 \lor 1 \land \Delta_{\xi}} & T(\nu \oplus -\xi) \land T\xi \land X \land X
\end{array}
\]

Thus we obtain two diagonals

\[ \Delta_{U}, \Delta_{L}: T(\nu \oplus -\xi \oplus \xi) \to T(\nu \oplus -\xi) \land T\xi \land X \land X \to S^{k+1} \land T\xi \land X \land X \]

corresponding to the upper and lower routes around the diagram respectively.

There are then two cap–products induced from \( \{X\}_{n+2k+1} \)

\[
S_{U}, S_{L}: H^{*}(X) \longrightarrow H_{n-*}(X)
\]

\[
x \longrightarrow \Delta_{U} \{X\}/U \land x, \Delta_{L} \{X\}/U \land x
\]

where in each case \( x \in H^{*}(X) \) is evaluated on the second term \( H^{*}(X) \). Following through the definition of \( \Delta_{U} \) we see that \( S_{U} \) is given by

\[
S_{U}(x) = [(\alpha \cap) \ast (\{X\} \cap x)] \cap U
\]

for \( x \in H^{*}(X) \). Similarly

\[
S_{L}(x) = [(\{X\} \cap ((D\alpha)^{*}(s^{k+1}) \cup U)] \cap x
\]

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where \( s^{k+i} \in H^{k+i}(S^{k+i}) \) is a generator. Thus if we let \([X] \in H_n(X)\) be given by

\[
[X] = \{X\} \cap ((D\alpha)^* (s^{k+i}) \cup U) = ((\alpha \cap) \{X\}) \cap U
\]

we conclude that

\[
\tilde{H}_{n+2k+i-1-\cdot}(T(\nu \oplus -\xi \oplus \xi)) \xrightarrow{(\alpha \cap) \cdot} \tilde{H}_{n+2k+i-1-\cdot}(\Sigma^{k+i} T\xi)
\]

\[
\begin{array}{ccc}
\{X\} \cap & \downarrow & \cap U \\
H^*(X) & \xrightarrow{[X] \cap} & H_{n-\cdot}(X)
\end{array}
\]

commutes and so \([X]\cap\) is an isomorphism. \(\blacksquare\)
CHAPTER TWO

§1 NORMAL SPACES AND KERNELS

DEFINITION A normal space \((X, \xi, \alpha_X)\) is a space \(X\) together with a \((k - 1)\)-spherical fibration \(\xi\) over \(X\), with a choice of Thom class, and a map \(\alpha_X : S^{n+k} \to T\xi, n \geq 0\). Then \(X\) is said to be of formal dimension \(n\).

REMARK We shall assume that \(X\) is of cellular dimension at most \(n + 1\), (see remark after Lemma 2.1.4.), and that \(k > n\). The latter is no restriction since we are only interested in \(\xi\) up to stable fibre homotopy.

By Theorem 1.4.1 a Poincaré space is an example of a normal space, but of course in general \(\alpha_X \cap : D^{n+2k+1}X_+ \to \Sigma^{k+1}T\xi\) need not be a homotopy equivalence.

DEFINITION A map \(f : X \to Y\) of normal spaces \((X, \xi, \alpha_X), (Y, \eta, \alpha_Y)\) is said to be a normal map if there is a fibre homotopy equivalence \(\Psi : \xi \cong f^*\eta\) such that

\[
\begin{array}{ccc}
S^{n+k} & \xrightarrow{\alpha_Y} & T\eta \\
\alpha_X \downarrow & & \uparrow Tf \\
T\xi & \xrightarrow{T\Psi} & T(f^*\eta)
\end{array}
\]
commutes up to homotopy. Furthermore the map is said to be of degree 1 if
\[ T\Psi^* \cdot Tf^*(U_i) = U_t. \]

**DEFINITION** If \((X, \xi, \alpha_X)\) is a normal space, define the kernels of \(X\), denoted \(K_*(X), K^*(X)\), by
\[
K_* (X) = H_{2k+l+1}^+ (\alpha_X \cap) = H_{*+2k+l} (C_{\alpha_X \cap})
\]
\[
K^* (X) = H^{2k+l+1+*} (\alpha_X \cap) = H^{*+2k+l+1} (C_{\alpha_X \cap})
\]
where \(\xi\) has a stable inverse with fibre \(S^{l-1}\).

Then the long exact sequence corresponding to the map \(\alpha_X \cap\) can be written in the form
\[
\cdots \rightarrow K_* (X) \rightarrow H^{n-k-*} (X) \xrightarrow{([X] \cap)_*} H_* (X) \rightarrow K_{*-1} (X) \rightarrow \cdots
\]
where \([X] = (\alpha_X)_* (s^{n+k}) \cap U_\xi\).

The kernels of a normal space satisfy a kind of duality.

**LEMMA 2.1.1.** If \(X\) is a normal space of formal dimension \(n\) then
\[ K_* (X) \cong K^{n-*}_{-1} (X) \]

**PROOF** We have the commutative diagram
\[
\begin{array}{cccccc}
\tilde{C}_* (T\nu^{n+k} (X)) & \xrightarrow{\{X\} \cap} & C^{n+k-*} (X) & \xrightarrow{UU} & \tilde{C}^{n+2k-*} (T\xi) \\
(\alpha_X \cap)_* \downarrow & & \downarrow ([X] \cap) & & \downarrow (\alpha_X \cap)^* \\
\tilde{C}_* (T\xi) & \xrightarrow{nU} & C_{*-k} (X) & \xrightarrow{\{X\} \cap} & \tilde{C}^{n+2k-*} (T\nu^{n+k} (X))
\end{array}
\]
Since the horizontal maps are homotopy equivalences, the result follows by taking mapping cones of the vertical maps.

The central problem that we study in this chapter is how to perform surgery on normal spaces in order to obtain Poincaré spaces. However, surgery at the middle dimension is not dealt with here. The kernels of a normal space give an immediate measure of how 'close' the space is to being a Poincaré space.

DEFINITION The normal space $X$ is said to be $j$--Poincaré if $K_*(X) = 0$, $* \leq j$.

Notice that $j$ may be negative, but that since $X$ is finite $j$ will always be finite. If $x$ is any real number let $[x]$ denote the integer part of $x$.

**LEMMA 2.1.2.** If $X$ is a normal space of formal dimension $n$ and $K_*(X) = 0$, $* \leq \lfloor \frac{n-1}{2} \rfloor$, then $K_*(X) = 0$ for all $*$, that is, $X$ is a Poincaré space.

**PROOF** If $K_*(X) = 0$, $* \leq \lfloor \frac{n-1}{2} \rfloor$, then by the universal coefficient theorem $K^*(X) = 0$, $* \leq \lfloor \frac{n-1}{2} \rfloor$, and so by Lemma 2.1.1. $K_{n-1-*}(X) = 0$, $* \leq \lfloor \frac{n-1}{2} \rfloor$, and the result follows.

We conclude that in order to obtain a Poincaré space we need only kill the kernels $K_*(X)$, $* \leq \lfloor \frac{n-1}{2} \rfloor$. The following theorem achieves just this, up to the
THEOREM 2.1.3 (SURGERY BELOW THE MIDDLE DIMENSION) Let $X$ be a simply-connected normal space of formal dimension $n$. Then there is a normal space $W$, which is $(\lfloor \frac{n-1}{2} \rfloor - 1)$-Poincaré, and a $(\lfloor \frac{n}{2} \rfloor + 1)$-connected normal map $f: W \to X$.

The rest of this chapter will be concerned with the proof of this theorem. The idea of the proof is to remove cells above the middle dimension which correspond to generators of the kernels. This is achieved by taking homotopy fibres and choosing an appropriate normal structure. The procedure is complicated by the fact that non-zero elements in lower kernels are introduced. The same procedure is used to kill these kernels giving rise to an induction argument.

We begin with the simplest case of 'kernel killing'.

LEMMA 2.1.4. If $(X, \xi, \alpha_X)$ is a normal space of formal dimension $n$, there is a normal space $W$ which is $(-2)$-Poincaré and an $n$-connected normal map $f: W \to X$. Furthermore if $K_{-1}(X) = 0$ then we may choose $W$ such that $K_{-1}(W) = 0$.

PROOF: Let $W$ be the homology truncation of $X$ at $n$, (see [2, Theorem 2.1]), that is, $H_\ast(W) = 0, \ast > n$ and there is a map $i: W \to X$ such that $i_\ast: H_\ast(W) \cong H_\ast(X), \ast \leq n$. Also $W$ is of cellular dimension $n + 1$. Form the spherical fibration $i^\ast \xi$ over $W$. Then $T(i): T(i^\ast \xi) \to T\xi$ is $(n + k)$-connected.
and so we may lift \( \alpha_x \) giving \( \alpha_W : S^{n+k} \to T(i^*\xi) \). This defines a normal structure on \( W \) for which \( i : W \to X \) is an \( n \)-connected normal map. Since \( H_* (W) = 0, * > n, W \) is \((-2)\)-Poincaré. Now suppose \( K_{-1}(X) = 0 \), then \( ([X] \cap)_0 \) is onto and \( H^{n+1}(X) = 0 \). Hence \( ([W] \cap)_0 \) is onto and \( H^{n+1}(W) = 0 \), so that \( K_{-1}(W) = 0 \).

As a consequence of Lemma 2.1.4. we assume from now on that every normal space of formal dimension \( n \) is \((-2)\)-Poincaré and of cellular dimension at most \( n + 1 \).

We now define a natural map between chain mapping cones and the singular chain complex of mapping cones. Consider a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\alpha \downarrow & & \downarrow \beta \\
A & \xrightarrow{f} & B
\end{array}
\]

of spaces. Then we have the commutative diagram

\[
\begin{array}{ccc}
C_* Y \oplus_{ C_{-1}X} C_{-1}X & \xrightarrow{\beta_* \oplus \alpha_*} & C_* B \oplus_{ C_{-1}A} C_{-1}A \\
\omega_g \downarrow & & \downarrow \omega_f \\
\tilde{C}_*(Y \cup_\omega CX) & \xrightarrow{(\beta \cup C\alpha)_*} & \tilde{C}_*(B \cup_\omega CA)
\end{array}
\]

where \( \omega_g(y \oplus x) = y + Cx \) for \( y \oplus x \in C_* Y \oplus_{ C_* } C_{-1}X, \) \( * \geq 1 \) and \( \omega_g(y) = y - * \) for \( y \in C_0 Y \) where \( * \) is a choice of base point. Similarly for \( \omega_f \). This defines chain maps. (For \( x \in C_{-1}X \), a generator we define \( Cx \in C_*(CX) \) to be the simplicial join of \( x \) and the base point of \( CX \). Let \( Cx \in C_*(Y \cup_\omega CX) \) be the image of
\( \odot x \) under the map \( CX \to Y \cup_x CX \). Then \( \omega_x \) and \( \omega_f \) are chain homotopy equivalences.

We shall have need of some standard chains. Let \( p < N \) and \( m \) be fixed positive integers. Let \( \{ s^p \} \in H^p(S^p) \) be a generator. Let \( d_{N+1} \in C_{N+1}(D^{N+1}) \), and \( s_N \in C_N(S^N) \) be any two singular chains such that \( \partial d_{N+1} = i_*(s_N) \) where \( i: S^N \hookrightarrow D^{N+1} \) is the inclusion of the boundary, and \( \{ s_N \} \in H_N(S^N) \) is a generator.

Consider the commutative diagram

\[
\begin{array}{ccc}
S^N & \xrightarrow{i} & D^{N+1} \\
\Delta_S \downarrow & & \downarrow \Delta_D \\
(V_s S^{N-p}) \wedge (V_t S^p) & \xrightarrow{(V_s, i) \wedge (V_t, 1)} & (V_s D^{N-p+1}) \wedge (V_t S^p)
\end{array}
\]

where \( 0 \leq s, t \leq m \) and \( \Delta_S \) is a sum of homeomorphisms \( S^N \cong S^{N-p} \wedge S^p \) over index pairs \( s = t \) and \( \Delta_D \) is the obvious extension of \( \Delta_S \). Let

\[
\Pi_r: (V_s S^{N-p}) \wedge (V_t S^p) \to S^{N-p} \wedge S^p
\]

be the projection onto the \( r \)-th term, \( 0 \leq r \leq m \) and \( \Sigma^p \) be

\[
\Sigma^p: C_\cdot(A \wedge S^p) \xrightarrow{AW} C_\cdot A \otimes C_\cdot S^p \xrightarrow{/s^p} C_\cdot_{-p} A
\]

where \( AW \) is the Alexander–Whitney map [12, p. 112] and \( / \) is the slash product.

Define elements

\[
d_{N-p+1}^r = \Sigma^p \cdot (\Pi_r)_* \cdot (\Delta_D)_* (d_{N+1}) \in C_{N-p+1}(D^{N-p+1})
\]

\[
s_N^{r-p} = \Sigma^p \cdot (\Pi_r)_* \cdot (\Delta_S)_* (s_N) \in C_{N-p}(S^{N-p})
\]
Then $\partial d'_{N-p+1} = i_*(s'_{N-p})$ and each $\{s'_{N-p}\}$ is a generator of $H_{N-p}(S^{N-p})$.

§2 THE $j$-FIBRE OF A $j$-POINCARE NORMAL SPACE

Suppose now that $(X, \xi, \alpha)$ is a normal space, with $\xi$ a $S^{k-1}$-fibration and $-\xi$ a $S^{l-1}$-fibration, which is $j$-Poincaré, where $j \leq \lfloor \frac{n-1}{2} \rfloor - 2$, that is

$$H_*(C_{\alpha x n}) = 0 \quad * \leq 2k + l + j + 1$$

Then there exists a map $\beta: V_+ S^{2k+l+1} \to \Sigma^{k+l} \nu^n+k(X)$, where $0 \leq r \leq m$, and a null-homotopy $H$ of $(\alpha x \cap) \cdot \beta$ such that

$$H \cup_{\alpha x n} C \beta: V_+ (D^{2k+l+2} \cup_{i} CS^{2k+l+1}) \to \Sigma^{k+l} \nu \cup_{\alpha x n} C \Sigma^{k+l} \nu$$

is onto on homology in dimension $2k + l + j + 2$. Here $\nu$ is the normal fibration of $X$. (Note that $K_{j+1}(X)$ is finitely generated and so $\nu$ is finite.)

**Remark** If $K_{j+1}(X)$ is free we may choose $\beta$ and $H$ such that $H \cup_{\alpha x n} C \beta$ is an isomorphism on homology.

The $n + 2k + l + p$-dual of $\beta$ is a map $\beta' \in \{X+, \nu, S^{n-j-1}\}$ such that for $p$ sufficiently large $T \cdot (\Sigma' \beta \wedge 1) \cdot (\Delta_S)_{k,0} \simeq (1 \wedge \beta') \cdot \{(X)_{n+2k+l+p}\}_{0,k}$. We may suppose $T \cdot (\Delta_S)_{k,0} = (\Delta_S)_{0,k}$ and so by naturality of $T$ we have $(1 \wedge \beta') \cdot \{(X)\}_{0,k} \simeq T \cdot (\beta \wedge 1) \cdot (\Delta_S)_{0,k}$. But $\dim X = n + 1 \leq 2(n - \lfloor \frac{n-1}{2} \rfloor) + 1 \leq 2(n-j-2)+1$ and so we may desuspend $\beta'$ to obtain $f: X_+ \to \nu, S^{n-j-1}$. Since
Let $k + l$ be arbitrarily large. We may desuspend the previous homotopy to obtain the homotopy commutative diagram

$$
\begin{array}{ccc}
S^{n+2k+l} & \xrightarrow{\Delta_x \equiv - \xi \otimes \xi} & \{X\}_{n+2k+l} \\
\downarrow \Delta_S & & \downarrow \beta \otimes 1 \\
(V_n S^{2k+j+1}) \wedge (V_n S^{n-j-1}) & \xrightarrow{\beta \otimes 1} & T(\nu \oplus - \xi \otimes \xi) \wedge (V_n S^{n-j-1})
\end{array}
$$

Taking the associated fibration of $f$ gives

$$
\begin{array}{ccc}
F_f & \xrightarrow{\Pi_x \cdot e} & F_f \\
\downarrow & & \downarrow \\
X & \xrightarrow{k} & \tilde{X} \\
\downarrow & \downarrow \\
V_n S^{n-j-1} & \xrightarrow{\beta \otimes 1} & V_n S^{n-j-1}
\end{array}
$$

where the top square commutes up to homotopy. Then $e$ is $n - j - 2 \leq (\lceil \frac{n}{2} \rceil + 1)$-connected. Let $F$ be the homology truncation of $F_f$ at $n$, and $i: F \to F_f \to X$. We shall call $F$ the $j$-fibre of the $j$-Poincaré space $X$.

We proceed to define a normal structure on $F$. Consider the diagram

$$
\begin{array}{ccc}
E \xi^* & \xrightarrow{e} & E \tilde{\xi} \\
\downarrow & & \downarrow \\
F & \xrightarrow{\Pi_x \cdot e} & \tilde{X}
\end{array}
$$

where all squares are pullback squares. Then there is a natural map $T\xi^* \to T\tilde{\xi}$.

**Lemma 2.2.4.** Let

$$
\psi: \Sigma F \to (\tilde{X}_+ \wedge V_n S^{n-j-1}) \cup (1 \wedge f) \cdot \Delta
$$

be the map $\phi_{*,e}$ where $e: F \to \tilde{X}$. Then $\psi$ is $(n + 1)$-connected.

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PROOF By [19, p. 319] there is a map
\[ \phi: F_+ \wedge \bigvee_i S^{n-j-1} \to \tilde{X} \cup CF \]
which is a homology equivalence. Since the spaces are simply-connected \( \phi \) is a homotopy equivalence. Let \( H \) be a homotopy from \( \phi \cdot \phi^{-1} \) to \( 1_{\tilde{X} \cup CF} \). The map \( \phi \) also fits into the following homotopy commutative diagram
\[
\begin{array}{ccc}
\tilde{X}_+ \wedge \bigvee_i S^{n-j-1} & \xrightarrow{\phi} & \tilde{X} \cup CF \\
\downarrow j \wedge 1 & & \downarrow \Delta \\
\tilde{X}_+ \wedge \bigvee_i S^{n-j-1} & &
\end{array}
\]
where \( \Delta \) is the diagonal map on \( \tilde{X} \) taking \( CF \) to the base point. Let \( G \) be the natural null-homotopy of \( i \cdot j: F \to \tilde{X} \cup CF \). We have the commutative diagram
\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{i} & \tilde{X} \cup CF \\
\downarrow 1 & & \downarrow \phi \\
\tilde{X} & \xrightarrow{\phi^{-1} \cdot i} & F_+ \wedge \bigvee_i S^{n-j-1} \\
\downarrow j & & \downarrow \phi^{-1} \cdot G \\
F & \xrightarrow{\phi^{-1} \cdot G} & CF \cup CF \\
\end{array}
\]
Now since \( \phi \) is a homotopy equivalence, by the five-lemma and simple-connectedness \( \phi \cup H \cdot i \cup C_\tilde{X} \) is a homotopy equivalence also. Furthermore
\[
(\phi \cup H \cdot i \cup C_\tilde{X}) \cdot (\phi^{-1} \cdot G \cup C_\tilde{j}) \simeq C_\tilde{X} \cup C_\tilde{j}: \Sigma F \to \tilde{X} \cup CF \cup C\tilde{X}
\]
by the homotopy \( C: (CF \cup CF) \times I) \to \tilde{X} \cup CF \cup C\tilde{X} \) where
\[
C(x \wedge t, s) =
\begin{cases}
H(G(x,t),s) & 0 \leq t \leq \frac{1}{2} \\
H(i \cdot j(x),s) & \frac{1}{2} \leq t \leq \frac{3}{4} \\
j(x) & \frac{3}{4} \leq t \leq 1
\end{cases}
\]
Hence $\phi^{-1} \cdot G \cup Cj : \Sigma F \to F_+ \wedge V_r S^{n-j-1} \cup \tilde{C}X$ is a homotopy equivalence.

We also have

\[
\begin{array}{ccc}
F & \xrightarrow{j} & CF \\
\downarrow & & \downarrow \phi^{-1} \cdot G \\
\tilde{X} & \xrightarrow{\phi^{-1} \cdot i} & F_+ \wedge V_r S^{n-j-1} \\
\downarrow & & \downarrow (j \wedge 1) \cup K \cup C\tilde{1}_X \\
\tilde{X} & \xrightarrow{\Delta_f} & \tilde{X}_+ \wedge V_r S^{n-j-1} \\
\end{array}
\]

where $K$ is a homotopy from $(j \wedge 1) \cdot \phi^{-1} \cdot i$ to $\Delta_f$. Note that $K$ may be written in the form $K' \cdot i$. Since $j \wedge 1$ is $(2n - 2j - 3)$–connected and $j \leq \lceil \frac{n-1}{2} \rceil - 2$ we conclude that $j \wedge 1$ is at least $(n + 2)$–connected. Hence by the five–lemma $(j \wedge 1) \cup K \cup C\tilde{1}_X$ is at least $(n + 1)$–connected.

Now $((j \wedge 1) \cup K \cup C\tilde{1}_X) \cdot (\phi^{-1} \cdot G \cup Cj) = ((j \wedge 1) \cdot \phi^{-1} \cdot G \cup K \cdot j) + (* \cup Cj)$ and so is homotopic to $\psi$ iff $(j \wedge 1) \cdot \phi^{-1} \cdot G \cup K \cdot j$ is null–homotopic. Such a null–homotopy is given by

\[
D(x \wedge t, s) = \begin{cases} 
(j \wedge 1) \cdot \phi^{-1} \cdot G(x, (1 - s)(1 - 2t) + s), & 0 \leq t \leq \frac{1}{2} \\
K' \cdot G(x, s), & \frac{1}{2} \leq t \leq 1 
\end{cases}
\]

Thus we conclude that $\psi$ is at least $(n + 1)$–connected.

**Lemma 2.2.5.** Let

\[
\Psi : \Sigma^{k+l+1} T\xi^* \to (\Sigma^{k+l} \tilde{T}\xi \wedge V_r S^{n-j-1}) \cup (1 \wedge \tilde{f}) \cdot \Sigma^{k+l} \Delta \xi \cdot C\Sigma^{k+l} \tilde{T}\xi
\]

be the map $\phi_{*,T(e)}$ where $T(e) : T\xi^* \to \tilde{T}\xi$. Then $\Psi$ is $(n + 2k + l + 1)$–connected.
PROOF It is sufficient to prove that $\Psi$ is a $(n+2k+l+1)$-homology equivalence.

We have the following commutative diagram

\[
\begin{array}{ccc}
\Sigma^{k+1}T\xi^* & \xrightarrow{\Psi} & (\Sigma^{k+1}T\xi \wedge \vee_r S^{n-j-1}) \cup C\Sigma^{k+1}T\xi \\
\downarrow & & \downarrow \\
\Sigma^{k+1}T\xi^* \wedge SF & \xrightarrow{T(i) \wedge \psi} & \Sigma^{k+1}T\xi \wedge (\tilde{X}_+ \wedge \vee_r S^{n-j-1} \cup \tilde{C}X)
\end{array}
\]

where the vertical maps are the diagonals induced on the mapping cones from $\Delta_{\xi^*}$ and $\Delta_{\xi}$. Since these diagonals induce Thom isomorphisms on homology the result follows immediately from the fact that $\psi$ is at least $(n+1)$-connected.

We also have homotopy equivalences

\[
\begin{align*}
A: & \quad (X_+ \wedge \vee_r S^{n-j-1}) \cup_{(1 \wedge f) \cdot \Delta} CX \rightarrow (\tilde{X}_+ \wedge \vee_r S^{n-j-1}) \cup_{(1 \wedge f) \cdot \Delta} \tilde{C}X \\
B: & \quad (\Sigma^{k+1}T\xi \wedge \vee_r S^{n-j-1}) \cup_{\Sigma^{k+1}T\xi} C\Sigma^{k+1}T\xi \\
& \quad \rightarrow (\Sigma^{k+1}T\xi \wedge \vee_r S^{n-j-1}) \cup_{\Sigma^{k+1}T\xi} C\Sigma^{k+1}T\xi
\end{align*}
\]

defined in the obvious way. For each map $\psi$, $\Psi$, $A$, $B$ we define chain maps $\tilde{\psi}$, $\tilde{\Psi}$, $\tilde{A}$, $\tilde{B}$ where, for example, $\tilde{\psi}$ is just

\[
C_*(*) \oplus C_{*-1}(F) \xrightarrow{\psi \oplus \psi_*} C_*(\tilde{X}_+ \wedge \vee_r S^{n-j-1}) \oplus C_{*-1}(\tilde{X})
\]

the right hand side being the chain mapping cone on $((1 \wedge f) \cdot \Delta)_*$. Furthermore

\[
\begin{array}{ccc}
C_*(*) \oplus C_{*-1}(F) & \xrightarrow{\tilde{\psi}} & C_*(\tilde{X}_+ \wedge \vee_r S^{n-j-1}) \oplus C_{*-1}(\tilde{X}) \\
\omega_* \downarrow & & \downarrow \omega_{(1 \wedge f) \cdot \Delta} \\
\tilde{C}_*(F) & \xrightarrow{\tilde{\psi}_{\cdot \cdot}} & \tilde{C}_*(\tilde{X}_+ \wedge \vee_r S^{n-j-1} \cup \tilde{C}X)
\end{array}
\]
commutes and similarly for $\Psi$, $A$, and $B$.

Let $N = n + 2k + 1$ and $p = n - j - 1$. Consider the diagram

\[
\begin{array}{cccccc}
S^N & \overset{\{X\}_N}{\longrightarrow} & T(\nu \oplus -\xi \oplus \xi) & \overset{\alpha_X \cap}{\longrightarrow} & \Sigma^{k+1}T\xi \\
\Delta_s \downarrow & & (1 \land f) \cdot \Delta_{\nu \oplus -\xi \oplus \xi} \downarrow & & \Sigma^{k+1}(1 \land f) \cdot \Delta_\xi \downarrow \\
(V, S_{N-p}) \land (V, S_\nu) & \overset{\beta \land 1}{\longrightarrow} & T(\nu \oplus -\xi \oplus \xi) \land (V, S_\nu) & \overset{(\alpha_X \cap) \land 1}{\longrightarrow} & \Sigma^{k+1}T\xi \land (V, S_\nu)
\end{array}
\]

where the second square commutes and the first square expresses the $S$–duality between $\beta$ and $f$, and so commutes up to homotopy. Let $H_1$ be such a homotopy.

Let $j: D^{N+1} \cup_i (S^N \times D^1) \to D^{N+1}$ be the obvious inclusion map. Now

\[
D^{N+1} \xrightarrow{\Delta_D} (V, D^{N-p+1}) \land (V, S_\nu) \xrightarrow{H \land 1} \Sigma^{k+1}T\xi \land (V, S_\nu)
\]

is a null–homotopy of $((\alpha_X \cap) \land 1) \cdot (\beta \land 1) \cdot \Delta_s$, and so letting $G' = ((H \land 1) \cdot \Delta_D) \cup ((\alpha_X \cap) \land 1) \cdot H_1$, the commutative diagram

\[
\begin{array}{ccc}
D^{N+1} \cup_i (S^N \times D^1) & \xrightarrow{j} & D^{N+1} \\
\downarrow G' & & \downarrow G \\
\Sigma^{k+1}T\xi \land (V, S_\nu) & \xrightarrow{1} & \Sigma^{k+1}T\xi \land (V, S_\nu)
\end{array}
\]

defines a null–homotopy $G$ of $\Sigma^{k+1}[(1 \land f) \cdot \Delta_\xi \cdot (\alpha_X \cap) \cdot \{X\} = \Sigma^{k+1}[(1 \land f) \cdot \Delta_\xi \cdot (\alpha_X$]. Hence we obtain a map

\[
D^{N+1} \cup_i CS^N \xrightarrow{G \cup CS^{k+1} \alpha_X} \Sigma^{k+1}T\xi \land (V, S_{n-j-1}) \cup_{\Sigma^{k+1}(1 \land f) \cdot \Delta_\xi} CS^{k+1}T\xi
\]

Let $\phi: S^{n+1} \to D^{N+1} \cup CS^N$ be any homeomorphism. Thus we have a homotopy class $\alpha_\phi : S^{n+k} \to T\xi^*$ where
\[ \Psi \cdot \Sigma^{k+1} \alpha_f \simeq B \cdot (G \cup C \Sigma^{k+1} \alpha_x) \cdot \phi \]

Furthermore

\[
\begin{array}{ccc}
\Sigma^{k+1} T \xi & \xrightarrow{B^{-1} \cdot \Psi} & \Sigma^{k+1} T \xi \wedge (V, \Sigma^{n-j-1}) \cup \Sigma^{k+1} T \xi \\
\downarrow & & \downarrow \Pi \\
\Sigma^{k+1} T \xi & \xrightarrow{\Pi} & \Sigma^{k+1} T \xi
\end{array}
\]

where \( \Pi \) is the collapse map, homotopy commutes.

Thus

\[
\Sigma^{k+1} T(i) \cdot \Sigma^{k+1} \alpha_f = \Pi \cdot B^{-1} \cdot \Psi \cdot \Sigma^{k+1} \alpha_x = \Pi \cdot (G \cup C \Sigma^{k+1} \alpha_x) = \Sigma^{k+1} \alpha_x
\]

which desuspending gives

\[ T(i) \cdot \alpha_f \simeq \alpha_x \]

and so \((F, \xi^*, \alpha_F)\) is a normal space for which \(i\) is a normal map.

§3 THE KERNELS OF THE \(j\)-FIBRE

We must now calculate \( K_* (F) \), and to this end we first evaluate \([F] \in H_n (F)\)

where \([F] = (\alpha_F)_* (s_{n+k})_* \cap U^*\). Now there is a commutative diagram

\[
\begin{array}{ccc}
C_* (D^{N+1}) \oplus C_{*-1} (S^N) & \xrightarrow{G_* \oplus (\Sigma^{k+1} \alpha_x)_*} & C_* (\Sigma^{k+1} T \xi \wedge (V, \Sigma^{n-j-1})) \oplus C_{*-1} \Sigma^{k+1} T \xi \\
\downarrow \omega_i & & \downarrow \omega_{\Sigma^{k+1} (1 \wedge j)} \\
\tilde{C}_* (D^{N+1} \cup_i C S^N) & \xrightarrow{(G \cup C \Sigma^{k+1} \alpha_x)_*} & \tilde{C}_* (\Sigma^{k+1} T \xi \wedge (V, \Sigma^{n-j-1}) \cup C \Sigma^{k+1} T \xi)
\end{array}
\]

Choosing a generator \( \{s_{N+1}\} \in H_{N+1} (S^{N+1}) \) such that \( \phi_* (s_{N+1}) = \omega_i (d_{N+1} \oplus s_N) \) we therefore have

\[ (G \cup C \Sigma^{k+1} \alpha_x)_* \cdot \phi_* (s_{N+1}) = \omega_{\Sigma^{k+1} (1 \wedge j)} \cdot (G_* (d_{N+1}) \oplus (\Sigma^{k+1} \alpha_x)_* (s_N)) \]

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Furthermore

\[
\begin{align*}
C_\ast(\ast) \oplus C_{\ast-1}(\Sigma^{k+i}T\tilde{\xi}) & \xrightarrow{\Omega \cdot \tilde{\psi}} (\otimes, C_{\ast-n+k+i+j+1}(\Sigma^{k+i}T\tilde{\xi})) \oplus C_{\ast-1}(\Sigma^{k+i}T\tilde{\xi}) \\
& \downarrow \oplus \cap U^* \\
C_\ast(\ast) \oplus C_{\ast-2k-i-1}(F) & \xrightarrow{\Omega \cdot \tilde{\psi}} (\otimes, C_{\ast-n-2k-1+i+j+1}X) \oplus C_{\ast-2k-i-1}X \\
\end{align*}
\]

commutes, where

\[
\Omega = (\otimes, \Sigma^{-j-1}(\Pi_1, \ldots, \Pi_r)) \oplus 1
\]

and so

\[
\begin{align*}
\tilde{A}^{-1} \cdot \Omega \cdot \tilde{\psi}([F]) &= \tilde{A}^{-1} \cdot \Omega \cdot \tilde{\psi}((\Sigma^{k+i}\alpha_F)_*(S_N) \cap U^*) \\
&= \tilde{A}^{-1} \cdot ((\otimes, \cap \tilde{U}) \oplus \cap \tilde{U}) \cdot \Omega \cdot \tilde{\psi}((\Sigma^{k+i}\alpha_F)_*(S_N)) \\
&= \tilde{A}^{-1} \cdot ((\otimes, \cap \tilde{U}) \oplus \cap \tilde{U}) \cdot \Omega \cdot \omega^{-1}_{(1\wedge f)} \cdot \psi_\ast \cdot \omega_\ast((\Sigma^{k+i}\alpha_F)_*(S_N)) \\
&= \tilde{A}^{-1} \cdot ((\otimes, \cap \tilde{U}) \oplus \cap \tilde{U}) \cdot \Omega \cdot \omega^{-1}_{(1\wedge f)} \cdot \psi_\ast((\Sigma\alpha_F)_*(S_{N+1})) \\
&= \tilde{A}^{-1} \cdot ((\otimes, \cap \tilde{U}) \oplus \cap \tilde{U}) \cdot \Omega \cdot \omega^{-1}_{(1\wedge f)} \cdot B_{\ast}(G \cup C\Sigma^{k+i}\alpha_x)_\ast \cdot \phi_\ast(S_{N+1}) \\
&= \tilde{A}^{-1} \cdot ((\otimes, \cap \tilde{U}) \oplus \cap \tilde{U}) \cdot \Omega \cdot \omega^{-1}_{(1\wedge f)} \cdot B_{\ast}(\omega_{(1\wedge f)} \cdot \Delta_\xi) \\
& \quad \cdot (G_{\ast}(d_{N+1}) \oplus (\Sigma^{k+i}\alpha_x)_*(S_N))
\end{align*}
\]
\[
A^{-1} \cdot ((\Theta, \cap \bar{U}) \oplus \cap \bar{U}) \cdot \Omega \cdot B(G_*(d_{N+1}) \oplus (\Sigma_{k+1}\alpha_X)_*(S_N))
\]
\[
= ((\Theta, \cap U) \oplus \cap U) \cdot \Omega(G_*(d_{N+1}) \oplus (\Sigma_{k+1}\alpha_X)_*(s_N))
\]

Now if \(EZ: C_*(S^N) \otimes C_*(D^1) \to C_*(S^N \times D^1)\) is the Eilenberg–Zilber map then

\[
[(\Sigma_{k+1}(1 \wedge f) \cdot \Delta_\nu \cdot \{X\}), - ((\beta \wedge 1) \cdot \Delta_S)_*(s_N) = \partial((H_1)_* \cdot EZ(s_N \otimes d_1))
\]

But \(j(d_{N+1} + EZ(s_N \otimes d_1)) \in C_{N+1}(D^{N+1})\) is just a triangulation of \(d_{N+1}\) and so

\[
G_*(d_{N+1}) \oplus (\Sigma_{k+1}\alpha_X)_*(s_N) =
\]

\[
((H \wedge 1)_* \cdot (\Delta_D)_*(d_{N+1}) + ((\alpha_X \cap) \wedge 1)_* \cdot (H_1)_* \cdot EZ(s_N \otimes d_1)) \oplus (\alpha_X)_*(s_N)
\]

up to a boundary. Thus

\[
\Sigma^{n-j-1} \cdot (\Pi_r)_* \cdot (H \wedge 1)_* \cdot (\Delta_D)_*(d_{N+1})
\]

\[
= \Sigma^{n-j-1} \cdot (H \wedge 1)_* \cdot (\Pi_r)_* \cdot (\Delta_D)_*(d_{N+1})
\]

\[
= H_* \cdot \Sigma^{n-j-1} \cdot (\Pi_r)_* \cdot (\Delta_D)_*(d_{N+1})
\]

\[
= H_*(d_{2k+1+j+2})
\]

Furthermore letting \(\gamma_r = \Sigma^{n-j-1} \cdot (\Pi_r)_* \cdot (H_1)_* \cdot EZ(s_N \otimes d_1)\) we have

\[
\partial \gamma_r = \Sigma^{n-j-1} \cdot (\Pi_r)_*[\Sigma_{k+1}(1 \wedge f)_* \cdot (\Delta_\nu)_* \cdot \{X\}_*(s_N) - (\beta \wedge 1)_* \cdot (\Delta_S)_*(s_N)]
\]

\[
= \beta_*([s_{2k+1+j+1}) - \{X\} \cap f^*([\Pi_r]^*(s_{n-j-1}))
\]

Then
\[
\tilde{A}^{-1} \cdot \Omega \cdot \tilde{\psi}([F]) = (\Theta_r (\cap U)(H_\ast(d_{2k+1+j+2}) + (\alpha \cap)(\gamma_r))) \oplus [X]
\]

from which it follows that \(e_\ast [F] = k_\ast [X]\).

We now calculate the kernels of \(F\). We have the Puppe sequence

\[
\begin{array}{ccc}
\Sigma^{k+1}T\nu & \xrightarrow{\alpha X \cap} & \Sigma^{k+1}T\xi \\
& \beta \downarrow & \downarrow H \cup C\beta \\
& j \downarrow & H \cup C\beta \\
& & \Sigma^{k+1}T\nu
\end{array}
\]

And hence long exact sequences

\[
\begin{array}{ccc}
H^{n-4}X & \xrightarrow{\{X\} \cap} & H_{q+2k+1}(\Sigma^{k+1}T\nu) \\
& \downarrow (\alpha X \cap) & \downarrow (\alpha X \cap) \\
H_{q}X & \xrightarrow{\cap U} & H_{q+2k+1}(\Sigma^{k+1}T\xi) \\
& \tilde{i} \downarrow & \downarrow i \\
K_{q-1}X & \xrightarrow{\tilde{p}} & H_{q+2k+1}(C_\ast(\Sigma^{k+1}T\xi) \oplus C_{\ast-1}(\Sigma^{k+1}T\nu)) \\
& \tilde{p} \downarrow & \downarrow p \\
H^{n-q+1}X & \xrightarrow{\{X\} \cap} & H_{q}(C_{\ast-1}(\Sigma^{k+1}T\nu))
\end{array}
\]

where \(i\) and \(p\) are the obvious inclusion and projection maps, and \(\tilde{i} = i \cdot (\cap U)^{-1}\), \(\tilde{p} = (\{X\} \cap)^{-1} \cdot p\). Consider Diagram (\(t\)) (overleaf) where \(q = (\Pi X)_\ast \cdot e_\ast \cdot ([F] \cap)\) and

\[
\omega: H^{n-j-1}(\bigvee_r S^{n-j-1}) \longrightarrow H_{j+2}(C_\ast(\Sigma^{k+1}T\xi) \oplus C_{\ast-1}(\Sigma^{k+1}T\nu))
\]

and the right hand sequence is the Serre exact sequence.
\begin{center}
\textbf{Diagram (†)}
\end{center}

\begin{align*}
K_{j+3}(X) & \quad \longleftarrow \quad 0 \\
\tilde{p} \downarrow & \\
H^{n-j-3}(X) & \quad \longleftarrow \quad H^{n-j-3}(\tilde{X}) \\
([X \cap])_{j+3} & \quad \downarrow \quad 1 \\
H_{j+3}(X) & \quad \longleftarrow \quad H^{n-j-3}(F) \\
\sim \tilde{\iota} \downarrow & \\
K_{j+2}(X) & \quad \longleftarrow \quad 0 \\
\tilde{p} \downarrow & \\
H^{n-j-2}(X) & \quad \longleftarrow \quad H^{n-j-2}(\tilde{X}) \\
([X \cap])_{j+2} & \quad \downarrow \quad 3 \\
H^{j+2}(X) & \quad \longleftarrow \quad H^{n-j-2}(F) \\
\sim \tilde{\iota} \downarrow & \\
K_{j+1}(X) & \quad \longleftarrow \quad H^{n-j-1}(\bigoplus S^{n-j-1}) \\
\tilde{p} \downarrow & \\
H^{n-j-1}(X) & \quad \longleftarrow \quad H^{n-j-1}(\tilde{X}) \\
([X \cap])_{j+1} & \quad \downarrow \quad 6 \\
H_{j+1}(X) & \quad \longleftarrow \quad H^{n-j-1}(F) \\
\sim \tilde{\iota} \downarrow & \\
K_{j}(X) & \quad \longleftarrow \quad 0 \\
\tilde{p} \downarrow & \\
H^{n-j}(X) & \quad \longleftarrow \quad H^{n-j}(\tilde{X})
\end{align*}
THEOREM 3.1.1. Diagram (†) is commutative

PROOF Clearly all the squares commute except possibly 1 through 7. Now
\[ q \cdot e^* = (\Pi_X) \cdot e \cdot ([F]\cap) \cdot e^* = \Pi_*([F]\cap) = \Pi_*([X]\cap) = ([X]\cap) \cdot k^* \]
and so squares 1, 3, 6 commute. It immediately follows that squares 2 and 7 commute. Also

\[ \tilde{p} \cdot \omega(\oplus n_r s^{n-j-1}) = ([X]\cap) \cdot p(\Sigma r_n_r(H_* (d_{2k+j+2} + (s_{2k+j+1})))
\]
\[ = ([X]\cap) \cdot (\Sigma r_n_r(\beta_*(s_{2k+j+1}))
\]
\[ = f^*(\oplus n_r s^{n-j-1})
\]
\[ = k^* \cdot f^*(\oplus n_r s^{n-j-1}) \]

hence square 5 commutes.

Now we need only show that square 4 commutes.

Now \( r^* : H^{n-j-2} F \to H^{n-j-1}(\bigvee_r S^{n-j-1}) \) is given by

\[ H^{n-j-2} F \xrightarrow{i^*} H^{n-j-2}(\Omega \Sigma \bigvee_r S^{n-j-1}) \xrightarrow{\Sigma \cdot (Id^*)^*} H^{n-j-1}(\bigvee_r S^{n-j-1}) \]

where \( Id^* : \bigvee_r S^{n-j-2} \to \Omega \Sigma \bigvee_r S^{n-j-1} \) is the adjoint map of the identity on \( \Sigma \bigvee_r S^{n-j-1} \). But \( (Id^*)^* \) in the Tor representation of \( H^*(\bigvee_r S^{n-j-1}) \) is known, (see [12, p.241]), and hence by the naturality of Tor we may find \( r^* \) in the Tor representation of \( H^* F \). We think of \( \text{Hom}(((\oplus n_{n+j+1} X) \oplus C_{-1} \tilde{X}), \mathbb{Z}) \)
as a subcomplex of the chain complex whose homology is Tor, (truncate the
canonical injective resolution at the second stage). Thus we may show that if the map

\[ \tilde{\tau} : H^{n-j-2}(\oplus C_{* - n + j + 1} W_X) \oplus C_{*-1} \tilde{X} \rightarrow H^{n-j-1}(V, S^{n-j-1}) \]

is defined by \( \tilde{\tau}(\oplus, n_r) \oplus x = \oplus, n_r, s^{n-j-1} \), where \( (\oplus, n_r) \oplus x \) is a cocycle in \( \text{Hom}(\oplus, C_0 \tilde{X}) \oplus C_{*-j-2} \tilde{X}, Z) \), then \( r^* = \tilde{\tau} \cdot \tilde{\psi} \).

We have the commutative diagram

\[
\begin{array}{ccc}
C_* F & \xrightarrow{\Omega \cdot \tilde{\psi}} & (\oplus, C_{*-n+j+2}(\tilde{X})) \oplus C_{*}(\tilde{X}) \\
(1 \otimes e) \cdot \Delta & \downarrow & (\oplus, \Delta) \oplus \Delta \\
C_*(F) \otimes C_{*}(\tilde{X}) & \xrightarrow{\Omega \cdot \tilde{\psi} \otimes 1} & (\oplus, C_{*-n+j+2}(\tilde{X})) \oplus C_{*}(\tilde{X}) \otimes C_{*}(\tilde{X})
\end{array}
\]

and so cap-products

\[
H^*(F) \xleftarrow{\tilde{\psi}^* \cdot \Omega^*} H^*(\oplus, C_{*-n+j+2}(\tilde{X}) \oplus C_{*}(\tilde{X})) \xrightarrow{A^*} H^*(\oplus, C_{*-n+j+2}(\tilde{X}) \oplus C_{*}(X))
\]

\[
\downarrow e_*([F] \cap) \quad \downarrow (\Omega \cdot \tilde{\psi}[F]) \cap \quad (A^{-1} \cdot \Omega \cdot \tilde{\psi}[F]) \cap
\]

\[
H_*(\tilde{X}) \quad \xrightarrow{(\Pi X)_*} \quad H_*(X)
\]

Thus we need only show that

\[
H^{n-j-2}(\oplus C_{*-n+j+2}(X)) \oplus C_{*}(X)) \xrightarrow{\tilde{\tau}} H^{n-j-1}(V, S^{n-j-1})
\]

\[
\downarrow ((A^{-1} \cdot \tilde{\psi}[F]) \cap)
\]

\[
H_{j+2}(X) \xrightarrow{\tilde{i}} H_{j+2}(C_{*}(\Sigma^{k+1} T\xi) \oplus C_{*-1}(\Sigma^{k+1} T\nu))
\]

commutes to show that square 4 commutes.

So suppose \( (\oplus, n_r) \oplus x^* \in (\oplus, C^0 X) \oplus C^{-j-2} X \) is any cocycle, then by definition \( \delta x^* = -\Sigma_r f^*(n_r s^{n-j-1}) \). Then

\[
\beta_*(\oplus, n_r, s^{2k+1+j+1}_r) = -\partial[\{X\} \cap x^* + \Sigma_r n_r \gamma_r]
\]

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Furthermore

\[
\tilde{\iota}(\Delta[(\oplus, \cap U)(H, d_{2k+1+j+2}) + (\alpha_X \cap \gamma_r)] \oplus [X]/(\oplus, n, \oplus x^*))
\]

\[
= \tilde{\iota}((\cap U) \cdot (\alpha_X \cap \gamma_r) \cdot \{X\} \cap x^* + \Sigma_r n_r (\cap U)(H, d_{2k+1+j+2}) + (\alpha_X \cap \gamma_r))
\]

\[
= \iota(\cap U)^{(-1)} \cdot (\cap U)((\alpha_X \cap \gamma_r) \cdot \{X\} \cap x^* + \Sigma_r n_r (H, d_{2k+1+j+2}) + (\alpha_X \cap \gamma_r))
\]

\[
= \iota(1 + \partial G + G\partial)((\alpha_X \cap \gamma_r) \cdot \{X\} \cap x^* + \Sigma_r n_r (H, d_{2k+1+j+2}) + (\alpha_X \cap \gamma_r))
\]

\[
= \iota((\alpha_X \cap \gamma_r) \cdot \{X\} \cap x^* + \Sigma_r n_r (H, d_{2k+1+j+2}) + (\alpha_X \cap \gamma_r))
\]

\[
+ \iota \cdot G((\alpha_X \cap \gamma_r) \cdot \partial(\{X\} \cap x^*) + \partial(\Sigma_r n_r (H, d_{2k+1+j+2}) + (\alpha_X \cap \gamma_r)))
\]

up to a boundary

\[
= [(\alpha_X \cap \gamma_r) \cdot \{X\} \cap x^* + \Sigma_r n_r (H, d_{2k+1+j+2}) + (\alpha_X \cap \gamma_r)] \oplus 0
\]

\[
+ \iota \cdot G([- (\alpha_X \cap \gamma_r) \cdot \beta_r n_r (d_{2k+1+j+2}) + \partial(\Sigma_r n_r H, d_{2k+1+j+2})])
\]

The term in the last square bracket is zero and so we are left with

\[
[(\alpha_X \cap \gamma_r) \cdot \{X\} \cap x^* + \Sigma_r n_r (H, d_{2k+1+j+2}) + (\alpha_X \cap \gamma_r)] \oplus 0
\]

\[
= H_r (\Sigma_r n_r, d_{2k+1+j+2}) \oplus \beta_r (\Sigma_r n_r, d_{2k+1+j+2}) - \partial(0 \oplus \{X\} \cap x^* + \Sigma_r n_r \gamma_r)
\]

\[
= \tau(\oplus n_r \oplus x^*)
\]

up to a boundary. Hence square 4 commutes and Theorem 3.1.1. is proved.

§4 Normal Surgery Below the Middle Dimension

In the following lemmas we assume \(X\) is a normal space which is \(j\)-Poincaré

and \(F\) is the \(j\)-fibre with the above normal structure together with the \(|\frac{n}{2} + 1|\)-
LEMMA A If \( j \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 2 \) and \( K_{-2}(X) = \cdots = K_j(X) = 0 \) then \( K_j(F) = K_{j+1}(F) = 0 \). Furthermore \( K_i(F) = K_i(X) \), \( j + 3 \leq i \leq n - j - 4 \) if \( j \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 3 \).

LEMMA B If \( j \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 3 \) and \( K_{-2}(X) = \cdots = K_j(X) = 0 \) and \( K_{j+2}(X) = 0 \), then \( K_j(F) = K_{j+1}(F) = 0 \) and \( K_{j+2}(F) \) is free. Furthermore \( K_i(F) = K_i(X) \), \( j + 3 \leq i \leq n - j - 4 \) if \( j \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 3 \).

LEMMA C If \( j \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 3 \) and \( K_{-2}(X) = \cdots = K_j(X) = 0 \), \( K_{j+1}(X) \) is free and \( K_{j+2}(X) = 0 \), then \( K_j(F) = K_{j+1}(F) = K_{j+2}(F) = 0 \). Furthermore \( K_i(F) = K_i(X) \), \( j + 3 \leq i \leq n - j - 4 \) if \( j \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 3 \).

PROOF OF LEMMA A Since \( K_j(X) = 0 \) and \( \omega \) is onto, the five-lemma implies that \( i_{j+1} \cdot ([F] \cap)_{j+1} \) is an isomorphism and so \( ([F] \cap)_{j+1} \) is an isomorphism since \( i \) is \( n - j - 2 > j + 2 \) connected. Further, by the five-lemma \( i_{j+2} \cdot ([F] \cap)_{j+2} \) is onto and so \( ([F] \cap)_{j+2} \) is onto. Hence \( K_{j+1}(F) = 0 \).

We also have the diagram

\[
\begin{array}{cccc}
0 = K_j(X) \rightarrow H^{n-j}(X) & \rightarrow & H_j(X) \rightarrow & K_{j-1}(X) = 0 \\
\downarrow i^* & & & \uparrow i_* \\
H^{n-j}(F) & \rightarrow & H_j(F)
\end{array}
\]

and so \( ([F] \cap)_j \) is an isomorphism. But \( ([F] \cap)_{j+1} \) is an isomorphism and so we conclude that \( K_j(F) = 0 \).
As for the second statement this follows from the fact that $e$ is $(n-j-2)$-connected.

**PROOF OF LEMMA B** By lemma A we have $K_j(F) = K_{j+1}(F) = 0$. Since $K_{j+2}(X) = 0$ it follows that $([X] \cap)_{j+3}$ is onto and $([X] \cap)_{j+2}$ is into. Also

\[
\begin{array}{cccc}
H^{n-j-3}(X) & \xrightarrow{([X] \cap)_{j+2}} & H_{j+3}(X) \\
i^* & & i_* & \\
H^{n-j-3}(F) & \xrightarrow{([F] \cap)_{j+2}} & H_{j+3}(F)
\end{array}
\]

commutes and so $([F] \cap)_{j+3}$ is onto and $K_{j+2}(F) \cong \ker([F] \cap)_{j+2}$. We have

\[
\begin{array}{cccc}
0 & \longrightarrow & H^{n-j-2}(X) & \xrightarrow{i^*} & H^{n-j-2}(F) & \xrightarrow{\tau^*} & \text{Im} \tau^* & \longrightarrow & 0 \\
& & ([X] \cap)_{j+2} & \downarrow & ([F] \cap)_{j+2} & \downarrow & \\
& & H_{j+2}(X) & \xleftarrow{i_*} & H_{j+2}(F)
\end{array}
\]

There is a splitting $r$ of $\tau^*$ since $\text{Im} \tau^*$ is free, and hence an isomorphism

\[
H^{n-j-2}(X) \oplus \text{Im} \tau^* \xrightarrow{(i^*, r)} H^{n-j-2}(F)
\]

Suppose $x \oplus y \in \ker(((F) \cap)_{j+2} \cdot (i^*, r))$ is a torsion element. Then $y = 0$ and so $([F] \cap)_{j+2} \cdot i^*(x) = 0$. Thus $([X] \cap)_{j+2}(x) = 0$, but $([X] \cap)_{j+2}$ is into and so $x = 0$. Hence $\ker(((F) \cap)_{j+2} \cdot (i^*, r))$ is free and so $\ker(((F) \cap)_{j+2}) \cong K_{j+2}(X)$ is also free.

**PROOF OF LEMMA C** Since $K_{j+1}(X)$ is free we may choose $H \cup C\beta$:

\[
\bigvee_
u S^{k+i+j+1} \to \Sigma^{k+i} T \sqcup \bigcup_{\alpha \in \Lambda} C \Sigma^{k+i} T \nu \to \text{induce an isomorphism on homology, and so } \omega \text{ may be taken to be an isomorphism. By Lemma A } K_j(F) = 0,
\]
\[ K_{j+1}(F) = 0. \] Further by the five-lemma \( i_{j+2} \cdot ([F] \cap)_{j+2} \) is an isomorphism and hence \( ([F] \cap)_{j+2} \) is also. Similarly \( ([F] \cap)_{j+3} \) is onto and so \( K_{j+2}(F) = 0. \]  

**Proof of Theorem 2.1.3** The proof is by induction \(-1 \leq r \leq \left[ \frac{n-1}{2} \right] - 2.\) The induction hypothesis at \( r \) is:

Let \( X \) be a normal space

\[(H_1(r)): \text{ If } i \leq r \text{ and } K_i(X) = K_{i+1}(X) = 0 \text{ then there is a normal space } W \text{ together with a } ([\frac{n}{2}] + 1)\text{-connected normal map } W \to X \text{ such that } K_*(W) = 0, \]
\[ * \leq i - 1, \text{ } K_i(W) \text{ is free, } K_{i+1}(W) = 0 \text{ and } K_*(W) = K_*(X) \text{ if } i + 2 \leq * \leq \left[ \frac{n-1}{2} \right] - 1. \]

\[(H_2(r)): \text{ There is a normal space } W \text{ together with a } ([\frac{n}{2}] + 1)\text{-connected normal map } W \to X \text{ such that } K_{-2}(W) = \cdots = K_{r-1}(W) = 0, \]
\[ K_*(W) = K_*(X), \text{ } r + 1 \leq * \leq \left[ \frac{n-1}{2} \right] - 1. \]

\[(H_3(r)): \text{ If } i \leq r - 1 \text{ and } K_i(X) = K_{i+1}(X) = 0 \text{ then there is a space } W \text{ together with a } ([\frac{n}{2}] + 1)\text{-connected normal map } W \to X \text{ such that } K_{-2}(W) = \]
\[ \cdots = K_{i+1}(W) = 0, \text{ and } K_*(W) = K_*(X), i + 2 \leq * \leq \left[ \frac{n-1}{2} \right] - 1. \]

where \( W \) is a space coming with a \( ([\frac{n}{2}] + 1)\text{-connected normal map } f : W \to X.\)

Using homology truncations it is easily seen that \( H_1(-1), H_2(-1), H_3(-1) \) are true. Suppose now that \( H_1(r), H_2(r), H_3(r) \) are true, \( r \leq \left[ \frac{n-1}{2} \right] - 2. \)
1) $H_1(r), H_2(r), H_3(r) \implies H_1(r + 1)$

Suppose $K_{r+1}(X) = K_{r+2}(X) = 0$, then $H_2(r)$ implies there is a $W_1$ such that $K_{r-2}(W_1) = \cdots = K_{r-1}(W_1) = K_{r+1}(W_1) = K_{r+2}(W_1) = 0$ and $K_*(W_1) = K_*(X) \ r + 1 \leq \ * \leq \left[\frac{n-1}{2}\right] - 1$. Lemma B applied at $j = r - 1$ implies there is a $W_2$ such that $K_{r-2}(W_2) = K_r(W_2) = 0$, $K_{r+1}(W_2)$ is free, $K_{r+2}(W_2) = 0$ and $K_*(W_2) = K_*(W_1) \ r + 2 \leq \ * \leq \left[\frac{n-1}{2}\right] - 1 \leq n - r - 3$. Finally $H_3(r)$ implies there is a $W$ such that $K_{r-2}(W) = \cdots = K_r(W) = 0$, $K_{r+1}(W)$ is free, $K_{r+2}(W) = 0$ and $K_*(W) = K_*(W_2) \ r + 1 \leq \ * \leq \left[\frac{n-1}{2}\right] - 1$.

2) $H_1(r), H_2(r), H_3(r) \implies H_2(r + 1)$

Suppose $K_{r-2}(X) = \cdots = K_{r-1}(X) = 0$, then Lemma A applied at $j = r - 1$ implies there is a $W_1$ such that $K_{r-1}(W_1) = K_r(W_1) = 0$ and $K_*(W_1) = K_*(X) \ r + 2 \leq \ * \leq \left[\frac{n-1}{2}\right] - 1 \leq n - r - 3$. $H_3(r)$ implies then that there is a $W$ such that $K_{r-2}(W) = \cdots = K_r(W) = 0$ and $K_*(W) = K_*(W_1) \ r + 2 \leq \ * \leq \left[\frac{n-1}{2}\right] - 1$.

3) $H_1(r), H_2(r), H_3(r) \implies H_3(r + 1)$

Suppose $K_r(X) = K_{r+1}(X) = 0$, then $H_1(r)$ implies there is a $W_1$ such that $K_*(W_1) = 0$, $* \leq r - 1$, $K_r(W_1)$ is free, $K_{r+1}(W_1) = 0$ and $K_*(W_1) = K_*(X)$.
\[ r + 2 \leq r + 2 \leq \ast \leq \left[ \frac{n-1}{2} \right] - 1. \] Lemma C applied at \( j = r - 1 \) implies there is a \( W_2 \) such that \( K_{r-1}(W_2) = K_r(W_2) = K_{r+1}(W_2) = 0 \) and \( K_*(W_2) = K_*(W_1) \).

\[ r + 2 \leq \ast \leq n - r - 4 \leq \left[ \frac{n-1}{2} \right] - 1. \] Finally \( H_3(r) \) implies there is a \( W \) such that \( K_{-2}(W) = \cdots = K_{r+1}(W) = 0 \) with the same equalities in the higher kernels.

By induction we conclude that \( H_1(\left[ \frac{n-1}{2} \right] - 1), H_2(\left[ \frac{n-1}{2} \right] - 1) \) and \( H_3(\left[ \frac{n-1}{2} \right] - 1) \) are true. But \( H_2(\left[ \frac{n-1}{2} \right] - 1) \) gives \( W_1 \) such that \( K_{-2}(W_1) = \cdots = K_{\left[ (n-1)/2 \right] - 2}(W_1) = 0 \), and then Lemma A applied at \( j = \left[ \frac{n-1}{2} \right] - 2 \) gives a \( W_2 \) such that \( K_{\left[ (n-1)/2 \right] - 2}(W_2) = K_{\left[ (n-1)/2 \right] - 1}(W_2) = 0 \). Finally \( H_3(\left[ \frac{n-1}{2} \right] - 1) \) gives a \( W \) such that \( K_{-2}(W) = \cdots = K_{\left[ (n-1)/2 \right] - 1}(W) = 0 \), that is \( W \) is \( (\left[ \frac{n-1}{2} \right] - 1) \)-Poincaré. \( \blacksquare \)
CHAPTER THREE

§1 EXTENSION OF COFIBRATIONS UP TO HOMOTOPY

The extension of cofibrations up to homotopy results in [13] were an essential step in the theory of Poincaré surgery in [14].

**DEFINITION** Let \( f: X \to Y \). We say \( f \) extends up to homotopy as a cofibration if there is a map \( g: A \to X \) and a null-homotopy \( H \) of \( f \cdot g \) such that the induced map \( \phi: C_\alpha \to Y \) is a homotopy equivalence.

Recall that as part of the proof of Poincaré surgery we wished to kill the generators of the lowest non-zero kernel of a normal space by 'removing' the dual cells \( f: X \to V_r S^{n-j-1} \). The approach in [14] was based on the recognition that 'removing' a cell, as required in surgery, was achieved by extending \( f \) up to homotopy as a cofibration, if this could indeed be done. Theorem 1.1 in [13] is

'THOREM 3.1.1.' If \( f: X \to Y \) is 2-connected, \( X, Y \) are 1-connected and \( \dim f \leq 2\text{con}Y + \text{conf} \), then \( f \) extends up to homotopy as a cofibration iff \( \alpha: X \to X \cup CF \) factors through \( f \).

Here \( F \) is the homotopy fibre of \( f \) and \( \alpha \) is the inclusion map.
The map \( f: X \to \vee_r S^{n-j-1} \) satisfies the connectivity and dimension conditions and the factoring condition can easily be seen to be equivalent to the condition that \((1 \land f) \cdot \Delta: X \to X \land (\vee_r S^{n-j-1})\) is null-homotopic, which is indeed true since \( f \) is dual to the generators of a kernel. Hence, if true, the theorem would be sufficient to extend \( f \) up to homotopy as a cofibration.

The map \((1 \land f) \cdot \Delta\) is just \( \bigoplus_r f^*(s_r^{n-j-1}) \cap \colon H_* X \to \bigoplus_r H_{*-n+j+1} X \) on homology, and can be thought of as detecting generalised Whitehead products of 'cells \( S^{n-j-1} \) with other cells. Such higher cells 'tie' the \( 'S^{n-j-1} \)-cells' into the space preventing their removal by extension of \( f \) as a cofibration. Cells whose attaching maps are triple Whitehead products of the \( 'S^{n-j-1} \)-cells' with other cells will in the same way also obstruct the extension of \( f \) as a cofibration. Such cells may be detected by the relevant Massey higher product in \( H^* X \). The conditions of theorem 3.1.1. make no reference to higher products, and in fact these products may be used to construct a counter-example to the theorem.

Our approach to the extension of cofibrations is motivated by the following theorem of Berstein and Hilton

\textbf{Theorem 3.1.2.} [2, Theorem A] If \( Y \) is a co-H-space and \( \dim Y \leq 3 \text{co} Y \) then \( Y \) is, up to homotopy, a suspension.

The desuspension of a space \( Y \) is equivalent to the extension of \( * \to Y \) as a cofibration, and so we think of desuspension as a special case of extension as a
cofibration. Similarly the property of \( Y \) being a co-H-space is a special case of the property of the map \( f \) being weakly-coprincipal.

**DEFINITION** Let \( f: X \to Y \). Consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow f & & \downarrow i_2 \\
Y & \xrightarrow{i_1} & M_{f,t} \\
p & \downarrow & \downarrow q \\
C_f & \xrightarrow{\phi_{i,t}} & C_i
\end{array}
\]

where \( p \) and \( q \) are the obvious inclusion maps. Then \( f \) is said to be weakly-coprincipal or w-coprincipal if there is a map \( r: C_f \to M_{f,t} \) such that 1. \( q \cdot r \simeq \phi_{i,t} \) and 2. \( \nabla \cdot r \simeq * \), where \( \nabla: M_{f,t} \to Y \) is the fold map \( 1_Y \cup (f \cdot \Pi_1) \cup 1_Y \).

See [20, 2.4.6.] for the dual notion of w-principality.

**LEMMA 3.1.3.** The map \( f: X \to Y \) is w-coprincipal if there is a homotopy pushout

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow f & & \downarrow i_2 \\
\tilde{Y} & \xrightarrow{\tilde{i}_1} & \tilde{M} \\
p & \downarrow & \downarrow \tilde{q} \\
\tilde{C}_f & \xrightarrow{\phi_{\tilde{i},t}} & \tilde{C}_i
\end{array}
\]
where $q$ is the obvious inclusion map, and a map $r: C_f \to \tilde{M}$ such that

1. $q \cdot r \simeq \phi_{i,f}$
2. $\nabla \cdot \Omega \cdot r \simeq *$

where $\Omega: \tilde{M} \to M_{f,f}$ is the homotopy equivalence induced by virtue of $\tilde{M}$ being a homotopy pushout.

**Proof** Let $r = \Omega \cdot r$. Then the proof consists of checking the definitions, and is omitted.

We obtain the following results dual to those in [20, pp. 54–57].

**Lemma 3.1.4.** The map $f: X \to Y$ is w-coprincipal iff there exists a homotopy equivalence $u: M_{f,f} \to Y \vee C_f$ such that $\Pi_1 \cdot u \simeq \nabla$ and $\Pi_2 \cdot u \simeq \phi_{i,f}^{-1} \cdot q$, where $\Pi_1 : Y \vee C_f \to Y$ and $\Pi_2 : Y \vee C_f \to C_f$ are the projection maps.

**Proof** $\Leftarrow$ Suppose $u: M_{f,f} \to Y \vee C_f$ is a homotopy equivalence as above. Let $v$ be the homotopy inverse of $u$ and define $r: C_f \to M_{f,f}$ to be $v \cdot j_2$ where

Then $\phi_{i,f}^{-1} \cdot q \cdot r \simeq (\Pi_2 \cdot u) \cdot v \cdot j_2 \simeq \Pi_2 \cdot j_2 = 1_{C_f}$, that is $q \cdot r \simeq \phi_{i,f}^{-1}$. Also $\nabla \cdot r \simeq (\Pi_1 \cdot u) \cdot v \cdot j_2 \simeq \Pi_1 \cdot j_2 = *$. Hence $f$ is w-coprincipal.

$\Rightarrow$ Suppose $f$ is w-coprincipal under $r: C_f \to M_{f,f}$. Let

$$v = \nabla \cdot (i_2 \vee r): Y \vee C_f \to M_{f,f}$$
where $\nabla: Y \vee Y \to Y$ is the usual fold map. Consider the commutative diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & Y \\
\downarrow j_1 & & \downarrow i_2 \\
Y \vee C_f & \longrightarrow & M_f, f \\
\downarrow \Pi_2 & & \downarrow q \\
C_f & \phi_{v,1} & C_i
\end{array}
$$

Then $\phi_{v,1} = q \cdot r \simeq \phi_{i, f}$ and so is a homotopy equivalence. Thus $v$ is a homotopy equivalence also. Let $u$ be the homotopy inverse of $v$. Then

$\bar{\nabla} \cdot v \simeq \nabla \cdot (\bar{\nabla} \vee \bar{\nabla}) \cdot (i_2 \vee r) \simeq \Pi_1$ and so $\Pi_1 \cdot u \simeq \bar{\nabla}$. Also $\phi_{i, f}^{-1} \cdot q \cdot v = \nabla \cdot (\phi_{i, f}^{-1} \vee \phi_{i, f}^{-1}) \cdot (q \vee q) \cdot (i_2 \vee r) \simeq \nabla \cdot (\ast \vee 1_{C_f}) \simeq \Pi_2$ and so $\phi_{i, f}^{-1} \cdot q \simeq \Pi_2 \cdot u$.

**Definition** Let $(A, \mu)$ be a co–$H$–space and $B$ a space. A coaction of $A$ on $B$ is a map $\eta: B \to B \vee A$ such that $\Pi_1 \cdot \eta \simeq 1_B$ and if $\delta = \Pi_2 \cdot \eta$ then

$$
\begin{array}{ccc}
B & \eta & \longrightarrow & B \vee A \\
\delta \downarrow & & \downarrow \delta \vee 1 \\
A & \mu & \longrightarrow & A \vee A
\end{array}
$$

commutes up to homotopy.

**Lemma 3.1.5.** If $f: X \to Y$ is $\ast$–coprincipal then $C_f$ is a co–$H$–space which coacts on $Y$.

**Proof** Let $\eta: Y \to Y \vee C_f$ be defined by $\eta = u \cdot i_1$ where $u: M_{f, f} \to Y \vee C_f$. 

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is as above. Consider the homotopy commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow f & & \downarrow j_i \\
Y & \xrightarrow{\eta} & Y \vee C_f \\
\downarrow p & & \downarrow p \vee 1 \\
C_f & \xrightarrow{\bar{\mu}} & C_f \vee C_f
\end{array}
\]

Here \( \bar{\mu} \) is defined on the mapping cone of \( f \) from the null-homotopy \((p \vee 1) \cdot f \simeq (p \vee 1) \cdot j_i \cdot f \simeq j_i \cdot p \cdot f \simeq \ast \). Let \( J : C_f \vee C_f \leftarrow C_f \times C_f \) and \( K : Y \vee C_f \leftarrow Y \times C_f \).

The natural coaction \( C_f \rightarrow C_f \vee \Sigma X \) induces an action \([C_f, Z] \times [\Sigma X, Z] \rightarrow [C_f, Z], (g, \alpha) \mapsto g^\alpha \) such that if \( g_1, g_2 \in [C_f, Z] \) satisfy \( g_1 \cdot p \simeq g_2 \cdot p \) then there exist an \( \alpha \in [\Sigma X, Z] \) such that \( g_1^\alpha \simeq g_2 \). But

\[
J \cdot \bar{\mu} \cdot p \simeq J \cdot (p \vee 1) \cdot \eta \simeq (p \times 1) \cdot K \cdot \eta \simeq (p \times 1) \cdot (1 \times p) = \Delta \cdot p
\]

and so there exists \( \bar{\alpha} : \Sigma X \rightarrow C_f \times C_f \) such that \( (J \cdot \bar{\mu})^\bar{\alpha} \simeq \Delta \). Since \( \Sigma X \) is a suspension \( \bar{\alpha} \) lifts into \( C_f \vee C_f \) giving a map \( \alpha : \Sigma X \rightarrow C_f \vee C_f \).

Let \( \mu = \bar{\mu}^\alpha : C_f \rightarrow C_f \vee C_f \). Then

\[
J \cdot \mu = J \cdot (\bar{\mu}^\alpha) \simeq (J \cdot \bar{\mu})^{\bar{\alpha}} \simeq (J \cdot \bar{\mu})^\bar{\alpha} \simeq \Delta
\]

and so \( (C_f, \mu) \) is a co-H-space.

Furthermore \( \Pi_1 \cdot \eta = \Pi_1 \cdot u \cdot i_1 \simeq \bar{\nabla} \cdot i_1 = 1_Y \) and \( \Pi_2 \cdot \eta = \Pi_2 \cdot u \cdot i_1 \simeq \phi_{i,f}^{-1} \cdot q \cdot i_1 \simeq p \). Finally \( \mu \cdot p \simeq \bar{\mu} \cdot p \simeq (p \vee 1) \cdot \eta \) and so \( C_f \) coacts on \( Y \). \( \blacksquare \)
Thus we may think of a $w$-coprincipal map as generalising the notion of a co-$H$-space. By analogy with theorem 3.1.2. we have

**THEOREM 3.1.6.** If $f: X \to Y$ is a 2-connected map of 1-connected spaces and $\dim f \leq 2\text{conf} + \text{conY} - 1$ then $f$ extends up to homotopy as a cofibration.

Theorem 3.1.1. is true within this range. (It fails in the iteration of the initial argument, which is unnecessary here.) We shall show that the factoring condition follows from $w$-coprincipality and so [13, Theorem 1.1] provides a proof of this theorem. (A proof along the lines of theorem 3.1.2. would also do of course — see [4, §4] for the necessary relative Hopf invariant.)

Notice that the connectivity conditions are by no means strong enough to show that $f: X \to \vee_s S^{n-j-1}$ extends as a cofibration. Again, by analogy with the desuspension problem we might expect that higher co-associativity conditions on the coaction of a $w$-coprincipal map would allow us to relax the connectivity conditions.

Suppose now that $f: X \to Y$ is a fibration with $j: F \to X$ the inclusion of the fibre. Let $\tau = \phi \cdot j: \Sigma F \to C_f$, and $\delta: C_f \to \Sigma X$ the map which collapses $Y$.

**LEMMA 3.1.7.** $f$ is $w$-coprincipal iff there is a map $s: C_f \to \Sigma F$ such that

1. $\tau \cdot s \simeq 1_{C_f}$

and 2. $\Sigma i \cdot s \simeq \delta$

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PROOF ⇐ Suppose such a map $s$ exists. Let $r = \omega \cdot s : C_f \to M_{f,i}$, where $\omega : \Sigma F \to M_{f,i}$ is defined by $\omega(x \wedge t) = \{x, t\} \in M_{f,i}$. Then

$$q \cdot r = q \cdot \omega \cdot s \simeq \phi_{i,f} \cdot \tau \cdot s \simeq \phi_{i,f}$$

Also $\nabla \cdot r = \nabla \cdot \omega \cdot s = \ast$. Thus $f$ is $w$-coprincipal.

$\Rightarrow$ Suppose $f$ is $w$-coprincipal under $r : C_f \to M_{f,i}$. Since $f$ is a fibration so is $\nabla$. (If $\lambda : \{(\alpha, x) \in PY \times X | \alpha(1) = f(x)\} \to X$ is a lifting function for $f$ then $\Lambda : \{(\alpha, \{x, t\}) \in PY \times M_{f,i} | \alpha(1) = \nabla\{x, t\}\} \to M_{f,i}$ defined by $\Lambda(\alpha, \{x, t\}) = \{\lambda(\alpha, x), t\}$ is a lifting function for $\nabla$. In fact $\omega$ is just the inclusion of the fibre of $\nabla$, and so the null-homotopy $\nabla \cdot r \simeq \ast$ gives a map $s : C_f \to \Sigma F$ such that $\omega \cdot s \simeq r$. But $\tau \cdot s \simeq \phi_{i,f}^{-1} \cdot q \cdot \omega \cdot s \simeq \phi_{i,f}^{-1} \cdot q \cdot r \simeq 1_{C_f}$. Furthermore

$$\Sigma i \cdot s = (/X \cup X) \cdot \omega \cdot s = \delta_2 \cdot q \cdot \omega \cdot s \simeq \delta_2 \cdot q \cdot r \simeq \delta_2 \cdot \phi_{i,f} \simeq \delta$$

where $(/X \cup X) : M_{f,i} \to \Sigma X$ collapses the ends and $\delta_2 : C_i \to \Sigma X$ collapses $Y$.

Hence the lemma is proved.

Lemma 3.1.8. If $f : X \to Y$ is $w$-coprincipal, and $2\text{con}Y \geq \dim X, \dim Y$ then $\alpha : X \to X \cup CF$ factors through $f$.

PROOF If $f$ is $w$-coprincipal there exists a map $s : C_f \to \Sigma F$ such that $\Sigma i \cdot s \simeq$
δ. Consider

\[ \Sigma F \xleftarrow{s} C_f \]
\[ \Sigma \downarrow \quad \downarrow \delta \]
\[ \Sigma X \xrightarrow{=} \Sigma X \]
\[ \Sigma \alpha \downarrow \quad \downarrow \Sigma f \]
\[ \Sigma X \cup C \Sigma f \xleftarrow{\phi_{1,*}} \Sigma Y \]

defines a factoring of \( \Sigma \alpha \) through \( \Sigma f \), and the connectivity conditions allow us to desuspend to give the required factoring.

Also \( \phi_{1,*} \) is a right homotopy inverse of \( \phi_{1,r} \) and so the amended extension condition in the Maths Review [11] of [13] is satisfied also.

§2 A COUNTER–EXAMPLE TO THEOREM 3.1.1.

We construct a map satisfying the conditions of Quinn’s theorem but which does not extend. (A similar counter–example has been constructed by J. Smith. See [15])

Consider the iterated Whitehead product \( \omega \)

\[ S^{p+q+r-2} \xrightarrow{[i_1, [i_2, i_3]]} S^p \vee S^q \vee S^r \]

and let \( p_{13} : S^p \vee S^q \vee S^r \to S^p \vee S^r \), \( p_{23} : S^p \vee S^q \vee S^r \to S^q \vee S^r \), \( p_2 : S^p \vee S^r \to S^r \) be the projection maps. Then from the properties of Whitehead products it is seen that \( p_{23} \cdot \omega \simeq * \) and \( p_{13} \cdot \omega \simeq * \). Choose null–homotopies \( H_{23} \) and \( H_{13} \) of \( p_{23} \cdot \omega \) and \( p_{13} \cdot \omega \) respectively so that the secondary operation
$p_2\cdot H_{23} \cup p_2'\cdot H_{13} : S^{p+q+r-1} \to S^r$ is null-homotopic. Define a null-homotopy $H$ of $p_3 \cdot \omega : S^p \vee S^q \vee S^r \to S^r$ to be either $p_2' \cdot H_{13}$ or $p_2'' \cdot H_{23}$, and use $H$ to define a map

$$f = p_3 \cup H : (S^p \vee S^q \vee S^r) \cup_\omega e^{p+q+r-1} \to S^r$$

If we restrict $p, q$ and $r$ so that $r \geq p + q - \min(p, q) + 2$ and $\min(p, q) \geq 2$ then the dimension and connectivity conditions of theorem 3.1.1. are satisfied.

To show that $f$ is w-coprincipal consider the commutative diagram

\[
\begin{array}{ccc}
(S^p \vee S^q \vee S^r) \cup_\omega e^{p+q+r-1} & \xrightarrow{f} & S^r \\
\downarrow f & & \downarrow i_3 \\
S^r & \xrightarrow{i_3} & S^{p+1} \vee S^{q+1} \vee S^r \vee S^{p+q+r}
\end{array}
\]

A homotopy $i_3 \cdot f \simeq i_3 \cdot f$ is sought making the diagram a homotopy pushout.

Firstly define $G : (S^p \vee S^q \vee S^r) \times I \to S^{p+1} \vee S^{q+1} \vee S^r \vee S^{p+q+r}$ by

- $G(x, t) = x \in S^r \subset S^{p+1} \vee S^{q+1} \vee S^r \vee S^{p+q+r}$ where $x \in S^r$ and $t \in I$
- $G(x, t) = x \land t \in S^{p+1} \subset S^{q+1} \vee S^{q+1} \vee S^r \vee S^{p+q+r}$ where $x \in S^p$ and $t \in I$
- $G(x, t) = x \land t \in S^{q+1} \subset S^{p+1} \vee S^{q+1} \vee S^r \vee S^{p+q+r}$ where $x \in S^q$ and $t \in I$

Let

$$Z = e^{p+q+r-1} \cup ((S^p \vee S^q \vee S^r) \times I) \cup e^{p+q+r-1}$$

where the two $p+q+r-1$-cells are attached at either end of the cylinder via $\omega$.

Extend $G$ over $Z$ by defining $G$ to be $i_3 \cdot f$ on these cells. There is a cofibration sequence

$$S^{p+q+r-1} \xrightarrow{\phi} Z \hookrightarrow ((S^p \vee S^q \vee S^r) \cup_\omega e^{p+q+r-1}) \times I$$
The map $\phi$ is a generalised Whitehead product. We shall show that $G$ can be extended over the whole of $\{(S^p \vee S^q \vee S^r) \cup \omega \} \times I$ by finding a null homotopy of $\chi = G \cdot \phi : S^{p+q+r-1} \rightarrow S^{q+1} \vee S^p \vee S^{p+q+r}$. It is seen that $\chi$ is given by

$$
\chi(x \wedge t) = \begin{cases}
  i_3 \cdot H(x, 3t) & 0 \leq t \leq \frac{1}{3} \\
  G(\omega(x), 3t - 1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\
  i_3 \cdot H(x, 3 - 3t) & \frac{2}{3} \leq t \leq 1
\end{cases}
$$

where $x \in S^{p+q+r-2}$, $t \in I$ and $i_3 : S^1 \hookrightarrow S^{p+q+r}$. By the Hilton-Milnor theorem [19, Thorem 8.1] to show that $\chi$ is null–homotopic it is sufficient to show that the components

a) $p_{12} \cdot \chi \in \Pi_{p+q+r-1}(S^{q+1} \vee S^p)$,

b) $p_3 \cdot \chi \in \Pi_{p+q+r-1}(S^r)$,

c) $p_{13} \cdot \chi \in \Pi_{p+q+r-1}(S^{p+1} \vee S^r)$ and

d) $p_{23} \cdot \Pi_{p+q+r-1}(S^{p+1} \vee S^r)$

are null-homotopic.

a) We have

$$
p_{12} \cdot \chi(x \wedge t) = \begin{cases}
  * & 0 \leq t \leq \frac{1}{3} \\
  p_{12} \cdot G(\omega(x), 3t - 1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\
  * & \frac{2}{3} \leq t \leq 1
\end{cases}
$$

but $p_{12} \cdot G(\omega(x), 3t - 1) = G(p_{12} \cdot \omega(x), 3t - 1)$ and $p_{12} \cdot \omega \simeq *$ and so $p_{12} \cdot \chi \simeq *$. 65
b) We have

\[
p_0 \cdot \chi(x \land t) = \begin{cases} H(x, 3t) & 0 \leq t \leq \frac{1}{3} \\ p_3 \cdot G(\omega(x), 3t - 1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ H(x, 3 - 3t) & \frac{2}{3} \leq t \leq 1 \end{cases}
\]

but \( p_3 \cdot G(\omega(x), t) = p_3 \cdot \omega(x) \) and so \( p_3 \cdot \chi \simeq * \).

c) We have

\[
p_{13} \cdot \chi(x \land t) = \begin{cases} i_2 \cdot H(x, 3t) & 0 \leq t \leq \frac{1}{3} \\ p_{13} \cdot G(\omega(x), 3t - 1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ i_2 \cdot H(x, 3 - 3t) & \frac{2}{3} \leq t \leq 1 \end{cases}
\]

Now define:

\[
\Sigma_p(x, t, \lambda, A) = p_{13} \cdot \omega(x, 3t - 1) i_2 \cdot H(x, \lambda, t)
\]

where \( \frac{1}{3} \leq t \leq \frac{2}{3} \) and \( \lambda \in I \), and extending over \( 0 \leq t \leq \frac{1}{3} \) and \( \frac{2}{3} \leq t \leq 1 \) using any null–homotopy of the secondary operation \( p_2 \cdot H_{13} \cup H \). Then \( \Phi \) is a null–homotopy of \( p_{13} \cdot \chi \).

d) The map \( p_{23} \cdot \chi \in \Pi_{p+q+r-1}(S^{q+1} \lor S^r) \) is zero in the same way as in (c).

Hence \( \chi \simeq * \). Now \( \Pi_3 \cdot \chi \colon S^{p+q+r-1} \to S^r \) is just the secondary operation \( H \cup H : e^{p+q+r-1} \cup e^{p+q+r-1} \to S^r \) and so is null–homotopic in the obvious way. (See b) above.) Let \( A \) be the null–homotopy. We may choose the null–homotopy \( \Sigma \) of \( \chi \) such that the secondary operation \( \Pi_3 \cdot \Sigma \cup A : S^{p+q+r} \to S^r \) is
null–homotopic. Thus we obtain a map
\[ G \cup \Sigma: \mathbb{Z} \cup e^{p+q+r} \to S^{p+1} \vee S^{q+1} \vee S^r \] which is a homotopy of \( i_3 \cdot f \) to \( i_3 \cdot f \).

Construct a second homotopy \( i_3 \cdot f \simeq i_3 \cdot f \)

\[ \Omega: \mathbb{Z} \cup e^{p+q+r} \xrightarrow{\mu} \mathbb{Z} \cup e^{p+q+r} \vee S^{p+q+r} \xrightarrow{(G \cup \Sigma) \vee 1} S^{p+1} \vee S^{q+1} \vee S^r \vee S^{p+q+r} \]

where \( \mu \) is the natural coaction, and so we have an induced map

\[ \chi_0: S_f \cup \mathbb{Z} \cup e^{p+q+r} \cup_f S^r \to S^{p+1} \vee S^{q+1} \vee S^r \vee S^{p+q+r} \]

Next we show that \( \chi_0 \) is a homotopy equivalence and it is sufficient to show that the map of cofibres induced by \( \Omega, \phi_{i,f} \), is a homotopy equivalence.

\[ (S^p \vee S^q \vee S^r) \cup e^{p+q+r-1} \xrightarrow{f} S^r \xrightarrow{\bar{q}} S^{p+1} \vee S^{q+1} \vee S^r \vee S^{p+q+r} \]

\[ S^r \xrightarrow{i_3} S^{p+1} \vee S^{q+1} \vee S^r \vee S^{p+q+r} \xrightarrow{\phi_{i,f}} S^{p+1} \vee S^{q+1} \vee S^{p+q+r} \]

Now the homology of \( C_f \) is

\[ H_*(C_f) = \begin{cases} \mathbb{Z} & \text{if } * = p+1 = q+1 \\ \mathbb{Z} & \text{if } * = p+1 \neq q+1 \\ \mathbb{Z} & \text{if } * = q+1 \neq p+1 \\ \mathbb{Z} & \text{if } * = p+q+r \\ 0 & \text{otherwise} \end{cases} \]

By choice of \( G \) on \( (S^p \vee S^q) \times I, (\phi_{i,f}) \), is an isomorphism in dimensions \( p+1 \) and \( q+1 \), and the choice of \( \Omega \) makes \( (\phi_{i,f}) \) an isomorphism in dimension \( p+q+r \).

Hence \( \chi_0 \) is a homotopy equivalence and we have a homotopy pushout.
We claim that under the map

\[ \overline{r} : C_{\phi} \rightarrow S^{p+1} \vee S^{q+1} \vee S^{r+t+r} = C_{i_{124}} \rightarrow S^{p+1} \vee S^{q+1} \vee S^{r} \vee S^{r+t+r} \]

\( f \) is \( w \)-coprincipal. Here \( i_{124} \) is the inclusion map. First we construct an inverse to \( \chi_{\nu} \) on \( S^{p+t+r} \). Consider the map \( e^{p+t+r} \)

\[ e^{p+t+r} : e^{p+t+r} \hookrightarrow Z \cup \phi e^{p+t+r} \hookrightarrow S^{r} \cup (Z \cup \phi e^{p+t+r}) \cup S^{r} \]

and

\[ \overline{\phi} : S^{p+t+r-1} \rightarrow Z \hookrightarrow S^{r} \cup Z \cup S^{r} \]

Now if \( j : S^{p+1} \vee S^{q+1} \vee S^{r} \hookrightarrow S^{r} \cup Z \cup S^{r} \) then \( j \cdot \chi_{\nu} = \overline{\phi} \). Thus we obtain the secondary operation

\[ \Gamma = (j \cdot \Sigma \cup e^{p+t+q+r+}) : S^{p+t+r} \rightarrow S^{r} \cup (Z \cup \phi e^{p+t+r}) \cup S^{r} \]

Claim \( \chi_{\nu} \cdot \Gamma \simeq i_{4} : S^{p+t+r} \hookrightarrow S^{p+1} \vee S^{q+1} \vee S^{r} \vee S^{p+t+r} \). We have the commutative diagram

\[ \begin{array}{ccc}
S^{p+q+r} & \xrightarrow{\Gamma} & S^{r} \cup (Z \cup e^{p+q+r}) \cup S^{r} \\
\nu \downarrow & & \downarrow \mu \\
S^{p+t+r} \vee S^{p+t+r} & \xrightarrow{\Gamma \vee 1} & S^{r} \cup (Z \cup e^{p+q+r}) \cup S^{r} \vee S^{p+t+r}
\end{array} \]

and

\[ (i_{3} \cup (G \cup \Sigma) \cup \iota_{3}) \cdot (j \cdot \Sigma \cup e^{p+t+r}) = \Sigma \cup \Sigma \simeq * \]

since \( (i_{3} \cup (G \cup \Sigma) \cup \iota_{3}) \cdot j = 1_{S^{r} \vee S^{q} \vee S^{r}} \). Thus \( \chi_{\nu} \cdot \Gamma \simeq i_{4} \). Now to show that \( f \) is \( w \)-coprincipal we have \( \overline{q} \cdot \overline{r} = \overline{q} \cdot i_{124} \cdot \phi_{i,f} = \phi_{i,f} \) and so the first condition
is verified. To verify the second condition we note that we may construct the fold map \( \nabla : S^r \cup (Z \cup e^{p+q+r}) \cup S^r \to S^r \), up to homotopy, by taking its restriction on \( S^r \cup Z \cup S^r \) and extending over the \( p+q+r \)-cell using the null–homotopy \( \Lambda \) given above. We must show that \( \nabla \cdot \chi^{-1} \cdot r \simeq * \). We have

\[
(\nabla |_{S^r \cup Z \cup S^r} \cup \Lambda) \cdot \chi^{-1} \cdot i_{124} \cdot \phi_{i,j} \simeq (\nabla |_{S^r \cup Z \cup S^r} \cup \Lambda) \cdot \Gamma \cdot \phi_{i,j} \\
= (\nabla |_{S^r \cup Z \cup S^r} \cup \Lambda) \cdot (j \cdot \Sigma \cup e^{l \kappa}) \cdot \phi_{i,j} \\
= (\Pi_3 \cdot \Sigma \cup \Lambda) \cdot \phi_{i,j} \simeq *
\]

by choice of \( \Sigma \). Hence \( f \) is \( w \)-coprinciple.

Next we show that \( f \) does not extend up to homotopy as a cofibration. Suppose in fact that \( f \) does extend, that is we have

\[
\begin{align*}
A \to (S^p \vee S^q \vee S^r) \cup_{\omega} e^{p+q+r-1} & \xrightarrow{f} S^r \\
\end{align*}
\]

which is a cofibration up to homotopy. Thus \( i^* : H^{p+q+r-1}((S^p \vee S^q \vee S^r) \cup_{\omega} e^{p+q+r-1}) \to H^{p+q+r-1}(A) \) is an isomorphism. Now let \( e^p, e^q, e^r, e^{p+q+r-1} \in H^*(S^p \vee S^q \vee S^r) \cup_{\omega} e^{p+q+r-1} \) be the obvious generators. Then since \( \omega \) is an iterated Whitehead product, the Massey product \( \langle e^p, e^q, e^r \rangle \) is non–zero, see [10, Lemma 7]. However \( i^* \langle e^p, e^q, e^r \rangle = \langle i^* e^p, i^* e^q, i^* e^r \rangle \\
= \langle i^* e^p, 0, i^* e^q \rangle = 0 \) since \( e^r \in \text{Im} f^* = \ker i^* \). This contradicts \( i^* \) being an isomorphism in dimension \( p+q+r-1 \).

This completes the counter–example.
In this section we prove

**Lemma 1.2.3.** Let \( \xi, \eta, \) and \( \omega \) be fibrations over \( X \). Then the following diagram commutes up to homotopy

\[
\begin{array}{ccc}
T(\xi \oplus \eta \oplus \omega) & \xrightarrow{\Delta_{\xi,\eta,\omega}} & T(\xi \oplus \eta) \wedge T\omega \\
\Delta_{\xi,\eta,\omega} & & \downarrow \Delta_{\xi,\eta} \wedge 1 \\
T(\xi) \wedge T(\eta \oplus \omega) & \xrightarrow{1 \wedge \Delta_{\eta,\omega}} & T\xi \wedge T\eta \wedge T\omega
\end{array}
\]

Suppose \( p, q \) and \( r \) are the projection maps of \( \eta, \xi \) and \( \omega \) respectively. Following through the definition of the diagonal \( \Delta_{\xi,\eta} : T(\xi \oplus \eta) \to T\xi \wedge T\eta \) we see it is given by

\[
\Delta_{\xi,\eta} : X \cup C(E\eta \cup \overline{W}_{p,q} \times I \cup E\xi) \longrightarrow T\eta \wedge T\xi
\]

where

\[
\Delta_{\xi,\eta}(x) = x \wedge x,
\]

\[
\Delta_{\xi,\eta}(((\alpha, \beta), s) \perp t) = \begin{cases} 
p(\alpha) \wedge (\beta \perp 2st) & 0 \leq t \leq \frac{1}{2} \\
(\alpha \perp (2t - 1)) \wedge (\beta \perp s) & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

\[
\Delta_{\xi,\eta}(\alpha \perp t) = \begin{cases} 
p(\alpha) \wedge p(\alpha) & 0 \leq t \leq \frac{1}{2} \\
(\alpha \perp (2t - 1)) \wedge p(\alpha) & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

\[
\Delta_{\xi,\eta}(\beta \perp t) = \begin{cases} 
p(\beta) \wedge (\beta \perp 2t) & 0 \leq t \leq \frac{1}{2} \\
* & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

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where \( \alpha \in E\eta, \beta \in E\xi, \) and \( s \in [0,1] \). (If \( x \in X, x \perp t \in CX \) is the image of \((x,t) \in X \times I \) under the identification map \( X \times I \to CX \).)

Let \( \sigma^2 \) be a 2-simplex. Let \( (\lambda_1, \lambda_2, \lambda_3) \) be the barycentric coordinates of \( \sigma^2 \). Thus each point of \( \sigma^2 \) is uniquely specified by such a triple with each \( \lambda_i \geq 0 \) and \( \lambda_1 + \lambda_2 + \lambda_3 = 1 \). It was claimed in lemma 1.2.1. that \( \eta \oplus (\xi \oplus \omega) = (\eta \oplus \xi) \oplus \omega \). In fact each is homeomorphic to the fibration \( \eta \oplus \xi \oplus \omega \) whose total space is

\[
\tilde{W}_{p,q,r} \times \sigma^2
\]

with the identifications

\[
(\alpha, \beta, \gamma, \lambda_1, \lambda_2, \lambda_3) \sim (\alpha', \beta', \gamma', \lambda_1', \lambda_2', \lambda_3')
\]

if

either

\begin{itemize}
  \item[a)] \( \alpha = \alpha', \beta = \beta', \lambda_3 = 0, \lambda_1 = \lambda_1', \lambda_2 = \lambda_2' \)
  \item[b)] \( \beta = \beta', \gamma = \gamma', \lambda_1 = 0, \lambda_2 = \lambda_2', \lambda_3 = \lambda_3' \)
  \item[c)] \( \alpha = \alpha', \gamma = \gamma', \lambda_2 = 0, \lambda_1 = \lambda_1', \lambda_3 = \lambda_3' \)
\end{itemize}

where \( \alpha \in E\eta, \beta \in E\xi \) and \( \gamma \in E\omega \). (This is just the homotopy pushout of the three projection maps \( \tilde{W}_{p,q,r} \to E\eta, E\xi, E\eta \).) The projection map \( \chi: E(\eta \oplus \xi \oplus \omega) \to X \) is given by \( \chi(\alpha, \beta, \gamma, \lambda_1, \lambda_2, \lambda_3) = p(\alpha) \). Then \( \chi \) is compatible with the identifications and is seen to be a fibration.

Now identifying \((\eta \oplus \xi) \oplus \omega \) and \( \eta \oplus (\xi \oplus \omega) \) with \( \eta \oplus \xi \oplus \omega \) gives meaning to the commutative diagram in Lemma 1.2.3.. Since \( \chi \) is onto we may represent elements of \( T(\eta \oplus \xi \oplus \omega) \) by tuples \( (\alpha, \beta, \gamma, \lambda_1, \lambda_2, \lambda_3, t) \) for \( t \in [0,1], \) \( t = 1 \) is
the base point of the cone). Then it is a straightforward but tedious calculation to show that \((\Delta_{\eta, t} \cap 1) \cdot \Delta_{\eta, t, \omega} \) and \((1 \cap \Delta_{\epsilon, t}) \cdot \Delta_{\eta, t, \omega} \) on \((\alpha, \beta, \gamma, \lambda_1, \lambda_2, \lambda_3, t)\) are

\[
\begin{cases}
p(\alpha) \land q(\beta) \land (\gamma \perp 2\lambda_3 t) & 0 \leq t \leq \frac{1}{2} \\
p(\alpha) \land (\beta \perp 2(\frac{\lambda_2}{1 - \lambda_3})(2t - 1)) \land (\gamma \perp \lambda_3) & \frac{1}{2} \leq t \leq \frac{3}{4} \\
(\alpha \perp (4t - 3)) \land (\beta \perp (\frac{\lambda_2}{1 - \lambda_3})) \land (\gamma \perp \lambda_3) & \frac{3}{4} \leq t \leq 1
\end{cases}
\]

and

\[
\begin{cases}
p(\alpha) \land q(\beta) \land (\gamma \perp 4\lambda_3 t) & 0 \leq 2(1 - \lambda_1)t \leq \frac{1}{2} \quad 0 \leq t \leq \frac{1}{2} \\
p(\alpha) \land (\beta \perp (4(1 - \lambda_1) - 1)) \land (\gamma \perp \frac{\lambda_3}{1 - \lambda_1}) & \frac{1}{2} \leq 2(1 - \lambda_1)t \leq 1 \quad 0 \leq t \leq \frac{1}{2} \\
(\alpha \perp (2t - 1)) \land q(\beta) \land (\gamma \perp 2\lambda_3) & 0 \leq 1 - \lambda_1 \leq \frac{1}{2} \quad 0 \leq t \leq \frac{1}{2} \\
(\alpha \perp (2t - 1)) \land (\beta \perp (2\lambda_1 + 1)) \land (\gamma \perp \frac{\lambda_3}{1 - \lambda_1}) & \frac{1}{2} \leq 1 - \lambda_1 \leq 1 \quad \frac{1}{2} \leq t \leq 1
\end{cases}
\]

respectively. Replace \(\lambda_1\) by \(1 - \lambda_1\). Consider the sets

\[
A = \{(w, t) \in E(\eta \oplus \xi \oplus \omega) \times I | 0 \leq 2\lambda_1 t \leq \frac{1}{2}, 0 \leq t \leq \frac{1}{2}\}
\]

\[
B = \{(w, t) \in E(\eta \oplus \xi \oplus \omega) \times I | \frac{1}{2} \leq 2\lambda_1 t \leq 1, 0 \leq t \leq \frac{1}{2}\}
\]

\[
C = \{(w, t) \in E(\eta \oplus \xi \oplus \omega) \times I | 0 \leq \lambda_1 \leq \frac{1}{2}, \frac{1}{2} \leq t \leq \frac{3}{4}\}
\]

\[
D = \{(w, t) \in E(\eta \oplus \xi \oplus \omega) \times I | \frac{1}{2} \leq \lambda_1 \leq 1, \frac{1}{2} \leq t \leq \frac{3}{4}\}
\]

\[
E = \{(w, t) \in E(\eta \oplus \xi \oplus \omega) \times I | 0 \leq \lambda_1 \leq \frac{1}{2}, \frac{3}{4} \leq t \leq 1\}
\]

\[
F = \{(w, t) \in E(\eta \oplus \xi \oplus \omega) \times I | \frac{1}{2} \leq \lambda_1 \leq 1, \frac{3}{4} \leq t \leq 1\}
\]

where \(w = (\alpha, \beta, \gamma, \lambda_1, \lambda_2, \lambda_3)\). Then since

\[
E(\eta \oplus \xi \oplus \omega) \times I = A \cup B \cup C \cup D \cup E \cup F
\]
parametrizes $T(\eta \oplus \xi \oplus \omega)$ we may define a null homotopy $(\Delta_{\eta, \xi} \wedge 1) \cdot \Delta_{\eta \oplus \xi \oplus \omega} \simeq (1 \wedge \Delta_{\xi, \omega}) \cdot \Delta_{\eta, \xi \oplus \omega}$ by defining one on the subsets $A_1, \ldots, F$ and checking that it is well defined on the intersections and compatible with the identifications. For example, on $A$ define a homotopy $H_A : A \times I \rightarrow T(\eta \oplus \xi \oplus \omega)$

$$H_A((w, t), r) = \begin{cases} \left( p(\alpha) \wedge q(\beta) \wedge (\gamma \perp ((2(1 - r) + 4r)\lambda_3 t) \right) & 0 \leq r \leq \frac{1}{2} \\ p(\alpha) \wedge q(\beta) \wedge (\gamma \perp 4\lambda_3 t) & \frac{1}{2} \leq r \leq 1 \end{cases}$$

then $H_A$ may be extended over $B$ by $H_B : B \times I \rightarrow T(\eta \oplus \xi \oplus \omega)$

$$H_B((w, t), r) = \begin{cases} \left( p(\alpha) \wedge q(\beta) \wedge (\gamma \perp ((1 - r)2\lambda_3 t + r\frac{\lambda_3}{\lambda_1}) \right) & 0 \leq r \leq \frac{1}{2} \\ p(\alpha) \wedge (\beta \perp r(4\lambda_1 t - 1)) \wedge (\gamma \perp \frac{\lambda_3}{\lambda_1}) & \frac{1}{2} \leq r \leq 1 \end{cases}$$

It should be checked that $H_A$ and $H_B$ agree on $A \cup B$ and that they are compatible with the identifications. Continuing in this way by defining such 'linear' homotopies we obtain an extension over $E(\eta \oplus \xi \oplus \omega)$ and the lemma is proved. ■
REFERENCES


