THRESHOLD IMPROVEMENT IN F.M. SYSTEMS BY THE USE OF NEGATIVE FREQUENCY-FEEDBACK

by

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Thesis presented for the Degree of Doctor of Philosophy of the University of Edinburgh in the Faculty of Science

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# CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>1</td>
</tr>
<tr>
<td>GLOSSARY OF PRINCIPAL SYMBOLS</td>
<td>2</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>6</td>
</tr>
<tr>
<td>CHAPTER 1. BASIC RELATIONSHIPS IN FM</td>
<td>8</td>
</tr>
<tr>
<td>1.1 Introduction</td>
<td>8</td>
</tr>
<tr>
<td>1.2 FM Wave. Spectrum and Bandwidth</td>
<td>8</td>
</tr>
<tr>
<td>1.3 The Conventional FM Receiver Above Threshold. Signal to Noise Ratio</td>
<td>17</td>
</tr>
<tr>
<td>1.4 Threshold Phenomenon. Preliminary Discussion</td>
<td>27</td>
</tr>
<tr>
<td>1.5 FM Feedback (FMFB) Receiver</td>
<td>30</td>
</tr>
<tr>
<td>1.6 Previous Work on FMFB Receivers</td>
<td>33</td>
</tr>
<tr>
<td>CHAPTER 2. MATHEMATICAL REPRESENTATION OF NOISE</td>
<td>37</td>
</tr>
<tr>
<td>2.1 Introduction</td>
<td>37</td>
</tr>
<tr>
<td>2.2 Stationary and Ergodic Processes</td>
<td>37</td>
</tr>
<tr>
<td>2.3 Random Gaussian Noise</td>
<td>38</td>
</tr>
<tr>
<td>2.4 FM Carrier Plus Random Gaussian Noise</td>
<td>53</td>
</tr>
<tr>
<td>2.5 Band-limited Noise. Noise Bandwidth</td>
<td>65</td>
</tr>
<tr>
<td>CHAPTER 3. FM NOISE CHARACTERISTICS</td>
<td>73</td>
</tr>
<tr>
<td>3.1 Introduction</td>
<td>73</td>
</tr>
<tr>
<td>3.2 Two-term Approximation of FM Noise Spectrum</td>
<td>74</td>
</tr>
<tr>
<td>3.3 Difficulties Associated with More Complex Analyses</td>
<td>87</td>
</tr>
<tr>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>------</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>The Effects of Modulation</td>
</tr>
<tr>
<td>3.5</td>
<td>Signal Suppression Effect</td>
</tr>
<tr>
<td>3.6</td>
<td>Output Signal to Noise Ratio</td>
</tr>
<tr>
<td>3.7</td>
<td>Conditions at the Threshold</td>
</tr>
<tr>
<td>4.1</td>
<td>Introduction</td>
</tr>
<tr>
<td>4.2</td>
<td>FM Analogue</td>
</tr>
<tr>
<td>4.3</td>
<td>Effect of Feedback on FM Noise</td>
</tr>
<tr>
<td>4.4</td>
<td>Phase Characteristics and System Stability</td>
</tr>
<tr>
<td>4.5</td>
<td>Threshold Improvement Using Feedback</td>
</tr>
<tr>
<td>4.6</td>
<td>Design Criteria and Design Example</td>
</tr>
<tr>
<td>4.7</td>
<td>Conclusions</td>
</tr>
<tr>
<td>A.1</td>
<td>Introduction</td>
</tr>
<tr>
<td>A.2</td>
<td>Noise Generator</td>
</tr>
<tr>
<td>A.3</td>
<td>Summing Amplifier</td>
</tr>
<tr>
<td>A.4</td>
<td>Mixer</td>
</tr>
<tr>
<td>A.5</td>
<td>IF Channel</td>
</tr>
<tr>
<td>A.6</td>
<td>Ratio Detector</td>
</tr>
<tr>
<td>A.7</td>
<td>Buffer Amplifier and Feedback Control</td>
</tr>
<tr>
<td>A.8</td>
<td>Voltage - Controlled Oscillator (VCO)</td>
</tr>
<tr>
<td>A.9</td>
<td>Gain and Phase Measurements</td>
</tr>
<tr>
<td>A.10</td>
<td>Noise Measurements</td>
</tr>
<tr>
<td>B.1</td>
<td>Single-pole Filter</td>
</tr>
<tr>
<td>B.2</td>
<td>Two-pole Filter</td>
</tr>
<tr>
<td>B.3</td>
<td>Gaussian Filter</td>
</tr>
<tr>
<td>B.4</td>
<td>Rectangular Filter</td>
</tr>
</tbody>
</table>
## CONTENTS

<table>
<thead>
<tr>
<th>APPENDIX C.</th>
<th>RESULTS OF FOURIER AND LAPLACE TRANSFORMATIONS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>C.1</td>
<td>Defining Relationships</td>
<td>222</td>
</tr>
<tr>
<td>C.2</td>
<td>Amplitude and Power Spectrum of Signal $f(t)$</td>
<td>223</td>
</tr>
<tr>
<td>C.3</td>
<td>Frequency-Domain Shift Theorem</td>
<td>224</td>
</tr>
<tr>
<td>C.4</td>
<td>Transfer Function of a Device</td>
<td>224</td>
</tr>
<tr>
<td>C.5</td>
<td>Distortionless Transmission of a Signal</td>
<td>225</td>
</tr>
<tr>
<td></td>
<td>Time-Domain Shift Theorem</td>
<td>226</td>
</tr>
<tr>
<td>APPENDIX D.</td>
<td>CHANGE OF VARIABLE</td>
<td>227</td>
</tr>
<tr>
<td>APPENDIX E.</td>
<td>REDUCTION OF VARIOUS RESULTS TO A COMMON FORM</td>
<td>229</td>
</tr>
<tr>
<td>APPENDIX F.</td>
<td>STUMPERS' NO-MODULATION FORMULA</td>
<td>230</td>
</tr>
<tr>
<td>APPENDIX G.</td>
<td>EVALUATION OF FUNCTIONS IN SEC. 3.4</td>
<td>235</td>
</tr>
<tr>
<td>G.1</td>
<td>Evaluation of Summation in Eqn. (3.30)</td>
<td>235</td>
</tr>
<tr>
<td>G.2</td>
<td>Remarks on the Evaluation of $n/E_n$</td>
<td>236</td>
</tr>
<tr>
<td>APPENDIX H.</td>
<td>ON THE SIGNAL SUPPRESSION EFFECT</td>
<td>239</td>
</tr>
<tr>
<td>H.1</td>
<td>The Differentiation in Eqn. (3.44a)</td>
<td>239</td>
</tr>
<tr>
<td>H.2</td>
<td>Jacobian of the Transformation in Eqns. (3.49) and (3.50)</td>
<td>240</td>
</tr>
<tr>
<td>H.3</td>
<td>The Integration in Eqn. (3.53)</td>
<td>241</td>
</tr>
<tr>
<td>APPENDIX I.</td>
<td>EVALUATION OF SOME MATHEMATICAL FUNCTIONS</td>
<td>243</td>
</tr>
<tr>
<td>I.1</td>
<td>The Error Function, $\text{erf} x$</td>
<td>243</td>
</tr>
<tr>
<td>I.2</td>
<td>The Bessel Function of the First Kind and Order $n$, $J_n(x)$</td>
<td>243</td>
</tr>
<tr>
<td>I.3</td>
<td>The Bessel Function of the First Kind and Imaginary Argument, $I_n(x)$</td>
<td>244</td>
</tr>
<tr>
<td>CONTENTS</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>1.4 The Confluent Hypergeometric Function.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F_{1}(a,b,x)$</td>
<td>244</td>
<td></td>
</tr>
<tr>
<td>REFERENCES</td>
<td>246</td>
<td></td>
</tr>
</tbody>
</table>
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### GLOSSARY OF PRINCIPAL SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>G(j\omega)</td>
</tr>
<tr>
<td>$\angle G(j\omega)$</td>
<td>angle of $G(j\omega)$</td>
</tr>
<tr>
<td>$n(t)$</td>
<td>time average or ensemble average of $n(t)$</td>
</tr>
<tr>
<td>$x(t)$</td>
<td>derivative with respect to time of $x(t)$</td>
</tr>
<tr>
<td>$\approx$</td>
<td>approximate equality</td>
</tr>
<tr>
<td>$A$</td>
<td>amplitude, Fourier coefficient</td>
</tr>
<tr>
<td>$AM$</td>
<td>amplitude modulation, amplitude modulated</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>phase angle, fraction of critical coupling</td>
</tr>
<tr>
<td>$\alpha(t)$</td>
<td>phase function</td>
</tr>
<tr>
<td>$B$</td>
<td>bandwidth, Fourier coefficient</td>
</tr>
<tr>
<td>$C$</td>
<td>capacitance, Fourier coefficient</td>
</tr>
<tr>
<td>$(\text{CNR})$</td>
<td>carrier to noise power ratio (measured at the intermediate frequency)</td>
</tr>
<tr>
<td>$\text{cerf } x$</td>
<td>complementary error function</td>
</tr>
<tr>
<td>$\Delta f, \Delta \omega$</td>
<td>frequency deviation</td>
</tr>
<tr>
<td>$\delta(x)$</td>
<td>Dirac delta function</td>
</tr>
<tr>
<td>$\text{erf } x$</td>
<td>error function</td>
</tr>
<tr>
<td>$F$</td>
<td>Farad</td>
</tr>
<tr>
<td>$FM$</td>
<td>frequency modulation, frequency modulated</td>
</tr>
<tr>
<td>$FMFB$</td>
<td>frequency modulation feedback</td>
</tr>
</tbody>
</table>
GLOSSARY OF PRINCIPAL SYMBOLS

\( _1F_1(a,b,x) \) = confluent hypergeometric function

\( f \) = frequency

\( \mathcal{F}\{f(t)\} = \mathcal{F}(j\omega) \) = Fourier transform of \( f(t) \)

\( \mathcal{F}^{-1}\{F(j\omega)\} = f(t) \) = inverse Fourier transform of \( F(j\omega) \)

\( G(s) \) = transfer function

\( \Gamma(z) \) = gamma function

\( \gamma, \gamma(t) \) = phase, phase function

\( H \) = Henry

\( IF \) = intermediate frequency

\( I_n(x) \) = Bessel function of the first kind and imaginary argument

\( J \) = Jacobian

\( J_n(x) \) = Bessel function of the first kind, nth order

\( j \) = \((-1)^{\frac{1}{2}}\)

\( K \) = voltage gain

\( L \) = inductance

\( L(s) \) = loop-function

\( \mathcal{L}\{f(t)\} = \mathcal{L}(s) \) = Laplace transform of \( f(t) \)

\( \mathcal{L}^{-1}\{F(s)\} = f(t) \) = inverse Laplace transform of \( F(s) \)

\( \lambda, \lambda(t) \) = phase, phase function

\( M \) = modulation index, mutual inductance

\( N \) = mean noise power

\( n \) = number of impulses in output

\( n(t) \) = noise voltage
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_1$</td>
<td>mean value of $n(t)$</td>
</tr>
<tr>
<td>$n_2$</td>
<td>mean square value of $n(t)$</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>ohms</td>
</tr>
<tr>
<td>$\omega$</td>
<td>frequency</td>
</tr>
<tr>
<td>$P$</td>
<td>average power of sinusoid</td>
</tr>
<tr>
<td>$p(x)$</td>
<td>probability distribution function of variable $x$</td>
</tr>
<tr>
<td>$\phi, \phi(t)$</td>
<td>phase, phase function</td>
</tr>
<tr>
<td>$\Phi(s)$</td>
<td>Laplace transform of $\phi(t)$</td>
</tr>
<tr>
<td>$Q$</td>
<td>quality factor</td>
</tr>
<tr>
<td>$R$</td>
<td>resistance</td>
</tr>
<tr>
<td>$RF$</td>
<td>radio frequency</td>
</tr>
<tr>
<td>$R(t)$</td>
<td>amplitude function of resultant wave</td>
</tr>
<tr>
<td>$r$</td>
<td>resistance, radius of gyration</td>
</tr>
<tr>
<td>$r(t)$</td>
<td>amplitude function of noise</td>
</tr>
<tr>
<td>$\rho(t)$</td>
<td>modulation phase function</td>
</tr>
<tr>
<td>$S$</td>
<td>signal power</td>
</tr>
<tr>
<td>$SNR$</td>
<td>signal to noise power ratio (measured at the output)</td>
</tr>
<tr>
<td>$s(t)$</td>
<td>signal voltage</td>
</tr>
<tr>
<td>$T$</td>
<td>time interval, time constant</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$\theta, \theta(t)$</td>
<td>phase, phase function</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>unit step function</td>
</tr>
</tbody>
</table>
GLOSSARY OF PRINCIPAL SYMBOLS

V = volts
VCO = voltage-controlled oscillator
V(s) = Laplace transform of v(t)
v(t) = voltage function
W = level of power density function
W(f) = power density function (power spectrum)
X(t) = voltage in phase with unmodulated carrier
X(s) = Laplace transform of x(t)
x(t) = voltage in phase with carrier, function related to intermediate-frequency wave
Y = admittance
Y(t) = voltage in quadrature with unmodulated carrier
Y(s) = Laplace transform of y(t)
y(t) = voltage in quadrature with carrier, function related to intermediate-frequency wave
Z = impedance
INTRODUCTION

In the establishing of communication links via satellites, the choice of a suitable transmission system is an important one. Because of the large distances involved, the path loss (reduction in carrier level between transmitter and receiver) is very high and economy of the received carrier power therefore becomes of primary importance. This is especially true for high bandwidth systems such as television and multi-channel telephony where the large bandwidths involved increase the amount of noise the system has to contend with. Large-index frequency modulation proves very suitable for such applications but exhibits a characteristic threshold phenomenon which impairs its usefulness. Consequently the problem of improving the threshold performance of FM systems assumes considerable importance and has received a great deal of attention from communication engineers in recent years.

FM receivers using negative frequency-feedback have been shown by several authors to give threshold improvement but, for the main part, rule of thumb designs have been used, and there seemed a need for a clear insight into the working of such demodulators. It is towards satisfying such a need that the work described herein was undertaken.

The approach to the problem was to develop a noise theory of the FM receiver which would embrace, unify, and
in some cases extend, the work of previous authors. This theory was then applied to the FM feedback (FMFB) receiver, and as far as time and resources permitted, the theory was verified. In the development of the noise theory, a general approach was adopted and the results were then applied to the specific case. Some of the more general results have been included because of their wider application.

Throughout the investigations, a qualitative description of the system's operation was sought and it was this endeavour which, in the end, proved most rewarding.
CHAPTER 1

BASIC RELATIONSHIPS IN FM

1.1 Introduction. In this chapter the basic relationships in FM are set out, primarily to provide a quick and easy reference, but also to provide the basic tools with which to carry out more complex analysis. We are also able to develop the analysis in such a way as to set the general method of approach and to obtain a qualitative description of the system. Here too the terminology and nomenclature are defined. Unless of particular significance, no distinction will be made throughout between angular frequency \( \omega \) (in radians per second) and cyclic frequency \( f \) (in cycles per second or Hertz), the relationship

\[
\omega = 2\pi f
\]

being assumed always.

1.2 FM Wave. Spectrum and Bandwidth. In FM the instantaneous frequency \( \omega_i(t) \) of a radio frequency (RF) carrier wave is made to vary in accordance with the information-bearing signal \( v_S(t) \). \( v_S(t) \) also goes under the various names of 'baseband signal', 'modulating signal', 'message', 'information', or simply 'signal'. If the frequency of the unmodulated RF carrier is \( \omega_R \), the instantaneous frequency
of the FM carrier is

\[ \omega_i(t) = \omega_R + v_s(t) \] (1.2)

The instantaneous phase \( \theta_i(t) \) of the FM carrier is the integral with respect to time of the instantaneous frequency. Thus

\[ \theta_i(t) = \int_{-\infty}^{t} \omega_i(t) \, dt = \omega_R t + \theta_0 + \int_{0}^{t} v_s(t) \, dt \] (1.3)

where \( \theta_0 \) is a constant of the integration and is the total phase angle at \( t = 0 \). If \( t = 0 \) is so chosen that \( \theta_0 \) is zero, the RF FM carrier wave \( v_R(t) \) is then given by

\[ v_R(t) = A_R \cos \theta_i(t) = A_R \cos \left( \omega_R t + \int_{0}^{t} v_s(t) \, dt \right) \] (1.4)

To obtain the bandwidth occupied by this signal, we need its frequency spectrum, which is obtained by Fourier analysis. Unfortunately, there is no simple general theorem that relates the Fourier transform of \( v_s(t) \) to the Fourier transform of \( v_R(t) \) in eqn. (1.4), but useful information may be obtained by considering the case where \( v_s(t) \) is a single sinusoidal tone. This will yield the relationships for what is usually called 'single-tone FM', but it must be borne in mind that this is a special and much simplified case. In general, the information signal \( v_s(t) \) may be resolved into the sum of several tones; but FM is a non-linear process, and it is therefore not possible to apply superposition. Even for double-tone FM, the analysis becomes quite cumbersome.
Returning then to single-tone FM, \( v_s(t) \) is given by

\[
v_s(t) = A_s \cos \omega_s t = A_s \cos 2\pi f_s t
\]

where \( A_s \), the amplitude of the modulating tone, is assigned the symbol \( \Delta \omega = 2\pi \Delta f \). The instantaneous radian frequency of the FM wave is obtained from eqn. (1.2), as

\[
\omega_i(t) = \omega_R + \Delta \omega \cos \omega_s t
\]

and the instantaneous cyclic frequency \( f_i(t) \) obtained from

\[
2f_i(t) = 2f_R + 2\Delta f \cos \omega_s t
\]

\( \Delta f \) is called the frequency deviation and represents the maximum excursion the instantaneous frequency makes away from \( f_R \) as the former varies sinusoidally about \( f_R \). From eqns. (1.5) and (1.4) the FM wave now becomes

\[
v_{R}(t) = A_R \cos \left( \omega_R t + \int_0^t \Delta \omega \cos \omega_s t \, dt \right)
\]

where

\[
\begin{align*}
A_R & = A_R \cos \left( \omega_R t + \frac{\Delta \omega}{\omega_s} \sin \omega_s t \right) \\
& = A_R \cos \left( \omega_R t + \frac{\Delta f}{f_s} \sin \omega_s t \right) \\
& = A_R \cos \left( \omega_R t + M \sin \omega_s t \right)
\end{align*}
\]
Sec. 1.2. BASIC RELATIONSHIPS IN FM

\[
M = \frac{\Delta \omega}{\omega_s} = \frac{\Delta f}{f_s} \tag{1.10}
\]

The modulation index \( M \), the ratio of the frequency deviation to the modulating frequency, is called the modulation index and is a very important parameter in FM analysis.

The average power associated with the FM wave given in eqn. (1.9d) is seen to be independent of the modulation, since this is contained in the frequency of the wave. Thus both modulated and unmodulated waves have the same average power \( P_R \) given by

\[
P_R = A_R^2/2 \tag{1.11}
\]

This may be established rigorously (see for example ref. 22 chap. 3) from

\[
P_R = \frac{1}{T} \int_0^T v_R^2(t) \, dt \tag{1.12}
\]

where \( T = 1/f_s \) is the time period of \( v_s(t) \), the modulating tone.

To find the frequency spectrum of the single-tone FM wave in eqn. (1.9d), it is more convenient to use the complex number representation.

\[
v_R(t) = A_R e^{j(\omega_R t + M \sin \omega_s t)} \tag{1.13a}
\]

\[
= x(t) A_R e^{j\omega_R t} \tag{1.13b}
\]

\[
x(t) = e^{jM \sin \omega_s t} \tag{1.14}
\]
From the result in appendix C.3, we expect the frequency spectrum of \( v_R(t) \) in eqn. (1.13b) to consist of the frequency spectrum of \( x(t) \) shifted upwards in frequency by amount \( \omega_R \). We therefore direct our attention to \( x(t) \) which is seen to be periodic with frequency \( \omega_s \); so we expect \( x(t) \) to have a discrete (line) spectrum rather than a continuous one.

Using the complex Fourier series representation therefore gives

\[
x(t) = e^{jM \sin \omega_s t} = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_s t} \quad (1.15)
\]

\[
C_n = \omega_s (2\pi)^{-1} \int_{-\pi/\omega_s}^{+\pi/\omega_s} e^{jM \sin \omega_s t} - jn\omega_s t \ dt \quad (1.16a)
\]

\[
= (2\pi)^{-1} \int_{-\pi}^{+\pi} e^{jM \sin x - nx} \ dx \quad , \ x = \omega_s t \quad (1.16b)
\]

\[
= J_n(M) \quad (1.16c)
\]

Eqn. (1.16b) is a standard integral (ref. 29, sec. 8.41) and in fact defines the Bessel function \( J_n(M) \) of the first kind, order \( n \) and argument \( M \). The relationship that

\[
J_{-n}(M) = (-1)^n J_n(M) \quad (1.17)
\]

enables us to express \( x(t) \) as
Sec. 1.2. BASIC RELATIONSHIPS IN FM

\[ x(t) = J_0(M) \]
\[ + J_1(M) \left[ e^{j\omega t} - e^{-j\omega t} \right] \]
\[ + J_2(M) \left[ e^{j2\omega t} - e^{-j2\omega t} \right] \]
\[ + \ldots \]
\[ + J_n(M) \left[ e^{jn\omega t} - e^{-jn\omega t} \right] \]
\[ + \ldots \] (1.18)

\( x(t) \) therefore comprises the set of cosines which have amplitudes \( J_n(M) \) and which are placed multiples of \( \omega_s \) on either side of zero frequency. The FM wave \( v_R(t) \) in eqns. (1.9d) and (1.13a) is obtained by shifting \( x(t) \) upward in frequency through \( \omega_{R} \), the frequency of the unmodulated carrier.

\[ v_R(t) = A_R J_0(M) e^{j\omega_{R} t} \]
\[ + A_R J_1(M) \left[ e^{j(\omega_{R} + \omega_s) t} - e^{j(\omega_{R} - \omega_s) t} \right] \]
\[ + A_R J_2(M) \left[ e^{j(\omega_{R} + 2\omega_s) t} + e^{j(\omega_{R} - 2\omega_s) t} \right] \]
\[ + \ldots \]
\[ + A_R J_n(M) \left[ e^{j(\omega_{R} + n\omega_s) t} + (-1)^n e^{j(\omega_{R} - n\omega_s) t} \right] \] (1.19)
The spectrum of the FM wave is in general seen to consist of a carrier term and an infinite number of side frequencies spaced $\omega_s$ apart on either side of the carrier, the $n$th side frequency having amplitude proportional to $J_n(M)$. In theory, then, an infinite bandwidth would be needed for the 100% distortionless transmission of an FM wave, but in practice, the amplitudes of the higher order side frequencies become negligibly small, and the FM wave approximates to a band-limited signal. A study of Bessel functions reveals that for a given argument $M$, $J_n(M)$ forms a rapidly decreasing sequence for $n = M, M + 1, M + 2$, etc. One method of obtaining the bandwidth necessary for the transmission of an FM wave is to determine the amplitudes of the side frequencies as far as what is taken to be, in accordance with some criterion, the first insignificant one. If $J_{M+2}(M)$ is taken as the first insignificant amplitude, then there are $M + 1$ significant ones, and since these are distributed at intervals of $\omega_s$ on either side of the carrier, the bandwidth $B$ (in Hz) required is

$$B = 2f_s(1 + M) \quad (1.20)$$

this equation being often referred to as 'Carson's rule'.

'Narrowband' FM arises when $M$, the modulation index, is sufficiently small, and $0 \leq M \leq \frac{1}{2}$ is usually taken as the range which satisfies this requirement. Over this range of $M$, $J_0(M)$ and $J_1(M)$ are found to be the only two significant amplitudes and so from eqn. (1.19)
v_R(t) = A_R^J_o(M)e^{j\omega_R t} + A_R^J_1(M) \left[ e^{j(\omega_R + \omega_s) t} - e^{j(\omega_R - \omega_s) t} \right] \quad (1.21)

This shows that only the carrier frequency and two side frequencies are produced, the side frequencies being \( \omega_s \) away from the carrier with one on either side of it. The result of eqn. (1.21) may be obtained without Fourier analysis by expanding \( x(t) \) in eqn. (1.14) into the exponential series. Thus

\[
x(t) = e^{jM \sin \omega_s t} = \sum_{n=0}^{\infty} \frac{[jM \sin \omega_s t]^n}{n!} \quad (1.22)
\]

\( M \sin \omega_s t \) is a sinusoid with amplitude \( M \) so that

\[
o \leq |M \sin \omega_s t| \leq \frac{1}{2} \quad , \quad 0 \leq M \leq \frac{1}{2}
\]

and \( x(t) \) in eqn. (1.22) may be approximated by the first two terms in the series expansion.

\[
x(t) \approx 1 + jM \sin \omega_s t \quad , \quad 0 \leq M \leq \frac{1}{2}
\]

Using the identity

\[
\sin b = (e^{jb} - e^{-jb})/(2j)
\]

leads to

\[
x(t) \approx 1 + \frac{M}{2} \left[ e^{j\omega_s t} - e^{-j\omega_s t} \right]
\]

(1.26)
Sec. 1.2. BASIC RELATIONSHIPS IN FM

and \( v_R(t) \) in eqn. (1.13b) becomes

\[
v_R(t) = A_R e^{j\omega_R t} + A_R^M \left[ e^{j(\omega_R + \omega_s) t} - e^{j(\omega_R - \omega_s) t} \right]
\]  

(1.27)

but since

\[
J_0(M) = 1; \quad J_1(M) = M/2, \quad 0 \leq M \leq \frac{1}{2}
\]  

(1.28)

eqns. (1.27) and (1.21) are seen to be the same.

In narrowband FM, then, each modulating tone in the baseband signal produces a single tone \( \omega_s \) above the carrier and a single tone \( -\omega_s \) below the carrier. A baseband signal having several frequency components within the band extending from \( \omega_a \) to \( \omega_b \) say, will therefore produce two sidebands: an upper one extending from \( \omega_R + \omega_a \) to \( \omega_R + \omega_b \) and a lower one extending from \( \omega_R - \omega_a \) to \( \omega_R - \omega_b \). Narrowband FM is thus very similar to amplitude modulation (AM), both requiring a bandwidth of twice the highest signal frequency for double sideband transmission. FM systems are, however, inherently more physically complex than AM systems, and narrowband FM is seldom, if ever, used.

For values of the modulation index \( M \) in the range \( \frac{1}{2} < M < \infty \), the number of significant side frequencies arising from a single baseband tone increases as \( M \) increases. This produces wideband FM which is quite widely used for reasons we shall discuss later. The bandwidth required for single-tone FM may be obtained from Carson's rule in eqn. (1.20) and, since \( M \) is the ratio of the frequency deviation \( \Delta f \) to the modulating frequency \( f_s \), eqn. (1.20) may be
B = 2f_0 (1 + \Delta f/f_s) = 2f_s + 2\Delta f \quad (1.29)

If it is assumed that in a multitone FM wave all the tones have equal frequency deviation \( \Delta f \), then from eqn. (1.29), it is the highest modulating frequency which dictates the bandwidth requirement. If \( M \) is large so that \( \Delta f >> f_s \), the bandwidth will be seen from eqn. (1.29) to be approximately twice the frequency deviation \( \Delta f \).

1.3 The Conventional FM Receiver Above Threshold. Signal To Noise Ratio. Having discussed some of the basic relationships in FM, it is now possible to assess the performance of the FM receiver operating 'above threshold'. To understand what is meant by 'above threshold', reference is made to fig. 1.1 which gives a schematic of the FM receiver.

The RF FM carrier wave \( v_R(t) \) enters the receiver along with wideband gaussian noise having power spectrum \( W_R(f) \). Wideband describes the power spectrum \( W_R(f) \) which is assumed to have constant level \( W_R \) watts/Hz over a large range of frequencies centered on \( f_R \). Gaussian describes the amplitude probability distribution of the noise, and this is dealt with in chap. 2. It is in fact irrelevant to the present discussion and need not be considered further at this stage.

The RF carrier and noise entering the receiver may or may not be passed through an RF amplifier, which raises the carrier amplitude \( A_R \) to a satisfactory level. The gain of the RF amplifier, when present, may
conveniently be included in $K_M$, the gain of the mixer. The mixer acts as a frequency reducer, producing at its output the intermediate frequency (IF) wave whose frequency $f_o$ is the difference between the frequency $f_R$ of the RF wave and the frequency $f_L$ of the local oscillator wave as shown in fig. 1.1. The IF wave is further amplified in the IF amplifier (assumed broadband) and is then passed through an IF filter with frequency response $G(j\omega)$ such that

$$|G(j\omega)| = 1, \quad f_o - \frac{B}{2} \leq \omega \leq f_o + \frac{B}{2} \quad (1.30a)$$

$$= 0 \quad \text{elsewhere} \quad (1.30b)$$

The bandwidth $B$ of this filter is that which is required for the transmission of the FM wave. Consequently the wave will be unaffected by its passage through the filter. The IF wave $v_o(t)$ at the output of the IF filter is given in fig. 1.1.

At the output of the mixer, the input noise will be raised in power level by the factor $K_M^2$ and shifted down in frequency by amount $f_L$ so that any imagined band of frequencies centered on $f_R$ will now be centered on $f_o$, the intermediate frequency. In the IF amplifier, the noise power level will be further raised by the factor $K_{IF}^2$, and, at the output of the IF filter, the noise will be 'band-limited' or 'narrowband'; that is, the noise spectrum will only have components in some finite bandwidth, $B$ in this case. The power spectrum $W_o(f)$ of the noise at the output of the IF filter is shown in fig. 1.1.

So far no consideration has been given to system noise,
i.e., noise which arises within the system itself. In modern communication systems, the use of masers and other low noise equipment makes it possible to reduce system noise to a level which is small compared to the noise which enters the system externally. In any event, at the high frequencies we are concerned with, system noise will either be thermal or shot noise and is therefore of the same type (wideband, gaussian) as the input electromagnetic noise. System noise would tend therefore to alter only the IF noise level and not its nature.

The mean noise power $N_o$ at the output of the IF filter is obtained by integration of $W_o(f)$, thus

$$N_o = \int_{f_o - B/2}^{f_o + B/2} W_o(f)df = W_o B$$

(1.31)

where $W_o$ is the constant level of $W_o(f)$. The mean carrier power $P_o$ at the output of the IF filter is $A_o^2/2$ [see eqn. (1.11)] where $A_o$ is the amplitude of the FM wave $v_o(t)$ at the IF filter output. The ratio (CNR) of mean carrier power to mean noise power is then given by

$$(CNR) = \frac{A_o^2}{2W_o B}$$

(1.32)

To investigate the interaction between the FM wave and the noise, we assume firstly that the noise may be represented by a large number of sinusoids distributed in the frequency range $f_o - B/2 \leq f \leq f_o + B/2$. Since the power spectrum of the noise is flat, we would expect each of the sinusoids to have the same mean power and therefore the same amplitude. Consider such a sinusoid at frequency $f_o + f_n$. The mean
noise power associated with this frequency is \( W_o(f_o + f_n)df = W_o df \) as shown in fig. 1.1. df covers an infinitely small band of frequencies centered on \( f_o + f_n \). The sinusoidal noise voltage \( n_o(t) \) at \( f_o + f_n \) may then be written

\[
n_o(t) = A_n \cos (\omega_o + \omega_n)t, \quad f_o - B/2 \leq f_o + f_n \leq f_o + B/2
\]

(1.33)

\[
A_{n/2}^2 = W_o(f_o + f_n)df = W_o df
\]

(1.34)

If it is now further assumed that the IF wave \( v_o(t) \) is unmodulated, the sum of unmodulated carrier plus a noise component at frequency \( f_o + f_n \) becomes

\[
v_o(t) + n_o(t) = A_o \cos \omega_o t + A_n \cos (\omega_o + \omega_n)t
\]

(1.35a)

\[
= (A_o + A_n \cos \omega_n t) \cos \omega_o t
\]

(1.35b)

\[
- (A_n \sin \omega_n t) \sin \omega_o t
\]

(1.35c)

\[
R(t) = (A_o + A_n \cos \omega_n t)^2 + (A_n \sin \omega_n t)^2
\]

(1.36)

\[
\phi(t) = \tan^{-1} \left[ \frac{A_n \sin \omega_n t}{(A_o + A_n \cos \omega_n t)} \right]
\]

(1.37)

Both amplitude \( R(t) \) and excess phase (phase above \( \omega_o t \)) \( \phi(t) \) of the resultant wave in eqn. (1.35c) are seen to vary with
Sec. 1.3. BASIC RELATIONSHIPS IN FM

time. If this wave is passed through a limiter (see fig. 1.1), the amplitude variations will be removed, producing then a wave with constant amplitude $A_c$. Limiter action is obtained with a device having the dynamic transfer characteristic (instantaneous output vs. instantaneous input) shown in fig. 1.1. Any variable-amplitude waveform, for example, applied to this device, will produce a square wave of the same frequency, and a tuned circuit in the limiter, tuned to this frequency (the fundamental), will remove the harmonics in the square wave to produce a constant amplitude sinusoidal waveform.

If the carrier to noise power ratio (CNR) in eqn. (1.32) is greater than about 10, then $A_n$, the amplitude of the noise voltage, will be small compared with $A_o$, the amplitude of the IF carrier, and $\phi(t)$ in eqn. (1.37) may be written

$$\phi(t) = \tan^{-1} \left( \frac{A_n}{A_o} \sin \omega_n t \right)$$  \hspace{1cm} (1.38a)

$$\approx \left( \frac{A_n}{A_o} \sin \omega_n t \right) , \quad A_n \ll A_o$$  \hspace{1cm} (1.38b)

The result in eqn. (1.38b) is obtained when it is remembered that for small $x$, $\tan^{-1} x \approx x$. Substitution of eqn. (1.38b) into eqn. (1.35c) gives

$$v_o(t) + n_o(t) = A_c \cos (\omega_o t + \left( \frac{A_n}{A_o} \sin \omega_n t \right))$$  \hspace{1cm} (1.39a)

$$= A_c \cos (\omega_o t + M \sin \omega_n t)$$  \hspace{1cm} (1.39b)

where $v_o(t) + n_o(t)$ is now the output of the limiter, $A_c$ is
the limited amplitude and \( \frac{A_n}{A_0} \) is set equal to \( M_n \). By comparison with eqn. (1.9d), the noise voltage at frequency \( f_0 + f_n \) is seen to frequency modulate the IF carrier in exactly the same way as a 'signal' voltage of frequency \( f_n \). Also since \( A_n << A_0 \), \( M \) is small so that the carrier is small-index modulated. It is this small-index modulation of the carrier by noise when the carrier to noise power ratio (CNR) is high that defines the above-threshold operation of the FM receiver.

We saw in sec. 1.2 that a baseband signal which small-index modulated a carrier, produced two sidebands, and the 'information' can therefore be retrieved from the sidebands. The band of noise centered on \( f_0 \) at the output of the IF filter may be regarded as these two sidebands from which the baseband noise may be retrieved.

From the limiter the IF wave enters the frequency detector or discriminator having the dynamic transfer characteristic shown in fig. 1.1. The slope or sensitivity \( K_D \) of the discriminator is usually dependent on the amplitude of the IF wave, and this creates the need for the limiter. The balanced discriminator (i.e. 'centered' on \( f_0 \)) produces an instantaneous output \( v_D(t) \) which is \( K_D \) times the instantaneous excess frequency (frequency above \( \omega_0 \)) of the input; thus

\[
v_D(t) = K_D (\text{inst. freq. of input} - \omega_0) \quad (1.40a)
\]

\[
= K_D \frac{d}{dt} (\text{inst. phase of input} - \omega_0 t) \quad (1.40b)
\]
Since frequency is the derivative with respect to time of phase [eqn. (1.40)], the discriminator may also be regarded as a phase differentiator.

The IF wave in eqn. (1.39a) entering the discriminator produces a noise output \( n_D(t) \) given by

\[
n_D(t) = K_D \omega_n \left( \frac{A_n}{A_0} \right) \cos \omega_n t
\]

(1.41)

when \( K_D \) is in volts per radian frequency. If \( W_D(f) \) is the power spectrum of the discriminator noise output, then the mean power associated with frequency \( f_n \) is \( W_D(f_n) df \) where \( df \) is the infinitesimally small band of frequencies centered on \( f_n \), and this is to be the same as \( K_D^2 \omega_n^2 \left( \frac{A_n}{A_0} \right)^2 \), the mean power associated with the noise voltage of frequency \( f_n \) in eqn. (1.41) Reference to fig. 1.1 and eqn. (1.33) will remind us that \( f_n \) lies in the range \(-B/2 \leq f_n \leq +B/2\) where \( B \) is the IF filter bandwidth. Thus

\[
W_D(f_n) df = K_D^2 \omega_n^2 \frac{A_n^2}{2A_0^2} = \left( K_D \omega_n / A_0 \right)^2 W_0(f_0 + f_n) df
\]

(1.42)

\[
\int_{-B/2}^{B/2} W_D(f) df = \int_{-B/2}^{B/2} \left( K_D \omega_n / A_0 \right)^2 W_0(f_0 + f) df
\]

(1.43a)

\[
= \int_{-B/2}^{B/2} \left( K_D \omega_n / A_0 \right)^2 2W_0(f_0 + f) \, df
\]

(1.43b)
Sec. 1.3. BASIC RELATIONSHIPS IN FM

where the relationship in eqn. (1.34) between the IF noise spectrum \( W_0(f) \) and \( A_n \) is used in eqn. (1.42), and the even-function property of \( W_0(f) \) is used in eqn. (1.43b). In eqns. (1.43) too the variable \( f_n \) has been replaced by \( f \) for convenience. \( W_0(f_0 + f) \) is the power spectrum \( W_0(f) \) translated downward in frequency by amount \( f_0 \) so that, as \( W_0(f) \) is even, \( W_0(f_0 + f) \) is also even. Thus the two sided integral (positive and negative frequencies) in eqn. (1.43a) can be made one-sided (positive frequencies only) by multiplying by 2. We have so far made no use of the fact that \( W_0(f) \) has constant level \( W_0 \), and indeed up to this point, no stipulations need be made regarding the nature of \( W_0(f) \).

From eqn. (1.43b), we obtain the power spectrum \( W_D(f) \) of the output noise as

\[
W_D(f) = K_D^2 \omega^2 2W_0(f_0 + f)/A_0^2 
\]

(1.44a)

\[
= K_D^2 (2\pi f)^2 2W_0(f_0 + f)/A_0^2 
\]

(1.44b)

and \( W_D(f) \) is seen to vary with frequency as \( f^2 \) (see fig. 1.1). For this reason \( W_D(f) \) is referred to as the 'parabolic' noise spectrum which characterises FM receivers operating above the threshold.
If $f_s$ is the highest frequency in the baseband signal for which the receiver is designed, the output filter's frequency response $G_{OUT}(j\omega)$ will be as in fig. 1.1. To obtain the total noise power $N_D$ at the output, we must then integrate $W_D(f)$ over the output bandwidth $f_s$. The gain factor $K_{OUT}^2$, by which $W_D(f)$ should be raised, is here omitted for convenience. Since $2f_s$ is the minimum IF bandwidth which can be used to transmit an FM signal (from sec. 1.2), $f_s$ will always be less than or equal to $B/2$. Thus

$$N_D = \int_0^{f_s} W_D(f)df = \int_0^{f_s} K_D \frac{2}{4\pi} f \frac{2}{2W_0/A_0} df$$

(1.45a)

$$= \frac{2}{8\pi} K_D W_0 f_s^3 / (3A_0^2)$$

(1.45b)

where use is now made in eqn (1.45a) of the fact that $W_o(f_o + f)$ has constant level $W_o$. It will be recalled that to obtain $N_D$, the total output noise power, we had assumed the carrier to be unmodulated; thus $N_D$ is the output noise power in the absence of modulation. We are also interested in the output signal power when there is modulation, and to obtain this we assume that noise is absent from the system. A baseband signal $\Delta \omega \cos \omega_s t$ will produce an IF FM wave with instantaneous frequency $\omega_o + \Delta \omega \cos \omega_s t$ as in eqn. (1.6). From eqn. (1.40a) a balanced detector will therefore produce a signal output $s_D(t)$ given by

$$s_D(t) = K_D \Delta \omega \cos \omega_s t$$

(1.46)
The mean signal power $S_D$ associated with this wave is given by

$$S_D = K_D^2(\Delta \omega)^2/2 = K_D^2 4\pi^2(\Delta f)^2/2$$  \hfill (1.47)

The output signal to noise ratio (SNR) is defined as the ratio of mean output signal power to output mean noise power. If we assume that the former is unaltered by the presence of noise, and that conversely the output noise power is unaltered by the presence of modulation, we derive

$$(\text{SNR}) = S_D/N_D = 3(\Delta f)^2 A_0^2/(4W_0 f_s^3)$$  \hfill (1.48)

The modulation index $M = \Delta f/f_s$ and the IF carrier to noise power ratio (CNR) $= A_0^2/(2W_0 B)$ may be introduced into eqn. (1.48) to yield a more familiar form

$$(\text{SNR}) = 3M^2(\text{CNR})B/(2f_s)$$  \hfill (1.49)

$B/(2f_s)$ is known as the bandwidth ratio of the system and as mentioned before has a minimum value of unity. (SNR) is frequently expressed in other forms. One form uses the ratio $(\text{CNR})'$ of mean IF carrier power $A_0^2/2$ to total mean noise power $W_0 f_s$ in an IF frequency window of $2f_s$ i.e., no matter what the IF bandwidth, the noise power is measured in a bandwidth $2f_s$. Thus

$$(\text{CNR})' = A_0^2/(4W_0 f_s)$$  \hfill (1.50)
and eqn. (1.48) becomes

\[
2 \frac{SNR}{SNR} = 3M (CNR)
\]

Another form is obtained from eqn. (1.49) by the use of Carson's rule, \( B = 2f_s(1 + M) \).

\[
\frac{SNR}{SNR} = 3M (1 + M) (CNR) \quad \quad \quad \quad \quad \quad \quad \quad (1.51)
\]

\[
\frac{3}{3M (CNR)} , \quad M >> 1 \quad \quad \quad \quad \quad \quad \quad \quad (1.52)
\]

So far we have been assuming that the noise output of the receiver is unaffected by the presence of modulation and that the signal output is unaffected by the presence of noise. We have been assuming implicitly as well that the receiver treats a complex modulating signal in the same way as a sinusoidal tone. Practical experience, it is found, tends to justify these assumptions, provided the receiver is operating above threshold. We shall, however, be taking a more critical look at these assumptions later.

1.4 Threshold Phenomenon. Preliminary Discussion. Typical plots of output \((SNR)\) vs. IF \((CNR)\) are shown in fig. 1.2. Such curves are derived from the classical mathematical analyses of various authors (ref. 1,3,4,5) and have been verified experimentally (ref. 7,8). Although the mathematical analyses have been quite rigorous, the authors make the assumption, as we have done, that noise and modulation do not interact. The \((SNR)\) curves are found to consist of a linear portion which corresponds to the above-threshold analysis;
but as (CNR) is decreased, the curves depart quite dramatically from linearity. It is this departure from linearity that defines the threshold in FM. The value of (CNR), for which (SNR) is 10 dB below linearity, is a common definition of the threshold.

Concerning ourselves for a moment with only the above-threshold operation of the receiver, we observe that for a given (CNR) measured in the IF bandwidth B, (SNR) improvement is obtained by increasing the modulation index M. This is in accordance with any of the expressions for (SNR) in eqns. (1.49) through (1.52). This useful property of FM systems may be explained by remarking that if we are dealing with the same baseband signal (therefore same $f_0$), increase in modulation index is brought about by increased deviation $\Delta f$, which, in turn, results in increased output signal. The increase in modulation index must, of course, be accompanied by a corresponding increase in IF bandwidth B, and if (CNR), measured in this bandwidth, is to remain unchanged, either the carrier power must be increased or the level of the IF noise spectrum must be decreased. Either event will reduce the output noise power $N_D$ in eqn. (1.45), and this, together with the increased signal output, results in increased output (SNR).

Turning now to the threshold, it will be remembered that the curves in fig. 1.2 have been derived by assuming that noise and modulation do not interact, so the threshold cannot be accounted for in terms of decreased signal output as (CNR) is decreased. We notice too, that the threshold worsens as the modulation index is increased, and by the same token, we
cannot assume that the noise output increases due to increased modulation. Much controversy has in fact arisen (see for example ref. 14) over the cause of the threshold. A widely held opinion is that, as (CNR) decreases in a system, the approximation in eqns. (1.38) becomes less valid, and the expansion of \( \tan^{-1} \left[ A_n \sin\omega_n t / (A_o + A_n \cos\omega_n t) \right] \) in eqn. (1.37) would contain additional noise terms. Consider, however, systems (1) and (2), characterised by curves (1) and (2) respectively in fig. 1.2. When both systems are operating at \((\text{CNR})_{T1}\), the threshold of system (1), we may write [see eqn. (1.32)]

\[
(\text{CNR})_{T1} = \left[ \frac{A_o^2}{(2W_o)} \right]_{1} / B_1 = \left[ \frac{A_o^2}{(2W_o)} \right]_{2} / B_2 \quad (1.53)
\]

where the subscripts 1 and 2 refer to quantities in systems (1) and (2) respectively. \( A_o \) is the IF carrier amplitude, \( W_o \) the level (assumed constant) of the IF noise spectrum, and \( B \) the IF bandwidth. Since system (2) has modulation index \( M_2 \) greater than \( M_1 \), it will also have IF bandwidth \( B_2 \) greater than \( B_1 \) so that \( \left[ \frac{A_o^2}{(2W_o)} \right]_2 \) will be greater than \( \left[ \frac{A_o^2}{(2W_o)} \right]_1 \). The validity of the approximation in eqns. (1.38) depends on the IF noise amplitude \( A_n \) [which is proportional to \( W_o \) as in eqn. (1.34)] being much smaller than the carrier amplitude \( A_o \). If this assumption holds in system (1), or if it is just about to break down, it should certainly hold in system (2) where \( \left[ \frac{A_o^2}{(2W_o)} \right] \) is larger. Thus the breakdown of the approximation in eqns. (1.38) cannot explain the threshold either.
Several authors (refs. 9,10,11,12) working on practical receivers have remarked that in the threshold region, the output noise takes on a different nature. Above threshold the noise has the usual hissing, thrashing characteristic of fluctuation noise, and as (CNR) is decreased, it increases in intensity. As the threshold is approached, however, the noise takes on a crackling, clicking, or popping-like nature which is subjectively quite alarming. A system just above threshold, having excellent (SNR) performance, will still have very good (SNR) performance just below threshold (1db below linearity). Yet because of the nature of the noise, the system becomes completely unusable in practical terms. The effect on a television picture is similar. The clicks or pops show up as intense black or white dots giving a characteristic sparkling effect. With a telephone multiplex signal, no distinct clicks or pops are discernible, the reason for which will be obvious once the threshold phenomenon has been examined in more detail.

1.5 FM Feedback (FMFB) Receiver. A schematic of the FMFB receiver is shown in fig. 1.3. For the present, the receiver will be assumed noise-free. Various time invariant phase angles $\phi$ have been included to illustrate that, as far as the signal output is concerned, they do not contribute to the system's performance. The relationships between the various $\phi$'s are of no importance at this stage and need not be considered here. They are in fact of concern from the point of view of system stability but that too is dealt with in a later section.

In the FMFB receiver, the demodulated output $v_D(t)$ is made
to frequency modulate the local oscillator wave $v_L(t)$. This is achieved by feeding back the demodulated signal through some feedback gain control network (Gain$K_F$) to a voltage controlled oscillator (VCO). The VCO then acts as an FM generator producing the output $v_L(t)$ given in fig. 1.3. $K_V$, the VCO's sensitivity or gain (in radians per second/volt), relates the frequency deviation $(\Delta \omega)_L$ of the oscillator wave to the input signal as shown in fig. 1.3. In the mixer the frequencies of the RF wave $v_R(t)$ and the local oscillator wave $v_L(t)$ subtract to produce the mixer output wave $v_M(t)$. This has carrier frequency $\omega_0$ but has modulation index $M_0$ and frequency deviation $(\Delta \omega)_0$ which are reduced as shown in fig. 1.3 because of the frequency subtraction of the two FM waves in the mixer.

On open loop i.e. without feedback, the demodulated output $[v_D(t)]_{OL}$ is

$$[v_D(t)]_{OL} = K_D (\Delta \omega)_R \cos \omega_s t$$  \hspace{1cm} (1.54)

and, on closed loop, the demodulated output $[v_D(t)]_{CL}$ is

$$[v_D(t)]_{CL} = K_D (\Delta \omega)_0 \cos \omega_s t$$  \hspace{1cm} (1.55)

but from fig. 1.3 we obtain that

$$(\Delta \omega)_O = (\Delta \omega)_R - (\Delta \omega)_L = (\Delta \omega)_R - K_V K_F K_D (\Delta \omega)_0$$  \hspace{1cm} (1.56)

$$(\Delta \omega)_O = (\Delta \omega)_R / \left(1 + K_V K_F K_D\right) = (\Delta \omega)_R / F$$  \hspace{1cm} (1.57)
where $F = 1 + \frac{K_v K_f K_D}{V_F}$ is called the feedback factor. From eqns. (1.54), (1.55) and (1.57), it will be seen that the application of feedback reduces the output signal amplitude by the feedback factor $F$, and the output signal power $S_D$ in eqn. (1.47) will be reduced by $F^2$.

To analyse, in an elementary way, the effect of feedback on the output noise, we assume, as before, that modulation is absent. The input to the receiver therefore consists of an unmodulated carrier plus wideband noise as described before. Reference to eqns. (1.39) will remind us that, when the receiver is assumed to be operating above threshold, the noise frequency-modulates the carrier in exactly the same way as the signal would. The application of feedback will therefore have the same effect on the noise as it had on the signal. The amplitude of each noise component in the output will be decreased by the factor $F$ and the noise power spectrum $W_D(f)$ by the factor $F^2$. The total output noise $N_D$ will also be reduced by $F^2$. Thus both signal and above-threshold noise are reduced to the same extent by feedback, and the signal to noise ratio remains the same.

Feedback may, however, be employed to improve the system's threshold performance in the following way. When feedback is applied, we have seen that the IF modulation index is reduced from $M_R$ to $M_o (= M_R/F)$. Thus, on feedback, the IF and baseband channel can be designed for this reduced modulation index. In the previous section we saw that the threshold performance of a system depended on the IF modulation index so that the feedback system will have the threshold performance of a system with modulation index $M_R/F$. The
signal to noise performance will, however, be that of a system with modulation index $M_R$, since feedback does not affect signal to noise ratio in the above-threshold region. These points are illustrated in fig. 1.4. Curve AB represents a system with open loop modulation index $M_R$ and curve DE, one with open loop modulation index $M_R/F$. When feedback of amount $F$ is applied to the system represented by curve AB, the signal to noise performance of a system with modulation index $M_R$ is retained, but the threshold occurs where it would for an open loop system with the lower modulation index $M_R/F$. Curve CB therefore represents the performance of the feedback system.

1.6 Previous Work on FMFB Receivers. The FMFB receiver was first proposed in 1939 by J.G. Chaffee (ref.9) who also made the first experimental investigations on the receiver. He discovered (without explaining) that starting with an open loop system, the application of feedback in itself could only achieve reduction of the output noise power if (1) the noise level was sufficiently small compared with the carrier level i.e. if the open loop system was above-threshold and if (2) the amount of feedback was not too great. The subject received comparatively little attention until the early sixties when satellite communications came into vogue. Practical receivers were built by several parties (refs. 10, 11, 12) none of whom provided any advances on the elementary theory of operation. The authors who did attempt to analyse the feedback receiver (refs. 13,15,16) still left much to be desired in terms of providing an understanding both of the feedback receiver and of the threshold phenomenon.
Schematic of Conventional FM Receiver

\[ v_R(t) = A_R \cos(\omega_R t + M \sin \omega_s t) \]
\[ v_L(t) = A_L \cos \omega_L t \]
\[ v_o(t) = A_R K_M K_IF \cos[(\omega_R - \omega_L)t + M \sin \omega_s t] \]
\[ = A_0 \cos(\omega_o t + M \sin \omega_s t); \quad A_0 = A_R K_M K_IF; \quad \omega_o = \omega_R - \omega_L \]
FIG. 1.2. Threshold Curves.
\[
\begin{align*}
\epsilon_r(t) &= A_R \cos(\omega_R t + M_R \sin \omega_s t); \quad M_R = (\Delta \omega)_R / \omega_s \\
\epsilon_L(t) &= A_L \cos(\omega_L t + M_L \sin \omega_s t) = A_L \cos(\omega_L t + K_V / K_F \epsilon_D(t)); \quad M_L = (\Delta \omega)_L / \omega_s \\
\epsilon_M(t) &= K_M A_R \cos[(\omega_R - \omega_L) t + (M_R - M_L) \sin \omega_s t - \phi_M] = K_M A_R \cos(\omega_o t + \phi_o)
\end{align*}
\]

\[
\begin{align*}
\omega_o &= \omega_R - \omega_L; \quad M_o = (\Delta \omega)_o / \omega_s = M_R - M_L = [(\Delta \omega)_R - (\Delta \omega)_L] / \omega_s \\
\epsilon_o(t) &= A_o \cos(\omega_o t + M_o \sin \omega_s t - \phi_o); \quad A_o = K_M K_{IF} A_R \\
\epsilon_D(t) &= K_D M_o \omega_s \cos \omega_s t = K_D (\Delta \omega)_o \cos \omega_s t
\end{align*}
\]

**FIG.1.3.** Schematic of FMFB Receiver
FIG. 1.4. Threshold Improvement Caused by Feedback
2.1 Introduction. In sec. 1.3, the band of noise at the output of the IF filter was resolved into several sinusoids distributed over the frequency range

\[ f_0 - B/2 \leq f \leq f_0 + B/2, \]

where B was bandwidth of the IF filter. Such an approach was justified at that stage since we were only concerned with the average properties of the IF and baseband noise voltages. In this chapter we concern ourselves with the instantaneous behaviour of the noise, and because of the incoherence of noise, this behaviour can only be quantified in statistical terms.

2.2 Stationary and Ergodic Processes. Consider the variables \( v_1(t), v_2(t), \ldots, v_k(t) \) shown in fig. 2.1, and let us assume that each of these variables is measured in one of \( k \) identical systems. \( v_1(t), v_2(t), \ldots, v_k(t) \) are said to form an ensemble. Let us say, for example, that we are interested in the average value of the ensemble variables \( v_1(t_1), v_2(t_1), \ldots, v_k(t_1) \) at time \( t_1 \); or we may be interested in the percentage of these ensemble variables which lie within the limits \( A \) to \( A + dA \) (fig. 2.1); or indeed we may require any other statistic of the ensemble variables at time \( t_1 \). Such statistics are called ensemble statistics.
If the ensemble statistics are the same whether they are measured at time $t_1$ or $t_2$ or at any other time, i.e., if the statistics are independent of the time at which they are taken, the variables are said to form a stationary process.

Instead of being interested in the ensemble statistics, we may be interested in the time statistics of any of the variables, and we may require the average of the values $v_1(t_1), v_1(t_2) \ldots v_1(t_n)$, these being the values of the same variable $v_1(t)$ at the different times $t_1, t_2 \ldots t_n$. This type of statistic is called the time statistic of the variable $v_1(t)$. If the time statistics of the variables are the same as the ensemble statistics, the latter being measured at any time, the variables are said to form an ergodic process. All ergodic processes are stationary, but not all stationary processes are ergodic.

The noise processes with which we shall be concerning ourselves, both here and in all subsequent chapters, will be assumed to be ergodic.

2.3 Random Gaussian Noise. Fig. 2.2 represents the oscillogram of a random gaussian noise voltage $n(t)$. Such an oscillogram could represent the shot noise voltage at the anode of a vacuum tube or the thermal noise voltage developed across a resistance or the electromagnetic radiation noise voltage at a receiver's input. $n(t)$ has mean value $n_1$ (also called average or d.c. value) defined by

$$n(t) = n_1 = \lim_{T \to \infty} \frac{1}{T} \int_0^T n(t)dt$$

(2.1)
where the bar () denotes 'average value of'. The mean square value $n^2$ of $n(t)$ is defined by

$$n^2(t) = n^2 = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} n^2(t) \, dt$$  \hspace{1cm} (2.2)

and the variance or average a.c. power $N$ of $n(t)$ is defined by

$$[n(t) - n_1]^2 = N = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} [n(t) - n_1]^2 \, dt$$  \hspace{1cm} (2.3)

The variable $n(t)$ represented in fig. 2.2 is said to be random in nature, since the two values $n(t)$ and $n(t + T)$ are unrelated for all $T$. The variable is said to be gaussian, because its probability density function $p(n)$ follows the gaussian or normal distribution.

$$p(n) = (2\pi N)^{-\frac{1}{2}} \exp[-(n-n_1)^2/(2N)]$$  \hspace{1cm} (2.4)

where $n_1$ and $N$ are defined in eqns. (2.1) and (2.3). The gaussian distribution is shown in fig. 2.3. The probability density function gives the probability $p(n) \, dn$ that the instantaneous value of $n(t)$ lies in the range $n$ to $n + \, dn$, and $p(n) \, dn$ is also the fraction of the total time that $n(t)$ spends in range $n$ to $n + \, dn$. $p(n) \, dn$ will be seen from fig. 2.3 to represent an element of area under the probability curve. Thus the probability $p(n_1 \leq n \leq n_2)$ that the instantaneous value of $n(t)$ lies between $n_1$ and $n_2$ is represented by the
area under the $p(n)$ curve, between the values $n_1$ and $n_2$.

$$p(n_1 \leq n \leq n_2) = \int_{n_1}^{n_2} p(n)dn \quad (2.5)$$

The probability, or fraction of the total time, $p(-\infty \leq n \leq +\infty)$, that the gaussian variable $n(t)$ lies between $-\infty$ and $+\infty$ is

$$p(-\infty \leq n \leq +\infty) = \int_{-\infty}^{+\infty} p(n)dn = 1 \quad (2.6)$$

as expected.

The mean value $\overline{n(t)}$, mean squared value $\overline{n^2(t)}$, and variance $\overline{[n(t)-n_1]^2}$ of $n(t)$ are obtained from the probability distribution thus -

$$\overline{n(t)} = \int_{-\infty}^{+\infty} np(n)dn = n_1 \quad (2.7)$$

$$\overline{n^2(t)} = \int_{-\infty}^{+\infty} n^2 p(n)dn = n_1^2 + N \quad (2.8)$$

$$\overline{[n(t)-n_1]^2} = \int_{-\infty}^{+\infty} (n-n_1)^2 p(n)dn = N \quad (2.9)$$

The results in eqns. (2.7) through (2.9) are all to be expected and may be obtained from tables (ref. 29) once the probability distribution $p(n)$ in eqn. (2.4) has been substituted into the integrals.
In the majority of cases, the noise with which we are concerned has zero mean, and we therefore assume this to be the case from here onward unless otherwise stated. When the noise voltage \( n(t) \) (or any variable) has zero mean, the mean squared value \( n^2(t) \) and variance \( [n(t) - n^1]^2 \) are obviously identical and have value \( N \). Since \( N \) may be regarded as the mean a.c. power content of the noise voltage \( n(t) \), if \( W_n(f) \) represents the power spectrum of \( n(t) \), \( N \) must also be given by

\[
N = \int_{0}^{\infty} W_n(f) df \quad (2.10)
\]

Because \( n(t) \) has a gaussian probability distribution, it may be regarded as the sum of a large number of random independent variables. [Investigations into the mechanisms of noise production show this to be true.] This is a corollary of the 'central-limit theorem' of probability, which states that if a random variable \( n(t) \) is the sum of \( K \) independent variables, then, as \( K \) approaches infinity, the probability distribution of \( n(t) \) approaches a normal or gaussian distribution, irrespective of the probability distribution of the individual constituent variables. Thus if

\[
n(t) = n_1(t) + n_2(t) + \ldots + n_K(t) = \sum_{k=1}^{K} n_k(t) \quad (2.11)
\]

where \( n_1(t), n_2(t), \ldots, n_K(t) \) are independent variables, then
irrespective of their individual probability distribution, the probability distribution $p(n)$ of $n(t)$ tends to the gaussian form as $K$ tends to infinity.

A random noise voltage $n(t)$ with zero mean lends itself to Fourier series representation of the form

$$n(t) = \sum_{n=1}^{N} A_n \cos \omega_n t + B_n \sin \omega_n t \quad (2.12)$$

$$A_n = \lim_{T \to \infty} \frac{2}{T} \int_0^T n(t) \cos \omega_n t \, dt \quad (2.13)$$

$$B_n = \lim_{T \to \infty} \frac{2}{T} \int_0^T n(t) \sin \omega_n t \, dt \quad (2.14)$$

$$\omega_n = 2\pi f_n \quad ; \quad f_n = n\Delta f \quad ; \quad \Delta f = 1/T \quad (2.15)$$

These are standard Fourier series equations, and in the limit, as $T$ tends to infinity, $\Delta f$ tends to $df$; the summation in eqn. (2.12) then becomes an integration, and the Fourier transform equations result. It is, however, more convenient at this time to manipulate the Fourier series representation.

Davenport and Root (ref. 18, chap. 6) have shown that since $n(t)$ is gaussian-distributed, $A_n$ and $B_n$ form a set of independent random gaussian variables, each having zero mean (in the ensemble sense). Their respective ensemble mean
squared values, \( A_n^2 \) and \( B_n^2 \) are given by

\[
A_n^2 = B_n^2 = W_n(f_n) df
\]  

(2.16)

where \( W_n(f) \) is the power spectrum of \( n(t) \) and \( f_n \) is defined in eqns. (2.12) and (2.15). Thus \( n(t) \) in eqn. (2.12) is seen to consist of a large number of random independent variables, and the central-limit theorem may be used to deduce the gaussian distribution of \( n(t) \). As far as the central-limit theorem is concerned, \( A_n \) and \( B_n \) need not be normally distributed, and Rice (ref. 2, sec. 2.8) suggests that if they could only take on the values \( \pm \sqrt{W}(f_n) df \) with equal probability of \( \frac{1}{2} \), they would still be random variables, and the central-limit theorem would still lead to the gaussian distribution for \( n(t) \).

Another useful Fourier series representation of \( n(t) \) is

\[
n(t) = \sum_{n=1}^{N} C_n \cos(\omega_n t - \alpha_n)
\]  

(2.17)

where \( \omega_n \) is defined in eqn. (2.15). This representation is used by Rice (ref. 2, sec. 2.8) where he assumes, possibly in keeping with his remarks about \( A_n \) and \( B_n \), that \( C_n \) has constant value \( [2W_n(f_n) df]^{\frac{1}{2}} \). \( \alpha_n \) he assumes to be a random phase angle which can have any value in the range \(-\pi\) to \(+\pi\) radians. The probability distribution \( p(\alpha_n) \) of \( \alpha_n \) would then be uniform in the range \(-\pi\) to \(+\pi\). Eqn. (2.17) then becomes

\[
n(t) = C_1 \cos(\omega_1 t - \alpha_1) + C_2 \cos(\omega_2 t - \alpha_2) + \ldots
\]  

(2.18)
where $C_1$, $C_2$ etc. are constants. However, since $\alpha_1$, $\alpha_2$ etc. are random independent phase angles, $n(t)$ comprises a large number of random independent variables and consequently has gaussian probability distribution.

Eqn. (2.17) will therefore provide $n(t)$ with its known statistical properties when the $C$'s are constant and the $\alpha$'s random and independent.

There should, however, be the normal Fourier series relationship between eqn. (2.12) and (2.17), but, as far as it could be ascertained, these relationships have not been developed in the literature and we therefore direct our attention to this matter. From eqn. (2.12) and (2.17)

$$C_n^2 = A_n^2 + B_n^2, \quad \alpha_n = \tan^{-1}(B_n/A_n) \quad (2.19)$$

We are required to relate the statistics of $A_n$ and $B_n$ to those of $C_n$ and $\alpha_n$. To do this, recourse is made to the theorem in probability that if $A_n$ and $B_n$ are two independent variables, having probability distributions $p(A_n)$ and $p(B_n)$ respectively, then the joint probability distribution $p(A_n, B_n)$ is given by

$$p(A_n, B_n) = p(A_n) \cdot p(B_n) \quad (2.20)$$

The joint probability distribution $p(A_n, B_n)$ gives the probability $p(A_n, B_n) \, dA_n \, dB_n$ that $A_n$ lies in the range $A_n$ to $A_n + dA_n$ and that simultaneously $B_n$ lies in the range $B_n$ to $B_n + dB_n$. $p(A_n, B_n) \, dA_n \, dB_n$ must be the same as the probability $p(C_n, \alpha_n) \, dC_n \, d\alpha_n$ that $C_n$ and $\alpha_n$ lie simultaneously in
the respective ranges \( C_n \) to \( C_n + dC_n \) and \( \alpha_n \) to \( \alpha_n + d\alpha_n \). Hence

\[
p(A_n, B_n) dA_n dB_n = p(C_n, \alpha_n) dC_n d\alpha_n
\]  

(2.21)

This equation raises the problem of changing from the variables \( A_n \) and \( B_n \) to the variables \( C_n \) and \( \alpha_n \), and this is dealt with in appendix D. We require the relations in eqn. (2.19) in the form

\[
A_n = C_n \cos \alpha_n ; \quad B_n = C_n \sin \alpha_n
\]  

(2.22)

and then write eqn. (2.21) as

\[
p(A_n, B_n) dA_n dB_n = f(C_n, \alpha_n) J dC_n d\alpha_n
\]  

(2.23)

where \( J \) is the Jacobian of the transformation (appendix D) and \( f(C_n, \alpha_n) \) is the function \( p(A_n, B_n) \) expressed in terms of the new variables \( C_n \) and \( \alpha_n \). For the case in point

\[
J = \begin{vmatrix}
\frac{\partial A_n}{\partial C_n} & \frac{\partial A_n}{\partial \alpha_n} \\
\frac{\partial B_n}{\partial C_n} & \frac{\partial B_n}{\partial \alpha_n}
\end{vmatrix} = \begin{vmatrix}
\cos \alpha_n & -C_n \sin \alpha_n \\
\sin \alpha_n & C_n \cos \alpha_n
\end{vmatrix} = C_n
\]  

(2.24)

Since \( A_n \) and \( B_n \) are random gaussian variables, each with zero mean and mean square value \( W_n(f) df \) where \( W_n(f) \) is the
power spectrum of the noise voltage \( n(t) \), it follows from eqn. (2.20) that

\[
p(A_n, B_n) = (2\pi W_n)^{-\frac{1}{2}} e^{-\frac{A_n^2}{2 W_n}} (2\pi W_n)^{-\frac{1}{2}} e^{-\frac{B_n^2}{2 W_n}}
\]

\[
= (2\pi W_n)^{-1} e^{-\frac{(A_n^2 + B_n^2)}{2 W_n}}
\]

(2.25a)

(2.25b)

where \( W_n \) is used in the equations as shorthand for \( W_n(f) \) df.

Substitution of eqn. (2.25b) into eqn. (2.23) gives

\[
p(A_n, B_n) dA_n dB_n = (2\pi W_n)^{-1} e^{-\frac{(A_n^2 + B_n^2)}{2 W_n}} C_n dC_n d\alpha_n
\]

\[
= (2\pi W_n)^{-1} C_n e^{-\frac{C_n^2}{2 W_n}} dC_n d\alpha_n
\]

(2.26a)

(2.26b)

Comparison of eqn. (2.26b) with (2.21) shows that the joint probability distribution \( p(C_n, \alpha_n) \) of \( C_n \) and \( \alpha_n \) is

\[
p(C_n, \alpha_n) = (2\pi W_n)^{-1} C_n e^{-\frac{C_n^2}{2 W_n}}
\]

(2.27)

with \( W_n = W_n(f) \) df. From this probability distribution, the statistics of \( C_n \) and \( \alpha_n \) may be found. The probability distribution \( p(C_n) \) of \( C_n \) may be found by integration of eqn. (2.27) with respect to \( \alpha_n \), over the entire range of \( \alpha_n \), and conversely, the probability distribution \( p(\alpha_n) \) of \( \alpha_n \) may be found by integration of eqn. (2.27) over the entire range \( C_n \). From eqn. (2.19), it will be obvious that \( \alpha_n \) lies in the range \(-\pi\) to \(+\pi\), since \( A_n \) and \( B_n \) are
random gaussian variables which can take on any value between $-\infty$ and $+\infty$. Hence

$$p(C_n) = \int_{-\infty}^{+\infty} \frac{1}{C_n} e^{-\frac{C_n^2}{2W_n}} \, d\alpha_n$$

$$= (\frac{1}{W_n}) C_n e^{-\frac{C_n^2}{2W_n}}$$

(2.28a)

(2.28b)

where $W_n \equiv W_n(f_n)df$. Eqn. (2.28b) defines the Rayleigh distribution, which is shown in fig. 2.4a. This distribution has a maximum value of $(W_n e)^{-\frac{1}{2}}$ occurring at $C_n = (W_n)^{\frac{1}{2}}$. $C_n$ ranges from zero to $\infty$ as indicated in eqns. (2.17) and (2.19) where it is seen to define an amplitude. Confirmation of this is obtained from the fact that the probability $p(0 \leq C_n \leq \infty)$ that $C_n$ lies between zero and $\infty$ is

$$p(0 \leq C_n \leq \infty) = \int_{0}^{\infty} p(C_n) dC_n = 1$$

(2.29)

This result may be obtained from tables (ref. 29, sec. 3.461) after the insertion of $p(C_n)$ into the integral.

The ensemble average value $\overline{C_n}$ and ensemble mean squared value $C_n^2$ of $C_n$ are obtained from

$$\overline{C_n} = \int_{0}^{\infty} C_n p(C_n) dC_n = (\frac{W_n}{2})^{\frac{1}{2}}$$

(2.30)

$$\overline{C_n^2} = \int_{0}^{\infty} C_n^2 p(C_n) dC_n = 2W_n$$

(2.31)
where \( W_n = W_n(f_n)df \). Both these results may be obtained from tables (ref. 29, sec. 3.461) after the insertion of \( p(C_n) \) into the integrals. The result in eqn. (2.31) may have been deduced from eqn. (2.19) since

\[
C_n^2 = A_n^2 + B_n^2 = A_n^2 + B_n^2 = 2W_n(f_n)df \quad (2.32)
\]

when \( A_n \) and \( B_n \) are random independent variables, each with mean square value \( W_n(f_n)df \).

The probability distribution \( p(\alpha_n) \) of \( \alpha_n \) is found by integration of eqn. (2.27) over the entire range of \( C_n \).

Thus

\[
p(\alpha_n) = \int_0^\infty (2\pi W_n)^{-1} e^{-C_n^2/(2W_n)} dC_n \quad (2.33a)
\]

\[
= 1/(2\pi) \quad (2.33b)
\]

This result follows from the result in eqn. (2.29) and shows that \( p(\alpha_n) \) is constant, i.e. \( \alpha_n \) is uniformly distributed. From eqn. (2.19) the range of \( \alpha_n \) must be \( -\pi \) to \( +\pi \) as we have already said. \( p(\alpha_n) \) is thus uniformly distributed in the range \( -\pi \) to \( +\pi \) and is shown in fig. 2.4b.

The implication of the uniform distribution is that \( \alpha_n \) can take on any value in the range \( -\pi \) to \( +\pi \) with equal probability of \( 1/(2\pi) \). The probability \( p(-\pi \leq \alpha_n \leq +\pi) \) that \( \alpha_n \) lies in the range \( -\pi \) to \( +\pi \) is, of course, unity.

\[
p(-\pi \leq \alpha_n \leq +\pi) = \int_{-\pi}^{+\pi} p(\alpha_n) d\alpha_n = 1 \quad (2.34)
\]
From eqns. (2.27), (2.28b), and (2.33b), it is seen that

\[ p(C_n, \alpha_n) = p(C_n) \cdot p(\alpha_n) \] (2.35)

showing that \( C_n \) and \( \alpha_n \) are independent random variables, \( C_n \) being Rayleigh distributed between zero and \( \infty \), \( \alpha_n \) being uniformly distributed between \(-\pi\) and \(+\pi\) radians. With these statistics, \( n(t) \) in eqn. (2.17) is again the sum of a large number of independent variables and will therefore have gaussian distribution.

Another important representation of noise is obtained from the Fourier representation in eqn. (2.12) as follows -

\[ n(t) = \sum_{n=1}^{N} A_n \cos \omega_n t + B_n \sin \omega_n t \] (2.36a)

\[ = \sum_{n=1}^{N} A_n \cos (\omega_n t - \omega_0 t + \omega_0 t) + B_n \sin (\omega_n t - \omega_0 t + \omega_0 t) \] (2.36b)

\[ = x(t) \cos \omega_0 t - y(t) \sin \omega_0 t \] (2.36c)

\[ = r(t) \cos [\omega_0 t + \gamma(t)] \] (2.36d)

\[ r^2(t) = x^2(t) + y^2(t) \] (2.37)

\[ \tan \gamma(t) = y(t)/x(t) \] (2.38)
Sec. 2.3. MATHEMATICAL REPRESENTATION OF NOISE

\[ x(t) = \sum_{n=1}^{N} A_n \cos \left( \omega_n - \omega_o \right) t + B_n \sin \left( \omega_n - \omega_o \right) t \tag{2.39} \]

\[ y(t) = \sum_{n=1}^{N} -B_n \cos \left( \omega_n - \omega_o \right) t + A_n \sin \left( \omega_n - \omega_o \right) t \tag{2.40} \]

\[ x(t) \text{ and } y(t) \text{ in eqns. (2.39) and (2.40) respectively have exactly the same form as } n(t) \text{ in eqn. (2.36a) but with all the Fourier terms shifted downwards in frequency by amount } \omega_o, \text{ where } \omega_o \text{ is some reference frequency. } x(t) \text{ and } y(t) \text{ are therefore random independent gaussian variables with the same statistics as } n(t) \text{ but with each having power spectrum } W_n(f + f_o) \text{ where } W_n(f) \text{ is the power spectrum of } n(t). W_n(f + f_o) \text{ is the power spectrum } W_n(f) \text{ shifted downward in frequency by amount } f_o. \]

The same conclusions about \( x(t) \) and \( y(t) \) may be deduced by using the representation in eqn. (2.17) for \( n(t) \).

\[ n(t) = \sum_{n=1}^{N} C_n \cos \left( \omega_n t - \alpha_n \right) = \sum_{n=1}^{N} C_n \cos \left( \omega_n t - \omega_o t + \omega_o t - \alpha_n \right) \tag{2.41a} \]

\[ = x(t) \cos \omega_o t - y(t) \sin \omega_o t \tag{2.41b} \]

\[ x(t) = \sum_{n=1}^{N} C_n \cos \left( (\omega_n - \omega_o) t - \alpha_n \right) \tag{2.42} \]
Sec. 2.3. MATHEMATICAL REPRESENTATION OF NOISE

\[ y(t) = \sum_{n=1}^{N} c_n \sin \left[ (\omega_n - \omega) t - \alpha_n \right] \]  \hspace{1cm} (2.43)

\( x(t) \) and \( y(t) \) are again seen to have exactly the same form as \( n(t) \), with all frequencies reduced by amount \( f_0 \).

Some of the more important properties of \( x(t) \) and \( y(t) \) are summarised in the following equations, in which the nomenclature is the same as we have been using.

\[ p(x) = \frac{1}{(2\pi)} e^{-x^2/(2N)} \]  \hspace{1cm} (2.44)

\[ p(y) = \frac{1}{(2\pi)} e^{-y^2/(2N)} \]

\[ x(t) = y(t) = n(t) = 0 \]; \[ x^2(t) = y^2(t) = n^2(t) = N \]  \hspace{1cm} (2.45)

The statistics of \( r(t) \) and \( y(t) \) in eqns. (2.36d) through (2.38) are obtained by observing the similarities (as far as the change of variable is concerned) between these equations and eqn. (2.19). By comparison with eqn. (2.22), we therefore write

\[ x(t) = r(t) \cos \gamma(t) \]; \[ y(t) = r(t) \sin \gamma(t) \]  \hspace{1cm} (2.46)

and, since \( x(t) \) and \( y(t) \) are both random gaussian variables with zero mean and mean square value \( N \), going through the steps contained in eqns. (2.23) through (2.33b), enables us to write
where \( p(r) \) and \( p(\gamma) \) are the probability distributions of \( r(t) \) and \( \gamma(t) \) respectively, these being random independent variables. \( r(t) \) is therefore Rayleigh distributed between zero and \( \infty \) and has mean value \( \bar{r}(t) \) and mean square value \( \bar{r}^2(t) \) given by [see eqns. (2.30) and (2.31)]

\[
\bar{r}(t) = (\pi N/2)^{\frac{1}{2}} \quad \bar{r}^2(t) = 2N
\]  

(2.49)

\( \gamma(t) \) from eqn. (2.48) will be uniformly distributed in the range \(-\pi \) to \( +\pi \).

In eqn. (2.36a), we saw that a random gaussian noise voltage \( n(t) \) may be regarded as a sum of sines and cosines, each having probable, rather than actual, amplitude. Eqn. (2.41a) suggests that \( n(t) \) may alternatively be regarded as a sum of cosines, each having a probable amplitude and random phase. For our purposes, however, the most useful concept of \( n(t) \) is obtained from eqn. (2.36d), which indicates that \( n(t) \) is a single cosine whose amplitude \( r(t) \) varies instantaneously with time, between the limits zero to \( \infty \), and whose excess phase \( \gamma(t) \) can instantaneously take on any value in the range \(-\pi \) to \( +\pi \) radians. Excess phase is used to mean the phase in excess of the reference, \( \omega_0 t \). Since \( r(t) \) is the instantaneous amplitude of \( n(t) \),
Sec. 2.4. MATHEMATICAL REPRESENTATION OF NOISE

$r^2(t)/2$ gives the instantaneous power of $n(t)$, and the average power $N$ of $n(t)$ is therefore $r^2(t)/2$, which, as eqn. (2.49) shows, is $N$.

2.4 FM Carrier Plus Random Gaussian Noise. A frequency modulated carrier $v_m(t)$ plus a random gaussian noise voltage $n(t)$ may be expressed as

\[ v_m(t) + n(t) = A \cos[\omega_0 t + \rho(t)] + n(t) \]  
\[ = A \cos[\omega_0 t + \rho(t)] + x(t) \cos \omega_0 t - y(t) \sin \omega_0 t \]  
\[ = A \cos[\omega_0 t + \rho(t)] + x(t) \cos \omega_0 t - y(t) \sin \omega_0 t \]  

where the representation in eqns. (2.36c) and (2.36d) is used here. Comparison of the modulated carrier $v_m(t)$ with eqn. (1.4) shows that the modulating signal $v_s(t)$ is given by

\[ v_s(t) = \frac{d}{dt}[\rho(t)] = \dot{\rho}(t) \]  

where the dot signifies differentiation with respect to time.

Concerning ourselves for a moment with the case of an unmodulated carrier $v(t)$, eqn. (2.50b) reduces to

\[ v(t) + n(t) = A \cos \omega_0 t + x(t) \cos \omega_0 t - y(t) \sin \omega_0 t \]  
\[ = [A + x(t)] \cos \omega_0 t - y(t) \sin \omega_0 t \]
Sec. 2.4. MATHEMATICAL REPRESENTATION OF NOISE

\[ X(t) \cos \omega_0 t - y(t) \sin \omega_0 t \]

(2.52c)

\[ = R(t) \cos [\omega_0 t + \phi(t)] \]

(2.52d)

\[ X(t) = A + x(t) \]

(2.53)

\[ R^2(t) = x^2(t) + y^2(t) \]

(2.54)

\[ \tan \phi(t) = y(t)/x(t) \]

(2.55)

The value of resolving the noise voltage \( n(t) \) into the components \( x(t) \cos \omega_0 t \) and \( -y(t) \sin \omega_0 t \) [eqn. (2.36c)], where \( \omega_0 \) is some reference frequency, now becomes obvious; when an unmodulated carrier of frequency \( \omega_0 \) is present with the noise, the noise may be regarded as comprising a component \( x(t) \) which is in phase with the carrier, and a component \( y(t) \) which is in lead-quadrature [\( \sin \omega_0 t \) lags \( \cos \omega_0 t \), therefore \( -\sin \omega_0 t \) leads \( \cos \omega_0 t \)] with the carrier. The statistics of \( x(t) \) and \( y(t) \), it will be remembered, are the same as those for \( n(t) \), and both \( x(t) \) and \( y(t) \) have power spectrum \( W_n(f + f_0) \) where \( W_n(f) \) is the power spectrum of \( n(t) \). The vector diagram of unmodulated carrier plus random noise is shown in Fig. 2.5a, which illustrates the representation of the noise voltage \( n(t) \) as a single cosine of time varying amplitude \( r(t) \) and time varying excess phase \( \gamma(t) \). The resolution of \( n(t) \) into in-phase and lead-quadrature components, \( x(t) \) and \( y(t) \) respectively, is also shown. The resultant \( v(t) + n(t) \) of unmodulated carrier
plus random noise, from eqn. (2.52a) and the vector diagram, may be interpreted as a single cosine with time varying amplitude \( R(t) \) and time varying excess phase \( \phi(t) \). This resultant therefore resembles an FM wave with carrier amplitude \( R(t) \), carrier frequency \( \omega_c \), and 'information' \( \phi(t) \).

If this resultant wave is put through, first a limiter to remove the amplitude variations, and then a balanced frequency-detector with sensitivity \( K_D \), the output voltage \( n_D(t) \) will be

\[
n_D(t) = K_D \phi(t)
\]  

(2.56)

where \( n_D(t) \) will be a noise voltage, since \( \phi(t) \) is 'produced' entirely by the noise voltage \( n(t) \).

We wish to investigate the statistical properties of \( R(t) \) and \( \phi(t) \), but we return to the case where the carrier is modulated for this. The vector diagram of modulated carrier \( v_m(t) \) plus random noise \( n(t) \) is shown in fig. 2.5b, the relevant equations being eqns. (2.50). The vector diagram suggests various other ways of expressing the sum of modulated carrier plus random noise, two of these being

1) \[
v_m(t) + n(t) = A \cos [\omega_c t + \rho(t)] + x_m(t)\cos[\omega_c t + \rho(t)]
\]

\[
= y_m(t) \sin [\omega_c t + \rho(t)]
\]  

(2.57)

2) \[
v_m(t) + n(t) = [A \cos \rho(t) + x(t)] \cos \omega_c t
\]

\[
- [A \sin \rho(t) + y(t)] \sin \omega_c t
\]  

(2.58a)
Sec. 2.4. MATHEMATICAL REPRESENTATION OF NOISE

\[ x(t) = x_m(t) \cos \omega t - y_m(t) \sin \omega t \quad (2.58b) \]
\[ = R_m(t) \cos [\omega t + \phi_m(t)] \quad (2.58c) \]

where

\[ x_m(t) = x(t) \cos \rho(t) + y(t) \sin \rho(t) \quad (2.59) \]
\[ y_m(t) = y(t) \cos \rho(t) - x(t) \sin \rho(t) \quad (2.60) \]
\[ X_m(t) = A \cos \rho(t) + x(t) \quad (2.61) \]
\[ Y_m(t) = A \sin \rho(t) + y(t) \quad (2.62) \]
\[ R_m^2(t) = X_m^2(t) + Y_m^2(t) \quad (2.63) \]
\[ \tan \phi_m(t) = \frac{Y_m(t)}{X_m(t)} \quad (2.64) \]

These various quantities are shown in the vector diagram, and any of the relations may be obtained by trigonometric manipulation or from the geometry of the vector diagram. Since \( \rho(t) \) is the phase angle due to the modulation and is therefore an independent variable, comparison of eqns. (2.59) and (2.60) with eqns. (2.41) through (2.45) enables us to deduce that \( x_m(t) \) and \( y_m(t) \) are random gaussian variables with zero mean and mean square value \( N \), \( N \) being the mean square value of \( x(t), y(t), \) and \( n(t) \). From eqns. (2.61)
Sec. 2.4. MATHEMATICAL REPRESENTATION OF NOISE

and (2.62), we see that $X_m(t)$ and $Y_m(t)$ are also random gaussian variables. The mean of $X_m(t)$ is $A \cos \rho(t)$ and that of $Y_m(t)$ is $A \sin \rho(t)$, both $X_m(t)$ and $Y_m(t)$ having variance $N$.

For the case of modulated carrier plus random gaussian noise, we again express the resultant as a single cosine, having time varying amplitude $R_m(t)$ and time varying excess phase $\phi_m(t)$ [eqn. (2.58c)]. This wave may again be likened to an FM wave with carrier amplitude $R_m(t)$, carrier frequency $\omega_o$, and 'information' $\dot{\phi_m}(t)$. The output $X_m(t)$ and $Y_m(t)$ of a balanced limiter-discriminator will, in this case, contain both the modulation $\rho(t)$ and noise, since from the vector diagram, say, $\phi_m(t)$ is 'produced' by modulation and noise.

To investigate the statistics of $R_m(t)$ and $\phi_m(t)$, we write eqns. (2.63) and (2.64) in the form

$$X_m(t) = R_m(t) \cos \phi_m(t) ; Y_m(t) = R_m(t) \sin \phi_m(t) \quad (2.65)$$

from which it will be apparent by comparison with eqns. (2.22) and (2.24) that the Jacobian $J$ of the transformation from the variables $X_m(t)$ and $Y_m(t)$ to $R_m(t)$ and $\phi_m(t)$ is

$$J = R_m(t) \quad (2.66)$$

$X_m(t)$ and $Y_m(t)$, we have said, are both random gaussian variables with variance $N$, and mean values $A \cos \rho(t)$ and $A \sin \rho(t)$ respectively. Since $X_m(t)$, $Y_m(t)$, $\rho(t)$ are independent variables, we write
Ecc. 2.4. MATHEMATICAL REPRESENTATION OF NOISE

\[ p(X_m, Y_m, \rho) = p(X_m)p(Y_m)p(\rho) \]  
(2.67a)

\[ = (2\pi N) e^{-\frac{1}{2} - [x_m - A \cos \rho]^2/(2N)} - \frac{1}{2} - [y_m - A \sin \rho]^2/(2N) p(\rho) \]  
(2.67b)

\[ = (2\pi N) e^{-1 - [x_m^2 + y_m^2 + A^2 - 2A(X_m \cos \rho + Y_m \sin \rho)]/(2N)} p(\rho) \]  
(2.67c)

and carrying out the change of variable, we have

\[ p(X_m, Y_m, \rho) \, dx_m \, dy_m \, d\rho \]

\[ = (2\pi N) e^{-1 - [R_m^2 + A^2 - 2AR_m \cos (\phi_m - \rho)]/(2N)} R_m p(\rho) dR_m \, d\phi_m \, d\rho \]  
(2.68a)

\[ = p(R_m, \phi_m, \rho) dR_m \, d\phi_m \, d\rho \]  
(2.68b)

from which

\[ p(R_m, \phi_m, \rho) \]

\[ = (R_m / N) e^{-(R_m^2 + A^2)/(2N)} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} AR_m \cos (\phi_m - \rho) \end{pmatrix} / N p(\rho) \]  
(2.69)

The probability distribution \( p(R_m) \) of \( R_m(t) \) is found by
integrating eqn. (2.69) over the ranges of \( \rho(t) \) and \( \phi_m(t) \) which will be \(-\pi\) to \(+\pi\) in both cases. Thus

\[
p(R_m) = \left( \frac{R_m}{N} \right) e^{-\left( \frac{R_m^2 + A^2}{2N} \right)} \int_{-\pi}^{+\pi} \{ p(\rho) \, d\rho \}.
\]

If the expression inside the brackets containing the integral in eqn. (2.70) is denoted \( A' \), then, bearing in mind that \( \rho(t) \) does not depend on \( \phi_m(t) \), the substitution

\[
x = \phi_m - \rho, \quad dx/d\phi_m = 1
\]

gives

\[
A' = (2\pi) \int_{-\pi}^{+\pi} e^{-\left( \frac{AR_m}{N} \right) \cos x} \, dx
\]

\[
= (2\pi) \int_{0}^{2\pi} e^{-\left( \frac{AR_m}{N} \right) \cos x} \, dx = I_0 \left( \frac{AR_m}{N} \right).
\]

The limits of the integration in eqn. (2.72a) can be changed from \(-\pi - \rho\) and \(+\pi - \rho\) to zero and \(2\pi\) as in eqn. (2.72b), since \( \cos x \) has the same periodicity in both ranges. \( I_0 \left( \frac{AR_m}{N} \right) \) is the modified Bessel function of the first kind and zero order, and eqn. (2.72b) is in fact a defining equation for this function (ref. 33, sec. 9.6.16).
The expression denoted by \( A \) is thus seen to be independent of \( \rho \) and may therefore be taken outside the integration with respect to \( \rho \) in eqn. (2.70), leaving the integral between \(-\pi\) and \(+\pi\) of \( p(\rho) d\rho \). The value of this integral is unity and \( p(R_m) \) becomes

\[
p(R_m) = (R_m/N) e^{-(R_m^2 + A^2)/(2N)} I_0(AR_m/N) \tag{2.73}
\]

Eqn. (2.73) reveals that the probability distribution \( p(R_m) \) of \( R_m(t) \) does not depend on the modulation \( \dot{\rho}(t) \). This may have been deduced by inspection of the vector diagram in fig. 2.5b from which it can be argued that the statistics of the amplitude function \( R_m(t) \) will not depend on the nature or presence of the phase angle \( \rho(t) \). Hence for modulated and unmodulated carrier, eqn. (2.73) gives the probability distribution of the amplitude of the resultant vector. From the property that \( I_0(0) = 1 \), eqn. (2.73) reduces to eqn. (2.47) when \( A = 0 \) (carrier absent), and \( R_m(t) \) becomes \( r(t) \).

The probability function in eqn. (2.73) has been investigated by Rice (ref. 2, sec. 3.10) in his treatment of unmodulated carrier plus noise, and he obtains the mean square value \( R_m^2(t) \) given by

\[
R_m^2(t) = \int_0^\infty R_m^2 p(R_m) dR_m = A^2 + 2N \tag{2.74}
\]

It will be noticed that this result gives
\[ R_m(t) = A^2, \quad N = 0 \]  
\[ = 2N, \quad A = 0 \]

which is to be expected, since \( R_m(t) \) is the amplitude of the resultant of carrier plus noise. The mean power associated with the resultant is thus \( R_m(t)/2 \). When only carrier is present \( (N = 0) \), the mean power of the resultant is \( A^2/2 \), and when only noise is present \( (A = 0) \), the mean power of the resultant is \( N \). When both carrier and noise are present, their mean powers add, and we observe too that the ratio of mean carrier power to mean noise power is \( A^2/(2N) \) as expected.

The probability distribution \( p(R_m) \) is best studied by changing (after Rice) to the variables

\[ v = R_m(N)^{-\frac{1}{2}}; \quad dv = dR_m(N)^{-\frac{1}{2}} \]

\[ a = A(N)^{-\frac{1}{2}}; \quad p(R_m)dR_m = p(v)dv \quad (2.76) \]

The new variable \( v \) has probability distribution given by

\[ p(v) = \frac{-(v^2 + a^2)/2}{I_0(\alpha v)} \quad (2.77) \]

and this probability distribution is reproduced from Rice, in fig. 2.6, for various values of the parameter \( a \). \( a \) may also be expressed as

\[ a = [2(CNR)]^{\frac{1}{2}} \quad ; \quad a^2 = 2(CNR) \quad (2.78) \]
where \((\text{CNR})\) is the ratio \(A^2/(2N)\) of mean carrier power to mean noise power. When \(a = 0\) [i.e., \(A = 0\) and the carrier is absent], \(p(v)\) gives the Rayleigh distribution as expected. Fig. 2.6 indicates that as \(a\) increases, \(p(v)\) tends to a gaussian distribution, and this may be shown to be so by using the asymptotic expression for \(I_0(\alpha v)\), as \(\alpha v\) tends to \(\infty\).

\[
I_0(\alpha v) = e^{-\frac{\alpha v}{2}}, \quad \alpha v \ll 1
\]  

\[\text{(2.79)}\]

Hence for \(\alpha v = (A^2/N)\) large, \(p(v)\) in eqn. (2.77) reduces to

\[
p(v) = v(2\alpha v) e^{-\frac{(v-a)^2}{2}}
\]

\[\text{(2.80)}\]

For large \(\alpha v\), \(N\), the mean noise power, must be small compared with \(A\). \(r(t)\) must therefore be small compared with \(A\). (t), and from the vector diagram in Fig. 2.5b, it is seen that when this is so, \(A\) and \(R_m\) are nearly equal. Assuming then, that in the limit, as \(\alpha v\) tends to \(\infty\), \(R_m\) and \(A\) are equal, we obtain that \(a\) and \(v\) in eqn. (2.76) must also be equal. \(p(v)\) in eqn. (2.80) and \(\alpha v\) then become

\[
p(v) = \left(2\pi\right) e^{-\frac{(v-a)^2}{2}}
\]

\[\text{(2.81)}\]

\[
\alpha v = A^2/N = 2(\text{CNR})
\]

\[\text{(2.32)}\]

Eqn. (2.81), by comparison with eqn. (2.4), is seen to define a gaussian distribution, but since we are more interested in the properties of \(R_m(t)\), we obtain, through the transformations in eqn. (2.76), that
This shows that when \( av \) [and therefore \((\text{CNR})\)] is large, \( R_m(t) \) is a random gaussian variable having mean value \( A \), mean squared value \( A^2 + 2N \), and variance or mean a.c. power \( N \). These conclusions may be interpreted qualitatively in the vector diagram of fig. 2.5b as follows: \( r(t) \), the amplitude of the noise vector \( n(t) \), we will remember, is Rayleigh distributed in the range zero to \( \omega \), and \( \gamma(t) \), the excess-phase angle of \( n(t) \), is uniformly distributed in the range \(-\pi\) to \(+\pi\). Thus \( r(t) \) can have any length between zero and \( \infty \) and may point with equal probability in any direction. However, when the ratio \((\text{CNR})\) of mean carrier power to mean noise power is large, \( r(t) \) is most of the time much less than \( A \), and the noise constitutes a small random perturbation about the tip of the carrier vector. The 'length' \( R_m(t) \) then has a mean value of \( A \) and varies randomly on either side of this mean. The symmetry of the variations suggests a gaussian distribution for \( R_m(t) \), provided, of course, \( r(t) \) is small compared with \( A \). The fact that \( r(t) \) may, at any instant, become very large, results in a departure from the exact gaussian distribution, and it is only in the limit as \((\text{CNR})\) tends to \( \infty \) that the distribution is gaussian [eqn. (2.83)]. However, from fig. 2.6 it will be seen that for \( \text{CNR} = 2 \) \((a = 2 = [2(\text{CNR})]^{1/2})\), the distribution is already nearly gaussian.

To obtain the probability distribution \( p(\phi_m) \) of the excess-phase angle \( \phi_m(t) \) of the resultant of modulated
carrier plus noise, we must return to eqn. (2.69), which gives the joint probability distribution \( p(R_m, \phi_m, \rho) \) of the variables \( R_m(t) \), \( \phi_m(t) \) and \( \rho(t) \), and integrate over the ranges of \( R_m \) and \( \rho \). \( R_m(t) \) ranges from zero to \( \infty \) and \( \rho(t) \) from \(-\pi\) to \( +\pi\); thus

\[
p(\phi_m) = \int_{-\pi}^{+\pi} \{p(\rho)d\rho\}.
\]

\[
(2\pi N)^{-1} \int_0^\infty \frac{e^{-\frac{2}{\lambda} - 2AR_m \cos(\phi_m - \rho)}}{(2N)} dR_m
\]

(2.84)

The result of the integration with respect to \( R_m \) is, in general, a function of \( \rho \) and therefore cannot be regarded as being outside the integration with respect to \( \rho \). Thus, unless the probability distribution of \( \rho \), \( p(\rho) \), is known, \( p(\phi_m) \) cannot be evaluated. Experiments with various probability distributions \( p(\rho) \) did not lead to any interesting results, and we consider then the case when modulation is absent. \( \rho \) then drops out of eqn. (2.84), and \( \phi_m \) reverts to \( \phi \), the value for unmodulated carrier. We then obtain, from eqn. (2.84), that

\[
p(\phi) = (2\pi N)^{-1} \int_0^\infty \frac{e^{-\frac{2}{\lambda} - 2AR_m \cos \phi}}{(2N)} dR_m
\]

(2.85)

The result of this integration (ref. 29, sec. 3.462) cannot be easily expressed in terms of elementary functions. However, if we assume, as before, that the carrier to noise power ratio (CNR) is large, so that the amplitude \( r(t) \) of the
noise vector is most of the time much less than the carrier amplitude \(A\), then from the vector diagram in fig. 2.5a, it will be seen that \(\phi(t)\) is most of the time small. Thus \(\cos \phi = 1\), and eqn. (2.85) reduces to (ref. 22, chap. 7)

\[
p(\phi) = (2\pi N')^{-1} e^{-\frac{1}{2} \phi^2/(2N')}, \quad -\pi \leq \phi \leq +\pi
\]

(2.86)

\[
N' = [2(CNR)]^{-1}
\]

(2.87)

which gives a gaussian distribution with zero mean and mean square value \([2(CNR)]^{-1}\). The gaussian distribution of \(\phi(t)\) for large \((CNR)\) is plausible from the vector diagram in fig. 2.5a, since for large \((CNR)\), \(r(t)\), as we have said, is most of the time much less than \(A\), and the noise constitutes a random perturbation about the tip of the carrier vector. \(\phi(t)\) will then fluctuate randomly about \(\phi(t) = 0\), the symmetry of the fluctuations suggesting a gaussian distribution. Indeed the situation is exactly analogous to the one in which \(R_m(t)\), the amplitude of the resultant of modulated carrier plus noise, had gaussian distribution for large \((CNR)\). For that case the distribution was in fact seen to be gaussian (fig. 2.6) for \((CNR)\) values as low as 2, and it is therefore expected that \(p(\phi)\) will retain an approximately gaussian distribution for \((CNR)\) values as low as this.

2.5 Band-limited Noise. Noise Bandwidth. So far in this chapter we have been discussing a noise voltage \(n(t)\), having any power spectrum \(W_n(f)\). We saw in sec. 1.3, however, that
in FM reception, the noise with which the system had to contend was band-limited or narrowband; that is, it had been passed through an IF filter and therefore only had frequency components in some finite bandwidth. We will then consider the properties of this noise. Let us assume that noise with power spectrum $W_n(f)$ is passed through an IF filter with frequency response $G(j\omega)$, which we will always assume to be symmetrical about the IF, $f_o$. For our purposes, $W_n(f)$ will always be wideband and have substantially constant level, $W_0$ watts/Hz, in the vicinity of $f_o$. Thus

$$W_n(f) = W_0$$

(2.88)

at the output of the IF filter, the power spectrum $W_0(f)$ of the noise will therefore be (see appendix C)

$$W_0(f) = W_0|G(j\omega)|^2$$

(2.89)

The mean noise power $N_o$ at the output of the IF filter is

$$N_o = \int_0^\infty W_0(f)df = W_0 \int_0^\infty |G(j\omega)|^2df = W_0 B_n$$

(2.90)

$$B_n = \int_0^\infty |G(j\omega)|^2df$$

(2.91)

$B_n$ is called the noise bandwidth or 'equivalent rectangular bandwidth' of the filter, the latter name being derived from the fact that a rectangular filter (as in sec. 1.3 and fig. 1.1)
with bandwidth $B_n$ would give the same output noise power as the filter under consideration. The noise bandwidth of some common filters is given in Table 3.1.

Random gaussian noise, after passing through the IF filter, will still be random and gaussian. This may be demonstrated by representing $G(j\omega)$ at a frequency $\omega_n$ as

$$G(j\omega_n) = k_n e^{j\lambda_n}$$

where $k_n$ and $\lambda_n$ are the modulus and phase respectively of $G(j\omega)$ at frequency $\omega_n$. If the noise input $n(t)$ to the filter is represented as in Eqn. (2.17), then the noise output $n_0(t)$ is

$$n_0(t) = \sum_{n=1}^{N} k_n C_n \cos (\omega_n t - \alpha_n + \lambda_n)$$

Hence $n_0(t)$, like the input $n(t)$, still comprises a large number of independent variables and will therefore be random and gaussian in consequence of the central-limit theorem. This means that all the statistical properties of the variables we have been considering in the previous sections remain unchanged when the noise is band-limited, but their frequency characteristics will change. For example, when noise is wideband, $\omega_n$ in Eqn. (2.36a) covers a large range of frequencies, and $x(t)$ and $y(t)$ in Eqns. (2.39) and (2.40) will comprise high-frequency components. ($\omega_o$ may here be thought of as the centre frequency of a large-bandwidth filter).

When however, the noise is band-limited, all the components
of the noise have frequencies near to \( \omega_0 \); that is, \( \omega_n \) is always near to \( \omega_0 \), and \( x(t) \) and \( y(t) \) then become low-frequency variables. \( x(t) \) and \( y(t) \) we saw, had respective power spectra \( W_x(f) \) and \( W_y(f) \) given by

\[
W_x(f) = W_y(f) = W_n(f + f_0)
\]  

(2.94)

where \( W_n(f + f_0) \) is the power spectrum \( W_n(f) \) of \( n(t) \) shifted down in frequency by amount \( f_0 \). If, as we have said, the IF noise power spectrum is symmetrical and centered on \( f_0 \), \( W_x(f) \) and \( W_y(f) \) will also be symmetrical but will be centered on zero frequency. Thus \( W_x(f) \) and \( W_y(f) \) in eqn. (2.94) are low-frequency, double-sided (positive and negative frequencies) and even-functioned distributions when \( n(t) \), having power spectrum \( W_0(f) \), is band-limited. In this case then, we may express \( W_x(f) \) and \( W_y(f) \) as

\[
W_x(f) = W_y(f) = 2W_0(f + f_0), \quad f > 0
\]  

(2.95)

In equation (2.36d), we saw that noise could be regarded as a single cosine with instantaneous amplitude \( r(t) \) and instantaneous phase \( \omega_0 t + \gamma(t) \). When \( x(t) \) and \( y(t) \) contain high-frequency components, \( r(t) \) and \( \gamma(t) \) will also contain high-frequency components [eqns. (2.37) and (2.38)], and wideband noise can be regarded as a single cosine whose instantaneous amplitude and instantaneous excess phase (above \( \omega_0 t \)) may vary rapidly with time. If \( \gamma(t) \) varies rapidly with time, \( \dot{\gamma}(t) \) will be large, and the instantaneous frequency \( [\omega_0 + \dot{\gamma}(t)] \) of the noise may therefore be large compared
When noise is band-limited, \( x(t) \), \( y(t) \), and therefore \( r(t) \) and \( \gamma(t) \), contain only low frequencies and band-limited noise may therefore be regarded as a single cosine whose instantaneous amplitude \( r(t) \) and instantaneous excess phase \( \gamma(t) \) vary only slowly with time. The instantaneous frequency \( \omega_o + \gamma(t) \) of band-limited noise will also be always near \( \omega_o \). These points are substantiated by reference to figs. 2.7a and 2.7b which show oscillograms of wideband and band-limited noise respectively. The oscillogram in fig. 2.7a shows the wideband noise output of a photomultiplier tube (appendix A.2). The level of the power spectrum of this noise voltage was found by measurement to be constant from d.c. up to about 120 MHz. Fig. 2.7b shows the band-limited noise output of a single pole IF filter (appendix B.1), having centre-frequency \( f_0 = 1.75 \) MHz and 3db-bandwidth 100 KHz.

When an RF carrier (modulated or unmodulated) of centre frequency \( \omega_o \) is present with band-limited noise centered on \( \omega_o \), \( x(t) \) and \( y(t) \) may be regarded as random low-frequency voltages in phase and in lead-quadrature respectively with the unmodulated carrier. All the variables with which we have been dealing in previous sections, whose time-varying properties depend on the time-varying properties of \( x(t) \) and \( y(t) \), must then be regarded as slowly-varying or low-frequency variables. Thus, for example, all the variables in eqns. (2.59) to (2.64) are low-frequency random variables, whether or not modulation, \( \rho(t) \), which is low-frequency, is present.
FIG. 2.1. Ensemble Variables.

FIG. 2.2. Random Gaussian Noise Voltage.

FIG. 2.3. The Gaussian Distribution.

FIG. 2.4. a. The Rayleigh Distribution
   b. The Uniform Distribution
FIG. 2.5. Vector Diagrams of Carrier Plus Noise
   a. Carrier Unmodulated,  b. Carrier Modulated
FIG. 2.6. Probability Distribution of $R_m(t)$

Vert. 50mV/cm; Horiz. 5μs/cm

FIG. 2.7. Oscillograms of Random Gaussian Noise

a. Wideband Noise

b. Bandlimited Noise
CHAPTER 3

FM NOISE CHARACTERISTICS

3.1 Introduction. The frequency spectrum of noise at the output of an FM receiver has been investigated by Rice, Middleton, Blachman, and Lawson and Uhlenbeck (ref. 3,4,6,5). These authors used correlation methods which involve very formidable mathematics, and indeed much of their analyses was carried out using numerical methods. These authors also confined themselves to considering only the case where the carrier is unmodulated. Stumpers (ref. 1), using a 'zero-crossing' method, has also obtained the output spectrum of FM noise, and he treats both the cases of unmodulated and modulated carrier. The zero-crossing analysis is even more mathematically complex than the correlation method, and it involves various approximations. Stumpers presents his result as a double-summation (appendix F) series for the case of unmodulated carrier, and as a triple-summation series for the case of modulated carrier, both series involving higher transcendental functions. Using an approximate method, Stumpers evaluates the double-summation series, and he thus obtains the output spectrum of the noise for various values of carrier to noise power ratio (CNR), when the carrier is unmodulated. He does not attempt to evaluate the triple-summation series.
Correlation and zero-crossing methods are of little use in helping to give an insight into the behaviour of FM systems under threshold and feedback conditions, but the results of these methods serve as excellent standards against which the results of a somewhat more simplified analysis may be compared. This simplified analysis represents the spectrum $W_D(f)$ of the output noise as the sum of two components. Thus

$$W_D(f) = W_1(f) + W_2(f)$$  \hspace{1cm} (3.1)

This two-term approximation has been used by Rice (ref. 21), who was led to this approach from observations on the behaviour of the output noise.

In this chapter, we consider some extensions of Rice's results, and discuss the significance and applications of the two-term approximation.

3.2. Two-term Approximation of FM Noise Spectrum. It has already been mentioned in sec. 1.4 that when an FM receiver is operating above threshold, i.e., with large carrier to noise power ratio (CNR), the output noise is fluctuation in nature. Crosby (ref. 7) discovered that the spectrum of this type of noise was characteristically parabolic, approaching zero level at zero frequency ($f = 0$). As (CNR) was decreased, the intensity of the fluctuation noise was found to increase, as did the level of the parabolic spectrum. A point was eventually reached where further decrease in (CNR) was found to introduce a clicking or popping-like type of noise. The output
noise at this stage was observed to contain large impulses which accounted for the pops; and examination of the spectrum revealed that a component having non-zero value at $f = 0$ had been introduced. Indeed this new component had an almost flat distribution, tending then to shift the parabolic component upward by a constant amount. This is illustrated in figs. 3.1a and 3.1b, which show oscillograms of FM noise and the corresponding measured spectra, for a) CNR = 10 and b) CNR = 5. For these measurements (see appendix A), the IF filter consisted of a single-pole filter centered on 1.75 MHz with 3db-bandwidth 100 KHz, and the carrier was unmodulated. The lower trace in each of the photographs shows the output of the IF filter. (This output is seen to have a slowly-varying amplitude, and on a different timebase, the frequency would have been seen to be nearly 1.75 MHz for most of the time. This is in fact the wave $v(t) + n_o(t) = R(t) \cos [\omega_o t + \phi(t)]$ dealt with in chap. 2). In fig. 3.1a the output noise is seen to be mainly fluctuation noise, and in fig. 3.1b the output is seen to contain large impulses. This behaviour of the noise lead Rice (ref. 21) to resolve the output noise $n_o(t)$ into two components, namely: 1) a fluctuation component $n_1(t)$, having power spectrum $W_1(f)$, and 2) an impulsive component $n_2(t)$, having power spectrum $W_2(f)$.

Rice (ref. 21) uses a well known approximation for the part $W_1(f)$ of the output spectrum, but closer investigation of $W_1(f)$ will reveal one of the fundamental properties of FM systems.

Returning to sec. 2.4 [eqn. (2.52d)] and fig. 2.5a, it will be remembered that the total phase angle of the resultant
of unmodulated carrier plus gaussian noise is \( \omega_0 t + \phi(t) \), where

\[
\phi(t) = \tan^{-1} \left\{ \frac{y(t)}{A + x(t)} \right\}
\]  

(3.2)

\( x(t) \) and \( y(t) \), the noise voltages in phase and in lead-quadrature respectively with the carrier voltage, have been discussed in sec. 2.4. Recalling that

\[
\tan^{-1} x = x - \left( \frac{x^3}{3} \right) + \left( \frac{x^5}{5} \right) \ldots \quad , \quad x^2 \leq 1 \]  

(3.3)

we are led to restricting the noise amplitude \( r(t) \) in eqn. (2.36d) and fig. 2.5a to having only values less than or equal to \( A \), the carrier amplitude. When this is done, it will be seen from eqn. (2.46) and fig 2.5a that whatever value the random phase angle \( \gamma(t) \) has, the argument of \( \tan^{-1} \) in eqn. (3.2) is less than one. The 'worst' case occurs when instantaneously \( \gamma(t) = \pi/2 \) and at the same time \( r(t) = A \); then, \( x(t) = 0, y(t) = A \), and the argument of \( \tan^{-1} \) becomes unity.

For all \( r(t) \leq \frac{A}{\sqrt{2}} \), we may therefore use the expansion in eqn. (3.3): thus

\[
\phi(t) = \left\{ \frac{y(t)}{A + x(t)} \right\} - \left( \frac{1}{3} \right) \left\{ \frac{y(t)}{A + x(t)} \right\}^3 + \ldots , \quad r(t) \leq \frac{A}{\sqrt{2}}
\]  

(3.4)

Eqn. (3.4) is exact, but if we now make the approximation that for \( r(t) \leq A \), \( x(t) \) and \( y(t) \) are most often small compared
with $A$, eqn. (3.4) simplifies to

$$\phi(t) = \{y(t)/(A + x(t))\}$$  \hspace{1cm} (3.5a)

$$\approx y(t)/A$$  \hspace{1cm} (3.5b)

Considering again the 'worst' case when $x(t) = 0$ and $y(t) = A$, the 2nd. and 3rd. terms in eqn. (3.4) are 33% and 20% respectively of the leading term. Even small improvements on the 'worst' case will bring about a rapid reduction in the contribution from these and higher order terms, because of the powers involved.

The significance of restricting the noise amplitude $r(t)$ to having values less than or equal to the carrier amplitude $A$ is seen from the vector diagram in fig. 2.5a. As $y(t)$ varies randomly, the noise vector may point in any direction, but the amplitude $r(t)$ must lie within the circle whose centre is the tip of the carrier vector and whose radius is the length $A$. For most of the time, the approximation in eqn. (3.5b) may be visualized to be true, and from this equation $\phi(t)$ is seen to be the variable $y(t)$ reduced in instantaneous value by factor $A$. Since $y(t)$ is a random gaussian variable with zero mean and mean square value $N_0$, where $N_0$ is the mean power of the IF noise voltage, $\phi(t)$ is also a random gaussian variable with zero mean and mean square value $N_0/A^2 \{= 1/[2(CNR)]\}$, $(CNR) = A^2/(2N_0)$. This is the result given in eqn. (2.86), and at that stage we implied that the result was valid for $(CNR)$ values as low as 2. It may therefore be expected that eqn. (3.5b) will
be a valid approximation for the excess-phase angle $\phi(t)$ when (CNR) is as low as 2.

The random gaussian fluctuation of $\phi(t)$ gives rise to the fluctuation component $v_1(t)$ in the output noise. If $K_D$ is the sensitivity (in volts/rad. freq.) of a balanced limiter-discriminator, $v_1(t)$ is given by

$$v_1(t) = K_D \dot{\phi}(t) = (K_D/A)\dot{y}(t) \tag{3.6}$$

The spectrum $W_1(f)$ of $v_1(t)$ is obtained by remarking that the discriminator acts as a phase differentiator (see also sec. 1.3) and therefore has transfer function $G_D(s) = s$. Since $y(t)$ has power spectrum $2W_0(f + f_0)$, where $W_0(f)$ is the IF noise spectrum [see eqn. (2.95)], from appendix C we obtain

$$W_1(f) = (K_D/A)^2 |j\omega|^2 2W_0(f + f_0) = (K_D/A)^2 \omega^2 2W_0(f + f_0) \tag{3.7a}$$

$$= 2\pi f^2 K_D^2 f^2 W_0(f + f_0)/A^2 , \quad f \geq 0 \tag{3.7b}$$

Since $W_0(f)$, the spectrum of the noise at the output of the IF filter, will be flat in the vicinity of $f_0$, the shifted-down version, $W_0(f + f_0)$, will be flat in the vicinity of zero frequency, and $W_1(f)$ in eqn. (3.7b) will vary as $f^2$ for low frequencies. This gives the well known parabolic spectrum we have met before.
The result in eqn. (3.7b) may be extracted from the classical analyses of Rice and Stumpers (ref. 1,3), for example. Both these results, however, contain a multiplying factor \([1 - e^{-(\text{CNR})}]^2\). To account for this factor, we return to eqn. (3.4) which gives an expression for \(\phi(t)\).

This expression is not true for all of the time, but only for the fraction of the total time, \(p(r \leq A)\), when \(r(t) \leq A\). We then say eqn. (3.4) has statistical probability \(p(r \leq A)\) of being true. The probability that the equation is true may be introduced into the equation by multiplying all the right-hand terms by \(p(r \leq A)\). To obtain the probability \(p(r \leq A)\) that \(r(t) \leq A\), we recall from eqn. (2.47) that \(r(t)\) is Rayleigh-distributed. Thus

\[
p(r \leq A) = \int_0^A \frac{r^2}{2N_0} e^{-\frac{r^2}{2N_0}} - \frac{A^2}{2N_0} e^{-\frac{A^2}{2N_0}} \text{d}r = 1 - e^{-\frac{A^2}{2N_0}} = 1 - e^{-\text{(CNR)}}
\]

(3.8)

where \(N_0\) is the mean IF noise power, and (CNR) is the IF carrier to noise power ratio. Eqns. (3.5b) and (3.6), for example, would contain this multiplying factor, \([1 - e^{-(\text{CNR})}]\), on the right-hand side, and \(W_1(f)\) in eqns. (3.7) would contain the multiplying factor \([1 - e^{-(\text{CNR})}]^2\). Thus

\[
W_1(f) = 8\pi K_D \frac{2}{[(1 - e^{-(\text{CNR})})^2]^2} \left[\pm W_0(f + f_o) / A\right]^2
\]

(3.9)

The impulsive component in the output noise of the FM receiver arises only when \(r(t) > A\). Under this condition, the noise vector in fig. 2.5a may again point in any
direction, but its amplitude \( r(t) \) must now lie outside the circle whose centre is the tip of the carrier vector and whose radius is \( A \). When this condition prevails, it is possible for the noise vector to very quickly sweep round in an approximate circle so that \( \gamma(t) \) very quickly changes by approximately ± 2\( \pi \). When this occurs, since \( r(t) > A \), \( \phi(t) \) will also change quickly by approximately ± 2\( \pi \). An impulse will therefore arise in the output \( \phi(t) \) as illustrated in fig. 3.2. The area under each of these impulses will approximately be 2\( \pi \). The positive impulses are produced when \( \phi(t) \) increases positively through 2\( \pi \), the negative ones being produced as \( \phi(t) \) increases negatively through 2\( \pi \).

When \( r(t) \leq A \), \( \gamma(t) \) may of course change very quickly by ± 2\( \pi \), but in this case the noise vector cuts the horizontal axis in the vector diagram of fig. 2.5a to the right of the origin, as it sweeps round in a circle. \( \phi(t) \) does not change unidirectionally through 2\( \pi \) (as in the case where \( r(t) > A \)), but performs a fluctuation instead. This is shown in fig. 3.2b where the shaded portion shows the behaviour of \( \phi(t) \) as \( \gamma(t) \), starting at zero, increases quickly through + 2\( \pi \). \( \phi(t) \) will similarly be a fluctuation, the area under \( \phi(t) \) \( [= \phi(t) \] being approximately zero for this event. A sudden change through ± 2\( \pi \) in \( \gamma(t) \) therefore contributes little d.c. in the output and therefore little low frequency noise. It does, however, contribute to the fluctuation noise \( n_{\perp}(t) \).

The occurrence of impulse noise disturbances in FM was mentioned as early as 1941 by Landon (ref. 19) and treated.
subsequently by several authors, the most notable of whom are Cohn and Rice (ref. 20, 21). These two authors obtain the result that, for an unmodulated carrier, the number \( n \) of impulses which occur per second is given by

\[
n = r \left[ 1 - \text{erf}(\text{CNR})^{\frac{1}{2}} \right] = r \text{cerf}(\text{CNR})^{\frac{1}{2}}
\]

(3.10)

\[
r = (2\pi)^{-\frac{1}{2}} \left( \frac{b_2}{b_0} \right)^{\frac{1}{2}}
\]

(3.11)

\[
b_n = (2\pi)^n \int_{-\infty}^{\infty} W_0(f) \left[ f - f_0 \right]^n df
\]

(3.12)

where \( W_0(f) \) is the power spectrum of the noise at the output of the IF filter. \( \text{erf} \) and \( \text{cerf} \) are the error function and complementary error function respectively.

\[
\text{erf} x = 2(\pi)^{-\frac{1}{2}} \int_0^x e^{-t^2} dt = 1 - 2(\pi)^{-\frac{1}{2}} \int_x^{\infty} e^{-t^2} dt
\]

(3.13)

The evaluation of this function is dealt with in appendix I.1. \( r \) will be seen to depend only on the shape of the IF noise spectrum \( W_0(f) \), which, from eqn. (2.39), depends only on the IF filter-shape. \( r \) may be regarded as the 'radius of gyration' of \( W_0(f) \) about the axis \( f = f_0 \). Values of \( r \) for some common IF filter-shapes are given in table 3.1.

Rice (ref. 21) conjectures that when the carrier to noise power ratio (CNR) is large, the impulses in the output occur randomly and independently of each other, and there are as many positive as negative impulses. The impulses at the
output may then be resolved into two sequences, one containing positive impulses, the other containing negative impulses. Each of these sequences is identical to a shot-noise type current. The mean of these two combined currents is zero, and the power spectrum of the combined currents is the sum of the individual power spectra. The power spectrum of each current may be deduced in exactly the same way it is done for shot noise. Thus, it is obtained that the power spectrum \( W_2(f) \) of the impulsive noise current \( n_2(t) \) in the output is

\[
W_2(f) = K_D \frac{2 \pi^2 n^2}{n} = K_D \frac{2 \pi^2}{r} \text{erf} \left( \frac{c}{\sqrt{R}} \right) \quad (3.14)
\]

which, like the power spectrum of shot noise, is flat, for low frequencies anyway. \( W_2(f) \), again like the power spectrum of shot noise, will 'tail' off at high frequencies due to the non-idealness of the impulses which make up the currents.

The two-term approximation to the FM noise spectrum we may now express as

\[
W_D(f) = W_1(f) + W_2(f) \quad (3.15a)
\]

\[
= 8 \pi K_D f \left[1 - e^{-\frac{(CNR)^2}{2}}\right] w_o(f + f_o)/A + K_D \frac{2 \pi^2}{r} \text{erf} \left( \frac{c}{\sqrt{R}} \right) \quad (3.15b)
\]

One way of regarding equations (3.15) is to take eqn. (3.15a) as exact and eqn. (3.15b) as approximate. \( W_1(f) \) and \( W_2(f) \) may then be regarded as partial spectra (like partial
pressures), \( W_1(f) \) accounting for the total spectrum when \( r(t) \leq A \), and \( W_2(f) \) accounting for the total spectrum when \( r(t) > A \). The two terms in eqn. (3.15b) are then the dominant terms in \( W_1(f) \) and \( W_2(f) \) for large (CNR). From eqn. (3.8), table 3.2, which gives the probability \( p(r \leq A) \) that \( r(t) \leq A \), may be drawn up. From this table it is seen that for (CNR) = 2, \( r(t) \leq A \) for 86% of the time, and the argument in sec. 2.4 that \( \phi(t) \) in fig. 2.5a has gaussian distribution for (CNR) values as low as this, is hereby strengthened. This makes eqn. (3.5b) nearly exact, and the 1st. term in eqn. (3.15b) is substantially the spectrum which arises when \( r(t) \leq A \). For (CNR) = 2, \( r(t) > A \) for only 14% of the time, and the random fluctuation of the noise vector for \( r(t) > A \) is not likely to manifest itself in much output power; only the sudden 'energetic' circular sweeps. The second term in eqn. (3.15b) therefore gives the major contribution to the entire power spectrum for \( r(t) > A \).

The range of (CNR) values for which eqn. (3.15b) is valid may in fact be established by comparing numeric evaluation of eqn. (3.15b) with numerical results obtained from the classical treatment of other authors. To express eqn. (3.15b) in a more convenient form, we write, from eqn. (2.39)

\[
W_0(f + f_0) = W_0 |G(j\omega + j\omega_0)|^2 = W_0 |G_{LP}(j\omega)|^2
\]

(3.16)

where \( W_0 \) is the level (assumed constant) of the power spectrum of the noise at the input to the IF filter. \( G(j\omega + j\omega_0) \) is the frequency response \( G(j\omega) \) of IF filter.
shifted down in frequency by amount \( \omega_0 \). Since \( G(j\omega) \) is bandpass in nature and centered on \( \omega_0 \), \( G(j\omega + j\omega_0) \) will be centered on zero frequency, and we then refer to \( G(j\omega + j\omega_0) \), both here and subsequently, as the low-pass analogue of the IF filter. It is also convenient to write \( G_{LP}(j\omega) \) for \( G(j\omega + j\omega_0) \) as in eqn. (3.16). From eqn. (2.90), we also express the carrier to noise power ratio (CNR) as

\[
(CNR) = \frac{2}{(2N_0)} = \frac{2}{(2\omega_0 B_n)}
\]  

(3.17)

where \( B_n \), the noise bandwidth of the IF filter, is defined in eqns. (2.90) and (2.91). Finally, by writing

\[
r = \frac{B_n}{a}
\]

(3.18)

where \( a \) is just a ratio, defined by eqn. (3.18), we are able to express eqn. (3.15b) as

\[
W_D(f)/(4\pi B_n K_D) = \frac{f}{B_n} [1-e^{-\text{erf} (CNR)^2}] (CNR)^{-1} |G_{LP}(j\omega)|^2 + (2\pi/a) \text{erf} (CNR)^\frac{1}{2}
\]

(3.19)

Table 3.3 shows the comparison of the numerical evaluation of eqn. (3.19) with the tabulated results of Rice, Stumpers, and Lawson and Uhlenbeck (ref. 3,1,5). The reduction of the results of these authors to the form given in eqn. (3.19) is given in appendix E. \( G(j\omega) \) and \( a \) are obtained from table 3.1 for the various filters.
Table 3.3 gives a clear indication that for (CNR) values as low as 1.5, (we had said 2), the agreement between eqn. (3.19) and more rigorous analyses is excellent and is acceptable for (CNR) as low as 1. We say this when it is remarked from table 3.3 that the agreement between eqn. (3.19) and the other results is as good as the agreement between the other results themselves. The difficulties which arise in evaluating results for the more rigorous work can be illustrated by Stumpers' (ref. 1) results. Stumpers' double-summation result for unmodulated carrier is dealt with in an appendix F. Stumpers' technique for evaluating the summation for a given (CNR), is to evaluate the first 10 terms \((k = 1, 2 \ldots 10)\) and to estimate the rest. A check on the accuracy of this method was obtained by programming the series for evaluation on a computer. The result for a gaussian filter \((G(j\omega)\) given in table 3.1) when \((CNR) = 2\) and \(f/B_n = 0.3\), is given in table F.1. Column 1 contains the individual terms of the series for \(k = 1, 2 \ldots 10\); column 2 contains the cumulative sum; and column 3 contains the square root of the value in column 2 times a factor of 2. Also shown in columns 2 and 3 are the relevant values for decade values of \(k\) up to 100. It will be seen that, contrary to Stumpers' estimated value of 0.43, the correct value is 0.46. Since other results with which eqn. (3.19) could be compared were available, and because of the heavy demand which would have had to be made on computer time, correct evaluation of Stumpers' double-summation series was not pursued.

The range of values of (CNR) for which eqn. (3.15b) is
valid was also established by measurement of the output noise spectrum using single-pole and two-pole filters. Figs. 3.3 and 3.4 show the comparison between the measured spectra and those computed from eqn. (3.19), from which it will be seen that at (CNR) = 1, the spectra computed from eqn. (3.19) are in agreement to within about 10% with the measured spectra.

The single-pole and two-pole filters are described in appendix B, and their relevant properties are summarised in table 3.1. Each filter had a 3dB-bandwidth of 100 KHz, and the system on which the measurements were made is described in appendix A. It will be seen from appendix B.1 that for a single-pole filter, \( r \), the radius of gyration, is theoretically infinite. In practice, this is of course not so, since practical filters only have an approximate single-pole response in the vicinity of resonance, and, at frequencies far removed from resonance, the approximations used to derive the single-pole response no longer hold. Additionally, at frequencies well outside the pass-band, parasitics and circuit non-linearities reduce the response of the filter to zero.

To determine \( r \) for the single-pole filter, it is assumed that eqn. (3.19) is correct for (CNR) = 10, and the spectrum of the noise is then measured at \( f = 0 \) for this value of (CNR). At \( f = 0 \), eqn. (3.19) reduces to

\[
\frac{W_D(f)}{(4\pi B K D^2)} = (2\pi/a) \text{ cerf} (\text{CNR})^{1/2}
\]

(3.20)

from which it was established that
\[ a = 0.752 \quad \text{and} \quad r = B_n/a = 1.33B_n \quad (3.21) \]

These values will not be the same for all single-pole filters but will depend on the particular filter and its application. Eqn. (3.21) does, however, give an indication of the order of magnitude that one might expect.

### 3.3 Difficulties Associated with More Complex Analyses

Two main difficulties are encountered when dealing with the more complex analyses we have already referred to. Firstly, as we saw in the case of Stumpers' no-modulation formula, there is the problem of numerical evaluation. The second difficulty is more important from our point of view, and it is that such analyses do not give us a mental picture of the behaviour of the noise in FM systems. The picture we have so far developed was obtained by relating the properties of noise in general to the properties of FM systems; this gave us the two-term approximation for FM noise. One is then led to enquire if it is possible to extract a two-term approximation from the more involved analyses. The short answer is that it is not at all obvious how this may be done. To illustrate this, we turn again to Stumpers' no-modulation formula in appendix F. There it is shown that each term in the series development may be expressed as the product of a (CNR)-dependent factor and a frequency-dependent factor; the former gives the 'amplitude' of each term, and the latter gives the 'frequency distribution' of each term. For the first 10 terms of the series, table F.2 gives, in columns 1 and 2, the (CNR)-
3.4 The Effects of Modulation. When we are dealing with a modulated carrier, we shall use the same form as in eqn. (3.15a) for the spectrum of the output noise; i.e., we shall again assume that the output noise spectrum $W_p(f)$ comprises the two partial spectra, $W_1(f)$ and $W_2(f)$, but now these must be modified to include the effects of modulation. By referring to fig. 2.5b, it will be seen that

$$\phi_m(t) = \rho(t) + \tan \left\{-1 \frac{y_m(t)}{[A + x_m(t)]}\right\}$$  \hspace{1cm} (3.22)

where $\phi_m(t)$ is the total phase above $\omega_0 t$ of the IF wave at the input to the balanced limiter-detector. $x_m(t)$ and $y_m(t)$, the noise voltages in phase and in lead-quadrature with the
modulated carrier, are dealt with in sec. 2.4. Differentiation of eqn. (3.22) with respect to time yields the required output $v_D(t)$. Thus

$$v_D(t) = K_D \dot{\phi}_m(t) - K_D \dot{\rho}(t) + K_D \left( \frac{d}{dt} \left[ \tan^{-1} \left( \frac{y_m(t)}{[A+x_m(t)]} \right) \right] \right)$$

(3.23)

and the output consists of the wanted modulation $\cdot \rho(t)$ plus unwanted noise. The power spectrum of the second term in eqn. (3.23) will therefore yield the spectrum of the output noise. Proceeding in exactly the same way as we did when the carrier was unmodulated, we arrive at the result that for $r(t) \leq A$

$$K_D \{ \dot{\phi}_m(t) - \rho(t) \} = (K_D / A) \left[ 1 - e^{-\text{CNR}} \right] \cdot \cdot y_m(t)$$

(3.24)

$$W_y(f) = (K_D / A)^2 \left[ 1 - e^{-\text{CNR}} \right]^2 (2\pi f)^2 W_y(f)$$

(3.25)

where $W_y(f)$ is now the power spectrum of $y_m(t)$. Eqns. (3.24) and (3.25) are to be compared with eqns. (3.6) and (3.7), the 'no-modulation' equations. The power spectrum of $y_m(t)$ has been obtained by Blachman (ref. 6) and Rice (ref. 21) for the case of single-tone modulation.

$$W_y(f) = 2 \sum_{k=-\infty}^{\infty} W_o(f + f_o + kf_s) J_k^2(M)$$

(3.26)
\[ \rho(t) = \Delta \omega \cos 2\pi f_s t ; \quad M = \Delta f/f_s \quad (3.27) \]

\( W_o(f) \) is the spectrum of the noise at the output of the IF filter, and \( J_n(M) \) has been met in sec. 1.2. From eqn. (2.89) we may write

\[ W_o(f + f_o + kf_s) = W_o |G(j\omega + j\omega_o + jk\omega_s)|^2 \quad (3.28) \]

and eqn. (3.26) becomes

\[ W_y(f) = 2W_o \sum |G|^2 J_k^2 \quad (3.29) \]

\[ \sum |G|^2 J_k^2 = \sum_{k = -\infty}^{+\infty} |G(j\omega + j\omega_o + jk\omega_s)|^2 J_k^2(M) \quad (3.30) \]

With the expressions in eqns. (3.29) and (3.30) and the expression for (CNR) in eqn. (3.17), we can express \( W_l(f) \) as

\[ W_l(f) = K D^2 (2\pi f)^2 [1-e^{-(CNR)}]^2 [(CNR)B_n]^{-1} \sum |G|^2 J_k^2 \quad (3.31) \]

The scheme for evaluating the summation in eqn. (3.30) is dealt with in appendix G.1. Table 3.4 shows the evaluation of the summation at different output frequencies for various values of modulation index \( M \) in the cases of single-pole, two-pole, and rectangular IF filters. From eqn. (3.29), the value of the summation at a particular frequency is proportional to \( W_y(f) \), since \( W_o \) is the (constant) midband
value of the IF noise spectrum. Table 3.4 therefore shows how the frequency distribution of $W_y(f)/(2W_o)$ varies with the modulation index. It may be shown analytically from eqn. (3.30) that, for a given frequency, $W_y(f)/(2W_o)$ decreases as the modulation index increases, and this is clearly indicated in table 3.4. The implication of this is that, for fixed (CNR), as the modulation increases, $W_y(f)$ and therefore $W_1(f)$ decreases. This can be explained, without going into too much detail, by reference to the vector diagram in fig. 2.5b. As the modulation index increases, say by an increase in $\Delta \omega$ [eqn. (3.27)], the phase angle $\rho(t)$ increases, and this would cause $y_m(t)$ to decrease. The output noise voltage in eqn. (3.23) would therefore be decreased, giving a reduction in $W_1(f)$.

The partial spectrum $W_2(f)$ of the output noise when modulation is present may again be written in the form [compare eqn. (3.14)]

$$W_2(f) = \frac{2}{8\pi n}$$

(3.32)

where $n$, the total number of impulses occurring per second, is affected by the modulation. Rice (ref. 21) has obtained the result (with some rearrangement here) that for single-tone modulation [see eqn. (3.27)],

$$\frac{n}{2} = A_1 + A_2 \quad (3.33)$$

$$A_1 = r(\pi) \int_{-\pi/2}^{\pi/2} (1+c \sin x)^2 \frac{1}{2} \text{erf} [(\text{CNR}) + (\text{CNR})^2 c \sin x] \, dx \quad (3.34)$$
Sec. 3.4. FM NOISE CHARACTERISTICS

\[ A_2 = r e^{-\frac{(CNR)}{[4\pi(CNR)]}} \{2 d e^{\frac{d}{[I_0(d) + I_1(d)]}} \} \tag{3.35} \]

\[ c = \Delta \omega/(2\pi r) = \Delta f/r ; d = (CNR)c^2/2 = (CNR)(\Delta f)^2/(2r) \tag{3.36} \]

\( r \) and \( r/Nr \) have been met in connection with eqn. (3.10), and \( (CNR) \) is the IF carrier to noise power ratio. \( I_0(d) \) and \( I_1(d) \) are the modified Bessel functions of the first kind and imaginary argument and are defined by

\[ I_n(d) = e^{-\frac{n}{d}} J_n(jd) = (\pi)^{-1} \int_0^\infty e^{-\frac{x}{d}} \cos nx \, dx , \text{ n integer} \tag{3.37} \]

The evaluation of these functions is dealt with in appendix 1.3.

The integration in eqn. (3.34) is best evaluated on a computer, using Romberg integration (ref. 32). Figs. 3.5 to 3.8 show plots of \( n/B_n \) for various values of modulation index \( M \), using different IF filters. Remarks concerned with the programming of these calculations are made in appendix G.2. From the curves, it will be seen that \( n/B_n \) increases with increase in \( M \) and that it is also greater for filters which have greater \( r/B_n (= 1/a \) in table 3.1) ratios. For the single-pole filter, the value of \( a \) given in eqn. (3.21) was used.

We may now combine the two partial spectra, \( W_1(f) \) in eqn. (3.31) and \( W_2(f) \) in eqn. (3.32), to give

\[ W_D(f)/(4\pi B_n K_D^2) = \pi(f/B_n)^2 -(CNR) 2^{-1} \left[1-e^{-\frac{2}{\pi[CNR]}}\right] (CNR) \sum G^2 J_k + 2\pi m/B_n \tag{3.38} \]
where both terms depend on the IF filter shape and the modulation index M. It is again desirable to test the accuracy of eqn. (3.38), but unfortunately results using classical methods have not anywhere been obtained, and it became necessary to justify eqn. (3.38) by experiment. Fig. 3.9 shows the plot of eqn. (3.38) for M = 0, 1, 20 and (CNR) = 10, 2, assuming a double-pole IF characteristic. Measurements on the system described in appendix A showed that at (CNR) = 10, the calculated results were accurate to within a few per cent, and that at (CNR) = 1, the calculated results were some 10% higher than the measured results. This is parallel to the no-modulation case (figs. 3.3 and 3.4), where the two-term approximation began to deviate from measured results as (CNR) decreased below 1. We therefore conclude that eqn. (3.38) may be used to describe the output noise spectrum when sine wave modulation is present and (CNR) is greater than 1.

The double-pole IF filter, used for the measurements just described, had a 3db-bandwidth B of 100 KHz, and, for the various modulation indices M, the frequency deviation Δf and modulating frequency f_s were so chosen that the various quantities would be related by Carson's rule

$$ B = 2f_s (1 + M) = 2f_s (1 + Δf/f_s) $$

(3.39)

For the numerical evaluation of eqn. (3.38), the value of the summation was obtained from table 3.4, and n/B_n was obtained by evaluation of the formulae in eqns. (3.33) through (3.36).
Signal Suppression Effect. If, as has been assumed, the modulating signal $v_s(t)$ is given by

$$v_s(t) = \rho(t) = \Delta \omega \cos \omega_s t$$  \hspace{1cm} (3.40)

then, for a noise-free channel or for a channel in which the noise does not interact with the signal, the output signal power $S_D$ would be given as in eqn. (1.47) by

$$S_D = \left( \frac{K_D}{2} \right) (\Delta \omega)^2$$  \hspace{1cm} (3.41)

It has, however, been known for some time that $S_D$ is instead given by

$$S_D = \left( \frac{K_D}{2} \right) (\Delta \omega)^2 \left[ 1 - e^{-\frac{(CNR)}{2}} \right]$$  \hspace{1cm} (3.42)

This result has been obtained from classical methods of analysis by Stumpers, Rice, and Lawson and Uhlenbeck (ref. 1, 21, 5) and has been observed experimentally by Crosby and Fuller (ref. 7, 8). From eqn. (3.42), the output signal power is seen to decrease as the carrier to noise power ratio $(CNR)$ decreases, and this effect has been called the 'suppression of the signal by the noise' or 'the signal-suppression effect'.

The mechanism of the signal-suppression effect may be explained in terms of our ideas about FM noise characteristics. It will be remembered [eqn. (3.8)] that the factor $[1-e^{-\frac{(CNR)}{2}}]$ is the probability that the amplitude $r(t)$ of the noise vector does not exceed the carrier amplitude $A$. If, as in the case of the partial spectrum $W_1(f)$, the signal only appears when $r(t) \leq A$, then the output signal power, like $W_1(f)$, would
contain the factor \[1 - e^{-(CNR)}\]². Reference to fig. 2.5b reveals that for \(r(t) \leq A\), the excess-phase angle \(\phi_m(t)\) fluctuates randomly about the average value \(\rho(t)\); alternatively we may say that for \(r(t) \leq A\), the resultant vector points, on average, in the \(\phi_m(t) = \rho(t)\) direction, having fluctuations about this value. The output \(\phi_m(t)\) of the detector will therefore contain the modulation \(\rho(t)\) plus fluctuation noise.

When \(r(t) > A\), \(\phi_m(t)\) again fluctuates randomly, but, in contrast to the previous case, the resultant vector can now point, with equal probability, in any direction. This is because, of the two amplitudes \(r(t)\) and \(A\), \(r(t)\) is the larger and \(\phi_m(t)\) is now 'governed' by \(\gamma(t)\). \(\gamma(t)\), we will remember, is the excess-phase angle of the noise vector and is equally distributed in the range \(-\pi\) to \(+\pi\). For \(r(t) > A\), then, \(\phi_m(t)\) has zero mean, and the modulation disappears completely. These qualitative arguments, in collaboration with the similar effect relating to \(W_1(r)\) are quite forceful, but they can also be supported mathematically.

From the vector diagram in fig. 2.5b, we obtain that

\[
y_m(t) = r(t) \sin [\gamma(t) - \rho(t)]; \quad x_m(t) = r(t) \cos [\gamma(t) - \rho(t)]
\]

(3.43)

and the output \(v_D(t)\) of the detector is, from eqn. (3.23) say,

\[
v_D(t) = K_D \dot{\phi_m} = K_D \rho + K_D (d\dot{\theta}/dt) \left[ \tan^{-1} \left( \frac{r \sin (\gamma - \rho)}{A + r \cos (\gamma - \rho)} \right) \right].
\]

(3.44a)
In eqn. (3.44a), we have dropped the argument, $t$, of the various quantities and shall continue to do so where convenient, bearing in mind that $K_D$ and $A$ are the only time-independent quantities in the equation. $v_D(t)$ in eqn. (3.44b) is expressed as the sum of the signal output $s_D(t)$ and a noise term $n_D(t)$. The ensemble average of $v_D(t)$ is expressed as

$$\bar{v}_D(t) = \bar{s}_D(t) + \bar{n}_D(t)$$

(3.45)

since $s_D(t)$ is independent of the noise. The average of the signal, in the ensemble sense, is just the signal, so

$$\bar{v}_D(t) = s_D(t) + n_D(t)$$

(3.46)

$n_D(t)$ is given by the second term in eqn. (3.44b) and performing the differentiation required (see appendix H.1) gives

$$n_D(t) = K_D \cdot \frac{Af \sin (\gamma-\rho) + [r^2 + Ar \cos (\gamma-\rho)](\gamma-\rho)}{A^2 + r^2 + 2Ar \cos (\gamma-\rho)} = f(r, \gamma, r', \gamma')$$

(3.47)
where \( f(r, \gamma, \dot{r}, \dot{\gamma}) \) indicates that \( n_D(t) \) must be regarded as a function of the variables in parentheses. To find the ensemble average \( \overline{n_D(t)} \), we must therefore average eqn. (3.47) over the ranges of these variables:

\[
\overline{n_D(t)} = \int \int \int f(r, \gamma, \dot{r}, \dot{\gamma}) p(r, \gamma, \dot{r}, \dot{\gamma}) \, dr \, d\gamma \, dr \, d\dot{\gamma} \tag{3.48}
\]

where \( p(r, \gamma, \dot{r}, \dot{\gamma}) \) is the joint probability distribution of the variables in parentheses. From eqn. (2.46), we write

\[
x = r \cos \gamma ; \quad y = r \sin \gamma \tag{3.49}
\]

\[
\dot{x} = \dot{r} \cos \gamma - r \dot{\gamma} \sin \gamma ; \quad \dot{y} = \dot{r} \sin \gamma + r \dot{\gamma} \cos \gamma \tag{3.50}
\]

and \( p(r, \gamma, \dot{r}, \dot{\gamma}) \) may be obtained from \( p(x, y, \dot{x}, \dot{y}) \) through the change of variables in eqns. (3.49) and (3.50). The Jacobian \( J \) of the transformation (appendix D) is obtained in appendix H.2 and is given by

\[
J = r^2 \tag{3.51}
\]

From our discussions of noise in chap. 2, \( x, y, \dot{x}, \dot{y} \), will be seen to be all random independent gaussian noise variables, each having zero mean. \( x \) and \( y \) each have variance \( N_0 \), and \( \dot{x} \) and \( \dot{y} \) will both have the same variance, \( N_1 \) say.

Then
\[ p(x, y, x', y') = p(x)p(y)p(x)p(y) \]

\[ = \frac{-\left(x^2 + y^2\right) / (2N_0)}{2\pi N_0} \cdot \frac{-\left(x'^2 + y'^2\right) / (2N_1)}{2\pi N_1} \]

\[ = \frac{e^{-r^2 / (2N_0)}}{2\pi N_0} \cdot \frac{e^{-\left(r^2 + r^2 y^2\right) / (2N_1)}}{2\pi N_1} \]

(3.52)

and eqn. (3.48) becomes

\[ n_D(t) = \int \int K_D \cdot \frac{A \sin (\gamma - \rho) + \{r^2 + A r \cos (\gamma - \rho)\}(\gamma - \rho)}{A^2 + r^2 + 2Ar \cos (\gamma - \rho)} \]

\[ \{ \frac{r^2 e^{-r^2 / (2N_0)}}{2\pi N_0} \cdot \frac{e^{-\left(r^2 + r^2 y^2\right) / (2N_1)}}{2\pi N_1} \} d\gamma d\gamma d\gamma \]

(3.53)

For this integration, the range of \( r \) is zero to \( \infty \), that of \( \gamma \) is \(-\pi\) to \(+\pi\), and both \( r \) and \( \gamma \) lie in the range from \(-\infty\) to \(+\infty\). The integration is carried out as described in appendix H.3, giving the result

\[ n_D(t) = 0, \quad r \leq A \]  

(3.54a)
\[ n_D(t) = 0, \quad r \leq A \quad (3.55a) \]
\[ = -K_D^\rho \left[ s_D(t) \right], \quad r > A \quad (3.55b) \]

In eqn. (3.46) then
\[ v_D(t) = s_D(t), \quad r \leq A \quad (3.56a) \]
\[ = 0, \quad r > A \quad (3.56b) \]

Thus, for \( r(t) \leq A \), the average (in the ensemble sense) of the output is just the required modulation, and, for \( r(t) > A \), the ensemble average of the output plunges to zero. In both cases, noise will also be present at the output, but the average of noise in the ensemble sense is zero.

The two conditions in eqns. (3.54) may be combined by observing that \( n_D(t) \) only exists for \( r \) in the range \( A \) to \( \infty \). Thus
\[ n_D(t) = -s_D(t) \int_A^\infty \frac{-r^2/(2N_o)}{(r/N_o)e - e} dr = -s_D(t)e^{-(\text{CNR})} \quad (3.57) \]
Sec. 3.6. FM NOISE CHARACTERISTICS

by straightforward integration. \((\text{CNR})\) is the ratio \(A^2/(2N_0)\) of mean carrier power to mean noise power. Eqn. (3.46) then becomes:

\[
\bar{v}_D(t) = s_D(t) \left[ 1 - e^{-(\text{CNR})} \right]
\]  

(3.58)

and the average output signal power then contains the factor \([1-e^{-(\text{CNR})}]^2\).

The signal-suppression effect is even more drastic than the reduction in signal power suggests. This may be seen from fig. 3.10, which represents the output when the modulation is sinusoidal. During the time from 0 to \(t_1\), \(r(t) \leq A\) and the modulation appears with superimposed noise. During the time interval from \(t_1\) to \(t_2\), \(r(t) > A\) and the output consists purely of noise. It is during this time, too, that impulsive noise arises. The intermittent disappearance of the signal, coupled with the simultaneous occurrence of impulses, will have drastic audible or visual effects.

3.6. Output Signal to Noise Ratio. If we assume, as in chap. 1, that a flat output filter, having a sharp cut-off at \(f_s\), is used in the output circuit, then the output noise power \(N_D\) is given by

\[
N_D = \int_{0}^{f_s} W_D(f) \, df
\]  

(3.59)

where \(W_D(f)\) is the output noise spectrum and \(f_s\) is the
highest frequency in baseband signal for which the receiver is designed. When the baseband signal is a single tone of frequency \( f_s \), the signal power \( S_D \), as we have just seen in the previous section, is given by

\[
S_D = \left( \frac{K_D^2}{2} / 2 \right) (\Delta \omega)^2 (1 - e^{-\frac{1}{2}(CNR)})^2
\]

(3.60)

and by the use of Carson's rule (see appendix G.2), this may be expressed as

\[
S_D = \left( \frac{K_D^2}{2} / 2 \right) \left[ 1 - e^{-\frac{1}{2}(CNR)} \right]^2 (\pi B)^2 \left[ M / (1 + M) \right]^2
\]

(3.61)

where \( B \) is the 3db-bandwidth of the IF filter, and \( M \) the modulation index.

\( W_D(f) \), the power spectrum of the output noise when modulation is present, may conveniently be expressed in the form [see eqn. (3.38)]

\[
W_D(f) = K_D^2 \pi^2 [1 - e^{-\frac{1}{2}(CNR)}]^2 \left[ (\text{CNR})_{B_n} \right]^{-1} f^2 |G|^2 J_k^2
\]

\[
+ K_D^2 \pi^2 \left[ n \right]^2
\]

(3.62)

where \( \sum |G|^2 J_k^2 \), [defined in eqn. (3.30)], and \( n \), the number of noise impulses per second, depend on the modulation index \( M \). From eqn. (3.59)

\[
N_D = \int_0^{f_s} W_D(f) \, df
\]
Sec. 3.6. FM NOISE CHARACTERISTICS

\[ \text{The output signal to noise ratio (SNR)} = \frac{S_D}{N_D} \text{ is found by dividing eqn. (3.61) by eqn. (3.63). Thus} \]

\[
(SNR) = \left[ \frac{\frac{2}{b(\text{CNR})_M}}{8b (1 + M) \left\{ \int_0^B \frac{2}{[2(1+M)]} \left( \frac{f}{B} \right)^2 |G| J_k \frac{df}{B} \right\}^2 + \frac{\delta(CNR)}{-(\text{CNR})_B^n (1 + M)} \right] \left[ \frac{1 - e^{-\frac{2}{B^n}}}{} \right] \]

where Carson's rule, \( B = 2f_s (1+M) \) and the ratio \( b = B/B_n \) have been used. \( B \) and \( B_n \) are the 3db-bandwidth and noise bandwidth respectively of the IF filter. Figs. 3.11 and 3.12 show the plot of eqn. (3.64) for single-pole and rectangular IF filters. The continuous lines represent the characteristics when the modulation is assumed not to affect the noise, in which case \( n \) is given as in eqn. (3.10) and \( \Sigma |G|^2J_k^2 \) reduces to \( |G_{LP}(j\omega)|^2 \). The dashed lines represent the characteristics when modulation is assumed to affect the noise; then \( n \) and \( \Sigma |G|^2J_k^2 \) are given as in
Whether or not the modulation is taken to affect the noise, the denominator in eqn. (3.64) will depend only on the IF filter shape and the modulation index. This will be obvious for the $n/B_n$ term by inspection of the relevant formulae. Some remarks on the evaluation of $n/B_n$ may be found in appendix G.2. For the integration in the brackets, the summation, which is dealt with in appendix G.1, again requires only the IF filter shape and modulation index. The integration is expressed with respect to a ratio, $f/B$, and is best carried out on a computer by Romberg integration [ref. 32].

Single-pole and rectangular IF filter shapes were chosen for the curves in figs. 3.11 and 3.12, because the former has the worst possible noise rejection properties and the latter has the best possible noise rejection properties.

**3.7. Conditions at the Threshold.** In sec. 1.4, the threshold was defined as the value of $(CNR)$ for which $(SNR)$ fell $1 \text{db}(= 1.259)$ below linearity. For a fixed modulation index and high $(CNR)$, the number of noise impulses $n$ in the output is very small. In eqn. (3.64), then, the second term in the denominator becomes negligible in comparison with the first term, which, in fact, does not depend on $(CNR)$. Thus, for high $(CNR)$, $(SNR)$ will have linear dependence on $(CNR)$ in accordance with the $(CNR)$ term in the numerator. As $(CNR)$ is decreased, the number of impulses in the output increases until the term with $n/B_n$ increases to a value which is $0.259$ times the value of the first term. $(SNR)$ is then $1 \text{db}$ below linearity, and the $(CNR)$ value for
this condition defines the threshold.

From the curves in figs. 3.11 and 3.12, it will be seen that the threshold performance of a rectangular filter is better than that of a single-pole filter and that the threshold is degraded by increase in modulation index. The threshold is also seen to be worse when the modulation is assumed to affect the noise output. The 'optimum' threshold may be thought of as the threshold (CNR) for a system in which a) the IF filter is rectangular and b) the modulation index is zero. The latter condition will mean that (SNR) in eqn. (3.64) will tend to zero, but the threshold condition, that the second term in the denominator is 0.259 times the first term, will still exist. For any other filter and any other value of modulation index, whether the modulation is assumed to affect the output noise or not, the threshold will be worse.

The application of the conditions defining the optimum threshold to the denominator in eqn. (3.64) gives

\[
(0.259) 8b^2 \int_0^{B/2} f^2 \, df \, B^{-3}
\]

\[
= 8(CNR) [1-e^{-(CNR)^{-2}} - \frac{1}{(12)} \text{erf} (CNR)]^\frac{1}{2}
\]

\[
(CNR) \text{erf} (CNR) [1-e^{-(CNR)^{-2}}] = 0.0374
\]

\[
(CNR) = 3.3
\]

The solution (CNR) = 3.3 was obtained from eqn. (3.66) by
'regula falsi' iteration (ref. 32). (CNR) = 3.3 is then our optimum threshold, and, for this value, the factor 
$[1 - e^{\frac{-1}{2}}]^{-2}$ is 1.07 and rapidly decreases to 1 as (CNR) increases. For all subsequent calculations, in which we are interested in (CNR) values above 3.3, we shall therefore omit this factor.

For the general case, the threshold, $(CNR)_T$, is obtained from the denominator of eqn. (3.64). Thus

$$(CNR)_T \left(\frac{n}{B_n}\right) = \left(0.259\right)^2 \frac{2}{(1+M)} \int_0^B \left[\frac{2(1+M)}{f/B} \right]^2 \frac{2}{G} \frac{2}{J \left(\frac{df}{B}\right)}$$

(3.68)

In this equation, the expression on the right-hand side is a constant for a fixed modulation index and is evaluated as we have described before. The expression on the left-hand side is the $(CNR)_T$ 'variable', $(n/B_n)$ being also evaluated as previously described. For the single-pole and rectangular IF filter shapes, the threshold was located from eqn. (3.68) using 'regula falsi' iteration (ref. 32). The results are shown in figs. 3.13 and 3.14, where the solid lines represent the case when the output noise is assumed to be unaffected by the modulation and the broken lines take into account the effect of modulation on the noise. $(CNR)_T$ is, as we have been assuming, the threshold carrier to noise ratio measured in the IF noise bandwidth $B_n$. Also plotted in figs. 3.13 and 3.14 is $(CNR)'_T$, which is the threshold carrier to noise ratio measured in a rectangular IF bandwidth.
Sec. 3.7. FM NOISE CHARACTERISTICS

In general

\[
\frac{(\text{CNR})}{W_0 B_n} = \frac{A^2}{(2W_0 B_n)} \quad (3.69)
\]

\[
(\text{CNR})' = \frac{A^2}{(2W_0 2f_s)} = (\text{CNR}) \left[ B_n/(2f_s) \right] = (\text{CNR})(1 + M) \quad (3.70)
\]

where \( W_0 \) is the midband level of the IF noise spectrum and \( B_n \) the noise-bandwidth of the IF filter. \( B/(2f_s) \), we said in sec. 1.3, is called the bandwidth ratio of the receiver and is related to the modulation index through Carson's rule, \( B = 2f_s(1+M) \). \( (\text{CNR})' \) is a more useful description of a system's threshold, because, for a given signal bandwidth \( f_s \), increase in modulation index is brought about by increased frequency deviation and therefore increased IF bandwidth requirement. Different modulation-index systems are then best compared by observing the threshold \( (\text{CNR}) \) in some fixed bandwidth, \( 2f_s \) being an arbitrary but convenient one.

From figs. 3.13 and 3.14, it will be seen that when the effect of modulation on the noise is taken into account, the threshold is never more than about 1db worse than when the noise is assumed to be unaffected by the modulation. This will lead us to assume that, for threshold and above-threshold conditions, the modulation and noise do not significantly interact. This effect, as mentioned in sec. 1.3, has frequently been observed in practice.

It is worth reiterating at this point that the threshold is far worse than the mathematics indicate. The threshold occurs when \( (\text{CNR}) \) becomes low enough for impulse noise to
be comparable with fluctuation noise. This means that the noise amplitude $r(t)$ has to be greater than the carrier amplitude $A$ for a significant proportion of the time, and, during all that time, signal output vanishes. It is the occurrence of impulses, coupled with the disappearance of the signal, that so drastically reduces the practical usefulness of a system on, or just below, threshold.
FIG. 3.1. Oscillograms of FM Noise and the Corresponding Spectra

FIG. 3.2. Discriminator Output for Circular Sweeps of the Noise Vector
FIG. 3.3. Measured and Calculated Noise Spectra for Single-Pole IF Filter
FIG. 3.4. Measured and Calculated Noise Spectra for Two-Pole IF Filter
FIG. 3.5. $n/B_n$ for Rectangular IF Filter
FIG. 3.6. $n/B_n$ for Gaussian IF Filter
FIG. 3.7. $n/B_n$ for Two-Pole IF Filter
FIG. 3.8. $n/B_n$ for Single-Pole IF Filter
FIG. 3.9. Effect of Modulation on Noise Spectrum

FIG. 3.10. The Signal Suppression Effect
FIG. 3.11. (SNR) for Single-Pole IF Filter
FIG. 3.12. (SNR) for Rectangular IF Filter
FIG. 3.13. Threshold (CNR) for Single-Pole IF Filter

FIG. 3.14. Threshold (CNR) for Rectangular IF Filter
### TABLE 3.1
Summary of Properties of Some IF Filters

| response                  | $|G(j\omega)|^2$ | $3\text{db}$ bandwidth | noise bandwidth | radius of gyration | $a=\frac{B_n}{r}$ | $b=\frac{B}{B_n}$ |
|---------------------------|----------------|-------------------------|-----------------|--------------------|-------------------|------------------|
| rectangular               | $B$            | $B_n$                   | $r$             | $\frac{B}{12^{\frac{1}{2}}}$ | $12^{\frac{1}{2}}$ | $1$              |
| $f_o-B/2\leq f\leq f_o+B/2$ | $B$            | $B$                     | $\frac{B}{12^{\frac{1}{2}}}$ | $12^{\frac{1}{2}}$ | $1$              |
| $0$, elsewhere            | $B$            | $B$                     | $\frac{B}{12^{\frac{1}{2}}}$ | $12^{\frac{1}{2}}$ | $1$              |
| gaussian                  | $B\left(\frac{4\log_2 e}{\pi}\right)^{\frac{1}{2}}$ | $B$                     | $\frac{B}{(2\pi)^{\frac{1}{2}}}\left(\frac{4\log_2 e}{\pi}\right)^{\frac{1}{2}}$ | $12^{\frac{1}{2}}$ | $1$              |
| two-pole                  | $\frac{1}{1+\left(\frac{f-f_o}{f'}\right)^2}$ | $2f'$                   | $\frac{4\log_2 e}{\pi}^{\frac{1}{2}}$ | $f'$ | $\frac{\pi}{2^{\frac{1}{2}}}$ | $\frac{23/2}{\pi}$ |
| single-pole               | $\frac{1}{1+\left(\frac{f-f_o}{f'}\right)^2}$ | $2f'$                   | $\frac{4\log_2 e}{\pi}^{\frac{1}{2}}$ | $\infty$ | $0$ | $\frac{2}{\pi}$ |

### TABLE 3.2
Probability that $r(t)\leq A$

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TABLE 3.4

\( W_y(f)/(2W_o) \)

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a. single-pole filter  
b. two-pole filter  
c. rectangular filter
4.1. Introduction. A preliminary study of the FMFB demodulator was made in sec. 1.5. We were then mainly interested in the output signal to noise performance of the feedback receiver when the IF carrier to noise power ratio (CNR) is high. In this chapter, we are concerned with a closer examination of the effects of feedback on signal and noise when (CNR) is not necessarily high. To do this, we consider the general signal transmission properties of FM receivers and develop their linear analogue. With this analogue it is possible to analyse the performance of the FMFB receiver from the point of view of output (SNR) performance and also from the important point of view of the system's stability. We will then be in a position to consider finally how best threshold improvement, using the FMFB receiver, may be effected.

We saw in the last chapter that, for the ranges of modulation index $M$ and carrier to noise power ratio (CNR) in which we were interested, we could assume that the noise and modulation did not significantly interact. For the requirements of this chapter, we will therefore make the assumption that modulation and noise may be dealt with independently.
4.2. **FM Analogue.** The FM receiver is schematically represented in fig. 4.1 and may be regarded as a conventional (open-loop) receiver or an FMFB receiver depending on whether the switch S is open or closed. We will refer to the 'IF channel' as the channel which starts at the mixer's output and ends at the discriminator's input, and the channel which starts at the discriminator output and ends at the voltage-controlled oscillator's input we will call the baseband channel. By assuming that all IF filtering is taken into account in the characteristic of a single IF filter and that similarly the low-pass baseband filter represents all baseband filtering, we may regard all the other devices in the FM receiver as broadband.

The most useful analogue of the FM receiver is the one we shall call 'the phase-function analogue'. This analogue deals with the excess-phase (phase in excess of the carrier phase) of the FM wave. In such an analogue, the mixer may be regarded as a conventional summing device with a negative-input terminal. This is shown in fig. 4.2. [In this figure \( \Phi(s) \) represents the Laplace transform of the excess-phase function \( \phi(t) \), and \( V_D(s) \) represents the Laplace transform of the output voltage \( v_D(t) \)]. The summing property of the mixer may be deduced quite simply by writing the total instantaneous phases \( \theta_R(t) \), \( \theta_L(t) \), \( \theta_I(t) \) of the RF, local oscillator, and IF waves respectively, in the form

\[
\theta_R(t) = \omega_R t + \phi_R(t) ; \quad \theta_L(t) = \omega_L t + \phi_L(t) ; \quad \theta_I(t) = \omega_I t + \phi_I(t) \quad (4.1)
\]
where $\omega_R$, $\omega_L$, $\omega_o$ are the carrier frequencies of the RF, local oscillator, and IF waves respectively. The total instantaneous phase of each wave may be regarded as comprising a linearly-varying 'carrier' term and an 'information' excess-phase term. The instantaneous frequency $\dot{\theta}_i(t)$ of the IF wave is the difference between the instantaneous frequencies $\dot{\theta}_R(t)$ and $\dot{\theta}_L(t)$ of the RF and local oscillator waves. Thus

$$\omega_o + \dot{\theta}_i(t) = \omega_R - \omega_L + \dot{\theta}_R(t) - \dot{\theta}_L(t) \quad (4.2)$$

and setting $\omega_R - \omega_L = \omega_o$ leads to

$$\dot{\phi}_i(t) = \dot{\phi}_R(t) - \dot{\phi}_L(t) \quad ; \quad \phi_i(t) = \phi_R(t) - \phi_L(t) \quad (4.3)$$

and the mixer is seen to produce an IF excess-phase, which is the difference between the excess-phases $\phi_R(t)$ and $\phi_L(t)$ of the RF and local oscillator waves.

Voltage amplification and voltage limiting in the IF channel will not directly manifest themselves in the excess-phase analogue of the receiver. This may be demonstrated by writing the IF wave, $v_i(t)$, at the output of the mixer, in complex form [see eqn. (1.13a)].

$$v_i(t) = A_i e^{j(\omega_o t + \phi_i(t))} \quad (4.4)$$

where $A_i$ is the amplitude of the IF wave at the output of the mixer. After broadband amplification, filtering, and
limiting in the IF channel, the wave $v_o(t)$ at the input to the discriminator is given by

$$v_o(t) = A_0 e^{j[\omega_0 t + \phi_0(t)]}$$

(4.5)

where $A_0$ is the output level which the limiter produces. For an ideal limiter (see fig. 1.1), $A_0$ will be constant for all values of $A_i$, the mixer's output amplitude. The gain $K_D$ of the discriminator will, in general, be dependent on the amplitude $A_0$. Thus for a given receiver, the effect on the excess-phase function of IF amplification and limiting is implicitly accounted for in the gain constant $K_D$ of the discriminator. The discriminator, acting as a balanced phase-differentiator, will produce at its output the derivative of the input excess-phase function. For the phase-function analogue of the receiver, we therefore regard the discriminator as a differentiator, having transfer function $K_D s$, as shown in fig. 4.2.

We have not yet considered the effect of the IF filter on the phase-function, but because of the relative complexity of this problem, the treatment is deferred for a moment.

In the discriminator, the excess-phase function is converted to an output voltage-function, which, in an open-loop system, is delivered to some output circuit. In the feedback receiver, this circuit must be placed outside the loop, as we shall see, and need not be regarded as part of the analogue.

The baseband channel provides conventional amplification

* It has been assumed that zeros in the envelope will be rare in this context so that $A_0$ will be maintained constant by the limiter.
and filtering of the baseband voltage. The broadband feedback-gain control unit regulates the amount of feedback applied, and the low-pass filter represents all the baseband filtering.

In our preliminary analysis of the feedback demodulator (sec. 4.2, fig. 1.3), we saw that the VCO acted as an FM generator. If the input to the VCO is \( v(t) \), the excess-phase function \( \phi_L(t) \) at the output and \( v(t) \) are related through

\[
\phi_L(t) = K_V \int_0^t v(t) \, dt \tag{4.6}
\]

where \( K_V \) is the gain of the VCO. This equation may be compared with eqns. (1.3) and (1.4). In the phase-function analogue of the receiver, the VCO performs the function of an integrator and may therefore be represented as a device having transfer function \( K_V/s \).

To determine the effect of the IF filter on the phase-function analogue, we express the filter's input \( v_i(t) \) and output \( v_o(t) \) in eqns. (4.4) and (4.5) in the form

\[
v_i(t) = x(t)e^{j\omega_0 t} \quad ; \quad x(t) = e^{j\phi_i(t)}
\]

\[
v_o(t) = y(t)e^{j\omega_0 t} \quad ; \quad y(t) = e^{j\phi_o(t)}
\]

where we have omitted the amplitudes of the \( v_i(t) \) and \( v_o(t) \) waves. If \( V_i(s) \), \( V_o(s) \), \( X(s) \), and \( Y(s) \) are the Laplace transforms of \( v_i(t) \), \( v_o(t) \), \( x(t) \), and \( y(t) \) respectively,
then, from appendix C, we may write

\[ V_i(s) = X(s - j\omega_0) \quad ; \quad V_o(s) = Y(s - j\omega_0) \]  \hspace{1cm} (4.9)

where \( X(s - j\omega_0) \) and \( Y(s - j\omega_0) \) are the Laplace transforms \( X(s) \) and \( Y(s) \), of \( x(t) \) and \( y(t) \) respectively, shifted upwards along the frequency axis by amount \( \omega_0 \). \( V_o(s) \) and \( V_i(s) \) are related, as dealt with in appendix C, through the transfer function \( G(s) \) of the IF filter. Thus

\[ Y(s - j\omega_0) = X(s - j\omega_0)G(s) \]  \hspace{1cm} (4.10)

and replacing the variable \( s - j\omega_0 \) with the variable \( s \) gives

\[ Y(s) = X(s)G_L(s) = X(s)G_{LP}(s) \]  \hspace{1cm} (4.11)

\( G(s + j\omega_0) \) \( [= G_{LP}(s)] \) is the transfer function \( G(s) \) of the IF filter, shifted downwards along the frequency axis by amount \( \omega_0 \), i.e., it is the transfer function of the low-pass analogue of the IF filter. That \( Y(s) \) and \( X(s) \) are related through the low-pass analogue of the IF filter may have been surmised from the representation of \( v_1(t) \) and \( v_o(t) \) in eqns. (4.7) and (4.8).

In general, the relation in eqn. (4.7) between the phase-function \( \phi_i(t) \) and \( x(t) \) is non-linear, and it is not possible to relate the Laplace transform, \( \Phi_i(s) \), of \( \phi_i(t) \) to the Laplace transform, \( X(s) \), of \( x(t) \) through a linear transfer function. Similar remarks must, of course, apply to \( \phi_o(t) \), \( y(t) \), \( \Phi_o(s) \),
and $Y(s)$. In the general case then, the effect of the IF filter on the phase-function analogue can only be taken into account by the inclusion of two 'function-generators', $G_1$ and $G_2$. This is shown in fig. 4.2 where $G_1$, having input $\phi_i(s)$, produces output $X(s)$. $G_2$ may be regarded as performing the inverse operation of producing an output $\phi_o(s)$ for an input $Y(s)$.

The phase-function analogue in fig. 4.2 is not very useful as it stands but may be simplified for specific applications. We consider firstly the reduction of the analogue for the 'normal' operation of the receiver. In this case, the 3db-bandwidth of the IF filter is so chosen that the essential components of the spectrum of the IF signal lie within it. Over the 3db-bandwidth of the filter, the amplitude response of the filter will be nearly constant, and the phase response will be nearly linear. This is the condition for the undistorted transmission of a wave through a device (see appendix C). If we disregard, for the time being, the relative phasing of the input and output waves, the IF wave is essentially the same before and after transmission through the IF filter. We may consequently omit the entire IF channel. In the analogue of fig. 4.2, $Y(s)$ and $X(s)$ are identical and also therefore are $\phi_o(s)$ and $\phi_i(s)$. The analogue then reduces to that shown in fig. 4.3. The low-pass filter has been omitted from fig. 4.3, since its effect is only important from the point of view of the system's stability, and this we will be considering later. When the system is on open-loop, we obtain
\[ V_D(s) = K_D \phi_R(s) \quad (4.12) \]

If it is also assumed that on closed-loop the IF signal is unaffected by its passage through the IF filter, we obtain

\[ V_D(s) = K_D \phi_i(s) = K_D \phi_R(s) - \phi_L(s) \quad (4.13) \]

\[ \phi_L(s) = (K_V F/s) V_D(s) \quad (4.14) \]

\[ V_D(s) = K_D \frac{\phi_R(s)}{(1 + K_D F K_V)} = K_D \frac{\phi_R(s)}{F} \quad (4.15) \]

where \( F = 1 + K_D K_V K_F \) is the feedback factor. Comparing eqns. (4.15) and (4.12) shows that on feedback the output signal amplitude is reduced by the feedback factor \( F \). This result has been obtained in sec. 1.5, where we had assumed the modulation was sinusoidal. From fig. 4.3 it will be seen that on open-loop the IF phase-function \( \phi_i(s) \) is just \( \phi_R(s) \), and from eqn. (4.15) it will be seen that on closed-loop \( \phi_i(s) = V_D(s)/(K_D S) \) is \( \phi_R(s)/F \). Thus on closed-loop, the IF phase-function is also reduced by the feedback factor as expected.

A very useful analogue of the FM receiver is the small-index analogue. This is the analogue which is applicable when the IF phase-function \( \phi_i(t) \) is such that

\[ |\phi_i(t)| \leq \frac{1}{\pi} \quad (4.16) \]
We encountered a similar situation in sec. 1.2, where, for a sinusoidal phase-function, we had said the modulation index $M$ needed to be less than $\frac{1}{2}$ in order to produce a 'small-index' FM wave. Under the small-index condition in eqn. (4.16), $x(t)$ in eqn. (4.7) becomes

$$x(t) = e^{j\phi(t)} = \sum_{n=0}^{\infty} \frac{[j\phi(t)]^n}{n!} \quad \text{(4.17)}$$

$$= 1 + j\phi(t) \quad \text{(4.18)}$$

We note from eqn. (4.7) that the IF wave $v_i(t)$ may be thought of as a low-frequency signal rotating with angular frequency $\omega_0$. The vector diagram representation of $v_i(t)$ is shown in fig. 4.4 from which it is observed that $v_i(t)$ may be regarded as the vector sum of an in-phase component and a lead-quadrature component, both components rotating with angular frequency $\omega_0$. The amplitude of $v_i(t)$ we have taken as 1 for convenience. When $\phi_i(t)$ is small, the in-phase component of $v_i(t)$ is 1 and the lead-quadrature component of $v_i(t)$ is $\phi_i(t)$, as obtained in eqn. (4.18).

The Laplace transform $X(s)$ of $x(t)$ is, from eqn. (4.18),

$$X(s) = \frac{1}{2} \delta(s) + j\phi_i(s) \quad \text{(4.19)}$$

where $\delta(s)$, the impulse function at $s = 0$, accounts for the
'd.c.' term in eqn. (4.18). The significance of eqn. (4.19) is that it indicates that when the phase-function $\phi_1(t)$ is small, the frequency spectrum of $x(t)$ consists of a 'd.c.' term plus the frequency spectrum of $\phi_1(t)$. When the spectrum of $x(t)$ is shifted upwards in frequency by amount $\omega_0$, we obtain the frequency spectrum of $v_i(t)$, which comprises the carrier term plus the sidebands of $x(t)$. These conclusions were drawn in sec. 1.2, when considering sinusoidal modulation.

From eqn. (4.11), $Y(s)$ is given by

$$Y(s) = \frac{1}{2} \delta(s) + j\Phi_1(s) G_{LP}(s) = \delta(s) + j\Phi_0(s)$$

(4.20)

$$\Phi_0(s) = \Phi_1(s) G_{LP}(s)$$

(4.21)

where $\delta(s) G_{LP}(s) = \delta(s)$, since $\delta(s)$ only exists at $s = 0$, and $G_{LP}(s) = 1$ at $s = 0$. Using $\mathcal{L}\{ \}$ to mean 'Laplace transform of $\{ \}$', eqns. (4.20), (4.19), and (4.17) lead us to write

$$Y(s) = \mathcal{L}\{e^{j\theta_0(t)} \}$$

(4.22)

so that $\Phi_0(s)$ and $\Phi_1(s)$ in the phase-function analogue of fig. 4.2 are related quite simply through the low-pass analogue of the IF filter as in eqn. (4.21). Thus when the IF phase-function is small, the small-index analogue shown in fig. 4.5 is obtained. From this analogue, we have that on feedback,

$$V_D(s) = \Phi_1(s) G_{LP}(s) K_D s = [\Phi_R(s) - \Phi_L(s)] G_{LP}(s) K_D s$$

(4.23)
\[ \Phi_L(s) = \left( K_V K_F / s \right) V_D(s) \]  \hspace{1cm} (4.24)

\[ V_D(s) / \Phi_R(s) = G_{\text{LP}}(s) K_D s / \left[ 1 + K_V K_F K_D G_{\text{LP}}(s) \right] \]  \hspace{1cm} (4.25)

\[ V_D(s) / \Phi_R(s), \text{ the ratio of output to input, we shall refer to as the system-function. } \]
It is a general theorem of linear control theory that the system-function has a bandwidth which is greater than the open-loop bandwidth [bandwidth of \( G_{\text{LP}}(s) \)] by the order of the feedback factor. This will be demonstrated at a later stage, but, for the time being, we assume that for frequencies which lie within the 3db-bandwidth, \( B/2 \), of \( G_{\text{LP}}(s) \), we may neglect the \( G_{\text{LP}}(s) \) terms in eqn. (4.25) and obtain

\[ V_D(s) / \Phi_R(s) = K_D s / F \]  \hspace{1cm} (4.26)

We are also interested in the response of the IF filter to a step function in phase. This situation arises in connection with the noise impulses discussed in chap. 3 and will be dealt with further at a later stage. For the moment we return to fig. 3.2a and observe that each time a noise impulse occurs, the phase jumps through \( \pm 2\pi \). \( \star \) We therefore consider the phase-function \( \phi_i(t) \) given by

\[ \phi_i(t) = U(t) \]  \hspace{1cm} (4.27)

\( \star \) Impulses of various types occur, but a reasonable assumption is to regard them as Heaviside step functions in the phase angle.
where $u(t)$ is the unit step-function occurring at time $t = 0$. From eqn. (4.7), $x(t)$ then becomes

$$x(t) = e^{jU(t)} \quad (4.28)$$

and this is the time-function which the low-pass analogue of the IF filter has to transmit. At this stage, it is convenient to change from complex-notation to real-notation and to express $x(t)$ in the form

$$x(t) = \cos \{U(t)\} = \cos \phi_1(t) \quad (4.29)$$

where, as we expect, $x(t)$ is a cosine whose phase, $\phi_1(t)$, is the step-function. The frequency of the cosine, $x(t)$, is obtained as the derivative $\dot{\phi}_1(t)$ of the phase. The derivative of the unit step is the unit impulse, and $x(t)$ is therefore a cosine with impulse frequency. The low-pass analogue of the IF filter, having finite bandwidth, [i.e., covering a finite range of frequencies], highly attenuates such a signal. We therefore conclude that when the input phase-function $\phi_1(t)$ consists of a step, the low-pass analogue of the IF filter produces zero output.

4.3. Effect of Feedback on FM Noise. Before considering the effect of feedback on FM noise, we must consider first how the feedback system functions as an amplitude system. The analogues we developed in the last section are only useful for considering the phase-function relations and are not applicable for the amplitude relations in the system.
Returning to the schematic, in fig. 4.1, of the system, it may be deduced that the amplitude conditions in the IF channel do not depend on the application of feedback. All amplitudes fed into the VCO are converted into frequency variations, the amplitude $A_L$ of the VCO's output being held at some constant level. Thus the IF noise spectrum $W_o(f)$, the IF carrier amplitude $A$, and therefore the IF carrier to noise power ratio (CNR) are not affected by the application of feedback. This was established by measurement of all 3 quantities on the experimental system described in appendix A.

Turning now to the noise characteristics of the feedback system, we recall from chap. 3 that the noise in the IF channel has two effects which are of importance from the point of view of causing interference at the output. One effect is to produce a perturbation of the carrier phase, giving rise to an excess-phase term, and the other effect is to produce jumps of $\pm 2\pi$ in the excess-phase of the IF wave.

From eqns. (3.5b) and (3.6), we see that the fluctuation phase-function $\phi_f(t)$ at the input to the detector, may be written

$$\phi_f(t) = \frac{y(t)}{A} \quad (4.30)$$

where $y(t)$ is the noise voltage in lead-quadrature with the unmodulated carrier (see sec. 2.4). The power spectrum $W_y(f)$ of $y(t)$ is given in eqn. (2.95)

$$W_y(f) = 2W_o(f + f_o) \quad (4.31)$$
and we have said that $W_o(f)$ is unaffected by feedback. $W_o(f + f_o)$ is similarly unaffected by feedback, and so we may regard $\phi_f(t)$ in eqn. (4.29) as a constant input at a point just in front of the detector. This is shown in fig. 4.6. The analogue used in this figure is the small-index analogue, and its use in this situation will be justified later. From eqns. (3.16), (4.31), and (4.30), we see that the power spectrum $W_\phi(f)$ of $\phi_f(t)$ may be written

$$W_\phi(f) = W_y(f)/A^2 = (2W_o/A^2)|G_{LP}(j\omega)|^2 \quad (4.32)$$

and, from appendix C, we are led to write the Laplace transform $\Phi_f(s)$ in the form

$$\Phi_f(s) = \Phi_1(s)G_{LP}(s) \quad (4.33)$$

where $\Phi_1(t)$ is the signal with power spectrum $2W_o/A^2$, $W_o$ being the midband level of the IF noise spectrum $W_o(f)$. From fig. 4.6, we have

$$V_1(s) = K_Ds\Phi_o(s) = K_Ds[\Phi_f(s) - \Phi_L(s)G_{LP}(s)] \quad (4.34)$$

$$\Phi_L(s) = V_1(s)K_FK_V/s \quad (4.35)$$

and using the relation in eqn. (4.33) gives

$$V_1(s)/\Phi_1(s) = G_{LP}(s)K_Ds/[1 + KVK_FKD_{LP}(s)] \quad (4.36)$$
Comparison of eqn. \((4.36)\) with eqn. \((4.25)\) suggests that the input \(\Phi_f(s)\) at the point just in front of the detector may be replaced by an input \(\Phi_1(s)\) at the RF input of the mixer. This is shown in fig. 4.7. As in eqn. \((4.26)\) we may neglect the \(G_{LP}(s)\) factors in eqn. \((4.36)\) and write

\[
\frac{V_1(s)}{\Phi_1(s)} = K_D s/F
\]  \((4.37)\)

We will recall from sec. 3.2, that eqn. \((4.30)\) is only valid when the amplitude \(r(t)\) of the IF noise vector is small compared with the IF carrier amplitude \(A\). Thus the phase-function \(\phi_f(t)\) in eqn. \((4.30)\) is small, and, from eqn. \((4.33)\), the phase-function \(\phi_1(t)\) at the input to the analogue in fig. 4.7 will also be small. We are therefore justified in using the small-index analogue.

The output time-function \(v_1(t)\) is the fluctuation noise component we have been discussing in chap. 3. The power spectrum \(W_1(f)\) of \(v_1(t)\) may therefore be obtained from eqn. \((4.37)\). We have already said that the input phase-function \(\phi_1(t)\) has power spectrum \(2W_0/A^2\); so from eqn. \((4.37)\)

\[
V_1(s) = \Phi_1(s)K_D s/F
\]  \((4.38)\)

\[
W_1(f) = \left(\frac{2W_0}{A^2}\right)K_D^2 \omega^2/F^2
\]  \((4.39)\)

and we see that the power spectrum \(W_1(f)\) of the output fluctuation noise is reduced by \(F^2\).

To investigate the effect of feedback on the spectrum due to impulsive noise, we recall from chap. 3 that the
number of impulses in the output depended only on the IF carrier to noise power ratio (CNR) and on the shape of the IF power spectrum, \( W_0(f) \). Neither of these, we have said, is affected by feedback. Thus, like the input \( \phi_f(t) \) for fluctuation noise, the input for impulse noise may be regarded as a sequence of steps of phase at a point just in front of the detector. We said in chap. 3 that these steps occurred randomly and independently of each other; so we may consider each one separately. If \( \phi_j(t) \) represents one such jump in phase, then

\[
\phi_j(t) = U(t) \quad ; \quad \phi_j(s) = 1/s
\]

where \( U(t) \) is the unit step-function. If we regard \( \phi_j(s) \) as replacing the input function \( \phi_f(s) \) in fig. 4.6, then the input \(-\phi_L(s)\) which arrives at the IF low-pass analogue, due to the single input \( \phi_j(s) \), is given by

\[
-\phi_L(s) = -(1/s)(KVKFK_D)
\]

and the input to the low-pass analogue of the IF filter is a negative step-function. We saw in sec. 4.2 that the low-pass analogue highly attenuates this function, so \( \phi_j(s) \) is the only input to the detector. Thus feedback does not act on the impulsive component of the output noise, and the power spectrum \( W_2(f) \) of the impulse noise output is unaffected by feedback.

We are now able to write down the expression for the
output noise spectrum, \( W_D(f) \), when the receiver is operating with feedback. \( W_1(f) \) is given in eqn. (4.39), and \( W_2(f) \) is given in eqn. (3.14). Thus

\[
W_D(f) = \left( \frac{2W_o}{A^2} \right) K_D^2 \left( \frac{\omega^2}{F^2} + K_D^2 \pi^2 r \operatorname{erf} (\text{CNR}) \right)^{1/2}
\]

(4.42)

\[
= 4\pi B_n K_D^2 \left\{ \left( \frac{\pi f/B_n}{\text{CNR}} \right)^2 \left( \frac{\text{CNR}}{1/F^2} + (2\pi a) \operatorname{erf} (\text{CNR}) \right) \right\}
\]

(4.43)

where, \( \text{CNR} \) \( \left[ = \frac{A^2}{(2W_o B_n)} \right] \) is the IF carrier to noise power ratio, \( B_n \) is the noise bandwidth of the IF noise filter, \( W_o \) is the midband level of the IF noise spectrum \( W_o(f) \), \( r \) is the radius of gyration of \( W(f) \), and \( a \) is a ratio defined in eqn. (3.18). Eqn. (4.43) is to be compared with eqn. (3.19), and it will be seen that eqn. (4.43) reduces to eqn. (3.19) when \( F = 1 \), i.e., when the system is on open-loop. The term containing the IF filter shape in eqn. (3.19) is not included in eqn. (4.43), since, as we have recently explained, it is not significant for output frequencies within the 3db-bandwidth of \( G_{LP}(j\omega) \). Eqn. (4.43) summarises the effect of feedback on the output noise spectrum, namely, that the fluctuation component of noise is reduced by the feedback factor, whereas the impulse component is virtually unaffected.

The performance of the feedback system was tested using the experimental set-up described in appendix A. For all the measurements on the feedback system, a single-pole IF filter, having a 3db-bandwidth of 100 KHz centered on 1.75 MHz, was used. Such a filter has a low-pass analogue with transfer function (see appendix B ) \( G_{LP}(s) \) given by
where the 3db-bandwidth $f'$ is $1/(2\pi T)$.

On the feedback system, it was ascertained, as we have already mentioned, that, for the feedback factors $F = 2$ and 4, the carrier amplitude, IF power spectrum, and (CNR) ratio, were unaffected. Oscillograms of the output noise for a carrier to noise power ratio (CNR) = 10 and feedback factors of $F = 1$ (no feedback), 2 and 4, are shown in fig. 4.8. The corresponding spectra of the noise are shown in fig. 4.9. From the oscillograms the noise will be seen to consist of both fluctuation and impulse noise. As feedback is applied, the oscillograms are seen to contain less fluctuation noise but the same amount of impulse noise. The measured spectra in fig. 4.9 confirm these observations. The measured spectra were found to agree, within the limits of experimental error, with eqn. (4.43). The form of this equation which involves the single-pole IF filter-shape, may be obtained from eqns. (4.36) and (4.44), giving

$$W_n(f) = \frac{n_B K_B^2}{\pi n_D^2}$$

$$= \pi (f/B_n)^2 (\text{CNR})^{-1} \left[ \frac{1/(1 + sT)}{1 + K_V K_P K_D [1/(1 + sT)]} \right]^2$$

$$+ (2\pi/a) \text{cerf} (\text{CNR})^{\frac{1}{2}}$$

$$= \pi (f/B_n)^2 (\text{CNR})^{-1} \left[ 1/[1 + s(T/F)] \right]^2 / F^2$$

$$+ (2\pi/a) \text{cerf} (\text{CNR})^{\frac{1}{2}}$$

(4.45)
The filter-shape in eqn. (4.46) has 3db-bandwidth \( f' = 1/[2\pi(T/F)] \), (see appendix \( B \)), which is seen to be extended by a factor \( F \) when compared with the open-loop bandwidth. A plot of this eqn. (4.46), with \( a = 0.752 \) as in eqn. (3.21), was found to agree, as we have said before, with the measured spectra in fig. 4.9.

4.4. Phase Characteristics and System Stability. The stability of a system depends on its phase characteristics with respect to sinusoidal signals. If we define the loop-function \( L(s) \) as the function which relates the 'output' \( \phi_L(s) \) to the input \( \phi_R(s) \) in fig. 4.2, Nyquist's stability criterion (ref. 27, chap. 4) is that the closed-loop system can only be stable if \( |L(s)| < 1 \) at the crossover frequency \( \omega_c \), which is the frequency at which the phase of \( L(s) \) is \(-180^\circ\). This is illustrated in fig. 4.10, in which the system is only stable if \( k < 1 \). Intuitively, this must be so, since, if an input signal reverses phase at the end of a loop, it becomes in-phase after going through the negative terminal of the adder, and if the loop-gain \( |L(s)| \) is greater than 1, the system is regenerative and oscillates at the crossover frequency \( \omega_c \). When a system oscillates, the oscillations can only be sinusoidal if there is only one crossover frequency, i.e., if there is only one frequency at which the system tends to regenerate. The analogue in fig. 4.2 may therefore be used to assess the stability performance of the system if we make all the phase functions sinusoidal.

If we stipulate that any oscillation which occurs in
the system will be such that the IF phase-function \( \phi_i(t) \) is small, then the loop-function \( L(s) \) may be obtained from the small-index analogue. Such an analogue is given in fig. 4.5, but here we are interested in the frequency response of the loop and must include the baseband filter as we have done in fig. 4.11. The fact that we are dealing with sinusoids leads us back to the concept of modulation index, (sec. 1.2), and we express \( \phi_i(t) \) in the form

\[
\phi_i(t) = M \sin \omega_i t
\]  

(4.47)

where, for the small-index analogue to be valid, \( M \), the modulation index, must lie in the range \( 0 \leq M \leq 1/2 \) [see eqn. (1.23)].

We must now investigate the phase characteristics of the system when the modulation index is not small. The baseband system will of course function as before. The effect of the IF system on the wave may be derived by assuming that the low-pass analogue of the IF filter has transfer function \( G_{LP}(s) \) given by

\[
G_{LP}(s) = k(s)e^{-\left(j\lambda_0 + st_1 + s^3t_2 + \ldots\right)}
\]  

(4.48)

where \( k(s) \) is the modulus of \( G_{LP}(s) \), and the exponential term gives the phase of \( G_{LP}(s) \). \( \lambda_0 \) is a constant term, \( st_1 \) a linear term, and so on. If \( X(s) \) is the Laplace transform of the input \( x(t) \) to this filter, then the output \( Y(s) \) is
Sec. 4.4.  FM ANALOGUE AND FMFB SYSTEMS  142

\[ Y(s) = k(s)X(s)e^{-j\lambda_o}e^{-st_1} - e^{-s^2t_2} \ldots \]  \hfill (4.49)

Since we are only interested in the phase of the output, we shall assume that \( k(s) \) is a constant, \( k_o \), bearing in mind that if it is not, it will cause the output to distort (but will not affect the phasing). The output time-function \( y(t) \) is then

\[ y(t) = k_o e^{-j\lambda_o} \int \{ X(s)e^{-st_1} - s^2t_2 \ldots \} \]  \hfill (4.50)

where \( \int \{ \} \) means the inverse Laplace transform of \( \{ \} \).

The output \( v_o(t) \) of the bandpass IF filter may be obtained from eqn. (4.3).

\[ v_o(t) = -1 \int \{ X(s)e^{-st_1} - s^2t_2 \ldots \}k_o e^{j(\omega_o t - \lambda_o)} \]  \hfill (4.51)

showing that a constant phase-delay term \( \lambda_o \) in the IF phase characteristic delays the carrier frequency in phase, but since the frequency detector is insensitive to the carrier phase, the loop stability is unaffected. By writing

\[ F_1(s) = (X(s)e^{-st_1}) \quad ; \quad F_2(s) = (e^{-s^2t_2} \ldots) \]  \hfill (4.52)

\[ f_1(t) = \int F_1(s) \quad ; \quad f_2(t) = \int F_2(s) \]  \hfill (4.53)

\( f_1(t) \) from appendix C.5 will be seen to be the function
x(t) delayed in time by amount $t_1$. Thus if the IF filter has constant gain and a phase characteristic comprising only a constant and a linear term, the carrier term will be delayed in phase by its passage through the filter, and the input function x(t) will be delayed in time. When other terms are present in the phase characteristic, the convolution theorem (appendix C.6) gives

$$v_o(t) = \left\{ \int_0^t f_1(\tau)f_2(t-\tau)d\tau \right\} e^{+j(\omega_0 t - \lambda_0)} \quad (4.54)$$

The term in brackets in eqn. (4.54) may be shown (see, for example, ref. 24, chap. 8) to be a distorted version of $f_1(t)$, no additional time delay being produced. In summary, then, x(t), in passing through the filter, is delayed in time by the linear component of phase and is distorted by all other frequency-dependent components. For a sinusoidal phase-function, x(t) in eqns. (4.7) and (4.47) becomes

$$x(t) = e^{jM \sin \omega_i t} \quad (4.55)$$

and when the IF phase characteristic contains only the linear term $-\omega t_1$, the output y(t) becomes the delayed version of x(t). Thus

$$y(t) = x(t-t_1) = e^{jM \sin \omega_i (t-t_1)} = e^{jM \sin (\omega_i t - \lambda_i)} ; \quad \lambda_i = \omega_i t_1 \quad (4.56)$$
Sec. 4.4. FM ANALOGUE AND FMFB SYSTEMS

and the sinusoidal phase-function, in passing through the IF filter, is delayed in phase by amount $\lambda_i$.

To determine the mode in which the feedback system oscillates, we direct our attention to the low-pass characteristic of a typical IF filter. This is shown in fig. 4.12. Within the 3db-bandwidth of this filter, the gain [i.e., modulus of $G_{\text{LF}}(s)$] is approximately constant and the phase approximately linear. Outside the 3db-bandwidth, both gain and phase begin to deviate from the 'ideal'. We consider the build up of an oscillation at frequency $\omega_i$ located within the 3db-bandwidth of the filter. Whatever the amplitude of the oscillation, it will be delayed in phase by amount $\lambda_i$. As the oscillation builds up at this frequency, spectral components are produced at intervals of $\omega_i$ according to the analysis in sec. 1.2 [see, for example, eqn. (1.18)], and the bandwidth $B_i$, occupied by the oscillation, increases. $B_i$, obtained from Carson's rule, is

$$B_i = 2f_i(1 + M) ; \quad M = [(B_i/2)/f_i]^{-1} \quad (4.57)$$

As soon as the bandwidth $B_i$ 'overspills' $B$, distortion begins to occur, producing a fundamental and harmonics. We have said that oscillations in a feedback system are sinusoidal; so the oscillation can only build up to the point where its spectrum occupies the 3db-bandwidth (or thereabouts) of the IF filter. Setting $B_i$ equal to $B$, where $B$ is the 3db-bandwidth of the IF filter, gives, from
and the IF filter can only sustain large-index oscillations if they are at frequencies which are small compared with the half-3db-bandwidth of the IF filter. When $B/(2f_1) = 1/2$, then $M = 1/2$, and $f_1 = B/3$ is the highest frequency at which the IF filter can sustain a large-index oscillation. At any higher frequency, the IF filter can only sustain a small-index oscillation ($M \leq 1/2$), this oscillation building up to such a maximum amplitude as will produce an IF modulation index of $M = 1/2$ (or thereabouts).

Using the system described in appendix A, the small-index loop-function was found to crossover at 90KHz. The 3db-bandwidth of the IF filter used was 100 KHz (half-3db-bandwidth = 50 KHz). When the loop-gain of the system was made large enough, the system was found to oscillate at 90 KHz, producing an amplitude at the detector output which corresponded with an IF modulation index of approximately $1/2$ (see appendix A).

4.5. Threshold Improvement Using Feedback. We are now in a position to investigate how feedback may be used to bring about threshold improvement when we are dealing with large-index FM reception. We saw in chap. 3 that the three components in the output which define the system’s performance are 1) the output signal, 2) fluctuation noise and 3) impulse noise. For a conventional (i.e., open-loop)
receiver, let us say that curve ABC in fig. 4.13 describes the output signal to noise performance of the system. The derivation of this curve is dealt with in chap. 3, where we saw that at the threshold point, \((\text{CNR})_T\), the impulsive component of the output noise began to assume precedence over the fluctuation noise. Now let us assume that feedback is applied to this receiver. We know that signal and fluctuation noise are reduced by the feedback factor \(F\) and that impulse noise is unaffected. At high carrier to noise ratios where impulse noise is negligible, the signal to noise performance of the system is therefore unaffected by feedback. However, as \((\text{CNR})\) is decreased, because fluctuation noise has been reduced by feedback, impulse noise assumes significance at a higher \((\text{CNR})\). Curve ADE in fig. 4.13 describes the performance of the system when feedback is applied, and the threshold \((\text{CNR})_T\) is higher than the open-loop threshold \((\text{CNR})_T\). From our previous discussions, we know that when feedback is applied, the modulation index \(M\) and frequency deviation \(\Delta f\) of the IF wave are decreased by the feedback factor (see, for example, secs. 1.5 and 4.2). Thus the IF bandwidth may be decreased to such an extent that it covers the essential components of the spectrum of the IF signal. From Carson's rule, \(B = 2f_s (1 + M) = 2f_s + 2\Delta f\), we observe that no matter how small \(M\) and \(\Delta f\) are made, \(B\) has a minimum value of \(2f_s\).

It will be seen from eqns. (4.30) through (4.39) that changing the IF filter-bandwidth affects only the bandwidth of \(W_1(f)\). \(W_1(f)\), the power spectrum of the fluctuation noise component, has bandwidth equal to the bandwidth
of the system-function, and we said [and demonstrated in eqn. (4.46)] that the system-function had greater bandwidth than the open-loop bandwidth. Thus, in the limiting case where enough feedback has been applied to shrink the IF bandwidth to \(2f_s\), the bandwidth of \(W_1(f)\) will be larger than \(2f_s\), and over the output range of frequencies, \(0\) to \(f_s\), \(W_1(f)\) will effectively be independent of the IF filter-shape. This is, of course, true on open-loop as well, where the bandwidth ratio \(B/(2f_s)\) is large for large-index systems, and \(W_1(f)\) is therefore independent of the IF filter shape over the range of output frequencies. In summary, then, when feedback is applied and the IF bandwidth reduced, the level of \(W_1(f)\) is reduced by \(F^2\) (\(F = \) feedback factor), and its shape is unaltered over the output range of frequencies. Thus, as in the case when the IF bandwidth was unchanged, output signal and fluctuation noise are reduced to the same extent by feedback.

From eqn. (4.42) it is seen that the power spectrum \(W_2(f)\) of the impulse noise is unaffected by feedback but depends on the radius of gyration \(r\) of the IF noise spectrum \(W_0(f)\), and depends on the IF carrier to noise power ratio (CNR). By writing \(r = B_n/a\), where \(a\) depends on the IF filter shape (see table 3.1) and where \(B_n\) is the noise bandwidth of the IF filter, and also by expressing (CNR) in the form

\[
(CNR) = \frac{A^2}{(2W_0 B_n)} = \frac{c}{B_n}, \quad c = \frac{A^2}{(2W_0)}
\]

(4.59)

where \(A\) is the IF carrier amplitude and \(W_0\) is the midband level of the IF noise spectrum, it is possible to express
Sec. 4.6. FM ANALOGUE AND FMFB SYSTEMS

\[ W_2(f) = K_D^2 8n^2 B_n \text{ cerf} (c/B_n)^{1/2} \]  (4.60)

Since \text{cerf}(x) decreases as x increases, decrease in the IF bandwidth will be seen to produce a twofold decrease in \( W_2(f) \). Over the threshold region, \((\text{CNR}) = 5 \) to 15, \text{cerf}(\text{CNR}) will be seen from tables (ref. 28) to decrease rapidly as \((\text{CNR})\) increases, and the decrease in \( W_2(f) \), brought about by decrease in \( B_n \), will, in practice, be greater than the decrease in \( W_1(f) \), which results from the application of feedback. If \( W_2(f) \) is depressed to a greater extent than \( W_1(f) \), it means that the system's threshold is better than it was on open-loop. This is illustrated in fig. 4.13 where curve AFG describes the performance of the system when feedback plus reduced IF bandwidth are used.

4.6. Design Criteria and Design Example. The points which were made in the last section can best be illustrated by considering a design example. In the foregoing section, it was assumed that all the filtering in the feedback loop was done at IF, and, in practice, as far as output signal to noise and threshold performance are concerned, this may be assumed to be the case, as we shall see. In establishing design criteria, however, we take into account the system-stability, and consideration must then be given to the entire loop-function. We saw in sec. 4.4 that large-index oscillations could only occur for frequencies over which the IF phase characteristic was linear and that no matter what
amplitude the oscillation had, as long as its spectrum was confined to the linear region of the IF filter, its phase delay would be the same as that of a small-index sinusoid. The loop-function \(L(s)\), in which we are interested, is then given in fig. 4.11.

If the loop-function \(L(s)\) contains only one single-pole or one double-pole filter at IF, the system is inherently stable, since for very high frequencies, the phase responses of the former and latter tend to \(-90^\circ\) and \(-180^\circ\) respectively. In practice, this is impossible to achieve for two main reasons.

1) Even with the use of balanced mixers and detectors, the output of the mixer will contain a certain amount of break-through of the local oscillator frequency (local oscillator break-through), and the output of the frequency detector will contain IF break-through. If the amount of break-through is considerable, overloading of subsequent amplifier stages will occur, and some method of rejecting both local oscillator and IF break-through must be included in the design. Both types of break-through may be reduced with a minimum of phase delay by series-resonant trapping circuits of the type shown in fig. 4.14. \(L\) and \(C\) resonate at the rejection frequency \(f_r\), \(\frac{1}{L}R\) being the resistance associated with \(L\). If \(R\) is much less than the impedance of \(C\) at \(f_r\), and if the circuit has very low loss (i.e., \(\frac{1}{L}R\) is small, and the circuit \(Q\) is large), the gain of this circuit will be nearly flat and the phase delay nearly zero for frequencies very nearly up to \(f_r\) (see, for example, ref. 37, chap. 15); that is, the 'activity' of the circuit may be confined
to a very narrow band of frequencies centered on $f_r$.

2) In addition to the phase delay which results from filtering, there is that which arises due to the time lag, $t_1$, of a signal in the transmission lines, transistors, etc., of the circuit. From

$$\lambda = \omega t_1 = 2\pi f t_1 \quad (4.61)$$

the contribution of this delay to the phase characteristic of the loop will be a linear one. The use of short lines and fast transistors will minimise this function.

The choice of the IF filter is dictated by a) its noise-rejection properties and b) its phase properties. Good noise-rejection is obtained by using filters with steep skirts, but the phase characteristics of such filters are poor. A single-pole filter has the worst possible noise-rejection properties, but, conversely, it has the best possible phase characteristics. A two-pole filter is a good compromise between the two requirements, and the practical realization of such a filter is easy (appendix B.2). The transfer function, $G_{LP}(s)$, of the low-pass analogue of the two-pole filter is given by (see appendix B.2)

$$G_{LP}(s) = \frac{1}{1 + 2^{\frac{1}{2}}sT + (sT)^2} \quad (4.62a)$$

$$= \left[1 + \left(\frac{f}{f'}\right)^2\right]^{\frac{1}{2}} \left[\frac{\tan^{-1}\left(2^{\frac{1}{2}}(f/f')\right)}{1 - \left(\frac{f}{f'}\right)^2}\right] \quad (4.62b)$$

where $f'$ is the 3db-bandwidth of the low-pass analogue.
The factor \( j \), which, as appendix B.2 shows, should be present in the numerator of eqn. (4.62a), has been here omitted, since its only effect is to introduce a constant phase angle of \( 90^\circ \) into the filter's phase characteristic, and we saw in sec. 4.4 that this term affects only the phase shift of the carrier and does not affect the phase shift of the signal.

For the feedback system, it is required that the IF, \( f_0 \), should be low compared with the local oscillator frequency \( f_L \), and should be high compared with the highest baseband frequency, \( f_s \). This is necessary so that the trapping circuits for \( f_L \) and \( f_0 \) do not introduce very much phase into the loop-function \( L(s) \). On the other hand, \( f_0 \) cannot be made too small compared with \( f_L \), since \( f_0 \), \( f_L \), and the RF, \( f_R \), are related through

\[
f_R - f_L = f_0 \tag{4.63}
\]

and small percentage variations in \( f_L \) will result in large percentage variations in \( f_0 \), if \( f_0 \ll f_L \).

The choice of \( f_0 \) also affects the Q requirement of the IF filter. For the two-pole filter, the circuit Q, the 3db-bandwidth, B, and \( f_0 \) are related, as shown in appendix B.2, through

\[
Q = 2^\frac{1}{2} f_0 / B \tag{4.64}
\]

and it is possible for \( f_0 \) to be so large that practical realizability of the circuit Q becomes difficult.

We will now apply these criteria to the design of a
feedback system for a given wideband FM transmission, and we will compare the performance of this demodulator with that of a conventional receiver. Let us say the FM transmission is specified by

\[ f_{RF} = 1000 \text{ MHz}; \ (\Delta f)_{RF} = 20 \text{ MHz}; \ f_s = 2 \text{ MHz} \quad (4.65) \]

where \( f_{RF} \) is the RF, \( (\Delta f)_{RF} \) is the frequency deviation of the RF wave, and \( f_s \) is the baseband signal bandwidth. The modulation index \( M_{RF} \) and the spectrum-bandwidth \( B_{RF} \) of the RF wave are given by

\[ M_{RF} = (\Delta f)_{RF}/f_s = 10; \ B_{RF} = 2f_s (1 + M_{RF}) = 44 \text{ MHz} \quad (4.66) \]

We proceed with the design in the following steps:

1) Choose a two-pole filter for the IF channel. The bandwidth \( B_{IF} \) of this filter is to be decided on the basis of the deviation \( (\Delta f)_{IF} \) and the modulation index \( M_{IF} \) in the IF channel.

\[ B_{IF} = 2f_s (1 + M_{IF}); \ M_{IF} = M_{RF}/F; \ (\Delta f)_{IF} = (\Delta f)_{RF}/F \quad (4.67) \]

\[ F = \text{feedback factor} = 1 + K_V K_F K_D = 1 + K_L \quad (4.68) \]

where \( K_L \) is called the loop-gain. We wish to apply as much feedback as will compress \( M_{IF} \) to the value of \( 1/2 \) (or thereabouts), at which point the IF wave will be small-index modulated and the IF bandwidth \( B_{IF} \) can be reduced to its
minimum possible value of $2f_s$. To achieve this, we need
a feedback factor of 20, i.e., a loop-gain $K_L$ of 19. It
is not always possible to obtain as much loop-gain as is
required to make $M_{IF} = \frac{1}{2}$ by increasing the feedback-gain
$K_F$. This is because if the voltage fed to the VCO is too
large, the VCO will saturate. Large loop gain may be
obtained, though, by designing the VCO to have high sensit-
itivity $K_V$, and we shall assume here that we may achieve a loop-
gain of 19, giving, as we have said, an IF 3db-bandwidth
$B_{IF} = 2f_s = 4$ MHz. (For any other choice of $B_{IF}$, the rest
of the design follows in the same way).

2) Choose $f_o = 100$ MHz. This makes $f_L = 900$ MHz, and a
$\pm 1\%$ ( = 9 MHz) change in $f_L$ results in a 9% change in $f_o$.
Strict control of $f_L$ is therefore required. The chosen
IF, $f_o$, is sufficiently low compared with $f_R$ and sufficiently
high compared with $f_s$ that the use of trapping circuits for
$f_L$ and $f_o$ will not introduce too much phase delay into the
loop-function $L(s)$ over the important range of frequencies.

3) With all the components in the loop designed, the loop-
characteristic $L(s)$ is measured as described in appendix A.9
By far the major contribution to the phase of $L(s)$ will come
from the two-pole IF filter (which, on its own, is inherently
stable). However, the trapping circuits and finite transit
time effects will introduce additional phase delay. The latter
source, we said, will contribute a linear term, and the former
source may be shown (see ref. 37, chap. 15) to also contri-
bute a small linear term for the range of low-frequencies we
are interested in. The effect of the additional phase will
be to make the phase of \( L(s) \) crossover the \(-180^\circ\) axis instead of making it tend towards it asymptotically.

4) The measured loop-function \( L(s) \) is tested for stable gain-margin and stable phase-margin (ref. 26, chap. 7), and if required, \( L(s) \) is compensated with one or more phase-lead networks of the type shown in fig. 4.15. Such networks are described by Thaler and Brown (ref. 26, chap. 7). By suitable choice of \( a \) and \( T_1 \), these networks may be used as powerful compensating tools, offsetting the linear phase-delay terms by providing approximately linear phase-lead.

This completes the design of the FMFB receiver, and we turn to a comparative assessment of its performance.

Turning to eqn. (4.36), we wish to show that \( W_1(f) \) has extended bandwidth on feedback. In this equation, then, we use the low-pass characteristic \( G_{LP}(s) \) of the two-pole filter given in eqn. (4.62a), and writing \( K_L = K \sqrt{F} K_D \) and \( F = 1 + K_L \) as before, we obtain

\[
\frac{G_{LP}(s)}{[1 + K_L G_{LP}(s)]} = \frac{1/F}{[1 - (\omega T)^2/F + 2^{1/2} sT/F]} \quad (4.69)
\]

The 3db-bandwidth \( f' [= \omega'/(2\pi)] \) of the function in eqn. (4.69) is obtained as the frequency at which the square of the modulus of the denominator has the value 2. Thus

\[
[1 - (\omega T)^2/F]^2 + [2^{1/2} \omega T/F]^2 = 2 \quad (4.70)
\]

\[
\omega' \doteq 7/T ; \quad f' \doteq 7/(2\pi T) \quad (4.71)
\]
when $F = 20$. From appendix B.2, the two-pole characteristic in eqn. (4.62a) has 3db-bandwidth $f' = 1/(2\pi T)$, so that with $F = 20$, the 3db-bandwidth is extended by a factor of 7.

The bandwidth of the two-pole filter in the IF channel of the feedback receiver is $2f_s$, and therefore, over the output range $f_s$, the spectrum $W_1(f)$ may be assumed to be independent of $G_{LP}(s)$. From eqn. (4.43), we therefore express the output noise spectrum $W_D(f)$ by

$$W_D(f) = 8\pi^2 K_D^2 \left[ f^2 \left( 2(CNR)B_n F^2 \right)^{-1} + \left( B_n/a \right) \operatorname{erf} \left( CNR \left( \frac{1}{2} \right) \right) \right] (4.72)$$

and from table 3.1, $a = 2.22$ for a two-pole filter. The output noise power $N_D$ in the baseband bandwidth is obtained from

$$N_D = \int_0^{f_s} W_D(f) df = 8\pi^2 K_D^2 \left[ f_s^3 \left( 6(CNR)B_n F^2 \right)^{-1} \right. \\
+ f_s \left( B_n/a \right) \operatorname{erf} \left( CNR \left( \frac{1}{2} \right) \right) \right] (4.73)$$

From sec. 3.7 we recall that at the threshold, the second term in eqn. (4.73) is 0.259 times the first term. If the threshold carrier to noise ratio is $(CNR)_T$, we therefore obtain from eqn. (4.73) that

$$f_s \left( B_n/a \right) \operatorname{erf} \left( CNR \left( \frac{1}{2} \right) \right)_T = 0.259 f_s^3 \left( 6(CNR)_T B_n F^2 \right)^{-1} (4.74)$$

For the given system under consideration,
where $a$ and $B_n$ for the two-pole filter are obtained from table 3.1. Substitution of these values into eqn. (4.74) gives

$$ (\text{CNR})_T \text{ cerf } (\text{CNR})_T^{\frac{1}{2}} = 4.81 \times 10^{-5} ; \quad (\text{CNR})_T = 10.5 \quad (4.76) $$

where the solution of the equation was obtained by the 'regula falsi' method (ref. 32). This is the threshold carrier to noise ratio measured in the noise bandwidth $B_n$ of the IF filter. The threshold carrier to noise ratio $(\text{CNR})'_T$, measured in an arbitrary rectangular bandwidth of $2f_s$, is then

$$ (\text{CNR})'_T = (\text{CNR})_T \times \left( \frac{B_n}{2f_s} \right) = 10.5 \times 2^{\frac{1}{4}} \pi/4 = 11.1 \quad (4.77) $$

To compare the threshold performance of the feedback system with that of a conventional receiver, we assume that the conventional receiver uses a rectangular IF filter, the bandwidth of which is given by $B_{RF} (= 44 \times 10^6 \text{ Hz})$ in eqn. (4.66). For the rectangular filter, $a = 1$ from table 3.1, so the relevant substitutions which must be made into eqn. (4.74) are

$$ f_s = 2 \times 10^6 \text{ Hz} ; \quad B_n = 44 \times 10^6 \text{ Hz} ; \quad a = 1 ; \quad F = 1 \quad (4.78) $$
which give

\[(\text{CNR})_T \text{ cerf} (\text{CNR})_T^{1/2} = 3.09 \times 10^{-4}; \quad (\text{CNR})_T = 8.5 \quad (4.79a)\]

\[(\text{CNR})_T = 8.5 \times 11 = 93.5 \quad (4.79b)\]

where \((\text{CNR})_T^{1/2}\) is, as before, the threshold carrier to noise ratio measured in a bandwidth of \(2f_s\). The threshold improvement \(I\), using the feedback system, is expressed as the ratio of the threshold \((\text{CNR})\) without feedback to the threshold \((\text{CNR})\) using feedback. Thus.

\[I = \frac{93.5}{11.1} = 8.42 = 9.2 \text{ db} \quad (4.80)\]

### 4.7. Conclusions

By a thorough investigation of the noise characteristics of FM systems, we have been able to explain exactly how feedback improves the threshold of large-index modulation systems. The greatest threshold improvement, and best output signal to noise ratio, are obtained with large RF modulation indices, provided of course that there is enough 'non-saturating' loop-gain in the system. Indeed, if there were limitless loop-gain, then, irrespective of the input modulation index, the IF wave could always be made a small-index wave; for the system we have just considered, the feedback threshold would always be given by eqns. (4.76) or (4.77). In practice there will be a limit on the available loop-gain and it will not always be possible to make the IF wave small-index modulated.
By properly designing the feedback receiver, the phase characteristics of the loop may be so tailored that the available loop-gain is the only factor which restricts the amount of feedback that may be applied.
FIG. 4.1. FMFB Receiver

\[ X(s) = j\phi_i(t) \]
\[ Y(s) = j\phi_o(t) \]

FIG. 4.2. Analogue of FMFB Receiver

FIG. 4.3. Signal Analogue
FIG. 4.4. Vector Diagram

FIG. 4.5. Small Index Analogue

FIG. 4.6. Fluctuation-Noise Analogue

FIG. 4.7. Equivalent Fluctuation-Noise Analogue
FIG. 4.8. OSCILLOGRAMS OF THE OUTPUT NOISE WHEN FEEDBACK IS APPLIED
FIG. 4.9. Spectrum of FM Noise when Feedback is Applied
FIG. 4.10. Polar Plot of $L(s)$

$L(s) = K_{VK}K_{FK}G_F(s)G_{LP}(s)$

FIG. 4.11. Loop-Function Analogue
FIG. 4.12. Low-Pass Characteristic of IF Filter

FIG. 4.13. Threshold Curves
FIG. 4.14. Trapping Circuit

\[ a = \frac{R_2}{R_1 + R_2} \]
\[ T_1 = R_1 C_1 \]
\[ G(s) = a(1 + sT_1)(1 + asT_1)^{-1} \]

a. Circuit Diagram

b. Frequency Response

FIG. 4.15. Phase-Lead Compensation
APPENDIX A

EXPERIMENTAL SET-UP AND MEASUREMENTS

A.1. Introduction. The objective of the experimental work was to test the basic theory of operation of the FMFB receiver, and towards this end, the experimental set-up schematically represented in fig. A.1 was designed and assembled. In the design, considerations of economy and ready-availability of components played a significant part. We shall discuss here the choice of the various frequencies and bandwidths.

11.75 MHz RF. The requirement here is that the RF should be much higher than the IF so that the former may easily be rejected by the IF filter. 11.75 MHz was found to be near the upper limit at which the available summing amplifiers would function satisfactorily.

1.75 MHz IF. The primary reason for choosing 1.75 MHz was that it is a commercial broadcast frequency (sound channel for TV), and IF amplifiers could therefore be easily obtained. 1.75 MHz is sufficiently low compared with the RF to make rejection of the RF by the IF filter satisfactory. The Q requirement of the IF filter at this fairly low IF is also non-critical.
100 KHz IF Bandwidth. One of the experimental objectives was to observe impulses at the detector's output, and for this purpose, a fairly wide output bandwidth was required. 50 KHz was chosen for the output bandwidth and 100 KHz for the 3db IF bandwidth. With this IF bandwidth, the shape of the IF filter would not manifest itself to any great extent in the output noise spectra.

With an IF bandwidth of 100 KHz and an IF of 1.75 MHz, the Q's of the single-pole and two-pole IF filters are (see appendix B)

\[ Q = \frac{1.75 \text{ MHz}}{100 \text{ KHz}} = 17.5 \text{ , single-pole filter} \quad (A.1) \]

\[ = \frac{2^{\frac{1}{2}} \times 17.5}{2} = 24.7 \text{ , two-pole filter} \quad (A.2) \]

and Q's of this order are easily obtained in practice.

A.2. Noise Generator. An RCA 931A photo-multiplier tube (PM, for short) was chosen as the noise generator because such tubes are known to be capable of producing high-level wideband noise. A high level of noise is required at this point so that the noise which arises within the system will be small by comparison. Sard (ref. 38), by considering the shape of the current impulses which arrive at the PM's anode, has calculated the spectrum of PM noise and obtains that it is flat to within 1db up to 200 MHz (3db point at 400 MHz approximately). Spangenberg (ref. 39, chap. 12) arrives at the following formula for the mean squared shot noise current \[ \frac{I_n^2}{2} \] at the anode of the PM tube.
\[
\frac{1}{n} = 2e_i A \left( g - 1 \right) / (g - 1)
\]

\( e = \) electron charge = 1.6 x 10\(^{-19}\) coulomb

\( i_a = \) d.c. anode current

\( g = \) gain per multiplier stage

\( n = \) number of multiplier stages

\( \Delta f = \) system bandwidth

(A.3)

The layout of the noise generator is schematically represented in fig. A.2, and the circuit diagram and equivalent circuit are shown in figs. A.3 and A.4 respectively.

For maximum noise output, it is required that the PM tube be run at maximum d.c. anode current and maximum gain, the latter being proportional to the anode-to-cathode voltage (see ref. 40). 100\(\mu A\) (= 1/10 of maximum rated) was chosen for the d.c. anode current, this being considered a good compromise between high output noise level and longevity of the tube. The tube was run at the recommended supply voltage of 1KV. The photocathode of the PM tube was illuminated by a small 24-volt bulb, which was run at approximately 12 volts so that it would produce a stable light intensity on both long and short term bases. The light intensity produced by the bulb was found to be much more than that required to produce a d.c. anode current of 100\(\mu A\), and a substantial area of the photocathode had therefore to be masked.

For the noise-generator application of the PM tube, it
is required that the dynode voltages do not fluctuate as the anode current fluctuates. This may be ensured by a) making the dynode-chain current (bleeder current) very much larger than the d.c. anode current (but this causes excessive heat dissipation in the dynode-chain resistors) and by b) a.c. decoupling the later dynode stages where the fluctuations are large. A compromise value of 1mA was chosen for the bleeder current, giving chain resistors of 100 kΩ and total power dissipation of 1 watt. The last two dynode stages were also a.c. decoupled as shown.

The a.c. equivalent circuit of the noise generator is shown in fig. A.4. The thermal noise-voltage produced by $R_a$ has been omitted from the equivalent circuit, since it is very much smaller than the noise-voltage produced by the tube. $R_a$, $R_L$, and $C_a$ are the anode resistance, external load resistance, and anode-to-earth capacitance respectively, and together they form a circuit having transfer characteristic $\frac{v_o}{i_n^2}$ which is low-pass in nature.

With $R_a = R_L = 50 \Omega$, $C_a$ does not have any appreciable shunting effect for frequencies up to about 1000 MHz, and preliminary measurements of the PM-noise spectrum under these conditions showed confirmation of Sard's (ref. 38) result. However, simultaneous measurement of the noise level showed a large deviation from Spangenberg's formula [eqn.(A.3)]. From fig. A.4, the matched power $P$ delivered to a 50Ω load is

$$P = \frac{i_n^2}{4} \cdot (50/4) \quad (A.4)$$

For the RCA 931A, the overall gain of the 9-stage tube, for
a supply of 1kV, is typically $8 \times 10^5$ giving a stage gain $g$ of 4.53. Substitution into eqn. (A.3), with $\Delta f = 1$ and $i_a = 100 \mu A$, therefore gives

$$P = 4.123 \times 10^{-16} \text{ watts/Hz} = -123.8 \text{ dbm/Hz} \tag{A.5}$$

This level is lower, by a factor of about $10^5$, than the level of -70 dBm which was measured. A possible (but unlikely) explanation of this is that the tube-gain is greater, by a factor of about $10^5$, than the quoted value.

Substitution of the measured value into eqn. (A.4) gives

$$\overline{i_n^2} = 8 \times 10^{-12} \text{ (Amp.)}^2/\text{Hz} \tag{A.6}$$

For use in the experimental set-up, the anode load was made 300Ω, and $R_L$, the input impedance of the summing amplifier, was made 2kΩ. The shunting effect of $C_a$ is then negligible for frequencies up to about 100 MHz.

Since signal and noise are fed into the system at the same point, namely at the input of the summing amplifier, we consider the noise power in a noise bandwidth equal to the IF noise bandwidth. For an IF bandwidth of 100 KHz, the noise bandwidth is 157 KHz for a single-pole filter-shape and 110 KHz for the two-pole filter-shape. Let us consider an IF noise bandwidth of 150 KHz. From eqn. (A.6), the mean squared noise current $\overline{i_n^2}$ and rms noise current $(\overline{i_n^2})^{\frac{1}{2}}$ in this bandwidth are
\[ \frac{1}{n} = 1.2 \times 10^{-6} \text{ (Amp.)}^2 \quad ; \quad \left(\frac{1}{n}\right)^{\frac{1}{2}} = 1.1 \text{ mA} \quad \text{(A.7)} \]

Neglecting the shunting effect of \( C_a \) (10 pF) and the loading effect of \( R_L \) (2kΩ) gives that the rms noise voltage \( (v_n^2)^{\frac{1}{2}} \) at the anode of the PM tube is

\[ (v_n^2)^{\frac{1}{2}} = 1.1 \times 300 = 330 \text{ mV rms} \quad \text{(A.8)} \]

This, then, is the rms noise-voltage that the PM gives in a rectangular bandwidth of 150 KHz (centered on the RF of 11.75 MHz, say) and it was found to be more than adequate for the experimental requirement. In fact the tube produced the required amount of noise at a d.c. anode current of 20\( \mu \)A and a supply voltage of \( \frac{1}{2} \text{ KV} \).

From eqn. (A.3), the mean squared noise current depends on the stage-gain of the tube and on the d.c. anode current. The noise output was varied by varying the supply voltage.

**A.3. Summing Amplifier.** PM signal and noise were added, using an RCA CA3015A operational amplifier as shown in fig. A.5. This device was chosen because of its high-frequency capability (see RCA data-sheet No. 310 and application notes ICAN 5290, 5213). Both summing resistors \( R_1 \) and \( R_2 \) were made equal (2kΩ) to create identical input impedance levels for both signal and noise. With \( R_f = 33 \text{kΩ} \), the gain for each input should be \( 33/2 = 16\frac{1}{2} = 24 \text{ dB} \). The measured saturation-curve in fig. A.6 confirms this. The load impedance is non-critical provided it is much larger than the
amplifier's output impedance (85Ω), and a load impedance of 2.2 kΩ was used. The frequency response of the summing amplifier is shown in fig. A.7. For both the curves in figs. A.6 and A.7, terminals 1 and 2 in fig. A.5 were used in turn as inputs, the output showing no significant difference with respect to the input terminal used. Fig. A.8 demonstrates the summing performance of the amplifier when a square wave (100mV p-p, 1/3 MHz) is fed to terminal 1 and a sine wave (30mV p-p, 4 MHz) is fed to terminal 2.

A.4. Mixer. An RCA CA3005 differential RF-amplifier (RCA data sheet No. 125, Application Note ICAN 5022) was used for the mixer as shown in fig. A.9. The application note for the device shows that the output voltage \( V_0 \) may be written

\[
V_0 = \text{constant} \cdot Z_L \cdot g_{m2} \cdot v_1 \cdot v_2
\]  

(A.9)

where \( g_{m2} \) is the transconductance of the 'current' transistor \( T_3 \) (fig. A.9). Thus, multiplicative mixing occurs, producing the sum-frequencies (upper sideband), difference-frequencies (lower sideband), and harmonic sidebands. The saturation characteristic of the CA 3005 indicates that the differential pair begins to saturate (and therefore limit) for about 100mV p-p input, and an input of 15mV p-p was therefore chosen. This was fed in across 200Ω which was found to be the maximum resistance which did not intolerably d.c. unbalance the differential pair. Too much d.c. unbalance resulted in decreased IF output and increased local-
oscillator breakthrough.

A 15mV p-p input to the mixer requires a 10mV p-p input to the summing amplifier, which is satisfactory, since it is well in excess of the summing amplifier's noise (noise figure = 12 db), and the noise generator is able to supply the required amount of noise.

The gain of the mixer \( \left( \frac{v_o}{v_i} = \text{amplitude of IF output/\text{amplitude of RF input}} \right) \) is proportional to \( v_2 g_{m2} Z_L \) from eqn. (A.9). The \( v_2 g_{m2} \) relationship of the CA3005 is linear up to 2.5v rms; so with an oscillator-drive of 1.5V p-p, good gain is achieved without saturation. In the design of the IF transformer, \( 4.3 \, \Omega = (1.2)^2 \times 3\,\text{K} \) was found to be the maximum convenient value of the tuned load \( Z_L \).

The IF transformer had to be designed to introduce as little phase into the system-response as is compatible with high attenuation of the oscillator frequency. A single-tuned transformer (giving single-pole response as shown in appendix B.1) was used, the \( Q \) and 3db-bandwidth being given by

\[
Q = \frac{R}{\omega L_1} = (1.2)^2 \times 3\,\Omega/(2\pi \times 1.75 \times 10^6 \times 3.1 \times 10^{-6})
\]
\[
= 4.375 \quad (A.10)
\]

\[
B = 1.75 \, \text{MHz}/4.375 = 400 \, \text{KHz} \quad (A.11)
\]

The single-pole response and 400 KHz bandwidth of the mixer was confirmed by feeding the mixer with a 10 MHz oscillator signal and sweeping the RF input over the range
0 to 50 MHz. In this way, it was also possible to observe that the two harmonic sidebands which were present, (one at 8.25 MHz and one at 11.75 MHz), were about 60 db down on the IF response.

A.5. IF Channel. The IF channel, shown in fig. A.10, uses two RCA CA3012 RF amplifiers (RCA data sheets No. 128, 129, Application Notes ICAN 5269, 5380) with an interstage filter which was either a single-tuned (single-pole response) or a double-tuned (two-pole response) transformer. Each filter was designed to present a load impedance of 2KΩ, to provide the same mid-band gain ($v_o/v_i$), and to have a 3db-bandwidth $B$ of 100 KHz.

For the single-tuned circuit (see appendix B.1), the coils are unity coupled. The parallel impedance reflected into the primary is $(2)^2 \cdot 3KΩ = 12KΩ$, and this, together with $R_1$ (the total effective parallel resistance), produces a parallel input impedance $R_{ll}$ of 2KΩ. The $Q$ and 3db-bandwidth $B$ of the single-tuned circuit are given by

\[
Q = R_{ll} / (\omega L_1) = 2KΩ / (2\pi \times 1.76 \times 10^6 \times 10.4 \times 10^{-6}) = 17.5 \quad (A.12)
\]

\[
B = 1.75 \times 10^6 / 17.5 = 100 \text{ KHz} \quad (A.13)
\]

The midband-gain $K_o$ is given (from appendix B.1) by

\[
K_o = g_m \frac{R_{ll}}{2} = g_m 1KΩ \quad (A.14)
\]

where $g_m$ is the transconductance of the first CA 3012.
In the double-tuned circuit (see appendix B.2), the coils were critically coupled. The total primary parallel resistance $R_1$ was made $4\Omega$; the total secondary parallel resistance $R_2$ was made $1\Omega$; and the inductances $L_1$ and $L_2$ were chosen so that the $Q$'s, $Q_1$ and $Q_2$, of the primary and secondary circuits were equal to 24.7.

$$Q = Q_1 = Q_2 = \frac{R_1}{(\omega L_1)} = \frac{R_2}{(\omega L_2)} = 24.7 \quad (A.15)$$

From appendix B.2, the midband gain $K_0$, driving point resistance level $R_{11}$, and 3db-bandwidth $B$ are given by

$$K_0 = g_m \frac{R_1 R_2}{2} = g_m \ 1\Omega \ ; \ R_{11} = \frac{R_1}{2} = 2\Omega \quad (A.16)$$

$$B = 2^{\frac{1}{2}} \times 1.75 \times 10^6/24.7 = 100 \text{ KHz} \quad (A.17)$$

The amplitude-responses of the two types of filters are shown in fig. A.11. To obtain the responses, the secondary of the mixer's output transformer was disconnected, and single frequencies were injected from a test source into the first CA3012 as shown in fig. A.10. The responses were also observed by using a test source whose frequency was swept from 0 to 50 MHz, and, in this way, it was possible to observe that, at the input to the second CA3012, there were no harmonics of the IF in the range 0 to -70 db.

The gain of the two filter circuits were compared and found equal, and the resistance levels were compared by
observing that when the first CA3012 was driven 'hard' enough, both filters produced the same limiting output (measured on the primary side).

The CA3012 RF amplifier consists essentially of several cascaded differential amplifiers, and the unit therefore has excellent limiting characteristics. For each unit, the input which gave an output that was 3db below the limiting output was verified to be about 5mV p-p. Limiting is undesirable in the first IF stage, as this would generate harmonics (sec. 1.3), and it was decided to run the first stage with an input of 0.48mV p-p. This was conveniently fed in across 10Ω as shown in fig. A.10. The d.c. balance between the differential inputs was also thus preserved. 0.48mV p-p was also found adequate for quieting requirements (signal input much larger than equivalent noise input).

With a load of 2KΩ, the units provided a gain of some 60 db (as specified), and 0.48mV p-p input produced 500mV p-p output on the primary side of the filters. On the secondary side, the voltage was 250mV p-p, and the second IF stage was thus driven well into limiting. The ratio-detector was designed to present a load impedance of 3.3 KΩ, and the limiting output into this impedance level was found to be 6.6 V p-p. The first CA3012, when made to limit, delivered 4 V p-p into the 2 KΩ impedance level.

The RF and IF systems, with peak-to-peak voltages, gain, and impedance levels, are shown in fig. A.12

A.6. Ratio-Detector. The circuit diagram of the ratio-detector is shown in fig. A.13a. The design and operation of the ratio-detector is very complex. It has been dealt
with in an original work by Seely and Avins (ref. 42), and the salient features of the design are summarised by Fournier and Lee (ref. 43) as well as by members of Texas Instruments' Staff (ref. 44). The amplitude modulation rejection (AM rejection) of the ratio-detector is treated in detail by Loughlin (ref. 45).

$L_1 - C_1$ and $L_2 - C_2$ form primary and secondary resonant circuits tuned to the IF, $f_0$. $C_{L1}$ and $C_{L2}$, the diode load-capacitors, present very low impedance at $f_0$ but high impedance (compared with the diode load resistors) at the highest baseband frequency. $C_s$, the stabilizing capacitor, is used to hold the voltage $v_s$ across $R_a$ and $R_b$ at a constant level. This is essential in enabling the detector to reject AM.

One source of AM sensitivity, which we shall call diode-AM sensitivity, arises on account of the fluctuations of diode impedance (or efficiency) with the applied signal level. To suppress diode-AM sensitivity, the diode efficiency is lowered by $r_a$ and $r_b$. The sum of $r_a$ and $r_b$ is usually made about 10% to 20% of the sum of $R_a$ and $R_b$, and the individual values are adjusted to optimise diode-AM rejection. The resistance $r_3$ in the tertiary circuit combats diode-AM sensitivity by limiting peak currents through the diodes, again reducing the fluctuation of diode impedance.

The circuit diagram of fig. A.13a may be reduced to those shown in figs. A.13b, A.13c, and A.13d. $R_L$ is given by

$$R_L = (r_a + R_a + r_b + R_b)/\eta$$

(A.18)
and is the total resistance seen between the diodes. \( \eta \) is the diode efficiency of each diode. If \( R_{L2} \) represents the parallel combination of the equivalent parallel resistance \( R_{L2} \) of \( L_2 \) and the damping resistance \( R_{D2} \), then

\[
1/R'_{L2} = 1/R_{L2} + 1/R_{D2} \quad (A.19)
\]

and similarly, for the primary circuit,

\[
1/R'_{L1} = 1/R_{L1} + 1/R_{D1} \quad (A.20)
\]

The ratio-detector has a tertiary winding \( L_3 \), connected to the centre-tap of the secondary, so that the load \( R_L/\eta \), seen by the diode current, is reflected across the tertiary winding as \( R_L/(4\eta) \). The primary and tertiary of the ratio-detector are unity-coupled; so with primary and secondary uncoupled, a resistance \( n^2R_L/(4\eta) \) is reflected into the primary, \( n \) being the primary-to-tertiary turns ratio. If \( R_1 \) and \( R_2 \) represent the total primary and secondary parallel resistances, then

\[
1/R_1 = 1/R'_{L1} + 1/[n^2R_L/(4\eta)] \quad ; \quad 1/R_2 = 1/R'_{L2} + 1/R_L \quad (A.21)
\]

The design of the ratio-detector may be approached in numerous ways, depending on the design criteria. For the present requirements, the following procedure proved most suitable.
1) Choose the uncoupled Q of the primary and secondary (Q₁ and Q₂ respectively) to be equal and to have value 15. The uncoupled Q of both circuits must be the same for a symmetrical detector characteristic. The frequency-separation between the peaks of the detector characteristic has not been analytically derived but is held to be of the order of $2f₀/Q$ and to depend on $\alpha$, where $f₀$ is the IF, Q is the value of the uncoupled primary and secondary Q, and $\alpha$ is the fraction of critical coupling (see appendix B.2).

2) Make $L₂$ as large as possible. Loughlin (ref. 45) shows that this assists in diode-AM rejection. $L₂ = 40\mu\text{H}$ was chosen as the maximum convenient value. $C₂$, the total secondary tuning capacitance, is found from $C₂L₂ = 1/\omega₀^2$; thus $C₂ = 205.1 \text{ pF}$.

3) Determine the total secondary parallel resistance $R₂$ from

\[
Q = 15 = R₂/\left(\omega₀L₂\right); \quad R₂ = 6.6 \text{ KΩ} \quad (A.22)
\]

4) If $R_L$ in eqn. (A.18) is regarded as the secondary loading and $R_L'/(L₂)$ as the effective parallel resistance of $L₂$, then the secondary loaded and unloaded Q's, $Q_{2L}$ and $Q_{2U}$ respectively, are given by

\[
Q_{2L} = 15 = R₂/\left(\omega₀L₂\right); \quad Q_{2U} = R_L'/(\omega₀L₂) \quad (A.23)
\]

The ratio $N = Q_{2U}/Q_{2L}$ must be high for good diode-AM
rejection (ref. 45). From eqns. (A.21) and (A.23) and from the definition of $N$, we obtain

$$N = \frac{R_L}{(R_L - R_2)} \quad (A.24)$$

With $R_L = 12$ kΩ, $N = 2.2$, which is suitable.

5) The unloaded secondary $Q_{2U}$ is found from eqns. (A.23) and (A.21)

$$Q_{2U} = 33 \quad (A.25)$$

6) Assume a diode efficiency $\eta$ of 90%. From eqn. (A.18)

$$r_a + R_a + r_b + R_b = \eta R_L = 12 \times 0.9 = 10.8 \text{ kΩ} \quad (A.26)$$

$r_a = 1.5$ kΩ and $r_b = 1$ kΩ are usual values (ref. 42) giving

$$R_a = R_b = \frac{(10.8-2.5)}{2} = 4.3 \text{ kΩ} \quad (A.27)$$

7) The impedance reflected into the tertiary due to diode loading $R_L$ is $R_L/4 = 3$ kΩ

8) $L_1$ is made less than $L_2$ to facilitate diode-AM rejection (ref. 45). Choose $L_1 = 25\mu$H

9) As in 4), the impedance, $n^2 R_L/4 = n^2 \frac{3}{2} \text{ kΩ}$, which is reflected into the primary on account of diode loading, is regarded as the primary loading, and $R_{L1}'$ is regarded as the
effective parallel resistance of $L_L$. The loaded and unloaded $Q_L$, $Q_{1L}$, and $Q_{1U}$ respectively, of the primary are then

$$Q_{1L} = 15 = R_1/(\omega L_1); \quad Q_{1U} = R_{L1}^f/(\omega L_1)$$  \hspace{1cm} (A.28)

From eqn. (A.28), $R_1 = 4.125$ KΩ, and from eqn. (A.21)

$$R_{L1}^f = n^2_3 \times 4.125/(n^2_3-4.125) \text{ KΩ}$$  \hspace{1cm} (A.29)

and $Q_{1U}$ from eqn. (A.28) becomes

$$Q_{1U} = 15/[1 - (1.375/n^2)]$$  \hspace{1cm} (A.30)

10) If $S$ is the voltage induced across half the secondary inductance (half-secondary voltage) and $T$ is the voltage induced across the tertiary winding, it may be shown (ref. 42) that

$$S/T = \alpha[1 + \left(n^2 Q_{2L} L_2/(4Q_{1U} L_1)\right)]^{1/2}$$  \hspace{1cm} (A.31)

Seely and Avins (ref. 42) indicate that a suitable compromise between good AM rejection and ease of circuit adjustment is obtained when $S/T = 0.8$. Eqn. (A.31) then gives

$$\alpha = 0.8 \left[1 + (6n^2/Q_{1U})\right]^{-1/2}$$  \hspace{1cm} (A.32)

11) From appendix B.2, two equal-Q mutually coupled
circuits have input resistance \( R_{11} \) given by

\[
\frac{1}{R_{11}} = \frac{1}{R_1} + \frac{1}{(R_1/\alpha^2)}
\]  
(A.33)

and since \( R_1 = 4.125 \text{ K\Omega} \), we have

\[
R_{11} = \frac{4.125}{1 + \alpha^2} \text{ K\Omega}
\]  
(A.34)

12) From eqns. (A.30), (A.32) and (A.34), the following table is drawn up:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( Q_{1U} )</th>
<th>( \alpha )</th>
<th>( R_{11} ) (K\Omega)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>22.9</td>
<td>.57</td>
<td>3.12</td>
</tr>
<tr>
<td>2.2</td>
<td>20.9</td>
<td>.52</td>
<td>3.25</td>
</tr>
<tr>
<td>2.3</td>
<td>20.3</td>
<td>.50</td>
<td>3.30</td>
</tr>
</tbody>
</table>

and the values in the last row were considered suitable.

The fraction of critical coupling \( \alpha \), the primary and secondary \( Q \)'s, and the \( S/T \) ratio were checked after the ratio-detector had been aligned. Seely and Avins (ref. 42) describe how these operations are performed. Fig. A.14 shows the detector's 'S' curve, from which it is seen that the peak-separation is about 200 KHz and the sensitivity \( K_D \) is \( 9.2 \text{ V/MHz} = \left[9.2/(2\pi)\right] \times 10^{-6} \text{ V/radian frequency} \).

The 'S' curve was obtained by a) sweeping the IF, b) varying the IF over a range centered on 1.75 MHz and recording the d.c. output, and c) observing the amplitude (at 1 KHz) of
Sec. A.7. Buffer Amplifier and Feedback Control. The circuit diagram of the baseband filter, buffer amplifier, and feedback control is shown in fig. A.15. The filter has a 3db-bandwidth \( f' = 100 \text{ KHz} \) and reduces IF breakthrough from 3.2mV rms to 0.75mV rms. The buffer stage follows the design of Towers (ref. 46) and has an input impedance of 5 M\( \Omega \) in parallel with 0.4 pF. The impedance in series with the output voltage is 15\( \Omega \), and the voltage gain is unity. The feedback control consists simply of a 100 K\( \Omega \) resistance in series with the output, the proportion \( R_f \) (see fig. A.15) of the output for different known values of IF frequency deviations \( \Delta f \).

The impedance level (3.3 K\( \Omega \)) at the input to the detector was obtained as that resistance which caused the voltage appearing at the detector's input terminals to decrease by half when the said resistance was shunted across the terminals.

The AM rejection of the limiting IF channel and detector was measured with the various voltage levels shown in fig. A.12. With the RF set at 11.75 MHz and the oscillator frequency set at 10 MHz, the deviation \( \Delta f \) of the RF wave was set at 15 KHz for a modulating frequency of 1KHz \([\text{this gives a modulation index of } 15, \text{ and gives FM-wave bandwidth } B = 2f_s(1 + M) = 32 \text{ KHz.}]\). The output peak-to-peak voltage \( V_{\text{FM}} \) was recorded. Under the same conditions but with 50\% AM instead of FM, the output \( V_{\text{AM}} \) was observed. The AM rejection \( [20 \log_{10} \frac{V_{\text{FM}}}{V_{\text{AM}}}] \) was found to be greater than 60 db.

The parallel network at the detector's output is a low-pass filter for filtering off IF breakthrough, and it has its 3db-point at 250 KHz.
which is called the feedback resistance.

A.8. Voltage-Controlled Oscillator. (VCO). The circuit diagram of the VCO is shown in fig. A.16. The oscillator consists of a wideband transistor stage with a resonant tank circuit formed by $L_1$ ($2.53\mu\text{H}$) in parallel with a capacitance of 100 pF, to give a centre frequency of 10 MHz. The 100 pF capacitance was provided by the varactors $D_1$ and $D_2$, which are both operated at a reverse d.c. bias of $\frac{1}{4}V$. At this bias, each varactor has capacitance 200 pF. Since $L_4$ has high a.c. impedance and $C_2$ ($=.001\mu\text{F}$, d.c. blocking) has low a.c. impedance, the varactors are in series for the a.c. circuit. $\frac{1}{4}V$ d.c. was supplied from a 3V zener via a potential divider ($R_7$, $R_8$), so that fine adjustment of varactor-bias could be made. The potentiometer voltage was fed to the varactors via $R_9$ (10 K$\Omega$) and $L_4$. The high impedance $R_9$ isolates the a.c. and d.c. circuits. $\frac{1}{4}V$ is the largest signal amplitude the varactors can handle, since larger amplitudes will drive them into conduction on one half cycle.

Feedback from the tank circuit to the emitter is made via $C_1$, whose a.c. impedance is low compared with the input impedance level at the emitter terminal. The output and input voltages are, therefore, in phase. The loading of the tank circuit is determined primarily by $R_5$ and $R_{10}$ and is an important property of the VCO. High output voltage requires high Q, but this would cause amplitude modulation of the output as the VCO's frequency was swept. $R_{10}$ is reflected into the tank circuit as $4R_{10} = 4 \text{K}\Omega$, and the value 2.2 K$\Omega$ for $R_5$ was chosen on the basis of measured performance.
Fig. A.17 shows the circuit seen by the baseband circuit at the terminals 11'. At baseband frequencies, \(L_4\) (100\(\mu\)H) constitutes a much lower impedance than either of the varactors, and the varactors are effectively in parallel, presenting a capacitance of 400 pF. \(R_{10}\) in parallel with this 400 pF, constitutes a low-pass filter with 3db-bandwidth \(f' = 200\) KHz.

The characteristic of the VCO is shown in fig. A.18, from which the sensitivity \(K_V = 1.2\) MHz/V = \(1.2 \times 2\pi \times 10^6\) radians per second/V is obtained. For the sensitivity measurement, a d.c. voltage was applied to terminals 11', and the VCO's output amplitude and frequency were observed. With -3V d.c. input, -1V appears at the junction of two varactors, and any larger negative d.c. voltage would cause the varactors to conduct. A 100 KHz, 6V p-p signal, applied at terminals 11', confirmed the AM characteristics of fig. A.18, but it was not possible to measure the frequency deviation.

A.9. Gain and Phase Measurements. The open-loop gain and small-index phase characteristic is shown in fig. A.20. The characteristic was obtained by feeding a baseband signal of known amplitude and frequency into the VCO (at terminals 11') and observing the amplitude and phase of the buffer amplifier's output, as shown in fig. A.19. For these measurements, \(R_F\), the feedback resistance, was made zero, so that the measured gain indicates the maximum loop-gain. It will be seen that the maximum loop-gain \(K_L\) (= 11) corresponds with the product \(K_K\) of the VCO's gain (1.2 MHz/V) and
the detector's gain (9.2 V/MHz), showing, as discussed in sec. 4.2, that the IF amplification does not contribute to the loop-gain.

The loop-function \( L(s) \) contains five single-pole low-pass filters, namely 1) the mixer output filter (3db-band-width \( f' = 200 \) KHz), 2) the IF filter \( f' = 50 \) KHz, 3) the ratio-detector output filter \( f' = 250 \) KHz, 4) the baseband filter \( f' = 100 \) KHz, and 5) the VCO's input filter \( f' = 200 \) KHz.

The loop-gain, calculated from the combined responses of these filters, corresponds with the measured loop-gain.

The phase characteristic, calculated from the response of the various filters, is shown in fig. A.20 along with the measured characteristic. It will be noticed that, at the higher frequencies, the measured phase delay is in excess of the calculated phase delay. The difference may be accounted for in terms of the linear phase delay term, which is contributed by the finite time lag in the lines, transistors, etc.

The phase characteristics of the system under large-index conditions were also investigated. For a given baseband-signal frequency, the frequency deviation of the VCO's output was increased from zero until the baseband output distorted. It was found that for frequencies less than 20 KHz, the VCO's wave could be made large-index (\( M > \frac{1}{4} \)) before the output distorted. During the transition from small-index to large-index and during distortion, the phase of the output remained the same.

From fig. A.20, it will be seen that at the crossover frequency (90 MHz), the loop-gain has fallen from 11 to 3.1,
so that an open-loop gain of \( \frac{11}{3} \cdot 1 = 3.55 \) is the maximum stable open-loop gain. A maximum value of 3 was chosen, giving a phase margin of 30° and a gain margin of 5 db (see ref. 26, chap. 7).

A.10. Noise Measurements. Measurements in the IF system were made at the IF filter output, using a true-rms wideband voltmeter in conjunction with a d.c. digital voltmeter.

The average rms value of a waveform was obtained by feeding the output from the rms-meter through a low-pass filter to the digital voltmeter. With the noise generator switched out of circuit, the rms carrier amplitude \( \overline{A_o} \) was measured. With the signal generator switched out of circuit and the noise generator switched in, the rms noise amplitude \( \overline{A_n} \) was measured. With both sources in circuit, the rms amplitude \( \overline{A_o + n} \) was also observed. The IF carrier to noise power ratio (CNR) is then given by

\[
(CNR) = \frac{(\overline{A_o})^2}{(\overline{A_n})^2}
\]  

(A.35)

\[
(\overline{A_o + n})^2 = (\overline{A_o})^2 + (\overline{A_n})^2
\]

(A.36)
eqn (A.36) being used to check that \( \overline{A_o} \) and \( \overline{A_n} \) are correct.

The frequency-distribution of the IF power spectrum was observed on a spectrum analyser, the resolution of which was not high enough for quantitative measurements. It was however, possible to observe that for various values of feedback, the level and distribution of the IF noise spectrum were unaltered.
The spectrum of the demodulated noise was obtained using a wave-analyser with a noise bandwidth of 8.2 Hz. This instrument is calibrated so as to convert the rectified mean of a sine wave into an rms value; i.e., it reads \(1.111\) times the rectified mean. We saw in sec. 2.4 that for the range of (CNR) values that we are interested in, the input phase-function to the detector is gaussian so that the detector's output noise will also be gaussian. For gaussian noise \(n(t)\) with zero mean and average a.c. power \(N\), the probability distribution \(p(n)\) is given in eqn. 2.4 (with \(n_1 = 0\)). The rectified d.c. mean \(|n(t)|\) of \(n(t)\) is found from

\[
|n(t)| = m = \int_{-\infty}^{\infty} |n|p(n) dn = 2 \int_{0}^{\infty} np(n) dn = (2N/\pi)^{1/2} \quad (A.37)
\]

where the integral may be evaluated from tables (ref. 30). Since the rms value of the noise is \((N)^{1/2}\), the form factor \(F_n (= \text{rms/m})\) is

\[
F_n = (\pi/2)^{1/2} = 1.253 \quad (A.38)
\]

Thus, if \(x_n\) is the reading indicated by the wave-analyser, the rms noise voltage \((v_{n})_{\text{rms}}\) is obtained from

\[
(v_{n})_{\text{rms}} = x_n \times 1.253/1.111 = 1.128x_n \quad (A.39)
\]

and the value \(W_n\) of the power spectrum is obtained from
The rms values of the noise [eqn. (A.39)] were checked by measurements with the true-rms wideband voltmeter. The measurements were made at the output of the wave-analyser's 8.2 Hz filter, after the gain of the wave-analyser, up to the filter output, had been measured. Values of $x_n$ were obtained by strapping a large capacitor across the terminals of the wave-analyser d.c. meter. The factor $4\pi K_D^2 B_n$, where $B_n$ is the noise bandwidth of the IF filter and $K_D$ the detector constant, is given by

$$4\pi K_D^2 B_n = 42.3 \times 10^7,$$ single-pole filter (A.41)

$$= 29.9 \times 10^7,$$ two-pole filter (A.42)

when $K_D = [9.2/(2\pi)] \times 10^{-6}$ V/radian frequency, and both IF filters have 3db-bandwidths of 100 KHz (noise-bandwidth found from table 3.1). The quantity $W_n/(4\pi K_D^2 B_n)$ is then obtained from eqn. (A.40) as

$$W_n/(4\pi K_D^2 B_n) = x_n^2 (3.67 \times 10^{-8}) \text{ watts/Hz}, \text{ single-pole filter}$$

(A.43)

$$= x_n^2 (5.18 \times 10^{-8}) \text{ watts/Hz}, \text{ two-pole filter}$$

(A.44)

when $x_n$ is in $\mu V$. 
FIG. A.1. The Experimental System
FIG.A.2. Schematic of Noise Generator

FIG.A.3. Circuit Diagram of Noise Generator

FIG.A.4. Equivalent Circuit
FIG. A.5. Summing Amplifier

Output

V p-p

FIG. A.6. Saturation Curve of Summing Amplifier at 11.75 MHz

Relative

Gain

db

0

-2

-4

-6

1

10

60

MHz

FIG. A.7. Frequency Response of Summing Amplifier

vert. 0.5V/cm

horiz. 0.5μs/cm

FIG. A.8. Summing Performance of Summing Amplifier
L₂ = 57.7 μH
L₁ = 83.1 μH
C₁ = 99.5 pF

FIG. A.9. Mixer
FIG.A.10. IF Channel
FIG. A.11. IF Filter Responses
All Voltages are peak-peak

FIG.A.12. Gain, Voltages and Impedance-Levels
$C_1, C_2 =$ primary and secondary capacitances (331, 206.9pF)

$R_{D1}, R_{D2} =$ primary and secondary damping resistances

$L_1, L_2 =$ primary and secondary inductances (25, 40μH)

$D_a, D_b =$ detector diodes (TI 1N914A)

$C_{L1}, C_{L2} =$ diode load capacitances (200, 200pF)

$R_a, R_b =$ diode load resistors (4.3, 4.3kΩ)

$r_a, r_b =$ diode stabilizing resistors (1.0, 1.5kΩ)

$C_s =$ stabilizing capacitance (8μF)

$L_3 =$ tertiary inductance

$r_3 =$ diode-current limiting resistor (100Ω)

$C_f, R_f =$ low-pass filter (200pF, 3.3kΩ)

**FIG. A.13. a. Ratio Detector**
FIG. A.13. Equivalents Circuits of the Ratio Detector
FIG. A.14. Detector 'S' curve

FIG. A.15. Baseband Circuit
\[ L_1 = L_2 = 2.5 \mu H \]

\[ D_1 = D_2 = \text{TRW V100 varactors} \]

\[ C = 0.1 \mu F \]

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FIG. A.16. Voltage-Controlled Oscillator (VCO)

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FIG. A.17. Baseband Equivalent of the VCO
FIG. A.18. VCO's Characteristic

FIG. A.19. Measurement of Loop Characteristic
FIG. A.20. Loop Characteristic
APPENDIX B
SOME IF FILTERS

B.1. Single-pole Filter. The single-tuned transformer circuit may be shown to have a single-pole response. We consider the circuit in fig. B.1, in which terminals aa' are the output terminals of the amplifier preceding the tuned circuit. \( v_i \) is the input voltage to the amplifier, and \( i = g_m v_i \) is the amplifier's output current, \( g_m \) being the transconductance of the amplifier. The tuned circuit consists of a primary inductance \( L_1 \) with associated series loss-resistance \( r_{L1} \) and a primary tuning capacitance \( C_{T1} \). \( R_o \) and \( C_o \) represent the parallel output resistance and capacitance respectively of the amplifier, and \( R_{D1} \) represents the primary damping resistance which is used to adjust the circuit \( Q \). On the secondary side, \( L_2 \), with associated loss resistance \( r_{L2} \), is unity coupled with \( L_1 \). \( R_1 \) and \( C_i \) represent the parallel input resistance and capacitance respectively of the amplifier into which the tuned circuit works.

For the IF amplifiers used (sec. A.5), \( C_i \) is typically 4 pF and \( R_i \) typically 3 k\( \Omega \). At the IF (1.75 MHz), \( C_i \) therefore has negligible shunting effect on \( R_i \), and the former may be omitted from the circuit. The secondary current \( i_2 \) then flows through \( r_{L2} \) and \( R_i \). Since \( r_{L2} \) will be much smaller than \( R_i \), the former may also be omitted from the
circuit. To the right of the terminals cc', the circuit in fig. B.1 reduces to those in fig. B.2. The Q of the $L_1 - n^2 R_1$ combination must be larger than the overall circuit Q, so if the latter is large (greater than about 3), the former must also be large. The circuit in fig. B.2b may therefore be reduced to the one in fig. B.2c (see, for example, ref. 36, chap. 2). To the right of terminals bb', the circuit in fig. B.1 therefore reduces to a resistance $r_b = r_{L1} + r_c$ in series with $L_1$, and calling the Q of this combination $Q_b$ gives

$$Q_b = \frac{\omega L_1}{r_b} = \frac{\omega L_1}{r_{L1} + r_c} \quad (B.1)$$

This Q must also be higher than the overall circuit Q; so the equivalent resistance $R_b$ in parallel with $L_1$ is (see ref. 36, chap. 2)

$$R_b = Q_b^2 r_b = \frac{\omega^2 L_1^2}{r_{L1} + r_c} \quad (B.2)$$

$$\frac{1}{R_b} = \frac{1}{(\omega^2 L_1^2/r_{L1})} + \frac{1}{(\omega^2 L_1^2/r_c)} = \frac{1}{R_{L1}} + \frac{1}{n^2 R_1} \quad (B.3)$$

$R_{L1}$ is the parallel 'loss' resistance of $L_1$, and $r_b$ is given in fig. B.2c. The circuit of fig. B.1 then reduces to that shown in fig. B.3, where $C_1$ is the total primary parallel capacitance and $R_{L1}$ is the 'driving-point' resistance.

$$C_1 = C_0 + C_{T1} ; \quad \frac{1}{R_{L1}} = \frac{1}{R_{L1}} + \frac{1}{n^2 R_1} + \frac{1}{R_o} + \frac{1}{R_{D1}} \quad (B.4)$$
The circuit shown in fig. B.3 is a parallel-resonant circuit with resonant frequency $\omega_0$ given by

$$\omega_0^2 = \frac{1}{L_1 C_1} \quad (B.5)$$

If $\delta$, the fractional detuning, is now defined by

$$\delta = \frac{\omega - \omega_0}{\omega_0} \quad (B.6)$$

it may be shown (ref. 35, chap. 4) that for small values of $\delta$ (i.e., for frequencies near to the resonant frequency), the admittance $Y_{11'}$, seen by the current source at the terminals 11', is given by

$$Y_{11'} = \left(\frac{1}{R_{11'}}\right)(1 + j2\delta Q) \quad (B.7)$$

$$Q = \frac{R_{11'}}{\omega_0 L_1} \quad (B.8)$$

At the resonant frequency (mid-band), $Y_{11'}$ is entirely real, and the driving-point impedance $Z_{11'}$ is $R_{11'}$.

The voltage $v_1$, appearing at the terminals 11', is

$$v_1 = \frac{i}{Y_{11'}} = g_m v_{11'} \left(1 + j2\delta Q\right) \quad (B.9)$$

and since $v_1 = n v_0$ where $n = 2$ is the primary-to-secondary turns ratio, the voltage gain $K_0 G(j\omega) = v_0 / v_1$ is given by
\[ K_0 G(j\omega) = \left( \frac{g_R}{l' l} \right) \frac{1}{2} \left[ \frac{1}{1 + j2\delta Q} \right] \]

where \( K_0 \) is the mid-band gain and \( G(j\omega) \) defines the filter response. The transfer function \( G_{LP}(s) \) of the low-pass analogue of the filter may be obtained by observing that in the low-pass version of the filter response, \( \delta \) in eqn. (B.6) becomes \( \omega/\omega_0 \). Thus, with \( s = j\omega \), we obtain

\[ G_{LP}(s) = \frac{1}{1 + s(2Q/\omega_0)} = \frac{1}{1 + sT} \quad (B.11) \]

and the response is seen to be single-poled. The frequency response \( G_{LP}(j\omega) \) may be written in polar form thus:

\[ G_{LP}(j\omega) = \left( 1 + \omega^2 T^2 \right)^{-\frac{1}{2}} \left( -\tan^{-1}(\omega T) \right) \]

and the power transfer function is

\[ |G_{LP}(j\omega)|^2 = \frac{1}{1 + \omega^2 T^2} \]

The 3db-bandwidth \( f' \) is obtained as the value at which the power transfer function has a value of \( \frac{1}{\sqrt{2}} \). Thus

\[ 2\pi f'T = 1 \quad f' = \frac{1}{2\pi T} = \frac{\omega_0}{2Q} \]

and the IF filter itself has 3db-bandwidth \( 2f' = \omega_0/Q \).
We may now express eqn. (B.13) as

\[ |G_{LP}(j\omega)|^2 = 1/[1 + (f/f')^2] \]  

(B.15)

The noise bandwidth \( B_n \) of the IF filter may be obtained from the low-pass analogue, if it is assumed that \( f_o \) is so large that at zero frequency the response of the IF filter has fallen to zero. Any integral between zero and \( \infty \) which involves the IF filter shape may then be replaced by one which goes from \(-\infty \) to \( \infty \) and which involves the low-pass shape. Similar remarks will of course be valid for calculation of the radius of gyration \( r \) [eqns. (3.11), (3.12) and (2.89)]. From eqn. (2.91), then, \( B_n \) is given by

\[
B_n = \int_{-\infty}^{\infty} |G_{LP}(j\omega)|^2 d\omega = 2 \int_{0}^{\infty} |G_{LP}(j\omega)|^2 d\omega = 2f'(\pi/2)
\]  

(B.16)

The integral is evaluated from tables (ref. 30) after substitution for \( |G_{LP}(j\omega)|^2 \). From eqns. (3.11), (3.12) and (2.89), \( r \) is given by

\[
r = (2\pi)^{-1} (b_2/b_0)^{\frac{1}{2}}
\]  

(B.17)

\[
b_0 = W_o B_n = W_o 2f'(\pi/2)
\]  

(B.18)

\[
b_2 = W_o (2\pi)^2 2 \int_{0}^{\infty} |G_{LP}(j\omega)|^2 f^2 df = \infty
\]  

(B.19)

\[
r = \infty
\]  

(B.20)
where \( W_0 \) is the midband level of the IF noise spectrum \( W_0(f) \). That \( b_2 \) is \( \infty \) may be deduced from the fact that as \( f \) tends to \( \infty \), \( |g_{LP}(j\omega)|^2 f^2 \) tends to 1. Thus \( r \), in theory, is infinite for the single-pole response.

**B.2. Two-pole Filter.** The double-tuned transformer, shown in fig. B.4, may be shown to have a two-pole response under certain conditions. The nomenclature has already been discussed in the previous section. If \( C_2 \) represents the total secondary parallel capacitance and \( R_c \) represents the total parallel resistance to the right of terminals \( cc' \), then

\[
C_2 = C_{T2} + C_o ; \quad 1/R_c = 1/R_{D2} + 1/R_1 \quad (B.21)
\]

If the \( C_2 - R_c \) parallel combination has high \( Q \), the combination may be resolved into a resistance \( r_c \) in series with \( C_2 \) (ref. 36, chap. 2), and the secondary circuit reduces to that shown in fig. B.5. Since \( r_c \) will be much smaller than the impedance of \( C_2 \), \( v_o \) effectively appears across \( C_2 \). The secondary \( Q \), \( Q_2 \), is given by

\[
Q_2 = \omega L_2/r_2 \quad (B.22)
\]

where \( r_2 \) represents the series combination of \( r_{L2} \) and \( r_c \). Considering the primary when uncoupled with the secondary, the \( r_{L2} - L_2 \) series combination may be replaced with \( R_{L2} \) in parallel with \( L_2 \), where \( R_{L2} \) is the effective parallel loss-resistance of \( L_1 \). The primary circuit may therefore
be reduced to the circuits shown in figs. B.6a and B.6b, where

\[ C_1 = C_{T1} + C_o \ ; \ 1/R_1 = 1/R_o + 1/R_{D1} + 1/R_{L1} \quad (B.23) \]

At the terminals \( dd' \) in fig. B.6b, \( R_1 \) in parallel with \( L_1 \) may be resolved into \( r_1 \) in series with \( L_1 \), and the primary \( Q, Q_1 \) is given by

\[ Q_1 = \omega o L_1 / r_1 \quad (B.24) \]

To the left of the terminals \( dd' \), Thévenin's theorem may be used to represent the current \( i = g_m v_i \) in parallel with \( C_1 \) as a voltage \( v_1 \) in series with \( C \), \( v_1 \) being given by

\[ v_1 = g_m v_i (1/j\omega C_1) \quad (B.25) \]

The equivalent series circuit is shown in fig. B.6c. The double-tuned circuit of fig. B.4 may now be represented by the equivalent series circuit in fig. B.7a, and the equivalent parallel circuit is shown in fig. B.7b in which

\[ R_1 = \omega o^2 L_1^2 / r_1 \ ; \ R_2 = \omega o^2 L_2^2 / r_2 \quad (B.26) \]

The dots in fig. B.7 indicate that with current flowing into \( L_1 \) and out of \( L_2 \) at the dotted ends, the mutual inductance \( M \) between the coils is negative. Thus

\[ M = -k(L_1 L_2)^{1/2} \quad (B.27) \]
k being the coefficient of coupling. The series circuit has been analysed by Seely (ref. 41, chap. 9), who shows that for frequencies near to the resonant frequency, the gain \( K_0 G(j\omega) = v_o/v_1 \) is given by

\[
K_0 G(j\omega) = j a g m Q_1 Q_2 (r_1 r_2) \frac{1}{2} [(1+a^2) + j2\delta (Q_1 + Q_2) - 4\delta^2 Q_1 Q_2]^{-1},
\]

\[
a = k(Q_1 Q_2) \frac{1}{2} ; \quad \delta = (\omega - \omega_0)/\omega_0 \quad (B.28)
\]

The two-pole response is obtained when the circuits have equal Q's and are critically coupled. Thus, with

\[
Q_1 = Q_2 = Q ; \quad k = k_c = 1/Q ; \quad R_1^2 = Q^2 r_1 ; \quad R_2^2 = Q^2 r_2
\]

(B.29)

eqn. (B.28) reduces to

\[
K_0 G(j\omega) = (g_m (R_1 R_2)^{1/2}) j (1 + j2Q\delta - 2\delta^2 Q^2)^{-1} \quad (B.30)
\]

\[
K_0 = g_m (R_1 R_2)^{1/2} \quad (B.31)
\]

\[
G(j\omega) = j/(1 + j2Q\delta - 2\delta^2 Q^2) \quad (B.32)
\]

The transfer function of the low-pass analogue is obtained as in the previous section.
which shows that $G_{LP}(s)$ has two-poled response.

To determine the impedance $Z_b$ presented to the current $i$ in fig. B.4, we cannot represent the primary by the equivalent series circuit, since the terminals $11'$ would then be inaccessible. We therefore consider the impedance $Z_b$ seen at the terminals $bb'$. The relevant circuit is shown in fig. B.8, from which (see ref. 37, chap. 18)

$$Z_b = r_{LL} + j\omega L_1 + \omega^2 M^2 \left[ j\omega L_2 + r_2 - j/\omega C \right]^{-1} \quad (B.34)$$

At secondary resonance, the imaginary terms cancel to give

$$Z_b = r_{LL} + j\omega_0 L_1 + \omega_0^2 M^2 / r_2 \quad (B.35)$$

where $\omega_0$ is the resonant frequency.

The mutual coupling $M$ between the coils is given in eqn. (B.27), and assuming the primary and secondary $Q$'s to be equal, the critical coupling $k_c$ is defined in eqn. (B.29). It is necessary to consider the general case for any value of coupling $k$, and we express $k$ as

$$k = \alpha k_c = \alpha / Q \quad (B.36)$$

where $\alpha$ is called the fraction of critical coupling. Eqn. (B.35) now reduces to
The equivalent circuit to the right of the terminals $bb'$ is shown in fig. B.9, and defining the $Q$ of this circuit as $Q_b$, we have

$$Q_b = \omega_0 L_1 / (r_{L1} + \alpha^2 r_1)$$

(B.38)

By comparison with eqns. (B.1) through (B.3), the equivalent resistance $R_b$ in parallel with $L_{11}$ is

$$R_b = \omega_0^2 L_{11}^2 / (r_{L1} + \alpha^2 r_1)$$

(B.39)

Thus

$$1/R_b = 1/R_{L1} + 1/(R_1 / \alpha^2)$$

(B.40)

where $R_{L1}$ is the parallel loss resistance of $L_{11}$ and $R_1$ is the total effective parallel resistance of the uncoupled primary circuit. $Z_b$ is therefore seen to consist of $L_{11}$, $R_{L1}$, and $R_1 / \alpha$ in parallel. At the terminals $ll'$, the circuit in fig. B.4 therefore reduces to those shown in fig. B.10, since $R_1$ represents the parallel combination of $R_{L1}$, $R_o$, and $R_{DL}$.

At resonance, then, the driving-point resistance $R_{ll'}$ is given by

$$1/R_{ll'} = 1/R_1 + 1/(R_1 / \alpha^2)$$

(B.41)
At critical coupling, \( \alpha = 1 \), and

\[ R_{11} = \frac{R_1}{2} \quad \text{(B.42)} \]

Returning now to the transfer function of \( G_{LP}(j\omega) \) for the two-pole filter, eqn. (B.33) may be used to express \( G_{LP}(j\omega) \) in polar form

\[
G_{LP}(j\omega) = \left( 1 + \omega^4 T^4 \right)^{-\frac{1}{2}} \left( -\tan^{-1} \left[ 2^\frac{1}{2} \omega T / (1 - \omega^2 T^2) \right] + \pi/2 \right) \quad \text{(B.43)}
\]

and the power transfer function \( |G_{LP}(j\omega)|^2 \) is

\[
|G_{LP}(j\omega)|^2 = 1 / (1 + \omega^4 T^4) \quad \text{(B.44)}
\]

which has 3db-bandwidth \( f' \) given by

\[
(2\pi f')^4 T^4 = 1 \quad ; \quad f' = 1 / (2\pi T) = f_o / (2^{\frac{1}{2}} Q) \quad \text{(B.45)}
\]

and the IF filter has 3db-bandwidth \( 2f' = 2^{\frac{1}{2}} f_o / Q \). We may now express eqn. (B.44) in the form

\[
|G_{LP}(j\omega)|^2 = 1 / \left[ 1 + \left( f / f' \right)^4 \right] \quad \text{(B.46)}
\]

and the noise bandwidth \( B_n \) of the IF filter is

\[
B_n = 2 \int_0^\infty |G_{LP}(j\omega)|^2 \, df = 2f' \pi 2^{-3/2} \quad \text{(B.47)}
\]
Sec. B.3.  APPENDIX B

The integral in eqn. (B.47) is dealt with by Edwards (ref. 31, chap. 5, art. 167). The radius of gyration \( r \) is given by

\[
r = (2\pi)^{-1} \left( \frac{b_2}{b_0} \right)^{\frac{1}{2}} \]  
\( \text{(B.48)} \)

\[
b_0 = W_o \cdot B_n = W_o 2f' \quad m \quad 2^{-3/2} \]  
\( \text{(B.49)} \)

\[
b_2 = W_o (2\pi)^2 \cdot 2 \int_0^\infty |G_{LP}(j\omega)|^2 f^2 df = W_o (2\pi)^2 (f')^3 m \quad 2^{-1/2} \]  
\( \text{(B.50)} \)

This integral is evaluated from Edwards (ref. 31, chap. 6, art. 166). Hence

\[
r = f' \]  
\( \text{(B.51)} \)

B.3. Gaussian Filter. The gaussian filter-shape has power transfer function given by

\[
|G(j\omega)|^2 = e^{-\pi(f-f_o)^2/f_1^2} \]  
\( \text{(B.52)} \)

and the power transfer function of the low-pass analogue is

\[
|G_{LP}(j\omega)|^2 = e^{-\pi(f/f_1)^2} \]  
\( \text{(B.53)} \)

The 3db-bandwidth \( f' \) of this filter is found from

\[
-\pi(f'/f_1)^2 \]  
\( \text{e} = \frac{1}{2} \); \quad f' = f_1(\log_e 2/m)^\frac{1}{2} \]  
\( \text{(B.54)} \)
and the IF filter has 3db-bandwidth $2f' = f_1 \left(4 \log_e 2/m \right)^{1/2}$.

The noise bandwidth $B_n$ and radius of gyration $r$ are given by

$$B_n = 2 \int_0^\infty |G_{LP}(j\omega)|^2 \, df = f_1 \quad (B.55)$$

$$r = (2\pi)^{-1} \left( \frac{b_2}{b_0} \right)^{1/4} \quad (B.56)$$

$$b_0 = W_0 B_n = W_0 f_1 \quad (B.57)$$

$$b_2 = W_0 (2\pi)^2 \int_0^\infty |G_{LP}(j\omega)|^2 f^2 \, df = W_0 (2\pi)^2 f_1^3/(4\pi) \quad (B.58)$$

$$r = f_1/2 \sqrt{\pi} \quad (B.59)$$

Both the integrals in eqns. (B.55) and (B.58) may be evaluated from tables (ref. 30).

**B.4. Rectangular Filter.** The power transfer function of the rectangular filter and its low-pass analogue are given respectively by

$$|G(j\omega)|^2 = 1, \quad f_0 - B/2 \leq f \leq f_0 + B/2 \quad (B.60)$$

$$|G_{LP}(j\omega)|^2 = 1, \quad 0 \leq f \leq B/2 \quad (B.61)$$

The 3db-bandwidth of the low-pass analogue is $f' = B/2$, and the IF filter has 3db-bandwidth $2f' = B$. Also

$$B_n = B \quad (B.62)$$
Sec. B.4. APPENDIX B

\[ r = (2\pi)^{-1} \left( \frac{b_2}{b_0} \right)^{\frac{1}{4}} \]  
(B.63)

\[ b_0 = W_0 B_{on} = 2W_0 f' \]  
(B.64)

\[ b_2 = W_0 (2\pi)^2 \int_0^\infty |g_{LP}(j\omega)|^2 f^2 df = W_0 (2\pi)^2 (f')^{3/3} \]  
(B.65)

\[ r = 3^{-\frac{1}{2}} f' \]  
(B.66)
FIG. B.1. Single-Tuned Circuit

FIG. B.2. Equivalent Circuit at the Terminals cc'

FIG. B.3. The Equivalent Circuit
FIG. B.4. Double-Tuned Circuit

\[ r_L = \frac{1}{\omega^2 C^2 R} \]

FIG. B.5. The Secondary Circuit
FIG. B.6. The Primary Circuit
a. Series Circuit

b. Parallel Circuit

FIG. B.7. Equivalent Circuits

FIG. B.8. Equivalent Circuit at the Terminals bb'
FIG. B.9. Equivalent Circuit at the terminals bb'

FIG. B.10. Equivalent Circuits for Driving-point Impedance
APPENDIX C
RESULTS OF FOURIER AND LAPLACE TRANSFORMATIONS

C.1. Defining Relationships. The deviation and proof of the results summarised in this appendix may be found in texts (refs. 23, 24, 25) on transform methods in signal analysis.

A function \( f(t) \), its Laplace transform \( F(s) \), and its Fourier transform \( F(j\omega) \) are related through

\[
F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st} \, dt \tag{C.1}
\]

\[
F(j\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt \tag{C.2}
\]

\[
f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} \, ds \tag{C.3}
\]

\[
= \mathcal{F}^{-1}[F(j\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)e^{j\omega t} \, d\omega \tag{C.4}
\]

\( \mathcal{L} \) and \( \mathcal{F} \) symbolically represent the operations of Laplace and Fourier transforming and \( \mathcal{L}^{-1} \) and \( \mathcal{F}^{-1} \) the inverse processes.

If the quadratic content \( F_q \) of \( f(t) \) is defined as the total signal energy, so that

\[
F_q = \int_{-\infty}^{\infty} f^2(t) \, dt \tag{C.5}
\]
then for a signal $f(t)$, which is zero for time $t$ less than zero and which has finite quadratic content, the Fourier transform may be obtained from the Laplace transform by replacing $s$ with $j\omega$.

$$F(j\omega) = [F(s)]_{s=j\omega} \quad (C.6)$$

C.2. Amplitude and Power Spectrum of Signal $f(t)$. If $f(t)$ has Fourier transform $F(j\omega)$, the modulus and phase of $F(j\omega)$ are called the amplitude and phase spectrum respectively of $f(t)$. Thus if

$$F(j\omega) = |F(j\omega)| \angle F(j\omega) \quad (C.7)$$

then the sinusoidal component to $f(t)$ at frequency $f$ is obtained from

$$C_n \cos (2\pi f + \lambda) = 2 |F(j\omega)| df \cos [2\pi f + \angle F(j\omega)] \quad (C.8)$$

where $df$ is an infinitesimally small band of frequencies centered on $f$.

The power spectrum $W(f)$ and the quadratic content $F_q$ of $f(t)$ are related through

$$F_q = \int_{-\infty}^{+\infty} f^2(t) dt = \int_{-\infty}^{+\infty} (2\pi)^{-1} |F(j\omega)|^2 d\omega = \int_{-\infty}^{+\infty} |F(j\omega)|^2 df$$

$$= \int_{0}^{\infty} 2(2\pi)^{-1} |F(j\omega)|^2 d\omega = \int_{0}^{\infty} 2 |F(j\omega)|^2 df$$

$$= \int_{0}^{\infty} W(f) df \quad (C.9)$$
|F(jω)|^2 and 2|F(jω)|^2 are the two-sided and one-sided power spectra respectively of f(t).

C.3. Frequency-Domain Shift Theorem. If f(t) is such that

\[ f(t) = x(t) e^{jω_0 t} \]  \hspace{1cm} (C.10)

the Laplace transform F(s) of f(t) is

\[ F(s) = \int_{0}^{\infty} x(t) e^{-st} e^{jω_0 t} \, dt = \int_{0}^{\infty} x(t) e^{-(s-jω_0) t} \, dt \]  \hspace{1cm} (C.11a)

\[ = X(s - jω_0) \]  \hspace{1cm} (C.11b)

where X(s) is the Laplace transform of x(t). The frequency spectrum of f(t) is therefore obtained as the frequency spectrum of x(t), which has been shifted upward in frequency by amount ω_0.

C.4. Transfer Function of a Device. Let f_i(t) and f_o(t) be the input and output of the device and let F_i(s) and F_o(s) be the Laplace transforms of f_i(t) and f_o(t) respectively. The transfer function G(s) of the device is the function which relates F_i(s) and F_o(s) through

\[ F_o(s) = F_i(s) G(s) \]  \hspace{1cm} (C.12)

F_i(jω)G(jω) is then the amplitude spectrum of the output,
and \(2|F_i(j\omega)G(j\omega)|^2 = 2|F_i(j\omega)|^2|G(j\omega)|^2\) is the power spectrum of the output. \(|G(j\omega)|^2\) is therefore the power transfer function of the device.

\[225\]

**C.5. Distortionless Transmission of a Signal. Time-Domain Shift Theorem.** If \(f_i(t)\) is transmitted through a device whose transfer function \(G(s)\) has constant modulus and linear phase response over the entire range of frequencies for which the frequency spectrum of \(f_i(t)\) exists, then \(G(s)\) may be expressed as

\[
G(s) = k_0 e^{-st_1}
\]  

(C.13)

and the Laplace transform \(F_o(s)\) of the output \(f_o(t)\) is

\[
F_o(s) = F_i(s)k_0 e^{-st_1}
\]  

(C.14)

where \(F_i(s)\) is the Laplace transform of the input \(f_i(t)\). \(f_o(t)\) is given by

\[
f_o(t) = k_0 \int_0^t \{F_i(s)e^{-st}\} = k_0 f_i(t-t_1)U(t-t_1)
\]  

(C.15)

\(U(t-t_1)\) is the unit step-function occurring at time \(t = t_1\), so that \(f_o(t)\) is an amplified version of \(f_i(t)\) which has been delayed in time by amount \(t = t_1\). \(f_i(t)\), on passing through the device, is delayed in time but is undistorted. Any other form of \(G(s)\) will cause distortion.
C.6. Convolution Theorem. If \( f_1(t) \) and \( f_2(t) \) are two functions having Laplace transforms \( F_1(s) \) and \( F_2(s) \), the convolution theorem states that

\[
\mathcal{L}^{-1}\{F_1(s)F_2(s)\} = \int_0^t f_1(\tau)f_2(t-\tau) \, d\tau \tag{C.16}
\]
APPENDIX D

CHANGE OF VARIABLE

Let \( f_x[x_1, x_2, \ldots, x_n] \) be a function of the several variables \( x_1, x_2, \ldots, x_n \), where these variables define some coordinate system. It is now required to change to another coordinate system defined by \( u_1, u_2, \ldots, u_n \) where

\[
\begin{align*}
x_1 &= f_1(u_1, u_2, \ldots, u_n) \\
x_2 &= f_2(u_1, u_2, \ldots, u_n) \\
&\vdots \\
x_n &= f_n(u_1, u_2, \ldots, u_n)
\end{align*}
\]

(D.1.1)  

The eqns. (D.1.1) through (D.1.n) define the change of variable. The function \( f_x[x_1, x_2, \ldots, x_n] \) will map into the function \( f_u[u_1, u_2, \ldots, u_n] \), written \( f(x_1, x_2, \ldots, x_n) \) for short, through the change of variable. It may be shown (ref. 31, chap.6) that

\[
\int \int \cdots \int f_x[x_1, x_2, \ldots, x_n] \, dx_1 \, dx_2 \cdots dx_n =
\]
\[ \int_R \cdots \int_{u_1,u_2, \ldots, u_n} f_u[u_1, u_2, \ldots, u_n] \ J \ du_1 \ du_2 \cdots \ du_n \]  
\( (D.2) \)

\[ \delta x_1 \delta x_2 \cdots \delta x_n = J \ du_1 \ du_2 \cdots \ du_n \]  
\( (D.3) \)

\[ J = \frac{\delta(x_1, x_2, \ldots, x_n)}{\delta(u_1, u_2, \ldots, u_n)} \equiv \det \left( \frac{\delta x_i}{\delta u_j} \right) \]

\[ = \begin{vmatrix} \frac{\delta x_1}{\delta u_1} & \frac{\delta x_2}{\delta u_1} & \cdots & \frac{\delta x_n}{\delta u_1} \\ \frac{\delta x_1}{\delta u_2} & \frac{\delta x_2}{\delta u_2} & \cdots & \frac{\delta x_n}{\delta u_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\delta x_1}{\delta u_n} & \frac{\delta x_2}{\delta u_n} & \cdots & \frac{\delta x_n}{\delta u_n} \end{vmatrix} = \begin{vmatrix} \frac{\delta x_1}{\delta u_1} & \frac{\delta x_1}{\delta u_2} & \cdots & \frac{\delta x_1}{\delta u_n} \\ \frac{\delta x_2}{\delta u_1} & \frac{\delta x_2}{\delta u_2} & \cdots & \frac{\delta x_2}{\delta u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\delta x_n}{\delta u_1} & \frac{\delta x_n}{\delta u_2} & \cdots & \frac{\delta x_n}{\delta u_n} \end{vmatrix} \]

\( R_x \) is the range of integration in the \( x \)-coordinate system and maps into \( R_u \) in the \( u \)-coordinate system. \( J \), the Jacobian of the transformation, is also regarded as the 'elemental magnification factor'. 
APPENDIX E

REDUCTION OF VARIOUS RESULTS TO A COMMON FORM

We are interested in expressing the numerical results of Stumpers, Rice, and Lawson and Uhlenbeck (refs. 1, 3, 5) in the form \( W_D(f)/(4\pi B_n K_D^2) = x \). All three authors assume \( K_D = 1 \), and the results of Lawson and Uhlenbeck are, in fact, in the required form.

Stumpers tabulates the function \( [E(u)]^{1/2}/\Delta \omega \), where \( \Delta \omega \) is half the noise bandwidth \( B_n \) of the IF filter (rectangular and gaussian), \( u = \omega/(2\Delta \omega) \) is the frequency variable, and \( E(u) \) is the noise spectrum expressed as a function of \( u \).

We therefore write \( [E(u)]^{1/2}/\Delta \omega = x_1 \). Then

\[
E(u)du = x_1^2(\Delta \omega)2du = x_1^2(2\pi B_n/2)^22\pi df/(2\pi B_n) = W_D(f)df \tag{E.1}
\]

\[
W_D(f)/(4\pi B_n) = x_1^2(\pi/4) = x \tag{E.2}
\]

Rice considers only the gaussian filter-shape and tabulates \( W_D(f)/(4\pi^2 \sigma) \), where \( \sigma(2\pi)^{1/2} \) is the noise bandwidth \( B_n \) of the gaussian shape he uses. We therefore write \( W_D(f)/(4\pi^2 \sigma) = x_2 \). Then

\[
W_D(f) = x_2 4\pi^2 \sigma = x_2 4\pi B_n (\pi/2)^{1/2} ; W_D(f)/(4\pi B_n) = x_2 (\pi/2)^{1/2} = x \tag{E.3}
\]
APPENDIX F

STUMPERS' NO-MODULATION FORMULA

Stumpers (ref.1), in his treatment of FM noise for the case when the carrier is unmodulated, arrives at the result

$$E(u)/[4(\Delta \omega)^2] = \sum_{k=1}^{\infty} k e^{-2(CNR)} \frac{1}{1-F_1[-k+1,1,(CNR)]} h_{2k}(u)$$

$$+ \sum_{k=1}^{\infty} \sum_{r=0}^{\frac{1}{2}(k-1)} \left\{ \binom{k-r}{r} e^{-2(CNR)} \frac{1}{[(k-r)^2(k-2r)!(CNR)]} \right\}.$$  \(F.1\)

where \(1_F_1[ \ ]\) is the confluent hypergeometric function (see appendix I.4) and \(h_k(u)\) and \(b_k(u)\) are functions which depend on the IF filter-shape. For gaussian IF filter-shape

$$h_k(u) = \frac{1}{(2\pi)^{k/2}} e^{-\frac{1}{2}u^2/k}, \quad k = 2,3,...; \quad h_1(u) = u e^{-\frac{1}{2}u^2}$$  \(F.2\)

$$b_k(u) = 2^k e^{-\frac{1}{2}u^2/k} (2u - k/m), \quad k = 2,3,...; \quad b_1(u) = 0$$  \(F.3\)
u is the frequency variable defined by

\[ u = \omega / 2\Delta \omega = f / B_n \]  \hspace{1cm} (F.4)

where \(2\Delta \omega\) is the noise bandwidth (in units of radian frequency) of the IF filter. \(E(u)\) is the noise spectrum expressed as a function of \(u\).

In the development of the series expansion of eqn. (F.1) it is convenient to express each term as the product of a \((\text{CNR})\)-dependent factor and a \(u\)-dependent factor. Thus for the gaussian filter-shape, we set out the first four terms of the series in the following scheme:

\[ k=1, r=0 \hspace{0.5cm} e^{-2(\text{CNR})} \frac{2}{(\text{CNR})} F_{1}[1,2,(\text{CNR})] \{ h_1(u) + b_1(u) \} \]

\[ k=1 \hspace{0.5cm} e^{-2(\text{CNR})} \frac{2}{(\text{CNR})} F_{1}[0,1,(\text{CNR})] \{ h_2(u) \} \]

\[ k=2, r=0 \hspace{0.5cm} [(2!)^{-1}(\text{CNR})^2] F_{2}[1,3,(\text{CNR})] \{ 2h_2(u) + 4b_2(u) \} \]

\[ k=3, r=0 \]

\[ k=3, r=1 \]

\[ k=4 \]

\[ k=4, r=0 \]

\[ k=4, r=1 \]

In this way, it is seen that the single-summation in eqn. (F.1) only contributes to the series for even-values of \(k\). The
expressions in the curly brackets \{\} contain the $u$-dependent terms, the other factor of each product being the $(\text{CNR})$-dependent part.

Table F.2 shows the evaluation of the first ten terms of the series for two values of $(\text{CNR})$, namely 5 and 10, and for three values of $u(=f/B_n)$, namely 0, 0.25, and 0.5.

Table F.1 sets out the sum of the series at different values of $k$, when $(\text{CNR}) = 2$ and $f/B_n = 0.3$. Column 1 contains the individual terms, and column 2 contains the cumulated sum. Column 3 contains the square root of the value in column 2, times a factor of 2.

The numerical work for both tables was done on a computer.
### TABLE F.1

Evaluation of Double-Summation.  \((\text{CNR})=2, \ f/B_n=0.3\)

<table>
<thead>
<tr>
<th>K=</th>
<th>1, R= 0</th>
<th>2.5a</th>
<th>2.5a</th>
<th>3.2a</th>
</tr>
</thead>
<tbody>
<tr>
<td>K=</td>
<td>2, R= 0</td>
<td>2.4a</td>
<td>2.8a</td>
<td>3.3a</td>
</tr>
<tr>
<td>K=</td>
<td>2</td>
<td>1.8a</td>
<td>3.0a</td>
<td>3.4a</td>
</tr>
<tr>
<td>K=</td>
<td>3, R= 0</td>
<td>1.2a</td>
<td>3.0a</td>
<td>3.5a</td>
</tr>
<tr>
<td>K=</td>
<td>3, R= 1</td>
<td>1.2a</td>
<td>3.4a</td>
<td>3.7a</td>
</tr>
<tr>
<td>K=</td>
<td>4, R= 0</td>
<td>0.8a</td>
<td>3.4a</td>
<td>3.7a</td>
</tr>
<tr>
<td>K=</td>
<td>4, R= 1</td>
<td>0.8a</td>
<td>3.7a</td>
<td>3.9a</td>
</tr>
<tr>
<td>K=</td>
<td>4</td>
<td>0.8a</td>
<td>3.9a</td>
<td>4.0a</td>
</tr>
<tr>
<td>K=</td>
<td>5, R= 0</td>
<td>1.6a</td>
<td>3.3a</td>
<td>3.9a</td>
</tr>
<tr>
<td>K=</td>
<td>5, R= 1</td>
<td>1.4a</td>
<td>3.6a</td>
<td>4.0a</td>
</tr>
<tr>
<td>K=</td>
<td>6, R= 0</td>
<td>0.0a</td>
<td>3.6a</td>
<td>4.0a</td>
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<tr>
<td>K=</td>
<td>6, R= 1</td>
<td>5.4a</td>
<td>4.0a</td>
<td>4.0a</td>
</tr>
<tr>
<td>K=</td>
<td>6, R= 2</td>
<td>5.1a</td>
<td>4.0a</td>
<td>4.0a</td>
</tr>
<tr>
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<td>6</td>
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<td>4.1a</td>
<td>4.0a</td>
</tr>
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<td>5.4a</td>
<td>4.1a</td>
<td>4.0a</td>
</tr>
<tr>
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<td>1.7a</td>
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<td>4.0a</td>
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<td>4.2a</td>
<td>4.1a</td>
</tr>
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<td>4.2a</td>
<td>4.1a</td>
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<td>4.1a</td>
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<td>2.8a</td>
<td>4.3a</td>
<td>4.1a</td>
</tr>
<tr>
<td>K=</td>
<td>9, R= 0</td>
<td>1.4a</td>
<td>4.3a</td>
<td>4.1a</td>
</tr>
<tr>
<td>K=</td>
<td>9, R= 1</td>
<td>1.2a</td>
<td>4.3a</td>
<td>4.1a</td>
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<tr>
<td>K=</td>
<td>9, R= 2</td>
<td>3.1a</td>
<td>4.3a</td>
<td>4.1a</td>
</tr>
<tr>
<td>K=</td>
<td>9, R= 3</td>
<td>3.2a</td>
<td>4.3a</td>
<td>4.2a</td>
</tr>
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<td>4.4a</td>
<td>4.2a</td>
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<td>2.1a</td>
<td>4.4a</td>
<td>4.2a</td>
</tr>
<tr>
<td>K=</td>
<td>10, R= 1</td>
<td>2.5a</td>
<td>4.4a</td>
<td>4.2a</td>
</tr>
<tr>
<td>K=</td>
<td>10, R= 2</td>
<td>1.2a</td>
<td>4.4a</td>
<td>4.2a</td>
</tr>
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<td>10, R= 4</td>
<td>1.3a</td>
<td>4.4a</td>
<td>4.2a</td>
</tr>
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<td>K=</td>
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<td>4.4a</td>
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<td>30</td>
<td></td>
<td>5.1a</td>
<td>4.5a</td>
</tr>
<tr>
<td>K=</td>
<td>40</td>
<td></td>
<td>5.1a</td>
<td>4.5a</td>
</tr>
<tr>
<td>K=</td>
<td>50</td>
<td></td>
<td>5.2a</td>
<td>4.6a</td>
</tr>
<tr>
<td>K=</td>
<td>60</td>
<td></td>
<td>5.2a</td>
<td>4.6a</td>
</tr>
<tr>
<td>K=</td>
<td>70</td>
<td></td>
<td>5.2a</td>
<td>4.6a</td>
</tr>
<tr>
<td>K=</td>
<td>80</td>
<td></td>
<td>5.2a</td>
<td>4.6a</td>
</tr>
<tr>
<td>K=</td>
<td>90</td>
<td></td>
<td>5.2a</td>
<td>4.6a</td>
</tr>
<tr>
<td>K=</td>
<td>100</td>
<td></td>
<td>5.3a</td>
<td>4.6a</td>
</tr>
</tbody>
</table>

**key:**  \(2.5a \equiv 2.5 \times 10^{-2}\)
TABLE F.2

Individual Terms of Double-Summation

<table>
<thead>
<tr>
<th>$k$</th>
<th>$n$</th>
<th>(CNR)</th>
<th>$f/B_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$k=1, n=0$</td>
<td>1.00 - 1</td>
<td>1.00 - 2</td>
<td>0.00 - 99</td>
</tr>
<tr>
<td>$k=2$</td>
<td>2.00 - 9</td>
<td>0.00 - 99</td>
<td>1.10 - 1</td>
</tr>
<tr>
<td>$k=2, n=0$</td>
<td>1.00 - 2</td>
<td>1.00 - 4</td>
<td>6.00 - 9</td>
</tr>
<tr>
<td>$k=3, n=0$</td>
<td>2.00 - 3</td>
<td>2.00 - 6</td>
<td>4.90 - 9</td>
</tr>
<tr>
<td>$k=3, n=1$</td>
<td>3.10 - 8</td>
<td>0.00 - 99</td>
<td>8.20 - 9</td>
</tr>
<tr>
<td>$k=4$</td>
<td>8.30 - 8</td>
<td>0.00 - 99</td>
<td>8.00 - 2</td>
</tr>
<tr>
<td>$k=4, n=0$</td>
<td>5.90 - 4</td>
<td>6.00 - 8</td>
<td>4.20 - 9</td>
</tr>
<tr>
<td>$k=4, n=1$</td>
<td>1.40 - 7</td>
<td>0.00 - 99</td>
<td>6.00 - 2</td>
</tr>
<tr>
<td>$k=5, n=0$</td>
<td>2.30 - 4</td>
<td>2.40 - 9</td>
<td>3.80 - 9</td>
</tr>
<tr>
<td>$k=5, n=1$</td>
<td>4.30 - 7</td>
<td>0.00 - 99</td>
<td>4.60 - 2</td>
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<tr>
<td>$k=5, n=2$</td>
<td>5.50 - 7</td>
<td>0.00 - 99</td>
<td>6.80 - 2</td>
</tr>
<tr>
<td>$k=6$</td>
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<td>0.00 - 99</td>
<td>6.50 - 2</td>
</tr>
<tr>
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<td>1.20 - 10</td>
<td>3.50 - 9</td>
</tr>
<tr>
<td>$k=6, n=1$</td>
<td>1.00 - 6</td>
<td>0.00 - 99</td>
<td>3.60 - 2</td>
</tr>
<tr>
<td>$k=6, n=2$</td>
<td>1.30 - 6</td>
<td>0.00 - 99</td>
<td>5.80 - 2</td>
</tr>
<tr>
<td>$k=7, n=0$</td>
<td>5.40 - 5</td>
<td>7.20 - 12</td>
<td>3.20 - 9</td>
</tr>
<tr>
<td>$k=7, n=1$</td>
<td>2.00 - 6</td>
<td>0.00 - 99</td>
<td>2.90 - 2</td>
</tr>
<tr>
<td>$k=7, n=2$</td>
<td>2.20 - 6</td>
<td>0.00 - 99</td>
<td>4.90 - 2</td>
</tr>
<tr>
<td>$k=7, n=3$</td>
<td>2.10 - 6</td>
<td>0.00 - 99</td>
<td>5.90 - 2</td>
</tr>
<tr>
<td>$k=8$</td>
<td>1.10 - 6</td>
<td>8.00 - 78</td>
<td>5.60 - 2</td>
</tr>
<tr>
<td>$k=8, n=0$</td>
<td>3.10 - 5</td>
<td>5.00 - 13</td>
<td>3.00 - 9</td>
</tr>
<tr>
<td>$k=8, n=1$</td>
<td>3.30 - 6</td>
<td>0.00 - 99</td>
<td>2.50 - 2</td>
</tr>
<tr>
<td>$k=8, n=2$</td>
<td>2.90 - 6</td>
<td>0.00 - 99</td>
<td>4.20 - 2</td>
</tr>
<tr>
<td>$k=8, n=3$</td>
<td>7.30 - 6</td>
<td>0.00 - 99</td>
<td>5.30 - 2</td>
</tr>
<tr>
<td>$k=9, n=0$</td>
<td>1.80 - 5</td>
<td>4.00 - 14</td>
<td>2.80 - 9</td>
</tr>
<tr>
<td>$k=9, n=1$</td>
<td>4.60 - 6</td>
<td>0.00 - 99</td>
<td>2.10 - 2</td>
</tr>
<tr>
<td>$k=9, n=2$</td>
<td>2.90 - 6</td>
<td>0.00 - 99</td>
<td>3.70 - 2</td>
</tr>
<tr>
<td>$k=9, n=3$</td>
<td>1.70 - 6</td>
<td>0.00 - 99</td>
<td>4.70 - 2</td>
</tr>
<tr>
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<td>1.20 - 6</td>
<td>3.40 - 76</td>
<td>5.20 - 2</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.00 - 8</td>
<td>3.40 - 75</td>
<td>5.00 - 2</td>
</tr>
<tr>
<td>$k=10, n=0$</td>
<td>1.10 - 5</td>
<td>3.60 - 15</td>
<td>2.70 - 9</td>
</tr>
<tr>
<td>$k=10, n=1$</td>
<td>5.70 - 6</td>
<td>0.00 - 99</td>
<td>1.80 - 2</td>
</tr>
<tr>
<td>$k=10, n=2$</td>
<td>2.30 - 6</td>
<td>0.00 - 99</td>
<td>3.20 - 2</td>
</tr>
<tr>
<td>$k=10, n=3$</td>
<td>6.80 - 7</td>
<td>0.00 - 99</td>
<td>4.20 - 2</td>
</tr>
<tr>
<td>$k=10, n=4$</td>
<td>1.90 - 7</td>
<td>0.00 - 99</td>
<td>4.80 - 2</td>
</tr>
</tbody>
</table>

key: see table F.1
G.1. Evaluation of Summation in Eqn. (3.30). We consider here the evaluation of the function \( \sum \), where

\[
\Sigma = \sum_{k=-\infty}^{+\infty} |G(j\omega_j + j\omega_0 + jk\omega_s)|^2 J_k^2(M) \quad \text{(G.1a)}
\]

\[
= ... + a_0 J_0^2 + a_1 J_1^2 + a_2 J_2^2 + ... \quad \text{(G.1b)}
\]

where the coefficients \( a_0, a_1, a_2, \ldots \) depend on the IF filter-shape. For a particular frequency \( \omega (=2\pi f) \), these coefficients are obtained as shown in fig. G.1. The filter-shape shown is the low-pass analogue of the IF filter. Thus \( a_k \) is given by

\[
a_k = |G_{LF}(j\omega_j + jk\omega_s)|^2 \quad \text{(G.2)}
\]

\( \omega_s, M \), and the 3db-bandwidth \( f' \) of the low-pass analogue are related through Carson's rule, \( 2f' = B = 2f_s (1 = M) \), where \( B \) is the 3db-bandwidth of the IF filter.

For the single-pole and two-pole filters, the summation is taken between the limits \( k = \pm (2M + 10) \), since at orders of
2M + 10 and above, the Bessel function of argument M has negligible value.

For the rectangular filter, fig. G.2 indicates that the summation is self-truncating. From Carson's rule,

\[ B/2 = f_s (1 + M) \]  \hspace{1cm} (G.3)

so that \( |g_{LP}(j\omega)|^2 \) covers the frequency band \( -f_s (1 + M) \leq \omega \leq f_s (1 + M) \), and from fig. G.2, the following scheme may be deduced:

<table>
<thead>
<tr>
<th>Range of ( f )</th>
<th>Range of ( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 \leq f &lt; f_s )</td>
<td>( 0 \leq \left</td>
</tr>
<tr>
<td>( f_s \leq f &lt; 2f_s )</td>
<td>( 1 \leq \left</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( nf_s \leq f &lt; (n+1)f_s )</td>
<td>( n \leq \left</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( Mf_s \leq f &lt; (M+1)f_s )</td>
<td>( M \leq \left</td>
</tr>
<tr>
<td>( f=(M+1)f_s )</td>
<td>( \left</td>
</tr>
</tbody>
</table>

The second column under 'Range of \( f \)' is obtained by dividing through the first column by \( f_s \), \( f_s \) being given in eqn. (G.3).

**G.2. Remarks on the Evaluation of \( n/B_n \).** From eqns. (3.33) through (3.35), it is seen that \( n \) has direct dependence on \( r \). Since \( r = B_n/a \), where \( B_n \) is the IF filter's noise bandwidth.
and a is a property of the IF filter-shape (see table 3.1), \( n \) is directly proportional to \( B_n \), and it becomes convenient to deal with \( n/B_n \). We are also interested in the quantities \( c \) and \( d \) in eqn. (3.36). From Carson's rule,

\[
B = 2f_s (1+M) = 2f_s + 2\Delta f ; \quad 2f_s/B = 1/(1+M) ;
\]

\[
\Delta f = B/2 - f_s = B/2[M/(1+M)] \quad (G.4)
\]

Thus, from eqn. (3.36)

\[
c = \Delta f/r = (ab/2)[M/(1+M)] \quad (G.5)
\]

where \( a \) and \( b \) are properties of the IF filter-shape (see table 3.1). From \( c, d \) may be obtained, as in eqn. (3.36), for a particular value of \( (CNR) \). Thus \( n/B_n \) may be regarded as a function of \( M, (CNR) \), and the IF filter-shape.
FIG. G.1. Scheme for the Evaluation of $F$

FIG. G.2. Rectangular Filter
APPENDIX H

ON THE SIGNAL SUPPRESSION EFFECT

H.1. The Differentiation in Eqn. (3.44a). We recall that

\[ \frac{-1}{(d/dt)\tan x} = (1 + x^2)^{-1} x \]  \hspace{1cm} (H.1)

and denoting the differentiation required in eqn. (3.44a) by the symbol D gives

\[ D = \frac{-1}{(d/dt)\tan \left\{ r \sin (\gamma-p)/[A + r \cos (\gamma-p)] \right\}} \]  \hspace{1cm} (H.2a)

\[ = \left\{ 1 + r^2 \sin^2 (\gamma-p)/[A + r \cos (\gamma-p)]^2 \right\}^{-1} \]

\[ (d/dt)[r \sin (\gamma-p)/[A + r \cos (\gamma-p)]] \]  \hspace{1cm} (H.2b)

Making the substitution

\[ \gamma-p = u \; ; \; \dot{\gamma-p} = \dot{u} \]  \hspace{1cm} (H.3)

and remembering that for a function \( f(r,u) \) of \( r \) and \( u \)

\[ \frac{(d/dt)[f(r,u)]}{\dot{r}} = \left( \frac{\partial}{\partial r} [f(r,u)] \right) \dot{r} + \left( \frac{\partial}{\partial u} [f(r,u)] \right) \dot{u} \]  \hspace{1cm} (H.4)
we form

\[
\frac{\partial}{\partial r} \left[ \frac{r \sin u}{(A + r \cos u)} \right] = A \frac{\sin u}{(A + r \cos u)} \quad (H.5)
\]

and

\[
\frac{\partial}{\partial u} \left[ \frac{r \sin u}{(A + r \cos u)} \right] = \frac{(A r \cos u + r^2)}{(A + r \cos u)^2} \quad (H.6)
\]

We then obtain that

\[
\frac{d}{dt} \left[ \frac{r \sin u}{(A + r \cos u)} \right]
= \left\{ A \frac{\sin u}{(A + r \cos u)} \right\} \left\{ A + r \cos u \right\}^{-2} \quad (H.7)
\]

and substitution into eqn. (H.2b) gives the result in eqn. (3.47).

**H.2. Jacobian of the Transformation in Egns. (3.49) and (3.50).** From appendix D, the Jacobian $J$ of the transformation is

\[
J = \begin{vmatrix}
\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial x}{\partial \gamma} & \frac{\partial y}{\partial \gamma} \\
\frac{\partial x}{\partial \gamma} & \frac{\partial y}{\partial \gamma} & \ldots \\
\frac{\partial x}{\partial \gamma} & \frac{\partial y}{\partial \gamma} & \ldots \\
\frac{\partial x}{\partial \gamma} & \frac{\partial y}{\partial \gamma} & \ldots \\
\end{vmatrix}
\]

\[
= \begin{vmatrix}
\cos \gamma & \sin \gamma & -\sin \gamma & \cos \gamma \\
-r \sin \gamma & r \cos \gamma & -r \sin \gamma & r \cos \gamma \\
o & o & \cos \gamma & \sin \gamma \\
o & o & -r \sin \gamma & r \cos \gamma \\
\end{vmatrix}
\]

\[
= r^2 \quad (H.8)
\]
H.3. The Integration in Eqn. (3.53). Setting $n_D(t) = A$ for simplicity, rearranging the terms in eqn. (3.53) leads to

$$A = K_D \int \int \int (B \cdot d\mathbf{r} \cdot d\mathbf{y}).$$

$$(2\pi N) e^{\frac{-r^2}{(2N)}} \int_{-\infty}^{\infty} [\{rA \sin (\gamma - \rho) - \rho (r^2 + Ar \cos (\gamma - \rho))]$$

$$+ \{r^2 + Ar \cos (\gamma - \rho) \} \cdot e^{\frac{-r^2}{(2N)}} dy \quad (H.9)$$

where $B = r^2 e^{\frac{-r^2}{(2N)}} /[2\pi N \{A^2 + r^2 + 2Ar \cos (\gamma - \rho)] \} \quad (H.10)$

From the result that

$$\int_{-\infty}^{+\infty} (a+bx)e^{-cx^2} = a(\pi/c)^{1/2}, \quad (\text{ref.} 29) \quad (H.11)$$

we set

$$a = \{rA \sin (\gamma - \rho) - \rho (r^2 + Ar \cos (\gamma - \rho))\};$$

$$b = \{r^2 + Ar \cos (\gamma - \rho)\}; \quad c = r^2/(2\pi N) \quad (H.12)$$

and integrating eqn. (H.9) with respect to $\dot{y}$ gives
A = K_D \int \int \int (B \, dr \, dy).

\begin{align*}
-1 \quad \frac{1}{2} \quad 2 \\
\frac{r}{(2\pi N_1)} \quad \{ -\rho [r + Ar \cos (\gamma - \rho)] + A \sin (\gamma - \rho) \} e^{-r^2/(2N_1)} \, dr
\end{align*}

(H.13)

The integration with respect to \( r \) is again of the form shown in eqn. (H.11) so setting

\begin{align*}
a &= -\rho [r^2 + Ar \cos (\gamma - \rho)] ; \\
b &= A \sin (\gamma - \rho) ; \\
c &= 1/(2N_1)
\end{align*}

(H.14)

leads to

\begin{align*}
A &= -K_D \rho \int \int \left. \frac{e^{-r^2/(2N_0)}}{2\pi N_0} \right. dr \cdot \frac{1 + (A/r) \cos (\gamma - \rho)}{1 + (A/r)^2 + 2(A/r) \cos (\gamma - \rho)} \, dy
\end{align*}

(H.15)

The substitution \( \gamma - \rho = x, \, dy = dx \), reduces the integration with respect to \( \gamma \) to a standard form (see, for example, ref. 30)

\begin{align*}
\int_{-\pi - \rho}^{+\pi - \rho} \frac{1 + (A/r) \cos x \, dx}{1 + (A/r)^2 + 2Ar \cos x} = \begin{cases} 0, & r < A \\
\pi, & r = A \\
2\pi, & r > A
\end{cases}
\end{align*}

(H.16)

and the final result of the integration is given in eqns. (3.54).
APPENDIX I

EVALUATION OF SOME MATHEMATICAL FUNCTIONS

I.1. The Error Function, $\text{erf} \, x$. The defining equations for $\text{erf} \, x$ are given in sec. 3.2 [eqn. (3.13)]. The function is treated in some detail in ref. 33. $\text{erf} \, x$ has series representation

$$\text{erf} \, x = 2(x) \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)} \quad (I.1)$$

and this series may be used for the evaluation for $\text{erf} \, x$ for $|x| \leq 1$. For values of $|x| > 1$ eqn. (I.1) is not very suitable for numerical evaluation. $\text{erf} \, x$ is then best evaluated by using the 'continued fraction' result

$$\frac{x}{2e} \int_{x}^{\infty} e^{-t^2} dt = \frac{1}{x^+} + \frac{1}{x^+} + \frac{3/2}{x^+} + \frac{2}{x^+} + ... \quad (I.2)$$

I.2. The Bessel Function of the First Kind and Order $n$, $J_n(x)$. This function is defined by eqns. (1.16) and is dealt with in refs. 30 and 33. $J_n(x)$ has series expansion

$$J_n(x) = (x/2)^n \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{[k!\Gamma(n+k+1)]} \quad (I.3)$$
where \( \Gamma(z) \) is the gamma (factorial) function. The series expansion in eqn. (1.3) is amenable to numerical evaluation for \( |x| \leq 1 \). For \( |x| > 1 \), the series expansion may also be used if \( |n| \geq |x| \). For \( |x| > 1 \) and \( |n| < |x| \), \( J_n(x) \) is obtained by evaluating \( J_{2n+1}(x) \) and \( J_{2n}(x) \) from the series expansion and then using repeatedly the recurrence relation

\[
(2n/x)J_n(x) = J_{n-1}(x) + J_{n+1}(x) \tag{1.4}
\]

1.3. The Bessel Function of the First Kind and Imaginary Argument, \( I_n(x) \). This function is defined in eqn. (3.37). It is also dealt with in refs. 30 and 33. \( I_n(x) \) has series representation

\[
I_n(x) = (x/2)^n \sum_{k=0}^{\infty} \frac{(x^2/4)^k}{k! \Gamma(n + k + 1)} \tag{1.5}
\]

The evaluation of this function is done in exactly the same way as for \( J_n(x) \), the recurrence relation here being

\[
(2n/x) I_n(x) = I_{n-1}(x) - I_{n+1}(x) \tag{1.6}
\]

1.4. The Confluent Hypergeometric Function, \( _1F_1(a, b, x) \). This function is dealt with in refs. 33 and 34. \( _1F_1 \) has series representation

\[
_1F_1(a, b, x) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n \left[ b_1 \right] \tag{1.7}
\]
where \( (a)_n = a(a+1)...(a+n-1); \ (a)_0 = 1 \) \hspace{1cm} (I.8)

Thus, \( _1 F_1 (-1, b, x) = 1-(x/b) \); \( _1 F_1 (0, b, x) = 1 \) \hspace{1cm} (I.9)

From these starting values, the function may be evaluated for any value of \( a \) by using the recurrence relation

\[
(b-a)_1 {}_1 F_1 (a-1, b, x)+(2a-b+x)_1 {}_1 F_1 (a, b, x)-a_1 {}_1 F_1 (a+1, b, x) = 0
\] \hspace{1cm} (I.10)
REFERENCES


REFERENCES


REFERENCES


