Deflection and Stress Analysis
of Multi-storey Shear Wall Structures

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DEFLECTION AND STRESS ANALYSIS
OF MULTI-STOREY SHEAR WALL STRUCTURES
The current trend towards tall buildings has illuminated the necessity for greater knowledge of the behaviour of these structures. The effect of lateral loading on high-rise buildings is very pronounced and the question of their stability may be dealt with in various ways. The most economic means of providing lateral stiffness is the use of shear-resistant walls. These walls usually contain openings for windows, doors, corridors, etc.

Exact analysis of multi-storey shear wall structures is difficult and the available methods, therefore, include various approximations. In these methods, emphasis is placed on the deflection analysis whereas the problems of the stress distribution across the walls and the stress concentration resulting from the presence of openings are usually overlooked.

The present investigation is divided into two parts. The first is aimed at providing an assessment of the methods of deflection analysis of shear walls, whereas the second part involves a study of the stress distribution in the walls due to lateral loading.

In the first part the so-called shear-connection method is reviewed and the theory extended to deal with multi-bay structures. The limitations of the method with respect to the shape of walls are found. The conclusions reached are based on the results using this method compared with results using the wide-column methods and perspex-model tests. The study in this part resulted in two efficient computer programs capable of analysing any type of multi-storey multi-bay shear wall structure.

The second part deals with the stress analysis of shear wall structures. The various theoretical procedures used may be categorised as:

a) approaches based on the fundamentals of the theory of elasticity; and

b) approaches based on numerical analysis.
Two approaches fall in category "a". These are the Elementary E.T.B., and the Stress Function methods.

Category "b" includes an extensive study of the Finite Element, and the Dynamic Relaxation methods.

Computer programs are written to carry out the calculations in each of these methods.

The experimental work involved in this part is based on photo-elastic analysis using the frozen-stress technique. Shear wall models of different shapes are tested and the results obtained are compared to the theoretical ones.

It should be noted that throughout these investigations the material used is assumed to be linearly elastic, isotropic and continuous. Thus, further modifications have to be made to the theories presented if the construction materials do not conform to these idealisations.
"All the effects of nature are only the mathematical consequences of a small number of immutable laws."
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REFERENCES
Tall buildings are generally used for both commercial and residential purposes throughout the world. The effect of horizontal loading on these buildings is profound and provision of lateral stability is therefore the most important structural requirement. Lateral stiffness is usually achieved by means of particular characteristics of the structure or the agency of shear-resistant components. In framed structures, lateral rigidity depends largely on the monolithic connection between beams and columns, whereas the overall strength of conventional load-bearing masonry buildings ensures stiffness against horizontal loading. On the other hand, structures such as shear walls, cores, truss assemblies, etc. are very stiff and may therefore be employed to withstand horizontal loading. Floor slabs are also very stiff in their own plane and their contribution to lateral stability can be considerable. However, the stiffening effect of these slabs is usually not considered and their function is assumed to be merely collecting and distributing lateral forces.

Hence, depending on their structural behaviour, modern buildings may be divided into three basic types:

a) Framed structures in which the component units are beams and columns.

b) Large-panel structures consisting mainly of large panels butt-jointed to form the entire structure. These panels may be in situ, prefabricated concrete or brick walls.

c) Composite structures which are combinations of a) and b).
a) Framed Structure

b) Large - Panel Structure

c) Composite Structure

Fig. 1, Types of Building.

TYPICAL FLOOR PLAN
Beside the structural requirements, economic considerations determine the suitability of a type of construction for a particular job. However, in most cases type b) has a better claim for pre-eminence. In addition to adequate lateral stability, large panels provide excellent sound and thermal insulation both vertically and horizontally. These panels are usually prefabricated under controlled factory conditions and jointed together on site. Finishes are usually excellent and plaster can be directly applied to the panels. Construction time is thus reduced to a minimum. In fact, the methods of industrialised building have extended to mass production of units comprising complete sections of rooms which incorporate all finishes and services. These units can be assembled easily on site to form the entire structure. Composite constructions, type c), are also very common. In this system horizontal loads are withstood by shear-resistant components, whereas dead and live loads are carried jointly by these components and columns. The shear components may be employed either separately or in conjunction with conventional frames to provide the required stiffness. These shear-resistant units should be located so as to carry as much of the weights of the floors as possible in order to offset the tensile stresses produced by the horizontal loading. Also, consideration should be given to the arrangement of these components so that torsional stresses are minimum.

The analysis of multi-storey buildings is difficult owing to the three dimensional nature of these structures. However, the complexity can be reduced considerably by neglecting the orthogonal interaction between the various components of the structure. Whilst this assumption renders two-dimensional structural components capable of analysis by refined methods, it results in an underestimation of the overall rigidity of the building.

Generally, the main object of elastic analysis is to provide an assessment of the structural stiffness, as well as information about any local stress concentrations. Exact analytical procedures produce exact values of
deflection and, of course, accurate insight of the stress distribution since stress is a derivative of deflection. This rule does not, however, follow for approximate methods. The stresses, being a very sensitive parameter, would suffer from the idealizations involved in approximate methods and the accuracy of their assessment will depend on how well the resemblance is between the idealized structure and the actual one.

On the other hand, deflections are less sensitive than stresses and their values can, in most cases, be evaluated fairly accurately by approximate methods which include crude idealizations.

On this basis, methods of analysis of shear wall structures may be categorised as:

(i) deflection analysis, and
(ii) stress analysis.

The first group of methods usually involves the solution of a substitute structure which leads to the direct determination of deflections. The stresses, if required, can be obtained by double-differentiation of the deflection equations. Most of the stress analysis methods, on the other hand, are by definition concerned with the calculation of stresses. Deflections may then be obtained by twice-integrating the expressions of stresses.

The present work comprises an investigation into the structural behaviour of one type of shear-resistant component, namely shear wall structures. Other shear-resistant systems, such as infilled frames, combined frame and wall assemblies are beyond the scope of this work. One phase of this investigation is concerned with the stiffness of multi-storey shear wall structures containing openings. This is embodied in Part I of the thesis. In this part, which is entitled "Deflection Analysis of Multi-Storey Shear Wall Structures", the relevant theories are reviewed and the new contribution presented is supplemented by model tests.
Part II of the thesis is entitled "Stress Analysis of Multi-Storey Shear Wall Structures". This part comprises a study of the stress distribution in shear walls, with particular reference to local stress concentrations resulting from the presence of openings. Various analytical procedures are used and the results thereby obtained are compared with photo-elastic models.

It is generally recognised that the problem of shear wall analysis requires involved mathematical treatment. Often the treatment is complicated and requires a considerable amount of time to execute the calculations manually. Thus, the use of such techniques in a design office would not be a practical proposition. Much of the tedious work associated with refined theoretical approaches can, however, be carried out by high-speed electronic digital computers. Hence, most of the work described in this thesis is presented in computer program form. These programs, written in Atlas Autocode for an English Electric KDF 9 machine, commence with the basic problem data and conclude with the required results, thus involving no prior preparations or hand calculations. Since the input information is merely the particulars of the structure, these programs can be handled by inexperienced personnel.

For convenience, the analysis assumes that the material of construction is homogeneous, isotropic, linearly elastic and continuous. Thus, the validity of the present work is restricted to materials such as concrete for which these idealizations may be regarded as reasonable.
PART I

DEFLECTION ANALYSIS
Model of a perspex model of a 12-storey box-type structure
2.1 INTRODUCTION

2.1.1 General

Shear wall assemblies are commonly used in multi-storey structures to resist wind loading and provide lateral stiffness to the building. These walls usually extend along the transverse direction of the building (Fig. 2.1) and the spacing between them is arranged so that wind loading can be resisted both efficiently and economically without producing excessive tensile stresses.

These shear-resistant structures may consist of simple built-in brick or reinforced concrete cantilevers or a number of walls interconnected by beams or floor slabs, as shown in Fig. 2.2.

The presence of openings between the component walls results in a reduction of the structural stiffness in addition to causing local stress-concentration effects. Exact analysis of multi-storey structures would inevitably involve the interaction between the walls and slabs, thus presenting a highly complex three-dimensional structural system. However, shear walls may be considered as two-dimensional plane-stress components acting independently of the rest of the structure. This idealization merely results in an underestimation of the overall stiffness of the building and hence an increase of the factor of safety.

In practice, only simple analytical approaches to structural engineering problems are adopted. The common methods of analysis of shear wall structures with openings are based on the elementary Engineering Theory of Bending and include major assumptions as to the behaviour of the structure. With slender connecting beams the component walls are usually treated as simple cantilevers jointly resisting the wind loading in proportion to their individual
Fig. (2.1), Interconnected Shear Walls.
Fig. (2.2), Methods of Deflection Analysis of Shear Wall Structures.
bending stiffnesses. On the other hand, when the connecting beams are stiff the walls are assumed to act as one built-in cantilever unit. It is apparent that the first assumption ignores the coupling effect of the connecting beams and hence underestimates the stiffness of the structure, whereas in the second hypothesis the exaggeration of the degree of interaction between the walls leads to an overestimation of the structural rigidity.

These assumptions may be justified when the height of the structure is not excessive, but will give erroneous results for taller buildings. However, the recent advances in methods of structural analysis in addition to the availability of modern electronic computers have given rise to extensive research work on the behaviour of shear wall structures. Review of this work may be found in various publications\(^2, 63, 64, 69, 90\). The latest and most comprehensive survey of the available methods was reported by Coull\(^36\).

2.1.2 Review of Previous Work

Due to the complexity of the formulation of the analysis of shear wall assemblies as a boundary value problem with multiple connections and mixed boundary conditions, all the existing methods are concerned with idealized substitute structures, the solution of which can be obtained fairly easily.

Depending on the idealization, the analytical treatments may be categorised as frame analogy methods and continuous media methods. To the first category belong Benjamin's method\(^{13, 14, 15}\), Green's Equivalent Frame method\(^56\), Amaratunga's Procedure\(^2\) and Frischmann et al. Wide Column Analogy\(^49\), whereas the shear connection method\(^{11, 12, 23, 24, 46, 98, 101}\) falls into the second category.

A brief review of these methods is contained in the following sections.
2.1.2.1 Frame Analogy Methods

The Portland Cement Association\(^{(122)}\) introduced a simple hand method of analysis of shear walls containing an opening. This procedure, which is based on simple strength of material theories, treats the wall as an assemblage of piers and spandrels. The distortion of the wall under lateral loads is assumed to be the combined distortion due to bending and shear stresses of the individual components. Benjamin and Williams\(^{(14, 15)}\) conducted an experimental investigation on brick and reinforced concrete walls with openings. The findings of this work lead to the introduction of the so-called Benjamin's method. This procedure, which includes the effect of the axial strain of the pier elements, is virtually a modification of the Portland Cement Association method.

Work by the author\(^{(63)}\) has shown that the assumptions involved in Benjamin's technique are crude and the procedure is therefore unsuitable for the analysis of shear wall structures. Yet, Sinha\(^{(127)}\) reported that Benjamin's approach gives good agreement with experimental results of 1/6th scale brick cross wall models. Sinha's findings may, however, be attributed to the fact that in brick walls the cracks in the mortar joints between the piers and spandrels render these joints incapable of transmitting moments and, therefore, the structure behaves in accordance with the idealization of Benjamin's method.

Green\(^{(53)}\) appears to be the first to treat high-rise bracing walls as an assemblage of beams and columns forming a multi-storey frame (Fig. 2.2b). This approach is termed "The Equivalent Frame Method". However, McLeod\(^{(90)}\) reported that the results obtained by using this method depend on the relative ratio between the width of the columns and their distance centre to centre and the method generally overestimates the deflection. Amaratunga\(^{(2)}\) adopted a similar technique to obtain an initial estimation of the redundant forces in the walls. This solution was subsequently refined to give accurate values. Similar analyses were reported by Candy\(^{(128)}\) and Zbirohowski-Koscia\(^{(116)}\).
Frischmann et al. (49) suggested that the accuracy of the equivalent frame method can be improved considerably if the effect of the finite width of the column is included in the analysis. In their approach the shear wall structure is idealized as a column interconnected by beams, part of which is of infinite rigidity (Fig. 2.2c). This procedure is termed the "Wide Column Analogy Method". Owing to its better representation of the structural stiffness, this method appears to be the most powerful of the frame analogy methods.

2.1.2.2 Methods of Continuous Media

Methods of continuous media were utilized for the analysis of bracing walls by Beck (11, 12) and others (46, 98). In this method the discrete connecting beams are replaced by continuous media in shear (Fig. 2.2d). After evaluating the elastic properties of the media, the behaviour of the structural system can be described by a second-order differential equation. The solution of this equation results in the determination of the unknown straining actions of the walls.

This elegant approach, which is termed the "Shear Connection Method", was originally suggested by Chitty and Pippard (23, 24) in connection with the analysis of a cantilever composed of parallel beams interconnected with cross-members. A similar technique was employed by Jaeger, Hendry and Hussein (129) for the estimation of the sway deflections of framed structures.

2.1.3 Objective and Scope

It would appear from the preceding section that both the shear connection and the wide-column analogy approaches are the most important of the methods of deflection analysis of shear wall structures. Both methods are nevertheless open to criticism for various reasons. As far as the shear connection method is concerned the following drawbacks may be noted:
(i) The theory is not extended to deal with multi-bay structures.

(ii) The shortening of the connecting beams and the shear deformation of the walls are ignored.

(iii) Lack of experimental evidence to determine the limitations of the technique.

The wide column analogy, on the other hand, although based on reasonable assumptions, lacks:

a. information about the effective length of the connecting beams, and

b. knowledge about the limitation of the method with regard to the shape of the walls.

Hence, these two methods merit further study and the present work is aimed at examining their validity for the analysis of shear wall structures having various shapes. In fact, the basic objectives of the present work were to provide information in answer to the questions posed by the above considerations. Specifically, the first objective was the improvement of the shear connection method. A solution of the general equation of this method was found thus extending the application of the technique to multi-bay structures.

Experimental techniques also serve a useful purpose in the analysis of shear wall structures and have often been used in the study of walls with irregular shapes. In fact, it was by means of experimental investigation that the limitations of the theoretical analysis outlined above in (iii), a. and b. were found. Several shear wall models of different shapes were made of Perspex and their deflection under uniformly distributed lateral loading was measured. These values were compared with computed results obtained by using the two analytical treatments. Furthermore, a three-dimensional 12 storey Perspex model consisting of walls of different sizes connected by floor slabs was constructed. The objective of this last model was to determine the degree of interaction between the walls and floor slabs.

The general conclusion and summary of the findings are embodied in Section 2.4.
2.1.4 **List of Notations**

- **x** longitudinal distance from the origin
- **l** height of building
- **a, b, c,...** width of walls A, B & C respectively
- **t** thickness of the walls
- **e₁, e₂,...** width of the openings
- **c₁, c₂,...** distance between the centre line of each two successive walls
- **Aₐ, Aₐ,...** cross sectional area of walls A & B respectively
- **I₀, I₀** M.O.I. for walls A & B
- **I₀** Sum of M.O.I. of walls = Iₐ + Iₐ +...
- **h** depth of the connecting beams
- **Iₐ** M.O.I. of the connecting beams
- **n** number of walls
- **m = z** number of rows of openings = (n - 1)
- **M₀** externally applied moment
- **R₁, R₂,...** vertical shear in the medium 1, 2 respectively
- **E** Young's Modulus of the material
- **ν** Poisson's Ratio of the material
- **G** Modulus of Elasticity in shear
- **Cₘ₁, Cₘ₂** shear constant of the medium 1, 2, ...

where

\[
Cₘᵢ = \frac{12 E Iₐ}{d \cdot eᵢ^3 \left[ 1 + 2 A(t⁺ν) \cdot (h/lᵢ) \right]^2}
\]

\[
d = \text{storey height.}
\]

- **aᵢ₁, aᵢ₂,...** constants depending on the geometry of the structure and defined as
- **B₁, B₂,...**

\[
aᵢᵢ = \frac{Cₘᵢ}{E I₀} \left[ \frac{eᵢ^2}{Aᵢ} + \frac{I₀}{Aᵢ} + \frac{I₀}{Aᵢ₊₁} \right] \quad \text{for } i = j
\]

\[
aᵢᵢ = \frac{Cₘᵢ}{E I₀} \left[ Cᵢ \cdot cᵢ - \frac{I₀}{Aᵢ} \right] \quad \text{for } i = j⁺₁
\]

- **9 a**
\[ a_{ij} = \frac{C_{mj}}{EI_0} \left[ c_i c_j \right] \quad \text{for} \quad i > j+1 \]

\[ a_{ij} = \frac{C_{mj}}{EI_0} a_{ji} \quad \text{for} \quad i < j \]

and

\[ B_i = \frac{C_{mi}}{EI_0} C_i \quad \text{for} \quad i = 1, 2, \ldots, m \]

the degree of statical indeterminacy of the frame.

the straining actions in the connecting beams.

\[ f_{ij} = \int \frac{m_i m_j}{EI} \frac{dS}{dl} + \int \frac{S_i S_j}{ArG} \frac{dS}{dl} + \int \frac{s_i s_j}{AE} \frac{dS}{dl} \]

\[ u_i = \int \frac{m_i m_i}{EI} \frac{dS}{dl} + \int \frac{s_i s_i}{ArG} \frac{dS}{dl} + \int \frac{n_i n_i}{AE} \frac{dS}{dl} \]

where \( m_i, s_i, n_i \) are the moment, shear and thrust diagrams for the released structure respectively for \( x_i = 1 \)

\( m_0, s_0, n_0 \) Moment, shear and thrust diagrams for released structure under applied loading.

\( A \) Cross sectional area.

Certain other quantities are defined in the text as they occur.

ABBREVIATIONS

SCM = the shear connection method.

WCA = the wide column analogy.
2.2 METHODS OF DEFORMATION ANALYSIS

2.2.1 The Generalised Shear Connection Method

2.2.1.1 Preamble

The so-called shear connection method, originally suggested by Chitty and Pippard (23, 24), was employed for the analysis of multi-storey shear wall structures by Beck (11, 12), Rosman (98-102), Eriksson (46) and others (16, 33, 37, 111, 117). In this method the discrete connecting members are replaced by continuous media in pure shear, as shown in Fig. 2.3. The elastic properties of these media can be readily evaluated (46). The analysis also assumes that:

(i) The axial deformation of the beams and shear deformation of the walls are negligible.

(ii) There is a point of contraflexure at the centre of the beams.

(iii) The medium is uniform and continuous.

However, the solution is approached by introducing hypothetical releases in the media at the assumed points of contraflexure. These releases render the structure statically determinate. Hence, the consideration of the conditions of compatibility of the structure results in a set of simultaneous differential equations. The derivation of these equations is given explicitly elsewhere (46, 107).

This set of equations may be expressed as:

\[
\begin{align*}
R_1'' &= R_1 a_{i1}^2 + R_2 a_{i2}^2 + \ldots + R_m a_{mi}^2 - M_0 B_1 \\
R_2'' &= R_1 a_{i2}^2 + R_2 a_{22}^2 + \ldots + R_m a_{m2}^2 - M_0 B_2 \\
&\vdots \\
R_i'' &= R_1 a_{ii}^2 + R_2 a_{2i}^2 + \ldots + R_m a_{mi}^2 - M_0 B_i \\
&\vdots \\
R_m'' &= R_1 a_{im}^2 + R_2 a_{2m}^2 + \ldots + R_m a_{mm}^2 - M_0 B_m
\end{align*}
\]

(2.1)
Fig. (2.3), The idealization in the SCM technique.
Where:

- \( R_i \) = the shear in medium "i"
- \( R_i'' \) = the second derivative of \( R \) with respect to \( x \)
- \( \alpha_{ij} \) = constants
- \( M_0 \) = externally applied moment
- \( m \) = order of matrix = No. of rows of openings

A detailed list of notation is given in Appendix I. In what follows, the set of equations given in (2.1) will be referred to as the general equation.

Most of the published work\((12,46,16,33)\) on the shear connection method deals with two-wall structures, i.e., one redundant. In this simple case the general equation degenerates to:

\[
R_i'' = R_i \alpha_{ii}^2 - M_0 B_i
\]  

(2.2)

The solution of which is:

\[
R_i = \tilde{A} \cosh \alpha_{ii} x + \tilde{B} \sinh \alpha_{ii} x + \frac{B_i}{\alpha_{ii}^2} \left( M_0 + m_0'' + \ldots \right)
\]

(2.3)

Where \( \tilde{A} \) and \( \tilde{B} \) are arbitrary constants.

The boundary conditions, when assuming the origin to be at the top of the building as shown in Fig. 2.3, are:

(i) \( R = 0 \) at \( x = 0 \), and
(ii) \( R^1 = 0 \) at \( x = 1 \)

The constant terms \( \tilde{A} \) and \( \tilde{B} \) may be evaluated by substituting the above boundary conditions into equation 2.3. However, Baehrle (10) assumed that proportionality exists between the redundants \( R \). He suggested that the complexity of the general equation may be reduced by assuming that the shear forces in the media are related to each other by appropriate functions of \( x \). Eriksson(46) adopted this technique for the analysis of a three-wall structure. Nevertheless, on examining Baehrle's treatment it was found that it lacks adaptability and is, therefore, unsuitable for repetitive and automatic computation. Furthermore, this approach results in an enormous increase in the arithmetic involved and hence its use is prohibitive.
Baehre's assumptions also result in a lower bound to the redundant shear in the media which means an under-estimation of the stiffness of the structure.

A solution using Fourier's series was introduced by Bachelor(9). This procedure is acceptable for automatic computation but it was reported by McLeod(90) that the solution tends to become unstable for more than three rows of openings.

An original method was developed by Soane(107) for solving the general equation by analogue computer. In this method the general equation was simulated by electronic circuits and the height of the building by time. Being restricted by the size of the analogue computer available at the University of Edinburgh, Soane managed only to utilize this technique in analysing structures of up to 6-symmetrical walls, i.e. 3-redundants. Irrespective of the errors resulting from the sensitivity of the electronic equipments used, Soane's method appears to be very powerful. No further assumptions are needed as the process of integration with respect to "x" is carried out by means of appropriate amplifiers and the initial conditions which satisfy the boundary requirements are set either manually or automatically by trial and error procedure. However, Soane's method requires, in addition to the availability of an analogue computer, specialized knowledge of analogue computing techniques. Furthermore, it is time-consuming since the preparation of the equations for the computer involves hand calculation of the constants of the equations as well as the trial and error method in scaling the problem. Moreover, this technique is inefficient for repetitive use.
2.2.1.2 Solution of the General Equation by Matrix Orthogonalisation Method

The general equation may be expressed as

\[
\begin{bmatrix}
R''_1 \\
\vdots \\
R''_i \\
\vdots \\
R''_m
\end{bmatrix} =
\begin{bmatrix}
\alpha_i^2 & \alpha_{2i} & \cdots & \alpha_{mi} \\
\vdots & \ddots & \vdots & \vdots \\
\alpha_{1i} & \alpha_{2i} & \cdots & \alpha_{mi} \\
\vdots & \ddots & \ddots & \vdots \\
\alpha_{1m} & \alpha_{2m} & \cdots & \alpha_{mm}
\end{bmatrix}
\begin{bmatrix}
R''_1 \\
\vdots \\
R''_i \\
\vdots \\
R''_m
\end{bmatrix} -
\begin{bmatrix}
B_1 \\
\vdots \\
B_i \\
\vdots \\
B_m
\end{bmatrix}
\]

or symbolically as

\[
\{R''\} = [A]\{R\} - M_0 \{B\}
\]  

(2.5)

Equation (2.4) is a set of simultaneous coupled differential equations of the second order. Since the coefficients of \{R\} are constants, this set of equations can be diagonalised through a linearly independent transformation matrix. The redundants \{R\} can be expressed in terms of independent functions as follows:

\[
\{R\} = [T]\{J\}
\]  

(2.6)

where \([T]\) is a transformation matrix of the order \(m \times m\) and \{J\} is a set of uncoupled functions of \(x\).

Substituting in (2.5) gives

\[
[T]\{J''\} = [A][T]\{J\} - M_0 \{B\}
\]  

(2.7)

which, after pre-multiplying by \([T^{-1}]\), becomes

\[
\{J''\} = [T^{-1}][A][T]\{J\} - M_0 \{T^{-1}B\}
\]  

(2.8)

or

\[
\{J''\} = [X]\{J\} - M_0 \{T^{-1}B\}
\]  

(2.9)

where

\[
[X] = [T^{-1}][A][T]
\]
Choosing the coefficients of \([T]\) such that \([X]\) is a diagonal matrix,
i.e. 
\[
[T^{-1}[A][T] = [X] = \text{diagonal matrix}
\]
Hence 
\[
[A][T] = [T][X]
\]  
(2.12)
A more familiar form of (2.12) is 
\[
\begin{bmatrix}
A - X_i I
\end{bmatrix} \{t_i\} = 0
\]
(2.13)
which is a characteristic value problem. \(X_i\) is the \(i^{th}\) Eigen value to which there is a corresponding Eigen vector \(\{t_i\}\). 
I being the unit matrix.
By Cramer's rule the determinant of coefficients \([T]\) must be zero, i.e.
\[
\begin{vmatrix}
A - X_i I
\end{vmatrix} = 0
\]  
(2.14)
Upon expanding the determinant (2.14) an \(m^{th}\) order equation in \(X\) is obtained. The roots of this equation are the Eigen values and the corresponding Eigen vectors can therefore be found. Having obtained the Eigen values the transformation matrix \([T]\) can be constructed. 
Putting \(\{U\} = [T^{-1}][B]\) in equation (2.9) it becomes 
\[
\{J\} = [X]\{J\} - M_0 \{U\}
\]  
(2.15)
Equation (2.15) is a set of simple second order differential equations with constant coefficients and uncoupled variables. The \(i^{th}\) equation of which is of the form
\[
J_i'' = X_i J_i' - M_0 U_i
\]  
(2.16)
The solution of which is
\[
J_i = C_i \sinh(\sqrt{X_i} \cdot x) + D_i \cosh(\sqrt{X_i} \cdot x) + \frac{U_i}{X_i} (M_0 + \frac{M_0''}{X_i^2})(2.17)
\]  
where \(C_i\) and \(D_i\) are arbitrary constants.
The boundary conditions, when assuming the origin to be at the top of the structure, are:

(i) \( J = 0 \) at \( x = 0 \); and
(ii) \( J' = 0 \) at \( x = 1 \) \hspace{1cm} (2.18)

The values of the arbitrary constants may now be found by substituting from (2.18) into (2.17). Having obtained the functions \( \{J\} \), the shear forces \( \{R\} \) in the media can be easily computed by substituting into equation (2.6).

Calculations for the bending moments, shear forces and deflection in each wall can therefore be carried out as follows:

\[
M_{ix} = \frac{I_i}{I_0} \left( M_0 - \sum_{i=1}^{m} R_i \cdot C_i \right)
\]

\[
Q_{ix} = \frac{I_i}{I_0} \left( M_0' - \sum_{i=1}^{m} R_i' \cdot C_i \right)
\]

\[
Y_{ix} = \int_{x}^{x} \frac{M_{ix}}{E L_i} \, dx \, dx
\]

\[\text{where } i = 1, 2, \ldots, m\]

\( M_{ix}, Q_{ix}, Y_{ix} \) are the bending moment, shearing forces and deflection respectively in the \( i \)-th wall at a distance \( x \) from its top.

Although the solution of the general equation (2.5) is feasible for hand calculation for up to 4-wall structures, automatic computation is undoubtedly advantageous as it provides the facility for repetitive use. A computer programme was, therefore, written to carry out the calculations. A description of the programme and instructions for its use are given in appendix I.
2.2.2 The Wide-Column Analogy Method

2.2.2.1 Preamble

The wide column analogy, which was suggested by Frischmann et al.\(^{(49)}\), is virtually a modification of the frame analogy methods\(^{(2,77)}\). This procedure permits account to be taken of the finite width of the columns. The idealised structure, Fig. 2.4, consists of:

(i) Columns - the flexural strength, \(EI_w\), of which is the same as the corresponding walls; and
(ii) Connecting beams - the length of which is the distance centre to centre of the adjacent walls.

These beams consist of two parts. The central part is the clear distance between the walls in length and having the same flexural rigidity of the connecting members. The outer parts are members with infinite stiffness.

With these assumptions, the procedure degenerates to the solution of a multi-storey frame with rigid members subjected to a given load pattern. In the analysis of such frames there are two basic sets of conditions which must be satisfied. These are the conditions of static equilibrium and compatibility. The first condition requires that the loads acting on the ends of each component must be such as to keep the member in equilibrium. Furthermore, the sum of the internal loads produced at a joint must equal the external loads at this joint, whilst the second condition requires that the displacements of the end of each member must be compatible with the displacements of the joints to which the member is attached.

Depending on the order in which the conditions of equilibrium and compatibility are applied, methods of frame analysis may be classified as:

a. Compatibility approach in which the static equilibrium of the structural system is satisfied first and the redundant forces are expressed by equations of displacement compatibility. This procedure is sometimes termed the force or the flexibility method.
b. Equilibrium approach in which compatibility conditions are used first to give rise to equations of joint equilibrium. This technique is also termed stiffness or displacement method.

Experience has, however, shown that there is no preference for one approach over the other. Yet while the compatibility method seems to be favoured for aircraft fuselage analysis (4, 5, 6), McLeod (90) suggested that the equilibrium approach is more readily adaptable for automatic solution.

In the present work, the compatibility method is utilized for, besides leading to the direct determination of the redundant forces in the connecting members, this technique lends itself to arrangements by which a well-conditioned flexibility matrix is obtained (see Appendix II). Similar techniques were employed for the analysis of shear walls by Frischmann et al. (49) and Jain and Chandra (66). In Frischmann's analysis the normal and shear deformations of the walls are ignored and the equations of compatibility are first worked out before the solution is sought by the use of digital computers. Jain and Chandra (66) included the effects of normal and shear deformation of the walls in their analysis and presented design charts for symmetrical walls.

2.2.2.2 Formulation of the Analysis

In the typical idealized statically indeterminate frame shown in Fig. 2.4 there are three degrees of freedom per beam. These degrees of freedom represent axial force, shearing force and moment (Fig. 2.4b). The degree of static indeterminacy "m" of the frame is, therefore, three times the number of connecting beams. The structure may, however, be rendered statically determinate by introducing releases at suitable points. The number of releases must be equal to the number of degrees of freedom of the structure.

For convenience, these releases are introduced at the mid-span of the connecting members. The application of releases will create displacement discontinuities which are removed by a set of straining actions \( x_1, x_2, \ldots x_m \).
Shear Wall Structure  WCA Idealization  Micheal's modified WCA method  The degrees of freedom in the beams.

Fig. (2.4), The Wide-Column Analogies.
These actions are, in fact, the unknown forces in the beams.

The condition of compatibility may be expressed as

\[
\begin{align*}
\sum_{j=1}^{m} f_{ij} x_j + f_{i1} + f_{i2} x_2 + \cdots + f_{im} x_m &= -u_i \\
\sum_{j=1}^{m} f_{ij} x_j + f_{i1} + f_{i2} x_2 + \cdots + f_{im} x_m &= -u_2 \\
\vdots & \vdots \\
\sum_{j=1}^{m} f_{ij} x_j + f_{i1} + f_{i2} x_2 + \cdots + f_{im} x_m &= -u_m
\end{align*}
\]

or symbolically as

\[
[f] \{x\} = -\{u\}
\]

where

- \([f]\) is the flexibility matrix of the order \((m \times m)\);
- \(\{x\}\) is the unknown vector \((m \times 1)\);
- \(\{u\}\) is the incompatible displacement vector \((m \times 1)\);
- \(m\) is the degree of statical indeterminacy;
- \([f]\) is essentially a square and symmetric matrix.

Once the straining actions \(\{x\}\) are evaluated the moments, shears and deflections of the walls can easily be computed.

Micheal\(^{(83)}\) suggested that the flexibility of the beam wall joint could produce a significant reduction in the degree of interaction of the wall. He has shown that account for the local yielding of the beams at the joint can be approximately included in the analysis by extending the span of the beam by an amount equal to its depth, as shown in Fig. 2.4c.

In Appendix II the properties of the flexibility matrix \([f]\) and the techniques of assembling and solving the set of simultaneous equations \((2.21)\) are stated. The same Appendix includes a description of the corresponding computer program and instructions on its use.
2.3. **EXPERIMENTAL WORK**

2.3.1 **Objective and Scope**

In the field of Engineering Design, when it is required to examine the suitability of a method of analysis of a particular structural system the theoretical calculations are often compared with experimental results based on model or full-scale tests. In other words, the experimental results are taken as the "exact" values. This, of course, is not true since the accuracy of the results usually depends on the model material, the loading and testing techniques. However, experience has shown that for the material and measuring equipment described below the experimental errors are unlikely to exceed ±1%. Hence, the experimental programme was designed to provide a criterion by which the accuracy of the analytical procedures is assessed.

Various shear wall models were made of Perspex and tested under idealized uniformly distributed load. The deflection profile was taken as the comparison parameter. These models are:

(i) Series A, which consists of three 2 wall structures with one row of openings. These are designated A1, A2 and A3 (Fig. 2.5).

(ii) Series B, which consists of three 3-wall structures with two rows of openings. These are designated B1, B2 and B3 (Fig. 2.6).

(iii) A 12-storey two-dimensional structure consisting of two- and three-wall assemblies interconnected by floor slabs (Fig. 2.7). This model is designated C.

The only varying parameters considered in this investigation are:

a) the number of interconnected walls.
b) the ratio of beam stiffness to wall stiffness.
c) the height of the building.
d) the effective width of the connecting floor slabs.
Series A and B served to establish the limitation of the method of analysis as regards the variables "a", "b" and "c". The effect of the height of the building was assessed by reducing the number of storeys of models A and B. Thus the number of tests conducted on the two-dimensional models was 18. These are designated as:


The figures following the stroke refer to the number of storeys of the model.

Model C was designed to furnish information about the effective width of the connecting slabs.

### 2.3.2 Preparation of the Specimen and Properties of the Material

Perspex lends itself as a suitable model material on the grounds of its cheapness and ease of workability. In fact, acrylic materials, besides being perfectly homogeneous and isotropic, possess a low elastic modulus and measurable deflections may thus be obtained at low loads. The disadvantage of such materials is their tendency to exhibit creep. Nevertheless, this may be overcome by using the on-off loading cycle suggested by Soane.

Series A and B were mirror-image models. These were supported at the centre and loaded on both sides, as shown in Fig. 2.8. Thus the fixed end condition was simulated.

Model C was constructed of Perspex panels joined together by gap-filling with Tensol cement No. 7. Fig. 2.9 shows various stages of the construction of the model. The structure was firmly held to a 1/4" inch Perspex sheet by means of square Perspex beams glued to both the sheet and the walls. The Perspex sheet was then screwed to a rigid steel base (Fig. 2.10). Machining and finishing of the models were carried out in accordance with the maker's recommendation.

The elastic constants of Perspex were taken as:
Fig. (2.5), Particulars of the Models - Series A

<table>
<thead>
<tr>
<th>MODEL</th>
<th>Number of Storeys</th>
<th>Total Height L</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1/15</td>
<td>15</td>
<td>15 in.</td>
</tr>
<tr>
<td>A1/10</td>
<td>10</td>
<td>10 in.</td>
</tr>
<tr>
<td>A1/5</td>
<td>5</td>
<td>5 in.</td>
</tr>
<tr>
<td>A2/15</td>
<td>15</td>
<td>15 in.</td>
</tr>
<tr>
<td>A2/10</td>
<td>10</td>
<td>10 in.</td>
</tr>
<tr>
<td>A2/5</td>
<td>5</td>
<td>5 in.</td>
</tr>
<tr>
<td>A3/15</td>
<td>15</td>
<td>15 in.</td>
</tr>
<tr>
<td>A3/10</td>
<td>10</td>
<td>10 in.</td>
</tr>
<tr>
<td>A3/5</td>
<td>5</td>
<td>5 in.</td>
</tr>
</tbody>
</table>
Fig. (26), Particulars of the models - Series B.
All walls and floor slabs 1/2" thick
Fig. 2.9(a), A typical storey-unit / Model C.
Fig. 2.10, Testing of Model C.
Young's Modulus "$E" = 0.42 \times 10^6$; and
Poisson's Ratio "$\nu" = 0.35

These figures were obtained from the maker's book\cite{125} and Soane\cite{107}.

2.3.3 Testing Techniques

The on-off loading technique used is as follows. The model is first set and left for an hour to assume its natural profile. Zero readings were then taken and a unit load was applied at each panel point. After half an hour the effective creep was complete and a set of readings was recorded and the loads removed. This process was repeated with increased load increments a sufficient number of times to get a linear load-deflection curve. Most of the tests were conducted at almost the same temperature and humidity level. The deflections were measured by means of Boulton-Paul inductive transducers type F55. Fig. 2.11 shows the loading and transducer arrangements in the various models. Deflection readings were directly displayed on a transducer-meter type C51 (Fig. 2.12).

2.3.4 Records of Results

The experimental and analytical deflection profiles of Models A are illustrated in Figs. 2.13 - 2.21. These figures also show the computed values of the moments and shear forces in the walls.

Fig. 2.22 gives the variation of the tip deflection "$\delta$" of these models with the number of storeys "$N$". The relationship between "$\delta$" and the depth of the connecting beams "$h$" for the two-wall models is plotted in Fig. 2.23.

Records of the results of the three-wall model (Series B) are also presented. Figs. 2.24 - 2.32 illustrate the experimental and theoretical deflection profiles of these models and show the computed moments on each of the component walls. The variation of the tip deflection "$\delta$" of the models with the number of storeys "$N$" and the depth of the connecting beams "$h$" is shown in Fig. 2.34 and Fig. 2.35 respectively.
The experimental and theoretical results* of the various walls of model C are collected in Fig. 2.36. In this figure the deflection of the walls obtained experimentally is compared with analytical results using the WCA method and assuming various "effective widths" of the connecting slabs. The results of this investigation are also listed in Table I. Column 3 of this table gives the tip deflection of the models as obtained by using Micheal's modified Wide Column method.

2.4 DISCUSSION OF RESULTS

2.4.1 Two-Wall Models (Series A)

Both methods of analysis (the shear connection and wide column) gave smaller values of deflection for all models tested. The results computed by the WCA, however, were better than those predicted by the SCM. The accuracy of the results seems to depend on two factors. These factors are:

(i) the height of the building; and
(ii) the stiffness of the connecting beams.

For the 15-storey model with slender connecting beams (A1/15) the error produced by the WCA and SCM analyses was -2.4% and -7.8% respectively. These values were as high as -5% and -13% for the model with stiff beams (A3/15). With low rise models the discrepancy between the experimental and the WCA and SCM results ranged from -6.25% and -36.50% respectively for slender beams (A1/5) to +2.87% and -37% for stiff connecting members (A3/5).

These results clearly indicate that the accuracy of the analytical solutions is sensitive to variations of the stiffness of the connecting beams and of the height of the building; the accuracy being high for tall structures and structures with slender beams and degenerates for buildings with stiff connecting members or low walls.

* The deflections of the component walls of Model C obtained by using the SCM and assuming various effective widths are listed in Table I. For clarity, these results were omitted from Fig. 2.36.
Fig. 2.11a, Testing of Model A2/10.
Fig. 2.11 C, Testing of Model 32/12.
Fig. 2.11-d, Positions of the transducers / Model C.
Fig. 2.11-E, Transducer Arrangement / Model C.
Fig. 2.12, The Transducer Meter.
The effect of these two factors on the accuracy of the results appears, however, to be more profound in the shear connection analysis. In fact, Fig. 2.21 reveals that the SCM is unsuitable for the analysis of low-rise structures with stiff connecting beams. The structural behaviour of such buildings is more of a wall with holes than walls coupled with connecting members, i.e. the structure behaves as one unit. Also, neglecting the shear and axial deformations in the shear-connection analysis may partly be responsible for the inaccuracy of the results. Nevertheless, the SCM is shown to produce reasonably accurate results for tall buildings (above 12 storeys) with slender beams. The trend of the results appears to indicate that the difference between the SCM and WCA calculations would decrease considerably as the structure gets taller.

It is evident that the shear connection analysis may be improved by accounting for the shear and axial deformations of the walls. Including the effect of local yielding deformation of the joint between the walls and the connecting beams, as suggested by Michael(83), may also enhance the accuracy of this procedure.

The wide column method, on the other hand, appears to be a very powerful approach. This may be demonstrated by the close agreement obtained in most cases between the experimental and the WCA results. The modified WCA suggested by Michael(83) gives even better assessment of deflections for low-rise models and models with slender beams (Table I). However, with stiff coupling beams Michael's correction becomes excessive and results in an overestimation of deflections.

It is worth mentioning that close agreement between the moments on the walls predicted by the WCA and SCM is obtained in all cases. In the meantime the computed values of the shear forces on the walls differ considerably (Figs. 2.14 - 2.21).
A discrepancy of as much as -62% was obtained between the shear forces on the walls calculated by the shear connection and the wide column procedures. This feature may be attributed to the idealisation included in the SCM.

Figs. 2.22 and 2.23, showing "\( \bar{\sigma} - N \)" and "\( \bar{\sigma} - d \)" relationship respectively, reveal some interesting points. Fig. 2.22 shows, as expected, that the flexibility of shear walls with slender beams is more sensitive to variations in the height of the structure than walls with stiff coupling members. Fig. 2.23, on the other hand, indicates that, for tall buildings, the structural stiffness increases rapidly with the beam stiffness. With low-rise structures it appears that changes of the width of the connecting beams result in an insignificant increase in the overall structural stiffness. Fig. 2.23 also shows that there exists a limiting value of the width of the connecting beams which gives optimum structural rigidity. With reference to the same figure, it appears that beam depths of 3/8 inch and 7/16 inch would result in optimum structural stiffness of the 15- and 10-storey models respectively.

2.4.2 Three-wall Models (Series B)

Similar to Series A the results of the three-wall models reveal that both analytical procedures underestimated the deflection of the model with the WCA giving a better assessment of the structural flexibility than the SCM. The discrepancy between the measured and computed results is large for structures coupled with stiff members and low-rise assemblies. Close agreement between the computed moments on the walls using the WCA and SCM approaches is obtained in the case of tall buildings with slender beams, B1/12. Figs. 2.27 and 2.30 indicate that for models with stiff connecting beams (B2/12 and B3/12) the SCM gives smaller values of moments than the WCA approach.
It appears that, with slender connecting beams the effect of the coupling action decreases with the height of the structure. This is demonstrated by Fig. 2.26 which reveals the ineffectiveness of the connecting members in the B1/4 model. The moments computed by the WCA method at the clamped end of model B1/4 are 1.55, 0.33 and 0.27 lb.in for the three component walls. When the walls are assumed to act as separate cantilevers the moment of each wall is 1.15, 0.335 and 0.335 lb.in. These figures suggest that the coupling action of the connecting beams ceased, for this model, to be effective and the walls behaved as separate cantilevers.

The discrepancy between the measured and the computed results for the models with stiff beams was large. For model B3/12 the WCA and SCM approaches underestimated the tip deflection of the structure by as much as -5.4%, and -16.5% respectively. The corresponding figures for B3/4 are -22% and -38.5%. Also, the moments computed by the SCM for the loaded walls of B3/12 and B3/4 are respectively 14% and 18.5% less than the WCA results.

Results using Micheal's modified WCA method, given in Table I, confirm that Micheal's correction, whilst slightly improving the accuracy of the analysis of walls with slender beams, renders erroneous results when the connecting beams are stiff.

2.4.3 Walls Connected by Floor Slab (Model C)

The results given in Fig. 2.36 and Table I suggest that the value of the "effective width" of the connecting slabs is dependent on the method of analysis employed. The close agreement between the deflection profile of the two-wall assembly obtained experimentally and by using the WCA and full effective width (3½") indicates that the assumption of fully effective bay width in conjunction with the WCA analysis produce accurate results. The SCM, on the other hand, is known to give smaller values of deflections than the wide-column approach. It can be expected, therefore, that the assumption of fully effective width will result in this method yielding an underestimated deflection value. In fact, it is shown in Table I that
when the effective width was taken as equal to the clear span between the adjoining walls, the SCM gave better assessment of the deflections.

The results of the three-wall assembly plotted in Fig. 2.36 are confusing. It appears that at the top of the model a very small part of the slab was contributing to the interaction between the walls. The effective width gradually increased in the lower parts of the model. In fact, in the lower half the effective width appears to be greater than the full bay width (2\frac{1}{2}\text{''}).

The SCM results are listed in Table I. The lack of conformity between the experimental and analytical results may be attributed to the fact that the three-wall structure is an edge wall, at which the stiffening effect of the connecting medium (due to the out-of-plane bending of the floor slabs) occurs in one side only. It appears that, for edge beams, the concept of constant-along-the-total height "effective width" does not hold. However, further experimental work is needed to establish the degree of interaction between the floor slabs and the walls at various levels of the structure and to explore the behaviour of these assemblies.

2.4.4 Summary and Conclusion

The present work comprises an investigation into the methods of analysis of multi-storey multi-bay shear wall structures. Two analytical approaches, namely the shear connection and the wide column analogy procedures, are presented and their suitability is examined by collating the theoretical results with experimental data. The testing programme was designed to cover specific variables. These are:

a. the height of the building;
b. the stiffness of the connecting beams; and
c. the effective coupling width in walls connected solely by floor slabs.

Due to limitation of time various other important factors, such as the stiffness of the walls, the position of openings etc. were not considered in the investigation.
Hence, within the limits of the experimental work the following conclusions may be drawn.

2.4.4.1 Methods of Analysis of Shear Wall Structures

a. The shear connection method

The mathematical solution of the differential equation of the SCM is a new contribution which allows the application of this method to multi-bay structures. This solution may also permit account to be taken in the analysis of the effects of the shortening of the beams and the shear deformation of the walls, which are neglected in the shear connection procedure.

In general the SCM results in an over-estimation of the structural stiffness. Most of the evidence indicates that the shear connection method is efficient for the analysis of tall structures with slender beams and its accuracy in estimating the deflections degenerates as the height of the building gets shorter and the beams become stiffer. Notwithstanding the difference in the assessment of the structural stiffness, the SCM appears to give, in most cases, accurate values of the moments in the various walls.

b. The wide column analogy

Test results show that this frame analogy approach is extremely powerful in the analysis of shear walls with openings. While the procedure overestimates the stiffness of the structure, its accuracy in assessing the deflections was, in most cases, better than the SCM. This may be attributed to the realistic idealisations included in this approach. Similar to the shear connection method, the accuracy of the WCA improves for taller structures with slender beams. The maximum error in estimating the deflection by the WCA for low walls with stiff beams (B3/4) was 22%.

The correction factor suggested by Micheal appears to slightly improve the accuracy of the solution of low-rise walls and walls with slender beams. However, this factor gave erroneous results when the connecting beams were stiff. It should be realized that Micheal's
analysis is based on monolithic structures (i.e. with perfect joints). In practice, the beam-wall joint is usually cast in-situ and is reinforced in two directions and hence Michael's treatment may not be representative of the actual behaviour of the joint.

It is shown that the WCA results in a large number of simultaneous equations and therefore this procedure is efficient only if a digital computer is available. The computer program (WCAP) written to carry out the calculations is capable of the analysis of multi-bay structures yet it is limited to walls with regular opening pattern.

2.4.4.2 Structural Behaviour

a. Effect of the beam stiffness on the structural behaviour

The results of the present investigation clearly indicate that the structural behaviour depends on the ratio of the stiffnesses of the connecting beams and the adjacent walls. The effectiveness of the coupling action produced by the connecting members appears to decrease for smaller values of beam stiffness/wall stiffness. For tall buildings with very flexible beams the structure tends to behave as separate cantilevers. Whereas with stiff connecting beams the coupling action increases and the structure behaves as interconnected walls. Very stiff beams, particularly in low-rise buildings, result in a structural behaviour similar to that of walls with holes rather than walls coupled with connecting members.

The results also show that for tall buildings the structural stiffness increases rapidly with the beam stiffness whereas in the case of low-rise walls changes in the width of the beams result in an insignificant increase in the overall structural stiffness.

b. Beam-wall stiffness ratio for optimum structural rigidity

Figs. 2.23 and 2.35 show that for a given shear wall structure there exists a limiting value of the beam-wall stiffness which produces optimum structural rigidity under lateral loading.

c. Effect of variations in the height of the structure

it appears that the flexibility of structures with slender beams is more sensitive to variations in the
height of the building than structures with stiff members.

d. **Walls connected with floor slabs**

The results of mode C indicate that the width of the floor slab which provides interaction between the walls depends on the position of the wall assembly. The assumption of fully effective bay width in both the WCA and SCM gave fairly accurate results for the internal walls. This assumption when employed for the analysis of the external wall assembly resulted in erroneous estimate of the structural stiffness which may be attributed to the effect of out-of-plane bending of the floor slabs. In a real structure the floor slabs are free to bend under vertical loadings. The bending configuration depends on the boundary conditions of the floor and varies from storey to storey. This may alter the mechanism of interaction between the walls and slabs and hence affects the structural behaviour.

The width of the slab contributing to the interconnection of the component walls of the edge structure appears to have varied along the length of the assembly. The measured deflections at the top of the wall reveal that a very small portion of the connecting slab has contributed to the interaction between the walls.

The effective width then gradually increased in the lower parts of the structure and the measured deflections of the bottom half show that the effective width was greater than the full bay width. This feature merits further investigation.
Model A1/15

$h = 0.125''$

$\omega = 0.10\text{ lb/in.}$

Fig. (2.13), Results of Model A1/15.
Model A1/10

\[ h = 0.125'' \], \[ w = 0.10 \text{ lb./in.} \]

Shear Force \(1\text{b. x }10^{-1}\)

Bending Moment \(1\text{b. in. x }10^{-1}\)

Deflection \(\text{in. x }10^{-4}\)

Fig. (2.14), Results of Model A1/10
Model A1/5

$h = 0.125"$, $\omega = 0.10$ lb/in.

Fig.(215), Results of Model A1/5
Model A2/15

$h = 0.25''$

$\omega = 0.10 \text{ lb./in.}$

Fig. (2.16), Results of Model A2/15
Model A2/10

\[ h = 0.25" \quad , \quad w = 0.10 \text{ lb/in.} \]

Fig. (2.17), Results of Model A2/10
Model A2/5

\( h = 0.25\,\text{"}, \quad w = 0.10 \, \text{lb/in.} \)

Fig. (2.18), Results of Model A2/5
Model A3/15

$h = 0.50''$, $w = 0.10$ lb/in.

---

**Fig. (2.19), Results of Model A3/15**
Model A3/10

\[ h = 0.50'' , \quad w = 0.10 \text{ lb/in.} \]

Shear force: \( 1 \text{ lb} \times 10^{-1} \)

Bending moment: \( 1 \text{ lb-in.} \times 10^{-1} \)

Deflection: \( 1 \text{ in.} \times 10^{-4} \)

Fig. (2.20), Results of Model A3/10
Model A3/5

$h = 0.50''$, $w = 0.10$ lb/in.

Shear Force 1b. x $10^{-1}$

Bending Moment 1b.in. x $10^{3}$

Deflection in. x $10^{-5}$

Fig. (2.21), Results of Model A3/5
Fig. (2.22), Deflection of top of the wall vs. Number of Storeys

Models A
$\omega = 0.10 \text{ lb/in.}$

Fig. (2.23), Tip-deflection vs. Depth of beams.

Series A
$\omega = 0.10 \text{ lb/in.}$
Wall A

\[ \omega = 0.10 \text{ lb./in.} \]

Deflection in. \( \times 10^{-3} \)

Wall B

Deflection in. \( \times 10^{-3} \)

Wall C

Deflection in. \( \times 10^{-3} \)

\[ \omega = 0.10 \text{ lb./in.} \]

Storey Number

Fig. (2.24), Results of Model B1/12
Wall A, Bending Moment 16 in.

Wall B, Bending Moment 16 in.

Wall "C", Bending Moment in. 16 in.

Fig. (22A b), Moments on the walls. Model B1/12
Fig. (2.28), Deflection of the walls - Model B1/8
Wall "A"

Wall "B"

Wall "C"

Moment 16 in. x 10^{-1}

Moment 16 in. x 10^{-1}

Moment 16 in. x 10^{-1}

Fig. (2.25 b). Moments on the walls - Model B1/3
Fig. (2.26), Deflection of the walls - Model B1/A
Fig. (2.26 b), Moments on the walls - Model B1A
Wall A

Wall B

Wall C

\[ \omega = 0.10 \text{ lb/in.} \]

Deflection in. \times 10^{-3}

Deflection in. \times 10^{-3}

Deflection in. \times 10^{-3}

Storey Number

Fig. (2.27), Deflection of the walls - Model B2/12
Fig. (2.27 b), Moments on the walls - Model B2/12
Fig. (2.28), Deflection of the walls - Model B2/8
Wall A

Wall B

Wall C

--- SCM ---

--- WCA ---

Moments 16 in. x 10^-1

Fig. (2.28-b), Moments on the walls - Model B2/8
Fig. (2.29), Deflection of the walls - Model B2/4
Figure 2.29b, Moments on the Walls - Model B2/4
Fig. (2.30), Deflection of the walls - Model B3/12
Fig. (2.30 b), Moments on the walls - Model B3/12
Fig. (2.31), Deflection of the walls - Model B3/8
Fig. (2.31 b), Moments on the walls – Model B3/8
Fig. (2.32), Deflection of the walls - Model B3/4
Wall A

Wall B

Wall C

--- SCM
--- WCA

Fig. (2.32 b), Moments on the walls - Model B3/4
Fig. (2.33), Tip-deflection $\delta$ vs. depth of connecting beam $h$.

Fig. (2.34), Tip-deflection $\delta$ vs. Number of stories $N$.

$e = 0.10 \text{ in.}$

$\omega = 0.10 \text{ in.}$

$\frac{h}{L} = 0.10$

Models B

Models B1

Models B2

Models B3
Walls connected by floor slabs

Model "C"

3-Wall Assembly

2-Wall Assembly

Fig. (2.36), Results of Model "C"
<table>
<thead>
<tr>
<th>SERIES</th>
<th>DESIGNATION</th>
<th>SCM (_{x10^{-3}}) in.</th>
<th>WCA (_{x10^{-3}}) in.</th>
<th>MOD. WCA (_{x10^{-3}}) in.</th>
<th>EXP (_{x10^{-3}}) in.</th>
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<td>A</td>
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<tr>
<td></td>
<td>(t_b = 2'')</td>
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<td>15.54</td>
<td>-</td>
<td>16.00</td>
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<tr>
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<td>(t_b = 3'')</td>
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<td>(t_b = 1'')</td>
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<td>(t_b = 2'')</td>
<td>9.05</td>
<td>9.42</td>
<td>-</td>
<td>12.60</td>
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PART II

STRESS ANALYSIS
Frozen stress patterns of a two-wall model made of Araldite 753 and subjected to UDL.
CHAPTER 3

STRESS ANALYSIS OF MULTI-STOREY SHEAR WALL STRUCTURES

3.1 INTRODUCTION

3.1.1 General

Besides estimating the structural stiffness, the design of shear walls involves assessing the strength of the building. The analysis of stresses in shear wall assemblies is extremely complex. These structures are, in mathematical terminology, classified as "boundary value problems with mixed boundary conditions". Furthermore, the presence of openings enhances the property of "multiple-connectivity". No exact analytical solution is available for such problems. Nevertheless, various approximate procedures capable of providing fairly accurate solutions can be found. In general, the approximations pertaining to these techniques may be of two different natures. These are:

analytical nature in which the governing equations are solved by approximate procedures such as the finite difference or the Line solution methods; or

physical nature in which the actual structural continua is substituted by an idealized system, a solution of which can be found. The finite-element grid analogy and lattice analogy methods fall into this category.

The experimental techniques can also serve a useful purpose in the analysis of shear walls and their merits cannot be overlooked. In this context, the science of photo-elasticity offers a simple but very powerful tool for the solution of interconnected walls with irregular shapes and subject to complex loading and boundary conditions. In addition, the ability of this method to render informations about local stress concentrations and regions of high stresses is prominent.
3.1.2 Review of Previous Work

The mathematical methods of stress analysis of shear wall structures may be divided into three main categories. These are:

(i) Elementary Methods

Prior to the development of high-speed computers, analytical treatment of shear wall structures was based on simple strength-of-material principles. These methods were aimed at first evaluating the loads on the individual walls. Interconnected assemblies were assumed to behave in a certain simplified fashion, e.g. separate cantilever or Benjamin's assumptions, thus enabling the determination of the straining actions resisted by each wall.

These methods generally underestimate the structural stiffness and consequently the bending moments are overestimated. Nowadays, with the aid of electronic computers various refined procedures for the calculation of the loads in the individual walls may be obtained (49, 90, 66). These methods give better assessment of the moments, shear forces, etc. resisted by the walls. Once the straining actions are evaluated, the stresses in the sections may be computed by using the Engineer's theory of bending, i.e.

\[ f = \pm \frac{N}{A} \pm \frac{M}{I} \]  \hspace{1cm} (3.1)

It should be noted that equation (3.1) is based on Bernoulli's assumptions that under loading the resulting deflections are small compared with the dimensions of the structure, and that the transverse plane sections remain plane after bending. Hence, the suitability of this method depends on how good the resemblance is between the actual behaviour of the structure and the behaviour assumed by Bernoulli.

(ii) Solution of the Governing Differential Equations

This approach involves formulating and solving the governing differential equations. A shear wall structure is essentially a two-dimensional plane-stress problem for which equations of static and kinematic equilibrium
may be found. The static equilibrium of the structure (assuming no body forces) may be expressed as follows:

\[ \frac{\partial f_x}{\partial x} + \frac{\partial f_{xy}}{\partial y} = 0 \quad \text{and} \quad \frac{\partial f_y}{\partial y} + \frac{\partial f_{xy}}{\partial x} = 0 \] (3.2)

Also, the condition of Kinematic equilibrium (compatibility) may be given by

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (f_x + f_y) = 0 \] (3.3)

Combining (3.2) and (3.3) gives the biharmonic equation

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 F = \nabla^2 F = 0 \] (3.4)

where \( F \) is a function of \( x \) and \( y \) and termed the "Airy Stress Function", and \( \nabla \) being the Laplace's operator.

The stresses may be expressed in terms of the function "\( F \)" as

\[ f_x = \frac{\partial^2 F}{\partial y^2} ; \quad f_y = \frac{\partial^2 F}{\partial x^2} ; \quad \text{and} \quad t_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \] (3.5)

Hence, the problem degenerates to finding a function \( F \) which satisfies the biharmonic equation (3.4) and subject to given boundary conditions. Generally, finding such a function is, except in the simplest cases, extremely complex. Nevertheless, there exist various approximate procedures capable of providing acceptable stress functions, e.g. selection of a function by trial and error method (58,86) minimization by variational methods of a series of functions with undetermined coefficients (86), expressing the function in terms of power series (68,86). A review of these methods was reported by Amaratunga (2) and Kazimi (69).

In regions where the stress concentration needs to be determined the analysis may be carried out further. The selected stress function may be expressed in terms of complex potentials and the region conformally transformed inside (or outside) a unit circle whereby the stresses may be calculated (105,2,86). This method was utilized in the analysis of shear walls with openings by Amaratunga (2).
A solution of the biharmonic function (3.4) may also be reached by means of the finite difference approximation (112), the line solution method (69) or Synge's space function (110).

The dynamic relaxation procedure which was suggested by Day (133) in 1960 is another technique which falls into this category. The innovation of this method lies in the introduction of dynamic terms into what is basically a static problem and treating the required static solution as a limiting case of the dynamic analysis. The derivation of the dynamic relaxation equations involves the separate consideration of:

a) the motion of an element of the body due to the imposed body forces; and
b) the elastic relation between the stresses and displacements during the course of the motion.

These equations are then expanded in finite difference form and in conjunction with the boundary conditions lead to the static solution by successive iterations.

(iii) Solution of a Substitute Structure

The methods which belong to this category involve idealizing the structural continua as an assemblage of a finite number of units, the properties of which can be evaluated. By solving the substitute system an approximate solution of the actual structure is obtained. Hrenikoff (62) and McHenry (88) independently derived a system of plane stress analysis based on a simple pin-jointed framework. In this approach, which is termed the lattice analogy, the wall is divided into a square grid and each element of the grid is replaced by a cell of the framework. These cells consist of main and diagonal bars which carry axial load. The behaviour of the actual plate is simulated by suitable proportioning of the framework members. McCormick (67) introduced a similar analysis in which the bending stiffness of the external members of the cells is included as an extra parameter. Grinter's grid analogy method (57) idealizes the actual structure as a square-grid framework consisting of members with rigid joints. Kazimi (69) has applied Grinter's approach to shear wall structures.
with openings. Clough et al.\cite{26} introduced the finite element method in 1956. Eversince this method gained popularity which overshadowed the lattice and grid analogies. For, besides its inherent ability to deal with two- and three-dimensional plane-stress problems this method permits account to be taken of material properties such as orthotropy, non-linearity etc. This has been demonstrated by work by Clough\cite{27,28} and others (7,92,95,96,96,119).

Another striking example of the pre-eminence of the finite element procedure was demonstrated by Rashid\cite{96}. In his work on the ultimate load analysis of prestressed concrete pressure vessels Rashid has shown that by successive iterative processes of the finite element method a visco-elastic analysis of complex structure can be obtained. By adopting this technique Rashid managed to locate the regions of high stresses in the pressure vessels and illustrated by the aid of computer graphic the propagation of cracking under loads.

In 1950, Ostuki\cite{93} reported a different approach in the analysis of reinforced concrete shear walls with openings. His method was based on idealizations similar to those used in the design of stiffened-skin fuselage structures. That is, that reinforced concrete walls may be considered as consisting of concrete panels resisting shear whereas the reinforcing bars resist direct stresses only.

Various other researchers utilized experimental techniques in the study of shear wall structures. Futami and Fujimoto\cite{57} carried out photoelastic investigations on two-storey walls containing a single opening in each storey. Results depicted by graphs of photoelastic study of single-storey shear walls with different shapes was also reported by Kokinopoules\cite{74}.
3.1.3 Objective and Scope

The present work is concerned with the elastic stress analysis of shear wall structures with openings. The analysis is primarily based on the fact that, by virtue of the wide column analogy method outlined in Part I of this thesis, the forces in the connecting members of the multi-bay shear wall structures may be determined fairly accurately. Hence, each of the component walls may be treated separately as a cantilever beam subject to the corresponding set of loading (shear, lateral forces and moments), Fig. 3.1.

Three factors necessitated the adoption of this technique. These are:

a. the size of computer available (16 K);
b. the difficulty of combining beam elements whose properties can adequately be described along a line and plane-stress two-dimensional wall elements; and
c. the size of the stiffness matrix of the whole structure is much larger than that of a single wall (e.g. for two wall structure \( K_{structure} \) is 4 times the size of \( K_{wall} \).

It may be argued that whilst the forces in the beams can be evaluated the stress variation resulting from these forces near the re-entrant corners would be difficult to assess. It is believed, however, that an error in describing the stress distribution near the joint between the beams and walls will have a slight local effect on the stresses in the adjacent walls.

Three various theoretical approaches are utilized. These are:

(i) The Engineer's theory of bending
(ii) The stress function method
(iii) The Dynamic Relaxation method
(iv) The finite element method.

The difference in the nature of these approaches may be noted. The first method is based on the strength-of-materials principles, the second procedure is based on the fundamentals of the theory of Elasticity and the last technique is one based on numerical analysis.
Fig. (3.1), Interconnected wall and a component cantilever

Fig. (3.2), Concentrated force on a large plate
With the view to establishing the comparative accuracy of each of these methods the three approaches were employed in the analysis of a typical shear wall structure with openings and other related two-dimensional plane-stress problems.

For brevity, the details of the mathematical derivation are omitted from the text. The sequences of the operations together with a description of the corresponding computer programs are embodied in the appendices. Appendix 4 on the finite-element method is particularly interesting. It is presented as a self-contained chapter comprising a historical review, formulation of the mathematical argument and various finer analytical and computational points associated with the finite element technique.

The experimental work presented in this part of the thesis is based on photo-elastic investigation using the frozen stress technique. A number of shear wall models of different shapes were tested and the results compared to the theoretical calculations.

Discussion of the results and summary of the findings are contained in Section 3.5.

3.1.4 List of Notations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$x, y$</td>
<td>Co-ordinate axes</td>
</tr>
<tr>
<td>$f$</td>
<td>Stress in lb/in$^2$</td>
</tr>
<tr>
<td>$f_x, f_y$</td>
<td>Stress in the x- and y- direction respectively</td>
</tr>
<tr>
<td>$t_{xy}$</td>
<td>Shear stress in lb/in$^2$</td>
</tr>
<tr>
<td>$F$</td>
<td>Airy Stress Function</td>
</tr>
<tr>
<td>$N$</td>
<td>Normal force</td>
</tr>
<tr>
<td>$M$</td>
<td>Applied bending moment</td>
</tr>
<tr>
<td>$Q$</td>
<td>Shear force</td>
</tr>
<tr>
<td>$A, I$</td>
<td>Cross sectional area and MOI respectively</td>
</tr>
<tr>
<td>$\vec{x}, \vec{y}$</td>
<td>Body forces in the x- and y- direction respectively</td>
</tr>
<tr>
<td>$\alpha_0, \alpha_i, b_i$</td>
<td>Constants</td>
</tr>
<tr>
<td>$\mu, \nu$</td>
<td>Displacement in the x- and y- directions respectively</td>
</tr>
</tbody>
</table>
Poisson's Ratio
Young's Modulus
Modulus of rigidity
Maximum and minimum principal stress respectively.

Abbreviations

ETB = Engineering Theory of Bending
(or elementary strength-of-materials theory)
SF = Stress Function Method
FE = Finite Element Method
DR = Dynamic Relaxation Technique
UFV = Unit Fringe Value
UDL = Uniformly distributed load

Certain other quantities are defined in the text as they occur.

3.2 METHODS OF STRESS ANALYSIS

3.2.1 The Elementary ETB Approach

Fig. 3.1 shows an individual wall of a typical shear-wall structure. Once the forces in the connecting beams are evaluated, the bending moments, shearing forces and thrust acting on the wall can be ascertained.

Hence, the stresses at any point \((x, y)\) may be given by:

\[
\begin{align*}
\sigma_x &= \sigma_j; \\
\sigma_y &= \pm \frac{N}{A} \pm \frac{M(x,y)}{I} \\
\tau_{xy} &= \pm \frac{Q}{2I} \left[ \alpha^2 - y^2 \right]
\end{align*}
\]  

(3.6)

This method is attractive and simple but its applicability is limited to structures behaving in accordance with Bernoulli's assumptions. In other words, erroneous results may be obtained for structures with low aspect ratio, i.e. deep beams. Furthermore, this method is incapable of providing information about local stress concentration.
Generally, regions of stress concentration in shear-wall structures occur locally at the re-entrant corners. The ratio between the spacing between these corners and the depth of the connecting beams is, in most cases, large. This justifies an analogy to be drawn between these forces acting on the wall and a concentrated force acting on a straight boundary of an infinitely large plate (Fig. 3.2). In such a case the stress distribution in the plate is known to be purely radial\(^{112}\). The general expression for these stresses is

\[
\sigma_r = \frac{P}{r} \cos \theta \tag{3.7}
\]

where \(\theta\) and \(r\) are the polar co-ordinates of the point in the plate; and \(A\) is a constant.

Resolving equation \(3.7\) in the \(x\)- and \(y\)- directions and noting that \(\cos \theta = \frac{x}{r}\) and \(\sin \theta = \frac{y}{r}\) gives

\[
\sigma_x = -\frac{2P}{\pi tr^4} x^3 \quad ; \\
\sigma_y = -\frac{2P}{\pi tr^4} xy^2 \quad ; \quad \text{and} \quad (3.8)
\]

\[
t_{xy} = -\frac{2P}{\pi tr^4} x^2 y
\]

where \(P\) is the applied force

\(t\) is the thickness of the plate.

These expressions are valid whether the force "P" is normal, parallel or inclined to the straight boundary provided the \(x\)- axis is taken along the direction of the force and the \(y\)- axis at 90° anticlockwise from it.\(^{50}\)
3.2.2 The Stress Function Approach

This analytical technique involves working out a stress function \( F \) which satisfies the biharmonic equation

\[
\frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \tag{3.9}
\]

and subjected to the boundary conditions given by

\[
\begin{align*}
lf_x + mt_{xy} &= \bar{x} \quad \text{and} \\
mf_y + lt_{xy} &= \bar{y}
\end{align*} \tag{3.10}
\]

where \( l \) and \( m \) are the direction cosines.

As mentioned above, exact determination of such a function would be very difficult due to the complexity of the boundary conditions. Nevertheless, approximate functions serve in providing reasonably accurate solutions. These functions must necessarily satisfy the biharmonic equation (3.9) since this equation ensures an overall state of equilibrium and compatibility in the structure. The functions need not, however, satisfy all the boundary conditions. In fact it is because of the inability of a stress function to satisfy all the boundary conditions that this function is an approximate one. Notwithstanding the violation of some boundary conditions, it can be shown\(^{(45,112,85,53)}\) that this approach gives accurate solutions in regions at reasonable distances from the boundaries at which equation 3.10 is not satisfied.

It is conceivable that if a stress function representing each set of the loads acting on the wall can be found, the stress function of the whole structure may be constructed by superimposing the functions of the individual sets of loadings.

The lateral loadings in the beams, Fig. 3.3, can easily be represented by a Fourier’s series. This series may be given by:

\[
P_{ij} = a_0 + \sum a_i \sin \frac{n\pi}{L} x + \sum b_i \cos \frac{n\pi}{L} x. \tag{3.11}
\]
Fig. (3.3), The individual sets of loading on the component cantilever.

Fig. (3.4), Stresses on an element - DR analysis.
Where $a_0$ represents a uniformly distributed load; 
$(a_1 \sin \frac{\pi x}{L})$ and $(a_2 \cos \frac{\pi x}{L})$ sin and cosine load 
intensity respectively;

$L = 2 x$ height of the wall;
$P_j$ the lateral loading.

A stress function for each component term in this series can be obtained (112).

Similarly, an approximate stress function for the coupling forces in the walls resulting from the shearing forces in the beams may be found. Hence, the stress function of the wall may be expressed as

$$F = F_0 + F_3 + F_C + F_{sh}$$

(3.12)

Where $F_0$ = Stress function for UDL on the structure
$F_3$ = Stress function for a beam loaded by $\sin \frac{\pi x}{L}$
$F_C$ = Stress function for a beam loaded by $\cos \frac{\pi x}{L}$
$F_{sh}$ = Stress function for the resulting shear forces.

The stresses may then be computed by using the following expressions

$$f_x = \frac{\partial^2 F}{\partial y^2}; \quad f_y = \frac{\partial^2 F}{\partial x^2}; \quad \text{and} \quad t_{xy} = -\frac{\partial^2 F}{\partial x \partial y}.$$  

(3.13)

Appendix 3 contains the detailed derivation of the stress functions (3.12). The computing process is also described in the same Appendix.

This approach is expected to give inaccurate results in the vicinity of boundaries where equation 3.10 is violated. Hence, normal stresses will be obtained at the top edge of the wall which is contrary to the fact that unloaded edges are free of normal stresses. However, it can be shown (Appendix 4) that these stresses, besides being self-equilibrating, are self-balanced, i.e. they produce no bending effect. Another source of error is the inability of the stress function $F_{sh}$ to represent all the boundary conditions. This function, whilst describing the state of bending of the walls resulting from the shearing forces in the connecting beams, does not account for the local shear stresses at the re-entrant corners.
3.2.3 The Dynamic Relaxation Method (DR)

Dynamic relaxation is essentially an iterative method for the analysis of the finite difference formulations of the equations of elasticity. This technique, originally suggested by Day (133) for the analysis of Tidal flow, was developed to deal with two- and three-dimensional boundary value problems. Otter (136) and others (139, 140) employed this procedure for the analysis of prestressed concrete pressure vessels. The application of the DR to other structural problems, e.g. arch dams, frame structures, etc., was made by Cassel et al. (131), Chaudhary et al. (132) and Rushton (138). Most of the mathematical basis of the method is outlined in various papers (133, 134, 135, 136).

A comprehensive account of the DR was recently reported by Welch (139). Welch's work is interesting for it was devoted to exploring the limitations of this procedure and examining the effect of various factors, such as the damping coefficient, the boundary conditions, etc., on the solution.

The concept of this technique is that the structure vibrates due to the sudden application of body forces. Then, by virtue of the structure possessing internal damping and the possible effect of any external damping factors, the structure settles whilst maintaining, during the course of this motion, the basic elastic relationship between the stresses and displacements. This sequence of events may be expressed mathematically as follows:

a. The equations of damped vibration of a structural system.

The equation of motion is: force = mass x acc.

This relationship can be expressed as, reference Fig. 3.4,

\[
\frac{\partial f_x}{\partial x} \Delta x \Delta y + \frac{\partial f_{xy}}{\partial y} \Delta x \Delta y = C_x \rho \Delta x \Delta y \dot{u} + P \Delta y = \rho \Delta x \Delta y \frac{\partial \dot{u}}{\partial t}
\]  

(3.14)

which may be rewritten as: 
\[
\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{\rho} \cdot \frac{\partial \mathbf{f}_x}{\partial x} + \frac{1}{\rho} \cdot \frac{\partial t_{xy}}{\partial y} - C_x \cdot \dot{\mathbf{u}} + \frac{1}{\rho} \cdot \frac{P}{\partial x}
\]

Similarly, the motion in the y-direction may be given by

\[
\frac{\partial \dot{\mathbf{v}}}{\partial t} = \frac{1}{\rho} \cdot \frac{\partial \mathbf{f}_y}{\partial y} + \frac{1}{\rho} \cdot \frac{\partial t_{xy}}{\partial x} - C_y \cdot \dot{\mathbf{v}} + \frac{1}{\rho} \cdot \frac{q}{\partial y}
\]

\[C_x\] and \[C_y\] are the damping coefficients in the x- and y-directions respectively.

b. The stress-strain relationship for two-dimensional problems.

The stress-strain relationship may be expressed as

\[
\begin{align*}
\mathbf{f}_x &= (\lambda + 2\mu) \frac{\partial \mathbf{u}}{\partial x} + \lambda \frac{\partial \mathbf{v}}{\partial y} \\
\mathbf{f}_y &= \lambda \frac{\partial \mathbf{u}}{\partial x} + (\lambda + 2\mu) \frac{\partial \mathbf{v}}{\partial y} \\
t_{xy} &= \mu \left( \frac{\partial \mathbf{u}}{\partial y} + \frac{\partial \mathbf{v}}{\partial x} \right)
\end{align*}
\]

\[\lambda, \mu\] are Lamé's constants.

\[\lambda = \frac{E\nu}{1-\nu^2}\] for plane-stress

\[= \frac{E\sigma}{(1+\nu)(1-2\nu)}\] for plane-strain

\[\mu = \frac{E}{2(1+\nu)}\] for both plane-stress and plane-strain.

By differentiating equation 3.16 with respect to time it becomes:

\[
\begin{align*}
\frac{\partial \mathbf{f}_x}{\partial t} &= (\lambda + 2\mu) \frac{\partial \dot{\mathbf{u}}}{\partial x} + \lambda \frac{\partial \dot{\mathbf{v}}}{\partial y} \\
\frac{\partial \mathbf{f}_y}{\partial t} &= \lambda \frac{\partial \dot{\mathbf{u}}}{\partial x} + (\lambda + 2\mu) \frac{\partial \dot{\mathbf{v}}}{\partial y} \\
\frac{\partial t_{xy}}{\partial t} &= \mu \left( \frac{\partial \dot{\mathbf{u}}}{\partial y} + \frac{\partial \dot{\mathbf{v}}}{\partial x} \right)
\end{align*}
\]

- 42 -
Expressions 3.15 and 3.17 are the basic equations in the dynamic relaxation analysis. In their finite difference form these equations become:

\[
\dot{u}_{ij}^{t+1} = \Delta t \left[ \frac{\dot{u}_{ij}^t}{\Delta t} + \frac{1}{\rho \Delta x} \left( f_{x,i,j}^t - f_{x,i-1,j}^t \right) + \frac{1}{\rho \Delta y} \left( f_{y,i,j+1}^t - f_{y,i,j}^t \right) \right.
\]

\[
- \frac{C_x}{2} \left( \dot{u}_{ij}^{t+1} + \dot{u}_{ij}^t \right) + \frac{p_{ij}}{\rho} \Delta x \right] \]

\[
\dot{v}_{ij}^{t+1} = \Delta t \left[ \frac{\dot{v}_{ij}^t}{\Delta t} + \frac{1}{\rho \Delta y} \left( f_{y,i,j}^t - f_{y,i,j-1}^t \right) \right.
\]

\[
+ \frac{1}{\rho \Delta x} \left( t_{xy,i,j}^t - t_{xy,i,j-1}^t \right) - \frac{C_y}{2} \left( \dot{v}_{ij}^{t+1} + \dot{v}_{ij}^t \right) + \frac{q_{ij}}{\rho \Delta y} \right] \]

and equation 3.8 becomes:

\[
f_{x,i,j}^{t+1} = \Delta t \left[ \frac{f_{x,i,j}^t}{\Delta t} + \frac{(\lambda + 2\omega)}{\Delta x} \left( \dot{u}_{x,i-1,j}^t + \dot{u}_{x,i,j}^t \right) + \frac{\lambda}{\Delta y} \left( \dot{v}_{x,i,j+1}^t + \dot{v}_{x,i,j}^t \right) \right] \]

\[
f_{y,i,j}^{t+1} = \Delta t \left[ \frac{f_{y,i,j}^t}{\Delta t} + \frac{\lambda}{\Delta x} \left( \dot{u}_{y,i,j+1}^t + \dot{u}_{y,i,j}^t \right) + \frac{(\lambda + 2\omega)}{\Delta y} \left( \dot{v}_{y,i,j}^t + \dot{v}_{y,i,j-1}^t \right) \right] \] from (3.17)

\[
t_{xy,i,j}^{t+1} = \Delta t \left[ \frac{t_{xy,i,j}^t}{\Delta t} + \frac{\omega}{\Delta x} \left( \dot{v}_{xy,i,j-1}^t + \dot{v}_{xy,i,j}^t \right) + \frac{\omega}{\Delta y} \left( \dot{v}_{xy,i,j}^t + \dot{v}_{xy,i,j+1}^t \right) \right] \]

where \(i,j\) are the positions of an element in the finite difference grid in the x- and y-direction respectively.

The damping coefficient per unit mass per unit velocity has been calculated by Otter(136,137). Its value was reported to be

\[
K = 2 \omega \cdot \Delta t \quad (3.20)
\]

where \(\omega\) = angular vibration frequency; and

\(\Delta t\) = time interval.
Once the damping factor has been determined, the calculations start by assuming that, at time $t = 0$, both the velocities and displacements are zero at all points of the structure. Substituting zero velocities in equation 3.19 gives the new stresses at time $t = t_1$. Inserting these values of stresses into equation 3.18 gives the corresponding new velocities. The iteration then continues until the accelerations and velocities approximate to zero, thus ensuring static equilibrium. Deflection of the various points may, at any stage, be computed by integrating the velocities, i.e.

$$u_{t+1} = u_t + \dot{u}_{t+1} \Delta t$$
$$v_{t+1} = v_t + \dot{v}_{t+1} \Delta t$$

(3.21)

Welch (139) developed a computer program for Taylor Woodrow Construction Ltd. to carry out these calculations on an IBM 7094. This program has been extensively used for the analysis of prestressed concrete pressure vessels (139, 140) and was employed from the solution of various elasticity problems in this thesis.

3.2.4 The Finite Element Method

The finite element procedure(5, 21, 25, 52) utilizes the matrix stiffness method for the analysis of two- and three-dimensional structural systems. The concept of this method is based on idealising the structural continuum as an assemblage of panel elements interconnected at nodal points and subjected to conditions of plane-stress (Fig. 3.5). The elements may be triangular, rectangular or, indeed, of any geometrical shape. However, experience(25, 90) has shown that rectangular elements yield better approximations of stresses and deflections in addition to presenting an easier programming task.

Generally, two degrees of freedom corresponding to the two translations in the co-ordinate directions are
Fig. (3.5), The Finite-Element Idealization.
allocated to each node. A third degree of freedom characterizing rotation of the nodal corners may also be added\(^ {90}\). Fig. 3.5c shows the directions of degrees of freedom at the nodal points of a typical rectangular panel element. The elastic characteristic of the individual elements may be expressed by the relationship between the forces acting at the nodes and the deflections resulting therefrom. This relationship is termed the "stiffness properties" of the element and may be expressed as

\[
[K_e] \{ \Delta_e \} = \{ S_e \}
\]

(3.22)

where \([K_e]\) is the stiffness matrix of the individual element (8 x 8 for rectangular panels); and \([S_e], [\Delta_e]\) are the vectors of the load applied at the nodal points and the resulting displacements respectively.

A pre-requisite to forming this relationship is prescribing the deformation pattern of the element. This may be obtained by expressing the displacements \((u, v)\) of any point within the element's boundaries as continuous functions of the distances \(x\) and \(y\) measured from the unstressed location. In other words, the displacement pattern may be defined as

\[
u = U_0 + \sum_{i=1}^{\infty} \left( A_i \cdot x + C_i \cdot y \right) i
\]

\[
v = V_0 + \sum_{i=1}^{\infty} \left( B_i \cdot x + D_i \cdot y \right) i
\]

(3.23)

Quadratic functions are usually employed to give deformation pattern of the form:

\[
u = A_1 + A_2 \cdot x + A_3 \cdot y + A_4 \cdot x^2 + A_5 \cdot y^2 + A_6 \cdot x y
\]

\[
v = B_1 + B_2 \cdot x + B_3 \cdot y + B_4 \cdot x^2 + B_5 \cdot y^2 + B_6 \cdot x y
\]

(3.24)
where $A_1, A_2, ... B_6$ are constants.

By adjusting these constants various interesting behaviour patterns may be obtained. The most widely used configurations are shown in Fig. 3.6, signifying:

a. linear edge displacement pattern (Fig. 3.6a);

b. linear edge stress pattern (Fig. 3.6b); and

c. uniform edge stress pattern (Fig. 3.6c).

It is interesting to note that each of these deformation patterns has a distinct feature. The first configuration ensures a state of compatibility within and at the boundary of the element. It does not, however, satisfy the conditions of static equilibrium between adjacent elements. On the other hand, pattern (b) satisfies the equilibrium conditions while pattern (c) violates both conditions of equilibrium and compatibility between the various elements. This feature raises the question of the accuracy of the overall analysis with respect to the prescribed deformation pattern. De Vubeke\(^{(43)}\) has shown that a lower bound to the strain energy of the actual continuum is reached if the prescribed element behaviour ensures compatibility along the element boundaries. The same work also shows that if only the static equilibrium between the adjacent elements is maintained an upper bound to the influence coefficients is obtained. Hence, it can be expected that pattern (a) will result in an underestimation of the structural deflection while pattern (b) gives higher deflection values. However, it can be shown\(^{(53)}\) that the three deformation pattern yield reasonable accuracy and the difference between their results decreases for finer meshes.

Once the deformation pattern is prescribed and the element stiffness properties established, the analysis proceeds by constructing and solving the equations of equilibrium of the entire assembly. These equations may be expressed as

$$[K]\{r\} = \{R\} \quad (3.25)$$
a) Linear edge displacement pattern  

**Stresses**

\[
\begin{align*}
  f_x &= \frac{E}{(1-\nu^2)} \left( a_1 + a_3 y + \nu a_6 + \nu a_7 x \right) \\
  f_y &= \frac{E}{(1-\nu^2)} \left( a_6 + a_7 x + \nu a_1 + \nu a_3 y \right) \\
  t_{xy} &= \frac{E}{2(1+\nu)} \left( a_2 + a_3 x + a_5 + a_7 y \right)
\end{align*}
\]

**Displacements**

\[
\begin{align*}
  u &= a_1 x + a_2 y + a_3 xy + a_4 \\
  v &= a_5 x + a_6 y + a_7 xy + a_8
\end{align*}
\]

b) Linear edge stress pattern  

c) Uniform edge stress pattern.

\[
\begin{align*}
  f_x &= a_1 + a_2 y \\
  f_y &= a_3 + a_4 x \\
  f_y &= a_2 + a_5 y \\
  t_{xy} &= a_3 - a_5 x - a_4 y.
\end{align*}
\]

\[ u = \frac{1}{E} \left[ x(a_1 - \nu a_3) + a_2 xy - \frac{a_4}{2}(\nu x^2 + y^2) + a_6 y + a_7 \right] \]

\[ u = \frac{1}{E} \left[ x(a_1 - \nu a_3) + a_2 y - \nu a_5 xy + \frac{a_4}{2} \left( x^2 - (2+\nu) y^2 \right) + a_7 \right] \]

\[ v = \frac{1}{E} \left[ x((2+\nu) a_5 - a_6) + y(a_3 - \nu a_1) + a_4 xy - \frac{a_8}{2} (x^2 + y^2) + a_8 \right] \]

\[ v = \frac{1}{E} \left[ x((2+\nu) a_5 - a_6) + y(a_3 - \nu a_1) - \nu a_5 xy + \frac{a_8}{2} (y^2 - (2+\nu) x^2) \right] \]

Fig. (3.6), Element behaviour patterns.
where \([K]\) is the Master Stiffness Matrix of the entire assembly. The size of \([K]\) is \(2n \times 2n\) where \(n\) is the number of nodal points.

\(\{R\}, \{r\}\) are the applied loads and nodal displacement vectors respectively.

\(K\) is essentially a sparse symmetric band matrix and is constructed by superimposing the stiffness matrices of the various elements of the substitute structure. A systematic method of assembling \(K\) which is thought to be more efficient and less demanding of computer space than Argyris's formal method is given in Appendix 4.

The solution of the set of equations 3.25 gives the nodal displacements. The stresses at the nodes may subsequently be obtained by substituting in the formulae

\[
\begin{align*}
 f_x &= E' \left( \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \right) ; \\
 f_y &= E' \left( \frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} \right) ; \text{ and} \\
 t_{xy} &= G \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)
\end{align*}
\]

(3.26)

where \(E' = \frac{E}{1-\nu^2}\)

Appendix 4 comprises the details of these operations and describes the method of solving the equilibrium equations 3.25.
3.3 ASSESSMENT OF THE COMPARATIVE ACCURACY OBTAINED
BY THE VARIOUS METHODS OF STRESS ANALYSIS

In this section assessment of the comparative accuracy obtained by the various analytical procedures is sought. As regards the solution of shear wall structures, the ideal approach would be to apply the various methods of analysis in turn to a range of walls obtained by permutations of the size and shape of the structure. The calculations obtained may then be related to experimental results. This approach is impractical for the following two reasons:

1. The large computing time that would be required to execute the analyses.
2. The massive experimental programme which need be conducted.

The same objective may, however, be reached by adopting a different approach. This approach involves employing the various procedures for the analysis of related plane-stress problems, the solution of which (experimental or analytical) is known. On the basis of the comparative accuracy of the results the most efficient method can, therefore, be selected. For this purpose the following two-dimensional plane-stress problems are chosen to establish the criteria of pre-eminence. These examples are:

(i) A rectangular plate subjected to uniform tensile loading.
(ii) Built-in cantilever beams with different aspect ratios subjected to UDL.
(iii) A large rectangular plate acted upon by a concentrated force.
(iv) A 5-storey shear wall model.

The procedures employed in the analysis of these structures are:
The Problem:

Total Load = 2 lb.

\[ t = 0.05'' \]

\[ E = 10.5 \times 10^6 \text{ psi} \]

\[ \nu = 0.35 \]

b1) Stress Expressions (ETB)

\[ f_x = 0 \]

\[ f_y = \frac{P}{A} \]

\[ t_{xy} = 0 \]

b2) Stress Expression (SF)

\[ f_x = 0 \]

\[ f_y = \frac{P}{A} \]

\[ t_{xy} = 0 \]

c) Dynamic Relaxation Grid. (arrows indicate direction of fixity)

d) Finite Element Idealisation

Fig. (3.7), Plate subjected to UDL.
b1) Stress Expressions (ETB)

\[
\begin{align*}
    f_x &= 0 \\
    f_y &= \frac{M_x}{I} \\
    t_{xy} &= \frac{3}{2I} [h^2 - x^2]
\end{align*}
\]

b2) Stress Expression (SF)

\[
\begin{align*}
    f_x &= \frac{3}{4} x \frac{\omega}{h^2} (h^2 - \frac{1}{3}x^2) - \frac{\omega}{2} \\
    f_y &= \frac{3}{4} y \frac{\omega}{h^2} (y^2 + \frac{2}{3} h^2 - \frac{2}{3} x^2) \\
    t_{xy} &= -\frac{3}{4} y \frac{\omega}{h^2} (x^2 - h^2)
\end{align*}
\]

Fig. (3.8), Cantilever beams under UDL.
a) The Problem

b1) Stress Expression (ETB)
\[ f_x = 0 \]
\[ f_y = \frac{P}{A} \]
\[ t_{xy} = 0 \]

b2) Stress Expression (SF)
\[ f_x = \frac{-2P}{\pi t (x^2+y^2)} x y \]
\[ f_y = \frac{-2P}{\pi t (x^2+y^2)} x y \]
\[ t_{xy} = \frac{-2P}{\pi t (x^2+y^2)} x y \]

c) DR Idealisation

d) F.E. Idealisation.

Fig. (3.9), Concentrated force acting on a plate
Expressions of Stresses

**ETB**

\[ f_x = 0 \]
\[ f_y = \pm \frac{MY}{I} \pm \frac{N}{A} \]
\[ t_{xy} = \pm \frac{Q}{2I}(b^2 - x^2) \]

**SF**

Stress Function \( F = F_0 + F_s + F_e + F_{sh} \).

\( F \) is given in Appendix 3.

\[ f_x = \frac{\partial^2 F}{\partial y^2} \]
\[ f_y = \frac{\partial^2 F}{\partial x^2} \]
\[ t_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \]

Particulars of the model and position of the selected critical sections.

*Fig. (3.10), Shear Wall Model (Model 2).*
Fig. (3.10/b), Idealisation of Shear Wall Model.

No. of Elements = 245
No. of Nodes = 288
HBW [K] = 20

F.E. Grid

D.R. Grid.

No fictitious elements
a. The elementary approach (ETB)
b. The stress function approach (SF)
c. The dynamic relaxation technique (DR)
d. The finite element method using the linear-edge displacement pattern (FE1)
e. The finite element method using the linear-edge stress pattern (FE2)

The loading and boundary conditions of these problems are shown in Figs. 3.7, 3.8, 3.9 and 3.10. The stress expressions as obtained by the various methods together with the structural idealisation adopted in the finite element analysis are also given in the same figures.

The size and shape of the elements are conceivably important factors in the finite element technique. To ascertain their effect on the accuracy of the analysis various idealisations of example (i) were considered and the results obtained compared to other analytical solutions. These idealisations are shown in Fig. 3.11 and the results plotted in Figs. 3.12 and 3.18.

For the purpose of examining the results, the comparison parameters must first be established. In general, stresses are far more sensitive than deflections and it is for this reason that they are utilized for the purpose of comparison. It should be borne in mind, however, that a criterion based on stresses may be inconsistent, e.g. shear stress prediction might be good while direct stress prediction may prove inaccurate and the degree of accuracy may vary from one cross-section to another.

The computed results of direct and shear stresses at various levels of structures i, ii, iii and iv are plotted in Figs. 3.12, 3.13, 3.14 and 3.15 respectively. Structure iv - the shear wall model - was analysed experimentally by the photo-elastic technique (Section 3.4) and the results obtained are also presented in Fig. 3.15.

* The linear edge stress pattern shown in Fig. 3.6c was omitted because to obtain accurate results very fine grid should be used. Also, this deformation pattern usually gives an ill-conditioned matrix.
Fig. (3.11), Various Idealisations of a Plate subjected to uniform tensile stress.
Fig. (3.12), Results – Plate subjected to tensile stress

Units of stress = lb./in.²
Fig. (3.13/1), Results - Cantilever under UDL.
Fig. 3.13(a), Results - Deep beam under UDL
Fig. (3.14), Results - Concentrated force on a plate.
Fig. (3.15/1), Isoclinic and Isochromatic Lines along A1-B1

Unit Fringe Value = 1.336 psi (Shear)

Fig. (3.15/2), Isoclinic and Isochromatic Lines along A2-B2
Fig. (3.15/3), Stresses in Section A-B.
Model 2

Unit Fringe Value = 1.336 psi (shear)

Wall Width = 2"

Fig. (3.15/4), Stresses in Section C-D.
Fig. (3.15/5), Isoclinic and Isochromatic Lines along C1-D1

Unit Fringe Value = 1.336 psi (Shear)

Fig. (3.15/6), Isoclinic and Isochromatic Lines along C2-D2.
Fig. 3.15/7, Stresses in Section 2-0 (Model 2)
Fig. (3.15/8), Stresses in Section 2-1. (Model 2)
Fig. (3.15/9), Stresses in Section 3-0 (Model 2)
Fig. (3.15/10), Stresses in Section 3-1
Fig. (3.15/11), Stress concentration near the re-entrant joint, (F.E.)
Frocht's shear difference method\(^{(50)}\) was used for the calculations of the normal and shear stresses at the various sections. The fringe pattern of this model obtained from a crossed polariscope is shown at the cover of this Part of the thesis. A view of the fringe pattern in a parallel polariscope is shown in Fig. 3.16. The isochromatic lines of the control specimen (circular disc) is given in Fig. 3.17.

3.3.1 Analysis of the Results

Plate under tensile stress

The results show the close agreement between the stresses "f\text{y}" (the major direct stress) predicted by the various methods of analysis at all the cross-sections. Both the ETB and the stress function methods assume no shear or lateral stresses occur under tensile loading and this explains the discrepancy obtained between the shear stress results. It is interesting to note that both the dynamic relaxation and finite element techniques, even with the coarse meshes used, gave adequately accurate results. Fig. 3.12 also reveals that whilst agreement between the results of f\text{y} and f\text{x} obtained by the methods of finite element with linear-edge displacement pattern (FE1) and linear-stress pattern (FE2) is secured, there exists a discrepancy between their prediction of the shear stresses "t\text{xy}". This is evidently attributed to the assumptions about the shear distribution along the element's boundaries incorporated in the prescribed element behaviour pattern. As expected, Fig. 3.18 shows that FE2 gives slightly higher values of deflection than FE1 and that the accuracy of this approach improves as the number of elements increases. Furthermore, this figure reveals that square elements yield better approximations of stresses and deflections than rectangular elements. The results of idealisations D and E show that with rectangular elements the accuracy of the FE analysis degenerates when the ratio between the loaded side and the unloaded side of the element decreases.
Fig. 3.16 a, Fringe Pattern (11-polariscope).
Model 2.
Fig. 3.16, Fringe Pattern - Model 2 (X-polariscope).
Crossed Polariscope.

Parallel Polariscope.
Built-in cantilevers under UDL

It is shown in Fig. 3.13 that near the top edge of the tall cantilever both the ETB and the stress function approaches gave higher values of stresses $f_y$ and $t_{xy}$ than the finite element prediction. This trend is reversed for the short beam with the SF, giving smaller stresses than the other procedures. This discrepancy may be attributed to the fact that the boundary conditions at the top edge of the cantilever are not satisfied by the stress function used in the analysis. The agreement between the DR and ETB predictions of normal stresses $f_y$ may be noted. However, in most cases the DR technique gave smaller values of shear stresses than both the ETB and SF approaches. As expected, the difference between the various results decreases at sections at a reasonable distance from the top edge of the beams. The close agreement between the finite element results using the two element deformation patterns reveals the little effect the variation in the prescribed element behaviour has on the overall results.

Concentrated force acting at the edge of a straight boundary

The ETB is shown in Fig. 3.14 to yield erroneous results at the vicinity of the applied load. However, this method gave accurate assessment of the stresses near the clamped end of the beam (or rather at a reasonable distance from the applied force). The stress function and the DR analyses on the other hand gave slightly smaller values of the stress concentration and higher shear stresses than the finite element procedure. However, it may be noticed that a close agreement between the DR and the FE results is obtained at section close to the clamped end. Also, the two element behaviour patterns used in the finite element analysis (linear edge displacement and linear stress patterns) gave almost identical results at all sections.
Shear wall model

The stresses in section AB (1/8th inch above foundation) shown in Fig. 3.15/3 show the close agreement between the experimental and the finite element results. The same figure also reveals that variation in the element deformation pattern in the finite element analysis produces insignificant difference in the results. The other methods of analysis, namely SF, DR and ETB, on the other hand yielded overestimated stresses, the percentage of error ranging from 19% to 88% for the SF and ETB respectively.

The same trend in the various results is also obtained for sections CD and 20, Fig. 3.15. The discrepancy between the SF and experimental shown in Figs. 3.15/3 and 3.15/4 may be attributed to the inability of the stress function to satisfy all the boundary conditions (Appendix 3). Yet, the large difference between the stresses predicted by the FE and the SF methods at section 20 which is at a reasonable distance from the foundation may be due to insufficient number of terms in Fourier's series. While the DR results for section AB are acceptable it is difficult to assess from this example the accuracy of this method in the analysis of shear wall structures. The bending effect in the walls produces lateral vibrations in the structure which requires small damping factor. This implies greater cycles of iterations and consequently large computing time. It was for this reason that only the lower part of the model, with computed forces acting upon it, was analysed. This idealisation, shown in Fig. 3.10, may have affected the accuracy of the DR analysis.
Based on the results of the above mentioned examples the following conclusions regarding the merits of the various analytical approaches may be drawn.

**The ETB**

This method is extremely simple and its validity is limited to structures behaving in accordance with Bernoulli's assumptions.

**The SF method**

This is very efficient when the stress function is known. However, the approximate function used in the shear wall problem required tedious arithmetic and produced overestimated results.

**The DR method**

This is a very elegant approach yet it only deals with square mesh and would require long computing time if the structure is subjected to bending mode.

**The FE procedure**

This method is extremely efficient and even with the coarse meshes used the accuracy obtained was very high. The element deformation pattern appears to have very little effect on the overall accuracy of the analysis (provided, of course, that the size of the element is small). However, experience has shown that the linear-edge displacement pattern gives a better conditioned matrix and it was used in the following analysis.
Fig. (3.18/1), Effect of the element shape on the F.E. results.
Fig. (3.18/2), Effect of the element shape on the F.E. results.
3.4 EXPERIMENTAL WORK

3.4.1 Object and Scope

Primarily the object of the experimental work was to establish a criterion by which the accuracy of the analytical solutions may be gauged. A second objective was to study the stress distribution in interconnected shear wall structures. For this purpose photo-elasticity lends itself as the most suitable experimental technique. It is known that the Unit Fringe Value (UFV) of photo-elastic materials reduces considerably at high temperatures*. This feature permits a larger number of fringes to be obtained at low load levels. The frozen stress technique was, therefore, chosen as the experimental tool. Models of shear walls with different shapes were made of Araldite, loaded at elevated temperatures in a heat-controlled oven and then examined in a polariscope. The choice of Araldite was made on the grounds of its cheapness, ease of machining and its relatively low unit fringe value (@ 1.35 p.s.i.).

Of the various grades of Araldite, the so-called CT200 is recommended by the makers for photo-elastic models. While the CT200 possesses high optical qualities it required considerable care in mixing the resin and hardener, casting at high temperature (120°C) and long curing time. In the mean time, Araldite MY753 appears to possess high optical and mechanical qualities without requiring elaborate precautions in the casting and curing processes and it has been used in this investigation.

Four five-storey shear wall models were made from stress-free cast-in-the-laboratory Araldite sheets. For each model control specimens, i.e. circular discs and built-in cantilevers, were also prepared from the same sheet. Each model and the corresponding control specimens were loaded simultaneously under similar testing environment.

* This is accompanied by a reduction in the elastic modulus "E".
The particulars of the models and control specimens are given in Fig. 3.19. Consideration of the lateral buckling of the models and the size of the oven used determined the height of the specimens (10") . The loading arrangement shown in Fig. 3.20 was designed to simulate uniformly distributed load along the height of the models without producing local stress effects.

3.4.2 Preparation of the Models

Araldite sheets were cast by using a 10:1 mix of MY753 resin and HY951 hardener. These components were thoroughly mixed and the solution kept in a vacuum chamber for half an hour to extract the air bubbles. The mixture was then poured from an outlet valve at the bottom of the mixing container into a vertically placed aluminium mould. Prior to casting, the inside surface of the mould was coated with a uniform thin layer of Releasile No. 7. Curing took place in the laboratory. Twenty-four hours after casting, the mould was dismantled and the cured sheet released and placed on a flat surface for another twelve hours. Following this, the sheet was examined in a polariscope to detect any local stress formations. Machining was then done and the specimens loaded in the oven on the same day to avoid age affects.

3.4.3 Testing Technique

Each model was placed horizontally with its base firmly clamped to the loading frame. Rubber pads were placed on one edge of the model along its entire length. Loading was directly applied on these pads. The loading frame was fitted with metal guides to eliminate buckling of the models (Fig. 3.20). The loaded models and the control specimens were placed in a thermo-controlled oven, shown in Fig. 3.21. A Perspex cam was designed to provide the required heating cycle. The heating cycle was as follows:

<table>
<thead>
<tr>
<th>Temperature</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>room temperature</td>
<td>2 hours</td>
</tr>
<tr>
<td>100°C</td>
<td>2 hours</td>
</tr>
<tr>
<td>100°C - room temperature</td>
<td>12 hours</td>
</tr>
<tr>
<td>TOTAL</td>
<td>16 hours</td>
</tr>
</tbody>
</table>
Fig. (3.19), Particulars of the models and control specimens.
Loading of the model.

Loading of the disc

Fig. 3.20, The loading arrangement.
The oven.

The loading rig.

Fig. 3.21, View of the oven and loading rig.
After the heating process the loads were removed and the moulds examined in a polariscope.

3.4.4 Properties of the Material

The unit fringe value of the material was computed by substituting the fringe order at the centre of the circular disc in the following expression:

\[ P - q = \frac{8P}{\pi td} = 2n \frac{F}{F_{sh}} \]

or

\[ F_{sh} = \frac{8P}{\pi td} \cdot \frac{1}{2n} \]

where

- \( P \) = the applied load.
- \( t, d \) = the thickness and diameter of the disc.
- \( n \) = number of fringe at the centre of the disc.
- \( F_{sh} \) = unit fringe value in shear in lb/in^2.

To determine the value of the materials Elastic Modulus \( E \), a control specimen consisting of a 2" x 10" cantilever was loaded with each model. The deflection of the cantilever at various points was carefully measured. These values were then substituted in the simple strength-of-material expression of cantilever deflections. The average value of \( E \) was found to be 1806 p.s.i. Poisson's ratio was taken as 0.44 as quoted by the makers.

More accurate values of \( E \) and \( \nu \) may be obtained by using a different technique which occurred to the author at the time of writing this thesis. This approach involves accurately measuring the distance between two diametrically opposite points on the circumference of the discs before and after loading. The deformation of the diameter may then be resolved into \( X \)- and \( Y \)-components and these values substituted into the following expressions obtained from Muskheilisville(86).
\[ u = \frac{P}{4G\pi t} \left[ \frac{2(\lambda+2G)}{\lambda+G} \log \frac{R^2}{r^2} + \frac{\cos 2\theta_1 - \cos 2\theta_2 - \frac{2G\cos\alpha}{\lambda+G}}{\lambda+G} \cdot \frac{y}{R} \right] \]

\[ v = \frac{P}{4G\pi t} \left[ \frac{2G}{\lambda+G} (\theta_1 + \theta_2) - \sin 2\theta_1 - \sin 2\theta_2 - \frac{2G\cos\alpha}{\lambda+G} \cdot \frac{x}{R} \right] \]

where \( G = \frac{E}{2(1+\nu)} \)

\( R = \) radius

Two simultaneous equations with the unknowns \( E \) and \( \nu \) may be obtained. Upon solving these equations the elastic moduli can be determined.

### 3.4.5 Analysis of the Models and Records of Results

For each of the models tested the photo-elastic results were compared to theoretical calculations based on the finite element analysis. It is well known that the informations obtainable from the photo-elastic technique are merely the loci of the difference of the principal stresses (isochromatic fringes) and their direction at every point (isoclinic lines).

From this experimental data the reduction of the separate principal stresses, the stress distribution across the various sections etc. need be performed. For this purpose the Frocht's shear difference method was adopted for its simplicity and adaptability to digital computation. This process is essentially a step-by-step integration, as shown in Fig. 3.15, of the equations of equilibrium, e.g.

\[ \frac{\partial f_x}{\partial x} + \frac{\partial t_{xy}}{\partial y} = 0 \]

or approximately

\[ (f_x)_{i+1} = (f_x)_i - \left( \frac{\Delta t_{xy}}{\Delta y} \right) \Delta x_{i,i+1} \]
Fig. (3.22), The F.E. Grid for the Araldite Models.
Fig. (3.23), The selected critical sections
and \[ t_{xy_i} = \frac{(P-Q)}{2} \sin 2\phi_i \]

where \( \frac{(P-Q)}{2\phi_i} \) is the maximum shear stress

and \( \phi_i \) is the direction of the principal stress at point "i".

The normal stress in the y-direction \( f_y \) may then be computed by using the expression:

\[ f_y = f_x \pm \sqrt{(P-Q)^2 - 4t_{xy}^2} \]

Details of the shear difference method may be found elsewhere (31, 50, 60, 61, 67).

The theoretical results were obtained by using the linear-edge displacement pattern of the finite element technique (Fig. 3.6b). For reasons outlined in Section 3.1.3, the interconnected wall models were first analysed by the WCA method to determine the forces in the connecting beams. Each of the component walls, acted upon by the calculated forces, was then considered separately as a built-in cantilever for the FE treatment. The idealised walls are shown in Figs. 3.10 and 3.22.

It is conceivable that with fine-mesh idealisations the size of the equilibrium equation gets bigger and the output data becomes enormous. In modern computer analysis these data can be classified and represented graphically by means of graph plotter (e.g. the stress distribution across the critical sections, the lines of principal stresses etc.). However, if this facility is unavailable a limited but equally informative result may be relied on.

Hence, in the present work comparison between the experimental and theoretical results at selected sections of the walls is shown. These sections, shown in Fig. 3.23, are understandably close to the clamped end of the walls and regions of high stress concentrations. The comparison parameters are the normal stress \( f_y \) and the loci of the maximum shear stresses \((P-Q)/2\).
Fig. 3.24, Fringe Pattern in a X-polariscope.
Model 1.
Fig. 3.24 a, Fringe Pattern in a II- polariscope
Model 1.
Fig. 3.25. A typical fringe pattern in a disc under diametral compression.

Crossed Polariscope.

Parallel Polariscope.
The fringe patterns of the models obtained by using crossed and parallel polariscope are shown in Figs. 3.24, 3.16, 3.29 and 3.32. A typical result of the control specimens is given in Figs. 3.17 and 3.25. The isoclinic lines of the models is presented in Figs. 3.26, 3.28, 3.30 and 3.33, while Fig. 3.35 shows the lines of principal stresses of model 2. The experimental and theoretical results of the four models are compared in Figs. 3.27, 3.15, 3.31 and 3.34.

3.4.6 Discussion of Results

The close agreement obtained between the theoretical and experimental results demonstrates two major points. The first is the pre-eminence of the finite element procedure in the analysis of boundary value problems. Even with the coarse meshes used, the accuracy of this method was extremely high.

The second point is the ability of the frozen stress technique of photo-elasticity in providing information on the stress variation and areas of stress concentration in complex model structures. The results are, of course, by no means exact and subject to various sources of inaccuracy which are associated with experimental and analytical work.

On the experimental side, errors may result from:

a) the difficulty in simulating the boundary conditions,
b) out-of-plane buckling of the model,
c) inaccurate recording of the experimental data; and
d) machining of the models and time-edge effects.

Elimination of (a) and (b) was achieved by firmly clamping the model to the loading frame and providing metal-guides along each side of the model, as shown in Fig. 3, to inhibit any tendency to out-of-plane buckling. Difficulty in recording accurately the experimental data is a drawback in photo-elastic testing. Isoclinics are difficult to locate accurately and tend to become diffused and indistinct near the base. A thin layer of a 2:1 one-Bromo naphthaline
and liquid paraffin which has the same reflective index as Araldite was applied to the model prior to viewing it in the polariscope to help locate the isoclinics.

Fractional fringes are also difficult to work out and their values in the present investigation were obtained by extrapolation, occasionally checked by colour matching.

To prevent formation of fringes in the specimens due to machining, the models were cut by saw slightly bigger than the final size. The openings were then drilled and the models filed down to the required size. The models were then examined in the polariscope. Loading took place immediately after machining to eliminate age effects.

Sources of error in the analytical treatment may result from the inherent difficulty in solving large ill-conditioned matrices. The method of successive over-relaxation \((48,126)\) appears to be very powerful in solving ill-conditioned matrices. In this method, the solution is obtained by successive iterative processes of an approximate solution vector, i.e.

\[
x_i^{r+1} = x_i^r + \beta \left( x_i^{r+1} - x_i^r \right).
\]

where

- \(x_i^{r+1}\) the solution of the set of equations at \((r+1)\) cycles of iterations
- \(x_i^r\) the solution at \(r\) cycles of iterations
- \(\beta\) over-relaxation factor

The rate of convergence of this equation, and indeed the length of computing time, depends on the over-relaxation factor \(\beta\). No exact method is known for determining the appropriate value of \(\beta\). Yet Carre(20) suggested that the optimum over-relaxation factor may be evaluated by using the following empirical formula:

\[
\beta = \frac{2}{1 + \sqrt{1 - \lambda_{\text{max}}}}
\]
where

\[ \lambda_{\text{max}} = \lim_{r \to \infty} \frac{n^{(r)}}{r} \]

- \( r \) = number of iterations
- \( n^{(r)} \) = the norm of the vector
  i.e. the modulus of the numerically
  largest element of this vector, or
  the arithmetic sum of the elements etc.

Carre pointed out that this formula is only valid for
matrices which possess Property A. This rules out the
suitability of the formula for the finite element analysis.
However, it can be shown (20, 126) that for large matrices
the value of \( \beta \) is \( 0 < \beta < 2 \) and Clough et al (25,26)
suggested that an arbitrary value between 1.85 and 1.95
would be sufficiently accurate in speeding up the convergence
of the solution.

As expected, non-linear stress distribution was obtained
in all sections, Figs.3.15/27/31, near the base. This
is attributed to the effect of stress concentration at the
fixed end. Linear variation of stresses and almost-parabolic
shear stress distribution were found at sections at a
distance from the base, Fig. 3.15

The presence of openings resulted in high stress
concentration at the re-entrant joints, the value of which
appears to depend on the beam stiffness. Figs. 3.15 - 3.34
show that the magnitude of the local stresses is greater
for flexible beams. The isochromatic lines of the four
models (Figs.3.16/24/29/32 ) show that large areas of
low stress occur at the upper part of the models and at
the end of the connecting beams.

The stress trajectories of Model 2 (Fig 3.35) show
that large parts of the shear wall are acting predominately
in shear with the principal stresses inclined approximately
at 45 degrees.
Model 1

Section 201 20 202

\[
\text{(p-9)/2} \quad \Theta
\]

UFV = 1.23 psi

Fig. (3.27), Stresses in section 11, Model 1
MODEL 1

\[ U_{FV} = 1.23 \text{ psi} \]

**Fig. (3.21/b), Stresses in section 10 (MODEL 1)**
Fig. (3.27/d), Stress concentration - Model 1, (F.E. Results).
Fig (3.28), Isodrome Lines/Model 2
Fig. 3.29, Fringe Pattern in a 11 el Polarizcope.
Model 4.
MODEL 4

Fig. 0.301, Isolines Lines/Model 4.
Fig. (3.31), Stresses in section 1-0 (Model 4)

$UFV = 1.511$ psi
MODEL A

Fig. (3.3) b, Stresses in Section II (Model A)
Fig. (3.27/c), Max. Shear Stress \((p-q)/2\), Section 20

MODEL 1

Fig. (3.31/c), Max. Shear Stress \((p-q)/2\) - Section 20

MODEL 4
Fig.(3.31/d), Stress concentration near the re-entrant joint, (F.E.)
Fig. 3.32, Fringe Pattern in a X-Polariscope, Model 8.
Fig. 3.32a, Fringe Pattern in a P®lariscope. Model B.
Fig. (3.34), Stresses in Sections 10 & 20 / Model 8
Fig. (3.34/b), Stresses near the re-entrant joint, (F.E. Results).
SUMMARY AND CONCLUSION

Based on the results of the present investigation, the following conclusions may be drawn.

4.1 METHODS OF ANALYSIS OF SHEAR WALL STRUCTURES

a) The Shear Connection Method

This method is suitable for the analysis of tall shear wall buildings with slender connecting beams and generally overestimates the structural stiffness. The mathematical solution of the governing differential equation is a new contribution which permits the application of this procedure to multi-bay structures. As well as being adaptable for automatic computation, the SCM is also feasible for hand calculation for up to 4-wall structures.

b) The Wide Column Analogy Method

The WCA is very powerful in the analysis of shear walls with any shape. It is only feasible, however, if a computer program is available. In most cases, this procedure overestimates the structural stiffness but gives better accuracy in the prediction of the deflection values than the SCM.

Michael's correction factor which accounts for the local yielding of the joints between the walls and beams slightly improves the results of low-rise walls and walls with slender beams. This factor appears, nonetheless, to be excessive in the case of stiff connecting beams.

c) The Elementary ETB

The ETB, which comprises major assumptions as to the structural behaviour, is a very simple approach. It can be used for first estimate of deflections and stresses in tall walls with very slender connecting beams. However, this method is incapable of providing information about local stress effects.
d) The Stress Function Method

This approach is powerful in the analysis of simple structures for which stress functions are known. Derivation of stress functions for complex structures is often difficult, if at all possible. Approximate SF's can, in some cases, give satisfactory results. However, the function derived in this thesis is for the analysis of shear walls with openings.

e) The Dynamic Relaxation Method

The close agreement obtained between the DR and the experimental results indicated that this procedure is extremely elegant and practical in solving boundary value problems. This approach may, however, be open to criticism for it requires regular square grid pattern, i.e. the grid cannot be made finer near areas of stress concentration.

f) The Finite Element Method

This technique is the most powerful of the methods of stress analysis. The merits of the FE procedure lie in its ability to account for varying material and structural properties of the component elements. Also, with this approach, irregular geometries and complex boundary conditions can be easily handled.

Furthermore, the effect of creep, temperature rise etc. on concrete structures can, as shown by Rashid's work, be readily obtained by this method. Generally, the accuracy of the FE method, even with coarse meshes, is high and the results converge to the exact solution as the number of elements increases. While more refined element deformation configurations may improve the results, the prescribed pattern appears to have insignificant effects on the overall accuracy of the finite element analysis.

g) Experimental Methods of Analysis

The low cost, ease of machining and consistent elastic properties of Perspex makes it a suitable material for plane-stress models. Nevertheless, it is only efficient for deflection analysis and not used in photo-elastic experiments owing to its high UFV. Araldite, on the other hand, is the recommended material for photo-elastic models.
The frozen-stress technique of photo-elasticity is shown to be a convenient and powerful tool in the analysis of shear walls with openings.

4.2 STRUCTURAL BEHAVIOUR

a) The behaviour of shear wall structures depends on the ratio of the stiffness of the connecting beams and the adjacent walls. Tall walls with very slender beams tend to behave as separate cantilevers, i.e. the effectiveness of the connecting beams decreases as the ratio "beam stiffness/wall stiffness" gets smaller. With very stiff connecting members the structure behaves as one unit (i.e. a wall with holes) rather than interconnected walls.

b) Most of the evidence indicates that there exists a limiting ratio of the stiffnesses of the beam and walls which gives optimum structural rigidity.

c) Floor slabs are effective in providing interaction between wall assemblies, with the full bay-width contributing in the coupling action.

d) Due to the effect of out-of-plane bending of the floor slabs, the behaviour of the edge wall did not conform to the response predicted analytically.

e) The photo-elastic testing of the Araldite models showed that there exist large areas of low stress at the upper part of the models and at the end of the connecting beams. Also, the stress trajectories shown in Fig. 3.35 reveal that large parts of the shear wall are acting predominately in shear.

4.3 SUGGESTION FOR FURTHER RESEARCH

a) It is conceivable that the accuracy of the SCM can be improved by considering the effect of the shortening of the connecting beams and the shear deformation of the walls. The resulting governing equation may easily be solved by the matrix orthogonalisation method described above.

b) Further study on the effect of local elastic yielding at the joint between the beams and walls on the accuracy of the WCA is needed. This may lead to establishing a correction factor which accounts for the relative stiffness of the beam and the walls.

c) Visco-elastic and visco-plastic analysis of shear walls by using the finite element method.

d) A correlation between experimental measurements based on full scale reinforced concrete (precast and insitu) and brick structures, and analytical results.
The Generalised Shear Connection Method - Digital Computation

A computer program, designated SCMP, was written in Atlas Autocode for an English Electric KDF/9 computer to carry out the calculations described in Part I, Section 2.2.1.2. A flow diagram of this program is given in Fig. A1. It can be seen that the program commences with the particulars of the structure and concludes with the required analysis (bending, shear, deflection of the various walls). The data input does not, therefore, involve prior preparations or hand calculations.

Sequence of the Computing Operations

The computation process is carried out in the following sequence:

1. Read in the data input, the order of which is given in the next section.
2. Compute the constants of the sections, i.e. $a_{11}, a_{21}, \ldots b_1$, etc.
3. Assemble matrix $[A]$ (equation 2.4).
4. Solve equation 2.13 to get the Eigen values and the corresponding Eigen vectors. The computer routine used was available at Edinburgh Regional Computing Centre and was written by J. McKay.
5. Assemble the transformation matrix $[T]$.
6. Solve the equation $[U] = [T^T][B]$ by using the Osborne routine for solving simultaneous equations.
7. Solve equation 2.15 and back substitute into equation 2.6 to get the values of $[R]$.
8. Compute the bending, shear forces and deflections of each wall using equation 2.19.
9. Check that the static equilibrium of the sections is maintained, i.e.
   
   The external moment = sum of the internal moments
   The applied shear = sum of the internal shear.
10. If (9) is satisfied, then print out the following informations at each storey level:
    The normal forces
    The shear forces
    The bending moments
    The deflection
Instructions for using the SCMP programme

The following points should be observed:

(i) The units used are pounds and inches.
(ii) The particulars of the structure should be given in the following order (these particulars are fed to the computer on a 7-hole paper tape and the figures are separated by one or more spaces or a new line):

- $n$ = number of walls
- $N$ = number of storeys
- $a, b, c, \ldots$ = width of the walls in inches
- $e_1, e_2, \ldots$ = width of the openings in inches
- $t$ = thickness of the walls
- $d$ = storey height
- $h$ = depth of the connecting beams or slabs.
- $E_w$ = Young's modulus of the material of the walls
- $E_b$ = Young's modulus of the material of the beams
- $\nu$ = Poisson's ratio of the material of the beams
- $w$ = intensity of wind pressure in lb/in of the height.
Fig. A1, Flow diagram of the SCMP program.
The Wide Column Analogy Method - Digital Computation

The WCA method involves assembling and solving a set of algebraic equations representing the conditions of kinematic compatibility. This condition may be expressed as

\[
[f] \{x\} = - \{u\}
\]

where

- \([f]\) is the matrix influence coefficients
- \(\{x\}\) the unknown actions in the beams
- \(\{u\}\) the incompressible displacement vector.

\([f]\) is essentially a square and symmetric matrix. The size of this matrix is 3 times the degree of static indeterminacy square, i.e. \(3m \times 3m\) where \(m\) is the number of the connecting beams.

Assembling the flexibility matrix

Fig. A2/1 shows a typical shear wall and the idealised structure. By suitably numbering the degrees of freedom in the beams (representing normal forces, shear and bending) a well-conditioned matrix is obtained. If the beams are numbered in the order shown in Fig. A2, the forces in these beams can be represented by:
Normal forces \((p)\) in the beams 1 to \(m = x_{p1} \) to \(x_{pm}\)
or \(x_1 \) to \(x_m\)

Shear forces \((Q)\) in the beams = \(x_{q1} \) to \(x_{q2m}\)
or \(x_{m+1} \) to \(x_{2m}\)

Bending moments \((M)\) = \(x_{2m+1} \) to \(x_{3m}\)

The set of equations ensuring compatibility may then be expressed as:

\[
\begin{bmatrix}
f_0 & f_1 & \cdots & f_m & f_{m+1} & \cdots & f_{2m} & f_{2m+1} & \cdots & f_{3m} \\
f_{m+1} & f_{m+2} & \cdots & f_{2m} & f_{2m+1} & \cdots & f_{3m} & & & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\
f_{2m,1} & f_{2m,2} & \cdots & f_{2m,2m} & f_{2m+1,2m} & \cdots & f_{2m+1,3m} & f_{2m+2,3m} & \cdots & f_{2m+3,3m} \\
f_{3m,1} & f_{3m,2} & \cdots & f_{3m,2m} & f_{3m+1,2m} & \cdots & f_{3m+1,3m} \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{m+1} \\
x_m \\
\vdots \\
x_{3m} \\
\end{bmatrix}
= 
\begin{bmatrix}
-u_1 \\
-u_2 \\
\vdots \\
-u_{m+1} \\
-u_m \\
\vdots \\
-u_{3m} \\
\end{bmatrix}
\]  

\( - (2) \)

It may be noticed that the flexibility matrix \([F]\) consists mainly of 6 sub-matrices, the dimensions of each of which is \(m \times m\). Equation (2) may therefore be rewritten as:

\[
\begin{bmatrix}
PP & PQ & PM & X_P \\
QP & QQ & QM & X_Q \\
MP & MQ & MM & X_M \\
\end{bmatrix}
= -
\begin{bmatrix}
U_P \\
U_Q \\
UM \\
\end{bmatrix}
\]
Where:

\[
[P_P] \quad \text{is an } m \times m \text{ submatrix representing the influence coefficients due to normal forces in the beams (P)}
\]

i.e. the displacement in the horizontal direction of beam "i" due to a unit horizontal load applied at beam "j".

\[
= \left[ \int \frac{x_i x_j}{EI} \, dl + \int \frac{N_i N_j}{AE} \, dl + \int \frac{Q_i Q_j}{ATG} \, dl \right]_{m \geq i \geq 1, \ m \geq j \geq 1}
\]

\[
[P_Q] \quad \text{the displacement in the horizontal direction of beam "i" due to a unit shear force in beam "j".}
\]

\[
= \left[ \int \frac{x_i x_j}{EI} \, dl + \cdots \right]_{m \geq i \geq 1, \ 2m \geq j \geq m+1}
\]

\[
= [Q_P]
\]

\[
[P_M] \quad \text{the displacement in the horizontal direction of beam "i" due to a unit moment applied at beam "j".}
\]

\[
= \left[ \int \frac{x_i x_j}{EI} \, dl + \cdots \right]_{m \geq i \geq 1, \ 3m \geq j \geq 2m+1}
\]

\[
= [M_P]
\]

\[
[Q_Q] \quad \text{the displacement in the vertical direction of beam "i" due to a unit shear force applied to beam "j".}
\]

\[
= \left[ \int \frac{x_i x_j}{EI} \, dl + \cdots \right]_{2m \geq i \geq m+1, \ 2m \geq j \geq m+1}
\]

\[
= [Q_Q]
\]

\[
[Q_M] \quad \text{the vertical displacement of beam "i" due to a unit rotation applied at beam "j".}
\]

\[
= \left[ \int \frac{x_i x_j}{EI} \, dl + \cdots \right]_{2m \geq i \geq m+1, \ 3m \geq j \geq 2m+1}
\]

\[
= [Q_M]
\]

\[
[M_M] \quad \text{the angle of rotation in beam "i" due to a unit moment applied at beam "j".}
\]

\[
= \left[ \int \frac{x_i x_j}{EI} \, dl + \cdots \right]_{3m \geq i \geq 2m+1, \ 3m \geq j \geq 2m+1}
\]

\[
= [M_M]
\]
\{x_p\}, \{u_p\} = \text{the redundant actions and the incompatible displacements for } m \geq i \geq 1

\{x_q\}, \{u_q\} = \text{the redundant actions and the incompatible displacements for } 2m \geq i \geq m+1

\{x_m\}, \{u_m\} = \text{the redundant actions and the incompatible displacements for } 3m \geq i \geq 2m+1

This arrangement results in a flexibility matrix with a predominant major diagonal and the values of the off-diagonal terms gradually decreasing in the direction of the minor diagonal. This feature characterises a well conditioned matrix. The subdivision of the flexibility matrix in this fashion also presents an easier programming task in the assembling of the various coefficients.

\textbf{Solution of the simultaneous equations}

A first estimate of the solution vector of equation (1) was obtained by using a standard routine for the solution linear equations (Eqn solve).

This routine is based on triangular factorisation of the flexibility matrix with row interchanges. The solution was then refined* by using the Gauss-Seidel iterative process (1,48,126).

\textbf{Computing Process}

The computer program written to carry out the wide column analysis is designated WCA/P. With a view to efficiently using the limited storage capacity of the KDF/9 machine (16K) the nested blocks technique was employed and informations were stored on magnetic tapes. The block begins with begin followed by the appropriate declarations and instructions and is terminated by end.

* Two iterations produced satisfactory results for all the models treated.
The declarations made at the head of the block are cancelled on reaching end, thus allowing the computer space used by this block to be employed for other operations.

The layout of the program and the arrangement of the blocks is given below.

```
begin
  * read input data.
  * compute constants of the sections

begin
  * compute influence coefficients
  * assemble flexibility matrix
  * store the equation on magnetic tape.
end

begin
  * read the set of equations from magnetic tape
  * Solve equation by using routine Eqn Solve)
  * Store solution vector on magnetic tape.
end

begin
  * iterate by using the Gauss-Seidel process to obtain a more accurate solution vector.
  * Store solution on magnetic tape.
end

begin
  * read solution from magnetic tape
  * compute B.M., S.F., normal forces and deflection on each wall
```
* print out results
end
end of program

Instructions for using the WCA/P

The program is capable of the analysis of symmetrical or non-symmetrical shear walls consisting of a number of rows of openings. However, the program is limited to walls with regular opening patterns. Due to the limited storage capacity of the KDF/9 machine the maximum size of the structure that can be analysed by this program consists of up to 400 connecting beams.

The units used in the program are pounds and inches and the order of the input data is as follows:

- \( n \) = number of walls
- \( N \) = number of storeys
- \( d \) = storey height
- \( t_w \) = thickness of the wall
- \( t_b \) = thickness of the beams
- \( h \) = depth of the beams
- \( E \) = Young's modulus of the material
- \( \nu \) = Poisson's ratio
- \( w \) = wind pressure in lb/in of the height
- \( a_1, a_2, \ldots \) = width of the component walls
- \( e_1, e_2, \ldots \) = width of the openings.
Stress Function Method - Formulation of the Function and Digital Computation

1. Evaluation of the forces acting on the individual component walls

Fig. 2.4 shows a shear wall structure subjected to uniformly distributed load (w lb/in'). The forces in the connecting beams may be evaluated by using the WCA method outlined in Part I of this thesis. The structure can then be treated as consisting of separate cantilevers acted upon by sets of forces at discrete points. The forces at each storey level include shear forces (q), lateral forces (p) and bending (m). Prior to finding a stress function for each of these forces, it is convenient for the analysis to express the latter two actions (p, m) in terms of Fourier's Series. This is given in the next two sections.

2. The lateral forces "P" expressed in terms of Fourier's Series

\[ f(p) = P_{op} + \sum a_{np} \cos \frac{n \pi x}{L} + \sum b_{np} \sin \frac{n \pi x}{L} x \quad (2.1) \]

It will be assumed that there exists a mirror-image of the structure acted upon by a similar loading pattern as shown in the above figure. The stress distribution "p" on the entire height of the structure may be expressed as:
Integrating this equation over the height, we get:

\[ \int_0^L f(p) \, dx = \int_0^L \alpha_{0p} \, dx + \int_0^L \alpha_{np} \cos \frac{n\pi}{L} x \, dx + \int_0^L \beta_{np} \sin \frac{n\pi}{L} x \, dx \quad (2.2) \]

\[ \text{RHS} = \alpha_{0p} \left| x \right|_0^L + \alpha_{np} \left| \frac{L}{n\pi} \sin \frac{n\pi}{L} x \right|_0^L - \beta_{np} \left| \frac{L}{n\pi} \cos \frac{n\pi}{L} x \right|_0^L \]

\[ \text{i.e. RHS} = \alpha_{0p} L - \beta_{np} \frac{L}{n\pi} \left[ \cos \frac{n\pi}{L} - 1 \right] \quad (2.3) \]

Hence, for the part that includes \( \cos \frac{n\pi}{L} \) to disappear, \( n \) should be even numbers,

i.e. \( n = 2, 4, 6 \ldots \)

Hence,

\[ \text{RHS} = \alpha_{0p} L \]

\[ \text{LHS} = \int_0^L f(p) \, dx = 2 \sum_{i=1}^{i=n} \frac{p_i \cdot \Phi}{i \cdot \pi} \]

i.e.

\[ \alpha_{0p} L = 2 \sum_{i=1}^{i=n} \frac{p_i \cdot \Phi}{i \cdot \pi} \quad (2.4) \]

from which we get

\[ \alpha_{0p} = \frac{2 \sum_{i=1}^{i=n} \frac{p_i \cdot \Phi}{i \cdot \pi}}{L} \quad (2.5) \]

where:

- \( N \) = number of storeys
- \( \Phi \) = depth of the beams
- \( p_i \) = stress in lb/in\(^2\) due to "P" at the \( i \)th storey.

Now, multiplying both sides of equation (1) by \( \sin \frac{n\pi}{L} x \) and integrating

\[ \int_0^L f(p) \sin \frac{n\pi}{L} x \, dx = \int_0^L \alpha_{0p} \sin \frac{n\pi}{L} x \, dx + \int_0^L \alpha_{np} \sin \frac{2n\pi}{L} x \, dx + \int_0^L \beta_{np} \sin \frac{2n\pi}{L} x \, dx \quad (2.6) \]

\[ \text{RHS} = - \alpha_{0p} \left| \cos \frac{n\pi}{L} x \right|_0^L - \frac{\alpha_{np}}{2} \left| \frac{L}{2n\pi} \cos \frac{2n\pi}{L} x \right|_0^L \]

\[ + \frac{\beta_{np}}{2} \left| x - \frac{L}{n\pi} \sin \frac{2n\pi}{L} x \right|_0^L \]

\[ = -74- \]
\[
\text{RHS} = bnp \cdot \frac{L}{2} \quad \ldots \quad (2.7)
\]
\[
\text{LHS} = \int_0^L f(p) \sin \frac{n\pi}{L} x \, dx = -\frac{L}{n\pi} \left[ f(p) \cos \frac{n\pi}{L} x \right]_0^L
\]

i.e. \quad \text{LHS} = -\frac{L}{n\pi} \left| \sum P_i \cos \frac{n\pi}{L} x \right|_{x_1}^{x_1+h}

= -\frac{L}{n\pi} \left| \sum \frac{P_i}{n} \left\{ \cos \frac{n\pi}{L} (x_1+h) - \cos \frac{n\pi}{L} x_1 \right\} \right|

= \frac{2L}{n\pi} \sum \frac{P_i}{n} \left\{ \sin \frac{n\pi}{2L} (2x_1+h) \cdot \sin \frac{n\pi}{2L} h \right\}

\text{but} \quad \text{RHS} = \text{LHS}

\text{i.e.} \quad bnp \cdot \frac{L}{2} = \frac{2L}{n\pi} \sum_{i=1}^{i=2N} \frac{P_i}{n} \left\{ \sin \frac{n\pi}{2L} (2x_i+h) \cdot \sin \frac{n\pi}{2L} h \right\}

\text{Hence,}

bnp = \frac{4}{n\pi} \sum_{i=1}^{i=2N} \frac{P_i}{n} \left\{ \sin \frac{n\pi}{2L} (2x_i+h) \cdot \sin \frac{n\pi}{2L} h \right\} \quad \ldots \quad (2.8)

Again, multiplying both sides of (1) by \( \cos \frac{n\pi}{L} x \) and integrating we get

\[
\int_0^L f(p) \cos \frac{n\pi}{L} x \, dx = \int_0^L f(p) \cos \frac{n\pi}{L} x \, dx + \int f(p) \cos^2 \frac{n\pi}{L} x \, dx + \int \frac{L}{n\pi} \sin \frac{2n\pi}{L} x \, dx \quad (2.9)
\]

\[
\text{RHS} = \frac{L}{n\pi}. \left. P \sin \frac{n\pi}{L} x \right|_0^L + \frac{anp}{2} \left. x + \frac{L}{n\pi}. \sin \frac{2n\pi}{L} x \right|_0^L - \frac{bnp}{2} \left. \frac{L}{2n\pi} \cos^2 \frac{n\pi}{L} x \right|_0^L
\]

\text{i.e.} \quad \text{RHS} = anp \cdot \frac{L}{2} \quad \ldots \quad (2.10)

- 75 -
The coefficients of equation (1) \((Acp, Anp, bnp)\) are now known.

3. Expressing the bending stresses resulting from the moments in the connecting beams in terms of Fourier's Series.
The stress \( f \) at any point between \( x = S, x = S + h \) may be given by:

\[
f_x = (S + \frac{h}{2} - x) \tan \theta_i
\]

where \( \tan \theta_i = \frac{2L}{h}. \)

The loading pattern shown in the above figure can be expressed as follows:

\[
f(x) = P_0 + \sum a_n x \cos \frac{n\pi x}{L} + \sum b_n x \sin \frac{n\pi x}{L} \quad \ldots (3.1)
\]

Integrating this expression over the whole length of the beam, we get:

\[
\int f(x) \, dx = \int P_0 \, dx + \int a_n x \cos \frac{n\pi x}{L} \, dx + \int b_n x \sin \frac{n\pi x}{L} \, dx
\]

\[
RHS = P_0 L + a_n \frac{L}{n\pi} \left( \sin \frac{n\pi x}{L} \right)_0^L - b_n \left( \frac{L}{n\pi} \right) \left( \cos \frac{n\pi x}{L} \right)_0^L
\]

\[
= P_0 L - b_n \frac{L}{n\pi} \left[ \cos n\pi - 1 \right]
\]

for the second term of this equation to disappear "n" should be even \( (n = 2, 4, 6, \ldots) \)

and

\[
RHS = P_0 L.
\]

\[
LHS = \int f(x) \, dx = 0
\]

but

\[
RHS = LHS
\]

thus

\[
P_0 L = 0 \quad (3.2)
\]

Now, multiplying both sides of (1) by \( \sin \frac{n\pi x}{L} \) and integrating, we get:

\[
\tan \theta_i \int (S + \frac{h}{2} - x) \sin \frac{n\pi x}{L} \, dx = \int a_n \frac{L}{2} \sin \frac{n\pi x}{L} \, dx
\]

\[
+ \int b_n x \sin^2 \frac{n\pi x}{L} \, dx.
\]

\[
\therefore \quad RHS = b_n \frac{L}{2}
\]

\[
LHS = \tan \theta_i \frac{L}{h} \int \sin \frac{n\pi x}{L} \, dx - \tan \theta_i \frac{L}{h} \int \sin \frac{n\pi x}{L} \, dx
\]
i.e. \[ \text{LHS} = \tan \theta_i \left[ \left( \frac{L}{n\pi} \right)^2 \sin \frac{n\pi}{L} x \right]^{5+h} \]

\[ = \tan \theta_i \cdot \frac{L}{n\pi} \left[ \cos \frac{n\pi}{L} x \left( 5 + \frac{h}{2} \right) \cos \frac{n\pi}{L} x \right]^{5+h} \]

\[ = \tan \theta_i \cdot \frac{L}{n\pi} \cdot \cos \frac{n\pi}{L} \left( \frac{2(5+h)}{2} \right) \left[ \frac{h}{n} \cos \frac{n\pi}{2L} \theta - \frac{2L}{n\pi} \sin \frac{n\pi}{2L} \theta \right] \]

but \[ \text{LHS} = \text{RHS} \]

Hence,

\[ \text{box} = \frac{2}{n} \sum_{i=1}^{2N} \frac{1}{n} \cdot \tan \theta_i \cdot \cos \frac{n\pi}{2L} \left( 2i + h \right) \quad \left[ \frac{h}{n} \cos \frac{n\pi}{2L} \theta - \frac{2L}{n\pi} \sin \frac{n\pi}{2L} \theta \right] \quad (3.3) \]
Again, multiplying equation (1) by \( \cos \frac{n \pi x}{L} \) and integrating we get:

\[
\tan \theta_i \int_0^L \left( s + \frac{b}{2} - x \right) \cos \frac{n \pi x}{L} \, dx = \int_0^L a_{nx} \cos \frac{n \pi x}{L} \, dx + \int_0^L \frac{b_{nx}}{2} \sin \frac{2n \pi x}{L} \, dx.
\]

\[
\text{RHS} = \int_0^L a_{nx} \cos^2 \frac{n \pi x}{L} \, dx + \frac{b_{nx}}{2} \int_0^L \sin \frac{2n \pi x}{L} \, dx
\]

ie. \( \text{RHS} = a_{nx} \frac{L}{2} \).

\[
\text{LHS} = \tan \theta_i \left[ \left( s + \frac{b}{2} \right) \int_0^L \cos \frac{n \pi x}{L} \, dx - \int_0^L x \cos \frac{n \pi x}{L} \, dx \right]
\]

\[
= \tan \theta_i \left[ \left( s + \frac{b}{2} \right) \frac{L}{n \pi} \sin \frac{n \pi x}{L} - \frac{Lx}{n \pi} \sin \frac{n \pi x}{L} - \left( \frac{L}{n \pi} \right)^2 \cos \frac{n \pi x}{L} \right]_0^L
\]

\[
= \tan \theta_i \cdot \frac{L}{n \pi} \cdot \sin \frac{n \pi x}{L} \cdot \frac{2s+b}{2}
\]

i.e. \( \text{LHS} = \tan \theta_i \cdot \sin \frac{n \pi x}{L} \cdot \frac{(2s+b)L}{2n \pi} \left[ \frac{2L}{n \pi} \sin \frac{n \pi x}{2L} - \frac{L}{2} \cos \frac{n \pi x}{2L} \right]
\]

but \( \text{RHS} = \text{LHS} \)

Hence,

\[
an_{nx} = \left\{ \frac{2}{n} \sum_{i=0}^{i=2N} \frac{1}{n} \tan \theta_i \cdot \sin \frac{n \pi x}{2L} \left( 2s+b \right) \right\}
\]

\[
\left[ \frac{2L}{n \pi} \sin \frac{n \pi x}{2L} - \frac{L}{2} \cos \frac{n \pi x}{2L} \right] \]

\[
\text{---(3.5) ---}
\]
The stresses in the walls may now be given by:

\[ f(y) + f(x) = A_0p + \sum \left\{ (a_n p + a_n x) \cos \frac{n \pi x}{L} + (b_n p + b_n x) \sin \frac{n \pi x}{L} \right\} \]

The coefficients in (3.6) are given in equations 2.5, 2.11, 3.5, 2.8 and 3.3.

4. Stress Functions of the Individual Loadings

4.1 A uniformly loaded cantilever

From statical considerations it is seen that the total shear at any section is proportional to \( x \). Consequently we may assume that

\[ t_{xy} = -Kx f(y) \] where \( K \) is constant but \( t_{xy} \) vanishes at both \( y = +c \) and \( y = -c \), so it may be reasonable to assume that

\[ f(y) = (c^2 - y^2) \]

i.e.

\[ t_{xy} = -Kx (c^2 - y^2) = -\frac{\partial^2 F}{\partial x \partial y} \]

or

\[ \frac{\partial^2 F}{\partial x \partial y} = Kx (c^2 - y^2) \].
Integrating this expression w.r.t. \( y \) we get

\[
\frac{\partial^2 F}{\partial x \partial y} = \int k x (c^2 - y^2) \, dy = k x (c^2 y - \frac{1}{3} y^3) + f_1(x)
\]

where \( f_1(x) \) is a function of \( x \) only,

i.e. \( \frac{\partial^2 F}{\partial x \partial y} = k y (c^2 - \frac{1}{3} y^3) + f_1'(x) = f_y \).

**Boundary Conditions**

(i) \( f_y = \frac{\partial^2 F}{\partial x \partial y} = -\omega(i-1) \) at \( y = -c \)

(ii) \( f_y = \frac{\partial^2 F}{\partial x \partial y} = -\omega(i) \) at \( y = +c \)

By substituting these boundary conditions into the expression for "\( f_y \)" we get:

\[
\begin{align*}
\frac{\partial F}{\partial y} &= \int 0 \, dx = k y (c^2 - y^2) + F_2(y) \\
\frac{\partial F}{\partial y} &= K \int x (c^2 - y^2) \, dx = \frac{K x^2}{2} (c^2 - y^2) + F_2(y)
\end{align*}
\]

where \( F_2(y) \) is a function of \( y \) only.

Hence, \( \frac{\partial F}{\partial y} = -\frac{\omega(i) + \omega(i-1)}{2} x \) + constant, and

\[
K = \frac{3}{4} \frac{\omega(i-1) - \omega(i)}{c^3}
\]

Now, by integrating the expression \( \frac{\partial^2 F}{\partial x \partial y} = k x (c^2 - y^2) \).

w.r.t. \( x \) we get

\[
\frac{\partial F}{\partial y} = \int k x (c^2 - y^2) \, dx = \frac{k x^2}{2} (c^2 - y^2) + F_2(y)
\]

where \( F_2(y) \) is a function of \( y \) only.

Hence, \( \frac{\partial F}{\partial y} = -k y x^2 + F_2'(y) \).

but \( \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = 0 \)

i.e. \( \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left( k y c^2 + F_2'(y) - \frac{1}{3} y^3 - \frac{\omega(i) + \omega(i-1)}{2} y^2 - k y x^2 \right) = 0 \)

i.e. \( [ -4 k y + F_2'''(y) ] = 0 \)

Hence,

\[
F_2'''(y) = 4 k y
\]

\[
F_2''(y) = 2 k y^2 + C \quad \text{and} \quad F_2'(y) = \frac{2}{3} k y^3 + C y + D
\]

where \( C \) and \( D \) are constants.
Now, \( f_x = \frac{\partial^2 F}{\partial y^2} = -K y x^2 + \frac{2}{3} K y^3 + C y + D. \)

we still have the following boundary conditions to be satisfied:

(i) \( \int f_x \, t \, dy = 0 \) at \( x = 0 \) (no axial force)
(ii) \( \int f_y \, t \, dy = 0 \) at \( x = 0 \) (no B.M. resultant)
(iii) \( \int f_y \, t \, dy = 0 \) at \( x = \) (no axial force)
(iv) \( \int f_y \, t \, dy = M \) at \( x = \) (ext. moment = internal moment)

By substituting into these conditions, the values of the different constants were found to be

\[ D = 0 \quad \text{and} \quad C = \frac{3}{10} \cdot \frac{\omega(3) - \omega(3-1)}{\epsilon} \]

The stress expression may now be given by

\[ f_x = \frac{\partial^2 F}{\partial y^2} = -\frac{3}{4} y \cdot \frac{\omega(3-1) - \omega(3)}{\epsilon^3} \left( x^2 + \frac{2}{5} \epsilon^2 - \frac{2}{3} y^2 \right) \]
\[ f_y = \frac{\partial^2 F}{\partial x^2} = \frac{3}{4} y \cdot \frac{\omega(3-1) - \omega(3)}{\epsilon^3} \left( \epsilon^2 - \frac{1}{3} y^2 \right) - \frac{\omega(3) + \omega(3-1)}{2} \]
\[ t_{xy} = -\frac{3}{4} \frac{\omega(3-1) - \omega(3)}{\epsilon^3} \cdot x \left( \epsilon^2 - y^2 \right) \]
4.2 A cantilever loaded at one end by a concentrated force

For this structure it may be reasonable to assume that $f_y = 0$ all over the structure, i.e. $f_y = \frac{\partial^2 F}{\partial x^2} = 0$
The biharmonic function may, therefore, be expressed as follows:

$$F = x \cdot f_1(y) + f_2(y)$$

where $f_1(y)$ and $f_2(y)$ are functions in $y$ only. But

$$t_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = 0 \quad \text{at} \quad y = \pm c$$

Hence,

$$t_{xy} = -f'_1(y) = 0 \quad \text{at} \quad y = \pm c$$

$f'_1(y)$ may be of the form $A(y^2 - c^2)$

where $A$ is a constant.

$$f_1(y) = A(\frac{1}{3}y^3 - c^2 y) + \alpha$$

and

$$F = Ax(\frac{1}{3}y^3 - c^2 y) + \alpha x + f_2(y)$$

but

$$t_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = -A(y^2 - c^2)$$

Differentiating this expression w.r.t. $y$ we get

$$f_x = \frac{\partial^2 F}{\partial y^2} = 2Ax + f''_2(y).$$
Since $f_x = 0$ at $x = 0$, then $f_2''(y) = 0$

i.e. $f_2(y) = \beta y + \gamma$

where $\beta, \gamma = \text{constants}$.

\[ t \int_{-c}^{c} t_{xy} \, dy = -P = -t \int_{-c}^{c} A(y^2 - c^2) \, dy \]

\[ A = -\frac{3}{4} \frac{P}{t \cdot c^3} \]

The stresses may then be given by

\[ f_x = -\frac{3}{4} \frac{P}{t \cdot c^3} \cdot xy \]

\[ f_y = 0 \]

\[ t_{xy} = \frac{3}{4} \frac{P}{t \cdot c^3} \cdot (y^2 - c^2) \]

4.3 A cantilever loaded with discontinuous shear forces (tangential) along its sides

Shear Wall Assembly

A Typical Component Wall.
It can be assumed that
\[ f_x = Dy + E \quad \text{and} \quad f_y = txy = 0 \]
where \( D \) and \( E \) are constants.

The boundary conditions are

(i) \[ t \int_{-c}^{+c} f_x \, dy \big|_{jd} = - \sum_0^{x=jd} P(l-j, j) - P(l, j) \]

(ii) \[ t \int_{-c}^{+c} f_x \, y \, dy \big|_{jd} = \sum_0^{x=jd} \{ P(l-j, j) + P(l, j) \} c \]

From (i) we get
\[ t \int_{-c}^{+c} (Dy + E) \, dy = - \sum_0^{jd} (P(l-j, j) - P(l, j)) = 2tc \]

Hence,
\[ E = -\left[ \left( \sum_0^{jd} P(l-j, j) - P(l, j) \right) \right] / 2tc \]

Condition (ii) gives
\[ t \int_{-c}^{+c} (Dy^2 + Ey) \, dy = c \sum_0^{jd} \{ P(l-j, j) + P(l, j) \} = \frac{2}{3} tc^3 D \]
\[ \therefore \quad D = \frac{\sum_0^{jd} (P(l-j, j) + P(l, j))}{2/3 tc^2} \]

The stresses may then be expressed as
\[ f_x = \frac{d^2 F}{dy^2} = \frac{\sum_0^{jd} (P(l-j, j) + P(l, j)) y - \sum_0^{jd} (P(l-j, j) - P(l, j))}{2/3 tc^2} \]
\[ f_y = 0 \]
\[ txy = 0 \]
The equation for the stress function is

\[ \frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \]  (1)

This equation may be satisfied by taking "F" in the form

\[ F = \sin \frac{n\pi x}{L} \cdot f(y) \]  (2)

where \( f(y) \) is a function of "y" only.

Substituting (2) into (1) gives

\[ \alpha f(y) - 2 \alpha^2 \frac{\pi}{L} f(y) + f(y) = 0 \]  (3)

where \( \alpha = \frac{n\pi}{L} \)

The solution of (3) is

\[ f(y) = C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y \]

and the stress function becomes

\[ F = \sin \alpha x \left( C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y \right) \]  (4)

where \( C_1, C_2, C_3 \) and \( C_4 \) are constants.
The corresponding stresses are:

\[ f_x = \frac{\partial^2 F}{\partial y^2} = \sin \alpha x \left[ C_1 \alpha^2 \cosh \alpha y + C_2 \alpha^2 \sinh \alpha y + C_3 \alpha (2 \sinh \alpha y + \alpha y \cosh \alpha y) + C_4 \alpha (2 \cosh \alpha y + \alpha y \sinh \alpha y) \right]. \]

\[ f_y = \frac{\partial^2 F}{\partial x^2} = -\alpha^2 \sin \alpha x \left[ C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y \right]. \]

\[ t_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = -\alpha \cos \alpha x \left[ C_1 \alpha \sinh \alpha y + C_2 \alpha \cosh \alpha y + C_3 (\cosh \alpha y + \alpha y \sinh \alpha y) + C_4 (\sinh \alpha y + \alpha y \cosh \alpha y) \right]. \]

(5)

The boundary conditions are

\[ t_{xy} = 0, \quad f_y = -B \sin \alpha x \quad \text{for} \quad y = +c. \]

\[ t_{xy} = 0, \quad f_y = -A \sin \alpha x \quad \text{for} \quad y = -c. \]

(6)

Substituting these conditions in equation (5) gives

\[ C_1 = \frac{A + B}{\alpha^2} \cdot \frac{\sinh \alpha c + \alpha c \cosh \alpha c}{\sinh 2\alpha c + 2\alpha c}. \]

\[ C_2 = -\frac{A - B}{\alpha^2} \cdot \frac{\cosh \alpha c + \alpha \cosh \alpha c}{\sinh 2\alpha c - 2\alpha c}. \]

\[ C_3 = \frac{A - B}{\alpha^2} \cdot \frac{\alpha \cosh \alpha c}{\sinh 2\alpha c - 2\alpha c}. \]

\[ C_4 = -\frac{A + B}{\alpha^2} \cdot \frac{\alpha \sinh \alpha c}{\sinh 2\alpha c + 2\alpha c}. \]

(7)

The stresses produced by \( \cos \frac{\pi x}{L} \) load intensity may be easily obtained from equation (5) exchanging \( \sin \alpha x \) for \( \cos \alpha x \) and vice versa, and by changing the sign of \( t_{xy} \).
6. Computation Process

Once the forces in the connecting beams are evaluated the lateral forces \( P \) and the bending stresses at the beam joints may be expressed in terms of Fourier's Series (Sections 2 and 3).

Thus,

\[
 f(p) + f(x) = A_{op} + \sum \left\{ (a_n p + a_n x) \cos \frac{n\pi x}{L} + (b_n p + b_n x) \sin \frac{n\pi x}{L} \right\} \tag{3.6}
\]

The stresses corresponding to the uniform load intensity \( A_{op} \) may be obtained by using the stress expression given in section (4.1). The stresses corresponding to the shear forces in the beams are given in section (4.3). The stress expression for the sinusoidal load intensities, \( i.e. \)

\[
 (a_n p + a_n x) \cos \frac{n\pi x}{L}, (b_n p + b_n x) \cos \frac{n\pi x}{L}
\]

are shown in Equation 5, Section 5.

A computer program using the Atlas Autocode was written to carry out the calculations. The number of terms in Fourier's Series was determined by the computer, such as the stresses at the beam joints approach the exact predetermined values.
APPENDIX IV

THE FINITE ELEMENT METHOD IN STRUCTURAL MECHANICS

1. INTRODUCTION

The finite element method is essentially a generalization of standard structural analysis procedures. This permits the calculation of stresses and deflections in two- and three-dimensional structures by the same techniques applied in the analysis of framed structures. The basic concept of the finite element method is the idealization of elastic continua as assemblages of discrete elements joined together at finite nodal points. The material properties of the original structure are, of course, retained in the individual elements. The essential elastic characteristics of a typical element may be expressed by the relationship between forces applied at the joints and the deflections resulting therefrom. Once this relationship is evaluated similar equations for the entire assembly can be constructed by superimposing the load-deflection relationship of the individual component elements.

It should be realized that the approximation employed in the finite element method is of a physical nature; the actual continuum is idealized as an assemblage of a finite number of elements. There need be no approximations in the mathematical analysis of the substitute system. This feature distinguishes the FE technique from the finite difference method, in which the exact governing equations of the actual structure are solved by approximate mathematical procedures.

The finite element method which was introduced in 1956 by Clough et al (26) was a natural development of two major factors. The first was the advent of electronic digital computers of considerable speed and memory capacity. The second factor was the advance in the matrix methods of structural analysis pioneered by Argyris(4) and Langefors (75).
Most of the literature on the finite element method is contained in the Aeronautical Engineering Journals, as it was for the need of the aircraft industry to more advanced analytical techniques that this method originated.

The application of the FE procedure for the analysis of modern aircraft was demonstrated by Argyris (5) and others (3,4,58,73).

However, this analytical technique was utilized in other engineering fields. The FE method was used by Paulling (94) in shipbuilding and by Zeinkiewicz (119) and others (21, 26, 28) in Civil Engineering problems.

In the sequel, the concept of the FE method is briefly outlined and a computer program for rectangular grid finite element analysis is given.

2. THE FINITE ELEMENT PROCEDURE

The finite element analysis may be divided into three basic phases. These phases are:

(i) Structural Idealization

This phase involves the subdivision of the actual continuum into an assemblage of finite segments. Broadly speaking, the element may have any arbitrary shape (i.e. triangular, rectangular, curved, etc.). Experience (4,5) has shown, however, that rectangular elements give higher accuracy and easier programming task.

(ii) Evaluation of the Element Elastic Properties

In this phase the physical and structural properties of a typical element is evaluated. This results in the load-deflection relationship of the element. To provide bases for the development of this relationship, simplifying assumptions as to the behaviour of the element under loading have to be adopted, e.g. a typical element in a two-dimensional structure, such as aircraft skin, walls and plates loaded in their own plane, is assumed to be in a state of plane-stress, whereas in the case of tunnels, dams and retaining walls the element is assumed to be subjected to plane-stress conditions.
Furthermore, the number of degrees of freedom assigned to each node must be fixed. In two-dimensional plane-stress problems two degrees of freedom characterising translation in the X- and Y- direction are usually allocated to each node. However, in structures where bending effect is prominent, a third degree of freedom representing notation may be assumed (90). Also, three translations in three perpendicular directions (x, y and z) are assigned to each node in three-dimensional problems.

Another prerequisite is that the physical behaviour of the element must be prescribed, thus providing a definable element displacement pattern. In general, the discretion concerning the element behaviour combined with the other assumptions usually results in violation of either (or both) the compatibility or equilibrium conditions within or at the boundary of the elements.

The effect of violating these basic structural requirements on the accuracy of the solution is discussed by De Veubeke (43). The force displacement relationship may be expressed most conveniently by the stiffness or the flexibility matrix of the individual element. These matrices are essentially symmetrical, square and the dimension of which is (the number of nodes times the number of degrees of freedom) square.

This relationship may be expressed as

\[
[K_e] \{r_e\} = \{S_e\} ; \text{ or } \{r_e\} = [f_e] \{S_e\}
\]

where

\([K_e]\) is the element stiffness matrix

\([f_e]\) is the element flexibility matrix

\([r_e], [S_e]\) are the displacement and load vectors respectively.
Analysis of the Entire Assemblage

Following phase (ii) the load-displacement relationship of the entire assemblage is constructed from the stiffness (or flexibility) matrices of the individual elements. The equilibrium equations of the whole structure may be given by

$$ [K] \{ r \} = \{ R \} $$

where $[K]$ is the master stiffness matrix of the entire structure.

$\{ r \}, \{ R \}$ the displacement and load vectors respectively.

3. ELEMENT DEFORMATION PATTERNS

At any point in a continuous medium subjected to some arbitrary load pattern the displacements $(\delta_0)$ of the point may be represented by functions of the distances $x$ and $y$ measured from the unstressed location, i.e.

$$ \mu = A + \sum_{i=1}^{n} (ax + by)^i \quad \text{and} \quad \nu = B + \sum_{i=1}^{n} (cx + dy)^i $$

where $A$ and $B$ represent rigid body displacements.

Quadratic functions are commonly used in which the displacements may be given by

$$ \mu = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy $$
$$ \nu = b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 y^2 + b_6 xy $$

where $a_1, a_2, \ldots, b_1, \ldots, b_6$ are constants.

By adjusting these constants various interesting patterns can be obtained.

Fig. 3.6d shows the linear edge displacement pattern suggested by Argyris (4). The deformation functions corresponding to this pattern are

$$ \mu = a_1 x + a_2 y + a_3 xy + a_4 $$
$$ \nu = a_5 x + a_6 y + a_7 xy + a_8 \ldots (3.1) $$

- 92 -
Clough et al. (26) employed a different element behaviour pattern in which linear stress variation along the boundaries of the elements was assumed (Fig. 3.6b), i.e.

\[
\begin{align*}
\sigma_x &= a_1 + a_2 y \\
\sigma_y &= a_3 + a_4 x \\
\tau_{xy} &= a_5 
\end{align*}
\]  

(3.2)

Another element stress pattern (Fig. 3.6c) given by

\[
\begin{align*}
\sigma_x &= a_1 + a_4 x \\
\sigma_y &= a_2 + a_5 y \\
\tau_{xy} &= a_3 - a_5 x - a_4 y 
\end{align*}
\]  

(3.3)

was used by Gallagher (52).

It can be shown that pattern 3.1 satisfies the conditions of compatibility at the boundary of the element while violating the equilibrium conditions at the nodal points. Pattern 3.2, on the other hand, only satisfies the equilibrium conditions, and pattern 3.3 violates both conditions of equilibrium and compatibility between the various elements.

De Veubeke's theory of bounds (43) serves to assess the overall accuracy of the analysis with respect to the prescribed deformation patterns. According to this theory, it can be expected that pattern 3.1 will result in an underestimation of the structural deflection whereas pattern 3.2 may give higher deflection values.

4. MATRIX FORMULATION OF THE ELEMENT STIFFNESS RELATIONSHIP

Once the element structural behaviour is prescribed, the formulation of the element stiffness matrix follows a sequence of simple matrix operations. These operations consist of:

(i) defining the element behaviour, i.e.

\[
\{e\} = f(c, x, y, z) 
\]  

(1)
where $f$ means a function of; and 
\[ c \]
is constant 
\[ \{ \hat{r}_e \} \] is the nodal displacement vector.

(ii) Expressing the nodal displacement $\{ \hat{r}_e \}$ in terms of generalised displacement $\{ a \}$ as

\[ \{ \varepsilon \} = [B] \{ a \} \]  \hspace{1cm} (2)

(iii) The strain at any point is also expressed in terms of $\{ a \}$ as

\[ \{ \varepsilon \} = [D] \{ a \} \]  \hspace{1cm} (3)

(iv) The stresses may, therefore, be given by

\[ \{ \sigma \} = [E] \{ \varepsilon \} = [E][D] \{ a \} \]  \hspace{1cm} (4)

(v) By definition the stiffness coefficient "$k_{ij}$" is given by

\[ k_{ij} = \int_v \sigma^t \varepsilon \ dV \]  \hspace{1cm} (5)

where $\sigma^t$ is the transpose of $\sigma$ 
\[ \varepsilon \] is the virtual strain.

Substituting from (2), (3) and (4) into (5) gives

\[ [K_e] = \mathcal{T} [B]^t \int_{xy} [D]^t [E][D] \ dxdy \]  \hspace{1cm} (6)

which is the element stiffness matrix.
5. ASSEMBLAGE OF THE MASTER STIFFNESS MATRIX $K$ OF THE ENTIRE STRUCTURE

The master stiffness matrix $K$ of the whole assembly may be constructed by using Argyris's formal co-ordinate transformation method (4). This is given by

$$ [K] = [a^T][k][a] $$

where $[a]$ is a matrix relating the generalised co-ordinates to the nodal point co-ordinates.

$[k]$ is the stiffness matrix of the unassembled unit panels written in diagonal partitioned form, i.e.

$$ [k] = \begin{bmatrix}
  k_a & & & \\
  & k_b & & \\
  & & \ddots & \\
  & & & k_m
\end{bmatrix} $$

where $k_a, k_b, \ldots k_m$ are the stiffness matrices of elements a, b, c, \ldots respectively.

$[K]$ is the assembled stiffness matrix of the entire structure, $K$ is a square, symmetric band and sparsely populated matrix.

This procedure requires large computing space for the storage of $[K]$, $[a]$ and $[a^T]$. Also, the matrix operation $[a^T][k][a]$ gives the complete stiffness matrix $K$ with its full band-width. However, advantage may be made of the properties of $[K]$, i.e. symmetric and band matrix, by merely storing a half band of the matrix, i.e.

$$ [K]_{2n \times 2n} = \begin{bmatrix}
\end{bmatrix} = \begin{bmatrix}
\end{bmatrix} $$

where $2n \times \frac{1}{2}$ bond width.
This process requires re-arrangement of the order of columns in the new stiffness matrix \([RK]\). It can be seen that the diagonal of the original matrix \([K]\) forms the first column in the half-bank matrix \(RK\), and consequently an element \(RK(i,j)\) corresponds to \(K(i,i+j-1)\), i.e.

\[
RK(i,j) = K(i,i+j-1)
\]

Two methods of assembling the matrix \(RK\) were devised. These are termed the method of blocks and the method of rows. The method of blocks involved superimposing the element stiffness matrices where elements meet at nodal points, as shown below,

\[
[RK] = \begin{bmatrix}
\vdots & \vdots \\
\vdots & \vdots \\
\end{bmatrix} = \begin{bmatrix}
RK
\end{bmatrix}
\]

whereas in the method of rows the master matrix is assembled in a regular fashion, i.e. row "i" is assembled before any element is inserted in row "i+1".

**Example**

Assemble the master stiffness matrix \(RK\) of the shown assembly. The half band width may be given by

\[
HBW = 2 \text{Npr} + 4 = 2 \times 3 + 4 = 10
\]

where \(HBW\) = half band width

\(\text{Npr}\) = Number of nodes per row.

The numbering of the nodal points and the degrees of freedom at each node is shown below. The following expression was used to number the degrees of freedom.
The code number of the degree of freedom =

\[ 2 \text{NoE} + 2 \left( \text{int. pt.} \frac{\text{NoE} - 0.2}{\text{Npr} - 1} \right) - 1 \]

where \( \text{NoE} \) = number of element
\( \text{Npr} \) = number of nodes per row
\( \text{Int. pt.} \) = integer part of

code numbers of nodes
code number of degrees of freedom

**Method of Blocks**

In this method the code numbers of the degrees of freedom of the typical element, the stiffness matrix of which is known and permanently stored in the computer, are stored in an array \( b(1:8) \), i.e.

\[
\begin{align*}
& b(1) \quad b(2) \quad b(3) \quad b(4) \quad b(5) \quad b(6) \quad b(7) \quad b(8) \\
& 3 \quad 7 \quad 4 \quad 8 \quad 1 \quad 5 \quad 2 \quad 6
\end{align*}
\]

Similarly, the code numbers of the degrees of freedom of the nodes of the various elements in the structures are in turn stored in another array \( a(1:8) \), i.e.
\[ a(1) \quad a(2) \quad a(3) \quad a(4) \quad a(5) \quad a(6) \quad a(7) \quad a(8) \]

Element A  
1  2  3  4  7  8  9  10

Element B  
33 4 5 6 9 10 11 12

The numbering of the degrees of freedom of the nodes should, of course, follow the same order adopted for the numbering of the degrees of freedom in the typical element.

The stiffness matrix is then assembled by substituting in the following expression for \( i = 1 \) to 8 for \( j = 1 \) to 8 (i.e. 64 times).

The expression for RK is

\[ RK(a(i), a(j) - a(i) + 1) = K_T(b(i), b(j)) + C \]

where \( K_T(\quad) \) = coefficient in the stiffness matrix of the typical element

\[ C \] = whatever is stored in the location \((a(i), a(j) - a(i) + 1)\) of RK.

The computer ignores any coefficient beyond the half band width (when \( a(i) > a(j) \)).

For the given assembly

\[ RK(1,1) = A(3,3) + 0 \]
\[ RK(1,2) = A(3,7) + 0 \]
\[ \vdots \]
\[ RK(3,3) = A(4,4) + 0 \]
\[ RK(3,4) = A(4,8) + 0 \]
\[ \vdots \]
\[ RK(3,3) = B(3,3) + A(4,4) \quad \rightarrow \quad C = A(4,4) \]
\[ RK(3,4) = B(3,7) + A(4,8) \quad \rightarrow \quad C = A(4,8) \]

where \( A(i,j), B(i,j) \) = coefficient in the stiffness matrix of element A and B respectively.

The complete stiffness matrix RK is shown below.

**Method of Rows**

For a structure with a given number of elements "N" and nodal points "n" the computer cycles through the nodes.
For a typical node "i", as shown below, the computer determines (by using a formula which relates the node number to the number of the elements) the code numbering of the surrounding nodes, i.e.

elements surrounding node 2 are numbered A and B.

Each of these elements is then considered separately and the degrees of freedom of each are given code numbers in the same order used for the typical element "T", for which the stiffness matrix is known, i.e. when considering node "1" the surrounding elements are element "A". The degrees of freedom of element A are shown below.

The code numbers of the typical element "T" are

The appropriate stiffness coefficient is then computed and inserted in the position in the matrix RK that corresponds to the nodal points, i.e.

\[
\begin{align*}
K(1,1) &= A(3,3) ; \quad K(2,2) = A(7,7) \\
K(1,2) &= A(3,7) ; \quad K(2,3) = A(7,4) \\
&\vdots \\
K(1,10) &= A(9,2) ; \quad K(2,10) = A(7,2)
\end{align*}
\]
For node "2" the surrounding elements are "A" and "B". The degrees of freedom of each are shown below.

Hence,

\[
\begin{align*}
K(3,3) &= A(4,4) + B(3,3) \\
K(3,4) &= A(4,3) + B(3,7) \\
K(3,7) &= A(4,5) + 0 \\
K(3,8) &= A(4,1) + 0 \\
K(3,9) &= A(4,6) + B(3,5) \\
K(3,10) &= A(4,2) + B(3,1) \\
K(3,12) &= 0 + B(3,2)
\end{align*}
\]

The master stiffness matrix of the given structure is shown in Fig. A4/1.

6. SOLUTION OF THE EQUILIBRIUM EQUATION

\[ [k] \{r\} = \{R\] \]

The Gauss-Seidel iterative procedure\(^{48,91}\) appears to be the most efficient method of solving the set of equilibrium equations. The displacement at node "m" after "s+1" iterations may be given by

\[
\begin{align*}
gr_m^{(s+1)} &= \frac{1}{K_{mm}} \left[ R_m - \sum_{i=m+1}^{N} K_{mi} \cdot \hat{r}_i^{(s+1)} - \sum_{i=N}^{m+1} K_{mi} \cdot \hat{r}_i^{(s)} \right] \\
\end{align*}
\] (1)

The change in the displacement value \(\Delta r_m\) is

\[
\Delta r_m^{(s)} = r_m^{(s+1)} - r_m^{(s)}
\] (2)
<table>
<thead>
<tr>
<th>(A_{33})</th>
<th>(A_{37})</th>
<th>(A_{34})</th>
<th>(A_{38})</th>
<th>0</th>
<th>0</th>
<th>(A_{31})</th>
<th>(A_{35})</th>
<th>(A_{32})</th>
<th>(A_{36})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_{77})</td>
<td>(A_{74})</td>
<td>(A_{78})</td>
<td>0</td>
<td>0</td>
<td>(A_{71})</td>
<td>(A_{75})</td>
<td>(A_{72})</td>
<td>(A_{76})</td>
<td>0</td>
</tr>
<tr>
<td>(A_{44})</td>
<td>(B_{33})</td>
<td>(A_{48})</td>
<td>(B_{37})</td>
<td>0</td>
<td>0</td>
<td>(A_{41})</td>
<td>(A_{45})</td>
<td>(A_{42})</td>
<td>(B_{31})</td>
</tr>
<tr>
<td>(A_{88})</td>
<td>(B_{77})</td>
<td>0</td>
<td>0</td>
<td>(B_{74})</td>
<td>(B_{78})</td>
<td>(A_{81})</td>
<td>(A_{85})</td>
<td>(A_{82})</td>
<td>(B_{71})</td>
</tr>
<tr>
<td>(B_{44})</td>
<td>(B_{48})</td>
<td>0</td>
<td>0</td>
<td>(B_{41})</td>
<td>(B_{45})</td>
<td>(B_{42})</td>
<td>(B_{46})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(B_{58})</td>
<td>0</td>
<td>0</td>
<td>(B_{51})</td>
<td>(B_{35})</td>
<td>(B_{32})</td>
<td>(B_{36})</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(A_{11})</td>
<td>(A_{15})</td>
<td>(A_{12})</td>
<td>(A_{16})</td>
<td>0</td>
<td>0</td>
<td>(A_{14})</td>
<td>(A_{16})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(A_{55})</td>
<td>(A_{52})</td>
<td>(A_{56})</td>
<td>0</td>
<td>0</td>
<td>(A_{54})</td>
<td>(A_{56})</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(A_{22})</td>
<td>(B_{11})</td>
<td>(A_{26})</td>
<td>(B_{15})</td>
<td>0</td>
<td>0</td>
<td>(A_{24})</td>
<td>(B_{16})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(A_{66})</td>
<td>(B_{55})</td>
<td>0</td>
<td>0</td>
<td>(B_{52})</td>
<td>(B_{56})</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(B_{22})</td>
<td>0</td>
<td>0</td>
<td>(B_{26})</td>
<td>0</td>
<td>0</td>
<td>(B_{24})</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(B_{66})</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Fig. A4/1, The stiffness matrix \([RK]\) of the shown structure.
Thus,

$$\Delta r_m^{(s)} = \frac{1}{k_{mm}} \left[ R_m - \sum_{i=1}^{N} K_{mi} r_i^{(s)} - \sum_{i=m-1}^{N} K_{mi} r_i^{(s)} \right]$$

(3)

The successive over-relaxation process, in which an over-relaxation factor is used to speed up the convergence of the set of equations, is a modification of the Gauss-Seidel process. In this procedure an improved estimate of the displacement of node "m" is obtained by multiplying the change of displacement $\Delta r_m^{(s)}$, equation (2), by an over-relaxation factor, i.e.

$$r_m^{(s+1)} = r_m^{(s)} + \beta \cdot \Delta r_m^{(s)}$$

(4)

The optimum value of $\beta$ varies with the type of structure and loading conditions. However, it was reported (25,26) that a value of 1.80 to 1.95 would provide nearly optimum results.

7. COMPUTER PROGRAM

A computer program was written in Atlas Autocode to carry out the Finite Element Analysis. This program utilizes magnetic tapes for the storage of the master stiffness matrix $[K]$.

The data input for the program is as follows:

- **NXY** = Number of elements
- **N** = Number of nodes
- **NXYPR** = Number of nodes per row
- **TYPE** = Number of element shapes
- **th** = Thickness of the structure
- **PARXY(i,j)** = A matrix consisting of the particulars of the various shapes of elements used. The size of the matrix is (1:TYPE)(1:5). The particulars are in the following order:
i1 = code number for the element
i2 = width of element
i3 = height of element
i4 = Young's modulus
i5 = Poisson's ratio

TYPE XY(N) = an array consisting of code numbers of each element with respect to its shape.

0 = Number of right hand side of the set of equations.

RXY(2N) = an array in the vertical and horizontal forces applied at each node is stored.

A print of the program is shown below.
BEGIN
INTEGER NXY, N, NXYP, NFN, TYPE, J, S, HSXY, CHANNEL, SECTION, PXX, GAPXY

CLAIM TAPE (1); SECTION = 1
READ (NXY, N, NXYP, TYPE)
NFN = N - NXYP + 1; HSXY = 2 * NXYP + 4; PXY = 4 * HSXY 40

BEGIN
AREAL ARRAY KXY(1:2PXY, 1:HSXY), PARXY(1:TYPE, 1:S)
AREAL XX, YY, EXY, EXY, V, CNXY, E

READ (PARXY(1,1))

AREPEAT
SECTION = 1

IF TYPE THEN
       WRITE (5, 1) FILE (1, SECTION, PARXY(1,1), PARXY(TYPE, S))

SECTION = SECTION + 1

BEGIN
READ WXY1(1:1:6, 1:VXY, WXY, VXY, XX, YY, 1:V); TYPE, J
AREAL ARRAY XX, YY, VXY, EXY, EXY, 1:V, E

AREUTINE ASPEC
   ASSEMBLE MATRIX (XX, YY, EXY, EXY, V, E)
   AREUTINE ASPEC
   N = S (AREUTINE NXY, VX, WXY, VXY)

ACYCLE I = 1; TYPE
U = PARXY(I, 2); V = PARXY(I, 3); EXY = PARXY(I, 4); V = 40

AREPEAT

END
READ (PARXY(1, J))

REPEAT
  1, SECTION = 1
  FILE(1, SECTION, PARXY(1, J), PARXY(1Y, J))
  SECTION = SECTION + 1

BEGIN
  INTEGER XXY1, XXY2, XXY3, XXY4, XXY5, XXY6, LXY, L, JJ
  REAL ARRAY <XTYPE, 1: 3, 1: 3>
  INTEGER X, Y
  ARDUINE ASPEC ASSEMBLE MATRIX X (ARRAY ANAME, K, INTEGER ANAME, AC
  INTEGER NAME X, Y, EXY, LXY, V, S
  ARDUINE ASPEC ASSEMBLE MATRIX X (K, I, D, L, TH, EXY, LXY, V, S)
  1CYCLE I = 1, 1, TYPE
  D = PARXY(1, 2) ; L = PARXY(1, 3) ; EXY = PARXY(1, 4) ; V = AC
  PARXY(1, 5)
  ASSEMBLE MATRIX X (K, I, D, L, TH, EXY, LXY, V, S)
  REPEAT
  CJXY = (1, 3, 6) : NEOLINE ; AORDITION CONJOEDUALS ; SPACES (3) ; PRINT (CJXY
  VXY, 3, 6)
  NEOLINE(2)
  AORDITION MEXOCONJOEDUALMATERIX
  NEOLINE(2)
  1CYCLE I = 1, 1, 3
  1CYCLE J = 1, 1, 3
  PRINT (K, 1, 1) / CJXY, 3, 2 : SPACES (3)
  REPEAT
  NEOLINE
  REPEAT
  NEOLINES (2)
  WRITE TO FILE (1, SECTION, TYPEXY(1), TYPEXY(1Y), XXY)
  SECTION = SECTION + 1
  FILE(1, SECTION, TYPEXY(1), TYPEXY(XXY))
  TION + 1
  LXY = INT PI (TYPEXY) ; X = FRAG PI (TYPEXY) ; XIF X > 0 AC
  THEN LXY = LXY + 1
  EXPXY = INT PI ((X - LXY * XXY) / 512 + 1) - INT PI (((X + XXY * XXY) / 512) + 1)
  1CYCLE II = 1, 1, LXY
  NULL X (XXY, TYPEXY, EXPXY)
  1CYCLE JJ = 1, 1, XXY
<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>PRINT</td>
<td>Prints the output to the console.</td>
</tr>
<tr>
<td>REPEAT</td>
<td>Repeats the previous command.</td>
</tr>
<tr>
<td>NLINL</td>
<td>Prints a newline character.</td>
</tr>
<tr>
<td>NLINES(2)</td>
<td>Prints two newline characters.</td>
</tr>
<tr>
<td>WRITLJFILE</td>
<td>Writes to a file with specific parameters.</td>
</tr>
<tr>
<td>ACYCLE</td>
<td>Advances the cursor to the next position.</td>
</tr>
<tr>
<td>READ</td>
<td>Reads data from the file.</td>
</tr>
<tr>
<td>WRITE</td>
<td>Writes data to the file.</td>
</tr>
<tr>
<td>ARoutine</td>
<td>Defines a routine with specific parameters.</td>
</tr>
</tbody>
</table>

**Example Code**

```plaintext
REPEAT
NLINL
NLINES(2)
WRITLJFILE(1, SECTION, TYPEXY(1), TYPEXY(NXY)) ; SECTION = 6
NLXY = INT PT((NXY)X + PXY); A = FRAC PT(NXY)X + PXY); IF A > 0 THEN NLXY = NLXY + 1
GAPXY = INT PT(((2 - BXY)X + PXY) / 512) + 1 - INT PT(((4 - BXY)X + PXY) / 512) + 1
ACYCLE 2 = 1, LXY
NULL (NXY, 2PXY, 4BXY)
ACYCLE 3 = 1, PXY
S = I1 + (I1 - 1) + PXY; IF S > NXY THEN -> 6
NXY1 = S - INT PT((S - 0.2) / NXYPR) ; NXY2 = NXY1 - 1 ; NXY3 = NXY6
2 - (NXYPR - 1) ; NXY4 = NXY1 - (NXYPR - 1)
NXY = 25 - 1 ; NXY = 2S
-> 4 IF FRAC PT (S/NXYPR) = 0
RUN S (NXY1, 3, 7)
4 IF NXY4 < 1 THEN -> 5
RUN S (NXY4, 1, 3)
5: 4 IF FRAC PT (((NXY1 - 1) / (NXYPR - 1)) = 0 AND NXY1 = 1 ATAG
THEN -> 6
4 : RUN S (NXY2, 4, 3)
4 IF NXY3 < 1 THEN -> 5
RUN S (NXY3, 2, 5)
6 : AREPEAT
WRITLJFILE(1, SECTION, NXY(X, 1), NXY2(X, 2), NXY3(X, 3))
AC = NXY1(X, 1), NXY0(X, 2), NXY1(X, 3)
A(1) = 2(NXY1 + 2 + INT PT((NXY - 0.2) / (NXYPR - 1))) ; A(2) = A(1) + 1 ; A(3) = A(2) + 1
A(4) = 2NXYPR; A(5) = A(4) + 2NXYPR
1 = TYPEXY(NXY)
```

**Notes**

- The code snippet is a part of a larger program, possibly for a graphics or print function.
- The code uses basic arithmetic and conditional statements to control the cursor's movement and file writing operations.
- The variables and functions are defined to manage the position of the cursor and the writing to the file.

This code snippet is written in a language similar to assembly or high-level programming languages used for graphics output systems.
AROUTINE RUW S  ( XINTERGE NXY, VNXY, HNXY)
XINTERGE N X, Y, VL, HL
A(1) = 2NxY - 1 + 2* (INT PT ((NXY - 0.2)/ (NXYPR-1))) ; A(2) = A(1) + 1 AC
A(3) = A(1) + 2 ; A(4) = A(1) + 3
A(5) = A(1)2NXYPR ; A(6) = A(2)2NXYPR ; A(7) = A(3)2NXYPR ; A(8) = A(4)2NXYPR
I = TYPEXY (NXY)

ACLYCLE J = 1, 1, 3

IF VXY > A(J) THEN -> 1
C = A(J) - VXY + 1 ; VL = VXY - 2* (II - 1)*PXY
RXXY(VL, C) = RXXY(VL, C) + X(I, VNXY, S(J))
I : IF HXY > A(J) THEN -> 2
C = A(J) - HXY + 1 ; HL = HXY - 2* (II - 1)*PXY
RXXY(HL, C) = RXXY(HL, C) + X(I, HNXY, S(J))
2 : AXREPEAT
ZEND

AROUTINE ASSEMBLE MATRIX K (ARRAY ANAME X, XINTERGE ANAME I, AREAL AC
ANAME X, Y, VL, HX, GXY, V, E)
XINTERGE T, B
AREAL X, Y, Z, N1, N2, N3

NEWLINES(2)
E = (GXY/(GXY / (E+1))) ; GXY = EXY/(2(1+V))
X1 = (GXY*HXY/LXY) / (LXY) ; N2 = (GXY*TXY/3LXY) / (3LXY) ; N3 = AC
V = (GXY*THXY) / (V+THXY)
(V*THXY) / (V+THXY)

K(1, 1) = N1 + N1 ; K(1, 2, 2) = K(1, 1) ; K(2, 3, 3) = K(1, 1, AC
1) ; K(1, 4, 4) = (1, 1)
K(1, 5, 5) = N2 + N2 ; K(1, 5, 6) = K(1, 5, 5) ; K(1, 7, 7) = K(1, 5, AC
5) ; K(1, 5, 5) = K(1, 5, 5)
K(1, 1, 2) = -N1 + (N1/2) ; K(1, 1, 3) = (41/2) - N1 ; K(1, 1, 4) =
K(1, 1, 3) = (41/2) - N1 ; K(1, 1, 4) =
K(1, 1, 3) = (41/2) - N1 ; K(1, 1, 4) =
K(1, 1, 3) = (41/2) - N1 ; K(1, 1, 4) =
K(1, 1, 3) = (41/2) - N1 ; K(1, 1, 4) =
K(1, 1, 3) = (41/2) - N1 ; K(1, 1, 4) =
\( K(1,3,4) = (N3 + (N1/2)) \)
\( K(1,3,5) = N3 - N3 \)
\( K(1,3,6) = N3 - N3 \)
\( K(1,3,7) = (N3 + N3) \)
\( K(1,3,8) = (N3 + N3) \)
\( K(1,4,5) = (N3 + N3) \)
\( K(1,4,6) = -N3 + N3 \)
\( K(1,4,7) = N3 - N3 \)
\( K(1,4,8) = N3 + N3 \)
\( K(1,5,6) = (N2/2 - N2) \)
\( K(1,5,7) = -N2 + (N2/2); \)
\( K(1,5,8) = -N2 + (N2/2) \)
\( K(1,6,7) = -((N2 + N2)/2) \)
\( K(1,6,8) = -N2 + (N2/2) \)
\( K(1,7,8) = (N2/2 - N2) \)

**CYCLE**
\( 3 = 1, 1, 3 \)
\( 4 = 1, 1, 3 \)

**IF**
\( B = 1, \text{THEN} \rightarrow 1000 \)

**1000 :**

**AREPEAT**

**END**

**END**

**END**

**END**

**END**

**END**

**END**

**APIGRAM**
**BEGIN**

**INTEGER** NXY, N, NXYPR, M, TYPE, J, S, KBXY, CHANNEL, SECTION, PXY, Z, R1, R2, C1, C2, J,

**REAL** D, L, EXY, V, E, CNXY, GXY

CLAIM TAP (1) ; SECTION = 1

CHANNEL = 1 ; NEMLINES(4)

READ (J, J)

ADAPTATION JBSOCNUMBER; SPACES(4); PRINT (J, J, 0); NEMLINES(2)

READ (NXY, N, NXYPR, TYPE)

M = N - NXYPR ; KBXY = 2* NXYPR + 4 ; PXY = KBXY * 4 + 1

**BEGIN**

**REAL** EPSXY

**INTEGER** SECT1, SECT2, SECT3, SECT, ITER, NSYARKER

**INTEGER** ARRAY TYPEXY(1:NXY)

**REAL** ARRAY RXY(1:2*N), KXY(1:5, 1:3), PARXY(1:TYPE, 1:3)

**ROUTINE** ASPEC

SYN PJS DEF BAND (ARRAYNAME XXY, AREALNAME EPSXY, A1, A2, A3, A4, A5, A6, A7, A8)

INTEGER NAME SECT1, SECT2, SECT3, SECT, 4C

TEM; **INTEGER** CHAN, NXY, RXY, NXYARKER; 4C

READ FROM FILE (1, SECTION, PARXY(1, J), PARXY(TYPE, 5))

SECTION = SECTION + 1

READ FROM FILE (1, SECTION, XXY(J, 1) , XXY(J, 3))

SECTION = SECTION + 1

READ FROM FILE (1, SECTION, TYPEXY(1, J), TYPEXY(NXY))

SECTION = SECTION + 1

SECT1 = 4 ; SECT2 = 1 ; SECT3 = 1 ; SECT = 140

NSYARKER = 1 ; NEMLINES(3)
ACYCLE = 2, 1, 1-1

\( s = \theta \rightarrow 1 \)
\( P(1, 0) / P(1, 0) \)
\( \rightarrow 3 \)
\( t = 2 \)
\( s + 1 \)
\( \text{ACYCLE} = 2, 1, 1-1 \)
\( \theta = \theta - 1 \)
\( s \rightarrow P(1, s + 1) \)
\( \rightarrow \theta = \theta - 1 \)
\( \text{ACYCLE} = 0, 1, 2 \times s - 2 \)

IF \( P(1, 0) < 0 \) AND \( A + 1 < N \) THEN \( \rightarrow 100 \)
IF \( P(1, 0) > 0 \) THEN \( \rightarrow 20 \)

\( A = (L - 1) \times (L - 1) \)
\( \text{READ FROM FILE (CHAN, S1, P(1, 1), P(2 \times s - 2, 2 \times s))} \)
\( S1 = S1 + 1 \)
\( \text{POSITION TAPE (CHAN, S2)} \)
\( \text{ACYCLE} = \theta + 1, 2 \times s - 2 \)
\( \text{ACYCLE} = \theta + 1, 2 \times s - 2 \)
\( \text{ACYCLE} = \theta + 1, 2 \times s - 2 \)
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\( \text{ACYCLE} = \theta + 1, 2 \times s - 2 \)
\( \text{ACYCLE} = \theta + 1, 2 \times s - 2 \)
\( \text{ACYCLE} = \theta + 1, 2 \times s - 2 \)

4: WRITE TO FILE (CHAN, S2, P(1, 1), P(2 \times s - 2, s))
S2 = S2 + 1
\( \theta = \theta + 1 \)
\( \text{POSITION TAPE (CHAN, S1)} \)
\( \theta = \theta + 1 \)
\( \text{ACYCLE} = \theta + 1, 2 \times s - 2 \)
\( \text{ACYCLE} = \theta + 1, 2 \times s - 2 \)
\( \text{ACYCLE} = \theta + 1, 2 \times s - 2 \)
\( \text{ACYCLE} = \theta + 1, 2 \times s - 2 \)
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\( \text{ACYCLE} = \theta + 1, 2 \times s - 2 \)
\( \text{ACYCLE} = \theta + 1, 2 \times s - 2 \)
\( \text{ACYCLE} = \theta + 1, 2 \times s - 2 \)
4: WRITE TO FILE (CHAN, S2, P(1, 1), P(R-1, 2*R))
S2 = S2 + 1
J = 1
POSITION TAPE (CHAN, S1)
CYCLE 1 = 1, 1, R-1
4 = 1 + R - 1
CYCLE J = 1, 1, 2*R
P(J, J) = P(J, J)
REPEAT
REPEAT
12: WRITE TO FILE (CHAN, S2, P(R, 1), P(2*R-2, 2*R))
SECT 3 = S2 + 1
ADAPTATION ~ ADJUSTED MATRIX STARTS SECT 3; WRITE (SECT2, 1)
ADAPTATION CANOGENOUS SECT 3; WRITE (S2, 1)
CYCLE 1 = 1, 1, N
X(1) = X(1)
C1 = 0
REPEAT
COMMENT ITERATIVE PROCESS STARTS HERE
9: SECT = S2 + 1
READ FROM FILE (CHAN, S2, P(1, 1), P(R-1, 2*R))
S2 = S2 + 1
DSUT
CYCLE 1 = 2, 1, R-1
4 = 1 + R - 1
***X(I) : ***R14, ***I : ***G14, ***G14
**P(I, I+1) : ***R15, ZERJ : ***X(I)
**F : **LOAD
**S21
**RJONP : ***X(I)
REPEAT
CYCLE L = 1, 1, R-1
READ FROM FILE (CHAN, S2, P(R, 1), P(2*R-2, 2*R))
S2 = S2 + 1
4 = 1 + R - 1
CYCLE 1 = R-1, 2*R-2
0 = 1 + 1
-> 6 AIF 0 < N
***X(0-R+1) : ***R14, **R
***G14 : ***G14
**P(I, 1) : ***R15, ZERJ : ***X(0)
**F : **LOAD
**S21
**RJONP
***X(0)
REPEAT
ZENDJFMOUCDE
-> 7 AIF L = R-1
CYCLE 1 = 1, 1, R-1
4 = 1 + R - 1
CYCLE J = 1, 1, 2*R
P(J, J) = P(J, J)
REPEAT
7: REPEAT
S2 = S2 - 3*ECECKS
S = S2
POSITION TAPE (CHAN, S2)
ADAPTATION BACKSUBSTITUTION
A = A - 1
C = C + 1
X(A) = X(A) / P(A, R)
R = R
A = A
CYCLE 1 = R-1, 1+1
*REPEAT
*CYCLE L=1,1,H-1
- READ FROM FILE (CHAN,S2,P(R,1),P(2*R-2,2*R))
S2=S2+1
L=L-1,R=R-1
*CYCLE L=2,1,2*R-2
G=L+1
*REPEAT A[R] O>S
**X(O)=O*R+1 ; **RM14 : **R
*C14 : *C014
**P(I,1) : **RM15 ; **RJ1 ; **X(O)
*PIX : *FLJAD
*JS21
*RUNDF
**=X(O)
*REPEAT
*ENDDF=CLUD
->7 AIF L=H-1
*CYCLE 1=1,1,R-1
L=L+R-1
*CYCLE 1=1,1,2*R
P(I,J)=P(I,J)
*REPEAT
*REPEAT
7: *REPEAT
5: S2=S2-3*SBLCKS
S=S2
PPOSITION TAPE (CHAN,S2)
LOCATION BACKSUBSTITUTJON
A=N-(K-2) *(R-1)
C=A+R+1
X(N)=X(N)/P(A,N)
G=G-1
*CYCLE 1=A-1,-1,C+1
G=G+1
*CLUD
**A : **X1 ; **I : **- ; **=R14
*JS27
*ENDDF=CLUD
*REPEAT
*CYCLE 1=G,-1,1
B=B+1
CLUD
**R : **NEG ; **NJ1 : **=R14
*JS27
*ENDDF=CLUD
*REPEAT
*CYCLE L=H-2,-1,1
READ FROM FILE (CHAN,S2,P(I,1),P(R-1,2*R))
S2=S2-3*SBLCKS AUNLESS L=1
S=S2
PPOSITION TAPE (CHAN,S2) AUNLESS L=1
O=(L-1) *(K-1)
*CYCLE 1=R-1,-1,1
G=G+1
CLUD
**R : **NEG ; **NJ1 : **=R14
*JS27
*ENDDF=CLUD
*REPEAT
*CYCLE L=I+2,-1,1
- READ FROM FILE (CHAN,S2,P(I,1),P(R-2,2*R))
S2=S2+1
S2=S2+2
*CYCLE L=I+1,1
*K(I)=X(I)+X(I) ; ->3 AIF IT=0
S1=S1+1
S1=S1+2
*REPEAT
S1=S1
K(I)=X(I)+X(I)
**S= **S017 (I)
*ENDDF=CLUD
*REPEAT
->11 AIF IT=0
*S017/S019 = **SRT(I)
*ENDDF=CLUD
*ENDDF=CLUD
ACAPTION ~ SOLUTION~VECTORS: START~SECTO: WRITE (SECT3, 2)
WRITE TO FILE (CHAM, SECT 3(N), 3(N))
ACAPTION ~ AND~END~SECTO: WRITE (SECT, 2)
SECT=SECT+1

ACAPTION ~
ACODE:
24: *ZERJ : *X(1,2,3) : *IX : *EMJDTD
*ZJ : *ZERJ
26: *O4150 : *O4140 : *OUXY : *X+F
*J2G14VZS : *UF : *X4XY
*X(3) : *EXIT
21: *ZERJ : *ZERJ
*J: 04150 : *O4140 : *OUXY : *X+F
*J2G14VZS : *UF : *X4XY
*ZERJ : *ZERJ:
**P(X,1,R) : *X : *X=E(X,3)
*EXIT
ACAPTION ~
ACODE:
ACAPTION ~ MATRIX~OUT SYM. OR, DEF. ~ DIAGONAL~TERM. OR, JAC.
ACAPTION ~
PRINT FL (P1, 1, 3)

ACAPTION ~ SYN. POS. OR, DEF. OR, AND~DATA~FAULT~OR; WRITE (R, 2) ; ASTOP
ACAPTION ~ SYN. POS. OR, DEF. OR, AND~DATA~FAULT~OR<2~X~1 ; ASTOP

ACAPTION ~
ACYCLE ~ I = 0, 2, 2N
PRINT FL (XY(1), 2)
-> 111 if FRAC PT(1/NXY) = 0
SPACES(2) : -> 222
111 : NEWLINE
222 : *X=PEAT

ACAPTION ~
ACYCLE ~ I = 1, 2, 2N-1
PRINT FL (XY(1), 2)
-> 101 if FRAC PT(1/NXY) = 0
SPACES(2) : -> 202
101 : NEWLINE
202 : *X=PEAT

ACAPTION ~
ACYCLE ~ I = 2, 2, 2N
PRINT FL (XY(1), 2)
-> 111 if FRAC PT(1/NXY) = 0
SPACES(2) : -> 222
111 : NEWLINE
222 : *X=PEAT

ACAPTION ~
BEGIN

ARRAY AXY(1:4, 1:3, 1:3)

FUNCTION SPEC STRESSES (INTEGER ANAME)

FUNCTION SPEC AVERAGE (INTEGER S, Z)

CYCLE I = 1, 1, 3

CYCLE J = 1, 1, N

STXY(I, J) = 0

REPEAT

REPEAT

CYCLE S = 1, 1, NXY

Z = TYPEXY(S) ; D = PARXY(Z, 2) ; L = PARXY(Z, 3) ; EXY = 46

PARXY(Z, 4) ; V = PARXY(Z, 5)

CYCLE I = 1, 1, 3

CYCLE J = 1, 1, 3

XY(I, J) = 0

REPEAT

REPEAT

XY(I, 1) = -(1/D) ; XY(I, 2) = (1/D)

XY(2, 1) = -(1/L) ; XY(2, 4) = (1/L)

XY(3, 1) = (1/2(L+L)) ; XY(3, 2) = -(1/(D+L)) ; XY(3, 3) = XYAC

XY(4, 1) = 1

CYCLE I = 1, 1, 4

CYCLE J = 1, 1, 4

XY(I, J) = XY(I, J+4)

REPEAT

REPEAT

M = 2*INT PI((S-2)/Z(NXYPR+4))-1 ; R2 = 2S+M1 ; G2 = R2+4L

2NXYPR

RR(3) = RR(2) ; RR(4) = RR(R2+1) ; RR(7) = RR(R2+2) ; 4L

RR(5) = RR(2+3) ; RR(1) = RR(G2+1) ; RR(6) = RR(2+2) ; 4L

RR(2) = RR(G2+3)
EC) Y(<f) ; •

CycIe i = 1,1,3
AXY(I) = 0
AXY(J) = AXY(I) + (AXY(I,J)) * AXY(J)
REPEAT

E = (AXY/(1-(v:2))) ; GXY = XXY/(2(1+v))
C1 = S + INT PT(K(S-0,2)/(AXYPR-1))
STXY(1,C1) = STXY(1,C1) + E*(AXY(1)+L+AXY(1)+V*AXY(1))
STXY(2,C1) = STXY(2,C1) + E*(AXY(1)*AXY(1)+L+AXY(1))
STXY(3,C1) = STXY(3,C1) + GXY*(AXY(2)+AXY(3)+L+AXY(7))
C1 = C1 + 1
STXY(1,C1) = STXY(1,C1) + E*(AXY(1)+L+AXY(3)+V*D*AXY(7)+V*AXY(6))
STXY(2,C1) = STXY(2,C1) + E*(AXY(5)+V*D*AXY(7)+V*L*AXY(3)+V*AXY(1))
STXY(3,C1) = STXY(3,C1) + GXY*(AXY(2)+AXY(3)+AXY(5)+L*AXY(7))
C1 = C1 + N*NYPR
STXY(1,C1) = STXY(1,C1) + E*(AXY(1)+L+AXY(6)+V*D*AXY(7))
STXY(2,C1) = STXY(2,C1) + E*(AXY(6)+V*D*AXY(7))
STXY(3,C1) = STXY(3,C1) + GXY*(AXY(2)+AXY(5))
C1 = C1 - 1
STXY(1,C1) = STXY(1,C1) + E*(AXY(1)+V*AXY(6))
STXY(2,C1) = STXY(2,C1) + E*(AXY(6)+V*AXY(1))
STXY(3,C1) = STXY(3,C1) + GXY*(AXY(2)+AXY(5))
REPEAT
C1 = N+1-N*NYPR
CYCLE i = 1,1,N
IF i = 1 ADR i = NYPR ADR i = N ADR i = C1
END
IF NYPR > i > 1 ADR n > i > C1 ATENC
END
IF FRAC PTK/(1/NYPR) = 0 ADR FRAC PTK/(1/NYPR) = 0 ATAC
END
END
AVERAGE (1,4) ; -> 1
2: AVERAGE (1,2) ; -> 1
1: REPEAT
CYCLE i = 1,1,N
ENDING (1)
YOUR EXPLANATION HERE...
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