INTERPRETATION OF FLOW AND PRESSURES IN FULL SCALE SILOS

by

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A RIGOROUS STATISTICAL TECHNIQUE FOR INFERRING CIRCULAR SILO WALL PRESSURES FROM WALL STRAIN MEASUREMENTS

8.1 Introduction

The pressures acting on the walls of a silo are needed as a starting point for the silo structural design. However the magnitudes and distribution of the pressures are not yet well understood, especially for conditions of eccentric discharge. The uncertainty about these pressures may be seen from the large discrepancies between national standards (over 100%) when applied to quite simple problems (Wilms et al, 1995). This paper outlines a technique by which the pressure distribution may be rigorously inferred from strain measurements made on a full scale silo, provided the silo wall is an isotropic thin axisymmetric shell.

Recent work on the interpretation of silo pressure experiments has shown that very many instantaneous measurements must be made to define the systematic and random components of the wall pressure distribution (Ooi et al, 1990; Ooi and Rotter, 1991c). Pressure cells are usually mounted in the silo wall to make such measurements. However, the cost of making and installing reliable cells is extremely high, and the number of cells needed to characterise a complex pressure distribution is large. Furthermore, for bulk solids with large particles, quite large cells are
required. The low stiffness of a large cell may disturb the pressure distribution significantly, especially in a stiff solid, but a stiffer cell may become so massive that it affects the pressure distribution by locally stiffening the structure (particularly in a thin-walled metal silo).

To overcome these shortcomings, strain gauges have sometimes been used in experiments to measure the strains in the silo walls. Attempts have been made to deduce the pressure distribution acting on the silo walls from simple strain readings (e.g. Blight, 1990a, 1990b; Bishara, 1992). Since a circular silo structure is normally a thin shell under unsymmetrical loading, both stretching and bending deformations of comparable magnitudes occur in the wall. Strains measured on the outside of the wall are thus considerably affected by local bending. Huge errors of interpretation can occur (Rotter and Ooi, 1991) if it is assumed that the local value of the wall pressure $p$ is simply related to the circumferential stress in the wall as $\sigma = pR/t$, or through two dimensional plane stress elasticity relations (e.g. Blight, 1990a). Instead, the strains and stresses at every point in the structure are affected by the normal pressure and frictional traction everywhere on the wall (Rotter, 1986b).

This chapter presents a new rigorous procedure by which the complete pressure distribution may be inferred from strains measured on the silo wall, by back-figuring from a series of structural analyses of the silo using characterised pressure distributions. Errors are minimised by a least-squares adjustment. Because the pressure pattern is defined in advance, the assumptions about pressure patterns, normally made by the choice of placement of pressure cells, are clearer to the user and also changeable in the light of the discovered distributions. The test silo must be quite free of defects and local complexities, so that the structural analysis used in the pressure inference process can be trusted.

The technique was developed for use in the interpretation of the strain measurements made on the specially built full scale test silo as described in Chapter 6. A simple example is presented here to illustrate the new procedure. The application of the
technique to a real silo involves many more considerations and will be discussed in the later chapters.

8.2 Strain measurements on a silo wall

If the pressure distribution and the silo structure are very symmetrical, there is less need for a large number of strain observations. The simple interpretations used before (Blight, 1990b; Bishara, 1992; Rotter and Ooi, 1991) are often reliable, and the new procedure is not needed, except near discontinuities. However, the conditions for this are quite strict: the silo must be a thin shell under axisymmetric or slowly varying loading, and with uniform base support. The strain measures must all be taken at points distant from structural and loading discontinuities to ensure that they are not influenced by local bending, so that interesting questions about local 'switch' loadings cannot be explored.

The membrane strains are strains at the middle surface of the shell, which are not easily measured directly. The strains which can easily be measured (surface strains on the outside and inside) are strongly affected by bending strains, which can be very large wherever local bending occurs. In unsymmetrically loaded silos with fabrication imperfections, plate thickness changes and stiffeners, local bending occurs in a very many places (Trahair, 1983) often in the very zones where the most dramatic pressure peaks are observed (e.g. near the hopper transition). Thus it seems to be a good general rule that both bending and membrane strains should always be measured at every station, even if the membrane strains alone are to be used to infer the pressure distribution.

Both experimental (Nielsen, 1983a; Hartlen et al, 1984; Rotter et al, 1986; Munch-Andersen and Nielsen, 1986, 1990) and theoretical (Rotter, 1986b; ACI313, 1989) studies indicate that the pressure patterns occurring in eccentrically discharging silos, or even in silos with minor unsymmetrical features, can be highly unsymmetrical. In
these circumstances, the number of observations required to characterise the pattern sufficiently, even to understand the behaviour qualitatively, is very large (Rotter et al., 1986).

If the measured strains on an unsymmetrically loaded shell are to be accurately translated into inferred wall pressures, the complete state of strain at any point on the wall must usually be considered. This complete state may be described in terms of six components (three membrane strains and three curvatures; or three strain components on each of the two surfaces). These six measures are needed at every station for every time step of the test.

Unfortunately, measurement of these six components normally means that rosette gauges must be placed on the inside surface of the silo wall (e.g. Bishara, 1992). An internal gauge inevitably causes a bump imperfection over which the stored bulk solid slides, which may in turn dramatically affect the local value of pressure (Rotter, 1983a), leading to a serious misinterpretation of the real pressure regime. The wires from the gauges must also be brought out of the silo, and these may interfere further with solids flow. In addition, correct translation of the measured strains into wall pressures involves a set of complicated structural analyses, which are essential if the inferred pressures are required to any useful accuracy under unsymmetrical loading patterns.

One remedy to the above dilemma is to use 'sandwich' or 'double deck' strain gauges (Fig. 8.1a) (Itoh, 1975; Kyowa, 1977), which can be placed on one surface but which measure both the local surface strain, and by extrapolation, the bending strain or curvature (Fig. 8.1b). The conditions under which these sandwich gauges are used are restricted to minimise local stiffening of the wall and shear lag in the readings (see Appendix A). A rosette of sandwich strain gauges can be set up at each measuring site to obtain the six measures of strain (Fig. 8.1c).

The six strain measures are here represented by the vector
\[
(\vec{\varepsilon})^T = [\varepsilon_m, \varepsilon_b]^T
\]

in which the membrane strains \(\varepsilon_m\) are given by the vector

\[
[\varepsilon_m] = [\varepsilon_{m\theta}, \varepsilon_{m\phi}, \varepsilon_{m\phi\theta}]^T
\]

and the bending curvatures by the vector

\[
[\varepsilon_b] = [\kappa_{b\theta}, \kappa_{b\phi}, \kappa_{b\phi\theta}]^T
\]

The values \(\varepsilon_{m\theta}, \varepsilon_{m\phi}\) and \(\varepsilon_{m\phi\theta}\) are the middle surface strains in the meridional, circumferential and shearing directions and \(\kappa_{b\theta}, \kappa_{b\phi}\) and \(\kappa_{b\phi\theta}\) are the meridional, circumferential and twisting curvatures respectively.

The complete set of measured strains at all measuring stations on the wall is then represented by the vector \((\vec{\varepsilon})_M\) (there being 6 strain measures at each of \(M\) stations). Although it is expected that six strain components would normally be used in the present analysis, the influence of the twisting curvature is almost always very small, and that of the bending curvatures may be minor in places. Thus, there may be occasions in which only 5 strain measures, or possibly only the 3 membrane strains, are used in the pressure inference procedure. In the rest of this paper, the number of strain measures at a point is therefore written as \(\beta\), where \(\beta\) can take values of 1, 2, 3, 4, 5 or 6 as appropriate (a distinct set of special practical circumstances can be identified for each of these possibilities).
8.3 Transformation of strains into inferred pressures

This section describes an analytical process by which the measured strains at a number of points on the silo wall at a given time can be transformed into inferred pressures, typically not at the same point.

8.3.1 Pressure distribution

When strain gauge readings are used to deduce the pressure pattern on the silo wall, there may or may not be a strong correlation between the strain observation at a point and the pressure at that point (e.g. Rotter, 1986). Thus, in interpreting the observations, it is better to make a clearly defined assumption about the nature of the pressure distribution. The assumption for the vertical distribution could be, for example, a piece-wise linear variation between observation stations (comparable with usual interpretations of pressure cell readings), a polynomial distribution down the wall (an obvious mathematical distribution, but not particularly well suited to silo pressures), a cubic spline function, or a Janssen (1895) expression with additional terms to display the departures from the best-fit Janssen curve (often the most useful description for silos).

The assumed variation of pressures around the circumference of the silo has received less attention, partly because it is often assumed to be invariant, as in most silo pressure theories. Here, it is probably best described by a harmonic decomposition (e.g. Ooi et al, 1990), giving constant pressure as the first term of the calculated set.

All the above possibilities and many others are available in the present formulation: the key concept is that the pressure distribution is formally assumed to take a particular form, and that the complete pressure distribution is characterised by the values of a limited number of relevant parameters aₚ.

To implement this procedure, the pressure distribution in the silo is first characterised as
\[ p_n(\theta,z) = \sum_{i=1}^{N} a_i f_i(\theta,z) \quad (8.4a) \]

\[ p_v(\theta,z) = \sum_{i=N+1}^{2N} a_i f_{i-N}(\theta,z) \quad (8.4b) \]

\[ p_\theta(\theta,z) = \sum_{i=2N+1}^{3N} a_i f_{i-2N}(\theta,z) \quad (8.4c) \]

in which the variables \( p_n, p_v \) and \( p_\theta \) represent the normal wall pressure, the vertical wall frictional traction, and the circumferential wall frictional traction at any point. The functions \( f_1, f_2, \ldots, f_N \) are used to represent the assumed variation of normal pressure and the loading coefficients \( a_1, a_2, \ldots, a_N \) define the magnitude of each normal pressure term. The coefficients \( a_{N+1}, \ldots, a_{2N} \) define the local vertical component of the wall frictional traction whilst \( a_{2N+1}, \ldots, a_{3N} \) define the horizontal (circumferential) frictional component.

Where this analysis is applied to a problem in which there is no wall friction, only the terms of Eq. 8.4a are required. Similarly, in a silo with completely axisymmetric wall loads, only the terms of Eqs 8.4a and 4b are needed. Thus, in the remainder of this paper, a generality of description is retained by defining the number of unknown parameters \( a_i \) as \( \alpha N \), where \( \alpha \) can take the value of 1, 2 or 3 as appropriate. It may be noted that the loading terms represented by Eq. 8.4c do not appear to have been included in any previous theoretical or experimental work on silo pressures.

### 8.3.2 Structural analysis of the silo

Since the strains measured on the silo wall are not direct measures of the wall pressure, a structural analysis is needed to relate the measured strains to the inferred
pressure distribution. If the structural behaviour is assumed to be very simple, with pressures related to wall strains through \( \sigma_n = \frac{E}{1-\nu^2} (\varepsilon_n + \nu \varepsilon_d) \) and \( \sigma_b = \frac{pR}{t} \), a structural analysis is still being used, though this analysis is trivially simple and is often in serious error.

The possible forms of structural analysis which could be applied in the present work include the membrane theory of shells (Rotter, 1987a) and the linear bending theory of shells (Rotter, 1987b). If the membrane theory of shells is applied to shells under non-symmetric loads, it yields only membrane strains for a given pressure distribution, and ignores the membrane strains arising from bending of the shell; nevertheless it may be adequate if all the strain observations are distant from sources of local bending (e.g. there are no rapid changes in the pressure distribution) and the shell support is uniform.

The bending theory of shells is much more complicated, and simple algebraic solutions of practical relevance to unsymmetrical loading distributions cannot, in general, be obtained. The bending theory does include all the phenomena of local bending. However, it suffers from the disadvantage that the bending strains which are calculated in many places may be only weakly influenced by the pressure distribution parameters being sought, making the inversion of the relevant matrix in the next section very difficult. For this reason, it is desirable that the technique presented here for interpreting pressures from strain readings includes some weighting features which permit the interpretation process to place more emphasis on some observations than on others.

The structural analysis consists of several steps: first the load must be defined as in Eq. 8.4, which indicates the target pressure distribution. The parameters \( a_1, a_2, \ldots, a_{\alpha N} \) each define the amplitude of a pressure distribution of a certain shape. For each of these \( \alpha N \) pressure distributions, a separate structural analysis must be conducted to determine the complete state of strain which that distribution would induce at each of
the locations in the shell at which the strains have been measured. To achieve this using the bending theory of shells, it is easiest to use a finite element analysis, which can include all necessary boundary conditions, stiffeners, adjacent structural elements etc. The wall stress resultants (three membrane stress resultants and three bending moments) are calculated at each gauge location by finite element analysis. These must then be transformed into stretching and bending strains at the gauge location using the constitutive equations for the shell. Alternatively, the wall strains are calculated directly from the displacements using the element strain-displacement relationships.

The $\alpha N$ structural analyses produce $M$ sets of $\beta$ strains $\{\varepsilon\}$ (there being $M$ gauge locations with $\beta$ gauges at each), induced by each of $\alpha N$ pressure distributions, giving the equation set

$$\{\varepsilon\}_M = [S]_{M \times \alpha N} \{a\}_\alpha N$$

where $[S]$ is termed the sensitivity matrix, and the element $S_{ij}$ is the value of the strain $\varepsilon_i$ which is caused by the load (Eq. 8.4) when $a_j = 1$ and all other load parameters equal zero. $[S]$ is also called Jacobian matrix (Gans, 1992) or design matrix (Press et al, 1988) in least squares method.

Equation 8.5 is the foundation of the following process for deducing pressures from the strain gauge readings.

**8.3.3 Inversion of pressure-strain relations with error minimisation**

The above analyses produce a total of $M$ sets of $\beta$ strain measures on the silo walls. The vector of observed strains in the silo test $\{\varepsilon\}$ is given by

$$\{\varepsilon\}_M = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M]^T$$

Equation 8.6
The goal of the present analysis is to find the load parameter vector \( \{a\} \) which has minimum errors in predicting the observations. To obtain this vector, a related vector of strains associated with \( \{a\} \) is defined as the vector of expected strains \( \{\varepsilon\} \). The discrepancies between the strain observations vector \( \{E\} \) and the expected strain vector \( \{\varepsilon\} \) may then be expressed as an error vector \( \{e\} \)

\[
\{e\} = [S]\{a\} - \{\varepsilon\}
\]  

(8.7)

The statistical processing of the observations seeks to minimise the error vector \( \{e\} \). This is achieved in a least squares manner by first expressing the errors as a target function

\[
E = \{e\}^T[W]\{e\}
\]  

(8.8)

in which \( [W] \) is a weighting matrix. If the weighting matrix is set as a unit matrix \( [I] \), the target function becomes simply the sum of the squares of the individual errors. The use of the weighting matrix permits different emphasis to be placed on different observations. The vector \( \{a\} \) can now be found by minimising Eq. 8.8 as

\[
\frac{\partial E}{\partial \{\varepsilon\}} = 0
\]  

(8.9)

which leads to the normal equations

\[
\]  

(8.10)

The solution of Eq. 8.10 may be written as

\[
\{a\} = ([S]^T[W][S])^{-1}[S]^T[W]\{\varepsilon\} = [U]\{\varepsilon\}
\]  

(8.11)

in which \( [U] \) is called the projection matrix and given by

### 8.4 Implementation of the pressure inference method

A general procedure, using the above calculations, may be described as follows:

a) Assume a set of functions with a finite set of unknowns to characterise the load distribution. This results in \( \alpha N \) unknown load coefficients (Eq. 8.4). The loads may be generalised loads which could include base settlements or temperature changes.

b) Calculate the sensitivity matrix \([S]\) (Eq. 8.5) for the structure with these gauge locations using a series of structural analyses of the shell. The structural calculations can only rarely be performed algebraically: finite element analysis is required in most useful applications.

c) Calculate the projection matrix \([U]\) using Eq. 8.12. The weighting matrix \([W]\) may be set to be equal to the unit matrix \([I]\) if no weighting of data is desired or needed, but it may be modified to reflect the different reliability or significance of different strain readings.

d) Calculate the load parameter vector \([a]\) for each time step of the experiment using the observed strain vectors \([\bar{e}]\) as in Eq. 8.11. In most laboratory experiments there are several hundred of these vectors and in a big experimental series there may be over \(10^5\), so the matrix \([U]\) is well worth calculating.

e) Calculate the magnitude of the fitting errors, which are given by \([e]\) in Eq. 8.7. Explore the sensitivity of the errors to the assumptions made about the load distribution. If necessary, repeat the procedure with a different assumed load distribution or a different weighting matrix \([W]\).
8.5 An example: inferring the pressure distribution on a silo wall

A simple example is presented in this section. It is used here simply to illustrate the procedure described above, not to verify the use of the technique for complex unsymmetrical loading conditions. The results of the interpretation of unsymmetrical strains observed in silo experiments are presented in the later chapters, and are not shown here because of space restrictions.

An on-ground cylindrical silo with a fixed base is shown in Fig. 8.2 under axisymmetric loading from the stored solid (Fig. 8.2). Because the loading is taken to be axisymmetric, only two strain observations are considered at each station. For simplicity, the normal pressure loading and the meridional traction on the silo walls (Eqs 8.4) are here approximated by a piece-wise linear variation with 6 segments of equal height, which can be represented by 6 triangular loading functions \( f_i \) as shown in Fig. 8.3. The normal pressure \( p_n \) and the vertical frictional traction \( p_v \) are then given by

\[
p_n(z) = \sum_{i=1}^{6} a_i f_i(z) \tag{8.13a}
\]

\[
p_v(z) = \sum_{i=1}^{6} a_i f_i(z) \tag{8.13b}
\]

The load parameter vector with \( \alpha N = 12 \) (\( N = 6 \)) is then defined by

\[
\{ a \} = \{ a_1, a_2, ..., a_{12} \}^T \tag{8.14}
\]

For this illustration, the silo was given a height \( H = 12 \text{m} \), radius \( R = 2 \text{m} \), wall thickness \( t = 5 \text{mm} \), Young's modulus \( E_w = 2 \times 10^5 \text{MPa} \), Poisson's ratio \( \nu_w = 0.3 \) and the wall was assumed to be isotropic. Suppose that pairs of gauges are installed at the heights of \( z_1 = 9.25, z_2 = 7.25, z_3 = 5.25, z_4 = 3.25, z_5 = 1.25 \) and \( z_6 = 0.60 \text{m} \). At each pair only the
circumferential and longitudinal strains on the outer surface of the shell wall are measured ($\beta=2$, $\beta M=12$). The strain vector $\{\varepsilon\}$ is given by

$$\{\varepsilon\} = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10}, \varepsilon_{11}, \varepsilon_{12}]^T.$$  

(8.15)

The numbering of the gauges is arranged so that strain observation $2i-1$ represents the circumferential strain at $z=z_i$ and strain observation $2i$ represents the longitudinal strain at $z=z_i$ (e.g. $\varepsilon_5$ presents the circumferential strain in the gauge at height $z_3$).

The measured strain at each gauge is a linear function of the load parameters (Eq. 8.14), and the coefficients in the matrix of these linear relationships are known as the sensitivity functions (for each strain with respect to each loading parameter). These sensitivity functions were evaluated using the FELASH suite of finite element programs for the analysis of axisymmetric shells (Rotter, 1989) with 12 load cases as shown in Fig. 8.3 (with $p_n=f_i$ for $i=1,6$ and $p_n=f_{i-6}$ for $i=7,12$).

The calculated strains or sensitivity functions for two loading functions, $f_1$ with $a_1=1$ kPa and $a_7=1$ kPa, are plotted in Fig. 8.4. For normal pressure loading ($a_1$), the circumferential strains follow the normal pressure loading closely with Poisson’s effect causing vertical strains. The meridional stress at a point in the wall is the integral of the frictional traction shears above that point. Thus for frictional traction loading ($a_7$), the compressive meridional strain increases with depth over the triangular loading patch (in bi-parabolic form) and remains constant thereafter. The associated circumferential strain is a Poisson effect. At the bottom boundary, strain variations due to local bending phenomena are seen.

The sensitivity matrix is given by
Following a concentric filling of a silo, the normal pressure distribution on the wall is close to that predicted by Janssen theory (1895):

\[ P_n(z) = \frac{\gamma R}{2 \mu} \left( 1 - e^{2\mu k(\gamma-z)/R} \right) \]  

(8.17a)

in which R is the radius of the silo, \( \mu \) is the wall friction coefficient and \( \gamma \) is the unit weight of the stored solid. The parameter k is the ratio of the normal wall pressure to the mean vertical stress in the solid (lateral pressure ratio) and is here assumed to be a material constant.

The meridional traction when friction is fully mobilised against the wall is

\[ p_v(z) = \mu p_n(z) \]  

(8.17b)

The technique of pressure interpretation can be further illustrated by studying an example in which it is supposed that the real load distribution is represented exactly by Eqs 8.17 with \( \gamma = 10 \text{ kN/m}^3 \), \( \mu = 0.4 \) and \( k = 0.5 \). The exact strains which would be observed at each gauge location may be calculated numerically as

\[ \{ \varepsilon \} = 10^6 \times [23.06, -12.69, 35.71, -26.04, 46.18, -41.57, 55.17, -58.58, 63.17, -76.58, 65.58, -82.56]^T \]  

(8.18)

<table>
<thead>
<tr>
<th>[ \text{[S]} ] =</th>
<th>[ \mu \varepsilon ] (8.16) ] kPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.25 0.749 -5.40E-04 -3.27E-06 -1.94E-08 -1.94E-10 0.483 0.0422 -2.21E-06 -1.32E-08 -7.93E-11 -4.80E-13</td>
<td></td>
</tr>
<tr>
<td>-0.376 -0.225 7.99E-05 4.79E-07 2.87E-09 1.74E-11 -1.61 -0.141 3.24E-07 1.94E-09 1.16E-11 7.92E-14</td>
<td></td>
</tr>
<tr>
<td>-1.97E-03 1.25 0.749 -5.40E-04 -3.27E-06 -1.94E-08 0.483 0.0422 -2.21E-06 -1.32E-08 -8.01E-11</td>
<td></td>
</tr>
<tr>
<td>2.85E-04 -0.376 0.225 7.99E-05 4.79E-07 2.87E-09 -2.00 -1.41 -0.141 3.24E-07 1.94E-09 1.17E-11</td>
<td></td>
</tr>
<tr>
<td>-1.19E-05 -1.97E-03 1.25 0.749 -5.40E-04 -3.27E-06 -2.00 -2.00 -1.41 -0.141 3.24E-07 -1.34E-08</td>
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</tr>
<tr>
<td>1.73E-06 2.35E-04 -0.376 -0.225 7.99E-05 4.79E-07 -2.18 -2.00 -2.00 -1.41 -0.141 3.24E-07 1.94E-09</td>
<td></td>
</tr>
<tr>
<td>-7.04E-08 -1.19E-05 -1.97E-03 1.25 0.749 -5.40E-04 0.000 0.000 0.483 0.0422 -2.21E-06</td>
<td></td>
</tr>
<tr>
<td>1.03E-08 1.73E-06 2.35E-04 -0.376 -0.225 7.99E-05 -2.00 -2.00 -1.61 -0.141 3.24E-07</td>
<td></td>
</tr>
<tr>
<td>-3.93E-10 -6.60E-08 -1.19E-05 -1.58E-03 1.25 0.749 0.000 0.000 0.483 0.0422 -2.21E-06</td>
<td></td>
</tr>
<tr>
<td>5.87E-11 9.81E-09 1.64E-06 2.34E-04 -0.376 -0.225 -2.00 -2.00 -2.00 -2.00 -1.41 -0.141</td>
<td></td>
</tr>
<tr>
<td>-6.82E-12 -9.05E-10 -1.35E-07 -2.21E-05 0.000 1.40 0.000 0.000 0.483 0.0422 0.573 0.147</td>
<td></td>
</tr>
<tr>
<td>-9.24E-12 -1.55E-09 -2.59E-07 -4.53E-05 -0.180 -0.420 -2.00 -2.00 -2.00 -2.00 -1.91 -0.400</td>
<td></td>
</tr>
</tbody>
</table>
which are here treated as the observed strains. The above technique may now be applied to these 'observed' strains to see how satisfactory the procedure is in inferring the "known" Janssen distribution under perfect conditions and when experimental errors are present.

The load parameter vector \( \{a\} \) was found by applying Eq. 8.11 (which describes the load distribution as piece-wise linear) to \( \{z\} \), and resulted in

\[
\{a\} = [8.54, 13.97, 17.60, 20.06, 21.66, 22.78, 3.46, 5.56, 7.02, 8.01, 8.68, 9.05]^T \text{ kPa}
\]  

(8.19)

These load coefficients were substituted into Eqs 8.13 to produce the predicted pressure and traction distributions. As expected, the predicted piece-wise linear normal pressure distribution is in very close agreement with the 'real' Janssen pressures as shown in Fig. 8.5. The slight differences are due to the multi-linear representation of the exponential relationship. Naturally, these differences can be further reduced by increasing the number of loading functions and the number of strain measures.

Since theoretical strains were used as observed strains in the above calculations and the number of unknown load parameters is equal to the number of 'observed' strains, a null error vector \( \{e\} \) (Eq. 8.7) is obtained. In real situations, discrepancies will arise between the expected and observed strains. These discrepancies and the sensitivity of the input data are briefly discussed next.
8.6 The requirements of experimental data

8.6.1 Number of strain observations

The total number of strain readings $\beta M$ must be equal to or greater than the number of unknown load parameters being sought $\alpha N$. In general, a larger number of strain observations and/or gauge positions may be expected to produce a better definition of the loading distribution.

8.6.2 Effects of random noise errors

The effects of noise in the observations and changes in the number of strain readings was explored by subjecting the simulated strains on the example silo to randomly generated noise. Three example cases with $\beta M=12$, 24 and 48 strain measurements were evaluated for the same axisymmetric problem ($\beta=2$). It was assumed that the pressure distribution to be found was the statistical best fit to a group of $\beta$ observations from gauges located at the positions defined in the previous section. Thus several independent observations were assumed at each gauged level. The expected strain set at each level was first determined by applying the loads of Eqs 8.17a and 17b, so the ideal strain vector remained unchanged (Eq. 8.18). However, noise in the signal was represented by perturbing the ideal observation with a randomly varying evenly distributed scatter with a peak amplitude of $e=\pm 5\%$ of the mean absolute strain $\varepsilon_p$ ($\varepsilon_p=\Sigma |\epsilon_1|/2M$, $i=1$ to $2M$) at that level superimposed on each strain. This variation was achieved using computer-generated evenly distributed random numbers. Figure 8.6 shows the simulated circumferential and meridional strain observations at $z=z_3=5.25m$.

For each of the perturbed strain observation sets, the load parameter vector $\{\alpha\}$ was determined using the inference procedure described above. The resulting load distribution was then calculated using Eqs 8.13. In the following statistical pressure-inference analysis, fifty such sets of strain observations were used for each of the
three cases $\beta M=12$, 24 and 48. For clarity, the deterministically-derived load
distribution for each of these 50 sets is shown in Fig. 8.7 for the case of $\beta M=12$.

Figures 8.8a and b show the mean deduced distributions for normal pressure and
meridional traction respectively. The mean deduced pressures for all three cases
approach the Janssen load distribution closely. It is significant that the inferred
frictional traction distribution is always worse than the inferred normal pressure
distribution: this is because, under axisymmetric loading inducing only membrane
strains, the strain observations for the former arise from the integral of the pressure
distribution whereas the latter arise directly from the pressures. The traction at the
bottom of the silo is further worsened by the small value of the sensitivity function
caused by $f_5$ in the last column of the sensitivity matrix Eq. 8.16 (that is, the vertical
strain near the bottom is the only measurement affected by the vertical friction near
the bottom, but it is not very sensitive to the value of this friction).

Table 8.1a shows the predicted mean pressure at each level $\bar{p}_n/p_{nj}$ (normalised by the
appropriate Janssen pressure) and its coefficient of variation $\sigma/p_{nj}$. Table 8.1b shows
the corresponding numbers for vertical frictional traction. For the same order of
signal-to-noise ratio in strain measurements (here 5%), the results suggest that the
loads can be determined with progressively increasing accuracy as the total number
of strain measurements increases. However, it appears that the improvements do not
increase proportionately with the number of strain readings $\beta M$ but with its square
root $\sqrt{\beta M}$:

$$\sigma \sqrt{\beta M} = \text{constant} \quad (8.20)$$

in which $\sigma$ is the standard deviation of the deduced pressures.
Table 8.1a Ratio of inferred mean normal pressure to assumed ideal (Janssen) value (e=5%)

<table>
<thead>
<tr>
<th>Height z</th>
<th>Mean pressure ratio</th>
<th>CoV</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta M=12$</td>
<td>$\beta M=24$</td>
<td>$\beta M=48$</td>
<td>$\beta M=12$</td>
<td>$\beta M=24$</td>
<td>$\beta M=48$</td>
</tr>
<tr>
<td>5H/6</td>
<td>1.06</td>
<td>1.01</td>
<td>1.05</td>
<td>20.3</td>
<td>15.0</td>
<td>10.6</td>
</tr>
<tr>
<td>4H/6</td>
<td>1.00</td>
<td>1.04</td>
<td>1.01</td>
<td>11.5</td>
<td>7.78</td>
<td>6.31</td>
</tr>
<tr>
<td>3H/6</td>
<td>1.02</td>
<td>0.984</td>
<td>1.02</td>
<td>10.3</td>
<td>6.51</td>
<td>5.11</td>
</tr>
<tr>
<td>2H/6</td>
<td>1.00</td>
<td>1.02</td>
<td>1.00</td>
<td>8.52</td>
<td>7.34</td>
<td>4.12</td>
</tr>
<tr>
<td>H/6</td>
<td>1.00</td>
<td>0.993</td>
<td>1.00</td>
<td>9.20</td>
<td>6.65</td>
<td>3.80</td>
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<tr>
<td>0</td>
<td>0.994</td>
<td>1.01</td>
<td>1.00</td>
<td>7.34</td>
<td>6.37</td>
<td>3.61</td>
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</table>

Table 8.1b Ratio of inferred mean frictional traction to assumed ideal (Janssen) value (e=5%)

<table>
<thead>
<tr>
<th>Height z</th>
<th>Mean traction ratio</th>
<th>CoV</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta M=12$</td>
<td>$\beta M=24$</td>
<td>$\beta M=48$</td>
<td>$\beta M=12$</td>
<td>$\beta M=24$</td>
<td>$\beta M=48$</td>
</tr>
<tr>
<td>5H/6</td>
<td>1.04</td>
<td>1.05</td>
<td>1.02</td>
<td>32.2</td>
<td>23.4</td>
<td>17.2</td>
</tr>
<tr>
<td>4H/6</td>
<td>1.04</td>
<td>0.987</td>
<td>1.03</td>
<td>32.5</td>
<td>24.7</td>
<td>18.2</td>
</tr>
<tr>
<td>3H/6</td>
<td>0.960</td>
<td>1.03</td>
<td>0.999</td>
<td>27.6</td>
<td>18.8</td>
<td>15.4</td>
</tr>
<tr>
<td>2H/6</td>
<td>1.02</td>
<td>0.992</td>
<td>1.02</td>
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<td>16.0</td>
<td>12.8</td>
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<tr>
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<td>0.972</td>
<td>30.1</td>
<td>22.2</td>
<td>13.1</td>
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<tr>
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<td>0.985</td>
<td>1.10</td>
<td>85.5</td>
<td>65.1</td>
<td>45.3</td>
</tr>
</tbody>
</table>

8.6.3 Quality of the strain observations

The quality of the source experimental data (i.e. the accuracy of the strain measurements) naturally has a strong influence on the inferred pressures. This depends on the gauges and experimental equipment as well as the data acquisition system.
The effect of the amplitude of errors in the observations on the inferred load distribution can be explored by subjecting the simulated strains in the example silo to different levels of randomly generated noise (5%, 10% and 20%). In this illustration, the procedure was repeated 50 times for each case. Figure 8.9 shows the deterministically-derived equivalent normal pressure distribution for each of these perturbed strain observation sets for the case of $\beta M=48$ with $e=10\%$. Table 8.2 shows the coefficient of variation of the inferred pressures with different levels of noise. The relationship between noise level and the standard deviation in the inferred pressure is quite linear and can be expressed as

\[ \frac{\sigma}{e} \approx \text{constant} \]  
(8.21)

Table 8.2 Standard deviation with different error levels ($\beta M=48$)

<table>
<thead>
<tr>
<th>Height z</th>
<th>$\text{CoV normal pressure}$ $\sigma/p_{nj}$ (%)</th>
<th>$\text{CoV frictional traction}$ $\sigma/p_{v_j}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$e=5%$ $e=10%$ $e=20%$</td>
<td>$e=5%$ $e=10%$ $e=20%$</td>
</tr>
<tr>
<td>5H/6</td>
<td>10.6 19.5 30.3</td>
<td>17.2 25.8 68.8</td>
</tr>
<tr>
<td>4H/6</td>
<td>6.31 13.3 22.3</td>
<td>18.2 27.9 67.9</td>
</tr>
<tr>
<td>3H/6</td>
<td>5.11 10.6 18.5</td>
<td>15.3 26.0 50.7</td>
</tr>
<tr>
<td>2H/6</td>
<td>4.12 9.09 17.3</td>
<td>12.8 24.2 51.9</td>
</tr>
<tr>
<td>H/6</td>
<td>3.81 10.4 15.7</td>
<td>13.1 29.4 58.8</td>
</tr>
<tr>
<td>0</td>
<td>3.60 8.04 13.5</td>
<td>45.3 94.9 171</td>
</tr>
</tbody>
</table>

8.6.4 Location of the measurement stations

The placing of the measurement stations affect the accuracy with which the load parameter vector can be evaluated. For this simple silo example, the influence may be explained using Eq. 8.5 which shows that each strain to be measured is a linear function of the load parameters: the proportionality factors are the sensitivity functions given in $\{S\}_{j\mu}$. Thus, in this example, where the outcome is known in
advance, larger strains are to be expected in the lower part of the silo, so measurements made in this zone should result in a better signal-to-noise ratio.

However, strain gauges should not be mounted too close to the boundary, because this area experiences significant local bending (Rotter, 1987a). This phenomenon is illustrated in Fig. 8.4, where the sensitivity functions can be seen to change dramatically with height near the bottom. Thus a small deviation from the intended position of a strain gauge may result in exaggerated interpretation errors through the sensitivity matrix $[S]$. Furthermore, minor differences between the experimental and analytically assumed boundary conditions can lead to very serious errors. This is an important restriction on the use of the method, and means that complex structural geometries, uncertain wall stiffnesses and uncertain boundary conditions all make the method unsuitable.

In general, the best locations for strain measurements can only be found through many analyses and careful study of the structural behaviour under a variety of expected load distributions: positions in the structure which display a poor sensitivity or which display dramatic variation with height should not be used as gauge sites.

8.6.5 Measurement of the strain components

As described above, the complete strain state requires six strain gauges at each station. The relative magnitude of each of the six strain components depends on the silo geometry, the boundary conditions, the choice of strain gauge sites and the loading conditions. A clear understanding of the shell behaviour is vital to achieving reasonable modelling of the strain state.

Because symmetrical loading was assumed in the above example silo, the only non-zero strains were the membrane and bending strains in the meridional and circumferential directions $\varepsilon_{m}, \varepsilon_{m}, \kappa_{b}$ and $\kappa_{b}$. 
Under these conditions, the circumferential strains are highly correlated with the normal pressures, while the meridional strains are sensitive to the frictional traction (Fig. 8.4). These simple connections become much weaker when the loading is non-symmetric, but a much more complicated example is needed to illustrate this, so it falls outside the scope of this paper.

8.6.6 Assumptions concerning the load distribution

When pressure cells are used to measure the pressure distribution, the pressure is deemed to have been reliably measured at each cell site, and the pressure between cells is inferred either by connecting observations with straight lines (Nielsen, 1983a; Ooi et al, 1990; Blight, 1990b) or, where the data permit it, with smooth curves. Both interpretations imply assumptions about the complete pressure distribution, though these are rarely stated. In the present process, the assumed pressure distribution is more clearly seen as an assumption, and is properly defined. However, since this assumption must be made at the outset of an interpretation, some comments on its form should be noted.

Any component of the assumed load distribution \( \{a_i\} \) (Eqs 8.4) which leads to small strains induced in the wall at all gauge stations will not be determined reliably. This may be stated alternatively as follows: if a column in the matrix \([S]\) (Eq. 8.16) contains only small entries, the load parameter \(a_i\) to which it corresponds will be poorly defined by the strain observations represented by the rows in \([S]\). If the strains are measured near all points at which the structural integrity is in question, then the unreliability of the load parameter evaluation is perhaps not important, since this parameter clearly influences the wall stresses very little, but the assumed load distribution should be designed so that all terms in the vector \(\{a\}\) are well conditioned.
8.7 Summary and conclusions

A procedure has been described for inferring the silo wall pressure distribution from measured strains on a silo wall. The matrices needed to implement the method have been defined, and some aspects of statistical techniques to investigate the quality of the outcome have been discussed. The theory has been presented in its fullest form, in which all strain readings are used, and the loading parameters include normal pressure, vertical wall friction and circumferential wall friction. A simple axisymmetrically loaded silo subject to Janssen normal pressures and frictional tractions has been used to illustrate the procedure.

Instrumentation restrictions in the number and placement of measuring stations have also been discussed. The number of strain observations and the level of noise in the strain measurements have been shown to have very significant effects on the inference process. The coefficient of variation of the inferred pressures can be reduced linearly by either reducing the level of noise or increasing the square root of the number of strain observations.

The simple exploratory analyses presented in this paper indicate that the instrumentation used in an experiment should be planned very carefully. When planning the experiment, it is important that a thorough structural analysis of the test silo should be made, followed by an investigation of the sensitivity of the loading parameters to the measuring station locations and data accuracy. Both the measuring station locations and the number of strain components measured at each location have a significant bearing on the usefulness of the data in inferring the pressures acting on the silo walls.

8.8 Notation

E  Young's modulus or target function
H  height of silo
M number of gauge locations
N number of load functions
R radius of silo
S sensitivity matrix
U projection matrix
W weighting matrix
a load parameter vector
e error vector or amplitude of noise
f load function
k lateral pressure ratio
p pressure or traction loading
t thickness of wall
z vertical coordinate
α number of load components
β number of strain measures at each gauge location
ε strain
ϕ meridional coordinate
γ unit weight of stored solids
κ curvature
μ wall friction coefficient
ν poison's ratio
σ standard deviation
θ circumferential coordinate

Subscripts
n normal
ν vertical
w wall
b bending
m membrane
Fig. 8.1 "Sandwich" strain gauge
$R = 2000\text{mm}$

$H/R = 6$

$R/t = 400$

$E_w = 2 \times 10^5 \text{MPa}$

$V_w = 0.3$

$\gamma = 1 \times 10^{-5} \text{N/mm}^3$

$\mu = 0.4$

$k = 0.5$

Fig. 8.2 Silo example

$$f_i(z) = \begin{cases} 
1-i+n(1-z/H) & \text{if } (n-i)H/n \leq z < (n-i+1)H/n \\
1+i-n(1-z/H) & \text{if } (n-i-1)H/n \leq z < (n-i)H/n \\
0 & \text{elsewhere}
\end{cases}$$

Fig. 8.3 Load functions

Fig. 8.4 Distribution of sensitivity functions
Fig. 8.5 Normal pressure distribution

Fig. 8.6 Simulated strain observations (e=5\%)

Fig. 8.7 Deduced normal pressures including random noise
Fig. 8.8 Distribution of predicted mean pressures and tractions (ε=5\%)  

Fig. 8.9 Predicted normal pressure distributions with random noise ε=10\%
Chapter 9

STATISTICS OF LOAD COEFFICIENTS
AND SELECTION OF LOAD FUNCTIONS

9.1 Introduction

A statistical procedure for inferring silo wall pressures from strain measurements on the walls was set out in Chapter 8. Because measured strains on silo walls inevitably contain observation errors or noise, the inferred loads also contain errors. Key questions which must be answered include how errors in the strain observations will propagate to the inferred loads, to what quality the inferred loads are reliable and how the results can be improved both in planning the experiment and in the interpretation process.

A simple example silo under axisymmetric loads was used to illustrate the procedure and to explore numerically the influence of the number and location of strain gauges and the quality of the strain measurements on the quality of the inferred load coefficients. However, the example is so specific that the conclusions cannot yet be generalised unless they can be theoretically proven. The theoretical verification of these conclusions is one of the objectives of this chapter.

In numerical regression of experimental data, a model is often considered as acceptable by engineering researchers if the fitted parameters seem to be reasonable to them. This is true in many cases, but is not always true because it lacks an objective measure of the quality of the fitted parameters. More serious researchers therefore look at the statistics of the fitted parameters. Fewer engineering researchers
also look at the overall acceptability of a model using a more formal statistical measure (e.g. F-test). For the proposed procedure of inferring wall pressures from strain measurements, general discussions on the statistics of the inferred load coefficients (e.g. the variances and covariances), the estimation of variances of strain observations from fitting and the goodness-of-fit are presented in this chapter.

The selection of load functions has naturally a significant effect on the quality of the inferred load coefficients. A bad choice of load functions may result in a failure of the inference process. However, in planning an experiment, the key concern is the placement of strain gauges at locations where the load functions take their values to form the sensitivity vectors. The relationships between the strain gauge locations and the load functions are subjected to discussion in this chapter. A dependence index is proposed to provide a simple tool to judge the quality of a proposed set of load functions and to help determine optimal strain gauge locations.

An example ring under harmonic pressures is used to illustrate the theoretical findings in this chapter. The influence of the strain gauge locations and similar load functions are further discussed through the analysis of the simple example.

Whilst many mathematical descriptions are involved, the chief objective of this chapter is not mathematical precision but the application of mathematical concepts in the proposed new method of inferring wall pressures from strain measurements in the previous chapter. The literature in Sections 9.2 and 9.3 refers to general statistical processing of data and not to strain observations or civil engineering or shell structures.
9.2 Statistics of a fitting

9.2.1 Variances and covariances of load coefficients

The statistics (e.g. the variances) of the estimated load coefficients are closely related to the inverse of the coefficient matrix on the left-hand side of the normal equations

\[ [S]^T[W][S][a] = [S]^T[W][\epsilon] \]  \hspace{1cm} (9.1) (8.10 bis)

There are several different ways of deducing the variance-covariance matrix of the inferred parameters in general theory of the least squares estimation (Bevington, 1969; Farebrother, 1988; Press et al, 1988; Gans, 1992). A general and slightly different deduction is briefly presented for deducing the variance-covariance matrix of the load parameters here.

The actual strain measurements \{\epsilon\} may be written as

\[ \{\epsilon\} = \{\epsilon\} + \{e\} \]  \hspace{1cm} (9.2)

in which \{e\} is the error vector in the strain readings \{\epsilon\} and \{\epsilon\} is the value that \{\epsilon\} would have taken if it had been observed without error. Note that \{\epsilon\} contains no error so that \(E[\{\epsilon\}] = \{\epsilon\}\) and \(\text{Cov}[\{\epsilon\}] = \text{Cov}[\{e\}]\).

Substituting

\[ \{e\} = [S][a] \]  \hspace{1cm} (9.3) (8.5 bis)

into Eq. 9.2 gives
\[ \{\bar{e}\} = [S]\{a\} + \{e\} \quad (9.4) \]

in which \{a\} is the value that the load coefficients would have taken if there were no observation errors in the strain readings.

The least squares estimator is given by

\[ \{\bar{a}\} = (\{S\}^T[R][S])^{-1}\{S\}^T[R]\{\bar{e}\} \quad (9.5) \]

Inserting Eq. 9.4 into Eq. 9.5 leads to

\[ \{\bar{a}\} = \{a\} + ([S]^T[R][S])^{-1}[S]^T[R]\{e\} \quad (9.6) \]

Errors of \{a\} due to reading errors of strain are then

\[ \{\delta_a\} = \{\bar{a}\} - \{a\} = ([S]^T[R][S])^{-1}[S]^T[R]\{e\} \quad (9.7) \]

It is assumed next that all the errors in strain readings are normally distributed with mean zero (\(E\{e\} = 0\)). Noting that both \([S]\) and \([R]\) are constant matrices, the expectation of the error in the load coefficients is given by

\[ E(\delta_a) = ([S]^T[R][S])^{-1}[S]^T[R]E\{e\} = 0 \quad (9.8) \]

Thus, \{\bar{a}\} is an unbiased estimator of \{a\}. The variance-covariance matrix of \{\bar{a}\} is given by
\[
\text{Cov}[[\bar{a}]] = E[[\delta_a]\{\delta_a\}^T]
\]
\[
= ([S]^T[W][S])^{-1}[S]^T[W]E[[\bar{e}]\{\bar{e}\}^T][W]^{-1}[S]^T[W][S])^{-1} \quad (9.9)
\]

where each diagonal element \(i_i\) represents the variance of \(\bar{a}_i\) and each off-diagonal element \(i_j\) represents the covariance between \(\bar{a}_i\) and \(\bar{a}_j\).

If the weighting matrix \([W]\) is chosen as the inverse of the variance-covariance matrix of strain observations, i.e.

\[
[W] = \text{Cov}^{-1}[[\bar{e}]] = E^{-1}[[\bar{e}]\{\bar{e}\}^T] \quad (9.10)
\]

Equation (9.9) is simplified as

\[
\text{Cov}[[\bar{a}]] = ([S]^T[W][S])^{-1} \quad (9.11)
\]

This is the inverse of the coefficient matrix on the left-hand side of the normal equations (Eq. 9.1).

If the variances and covariances of strain observations are unknown, but their relations to each other are known or can be estimated, it may then be assumed that the weighting matrix is proportional to the inverse of the variance-covariance matrix of the strain observations

\[
[W] = \sigma^2(\bar{e})E^{-1}[[\bar{e}]\{\bar{e}\}^T] \quad (9.12)
\]

where \(\sigma^2(\bar{e})\) is the variance of strain observations with unit weightings.
The variance-covariance matrix of the inferred load coefficients (Eq. 9.9) becomes

$$\text{Cov}[\{\bar{a}\}] = \sigma^2(\bar{e})[S]^T[W][S]^{-1}$$  \hspace{1cm} (9.13)

which is the inverse of the coefficient matrix on the left-hand side of the normal equations multiplied by \(\sigma^2(\bar{e})\), which may be estimated as shown in the next section.

If it is assumed that there is no correlation between the strain observations and they have equal variance \(\sigma^2(\bar{e})\), Equation 9.12 reduces to

$$[W] = \sigma^2(\bar{e})E^{-1}[[\bar{e}]\{\bar{e}\}^T]=[I]$$  \hspace{1cm} (9.14)

The variance-covariance matrix of inferred load coefficients then becomes

$$\text{Cov}[\{\bar{a}\}] = \sigma^2(\bar{e})[S]^T[S]^{-1}$$  \hspace{1cm} (9.15)

where \(\sigma^2(\bar{e})\) may also be estimated from the fitting. Because the estimation of \(\sigma^2(\bar{e})\) is itself subject to error, the estimated errors of the inferred load coefficients \(\{\bar{a}\}\) have a greater uncertainty in the last two cases.

In Chapter 8, the conclusion was drawn from numerical analysis, using a simple example, that the coefficient of variation of inferred loads varies linear with the noise level in the strain observations. From Eqs 9.11, 13 and 15, this is now theoretically verified.
9.2.2 Estimation of variances of strain observations

The variance of the strain observations with unit weight \( \sigma^2(\varepsilon) \) must be estimated from the fitting, if it has not been determined experimentally, to estimate the variances and covariances of load coefficients in the last two cases above. The estimation of \( \sigma(\varepsilon) \) itself also provides a control bar for the quality of strain observations.

From general theory of statistical processing of data (Farebrother, 1978; Gans, 1992), it may be found that the expectation of the sum of weighted strain residual squares has the value \( M - N \) if the weighting matrix equals the inverse of the variance-covariance matrix of strain observations,

\[
E[(\delta_i)^T \mathbf{W}^{-1} (\varepsilon_i) \varepsilon_i^T] = E((\delta_i)^T \mathbf{W} (\delta_i)) = M - N \tag{9.16}
\]

in which \( M \) is the number of strain observations, \( N \) is the number of load coefficients and the strain residual \( \{ \delta_i \} \) is defined as

\[
\{ \delta_i \} = \{ \varepsilon_i \} - [S] \{ \bar{a} \} \tag{9.17}
\]

If the weighting matrix is proportional to the inverse of the variance-covariance matrix of strain observations, substituting (9.12) into (9.16) leads to

\[
E\left[ \frac{(\delta_i)^T \mathbf{W} (\delta_i)}{\sigma^2(\varepsilon)} \right] = M - N \tag{9.18}
\]

The variance of the strain observations with a unit weighting matrix \( [\mathbf{W}] \) can then be estimated by
Thus, for the unweighted least squares method, this equation reduces to

$$ \sigma^2(\bar{e}) = \frac{\{\delta_e\}^T[W]\{\delta_e\}}{M-N} $$

(9.19)

9.2.3 Goodness-of-fit

Substituting Eq. 9.5 into Eq. 9.17, the strain residuals can be found as

$$ \{\delta_e\} = \{\bar{e}\} - \{S\}^{-1}\{S\}^T[\bar{W}][S]\{\delta_e\} $$

(9.21)

The strain residuals are thus linear combinations of strain observations. As the strain observations are assumed to belong to a multivariate normal distribution with mean zero vector, the strain residuals also belong to a multivariate normal distribution with mean zero vector according to general theory of statistics (von Mises, 1964; Mardia et al, 1979). When the weighting matrix is the inverse of the variance-covariance matrix of strain observations, the probability distribution for the sum of the weighted strain residual squares

$$ \chi^2 = \{\delta_e\}^T[W]\{\delta_e\} $$

(9.22)

may be found to be a chi-square distribution with M-N degrees of freedom (Press et al, 1988; Gans, 1992). The probability \( Q \) that the chi-square should exceed a particular value \( x=\chi^2 \) by chance may be computed using the incomplete gamma function as

$$ Q = 1 - \frac{1}{\Gamma(\xi)} \int_0^\infty e^{-t^\xi} dt \quad (\xi>0) $$

(9.23)
where $\zeta=M-N$ is the number of degrees of freedom and the gamma function is defined by

$$\Gamma(\zeta) = \int_0^\infty e^{-t^\zeta} dt \quad (9.24)$$

Because $\zeta=M-N=n$ here is an integer, the gamma function is the factorial function offset by one (Abramowitz and Stegun, 1964; Press et al, 1988),

$$\Gamma(n+1) = n! \quad (9.25)$$

The quantity $Q$ or its complement $P=1-Q$ is tabulated in many statistics books. This computed probability gives a quantitative measure for the goodness-of-fit of the model. If $Q$ is very small, then either the model is wrong and can be statistically rejected or the variance-covariance matrix of observations used is not right. It is also possible that observation errors may not be normally distributed as assumed. Because the mean of the weighted sum of residual squares is $M-N$, it may be deemed a good fit if $\chi^2=M-N$ without computing the quantity $Q$.

When the weighting matrix is proportional to the inverse of variance-covariance matrix of the strain observations, it is not possible to do an independent assessment of goodness-of-fit as above because the variance of the strain observations with unit weightings are estimated through the minimisation of $\chi^2$. However, the lower and upper bounds of the variance of strain observations with unit weight may be estimated by the experimenter. If the estimated variance from the fitting falls between the lower and upper bounds, the fitting may be acceptable.

While the quantity $Q$ or the estimated variance of strain observations provides a measure of the goodness-of-fit of the model, it is important to note that they are not the only criterion by which the acceptability of a model may be determined. It is
equally important that the variances of the load coefficients should be small and that the strain residuals should show no systematic trends.

9.3 Lower bound of variances of load coefficients

The variance-covariance matrix of load coefficients represents the quality of the inferred load coefficients. However, it cannot be obtained before a fitting is done. The objective of this section is to estimate a lower bound to the errors in the load coefficients and to investigate the conditions needed to achieve a lower bound.

The variance-covariance matrix of load coefficients for the unweighted least squares method is given by Eq. 9.13, which may be written in full as

\[
\begin{bmatrix}
\sigma^2(a_1) & \text{Cov}(a_1,a_2) & \ldots & \text{Cov}(a_1,a_N) \\
\text{Cov}(a_2,a_1) & \sigma^2(a_2) & \ldots & \text{Cov}(a_2,a_N) \\
\vdots & \vdots & \ddots & \vdots \\
\text{Cov}(a_N,a_1) & \text{Cov}(a_N,a_2) & \ldots & \sigma^2(a_N)
\end{bmatrix} = \sigma^2(\mathcal{E}) \begin{bmatrix}
S_1^T S_1 & S_1^T S_2 & \ldots & S_1^T S_N \\
S_2^T S_1 & S_2^T S_2 & \ldots & S_2^T S_N \\
\vdots & \vdots & \ddots & \vdots \\
S_N^T S_1 & S_N^T S_2 & \ldots & S_N^T S_N
\end{bmatrix}^{-1}
\]

(9.26)

where \( S_i \) is the i-th column vector in the sensitivity matrix \([S]\) which is produced by the i-th load function with unit amplitude.

If all the sensitivity vectors are orthogonal to each other, i.e. \( S_i^T S_j = 0 \) for all \( i \neq j \), all the off-diagonal elements disappear in the matrix \([S]^T[S]\) and so in its inverse. Thus, all the covariances between the load coefficients are zero while their variances become

\[
\sigma^2(a_i) = \sigma^2(\mathcal{E}) \ (S_i^T S_i)^{-1} = \sigma^2(\mathcal{E}) \left[ \sum_{j=0}^{M} \delta_{ji}^2 \right]^{-1}
\]

(9.27)
In general, sensitivity vectors may not be orthogonal to each other. The off-diagonal elements in the matrix $[S]^T[S]$ are therefore generally not zero. Considering the simplest example with $N=2$, the matrix $[S]^T[S]$ written in full becomes

$$
[S]^T[S] = \begin{bmatrix}
S_1^T S_1 & S_1^T S_2 \\
S_2^T S_1 & S_2^T S_2
\end{bmatrix}
$$

(9.28)

and the diagonal elements of its inverse are

$$
\begin{bmatrix}
\sigma^2(a_1) \\
\sigma^2(a_2)
\end{bmatrix} = \frac{\sigma^2(\xi)}{S_1^T S_1 S_2^T S_2 - (S_1^T S_2)^2} \begin{bmatrix}
S_2^T S_2 \\
S_1^T S_1
\end{bmatrix}
$$

(9.29)

where the symmetry of $[S]^T[S]$, i.e. $S_i^T S_i = S_i^T S_i$ has been used.

Note that $[S]^T[S]$ is symmetric positive definite (if there is no linear dependence between sensitivity vectors). Its inverse is also symmetric positive definite. It can also be proven that

$$
S_1^T S_1 > 0 \quad \text{and} \quad S_1^T S_1 S_2^T S_2 \geq [S_1^T S_1]^2
$$

(9.30)

From Eq. 9.29, it may be shown that

$$
\begin{bmatrix}
\sigma^2(a_1) \\
\sigma^2(a_2)
\end{bmatrix} \geq \sigma^2(\xi) \begin{bmatrix}
\frac{1}{S_1^T S_1} \\
\frac{1}{S_2^T S_2}
\end{bmatrix}
$$

(9.31)

Thus, $\sigma^2(a_i)$ has a lower limit of $\sigma^2(\xi)/S_1^T S_1$ and it achieves this minimum value when $S_1^T S_2 = 0$. From Eq. 9.29, $\sigma^2(a_i) \rightarrow \infty$ if $(S_1^T S_2)^2 \rightarrow S_1^T S_1 S_2^T S_2$. 

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These conclusions may be generalised. If the weighting matrix is the inverse of the variance-covariance matrix of strain observations (Eq. 9.10), the variance of load coefficient $a_i$,

$$\sigma^2(a_i) \geq \frac{1}{S_i^T[W]S_i}$$  \hspace{1cm} (9.32a)

If the weighting matrix is proportional to the inverse of the variance-covariance matrix of strain observations (Eq. 9.12)

$$\sigma^2(a_i) \geq \frac{\sigma^2(E)}{S_i^T[W]S_i}$$  \hspace{1cm} (9.32b)

If no weighting matrix is used (Eq. 9.14)

$$\sigma^2(a_i) \geq \frac{\sigma^2(E)}{S_i S_i}$$  \hspace{1cm} (9.32c)

Equations 9.32 have equalities (lower limits) if and only if sensitivity vector $S_i$ is orthogonal to all the other sensitivity vectors with weight $[W]$, or

$$S_i^T[W]S_j = 0 \hspace{1cm} \text{for all } j \neq i$$  \hspace{1cm} (9.33)

The sum of the weighted squares of all the elements for each sensitivity vector (the square of the length of the sensitivity vector for the unweighted least squares method) may be calculated and they may be plotted against their load function number. This curve represents the lower bound of the variance of load coefficients. The variance of a load coefficient reaches this lower bound if and only if it is orthogonal to all the other sensitivity vectors.
The above can be exploited to optimise the selection of gauge locations and load functions in the present analysis. This is shown in the following sections.

### 9.4 Dependence analysis of sensitivity vectors

#### 9.4.1 Criteria for selection of load functions

The following criteria should be considered in the selection of load functions.

a) The set of selected load functions should be able to describe all the required possible load distributions. This is essential because otherwise the real load distribution cannot be found whatever is done in later stages of the interpretation process. However, such a set of load functions may only be discovered after many trials. In the first instance, all the possible load functions may be listed as possible candidates. If the number of these load functions $N$ is small (e.g. $N<10$), it is possible to search all the $2^N$ possible combinations in a fitting. However, this is not possible either if a large number of load functions is involved (e.g. more than 100 in the experiments processed in Chapter 12), or before an experiment is completed.

b) The number of load functions should be kept to a minimum. This would not only useful for ease of interpretation of the results, but would also improve the statistics of the inferred load coefficients.

c) Linear dependencies among the sensitivity vectors should be avoided. The normal equations (Eq. 9.1) will be singular if such a linear dependence exists among the sensitivity vectors. A huge error may also arise if two or more sensitivity vectors are similar even though they are not linearly dependent. The ideal case is one in which all the sensitivity vectors are orthogonal to each other so that their variances are reduced to their lower limits as described in the previous section.
Because a sensitivity vector consists of values of the load functions at the sampling positions (strain gauge locations), the quality of a load function is essentially dependent on the locations of the strain gauges. It is therefore desirable to avoid using those load functions that produce similar sensitivity vectors or else to change the strain gauge locations so that all the selected load functions produce acceptable sensitivity vectors. This is discussed in the following sections through error analyses of load coefficients.

9.4.2 Factorisation of a vector

The least squares solution of \([\mathbf{S}] \{a\} = \{\mathbf{e}\}\) (Eq. 9.5) is the orthogonal projection of \(\{\mathbf{e}\}\) in an \(M\) dimensional space onto an \(N\) dimensional subspace that is formed by \(S_1, S_2, ..., S_N\). This orthogonal projection of \(\{\mathbf{e}\}\) is further factorised into components of base vectors \(S_i\) for \(i=1, 2, ..., N\). The load coefficients \(a_i\) are obtained for the appropriate component by dividing by the length of the \(i\)-th sensitivity vector \(\|S_i\|\).

If the first criterion for the selection of load functions is satisfied and there is no error in strain observations, the \(M\) dimensional strain vector \(\{\mathbf{e}\}\) is actually in the \(N\) dimensional sensitivity space which is formed by \([\mathbf{S}]\). Due to small errors in strain observations, \(\{\mathbf{e}\}\) may be slightly off the \(N\) dimensional sensitivity space. The orthogonal projection of \(\{\mathbf{e}\}\) onto the \(N\) dimensional sensitivity space minimises the errors. Therefore, errors in strain observations would not increase during the orthogonal projection, but may do so during factorisation.

An \(N\) dimensional vector

\[
\{a\} = [a_1, a_2, ..., a_N]^T
\]  

(9.34)
may represent a directional line (geometrical vector) from the origin O to a point \( A(a_1, a_2, ..., a_N) \) in an N dimensional coordinate system.

A simple example is used here to explore the error propagation during the factorisation of a vector. For given three vectors \( \{a\} = \{a_1, a_2\} \), \( \{b\} = \{b_1, b_2\} \) and \( \{c\} = \{c_1, c_2\} \) in two dimensional Cartesian coordinates, as shown in Fig. 9.1, it is necessary to factorise \( \{c\} \) into \( \{a\} \) and \( \{b\} \) so that

\[
\{c\} = \beta_a\{a\} + \beta_b\{b\} \tag{9.35}
\]

The solution can easily be obtained from geometry as

\[
\beta_a = \frac{\sin(\alpha_c - \alpha_b)}{r_a \sin \alpha_c} \quad r_c \quad \beta_b = \frac{\sin(\alpha_a - \alpha_c)}{r_b \sin \alpha_c} \quad r_c
\]

in which \( \alpha_a \), \( \alpha_b \) and \( \alpha_c \) are the angles between \( \{a\} \), \( \{b\} \) and \( \{c\} \) and the x-axis, \( \alpha \) is the angle between vectors \( \{a\} \) and \( \{b\} \) \((\alpha = \alpha_a - \alpha_b)\), and \( r_a \), \( r_b \) and \( r_c \) are the lengths of \( \{a\} \), \( \{b\} \) and \( \{c\} \) respectively:

\[
\alpha_a = \tan^{-1} \frac{a_2}{a_1} \quad \alpha_b = \tan^{-1} \frac{b_2}{b_1} \quad \alpha_c = \tan^{-1} \frac{c_2}{c_1} \tag{9.37a}
\]

\[
r_a = \sqrt{a_1^2 + a_2^2} \quad r_b = \sqrt{b_1^2 + b_2^2} \quad r_c = \sqrt{c_1^2 + c_2^2} \tag{9.37b}
\]

If the angle between \( \{a\} \) and \( \{b\} \) is zero, they are linearly dependent on each other.
In this case there is either no solution (when \( \alpha_c - \alpha_b \neq 0 \)) or an infinite number of solutions (when \( \alpha_c - \alpha_b = \alpha = 0 \)).

If \( \{c\} \) is subjected to error and it is assumed that \( \sigma^2(c_1) = \sigma^2(c_2) = \sigma^2(c) \) and \( \text{Cov}(c_1, c_2) = 0 \), the variances and covariance of \( \alpha_c \) and \( r_c \) may be found through partial differentiation of Eqs 9.37:

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\[ \sigma^2(\alpha_c) = \left(\frac{\partial \alpha_c}{\partial c_1}\right)^2 \sigma^2(c_1) + \left(\frac{\partial \alpha_c}{\partial c_2}\right)^2 \sigma^2(c_2) + 2 \frac{\partial^2 \alpha_c}{\partial c_1 \partial c_2} \text{Cov}(c_1, c_2) = \frac{\sigma^2(c)}{r^2_c} \]  (9.38a)

\[ \sigma^2(r_c) = \left(\frac{\partial r_c}{\partial c_1}\right)^2 \sigma^2(c_1) + \left(\frac{\partial r_c}{\partial c_2}\right)^2 \sigma^2(c_2) + 2 \frac{\partial^2 r_c}{\partial c_1 \partial c_2} \text{Cov}(c_1, c_2) = \sigma^2(c) \]  (9.38b)

\[ \text{Cov}(\alpha_c, r_c) = \frac{\partial \alpha_c \partial r_c}{\partial c_1 \partial c_1} \sigma^2(c_1) + \frac{\partial \alpha_c \partial r_c}{\partial c_2 \partial c_2} \sigma^2(c_2) + \left[ \frac{\partial \alpha_c \partial r_c}{\partial c_1 \partial c_2} + \frac{\partial \alpha_c \partial r_c}{\partial c_2 \partial c_1} \right] \text{Cov}(c_1, c_2) = 0 \]  (9.38c)

The variances of \( \beta_a \) and \( \beta_b \) and the covariance between them can be found through partial differentiation of Eq. 9.36:

\[ \sigma^2(\beta_a) = \frac{\sigma^2(c)}{r^2 \sin^2 \alpha} \]  (9.39a)

\[ \sigma^2(\beta_b) = \frac{\sigma^2(c)}{r^2 \sin^2 \alpha} \]  (9.39b)

\[ \text{Cov}(\beta_a, \beta_b) = \frac{-\cos(\alpha)}{r^2 \sin \alpha} \sigma^2(c) \]  (9.39c)

It is evident that the variances of \( \beta_a \) and \( \beta_b \) and the covariance between them are linear in \( \sigma^2(c) \). Equations 9.39 show again what was shown in Section 9.3: that the variance of coefficient \( \beta_i \) is inversely proportional to the square of the vector's length. It is then easily understood that the covariance is inversely proportional to the product of the two vectors' lengths. What is of most interest here is that variances and covariance are inversely proportional to \( \sin^2 \alpha \), or that their standard deviations are inversely proportional to \(|\sin \alpha|\). These conclusions may be extended to two N dimensional vectors.
9.4.3 Dependence index between two vectors

Given two M dimensional column vectors

\[ S_i = [s_{i1}, s_{i2}, ..., s_{Mi}]^T \]  
(9.40a)

\[ S_j = [s_{j1}, s_{j2}, ..., s_{Mj}]^T \]  
(9.40b)

The angle \( \alpha \) between them is given by

\[ \alpha = \cos^{-1} \frac{S_i^T S_j}{||S_i|| ||S_j||} = \cos^{-1} \frac{\sum_{k=1}^{M} s_{ki} s_{kj}}{\sqrt{\sum_{k=1}^{M} s_{ki}^2} \sqrt{\sum_{k=1}^{M} s_{kj}^2}} \]  
(9.41)

Thus,

\[ |\sin \alpha| = \sqrt{1 - \left( \frac{\sum_{k=1}^{M} s_{ki} s_{kj}}{\sqrt{\sum_{k=1}^{M} s_{ki}^2} \sqrt{\sum_{k=1}^{M} s_{kj}^2}} \right)^2} \]  
(9.42)

Due to the significant effect of \( |\sin \alpha| \) on the variances and covariances between sensitivity vectors (see Eq. 9.39), a dependence index between \( S_i \) and \( S_j \) may be defined as

\[ D(S_i, S_j) = 100 (1 - |\sin \alpha|) \]  
(9.43)

100 is introduced in Eq. 9.43 for the convenience of presenting the results so that \( D(S_i, S_j) \) has values from 0 to 100. If \( D(S_i, S_j)=0 \) (\( \alpha=90^\circ \)), \( S_i \) is orthogonal to \( S_j \). If \( D(S_i, S_j)=100 \) (\( \alpha=0^\circ \)), \( S_i \) is linearly dependent on \( S_j \). The more \( S_i \) depends on \( S_j \), the larger the value of \( D(S_i, S_j) \) would be. Another property of the dependence index is
that its value remains unchanged if any one of the vectors is multiplied by a non zero factor. The proof is straight-forward from the definition.

Following the definition, the dependence indices between the last two rows of Hilbert matrices $H_N$ (Hilbert, 1894), whose elements are of the form $h_{ij}=1/(i+j-1)$, $i,j=1,2,...,N$, are presented in Table 9.1 for orders up to 10. Hilbert matrices are ill-conditioned in nature. The inverse of a sixth order Hilbert matrix using single precision has only two or three significant digits (Forsythe and Moler, 1967).

Table 9.1 Dependence index between the last two rows in Nth order Hilbert matrices

<table>
<thead>
<tr>
<th>N</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>87.60</td>
<td>92.66</td>
<td>94.95</td>
<td>96.19</td>
<td>96.96</td>
<td>97.47</td>
<td>97.84</td>
<td>98.12</td>
<td>98.33</td>
</tr>
</tbody>
</table>

It may be noted that the correlation coefficient

$$ r_{ij} = \frac{(S_i - \bar{S}_i)^T(S_j - \bar{S}_j)}{||S_i - \bar{S}_i|| \ ||S_j - \bar{S}_j||} \quad (9.44) $$

has similar properties to $D(i,j)$ in many senses. However, it is independent of the mean values of $\bar{S}_i$ and $\bar{S}_j$. A shift of origin can sometimes remarkably improve a fitting (Jennings and McKeown, 1992). This may be reflected in $D(i,j)$ but not in $r_{ij}$. Furthermore, for a constant sensitivity vector $S_i$, $r_{ij}$ would always be zero even though $S_j$ is also close to constant. $D(i,j)$ does not have this defect. Moreover, for any of two rows in Hilbert matrices $H_N$, $r_{ij}$ is always 1.0 for any order $N$, indicating that they are totally correlated. Because lower order Hilbert matrices can still be factorised without special treatment, $r_{ij}$ is clearly not a good indicator of the dependence between two vectors.

For the weighted least squares method, Eq. 9.41 may be accordingly modified to
9.4.4 Dependence index between harmonic vectors

9.4.4.1 Dependence index

Given a set of M strain gauges (samplings) at \( \theta = \theta_k \) around the circumference of the silo, for \( k = 1, 2, \ldots, M \), only harmonic functions are used in the present treatment to fit the observations. The sensitivity vectors may be of the form

\[
S_n = f(n) [\cos(n\theta_1), \cos(n\theta_2), \ldots, \cos(n\theta_M)]^T
\]  

(9.46)

The dependence index between harmonics \( m \) and \( n \) is therefore given by

\[
D(m,n) = 100 \left[ 1 - \sqrt{1 - \frac{\sum_{k=1}^{M} \cos(m\theta_k) \cos(n\theta_k)^2}{\sum_{k=1}^{M} \cos^2(m\theta_k) \sum_{k=1}^{M} \cos^2(n\theta_k)}} \right]
\]  

(9.47)

Here \( f(n) \) does not appear in the equation because a non zero factor in a sensitivity vector does not affect the dependence index. If all the sampling intervals are integer multiples of the smallest sampling interval \( \Delta \theta_1 \) (i.e. \( \theta_k = i\Delta \theta_1 \), for \( k = 1, 2, \ldots, M \) and \( i \) being any integer), giving an integer number of intervals around the circumference (a very natural arrangement) \( K_1 = 2\pi/\Delta \theta_1 \), it may be found that the sensitivity vectors for harmonics \( m \) and \( n \) are linearly dependent if

\[
m = jK_1 \pm n \quad \text{for any integer } j
\]  

(9.48)

The proof is very simple because \( \cos(m\theta_k) = \cos(jK\pm n)\theta_k = \cos(2\pi j\pm n\theta_k) = \cos(n\theta_k) \). Substituting this into Eq. 9.47 gives \( D(m,n) = 100 \).
Thus if the gauges are placed at equal 30° spacings around the circumference 
(Δθ1=30°), K1=360°/30°=12, and harmonics are completely coupled if m−n=12 or 
m+n=12. Thus, the use of harmonics greater than half the number of equal spacings 
(6 here) is immediately ruled out. This is, perhaps, a relatively natural conclusion.

For the above findings, it is not necessary that all the samplings should be equally 
positioned. If the second smallest sampling interval is Δθ2 and the third smallest one 
is Δθ3, etc. with specific integers K2=2π/Δθ2, K3=2π/Δθ3 and so on, the dependence 
index between the sensitivity vectors for Harmonics m and n is larger if

\[ m = jK_i \pm n \quad \text{for } i=1,2, \ldots \text{ and any integer } j \]  

This is because \( \cos(mθ_k)=\cos(nθ_k) \) for all the \( θ_k=jΔθ_i \). Furthermore, the more \( K_i \) the 
pair m and n satisfy with \( m=jK_i\pm n \), the larger the dependence index between 
Harmonics m and n becomes.

### 9.4.4.2 Gauges at different radii

If there are a pair of strain gauges at a circumferential position \( θ \) with different radii 
r1 and r2 (e.g. the examples in the next section), the sensitivity vectors become

\[ S_n=\{f(n,r_1)[\cos(nθ_1),\cos(nθ_2),\ldots,\cos(nθ_M)],f(n,r_2)[\cos(nθ_1),\cos(nθ_2),\ldots,\cos(nθ_M)]\}^T \]  

The dependence index between Harmonics m and n satisfying Eq. 9.48 is

\[ D(m,n)=100 \left[ 1 - \sqrt{\frac{f(n,r_1)f(m,r_2)+f(n,r_2)f(m,r_1))^2}{f^2(n,r_1)+f^2(n,r_2)]\{f^2(m,r_1)+f^2(m,r_2)\}} \right] \]  

This index is smaller than 100 unless \( f(n,r_1)=f(n,r_2) \) and \( f(m,r_1)=f(m,r_2) \) (e.g. under 
pure tension in classical thin shell theory).
9.4.4.3 Other harmonics

For the Harmonics m and n which do not satisfy Eqs 9.48 and 9.49, if there are enough strain locations between 0 and \(\pi\) (or \(2\pi\)), it may be reasonable to expect that the value of \(\sum_{k=1}^{M} \cos(m\theta_k)\cos(n\theta_k)\) is close to \(\int_0^\pi \cos m\theta \cos n\theta \, d\theta\). Due to the orthogonal feature of cosine,

\[
\int_0^\pi \cos m\theta \cos n\theta \, d\theta = 0 \quad \text{for} \ m \neq n \tag{9.52}
\]

\([\sum_{k=1}^{M} \cos(m\theta_k)\cos(n\theta_k)]^2\) is then expected to be small compared with \(\sum_{k=1}^{M} \cos^2(m\theta_k)\sum_{k=1}^{M} \cos^2(n\theta_k)\). Thus all pairs of sensitivity vectors not covered by the criteria of Eqs 9.48 and 9.49 should be almost orthogonal, leading to well-conditioned interpretations in statistical terms.

It follows that if Harmonics m, n and i do not satisfy Eqs 9.48 and 9.49, \([\sum_{k=1}^{M} \cos(m\theta_k)[\lambda_1\cos(n\theta_k)+\lambda_2\cos(i\theta_k)]]^2\) is also small compared with \(\sum_{k=1}^{M} \cos^2(m\theta_k)\sum_{k=1}^{M} [\lambda_1\cos(n\theta_k)+\lambda_2\cos(i\theta_k)]^2\) for all non-zero values of \(\lambda_1\) and \(\lambda_2\). Therefore, the dependence index between the vector of Harmonic m and any new vector formed as a linear combination of the vectors of Harmonics n and i is small and close to zero. More harmonics may be added in the linear combination without loss of validity in the statement.

A full dependence analysis between sensitivity vectors may prevent the use of two similar load functions, but cannot guarantee that one sensitivity vector is not close to a linear combination of others, which may also cause the normal equations to be singular or close to singular. For example, a third line (vector) may lie on a plane formed by two other lines even though the angles between the three are all not very small. The significance of the characteristic above is that it assures that when
harmonics are used as load functions such phenomena may not occur if a full
dependence analysis is done and no two similar harmonics are used at the same time.

If all the cosines are changed to sines, the above conclusions naturally remain valid.
However, the properties are different if both \( \sin m \theta \) and \( \cos n \theta \) appear at the same
time. If the strain gauges are located reasonably uniformly over 0 to \( 2 \pi \), all the
indices between \( \sin m \theta \) and \( \cos n \theta \) are expected to be close to zero because
\[
\int_0^{2 \pi} \sin m \theta \cos n \theta = 0.
\]
If the strains are sampled over 0 to \( \pi \) only, the dependence index
between \( \sin m \theta \) and \( \cos n \theta \) may be expected to be close to zero if \( m \pm n \) is even, but
may take large values if \( m - n \) is odd, especially if \( m \pm n = k \pm 1 \) because
\[
\int_0^\pi \sin m \theta \cos n \theta = \begin{cases} 0 & \text{if } m \pm n \text{ even} \\ \frac{-2m}{m^2 - n^2} & \text{if } m \pm n \text{ odd} \end{cases}
\tag{9.53}
\]

9.4.5 Dependence matrix

The dependence indices between all the \( N \) sensitivity vectors which are of interest
may be calculated as defined by Eq. 9.43. For rapid assimilation, these dependence
indices are best arranged in a table as shown in Table 9.2. Here the first column and
the last row both represent the load function number. The entry for the \( i \)-th row and
\( j \)-th column \((i,j)\) represents the dependence index between the load functions at \((i,1)\)
and \((N+1,j)\). Because the matrix is symmetric, only the lower triangle is needed. For
ease of use, the diagonal elements may also be used to indicate the load function
numbers. Such a matrix may be useful in selecting load functions for a pressure
interpretation (see next section).
9.5 An example: a ring under harmonic pressures

9.5.1 The example ring

A ring (Fig. 9.2) subject to internal pressures acting normal to the ring and varying around the circumference in a harmonic form was used to illustrate the findings above. The ring was a Tee section consisting of a vertical leg with an annular plate outside it. The internal radius of the ring was 2100mm. The height and the thickness of the vertical leg were 123.5mm and 3mm respectively. The annular plate was located at the mid-height of the vertical leg and its cross-section was 100×10mm (Fig. 9.2a).

It was assumed that there are a pair of strain gauges at each location labelled from a to 1 (θ=0°, 15°, 30°, 45°, 60°, 90°, 120°, 150°, 180°, 225°, 270° and 315°) (Fig. 9.2b). Each pair of strain gauges was placed at an inner and an outer position in the annular plate with radii equal to 2121 and 2191 mm respectively to measure the circumferential strains (Fig. 9.2a). These strain gauges may be coded as Aai, Aao,...,
Ali and Alo respectively with respect to their positions. The total number of strain gauges was 24.

Thirty-two load functions were considered. Internal normal pressures varying as Harmonics 0, 2, 3, ..., 31 and 32 were chosen to represent these load functions:

\[ p_n(\theta) = a_n \cos(n\theta) \quad n=0, 2, 3, ..., 32 \]  

(9.54)

Here Harmonic 1 was excluded because it is not self-equilibrating so that the problem does not exist. The ring was modelled using finite elements in the program suite SIMA which is described in the next chapter. Applying each of these load functions with unit amplitude \( a_n = 1 \) to the ring induced a strain in each strain gauge. Some of these strains were then used to form the sensitivity matrix \( S \).

Two examples with different load functions were analysed here. The first 16 harmonics (number 0 to 16 excluding 1) were chosen as load functions in the first example. There are 16 unknowns and 24 strain observations so the sensitivity matrix \( [S] \) has the order 24x16 for each case. Each of the first 16 strain sets produced above under unit harmonic pressure was then input into the program as strain observations (referred to as the “observed” strain Cases 0 to 16 hereafter) respectively to infer the load coefficients. The unweighted least squares method was used to back-calculate the load coefficients. Because the \( i \)-th set of “observed” strains (Case \( i \)) is the same as the \( i \)-th column of the sensitivity matrix \( [S] \), which was produced by the \( i \)-th load function with unit amplitude, it is to be expected that the \( i \)-th inferred load coefficient should be 1.0 while all other load coefficients should be zero and the inferred strains \( \{E\} = [S] \{a\} \) should equal the input strains (i.e. strain residuals should be zero).

In the second example, the last 16 harmonics (17-32) were chosen as load functions by performing a similar analysis. The input strain observations were changed to the strain sets produced under Harmonics 17 to 32 with unit amplitude (“observed” strain Cases 17-32). The expected load coefficients are similar to those in the first
example: the load coefficient for Harmonic i should equal 1.0 while all others should be zero under strain observation Case i.

9.5.2 Estimated standard deviation of strain observations

The use of the unweighted least squares method in these examples means that the standard deviation or variance of the strain observations can be estimated but an independent assessment of the goodness-of-fit cannot be performed. The estimated standard deviation may then be used as an indicator of the quality of the interpretation and the measured data. A large estimated standard deviation may imply either that the interpretation is poorly planned or that the test data are poor.

It is implied in these analyses that all the strain observations in a set of input data are normally distributed and have the same standard deviation. The noise in the data came from round-off errors between the strain observations and the sensitivity vectors. The strain observations were read in from an ASCII data file with 5 significant figures whilst the sensitivity vectors were read in from a double precision binary file that was much more precise. These differences led to some "errors" in the inferred load coefficients.

Figures 9.3a and b show the estimated standard deviations for the two examples under each "observed" strain case. Figure 9.3a (Harmonics 0-16) shows that the standard deviation decreases as harmonic number increases. This is because the strain observations are smaller for higher order harmonics. Figure 9.3b (Harmonics 17-32) does not show this a phenomenon because most of the strains for these cases have two decimals. For numbers with two decimals, the error range is -0.005 to 0.005 and the standard deviation is about 0.0029. For numbers with one decimal (e.g. Harmonic 4), their error range is -0.05 to 0.05 and the standard deviation is about 0.029. Figure 9.3 shows that the estimated standard deviations of "observed" strains are within these ranges.
9.5.3 Dependence indices between sensitivity vectors

The dependence matrix for all the 32 load functions (Harmonics 0-32 excluding 1) is presented in Table 9.3. For clarity, only the integer part of the dependence index is shown. The elements in the first column, the last row and the diagonal line represent the harmonic numbers. Noting that the strain gauge positions are at $\theta=0^\circ$, $15^\circ$, $30^\circ$, $45^\circ$, $60^\circ$, $90^\circ$, $120^\circ$, $150^\circ$, $180^\circ$, $225^\circ$, $270^\circ$ and $315^\circ$, their intervals are $\Delta \theta_1=15^\circ$, $\Delta \theta_2=30^\circ$ and $\Delta \theta_3=45^\circ$. The specific integers in Eq. 9.49 are then $K_1=24$, $K_2=12$ and $K_3=8$. The table shows that the dependence indices between Harmonics $m$ and $n$ are very large on lines $m=24j\pm n$ for any integer $j$. These indices would have been 100 if only one strain were measured at a circumferential position, but are slightly lower because the two gauges were at different radii. In addition, the dependence indices between Harmonics $m$ and $n$ on lines $m=12j\pm n$ and $m=8\pm n$ for any integer $j$ have slightly larger values. The dependence index at the intersections of these two sets of lines are larger than those on only one line.
9.5.4 Inferred load coefficients

For all the 32 cases in the two examples the inferred load coefficient for Harmonic i is 1.000 for strain Case i. However, there are some small values for other load coefficients instead of the expected zeros. These inferred load coefficients are plotted against harmonic number in Figs 9.4a, b, c and d with all the expected ones omitted. Although all of the values shown in Fig. 9.4 are small compared with 1.0, their distributions indicate something interesting. These values may be termed "errors" because they are unexpected. They are caused by both roundoff errors from solving the equations and the differences between the input strain observations and the sensitivity vectors.
Figure 9.4 shows that the error generally increases with the harmonic number in each of the examples. This may be because the variance of a load coefficient is inversely proportional to the square of its length (or norm). As a lower order harmonic usually produces a sensitivity vector with greater length, it has therefore smaller error than the higher order harmonic.

It may also be seen that errors arising from “observed” strains of higher order harmonics (9-16 in Fig. 9.4b) are smaller than those from lower harmonics (0-8 in Fig. 9.4a). Note that the scale of the vertical coordinate in Fig. 9.4a is 20 times that in Fig. 9.4b. This difference may be because the “observed” strains of higher order harmonics have more decimals (smaller errors) here.

Figure 9.4 also shows that errors in the load coefficients are related to the “observed” strains. For example, the load coefficients of Harmonics 10 and 14 have much bigger errors under the “observed” strain Case 2 (Fig. 9.4a) and so does Harmonic 14 under strain Case 10 (Fig. 9.4b). Many more such examples can be found in Figs 9.4c and d. This phenomena may be explained using the dependence index matrix (Table 9.2). All of these harmonics can be found to have bigger dependence indices between each other (e.g. the dependence index between Harmonics 10 and 14 is 92).

Figure 9.4 further shows that load coefficients of odd harmonic numbers are more likely to have large errors under “observed” strains of odd harmonic numbers and even harmonic numbered load coefficients are sensitive to “observed” strains of even harmonic numbers. Furthermore, a few harmonics such as Harmonics 0, 12 and 24 are almost “error free”. The reasons for these phenomena will be discussed in the following sections.

9.5.5 Strain residuals

Table 9.4 presents the “observed” strains \( \{ \varepsilon \} \) and strain residuals \( \{ \delta \} \) for “observed” strain Cases 0, 2, 10 and 14. The strain residuals represent the goodness-of-fit for
individual strain observations. The strain residuals in Table 9.4 are smaller than the last significant digit of the "observed" strains, so the fitting is always good in terms of strain residuals.

In practice however, a careful examination of the residuals may be rather important. They may be either plotted or tabulated so that the examination can be carried out easily. As the strain residuals are expected to belong to a multivariate normal distribution with zero mean, it is expected that there should be roughly equal numbers of positive and negative residuals. If the weighted residuals \([w][\delta]\)
\([w]^T[w]=[W]\) are plotted against an independent variable (e.g. \(\theta\) and \(r\) here) or a dependent variable (\(\varepsilon\) here), a horizontal band should be formed as their variances are expected to be constant. If a systematic trend is found in such a plot of strain residuals, the assumptions may need to be queried and modified. A clear systematic trend may indicate that a particular term has been omitted in the fitting. The plot of strain residuals may also identify the presence of "outliers" that appear to belong to an alternate data set. The outliers may be either strain observations with larger systematic errors or observations with high leverages. Once the presence of outliers is revealed, they may need to be treated cautiously. Methods for dealing with outliers may be found in Hoaglin and Welsch (1978), Barnett and Lewis (1978) and Barnett (1983).

Under the constant internal normal pressure of Harmonic 0, there is a significant difference between the strain at the inner gauge (841.2\(\mu\varepsilon\)) and that at the outer gauge (805.9\(\mu\varepsilon\)) (Table 9.4). This is not expected in classical thin ring theory. This difference is, however, not due to an error in the finite element analysis but is caused by the effect of a thick ring, which is discussed further in Chapter 12.

### 9.5.6 Variance and covariance of load coefficients

In the above examples, the genuine load coefficients were known, so all other deduced load coefficients could be treated as "errors", since it was known that they should be identically zero. However, in practice it is not possible to distinguish the errors from the genuine inferred load coefficients. Only the variances and covariances of the inferred load coefficients may be used to represent their quality.

Figure 9.5 shows the lower bound of the normalised variance for all the 32 load functions. Here the normalised variance is defined as

\[
\sigma^2_{\varepsilon}(a_i) = \frac{\sigma^2_i(a_i)}{\sigma^2(\varepsilon)} \quad (9.55)
\]
in which subscripts \( l \) and \( n \) indicate lower limit and normalised respectively, the lower limit value \( \sigma^2_l(a_1) \) is defined by Eqs 9.32 and \( \sigma^2(\varepsilon) \) is the estimated variance of the strain observations (Fig. 9.3). It is seen that this lower limit value increases quickly as harmonic number increases and most even harmonics have smaller values than their neighbouring odd harmonics. This may contribute to that the errors of inferred load coefficients increase with harmonic number in Fig. 9.4.

Figure 9.6 shows the normalised variances of load coefficients \( \sigma^2_n(a_i) \) for the two examples with Harmonics 0-16 and 17-32 respectively. The lower bound is also shown for comparison. Here the normalised variance is defined as

\[
\sigma^2_n(a_i) = \frac{\sigma^2(a_i)}{\sigma^2(\varepsilon)} \tag{9.56}
\]

It is seen that Harmonics 9-11 and 13-15 in Fig. 9.6a and Harmonics 23 and 25 Fig. 9.6b have remarkably larger variances (less accuracy) than other harmonics. The reasons for this phenomenon may easily be found in the dependence matrix (Table 9.3), where the dependence indices between Harmonics 9 and 15, 10 and 14, 11 and 13 and 23 and 25 are all very big (88, 92, 96 and 99 respectively). Actually there are more such pairs with big dependence indices such as Harmonics 7 and 17. They do not appear abnormal either in the inferred load coefficient figures or in the variance figures because Harmonics 0-16 and 17-32 are separately grouped in the examples so that they do not appear in the same normal equations.

Similarly, the covariances may also be normalised by dividing them by the estimated variance of the strain observations \( \sigma^2(\varepsilon) \).
\[
\text{Cov}_n(a_i, a_j) = \frac{\text{Cov}(a_i, a_j)}{\sigma^2(e)}
\] 

(9.57)

The covariances are shown in Figs 9.7a and b for i, j=0-16 and Figs 9.7c and d for i, j=17-32. The value of \(\text{Cov}_n(a_i, a_j)\) can be found on the curve that is labelled i in the legend at harmonic number j on the x-coordinate. It may be seen that the covariance between higher order harmonics is much bigger than that between lower order harmonics. This may be because higher harmonics usually produce smaller sensitivity vectors. A large covariance between \(a_i\) and \(a_j\) means that an error in \(a_i\) may cause a large error in \(a_j\) and vice versa. If the covariance between \(a_i\) and \(a_j\) is zero, then an error in one load coefficient has no effect on the other. Many of the big covariances in Fig. 9.7 may be found to have corresponding large dependence indices in Table 9.2. In fact the dependence matrix does represent the covariances to some degree as discussed in Section 9.4.

The almost "error free" harmonics such as Harmonic 0, 12 and 24 in Fig. 9.4 are also seen here. If the variance (Fig. 9.6) and covariance (Fig. 9.7) are put together, very many similarities can indeed be found between them and the inferred load coefficients shown in Fig. 9.3, which represent the actual errors of inferred load coefficients in this case. The very low covariances between these specific "error free" harmonics and others mean that they are hardly affected by errors in other load coefficients so that their variances reach down close to their lower limit values.

Figure 9.7 also shows that many covariances \(\text{Cov}_n(a_i, a_j)\) have larger values if both i and j are odd or both are even. This means that errors in odd harmonics are more likely to propagate among odd harmonics and those in even harmonics are more likely to propagate among even harmonics. This phenomenon may also be explained by using the dependence matrix as follows.

Table 9.3 may be rearranged so that the left-upper triangle represents the dependence indices between even harmonics, the right-lower triangle represents those between
odd harmonics and the left-lower square represents those between even and odd harmonics as shown in Table 9.5. It shows that all the dependence indices between odd and even harmonics are very small.

Table 9.5 Dependence matrix (rearrangement of Table 9.3)

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9.5.7 Effect of similar sensitivity vectors

In the first example above it was found that the patterns of sensitivity vectors for Harmonics 9 and 15, 10 and 14, 11 and 13 are very similar to each other (Table 9.4). The big dependence indices between them (Table 9.3) should be the main cause for the large variances of load coefficients in Fig. 9.6a. To verify this conclusion, Harmonics 13-16 were omitted here so that Harmonics 0-12 (excluding 1) form the load functions. The normalised variances are shown in Fig. 9.8. Their lower bound...
and those obtained in the first example (the chief visible result in the figure) are also shown for comparison. It is seen that the variances have been effectively reduced by about 30 times to close to their lower limits. The covariances and the inferred “error” load coefficients were also reduced by the same order. Note that the ratio of gauge number to unknowns is \( M=N=24/12=2 \) here compared with \( M/N=24/16=1.5 \) in the first example. However, this difference can only be expected to reduce the error by a factor of \( \sqrt{2/1.5}=1.15 \) (see Chapter 8). Therefore the decrease in errors is directly attributable to the removal of similar load functions.

9.5.8 Effect of strain gauge locations

The strain gauges at \( \theta = 15^\circ, 45^\circ, 225^\circ \) and \( 315^\circ \) in the example in Section 9.5.7 were removed to explore the effect of strain gauge patterns.

The strain gauge positions are now at \( \theta = 0^\circ, 30^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ, 180^\circ \) and \( 270^\circ \). The intervals between gauge positions are now \( \Delta \theta_1=30^\circ \) and \( \Delta \theta_2=90^\circ \). It follows that \( K_1=12 \) and \( K_2=4 \). Therefore, the dependence indices between \( m \) and \( n \) are larger if \( m=\pm n \) for \( K_2=4 \) and \( K_2=12 \) and any integer \( i \) (Eq. 9.49). Because all but one of the intervals between the gauge positions is \( 30^\circ \), the dependence indices on the line \( m=4i \pm n \) are slightly larger than the others but much smaller than those on \( m=12i \pm n \) (Table 9.6).

Figure 9.9 shows that the removal of these gauges increases the variance of load coefficients by about 5 times. Again, the difference of \( M/N=16/12=1.33 \) here from \( M/N=24/12=2.0 \) in the first example in the previous section is not a sufficient reason for such a large increase in the error. The load coefficient variances are big in this example because there is a strong dependence between Harmonics 5 and 7, 4 and 8, 3 and 9, and 2 and 10 as shown in Table 9.6.
Table 9.6 Dependence matrix: effect of strain gauge locations

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9.6 Conclusions

The statistics of the method developed in Chapter 8 for inferring silo wall pressures from wall strain measurements have been discussed in this chapter, including the statistics of the inferred load coefficients, the estimation of the variance of strain observations, the analysis of strain residuals and the goodness-of-fit. A lower bound to the variances of the load coefficients has been established. A dependence index between two sensitivity vectors has been proposed, which is very useful in the selection of load functions when a large number of load functions are involved, and is also useful for determining the strain gauge locations when planning an experiment.
The selection of the load functions has a significant effect on the quality of the inferred loadings. The use of sensitivity vectors which are nearly linearly dependent will result in the ill-conditioning of the normal equations, leading to a failure in the interpretation. The use of orthogonal load functions minimises the errors in the load coefficients and is most ideal.

The quality of the load functions is also dependent on the strain gauge patterns. If harmonic loading functions are used and the strain gauges are placed at equal intervals of Δθ, the dependence index between Harmonics m and n becomes very large if m=jK±n for K=2π/Δθ with any integer j. These harmonic pairs should always be avoided in the same inference.

The application of these findings has been illustrated by the analysis of a simple example ring under harmonic pressures.

### 9.7 Notation

- **Cov**: covariance
- **D**: dependence index between two vectors (Eq. 9.43)
- **E**: expectation of a random variable/vector
- **H**: Hilbert matrix
- **I**: unit matrix
- **K**: integer
- **M**: number of strain gauges
- **N**: number of load functions
- **Q**: probability of chi-square distribution
- **S**: sensitivity matrix
- **U**: projection matrix
- **Var**: variance
W  weighting matrix
a  load coefficient
e  error in strain observations
f  load function
p  internal normal pressure
r  length of a 2D vector (Eq. 9.37)
α  angle
β  coefficient (Eq. 9.36)
δ  strain residuals/error of load coefficients
ε  strain
Γ  gamma function
σ  standard deviation
θ  circumferential coordinate
ζ  number of degrees of freedom in the incomplete gamma function

Subscripts
a  load coefficient
l  lower bound
n  normal/normalised
ε  strain observations
Fig. 9.1 Factorisation of vector C into A and B

Fig. 9.2 An example ring

a) Cross-section  b) Circumferential positions of strain gauges

Fig. 9.3a Estimated standard deviation of strains
(Load functions = Harmonic No. 0-16)
Fig. 9.3b Estimated standard deviation of strains
(Load function = Harmonic No. 17-32)

Fig. 9.4a Inferred load coefficients: "Observed" strain case 0-8
(Load function = Harmonic No. 0-16)

Fig. 9.4b Inferred load coefficients: "Observed" strain case 9-16
(Load function = Harmonic No. 0-16)
Fig. 9.4c Inferred load coefficients: “Observed” strain case 17-24
(Load function = Harmonic No. 17-32)

Fig. 9.4d Inferred load coefficients: “Observed” strain case 25-32
(Load function = Harmonic No. 17-32)

Fig. 9.5 Lower bound of load coefficient variance
Fig. 9.6a Variance of load coefficients (load function = harmonic No.0-16)

Fig. 9.6b Variance of load coefficients (load function = harmonic No.17-32)

Fig. 9.7a Covariance of load coefficients: between harmonics 0-8 and 0-16
Fig. 9.7b Covariance of load coefficients: between harmonics 9-16 and 0-16

Fig. 9.7c Covariance of load coefficients: between harmonics 17-24 and 17-32

Fig. 9.7d Covariance of load coefficients: between harmonics 25-32 and 17-32
Fig. 9.8 Variance of load coefficients: effect of similar load functions

Fig. 9.9 Variance of load coefficient: effect of strain gauge patterns
Chapter 10

DEVELOPMENT OF A PROGRAM SUITE
FOR INFERRING WALL PRESSURES IN
AXISYMMETRIC SILOS FROM WALL
STRAIN MEASUREMENTS

10.1 Introduction

The purpose of this chapter is to develop a general tool to infer wall loadings in
axisymmetric thin elastic silos from wall strain measurements using the technique
developed in the last two chapters. The tool consists of a series of programs. Some
technique details are discussed, including the calculation of strains at the strain gauge
locations under load functions of unit amplitude, imposing linear constraints on the
load functions and the solution of the normal equations. The main features of the
program suite and its functions are briefly described. Full details of its use are given
in Chen (1995b).

In the procedure developed here, three basic steps are used to infer silo wall loadings
from wall strain measurements:

a) A set of load functions is chosen and the strains which each would induce in all
the strain gauges for a unit amplitude of the load function are calculated to set up
the sensitivity matrix;
b) The calculated strains are inverted to determine load coefficients, minimising errors using a least squares procedure; and

c) The quality of the results is verified and the pressure pattern is presented.

10.2 Structural analysis of axisymmetric thin elastic shell structures

One of the essential steps in inferring wall pressures from strain measurements is to carry out a series of structural analyses under characterised load distributions (load functions) of unit amplitude. The strains which would occur at the strain gauge locations under ideal conditions may then be obtained from the results of the structural analysis.

Because the accuracy of the inferred loadings depends on the accuracy of the structural analysis in many senses, the linear bending theory of shells (Rotter, 1987b) is used in this study. Membrane theory (Rotter, 1987a) is not acceptable unless it is certain that the load distribution is almost axisymmetric, the structure is uniformly supported and all gauge locations are distant from discontinuities. Due to the complexity of the bending theory of shells under non-symmetrical loading, analytical solutions can only be obtained for a very few simple structures under the simplest loadings (e.g. Flugge, 1973; Lukasiewicz, 1979; Calladine, 1983). It is therefore easiest to use finite element analysis to obtain result of wide application.

In this thesis, the silo structure is assumed to be a linear elastic axisymmetric perfect shell. A doubly curved isoparametric "ring" element, which can model cylinders, spheres, cones, annular plates, other axisymmetric shapes and any combinations of these assembled together, is used to model the structure (Rotter, 1989, 1993). Each element has a node at either end, with six displacements (three translations and three rotations) at each node. Both displacements and loads are expanded as harmonic...
series. Due to the orthogonal property of trigonometric terms (cosine and sine), each
term of the harmonic series is separated and can be analysed by performing a separate
finite element analysis. General descriptions of the finite element formulations and

10.3 Calculation of strains at strain gauge locations

Finite element programs generally output displacements at nodes and internal forces
or strains at Gaussian integration points. However, the strains at the strain gauge
locations are required here. The locations of strain gauges may be neither at
Gaussian integration positions in an element nor at nodes. Although these strain
components can be interpolated or extrapolated from those at the Gaussian
integration points, errors may arise during interpolations and extrapolations.

A more accurate method is to calculate the strain-displacement relation matrix [B] at
each strain gauge position. The explicit expression of [B] may be found in Rotter
(1993) and Teng (1990). The dimensionless coordinate of the strain gauge
location in the element is required in calculating matrix [B],

\[ \bar{s}_g = \frac{l_g}{l} - 1 \]  

(10.1)

where \( l_g \) is the curvilinear meridional separation between node 1 and the strain gauge
location, and \( l \) is the half length of the element in which the strain gauge sits.

The strains at each strain gauge location can be obtained from the strain-displacement
relation

\[ \{\varepsilon\} = [B]\{\delta\} \]  

(10.2)
where \( \{ \delta \} \) is the nodal displacement vector for the element in which the strain gauge sits and \( \{ \varepsilon \} \) consists of three membrane strains and three bending curvatures

\[
\{ \varepsilon \} = [\varepsilon_\theta, \varepsilon_\phi, \tau_{\theta \phi}, \kappa_\theta, \kappa_\phi, \kappa_{\theta \phi}]^T
\]  

(10.3)

in which \( \varepsilon_\theta, \varepsilon_\phi, \tau_{\theta \phi}, \kappa_\theta, \kappa_\phi \) and \( \kappa_{\theta \phi} \) are meridional membrane strain, circumferential membrane strain, membrane shear strain, meridional bending curvature, circumferential bending curvature and twisting curvature respectively.

It may be noted that under harmonic loadings, the strain vectors are only needed at \( \theta = 0^\circ \) or \( 90^\circ \) for each point with different \( r \) and \( z \) coordinates using Eq. 10.2. Strain vectors at other locations around the circumference can be found from the known circumferential variation of strains and curvatures (Rotter, 1987b). For example, if the normal pressure is

\[
p = p_n \cos \theta
\]  

(10.4)

The membrane strains and bending curvatures vary as

\[
\varepsilon_\theta = \varepsilon_{\theta 0} \cos \theta
\]  

(10.5a)

\[
\varepsilon_\phi = \varepsilon_{\phi 0} \cos \theta
\]  

(10.5b)

\[
\gamma_{\theta \phi} = \gamma_{\theta \phi 0} \sin \theta
\]  

(10.5c)

\[
\kappa_\theta = \kappa_{\theta 0} \cos \theta
\]  

(10.5d)

\[
\kappa_\phi = \kappa_{\phi 0} \cos \theta
\]  

(10.5e)

\[
\kappa_{\theta \phi} = \kappa_{\theta \phi 0} \sin \theta
\]  

(10.5f)
where $\varepsilon_{m0}$, $\varepsilon_{c0}$, $\kappa_{m0}$ and $\kappa_{c0}$ are meridional membrane strain, circumferential membrane strain, meridional bending curvature and circumferential bending curvature at $\theta=0^\circ$ respectively, and $\gamma_{m0}$ and $\kappa_{\phi0}$ are membrane shear strain and twisting curvature at $\theta=90^\circ$ respectively.

10.4 Local stiffening effect of "double deck" bending strain gauges

As mentioned in Chapter 8, "double deck" (DD) or "sandwich" bending strain gauges are able to measure both bending and tension strains from one side of the silo wall. However, the stiffness of the filling material of the sandwich gauge may cause a local stiffening effect if the stiffness of the gauge is significant comparing with that of the wall to which it is attached. The author is unaware of any previous study of this local stiffening effect.

An approximate analytical solution for the local stiffening effect of a DD gauge is obtained in this thesis. The solution has been verified by three dimensional finite element calculations, which are presented in Appendix A. The local stiffening effect on the strain readings may be described by relative reading errors under pure tension and pure bending respectively. The strain in the lower gauge (the gauge on the shell surface) $\varepsilon_l$ and that in the upper gauge (the gauge off the shell surface) $\varepsilon_u$ are found to be

$$\varepsilon_l = \varepsilon_m(1-\varepsilon_{bs}) \pm \frac{t_b}{2} \kappa(1 - \varepsilon_{bs})$$  \hspace{1cm} (10.6a)

$$\varepsilon_u = \varepsilon_m(1-\varepsilon_{uhs}) \pm \left(\frac{t_b}{2} + t_g\right) \kappa(1 - \varepsilon_{uhs})$$  \hspace{1cm} (10.6b)
where $\varepsilon_m$ is the membrane strain in the plate or wall without the local stiffening effect, $\kappa$ is the bending curvature at the gauge centre without the local stiffening effect, $t_p$ is the thickness of the plate (wall) to which the DD gauge is attached, $t_g$ is the thickness of the DD gauge, and $e_{lts}$ and $e_{uts}$ are the relative errors in the lower and upper gauges under pure tension and $e_{lbs}$ and $e_{ubs}$ are the relative errors in lower and upper gauges under pure bending respectively. The sign before the thickness term in Eqs 10.6 should be plus if the strain gauge is on the outside surface of the shell and minus if it is on the inside.

The size of the relative errors $e_{lts}$, $e_{uts}$, $e_{lbs}$ and $e_{ubs}$ varies with the DD gauge aspect ratio, the ratio of DD gauge thickness to plate thickness, the modular ratio of DD gauge to plate and the Poisson’s ratio of the plate. Their values may be obtained as shown Appendix A. They should be assigned values of zero if local stiffening is not considered.

### 10.5 Calculation of strains in strain gauges

A sensitivity vector consists of a set of ideal calculated strains at all the strain gauge locations under a load function with a unit maximum value. The strain at each strain gauge location under each load function must be calculated before the sensitivity matrix can be formed.

To calculate the strain at a strain gauge location, the direct membrane strains and the bending curvatures in the direction of the strain gauge must be found. If the direction of the strain gauge is either meridional or circumferential, the direct membrane strain and bending curvature in the strain gauge direction has been obtained in Section 10.3. However, the direction of a strain gauge may be neither meridional nor circumferential (there is at least one such gauge in every strain gauge rosette). In general, it is in a direction inclined at an angle $\alpha$ to the circumferential axis $\theta$. 
The direct membrane strain $\varepsilon_m$ and the direct bending curvature $\kappa$ can be found as

$$
\varepsilon_m = \frac{1}{2} \left[ \varepsilon_o + \varepsilon_n + (\varepsilon_o - \varepsilon_n)\cos 2\alpha + \gamma_{e0}\sin 2\alpha \right]
$$

(10.7a)

$$
\kappa = \frac{1}{2} \left[ \kappa_o + \kappa_n + (\kappa_o - \kappa_n)\cos 2\alpha + \kappa_{e0}\sin 2\alpha \right]
$$

(10.7b)

The linear strain in any gauge is given by

$$
\varepsilon_i = \varepsilon_m \pm \frac{l_0}{2} \kappa
$$

(10.8)

Strains in DD gauges can be obtained using Eqs 10.6.

### 10.6 Imposing linear constraints

Some parameters in a model may be related to others and are thus constrained. A typical example is the load coefficients of vertical wall frictional traction, which may be set equal to the corresponding coefficients multiplying by the wall friction factor $\mu$, if the wall friction is assumed to be fully mobilised everywhere:

$$
q_i = \mu p_i \quad \text{for } i = 1, 2, ..., N_c
$$

(10.9)

Equations (10.9) may be written in a matrix form as the constraint equation

$$
\{q\} = \mu [I]\{p\}
$$

(10.10)

However, the constraint equations may be much more complicated in practice. More than two parameters may be involved in a constraint equation and they may be non-
linear. Only linear constraints are considered here. The general linear constraint equations may be expressed as (Farebrother, 1988; Gans, 1992)

\[ \{a\} = [C]\{a_i\} \quad (10.11) \]

where \(\{a\}\) is the load coefficient vector that includes both constrained and unconstrained parameters, \(\{a_i\}\) is the unconstrained (independent) parameter vector and \([C]\) is called the constraints matrix.

If there are \(N_c\) linear constraint equations (\(N_c\) dependent parameters) in the total of \(N\) parameters, the constraints matrix \([C]\) has the order of \(N \times (N - N_c)\). If there are no constraints, \([C]\) becomes a unit square matrix \([I]\) of order \(N\).

From Chapter 8,

\[ [S]\{a\} = \{\varepsilon\} \quad (10.12) (8.5 \text{ bis}) \]

in which \([S]\) is the sensitivity matrix and \(\{\varepsilon\}\) is the strain vector.

Combining Eqs 10.11 and 10.12 leads to

\[ [S][C]\{a_i\} = \{\varepsilon\} \quad (10.13) \]

or

\[ [S_c]\{a_i\} = \{\varepsilon\} \quad (10.14) \]

in which

\[ [S_c] = [S][C] \quad (10.15) \]
is the constrained sensitivity matrix. Its order is reduced from MxN to Mx(N-Nc).
If there are no constraints, \( \{a_1\} = \{a\} \) and \([S_c] = [S]\). Equations 10.14 may be solved using the linear least squares method.

The variance and covariance of all load parameters can be derived from Eq. 10.11:

\[
\text{Cov}\{a\} = [C]\text{Cov}\{a_1\}[C]^T
\]  

where \(\text{Cov}\{a_1\}\) is the variance-covariance matrix of \(\{a_1\}\), which is obtained from the least squares estimation of the constrained equations (Eq. 10.14). If \(\{a\}\) is arranged in such a manner that the first N-Nc elements represent the independent parameters and the last Nc represent the constrained parameters, the constraints matrix can be written as

\[
[C] = \begin{bmatrix} [I] \\ [c] \end{bmatrix}
\]

Substituting Eq. 10.17 into Eq. 10.16 gives

\[
\text{Cov}\{a\} = \begin{bmatrix} \text{Cov}\{a_1\} & \text{Cov}\{a_1\}[c]^T \\ [c]\text{Cov}\{a_1\} & [c]\text{Cov}\{a_1\}[c]^T \end{bmatrix}
\]  

If only one independent parameter is involved in each constraint equation, there is only one non-zero element in each row of \([c]\). In this case, all the calculations can be done without explicitly forming the full constraints matrix \([C]\).

### 10.7 Solution of the normal equations

Many methods are available to solve the normal equations
They can be roughly classified into two groups: the normal equations method and the orthogonal decomposition method. The normal equations method forms the normal equations (Eq. 10.19) explicitly. It may be solved using methods such as LU decomposition, Choleski decomposition, etc.

Orthogonal decomposition methods solve the equations without forming the normal equations explicitly, but by converting the sensitivity matrix \([S]\) (for the unweighted least squares method) or \([w][S]\) (for the weighted least squares method, \([w]^T[w]=[W]\)) into upper triangular form by means of orthogonal decomposition. These methods include LU decomposition of \([S]\) or \([w][S]\), Schmidt's orthogonalisation procedure (Schmidt, 1907 and 1908; Gram, 1883) which is often known as classical Gram-Schmidt procedure (Farebrother, 1988), Laplace's orthogonalisation procedure (Laplace, 1816) which is widely known as the modified Gram-Schmidt procedure (Farebrother, 1988), Householder's orthogonalisation procedure (Householder, 1958, 1964), Givens rotations (Givens, 958) and singular value decomposition (Forsythe et al, 1977; Golub and Loan, 1983; Lawson and Hanson, 1974; Wilkinson and Reinsch, 1971), etc.

When the sensitivity matrix \([S]\) is singular or nearly singular, computational problems arise in the normal equations method leading to serious errors. The more sophisticated orthogonal decomposition methods are supposed to be better at handling this situation, at the cost of more computation, more storage and more complicated coding than is required for the normal equations method. Apart from the technique using Givens rotations that is somewhat more costly, all of the methods involving decomposition of \([S]\) or \([w][S]\) require approximately twice as much computation as the normal equations method except when \([S]\) is almost square (Jennings and McKeown, 1992). Lawson and Hanson (1974) suggested that using the orthogonal decomposition method is equivalent to working in double precision as compared to the normal equations method. However, Gans (1992) concluded that

\[
[S]^T[W][S][\vec{a}] = [S]^T[W][\vec{e}]
\]

(10.19)(8.10 bis)
the orthogonal decomposition method shows no great advantage or disadvantage compared to the normal equations method, even when the sensitivity matrix is approaching singularity.

The normal equations method was adopted in this study for simplicity. If the strain gauge locations and load functions are carefully selected so that linear dependence among the sensitivity vectors is avoided, the solver may be used with confidence. If linear dependence does occur among the sensitivity vectors so that \( [S] \) is singular or close to singular, the program may be able to identify the problem and prompt a message to notify the user in some cases. If this happens, a careful re-examination of the load functions should be carried out and any load function that can be closely represented by a linear combination of others should be removed. Otherwise the errors in the load coefficients will be large and the model may well be rejected even when a sophisticated solution method is used.

The normal equations (Eq. 10.19) may be solved either by decomposing the matrix \([S]^T[W][S]\) and performing forward and back substitutions of \( \{b\} \), or by computing the inverse of \([S]^T[W][S]\) and \( \{U\}=([S]^T[W][S])^{-1}[S]^T[W] \) so that \( \{a\}=[U]\{e\} \). If there are \( L \) sets of strain readings which need to be processed, the required basic operations (one multiply plus one addition) are approximately \( N^3/3+N^2L+MNL \) for the LU decomposition method and \( N^3+N^2L \) for the matrix inversion method. Matrix inversion is therefore more efficient if \( L>2N^2/3M \). In the experimental silo under study here, \( N=100 \) and \( M=300 \). So matrix inversion is more efficient if more than 20 sets of strain are to be processed. Naturally, this is normally the case. Both routines are available currently in the program and it can automatically select an efficient routine. However, if the variance-covariance matrix of the load coefficients is required, the matrix inversion method must be used.
10.8 Program design and functions

10.8.1 General

The program suite SIMA (Silo pressures Inferred from strain Measurements on Axisymmetric silo walls) was developed to infer silo wall pressures and tractions from strain measurements on axisymmetric silo walls. It consists of several programs to accomplish the task. The main programs are listed in Table 10.1. There are also a few support programs to process the test data for the experiments carried out in the BMHB silo at British Steel (see Chapters 6 and 11).

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<tr>
<th>Program</th>
<th>Main functions</th>
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<td>SIMAP</td>
<td>Pre-processor of SIMAS: generating FE mesh and loadings</td>
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<tr>
<td>SIMAS</td>
<td>FE analysis of axisymmetric shells, calculation of sensitivity vectors</td>
</tr>
<tr>
<td>SIMAI</td>
<td>Inversion of strain measurements to load coefficients</td>
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<tr>
<td>SIMAR</td>
<td>Post-processor for presentation of results</td>
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<tr>
<td>SIMAD</td>
<td>Dependence analysis of sensitivity vectors</td>
</tr>
</tbody>
</table>

Strains may be measured at the inside, the outside or on both sides of the shell in any direction. If strains on both sides of the shell are measured, or DD gauges are used, the raw strain measurements may either be used directly or converted into membrane strains and bending curvatures first before inferring wall loadings.

Much care has been taken during the development of the program suite to make it flexible for users with minimum work required to process the raw test data. The program suite should also be able to be modified and updated easily.

The source code of the programs is about 15,000 lines in FORTRAN 77. The main features of the program suite are briefly described in this section. The use of the suite is illustrated in Fig. 10.2. Full details are given in Chen (1995b).
10.8.2 The silo structure

The silo structure is currently restricted to those that can be treated as axisymmetric linear elastic thin shells. Complex branched shell structures may be used. Because the technique is developed based on the assumption that strains in the shell structure can be uniquely determined under a given load function, it should also be perfect. The program suite should not be applied to concrete silos because strains in the structure may not be accurately represented due to defects such as cracks. Further, strains in thick concrete silos would be very small so that the signal-to-noise ratio would be small (lower quality) for the measured strains.

10.8.3 Program functions

The main programs in the program suite SIMA are SIMAP, SIMAS, SIMAI, SIMAR and SIMAD.

Program SIMAP (the SIMA suite of programs for inferring silo pressures from strain measurements in axisymmetric shells: Pre-processor) was derived from program PASHA (Pre-processor for Axisymmetric SHEll Analysis), which was developed by Professor J.M. Rotter in the Department of Civil Engineering at the University of Edinburgh. Although SIMAP was designed to meet the special requirements for the SIMA suite, it may also be used for general purposes. This pre-processor generates the finite element mesh from a limited definition of shell segments and can set up a great variety of different loading conditions on the shell. The structure can be a complex branched shell with multiple closed boxes. Orthotropic stiffening can be modelled by the smearing technique. Complex loadings can be applied by very limited definition of the basic parameters. Some special loadings that would very often be used in inferring loadings from strain measurements have been added in SIMAP.

Program SIMAS (the SIMA suite of programs for inferring silo pressures from strain measurements in axisymmetric shells: Sensitivity analysis) was derived from
program LEASH (Linear Elastic analysis of Axisymmetric Shells using Harmonics),
which was developed by Professor J.M. Rotter in the Department of Civil 
Engineering at the University of Edinburgh. This is a linear elastic shell analysis 
program using harmonics. It can be used to determine the deformation and strain 
distributions in axisymmetrically or non-symmetrically loaded orthopically stiffened 
branched doubly curved shells of revolution. General descriptions of the finite 
element formulation encoded in LEASH may be found in Rotter (1981, 1986, 
1987b). Some special functions are added in SIMAS to meet the requirements by 
SIMA suite. It may be used to calculate strains in any direction at any point of the 
shell. The local stiffening effect of "Double Deck" or "Sandwich" bending strain 
(DD) gauges may be considered. For DD gauges, it may output either strains at 
lower and upper gauges or tension strain and bending curvature. For a given set of 
strain gauge locations, the program forms a Strain Sensitivity Vector and writes it 
onto a .SSV file for program SIMAI to invert the strain evaluations into pressures.

Program SIMAI (the SIMA suite of programs for inferring silo pressures from strain 
measurements in axisymmetric shells: Inferring load parameters) was designed to:

- Input the appropriate records from the above binary .SSV file and form the 
sensitivity matrix;
- Read the experimental strain observations and form observation vectors;
- Generate random strain observation vectors;
- Form a weighting matrix as required;
- Solve the resulting equations to obtain load parameters and
- Perform statistic analysis of the results.

The power of modern PCs presents no problem to solve the sparse linear equations 
resulting from methods such as finite element analysis with an order up to 1000 in 
core. However, the matrices involved here are full except the weighting matrix W, 
which may be diagonal. If there are 1000 strain observations and 200 unknown load 
parameters to be found, the sensitivity matrix S alone is of order 200×1000. If it is
processed using real*8, the memory needed is $200 \times 1000 \times 8 = 1.6$ Mb. The required memory dramatically increases when matrix multiplications and inversions are involved. Therefore, it is necessary to pay great attention to the efficient use of computer memory. To make full use of the computer memory, only a single big one dimensional array is declared in the program SIMAI. All other arrays are dynamically allocated to this single array.

Program SIMAR (the SIMA suite of programs for inferring silo pressures from strain measurements in axisymmetric shells: Result presentation/post processor) prepares a data file for plotting the 2D and 3D load distribution. It may create data for plotting down meridional lines, on circumferential lines and a 3D picture or contours on the whole shell. The plots are done using either SURFER for Windows or EXCEL.

Program SIMAD (the SIMA suite of programs for inferring silo pressures from strain measurements in axisymmetric shells: Dependence analyses of load functions) is used to analyse the dependence among the load functions. Load functions which are dependent or close to dependent on each other should not be used in an interpretation. The dependence analysis is based on a given strain gauge pattern. The weighting method may also have an influence on the results. The formulations used in this program are given in Chapter 9.

10.9 Summary and conclusions

Some technical details of the development of a program suite for inferring wall loadings in silos from strain measurements on the walls have been discussed in this chapter. They include a brief introduction to the linear elastic finite element analysis of axisymmetric shells using harmonics, the calculation of strains at strain gauge locations and in strain gauges, the imposing of linear constraints on load parameters, the solution of normal equations and a brief summary of the program suite. The local
stiffening effect of "double deck" bending strain gauges is considered in the calculation of strains in strain gauges under characterised load functions.

A program suite, SIMA, has been developed for inferring loadings exerted on silo walls by the stored solids from strain measurements on silo walls. It consists of many programs to generate finite element meshes and loadings, perform linear elastic finite element analysis of axisymmetric shells, back-calculate the load coefficients and generate appropriate data files for the presentation of results. The program suite was designed for general use and could have many applications both for inferring wall pressures in silos from strain measurements on walls and for other more diverse applications.

10.10 Notation

\begin{itemize}
  \item \([B]\) \quad \text{strain-displacement relation matrix}
  \item \([C]\) \quad \text{constraints matrix}
  \item \([S]\) \quad \text{sensitivity matrix}
  \item \([W]\) \quad \text{weighting matrix}
  \item \(M\) \quad \text{number of strain observations}
  \item \(N\) \quad \text{number of load coefficients}
  \item \(N_c\) \quad \text{number of dependent load coefficients}
  \item \(a\) \quad \text{load coefficient}
  \item \(e\) \quad \text{relative error due to local stiffening effect}
  \item \(p\) \quad \text{normal pressure}
  \item \(q\) \quad \text{meridional traction}
  \item \(s_g\) \quad \text{coordinate of a strain gauge position in the element in which it sits}
  \item \(l_g\) \quad \text{meridional distance from the first node of an element in which the strain gauge sits to the strain gauge location}
  \item \(l\) \quad \text{half length of a shell element}
  \item \(\varepsilon\) \quad \text{strain}
\end{itemize}
\( \kappa \) bending curvature
\( \delta \) nodal displacement
\( \gamma \) membrane shear strain

Subscript

\( c \) constraint
\( l \) lower gauge (gauge on the shell surface)
\( u \) upper gauge (gauge off the shell surface)
\( t \) pure tension
\( b \) pure bending
\( \phi \) meridional
\( \Theta \) circumferential
\( p \) plate
\( g \) gauge
\( s \) stiffening effect
Fig. 10.1 Orientation of a strain gauge in an element

Fig. 10.2 Use of the program suite SIMA
Chapter 11

WALL STRAIN MEASUREMENT IN THE BRITISH STEEL SILO

11.1 Introduction

In earlier experiments, pressure cells have most commonly been used to acquire data on the loads on silo walls. However, pressure cells have a number of disadvantages (Rotter et al., 1995). The chief disadvantages are that they are expensive and that the interpretation of the pressure cell readings is fraught with difficulties. The cell reading varies with time, and is affected by the varying discrete forces from individual particles (c.f. Thornton, 1979). In addition, the mean pressure changes systematically with time, so the single particle effects must be separated from the global mean solid stress effects. Further, the pressure at points even a small distance apart can be quite different at some times (Rotter et al., 1986), so that widely spaced cells may reveal nothing about the pressure variation between their positions. This makes the interpretation of the pressure cell readings particularly difficult in flow situations. When eccentric discharge is expected to produce high local pressures at the edges of the flowing channel, the chance of detecting these local peaks is very small in a full scale silo. The deduction of global pressure patterns from these cell readings is a further major difficulty (Rotter et al., 1986). Many good experiments have become useless for structural design because of the simplifications introduced during the averaging process of interpretation.

Strain gauges present an alternative means of measuring silo wall pressures. Although they also have difficulties, the advantages of using strain gauges far outweigh those of using pressure cells (Rotter et al., 1995).
The purpose of silo pressure measurements is to deduce the stresses which will be induced in the silo walls and the technique using strain gauges solves this problem very effectively. Measuring the stresses which occur in the silo's walls directly, no important local high pressures can be omitted from the description: if they were important, they would produce wall stresses which would be observed, so if high pressures remain undetected it is because they are unimportant for the final goal of the pressure measurement process.

This gives the technique an advantage in the certainty of its direct application to silo design. It also allows the important features of the pressure pattern to be presented in a form which is directly useful for structural assessments, thus eliminating the further problem of envelopes and approximations to pressure distributions found from pressure cells (Rotter et al, 1986).

Strains on the silo's walls were measured for the experiments conducted in the British Steel silo (see Chapter 6). This chapter briefly summarises the silo structure and the strain measurement system. This information is needed to infer the wall pressures from the strain measurements in the next chapter. The experiments conducted in the silo and the properties of the solids used have been summarised in Chapter 6.

11.2 The silo structure

11.2.1 The upper structure

The flat bottomed silo was designed by Prof. J.M Rotter and was fabricated by J.T. Scotney Ltd, Hull. It was erected at the British Steel Research laboratories in Teesside. The overall geometry of the silo is shown in Fig. 11.1. The silo barrel, which was 9.5m in height, 4.2m in diameter and 3mm thick, was supported on a skirt of height 2.4m and 6mm thickness. Five ring stiffeners were positioned on the barrel. The rings were fabricated from 100x65x10mm angle section and were positioned at 1536, 3038, 4540, 6042, and 7544mm above the silo floor. A channel section 203x89mm was attached at floor level. The lower flange of this channel projected just above the floor plate with a vertical clearance of about 5mm and a horizontal clearance of 30mm, to permit relative movement between the barrel and floor. The roof was manufactured from 6mm plate which was at a slight camber to shed water. Details of parts of the silo structure are shown in Figs 11.2.
11.2.2 The foundation
The foundation was a hexagonal reinforced concrete slab. By the end of the series of experiments at British Steel (see Chapter 6), the foundation slab had suffered a significant uniform settlement of about 10mm, indicating that consolidation of the soil sub-strata was significant. However, the uniform settlement does not present a problem to the silo or the experiments being conducted in it. No cracking or uneven settlement of the slab was observable.

11.2.3 The floor system for independent support
The silo floor was designed as an independent structure so that pressures on the floor did not influence the stresses in the silo wall. The silo floor was 12mm thick and accommodated three outlets (see Chapter 6). The floor support was 457×152mm beams running in both directions across the silo. The spacing of the beams was equal in both directions, on 850mm centres, passing symmetrically between the slide valves attached to the outlets (Fig. 113).

Columns supported the floor loads at the outer beam intersections and at the beam ends. The outer columns were steel channels 102×51mm. These channels were arranged with their two flanges pointing radially towards the silo centre. The top of the channel was welded to the floor beams. Vertical stiffeners were placed between the flanges of each beam above the channel column web.

11.2.4 The openings in the silo skirt
There were two access openings opposite each other in line with the outlets 750mm wide by 1650mm high in the silo skirt, one for the belt feeder and one for access. These openings were stiffened down the sides with welded channel sections of size 152×76mm. Above each opening, stiffening was used to eliminate the effect of the opening on the rest of the silo by using a novel design concept (Rotter et al, 1995), so that the strains developing in the silo walls were caused only by wall pressures from the stored solids. This consisted of a horizontal lintel made of 152×76mm channel, and above it two inclined channel sections 76×38mm starting above the vertical stiffening beside the opening and each sloping at 30° to meet over the centre of the opening. This optimal design using practical sections has a maximum variation of
deflection from the ideal of 3.7%. This small error is in a displacement which is still somewhat removed from the gauge locations, so the problem is not great (Rotter et al., 1995).

11.3 Measurements of imperfections in the silo after its fabrication

After the silo had been constructed, measurements were made of the imperfections in the silo. These measurements involved checks on the diameter (Table 11.1), the circumference (Table 11.2) and local wall imperfections.

The following method was used whilst the silo was lying on its side on rollers: the horizontal diameter was measured using a steel band tape; the silo was then rotated on the rollers through 45° and the next (horizontal) diameter measured. This process was repeated four times. The results naturally include self-weight deformations, but all should be affected to the same extent, as all diameters were measured in the horizontal position. Tables 11.1 and 11.2 show little variation, apart from measurements near the joint between the silo barrel and the skirt.

The areas of the silo wall around the circumferential welds joining the plates together at the rings were checked according to the procedure laid down in the ECCS (1988) code. The measuring stick length used was 367mm. The local geometry of the silo was checked at the welds directly above the eccentric outlet and also opposite this outlet. Any noticeable defects elsewhere were also checked. The results show that the silo was generally well within the ECCS tolerances for the full strength rule. However, there were a few measurements which lie just outside the limit or are borderline. These were mostly on the opposite side of the silo to the eccentric discharge generator, where the greatest danger of buckling exists. Details are given in Rotter et al. (1995).
<table>
<thead>
<tr>
<th>SECTION</th>
<th>Diameter 1 (mm)</th>
<th>Diameter 2 (mm)</th>
<th>Diameter 3 (mm)</th>
<th>Diameter 4 (mm)</th>
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<tr>
<td>5th*</td>
<td>4,200</td>
<td>4,204</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4th*</td>
<td>4,208</td>
<td>4,200</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3rd*</td>
<td>4,212</td>
<td>4,206</td>
<td>4,208</td>
<td>4,206</td>
</tr>
<tr>
<td>2nd*</td>
<td>4,208</td>
<td>4,204</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1st*</td>
<td>4,214</td>
<td>4,200</td>
<td></td>
<td></td>
</tr>
<tr>
<td>At 1st ring</td>
<td>4,200</td>
<td>4,194</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Base</td>
<td>4,199</td>
<td>4,196</td>
<td>4,200</td>
<td>4,200</td>
</tr>
</tbody>
</table>

(*i.e. in barrel at a point half way between the ring with this number and the one above)

<table>
<thead>
<tr>
<th>No.</th>
<th>Location</th>
<th>Circumference (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>300mm below top of barrel</td>
<td>13,212</td>
</tr>
<tr>
<td>2</td>
<td>150mm above 5th ring</td>
<td>13,210</td>
</tr>
<tr>
<td>3</td>
<td>centrally between 4th and 5th rings</td>
<td>13,212</td>
</tr>
<tr>
<td>4</td>
<td>150mm above 4th ring</td>
<td>13,210</td>
</tr>
<tr>
<td>5</td>
<td>centrally between 3rd and 4th rings</td>
<td>13,212</td>
</tr>
<tr>
<td>6</td>
<td>150mm above 3rd ring</td>
<td>13,212</td>
</tr>
<tr>
<td>7</td>
<td>centrally between 2nd and 3rd rings</td>
<td>13,214</td>
</tr>
<tr>
<td>8</td>
<td>150mm above 2nd ring</td>
<td>13,214</td>
</tr>
<tr>
<td>9</td>
<td>centrally between 1st and 2nd rings</td>
<td>13,214</td>
</tr>
<tr>
<td>10</td>
<td>150mm above 1st ring</td>
<td>13,212</td>
</tr>
<tr>
<td>11</td>
<td>centre of section beneath 1st ring</td>
<td>13,208</td>
</tr>
<tr>
<td>12</td>
<td>150mm above base of section beneath 1st ring</td>
<td>13,207</td>
</tr>
<tr>
<td>13</td>
<td>30mm above base of section beneath 1st ring</td>
<td>13,196</td>
</tr>
<tr>
<td>14</td>
<td>100mm below base of section beneath 1st ring</td>
<td>13,214</td>
</tr>
</tbody>
</table>

### 11.4 Wall strain measurement

#### 11.4.1 General

The gauges attached to the silo barrel were "double deck" or "sandwich" gauges which can measure the strains on both the inner and outer surfaces of the silo wall without any installation within the silo which might interfere with the solids flow.
The gauges installed on the rings were designed to measure the tension and bending occurring in each ring at a large number of positions around the silo circumference. In addition, a number of thermocouples were installed to measure thermal changes, so that these could be independently checked if the strain gauges appeared to be registering large thermal strains.

11.4.2 Active gauges

A total of 384 strain gauges were attached to the silo wall whilst it was still in the factory where it had been fabricated. Of the 384 strain gauges, 160 were placed on the ring stiffeners, and the remaining 224 gauges on the silo barrel. These gauges were carefully protected from both moisture and the heating effect of solar radiation. A large array of dummy gauges, placed at different heights and different orientations around the silo wall were also used to monitor thermal effects. The overall strain gauge layout on the silo wall is shown in Fig. 11.4.

All the gauges were carefully protected with high density rubber foam covered with aluminium taping. After all of the gauges were installed, each was checked for the correct resistance and the gauge factor.

11.4.2.1 Ring gauges

Of the 160 ring gauges, only 120 were used as active gauges to measure stress-induced strains. The others ('dummy gauges') were installed to permit compensation for strains due to temperature fluctuations.

The ring gauges were located on five rings, A to E. Each ring had a horizontal flange of width 100mm. The upper surface of the flange of these rings was at the heights above the silo floor of 1536, 3038, 4540, 6042 and 7544mm respectively.

There were 24 gauges on each stiffening ring which were attached at 12 positions a, b, c, d, e, f, g, j, k, l, m and n around the circumference with θ=0°, 15°, 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°, 225°, 270° and 315° respectively (Fig. 11.4). These circumferential coordinates were defined by the angles measured anti-clockwise from the eccentric outlet when viewed from above. Each of these positions was occupied by two gauges so that a mean strain and a strain gradient could be measured across the ring. The gauges were placed at distances of 19.5mm from the wall (inside gauge) and 10.5mm from the outer edge of the ring (outside gauge). Each gauge was
uniquely coded by three characters representing the ring on which the gauge was located (A-E), the circumferential position (a-n) and inside ('i') or outside ('o') gauge. For example, the inside and outside gauges at Position b on Ring D were coded by Dbi and Dbo respectively.

A high resistance gauge (350 ohms) was used on the rings because it would be less affected by the resistance of the cables needed to connect it to the data logger. It also permits higher bridge voltages and thus larger output signals whilst minimising gauge heating from excitation. The ring gauges were self-compensated for mild steel.

11.4.2.2 Barrel gauges

Of the 224 barrel gauges, 192 were active gauges and the remaining 32 were dummies. They were placed at four levels, 1 to 4, with heights above the silo floor of 933, 2255, 3757 and 5259mm respectively.

At each level at which barrel gauges were installed, there were 8 positions a, b, c, e, f, h, k and m with circumferential angles of \(0=0^\circ, 15^\circ, 30^\circ, 60^\circ, 90^\circ, 135^\circ, 180^\circ\) and \(270^\circ\) respectively (Fig. 11.4). At each of these locations, a rosette of "sandwich" gauges was placed (six gauges total), so that there were 48 linear strain measures per level on the barrel.

Each rosette consisted of three gauges: 1 measured strains in the vertical direction, 2 at 45 degrees and 3 measured strains in the circumferential direction, with 2 rotated clockwise relative to 1. Gauge 1 was situated slightly above the intersection of the three gauges. Gauge 3 was situated slightly anti-clockwise (when looked at from above) from the intersection of the three gauges. The centre of each individual gauge was 50mm from where the three lines meet (Fig. 11.5).

The "sandwich" gauges used were of 30mm length, 7mm width and 2mm thickness. Each "sandwich" gauge consisted of two foil gauges, separated by a plastic sheet of 2mm in thickness. The two foil gauges in each of the double deck bending gauges were identified by the characters 'b' for the lower gauge in contact with the silo wall, and 't' for the top gauge, separated from the silo wall by the standard plastic sheet.
Each of the barrel gauges was uniquely coded by four characters representing the level the gauge located (1-4), the circumferential position (a-n), the orientation in the rosette (1-3) and top or bottom gauge (t or b). For example, the vertical top and bottom gauges at Position b of Level 3 were coded by lb1t and lb1b respectively.

11.4.3 Dummy gauges

The dummies were placed on a steel backing plate of thickness 3mm and dimensions 40mm x 80mm that was then attached to the silo wall. Each gauge was protected by foam and aluminium taping as described previously.

The dummy gauges were located in positions which could allow for differences in temperature with position around the circumference and with height up the silo wall.

11.4.3.1 Barrel dummy gauges

The 'active' barrel gauges were DD gauges to enable the strain gradient in the silo wall to be determined by extrapolation. Gauges from the same production batch were used as dummies. Because the thermal effects of the wind and sunlight varied around the silo circumference, dummies were placed at 90° spacings around the silo (Fig. 11.6). Each dummy gauge consisted of a single DD gauge and readings were taken of both the top and bottom gauges of the DD pair. These two readings allowed the effects of thermal differences between the top and bottom gauges on the silo wall, caused by the lower thermal conductivity of the sandwich filling, to be taken into account.

These gauges were not temperature compensated, so they indicate strains when the temperature changes even though they remain unstressed.

11.4.3.2 Ring dummy gauges

Dummy gauges were placed at 45° spacings all around the circumference of each ring (Fig. 11.6). Where possible, they were placed within 50mm of the corresponding ring gauge.
11.4.4 Logging system

Six data logging frames, each capable of logging 64 gauges, were used to log the strain readings. The frames were connected to a PC running a data logging program, SCAN1000 written by Hexatec Ltd. The data logging frames incorporated 13 bit analogue to digital converters that were capable of recording a maximum of 12 strain readings a second. In practice it was found that they were able to average 8 readings a second which corresponds to a total time of 48 seconds to log all the gauges on the silo and a scan period of 60 seconds was subsequently adopted. Ideally all the gauges should be logged simultaneously, but this was not possible and it was assumed that the differences in the state of the discharge between the start and end of a scan were negligible. It was however, important to ensure that all the gauges at a particular level were scanned sequentially and this was made possible by the appropriate calibration of the data logging program.

11.5 Water test for verification

Before any solid tests were done, the silo was filled with water, strain data being acquired during the filling process. The silo was then allowed to stand for two hours, whilst full, and then it was emptied, all the time the gauges being logged. This was done so that the readings from the strain gauges could be compared to predictable pressures on the silo walls, and hence to verify the technique for gaining pressures from strains.

11.6 Summary

The silo structure and the strain measurement system have been summarised in this chapter. The inference of wall pressures from the strain measurements will be presented in the next chapter.
Fig. 11.2a Detail of floor / column / skirt / barrel / ring / joint in silo

Fig. 11.2b Typical stiffening ring on barrel wall

Fig. 11.2c Detail of silo roof
Fig. 11.3 Silo floor beam grillage layout

Fig. 11.4 Overall strain gauge layout
Fig. 11.5 Numbering of rosette strain gauge

Fig. 11.6 Positioning of dummy gauges on the barrel and rings
Chapter 12

PRESSURE DISTRIBUTIONS IN THE
BRITISH STEEL SILO INFERRRED FROM
WALL STRAIN MEASUREMENTS

12.1 Outline

The silo in the British Steel Laboratories at Teesside was instrumented with 384 strain gauges, and these were logged at 60 second intervals throughout the tests (see Chapter 11). In this chapter, storing and incipient discharge pressures in a few of these experiments are inferred from the strain measurements. As most tests took several days to complete, the total body of data is extremely large. Its complete processing and interpretation will take some years and is therefore beyond the scope of this thesis.

Because accurate structural analyses are the basis of the strain pressure inference process developed in Chapters 8 to 10, a comprehensive study of the finite element modelling of the silo is carried out first. The structural behaviour of the silo under harmonic pressures is also studied to achieve a better understanding on the process. Load functions are carefully studied and chosen so that the best result can be achieved.

Under the conditions of concentric filling and discharge, the pressures exerted by the stored solids on the silo walls are expected to be substantially axisymmetric. The structure is also expected to respond in a substantially axisymmetric manner, which
is unlikely to cause a structural failure. By contrast, under conditions of eccentric discharge, the silo wall is subjected to unsymmetrical local pressures. The risk of structural failure is high in this case.

As the rigorous statistical method is very onerous to implement because it involves the establishment and manipulation of enormous sets of equations, it is only used to infer storing and incipient discharge pressures for one fully eccentric discharge Test PFA. Simple and quick techniques are used to interpret the strain gauge readings for Test PFA and a few other tests. The results are compared with those inferred using the most rigorous method.

12.2 Finite element modelling of the silo structure

12.2.1 Introduction

The program SIMAS developed in Chapter 10 was used to carry out finite element analyses for the silo shell structure. The silo structure was assumed to be linear elastic under all loadings. The convergence of the finite element modelling and the finite element modelling of the angle ring are studied in this section.

To limit the amount of computation, only a piece of the silo structure was studied here (Fig. 12.1a). It consisted of a 1502×3mm vertical wall with a radius of 2101.5mm and a 100×60×10mm unequal angle stiffener attached to its outside surface (see Chapter 11). Under axisymmetric constant pressures, there is substantially no meridional bending in the silo barrel far away from the ring stiffeners. The boundary conditions were therefore chosen as in Fig. 12.1.

12.3.2 Convergence of finite element mesh: conventional modelling

Four finite element meshes with 6, 8, 14 and 43 elements respectively were used to carry out a convergence study of the FE mesh (Models 1 to 4 in Figs 12.1b, c, d and
e). The element size in Model 4 was about $\lambda/10$ near the discontinuities and increased gradually to about $\lambda/5$, which should result in precise predictions. Here $\lambda=2.444\sqrt{\text{t}}=194\text{mm}$ is the bending half-wavelength of the vertical wall. The horizontal flange of the unequal angle was modelled by one element in Models 1-3 and three elements in Model 4. The vertical flange was assumed to deform compatibly with the wall and the slight difference between its radius and that of the wall was ignored. The modelling of the ring using line elements increases the cross-sectional area at the intersection point C between the vertical wall and the horizontal flange of the ring (Fig. 12.1b), which strengthens the structure. This strengthening may significantly affect the strains in the ring. To overcome this effect, the thickness of the flanges of the angle was reduced to $1560/(101.5+60)=9.66\text{mm}$, where $1560\text{mm}^2$ is the cross-sectional area of the 100x60 x10mm angle (SCI, 1987).

The radial displacements at A and B, the strains at the ring gauge positions and the barrel gauge position A (Fig. 1a) were used to monitor the convergence. The strains at the gauge positions were used to monitor the convergence because they are important items that will be used to infer load distributions on the silo walls. Table 12.1 shows that the differences between the predictions from Models 3 and 4 are very small (about one part in one thousand). It may therefore be concluded that the mesh in Model 3 is fine enough to give reasonably precise predictions.

It may be noted that the horizontal flange of the angle is small and the vertical flange together with the appropriate wall is about 0.15 bending half-wavelength. Therefore, each can be modelled by one element with high precision. To verify it, the horizontal flange in Model 3 was modelled by 3 elements while the others remained unchanged. The results made no difference with five significant digits.
Table 12.1 Convergence of FE modelling: the silo section under constant internal pressure p=1MPa

<table>
<thead>
<tr>
<th>Model Description</th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
<th>Model 4</th>
<th>Model 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial disp. at A, mm</td>
<td>7.6140</td>
<td>7.3968</td>
<td>7.3603</td>
<td>7.3605</td>
<td>7.3603</td>
</tr>
<tr>
<td>Radial disp. at B, mm</td>
<td>1.8353</td>
<td>1.5752</td>
<td>1.5567</td>
<td>1.5553</td>
<td>1.5602</td>
</tr>
<tr>
<td>Inner ring strain, με</td>
<td>832.49</td>
<td>702.46</td>
<td>697.57</td>
<td>696.82</td>
<td>697.84</td>
</tr>
<tr>
<td>Outer ring strain, με</td>
<td>797.14</td>
<td>672.65</td>
<td>667.95</td>
<td>667.23</td>
<td>668.21</td>
</tr>
<tr>
<td>Circum. barrel strain, με</td>
<td>3623.1</td>
<td>3519.8</td>
<td>3502.4</td>
<td>3502.5</td>
<td>3502.4</td>
</tr>
<tr>
<td>Merid. barrel strain, με</td>
<td>-1068.6</td>
<td>-1053.5</td>
<td>-1051.0</td>
<td>-1050.9</td>
<td>-1051.0</td>
</tr>
</tbody>
</table>

12.2.3 Modelling of ring stiffeners

12.2.3.1 Under constant pressure

The slight difference of radius between the vertical flange of the angle and the barrel was ignored above. The error due to this approximation was studied here. An orthotropic joint element was introduced for this purpose. Model 5 (Fig. 12.1f) has the same mesh as Model 3, but the angle is connected to the wall by three joint elements. The joint elements have a finite thickness t, a high meridional modulus E_v, zero circumferential modulus E_θ, zero shear modulus G and zero Poisson’s ratios v for both membrane and bending deformations so that the angle deforms compatibly with the wall while the structure is not strengthened circumferentially by the joint elements. In practice, it was found that very small values for E_v and G were required to avoid singularity during element matrix condensation (Rotter, 1989; Teng, 1990). The results for Model 5 presented in Table 12.1 were obtained by using t=10mm, E_v=10^5E, E_θ=10^-5E and G_v=G_θ=10^-5G. Here E and G are the Young’s modulus and shear modulus of mild steel. The results were different by less than one part in one thousand from those from Model 3. It would have been easy to conclude that Model 3 was good enough to accurately model the structure if no further studies had been carried out as described below.
12.2.3.2 **Comparison with FEA results of axisymmetric elastic body**

To assess the accuracy of the above results, the silo section was modelled as an axisymmetric elastic body using the finite element analysis program AXIPH which was written by Professor J. M. Rotter. It was a surprise to discover that all of the above models significantly (about 7%) underestimated the ring strains (Table 12.2). The cause was found to be the slight difference in the position of the vertical connection between the ring and the barrel in different models: the annular plate of the ring was modelled by line elements and connected to the barrel at its middle height in all the above models (Fig. 12.1), but the full height of the ring was assumed to deform compatibly with the barrel in the axisymmetric elastic body analysis here.

Meshes in Models 3 and 5 were slightly modified to accommodate this effect (Models 3a and 5a in Fig. 12.2). The results of the modified meshes agree well with those obtained by using axisymmetric elastic body analysis, with less than 1% error for Model 5a (Table 12.2).

**Table 12.2** FE modelling of the ring stiffener: the silo section under constant internal pressure $p=1\text{MPa}$

<table>
<thead>
<tr>
<th>Model</th>
<th>Model 3</th>
<th>Model 3a</th>
<th>Model 5a</th>
<th>AXIPH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Description</td>
<td>14 convention line elements</td>
<td>As Model 3, modelling the full height of ring</td>
<td>As Model 3a, using joint elements</td>
<td>Axisymmetric elastic body (exact)</td>
</tr>
<tr>
<td>Strain in inner ring gauge, $\mu$e</td>
<td>697.6</td>
<td>756.6</td>
<td>753.9</td>
<td>747.1</td>
</tr>
</tbody>
</table>

12.2.3.3 **Under harmonic pressures**

In this section, a pressure varying with harmonics was applied inside the silo section to examine whether the above conclusions can be extended to cases under unsymmetric loadings. The variation of the pressure with height was again assumed
to be constant. Both computer programs SIMAS and AXIPH were used to carry out finite element analyses.

Typical strain predictions in the ring gauges at $\theta=0^\circ$ are shown in Fig. 12.3. It is seen that very significant differences exist between different models. In comparison with the exact results of axisymmetric elastic body analysis, Model 5a is the best with errors less than 2% in the inner ring gauge and 1% in the outer one (Fig. 12.3c). For the conventional Model 3, the error in the outer ring gauge increases with harmonic number and reaches -10% at Harmonic 12, but that in the inner ring gauge is quite stable between Harmonics 6 and 12 and has a maximum error of -6%. When joint elements were introduced (Model 5), the errors have the similar magnitude but opposite sign. Models 3a and 5a (Fig. 12.2) produced very close predictions under axisymmetric pressures above because there is no circumferential bending in this case. However, Model 3a is the worst under harmonic pressures with up to 23% errors for harmonic numbers lower than 15. These results demonstrate clearly that very significant errors can result from incorrect modelling of the radial eccentricity of a ring stiffener.

To model the ring stiffeners with good accuracy, it may be concluded that

a) orthotropic joint elements should be used to connect the ring stiffener and the barrel to correctly model the radial eccentricity;
b) the joint elements should be connected to the barrel corresponding to the full height of the ring.

12.2.4 Effect of circumferential imperfection at ring levels

In the light of later debates during the interpretation of the measured data, it is unfortunate that accurate measurements were not made of the diameter at the locations of the rings after fabrication. However, careful measurements were made of the departure of the shell wall from verticality near the rings after the discovery of higher pressures during the data interpretation. Some results are plotted in Figs 12.4 and 12.5, where a local inward imperfection of the wall at each ring is seen. The
detail of the imperfect shape is shown in Fig. 12.5, where it is compared with the theoretical model used by Rotter and Teng (1989) to model the weld depression imperfection in a silo wall, based on shell bending theory. The readings bear a surprisingly close relationship to these proposed forms, though it may be fortuitous rather than evidence that these adopted imperfections are normally realistic.

The effect of the axisymmetric imperfection on the wall strains was also investigated. For a magnitude of 4mm in the local inward imperfection, Figure 12.6a shows the departure of wall strains from that of a perfect silo section under harmonic pressures. The relative errors due to ignoring the imperfection are shown in Fig. 12.6b. The error in the circumferential barrel gauge is smallest in all the gauges and has a peak value of 9% at Harmonic 11. By contrast, the error in the vertical barrel gauge increases with harmonic number with a power greater than 2 and reaches about 60% at Harmonic 12. The error in the inner ring gauge decreases from 19% at Harmonic 4 to -7% at Harmonic 12, and that in the outer ring gauge increases from -5% at Harmonic 2 to 35% at Harmonic 12.

Although the barrel wall between rings was found to be generally too smooth to measure departures from the perfect shape using simple methods, small local imperfections do exist on the silo (see Chapter 11). Because of the complexity in modelling local barrel imperfections, their influence on the wall strains are not clear. However, a reasonable speculation might be that local barrel imperfections strongly effect the local bending strains but have much less effect on the membrane strains. On bending strains, they may have a stronger influence under fairly axisymmetric loadings than under highly unsymmetric loadings.

12.2.5 Modelling of the whole silo structure

From all the discussions above, it may be concluded that the mesh in Model 5a (Fig. 12.2b) is good enough to model the structure precisely. A similar mesh was used in modelling the whole silo structure, which consisted of 164 elements (Fig. 12.7). A 5mm welding fillet between the ring stiffeners and barrels and a 4mm
inward axisymmetric imperfection adjacent to the rings were also considered. The local stiffening effect of DD strain gauges were included (Appendix A) in calculating barrel strains under characterised loadings. For the given geometries of the DD gauge and the wall thickness, the relative stiffening errors in the lower and upper foil gauges are $e_{ls}=0.780\%$ and $e_{uts}=1.39\%$ under pure tension and $e_{ls}=1.34\%$ and $e_{uts}=1.04\%$ under pure bending respectively. These corrections are therefore relatively small, but nevertheless important in attempts to eliminate as many sources of error as possible.

12.3 Structural behaviour of the silo under axisymmetric pressure

12.3.1 Effect of ring stiffener on the nearby shell deformation

Figures 12.8 and 9 show the deformed shape and strain distributions in the silo section (Fig. 12.1a) under an internal normal pressure $p=1\text{MPa}$. The deformations in Fig. 12.8 have been enlarged by a factor of 10. The presence of the ring stiffener significantly reduces the deformations of the adjacent wall. This leads to significantly smaller circumferential strains, but large meridional curvatures are also caused as a consequence (Fig. 12.9). Due to the asymmetry of the ring cross section, it is also subjected to strong rotation (Fig. 12.8b). The rotation causes bending strains in the horizontal flange of the ring (Fig. 12.10), which is about 11\% of the circumferential membrane strain.

12.3.2 Thick ring effect

Figure 12.10 also shows a significant thick ring effect. The circumferential membrane strain at the outer edge (792.3$\mu\text{e}$) of the ring is 6.05\% smaller than that at the inner edge (843.4$\mu\text{e}$). The relative difference between the total strain at the outer edge on the upper surface of the ring (711.3$\mu\text{e}$) and that at the inner edge (756.9$\mu\text{e}$) is almost the same as the membrane strains. This thick ring effect agrees very well with the predictions of elasticity theory (Appendix 12A).
12.3.3 Comparison with effective ring theory

The deformation of the ring may also be calculated using effective ring theory (Appendix B). Figure 12.11a shows the effective cross section of the ring, where the value \( l_e = \lambda / \pi = 61.75 \text{mm} \) represents the effective sections of the wall which act with the ring. The area and the moments of inertia of the cross section are \( A = 2116 \text{mm}^2 \), \( I_{xx} = 2.080 \times 10^6 \text{mm}^4 \) and \( I_{zz} = 2.302 \times 10^6 \text{mm}^4 \) respectively. The position of its centroid is at \( x_e = 27.53 \text{mm} \) and \( z_e = 105.92 \text{mm} \). The radius of the effective ring centroid is therefore \( r_e = \sqrt{x_e^2 + z_e^2} = 2127.53 \text{mm} \). Both upper and lower ends of the effective ring cross section (Fig. 12.11) are subjected to a rotational restraint of \( K_e = \frac{2E}{6r_e^2} = 5139 \text{N-mm/mm} \) (Appendix B). Due to the asymmetry of the ring section, the centroid has a vertical eccentricity to the middle of the effective wall section, \( e = z_e - \frac{(65 + 61.75 \times 2)}{2} = 11.67 \text{mm} \). The normal pressure (Fig. 12.11a) consequently produces not only a distributed radial loading \( P \) (Fig. 12.11b and c) but also a distributed moment \( m \) (Fig. 12.11b and d) about the effective ring centroidal axis.

For a unit normal pressure \( p = 1 \text{ MPa} \), they have the values of

\[
P = \frac{2100}{r_e} \left( \frac{\lambda}{2\pi} + 65 \right) = 186.1 \text{ N/mm} \quad (12.1a)
\]

\[
m = Pe = 2172 \text{ N-mm/mm} \quad (12.1b)
\]

where the factor \( 2100/r_e \) represents the transformation of the loadings on the inner surface of the wall to the effective ring centroidal axis.

The distributed radial force \( P \) is resisted by the ring tension \( N \) and the distributed moment \( m \) is resisted by a bending moment \( M \) about \( x-x \) axis (Fig. 12.11b):

\[
N = Pr_e \quad (12.2a)
\]
M = m_r_e \quad (12.2b)

where M is positive when it produces tensile stresses in the lower part of the cross section.

The strain on the upper (\(\varepsilon_u\)) surface of the horizontal flange of the angle ring stiffener due to N and M is given by

\[
\varepsilon_u = \frac{N}{EA_e} - \frac{Mz_u}{EI_{xx} + 2K_{ei}^2} = 935.6 - 207.9 = 727.7 \mu e
\quad (12.3)
\]

in which \(z_u=20.83\text{mm}\) is the vertical distance from the upper surface of the horizontal flange to the ring centroid. This agrees closely with the finite element predictions (711.3\(\mu e\) at the outer edge and 756.9\(\mu e\) at the inner edge).

The result shows that the effective ring theory (Rotter, 1983b) may be extended to unsymmetrical ring stiffeners by considering the rotation of the ring with a good accuracy (Appendix B). However, large errors (22% in this case) may result from ignoring the moment term as in some early work (e.g. Mackintosh, 1994).

12.4 Structural behaviour of the silo under harmonic pressures

Unit rectangular patch pressures varying with harmonics were applied to the silo to investigate the behaviour of the structure under unsymmetrical pressures. The pressure zone was centred on Ring C and extended over a height of 1502mm (Fig. 12.12).

Figure 12.13 shows the circumferential strains on Rings A-E at a point 9.5mm from the outside edge at \(\theta=0^\circ\). Ring C experiences the largest strain in all the cases because the pressures are directly applied near it. Other rings are also deformed,
though their deformation decreases quickly as harmonic number increases: asymmetric pressures at one position in the silo produce strains in the whole silo because this is a long wave bending problem (Calladine, 1983). Under axisymmetric pressures, only Ring C is deformed because this is a short wave bending problem.

The effective ring cross section (see Section 12.3.3) may possibly be used to calculate the ring deformation. The internal forces and strains of a ring under harmonic pressures can be calculated using Flugge’s equations (1973). However, the results were totally different from the finite element predictions (Fig. 12.14a). To examine whether this was due to the effect of the asymmetry of the ring here, the annular plate in Ring C was moved to the mid-height of the vertical flange so that the ring became symmetric (Fig. 12.12). The results again showed that the finite element predictions were totally different from those obtained from an effective ring cross section. It was concluded that either the effective ring theory does not work here or that Flugge’s (1973) ring equations are wrong.

To verify Flugge’s ring equations (1973), the circumferential strains at the inner and outer edges of a simple ring were calculated using both finite element analysis and Flugge’s equations. The inner and outer radii of the simple ring were 2100mm and 2200mm respectively and its thickness was 10mm. A harmonic pressure with a magnitude of $p=100N/mm$ was applied inside the ring. The match between the results from the two methods was very good (Fig. 12.15). Thus Flugge’s equations are clearly good for isolated ring under asymmetric loads, but when the ring is attached to a silo wall, more effects are present.

Therefore, the structural behaviour of the silo under asymmetric pressures is a complex matter. An asymmetric pressure patch may result in significant strains far away from the location of the pressure. Effective ring cross section theory, derived under axisymmetric pressures, should not be applied to asymmetric pressure conditions.
12.5 Verification of the instrumentation: water test

12.5.1 Strain observations

The first experiment undertaken in the silo was to fill it with water to test the instrumentation. The first test of this kind revealed the imperfect operation of dummy gauges, so this was remedied and the test repeated. The results shown here are for the second test, termed WFB (see Chapter 6).

The height of filling is marked as if it were a strain observation to indicate the point clearly. Significant thermal changes occurred at the instant that the water passed each level, and it was necessary to wait for these to stabilise and to carefully adjust the observations using the dummy gauges in order to achieve the correlation shown.

Because the water filling test was entirely axisymmetric and had very smoothly varying pressures, the strains observed at a given level were expected to be uniform around the circumference and there should have been no significant bending curvatures for the elastic axisymmetric shell structure. Figure 12.16 shows typical strain observations in the lowest Barrel 1. At first, it was astonishing that the observed barrel strains had huge departures from the expected uniform membrane strain distribution. It immediately led the author to question whether the whole strain measurement system was working well.

Confidence was restored when it was found that the membrane strains, which were deduced from the strain observations in the DD gauges, were rather uniform circumferentially (Fig. 12.17a). The deduced bending curvatures are large and irregular (Fig. 12.17b). For the known pressure distribution, these bending curvatures may result from local imperfections in the barrels (see Chapter 11 and Rotter et al., 1995). The orientation of the principal membrane strains at almost all gauge locations were vertical/circumferential (0° in Fig. 12.17c) as might be expected. The slightly non-vertical principal strain at location 1f was due to an
abnormal circumferential strain reading (Fig. 12.17a). A few similar strains were assigned smaller weightings during later processing. The orientations of the principal bending strains (e.g. Fig. 12.17c) vary in a complicated manner.

The observed strains in rings were much more uniform than those in the barrel, indicating that imperfections have much less effect on them.

In conclusion, the bending strains in the barrel sections under axisymmetric pressures were caused by wall imperfections, which cannot be predicted in the current case, so these should be excluded in inferring wall pressures. They may be used to infer the distribution of wall imperfections in future studies.

### 12.5.2 Simple interpretation

Under known smoothly varying axisymmetric pressures, the magnitude of the pressure at a given level can simply be related to the strains/stresses measured at the same level without serious errors. At barrel levels, the pressure \( p \) is related to the mean circumferential stress \( \overline{\sigma}_c \) as

\[
p = \frac{\overline{\sigma}_c t}{R}
\]  

in which \( t \) is the thickness and \( R \) is the radius of the cylindrical shell.

Under unit (1MPa) axisymmetric pressure, finite element analysis gives 787.3\( \mu \)e and 753.8\( \mu \)e in the inner and outer ring gauges respectively. The pressure at a ring level may be obtained by dividing the mean strain observations by these predicted strains.

The outcome of the strain gauge pressure inference procedure for Test WFB is shown in Fig. 12.18. The interpreted pressures are very close to the expected values at each level, using the height of water filling and the density of fresh water. It is seen that
the ring and barrel gauges produced compatible results in this test, so the differences found in later tests need explanations which are other than differences in the strain measuring system.

12.5.3 Estimation of the order of errors: rigorous interpretation

To estimate the order of error in the strain measurements and the inferred pressures, the rigorous procedure developed in Chapters 8 to 10 was implemented. A linearly increasing pressure distribution from zero at the top to a unit value (1MPa) at the silo floor was used as the load function. The results are presented in Table 12.3, where the percentage of fit (PoF) was defined as

\[
\text{PoF} = \frac{(\bar{\epsilon})^T[W][\epsilon] - (\epsilon)^T[W][\epsilon]}{(\epsilon)^T[W][\epsilon]}
\]

(12.5)

in which \([W]\) is the weighting matrix, \(\{\bar{\epsilon}\}\) is the observed strains and \(\{\epsilon\}\) is the strain residuals

\[
\{\epsilon\} = [S]\{a\} - \{\bar{\epsilon}\}
\]

(12.6) (8.7bis)

A good fit would have a PoF close to 100%.

Table 12.3 Results of the strain gauge pressure inference for water test WFB

<table>
<thead>
<tr>
<th>Strain observations</th>
<th>Estimated standard deviation, (\sigma_e)</th>
<th>16.5(\mu)e</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage of Fit, PoF</td>
<td>98.1 %</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Inferred pressure at silo floor</th>
<th>Value (Expected = 9.5x9.8 = 93.1kPa)</th>
<th>92.9kPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard deviation, (\sigma_a)</td>
<td>1.09kPa</td>
<td></td>
</tr>
<tr>
<td>Coefficient of variation, CoV</td>
<td>1.18%</td>
<td></td>
</tr>
</tbody>
</table>
The estimated standard deviation of strain observations $\sigma$, is an indicator of both the
goodness of the fit and the quality of the strain measurements. Bad strain
measurements certainly lead to a large $\sigma$, but good strain measurements may only
produce a small $\sigma$ if the fit is also good. The value of $\sigma=16.5\mu e$ for the water test
should provide a reference for other tests in the following inferences.

The inferred pressure and its error for the water filling experiment in Table 12.3
demonstrate the high accuracy of the strain measuring technique. Such a high
accuracy is later found to be much more difficult in the bulk solids tests where
conditions were much more complicated, but water test provides a reference for the
order of errors in pressures inferred in this manner in the experiments with bulk
solids.

12.6 Wall pressures after filling with iron ore pellets

12.6.1 Introduction

Two methods were used in this section to infer the iron ore filling pressures. The
simple interpretation method described in Section 12.5.2 was used to infer the mean
pressure (axisymmetric component) distribution at the end of filling in some of the
experiments. It should be noted that the method is accurate in inferring the water
pressure, but is less so for pressures exerted by bulk solids because the pressure
variation may not be as smooth as water pressure and frictional tractions influence
the ring strains. Further, the presence of non-zero unsymmetrical terms may also
effect the inferred axisymmetric term because the strain gauges were not equally
placed circumferentially. Nevertheless, it is a quick and simple method to estimate
the mean pressures. The rigorous statistical inference method (see Chapters 8 to 10)
was later used to infer both symmetrical and unsymmetrical pressures in Test PFA.
12.6.2 Strain observations

Figure 12.19 shows typical histories of strains during the whole filling and discharge process in Test PFA. Both barrel strains (Figs 12.19a and b) and ring strains (Figs 12.19c and d) increase (or decrease) steadily with the progress of the filling process. However, barrel strains increase much faster than ring strains because ring strains are much smaller than barrel strains under axisymmetric pressures (see Section 12.2).

When the discharge starts, both barrel and ring strains change dramatically. Figures 12.19a and b show that the magnitude for all the barrel strains at Positions a and b at Level 1 decrease quickly, because both are within the flow channel and the solids flow in the channel reduces the local pressure. By contrast, the magnitudes of the ring strains within or close to the flow channel increase dramatically (Figs 12.19c and d) because the rings need to resist most of the bending moments induced by pressure drops within the flow channel.

The strains caused by the stored solids after filling are termed storing strains here. They can be obtained by subtracting either the strains at the beginning of the filling (initial zero) or that at the end of discharge (end zero) from the strains after filling. The differences between the initial zero and the end zero are due to drift. By assuming that the drift is linear in time, a good quality zero can be subtracted from the storing strains. Figure 12.20 shows some typical storing strain distributions around the circumference. The thermal effects have been deducted using the dummy gauges. Although relatively axisymmetric pressures might have been expected under the condition of axisymmetric filling, both the membrane strains in the barrel (Figs 12.20a and b) and the ring strains (Figs 12.20c and d) show that the distributions are not axisymmetric. Figures 12.20c and d also show that the strains in Ring C are much smaller than those in the lowest Ring A, indicating the increase in pressure with depth below the solid surface.
12.6.3 Mean storing pressure: simplified interpretation

The inferred mean pressure (axisymmetric components) distribution at the end of filling in Test PFB is shown in Fig. 12.21a. It is compared with the Janssen pressure distribution which may be expected with a wall friction coefficient $\mu$ of 0.492 (wall friction angle of 26.2° from control tests: see Chapter 6) and a lateral pressure ratio of 0.426 (derived from $k = 1 - \sin \phi$, with $\phi = 35^\circ$).

The value of wall friction used in this calculation is relatively accurate. However, the appropriate lateral pressure ratio $k$ is dependent on the relations adopted to predict it. The relationship $k = 1 - \sin \phi$ (Jaky, 1948) is widely used to predict the lateral pressure ratio, though other relationships have also been proposed. The more difficult problem of interpretation here lies in the value of $\phi$ to adopt. The Jenike shear cell test results (see Chapter 6) were all at quite low stress levels, and produced a large effective internal friction angle of 52°. Using such a high value in this expression leads to an assessed $k$ value of 0.212, which is scarcely credible. If alternative expressions for $k$ are used from other sources (e.g. Walker, 1966; Hartlen et al, 1983; Rotter, 1992), or the traditional Rankine active pressure ratio is used, extremely low values of $k$ are found. In this context, it may be noted that the Australian code (AS3774, 1990) does not permit a value of $k$ below 0.35 to be adopted, and Jenike (1973) and Arnold et al (1980) proposed that the value of 0.40 should be used for all solids. Similarly, Gaylord and Gaylord (1984) quoted Homes as recommending a value of 0.45 for universal adoption. In this situation, the value chosen for comparison purposes here has been taken as $k = 1 - \sin \phi$ applied to the mean value of the angle of repose measured from the top surface of all the tests, as this value is accurately known. This angle was found to be 35.0°, giving a value of $k$ of 0.426.

The appropriate values of $k$ and $\mu$ naturally remain uncertain, but the Janssen line provides a useful reference. The top surface contact of the solid with the wall is marked by a point, and the Janssen curve naturally does not pass through this point
because it assumes a flat top surface, which must be interpreted as the effective surface.

For the barrel gauges, the pattern is very similar to the Janssen expectation. There is a reduction in pressure towards the bottom, which may be expected from finite element predictions involving flat bottomed silos (e.g. Ooi and Rotter, 1990b), and the slope of the curve is marginally higher than this Janssen prediction would suppose, but in general there is good support for the approximate theoretical pressure distribution. However, at the two bottom Rings A and B, the inferred pressure is very much higher. The reason for this is discussed below when the other test results have been shown.

The mean inferred pressures at the end of filling in Tests PFA and PFC are shown in Figs 12.21b and c. These distributions have the same characteristics as those of Test PFB, with the changes that the reduction in pressures towards the silo floor begin earlier, and the inferred ring pressures are considerably larger.

Finally, the same results for Test PHA are shown in Fig. 12.21d, which show the same effects again. The four tests are, in a general sense, repeatable, but the level of statistical variation from test to test is quite large. This is, however, to be expected in silo tests, as is known from the standard deviation of pressure cell readings (Ooi et al, 1990).

Of course, the four tests are all part of a single population, because the discharge eccentricity is not known at the filling stage. However, there can be systematic changes to filling pressure distributions throughout a test series as a result of material degradation, as was graphically shown by Ooi et al (1990). Some of the changes between the filling pressure distributions shown in this thesis may therefore be found on later processing to be caused by systematic changes rather than by stochastic variations.
The high pressures which were found on the rings may be caused by the systematic inward deviations of the silo wall at the ring locations (see Section 12.2.4), as these geometric imperfections have been postulated by others as a source of local high pressure (Hartlen et al, 1984; Nilsson, 1986). Similar phenomena are also seen in other experiments (e.g. Jenike et al, 1973a).

The magnitude of the local inward imperfection is significant in this silo. The wall has a thickness of 3mm and the maximum inward imperfection is of the order of 4mm. On a radius of 2.1m, this corresponds to a radial strain being imposed on the solid of the order of 0.004/2.1 = 0.2%. This is a very significant strain in a solid as stiff as iron ore pellets, so it is not surprising that quite large pressure changes are associated with it.

Because the ring strains are very small under conditions of concentric filling, they inevitably have a smaller signal-to-noise ratio than the barrel gauges. The inferred pressures at the ring levels here are therefore more uncertain. Better results can be achieved using the rigorous interpretation method in the next section.

12.6.4 General load functions for rigorous interpretations

From the discussions in Chapter 9, the selection of load functions is one of the key factors influencing the outcome of a pressure inference calculation. Many load functions have been tried during the course of this work for inferring both filling and discharge pressures, and evidently orthogonal functions produced better results. The load functions finally adopted are of the form

\[ f(z,\theta) = \sum \sum f_i(z)f_j(\theta) \] (12.7)

in which \( f_i(z) \) describes the meridional variation of pressures and \( f_j(\theta) \) describes the circumferential variation.
The functions used to describe the meridional variation of pressures consist of an exponential function at the top and a set of sinusoidal functions in the main body of the silo. The exponential function increases exponentially from the top surface (which varies for each experiment) to Ring D and then decreases linearly to zero at Ring C (Fig. 12.22a). The mathematical expression for the exponential part is

\[ f(z) = f_0 \left[ 1 - e^{-\frac{2kuz}{R}} \right] \quad (12.8) \]

where \( R=2.1 \text{m} \) is the radius of the cylinder; \( z \) is the depth below the top surface; \( k=0.426 \) and \( \mu=0.492 \) as defined in the previous section and \( f_0 \) is a coefficient so that the maximum value of the function at Ring D is unit (e.g. \( f_0=0.352 \) for Test PFA).

This function is necessary because there are almost no strain observations above Ring D (almost all the gauges on Ring E had been disconnected) making it impossible to infer a rigorous pressure distribution above Ring D. It is therefore best to assume that the pressure variation from the top surface to Ring D is similar to the Janssen pressure distribution while its magnitude is an unknown to be found. As there is only about 2m of fill height above Ring D, this assumption should not be far from the reality. The linear variation between Rings D and C in this load function is designed to connect the sinusoidal functions below it so that the inferred pressure distribution varies smoothly everywhere.

Sinusoidal loading functions (e.g. Fig. 12.22b) were used to describe the pressure variation between Ring D and the silo floor,

\[ f(z) = \sin \left( \frac{N\pi z}{L} \right), \text{ for } N=1, 2, 3, \ldots \quad (12.9) \]

in which \( L \) is the height of Ring D above the silo floor.
Harmonic functions were used to describe the circumferential variations

\[ f(\theta) = \begin{cases} 
\sin(-n\theta) & \text{for } n = -1, -2, \ldots \\
\cos(n\theta) & \text{for } n = 0, 1, 2, 3, \ldots 
\end{cases} \]  \hspace{1cm} (12.10)

The same load functions were used to describe the vertical frictional tractions. Circumferential frictional tractions were ignored here.

12.6.5 Filling pressure for Test PFA: rigorous interpretation

12.6.5.1 Load functions: circumferential variation

As mentioned above, harmonic functions were used as load functions to describe the circumferential variation of wall pressures. The pressures were assumed symmetrical about the diametral section through \( \theta = 0^\circ \). Therefore, only positive harmonic terms (cosine terms in Eq. 12.10) were used.

Under conditions of concentric filling, the thin barrel wall experiences moderate strains which are well related to the pressures, but the stiffening rings represent points at which the pressure is inadequate to induce even moderate strains (Figs 12.20). Under these conditions, the barrel gauge readings are the most useful and they play a key role in the strain pressure inference.

For each barrel strain gauge location with minimum 15° intervals and maximum 45° intervals, the highest harmonic number should not exceed 12 and preferably should not be greater than 4 according to the discussions in Chapter 9. The best number should be found numerically.

By using different circumferential harmonic terms, the percentage of fit and the estimated standard deviation of strain observations can be found in each inference calculation (Fig. 12.23). The load functions used for the vertical pressure variation were seven sinusoidal functions (\( N=1, 7 \) in Eq. 12.9) and the exponential function...
(Eq. 12.8). It was assumed that the vertical frictional tractions were fully mobilised so these were not treated as independent variables but considered using the linear constraints technique (see Chapter 10). More discussion on this matter is given in the next section.

The percentage of fit can naturally be improved by using more load functions, but its rate of increase significantly slows after harmonic 3 (Fig. 12.23a), which may imply that it is best to use harmonics up to number 3. This is confirmed by the estimated standard deviation of the strain observations, which has the smallest value when Harmonics 0 to 3 are used (Figure 12.23b).

Thus, Harmonics 0 to 3 were used to infer the storing pressures for Test PFA below.

12.6.5.2 Load functions: vertical variation

Figures 12.24 show the percentage of fit and the estimated standard deviation of the strain observations when different numbers of vertical sinusoidal functions were used. The load functions for the circumferential variation used here were Harmonics 0 to 3. Again, it was assumed that the wall friction was fully mobilised against the wall.

With more vertical sinusoidal terms, both the percentage of fit and the estimated standard deviation of strain observations decrease steadily. In this sense, it may be preferable to use even more terms. However, it was found that using any term of \( \sin(N\pi z/L) \) with \( N>7 \) or using more than 8 terms resulted in unstable or unacceptable pressure distributions. The reason is that strains were measured at only 8 levels (Rings A to D and Barrels 1 to 4). With 8 measured points on a meridian, the maximum number of unknowns which can be used is 8.

Because four vertical load functions are inadequate to describe the vertical pressure variations while the number is not allowed to exceed eight, the vertical frictional
tractions cannot be treated as independent variables. This is the reason why the assumption of fully mobilised friction against the silo wall is retained.

The load functions which were finally adopted for inferring pressures after filling in Test PFA were therefore Harmonics 0 to 3 for describing the circumferential variation and \(\sin(N\pi z/L)\) with \(N=1\) to 7 plus the exponential function (Eq. 12.8) at the top for describing the vertical variation of the normal pressure distribution. The percentage of fit and estimated standard deviation of strain observations are 96.5\% and 16.0\(\mu e\) respectively. These numbers are very close to the results in the water Test WFB (98.1\% and 16.5\(\mu e\) respectively), so the assumption of fully mobilised solids against the silo wall appears to be valid.

12.6.5.3 Observed versus inferred strains

Typical observed and inferred strains on the barrel sections and rings are shown in Figs 12.25a and b respectively. On the barrel sections, the inferred strains match very well with the observed values (Fig. 12.25a). By contrast, most of the inferred ring strains are about 20\% smaller than the observed values (Fig. 12.25b). This may imply that the actual pressures at ring levels were about 20\% higher than the inferred values but more localised. More discussion of this matter is given below.

12.6.5.4 Mean pressures

The inferred mean pressure at each level (axisymmetric component of the pressure distribution) on the wall is shown in Fig. 12.26. Pressures at the levels of Ring A and B are as twice high as those at adjacent barrel gauge levels. The peak of the high pressure appears to be slightly (about 100mm) above the horizontal leg of the ring stiffener. High pressures are also seen at Ring C, but the peak is about 0.5m above its horizontal leg. This may possibly be caused by the larger inward axisymmetric imperfection at that level (Fig. 12.4). Note that high pressure does not appear at the level of Ring D because restrictions of the adopted load functions can only describe a smooth variation there.
At a given barrel level, the mean measured circumferential stress governs the
interpreted value of the normal pressure at the same level, but the mean measured
vertical stress determines the cumulative pressures (frictional tractions) above it.
Therefore, the pressures at ring levels are not only inferred from the ring strains, but
also derive from barrel strains.

In fact, it may be suggested that the actual pressures at the ring levels are even higher
than is shown because the inferred ring strains are always smaller than the observed
values here. Because the adopted load functions are not able to describe more
localised zones of high pressure (which need many harmonics), their peak values are
reduced to retain the integral of the pressure correctly to match the observed vertical
membrane strains in the lower barrel sections.

As the ring strains are rather small compared with the barrel strains, the value of the
inferred pressure on each barrel section is constrained so that the inferred
circumferential membrane strains match the observed strains. The integral of the
inferred pressure from the top surface to each of the barrel levels is also constrained
by the wall friction assumption to match the observed vertical membrane strains.
The assumption of a smooth variation of pressure means that the value of the
pressure at the ring levels cannot rise high enough to match the observed strains if the
pressure peak is very localised. One possible remedy to this problem is to use four
terms for the load functions to model the vertical variation of pressures on the barrel
sections and another four terms to interpret the values of pressures at the four ring
levels by assuming their vertical variation. However, the inferred results are still
somewhat uncertain because the actual distribution of pressure is unknown. More
studies are required on this topic in the near future. However, it should be noted that
the present method of pressure inference produces far more information than pressure
cell readings since the inference procedure places correct restrictions on the
magnitudes of pressures between observation stations. Further, the local high
pressures found here on the rings could not easily be discovered using pressure cells.
Because the structure is particularly strong at the ring locations, the high filling pressures at these levels do not affect the silo design in any significant way. However, they may indicate that if a silo instrumentation system depends only on instrumented rings, the deduced pressures may be unrepresentative of the pressure on the remainder of the wall. Such instrumentation has been used in some previous studies.

The standard deviations shown in Fig. 12.26 are small (typically between 3 and 4 kPa) compared with the normal pressures. This indicates that the inferred pressures are a good representation of the actual pressures.

12.6.5.5 Unsymmetrical pressures

Figure 12.27a shows the 3D surface plot of the inferred pressures on the whole silo wall, including both axisymmetric and asymmetric terms. The pressure distribution is essentially axisymmetric in the higher parts of the silo (above Ring C), but quite large perturbations are seen in the lower parts (around Rings A and B). Higher pressures occur near $\theta=0^\circ$ and $\theta=120^\circ$ and lower pressures occur near $\theta=60^\circ$ and $\theta=180^\circ$ at the ring levels, indicating that Harmonic 3 is an important term. However, a lower pressure is found in the barrel sections near high pressures on the adjacent rings, and vice versa. Figures 12.27b shows the same surface plot viewed from a different angle. The contours for the same pressures are shown in Fig. 12.27c.

The standard deviation of the wall pressure is shown in Figs 12.28a (surface) and b (contours). The error distribution is quite uniform in the main body of the silo and then decreases with the height above Ring D. Larger errors are seen at the ring levels and on the meridians $\theta=0^\circ$ and $\theta=180^\circ$. The standard deviation is typically between 10% and 20% of the corresponding pressure in the lower parts of the silo. It increases to between 20% and 30% in the higher parts because the wall pressures
decrease. These values match well those found from pressure cell readings (Ooi et al., 1990).

Figures 12.29a and b show normal wall pressures and their corresponding standard deviations on meridians at θ=0°, 45°, 90°, 135° and 180°. The axisymmetric components of the pressure are also shown for comparison. The pressures vary in a very similar pattern on all these lines, with higher values at the ring levels and lower values on the barrel.

The standard deviation of the wall pressures including all terms is significantly larger than that for the axisymmetric terms alone (Fig. 12.29b) because of the additive feature of variance. It is essentially uniform at all circumferential positions except those at θ=0° and 180°. The standard deviations at θ=0° and 180° are significantly larger than at any other positions, indicating that the results for Harmonic 1 are of lower accuracy. However, this is to be expected as an instrumentation system of this kind is not very sensitive to Harmonic 1.

The circumferential variation of the normal pressure at the ring levels is shown in Fig. 12.30a. As already seen in Figs 12.27 and 29, the maximum fluctuation occurs on Ring A, with a maximum value differing by about 30% from the mean pressure. The standard deviations on Rings A, B and C are surprisingly close to each other, but that on Ring D, which is just off the plateau (Fig. 12.28a), is smaller (Fig. 12.30b).

12.6.6 Comparison of axisymmetric component with different predictions

12.6.6.1 Fit to Janssen distribution

The inferred axisymmetric storing pressure may be fitted to the Janssen distribution. The results are shown in Fig. 12.31. The wall frictional angle of 26.2° from control tests was assumed. The lateral pressure ratio was deduced using the criterion that the area under the Janssen distribution curve is equal to that under the inferred curve of pressure distribution. This ensures that the vertical wall forces are the same near the
silo bottom. The reason for this choice is that the vertical stress in the silo wall near the silo bottom dominates the buckling strength in most metal silos. The value of lateral pressure ratio derived from the inference was \( k = 0.427 \), which matches astonishingly well the adopted value of \( k = 0.426 \) in Section 12.6.3 (calculated from \( 1 - \sin \phi \)). This match is naturally fortuitous, but the inferred value for the whole wall is certainly about 0.43.

12.6.6.2 Comparison with results from simple interpretation

Figure 12.32 shows a comparison of the axisymmetric filling pressures inferred using the rigorous method with those found using the simple method of Section 12.6.3. While the simple method produces quite accurate pressures in the barrel sections, it tends to indicate higher pressures at the ring levels. Furthermore, it is not able to infer the pressure distributions between them. Nevertheless, it is a quick and simple method to use.

12.6.6.3 Comparison with predictions by classical theories and design codes

The normal pressure distributions using various classical theories and design codes are shown in Figs 12.33a and b respectively. The maximum pressures predicted by the classical theories vary from 18 to 40kPa: the maximum discrepancy between different predictions can be over 100%. The observed pressures are close to the largest theoretical prediction. The maximum pressures predicted by the design codes vary from 26 to 36kPa. The maximum discrepancy between them is about 40% and the observed pressures are about 10% higher than the highest prediction.

Most of these differences arise from differences in the adopted value of the lateral pressure ratio \( k \), which is generally calculated from the effective internal friction angle \( \phi \) (=52° for iron ore pellets) in many of the predictions. Although these predictions depend on the values of the input parameters, it is clear that different theories and different codes still give a wide scatter of pressure predictions.
12.7 Incipient discharge pressures under eccentric discharge

12.7.1 Introduction

Two methods were used in this section to infer the pressures at the beginning of discharge, which are referred to here as the incipient discharge pressures. A simple method was used to infer approximate values for these incipient discharge pressures at each ring level from the strain observations in the ring. This method was applied to each of the three fully eccentric discharge Tests PFA, PFB and PFC. The rigorous statistical interpretation method (see Chapters 8 to 10) was used to infer the complete pattern of the incipient discharge pressure accurately for the whole silo for Test PFA.

12.7.2 Strain observations

In Chapter 7, the flow pattern during eccentric discharge was found to be a narrow pipe funnel flow rising up the silo wall to the roof. This pattern certainly influenced the pressure distribution strongly, and led to pressure patterns which were not very different from those assumed in the design, where the stiffening rings had a vital role. For this reason, the stress state of the stiffening rings is very important during eccentric discharge.

The silo wall is subject to a reduction in pressure in the channel of flowing solid, and this induces a local inward movement in the wall throughout the silo height. The wall stresses associated with it are complicated, but the rings are predominantly subjected to inward bending in the flow channel, outward bending near the edge of the flow channel, and lesser effects elsewhere.

Figures 12.34a-e show the strain development on the rings for the first 25 minutes of discharge in Test PFA. All the strains take from 5 to 10 minutes to reach their peak values, indicating the formation stage of a stable flow channel. After that, most of the strain readings pass their peak values and then decline slightly, so that a 'worst
condition' may be sought. However, the peak strains at different points are not coincident in time, so the condition at given instants should really be studied, since there is no real pressure distribution corresponding to the peaks of all the strain observations.

The maximum strains occur at \( \theta=0^\circ \) (Position a) in all the rings, above the eccentric discharge outlet. Within the flow channel, the outer ring strains are compressive and the inner strains are tensile due to the inward moment, which is associated with a relative decrease in the wall pressure. At Positions c (\( \theta=30^\circ \)) and d (\( \theta=45^\circ \)), which are just outside the flow channel, the strains are tensile at the outer edge and compressive at the inner edge of the rings. The magnitude of the strains in all the rings is quite similar (Figs 12.34a-d).

The development of strains in the outer strain gauges at \( \theta=0^\circ \) in Rings A to D is compared in Fig. 12.34e. The first to develop strains is Ring A, closest to the floor, but the strain in this ring does not reach a value as high as that in the other rings because the size of the flow channel is small at this low level (see Chapter 7). The strain in Ring A remains stable once the peak value has been reached. Strains begin to develop in Ring B shortly afterwards. In the higher Rings C and D, where strains start to develop slightly later, a peak is reached and the strains decline again. When the flow channel is well developed, the terminal values are considerably lower than the peak values, probably because the widening flow channel has a less serious structural effect.

Typical circumferential variations of strain at various times are shown in Figs 12.35a and b. It is clearer in these figures that the greatest inward bending moment occurs at \( \theta=0^\circ \) and \( \theta=15^\circ \) while large outward bending moment occurs at \( \theta=30^\circ \) and \( \theta=45^\circ \). Beyond \( \theta=90^\circ \), all the strain observations are very small. Figures 12.35a and b also show that the strains are quite nicely (though not exactly) symmetrical about \( \theta=0^\circ \) (or \( \theta=180^\circ \)).
The membrane strains in the barrel sections (Fig. 12.36a) are much smaller and less stable than the ring strains. This was expected from the design calculations, where the rings were provided to support the asymmetric pressures (Rotter et al., 1995). Larger values are found in the area between $\theta=0^\circ$ and $\theta=30^\circ$ (Fig. 12.36b).

The strains observed in Test PFB (Fig. 12.37) were very similar to those observed in Test PFA. However, the strains observed in the third fully eccentric Test PFC (Fig. 12.38) were quite different and smaller than those in Tests PFA and PFB.

The flow channel in the half eccentric discharge Test PHA is much bigger than that in the fully eccentric discharge tests (see Chapter 7). Consequently, the discharge strains (Fig. 12.39) are much smaller than those under fully eccentric discharge, with a much less severe structural effect.

### 12.7.3 Approximate inference from ring strains

#### 12.7.3.1 The method

A simple and quick method was used here to infer the approximate changes in pressure from the recorded ring strains. The same procedure as the rigorous statistical method described in Chapter 8 was followed, but it was assumed that the strains in a ring are caused only by the local pressure changes near it. In this way the changes in pressure at each ring level can be approximately inferred from the strain readings from the ring alone.

Some assumptions must be made about the form of the pressure distribution: a simple one might be to suppose that the pressure varies linearly between values at fixed points (as in pressure cell results), but this produces pressure distributions which are hard to believe. More smoothly varying pressures are obtained by using harmonic series around the silo circumference: this leads to distributions which have more complexity than many engineers would like to accept, but the essential features
are captured in a least squares optimal fit. The sensitivity matrix was therefore obtained by applying characterised harmonic patch pressures around each of the rings. The patch pressures extend over a height of 1502mm on the silo wall and are centred at each of the rings.

The inferred pressures are naturally much less accurate than those found using the rigorous method applied to the complete silo, but the main features of the pressure distributions are expected to be captured.

### 12.7.3.2 Test PFA

The inferred pressure distribution at the level of Ring A at several instants after the start of discharge in Test PFA are shown in Fig. 12.40a. The pressure is assumed to be symmetrical about the discharge outlet (symmetry of the flow and the symmetry checks on strain readings mentioned above suggest that this is a reasonable assumption). Thirteen harmonic terms were used to give a precise description whilst maintaining good numerical stability. The harmonic term 1, which corresponds to global bending of the silo, or translation of the rings, is not well defined by these pressure inferences from the rings, and has been omitted. This term is also the subject of uncertainty and debate in the silos community about whether it should be included in a wall pressure analysis, as the pressures are not self-equilibrating and must be balanced by a global shear within the stored bulk solid. The other harmonic terms are all self-equilibrating. The simplest course of action is to eliminate the term, until the strain data processing of the barrel gauge observations can provide some concrete evidence in this debate.

The first curve in Fig. 12.40a, for time 23:06 is unaffected by the discharge. A minute later, the pressure at the channel centre has reduced by 17 kPa and after two minutes has fallen by 27 kPa. The low pressure region is widening steadily, and small peaks begin to develop at the sides of the pressure depression. The pattern has become quite stable towards the end of this time period, with a maximum pressure reduction of 34 kPa. It is noteworthy that the pressure at all points around the silo
circumference are affected by the formation of the channel. At the bottom Ring A, the pressure in the rest of the silo falls by about 15 kPa. At higher levels, the changes are smaller and more complex. This analysis is accurate enough to detect the real distribution of pressure in the channel, so it is a real finding that the pressure at the centre of the channel is lower than that at points towards the edge: this is seen to be slightly different in a later experiment.

The corresponding pressure distributions for Ring B are seen in Fig. 12.40b. The same phenomena occur, but a greater pressure reduction, to a peak of 47 kPa below the filling value, is seen. The same little waves at the edges of the channel are also found. In Fig. 12.40c (Ring C), the pressures are slower to reach their peak values, and the final pressure drop at the centre of the channel is only 33 kPa, a significantly smaller change than that at Ring B. However, the component for differential pressure changes across the diameter (Harmonic 1) has been omitted, so the overall shape of these curves may change a little with more careful analysis involving the barrel gauges too.

The pressure pattern in Ring D is shown in Fig. 12.40d. This again has the same shape, with a clearly peaking distribution in the channel, which is now quite clearly wider than the images for lower levels. Initially, the changes around \( \theta=0^\circ \) are small, indicating that the flowing solid has not reached Ring D. The pressure at the centre of the flow channel then progressively falls, and finally attains a maximum pressure drop of 36 kPa.

The changes in response at different levels are shown very clearly in these figures, with the phenomena of delayed pressure drop at higher levels and the little peaks of increased pressure at the sides of the channel both showing clearly. The final stable pressure distributions at the four rings are compared in Fig. 12.41. The little peaks at the sides of the channel are present at all levels, but these are quite small compared with some previous speculative suggestions (Jenike, 1967; Wood, 1983; Rotter,
At the higher levels the pressure variation within the channel is effectively triangular, whilst at lower levels it is more parabolic.

The changes with time are quite smooth. The variation of the pressure at the centre and at the edge of the channel are shown for Ring A in Fig. 12.42. It is evident that the stable condition is reached quickly. Similar results are seen in the other rings, with slower changes. In Fig. 12.43, the development of the pressure at the centre at each level illustrates this well, with the observations about the pressure rising slightly at Ring D before falling shown more clearly.

The stability of the analysis is illustrated in Fig. 12.44, where the magnitude of the different harmonic terms at each level in the 'peak' pressure distribution is shown. A good harmonic series is one which converges well with smoothly varying and declining higher terms. The patterns shown in Fig. 12.44 support the decision to truncate the series at twelve terms, and indicate that the pressure inference analysis is not in any sense numerically unstable. For further demonstration of this point, and to assist with the implementation of a harmonic model of eccentric discharge pressure patterns, the progressive changes in each harmonic at each of the four rings are shown in Figs 12.45a-m.

Finally, many designers would prefer to have these complicated pressure patterns simplified into something akin to a three-parameter model (Fig. 12.46): one pressure to characterise the value in the channel, a second to represent the width of the channel and the third for pressures elsewhere in the silo, as in Rotter (1986). A rigorous procedure has been followed to undertake a best-fit transformation in this manner, as it also provides a good measure of the channel edge at each level. The resulting pressure distributions are shown in Fig. 12.47. The rigorous transformation produces an image in which the width of the central rectangular pressure drop does not vary much with height, and is shown Fig. 12.48. The precise position of the edge peak of Fig. 12.41 is shown for comparison. The flow channel boundary derived from the residence time measurements (see Chapter 7) is also shown. Although none
of these estimations can be compared with a direct observation, it appears that the width of the central rectangular pressure matches surprisingly well with the flow channel boundary derived from residence time measurements. The edge peak of pressures is naturally slightly outside the flow channel boundary. This image is one of the most important of the study.

The changing values with time of the three parameters which characterise this pressure pattern are shown in Fig. 12.49a-c. The width is very ill-defined during formation of the channel, affecting the other values, but all the results are stable for the period beginning about 5 minutes after the start of discharge.

The results have been shown in great detail for this experiment, to illustrate the depth and complexity of the information behind the simplified results which will eventually be used from the tests. The other tests are given a briefer treatment.

12.7.3.3 Test PFB

The second fully eccentric test was conducted with the same initial flow rate as Test PFA. However, due to mechanical handling problems, the experiment had to be stopped after 5 minutes of discharge, before being restarted another 5 minutes later. This disruption in the discharge does not appear to have affected the results at all, indicating that arrested flow is not important in these tests.

Test PFB was very similar to Test PFA in the flow channel formation as shown in Chapter 7. It is therefore no great surprise that the inferred pressure distributions which are found are also very similar to those of Test PFA. As a result, only the complete changing pattern of pressure at each level is shown here (Fig. 12.50a-d). The differences between Figs 12.50 and 12.40 are generally trivial, and an initial view might be that they could be confused. However, there are tiny but notable differences, such as the magnitude of the little compressive zone at the edge of the flow channel in Ring B.
12.7.3.4 Test PFC

The third fully eccentric discharge test was performed later in the experimental sequence (see Chapter 6) when the proportion of fines in the solid were growing. Test PFC was undertaken without placing markers. It is therefore not possible to relate the different inferred pressures to what was possibly a small change in the flow channel.

The inferred pressures in Test PFC at the four ring heights are shown in Figs 12.51a-d. The first and most noticeable feature is the change to the pressure at the centre of the flow channel, which has become relatively constant at all levels. In some images (e.g. Fig. 12.51c), some of the later pressure patterns even develop an apparent lump at the centre of the flowing channel: this is however a normal consequence of truncated Fourier series, and will not affect the simplified interpretations at all. It could be smoothed by taking additional terms, or by modifying the simple assumption that all higher terms than those adopted are zero.

The general form of the pressures in Test PFC is similar to those of Tests PFA and PFB, but the reduced pressure zone is certainly wider at all levels, with the side peak pressures considerably further apart. The different flow channel interpretations derived from these pressure distributions are shown in Fig. 12.52.

12.7.4 Rigorous inference: Test PFA

12.7.4.1 Load functions

The general load functions described in Section 12.6.4 were also used to infer the incipient discharge pressures here. As discussed in Section 12.6.5.2, functions of \( \sin(N\pi z/L) \) with \( N=1 \) to 7 plus the exponential varying function (Eq. 12.8) at the top are the minimum requirement for describing the vertical variation of pressures after filling and this is also the maximum possible number which can be used. They were therefore also adopted in inferring the incipient discharge pressure.
Because of local pressure changes during discharge, more harmonic terms are required to describe the circumferential variation of discharge pressures than after filling. Figure 12.53 shows that both the percentage of fit and the estimated standard deviation of strain observations improve quickly with an increase in the number of harmonic terms used until Harmonic 6. After Harmonic 6, the improvement significantly slows down. Therefore, seven harmonic terms (Harmonics 0-6) are the minimum to capture the essential features of the discharge pressures. To accurately infer the peak values of local pressure changes, higher harmonics are desirable. In the following inferences, the maximum possible number of harmonic terms (Harmonics 0-12) were used.

Again, the pressures were assumed to be symmetrical about the diametral section through $\theta=0^\circ$ and the wall friction fully mobilised against the wall.

12.7.4.2 Observed versus inferred strains

Figure 54 shows typical comparisons of observed strains with inferred strains at 10 minutes after discharge began. The strains in the rings are quite large under discharge because the bending caused by local pressure changes are mainly sustained by the rings. These strains are dominant in the inference process and are therefore matched very well by the inferred strains (Fig. 12.54a). The match between the observed and inferred membrane barrel strains are slightly less good because of the small strain values (Fig. 12.54b). Interpolated values have been used between $\theta=80^\circ$ to $180^\circ$ to constrain the strains in this area.

12.7.4.3 Incipient discharge pressures: overall description

The inferred incipient discharge pressures in Test PFA are shown in Fig. 12.55 for the first 25 minutes of discharge. At each instant, the pressure changes are presented in the figures as a 3D surface plot at the top and a contour plot at the bottom. At 1 minute after the start of discharge, there is little disturbance of pressures in the silo
A major pressure decrease (about -40kPa) occurs directly above the outlet near Ring A (z=1.5m) at the next instant (Fig. 12.55b). There is a small pressure drop at Ring B (z=3.0m) and a small increase above Ring C (z=4.5m). It may be noted that the pressures are dropping in most parts of the silo below z=3.5m and increasing above it. This may indicate that the flow channel was about half way up the silo by this time. The increase of pressure in the high parts of the silo may be possibly due to a "switch" from an active plastic stress field after filling to a passive field during discharge (Nanninga, 1956). Nevertheless, the pressure increase is small and therefore not significant structurally.

At 3 minutes after the start of discharge, the pressure drop zone has moved up to about z=5m (Fig. 12.55c). The pressure reduction at Ring B is also very significant now. The pressure near Ring D (z=6m) has significantly increased to its peak value (about 20kPa). The higher pressure peaks at the edges of the flow channel have started to form.

With the development of the flow channel, the higher pressures in the higher parts of the silo start to decrease steadily to about 15kPa over the next two minutes (Figs 12.55d and e). The flow channel reaches the top surface at about 6 minutes after discharge began (Fig. 12.55f), but it has still not fully formed yet. The pressure continues to drop within the flow channel and to increase at the edges until about 8 minutes after discharge began (Fig. 12.55h). The pressures are quite stable after then with little variation (Figs 12.55i-m).

Similar pictures can be drawn for the standard deviations of the inferred discharge pressures at each instant. Some examples are shown in Figs 12.56a-b. After stabilisation, the standard deviation of the discharge pressure is typically between 4 and 5 kPa.
12.7.4.4 Incipient discharge pressures: mean pressure reductions

The axisymmetric components of the discharge pressures over the first few minutes are shown in Figs 12.57. They drop continually in the lower parts of silo in the first five minutes. Above Ring D, the overall pressure increases continuously in the first three minutes and then starts to decrease during the fourth minute, but is still positive at the fifth minute (Fig. 12.57a). In the next five minutes (Fig. 12.57b), there is little change in the lower parts of the silo. The pressure above Ring D continues to drop and becomes negative at $t=7$ minutes. After that, the discharge pressure pattern is very stable in most parts of the silo (Fig. 12.57c), with a little fluctuation above Ring D (Fig. 12.57d). These fluctuations may possibly be caused by the solids sloughing off the top surface and hitting the wall above the outlet.

By comparing the discharge pressures in Figs 12.57a-c with the filling pressures in Fig. 12.26, it is seen that large pressure drops occur at places where the filling pressure is large. There is an overall pressure increase around $z=4$ m because the filling pressure is very low in this zone. This could possibly be caused by the collapse of arching (higher pressures) directly above and below this zone. Figure 12.57e shows the axisymmetric wall pressure during discharge at $t=10$ min, which was produced by superimposing the discharge pressure (Fig. 12.57b) onto the storing pressure (Fig. 12.26). It is seen that the higher and lower pressures after filling have been significantly smoothed. The pressure at the level of Ring A is still very high possibly because many parts of the ring are in the stationary zone of solids.

The standard deviation for the axisymmetric components of the discharge pressures is shown in Fig. 12.58. The errors are small compared with the values of discharge pressures, indicating that the inferred pressures are well defined.

12.7.4.5 Incipient discharge pressures: components of Harmonic 1

The inferred discharge pressures show that the Harmonic 1 term does exist (Fig. 12.59a), though it is the subject of uncertainty and debate in the silos community about whether it should be included in a wall pressure analysis.
However, its vertical variation is not simple to describe. It is mainly negative in the lower parts of the silo and positive in the higher parts, which is similar to the distribution of the axisymmetric term in the first few minutes of discharge. The corresponding errors in this term are shown in Fig. 12.59b.

12.7.4.6 Incipient discharge pressures in different sections

The development of discharge pressure on the meridian directly above the outlet (θ=0°) is shown in Figs 12.60a and b. At the beginning of the discharge (t=1min), there is a small pressure increase in all the ring levels and a small reduction in the barrel levels. Although this is the very beginning of discharge, pressure changes are seen throughout the entire height of the silo. Nevertheless, all the changes are very small.

One minute later (t=2min), there is a major pressure drop near Ring A (z=1.5m) but still a very small increase at Ring B (z=3m) and moderate increase above z=4m. This may indicate that the flow channel has gone up to somewhere near z=2m. The magnitude of the pressure depression near Ring A is reduced to a value close to its stable value at the next strain sampling (t=3min), indicating that the flow channel has been stabilised at this height. The flow channel may have risen to a point close to Ring B (z=3m) now, which induces a major pressure drop there. The development of the flow channel also induces a peak pressure increase around Ring D.

As the flow channel reaches Ring D (t=4min), the higher pressure in the higher parts of the silo starts to reduce, and pressures in the middle of the silo continue to drop. The pressure reduction at Ring B reaches its greatest value at t=5min and becomes stable at t=7min (Fig. 12.60b). From Fig. 12.60b, it may be suggested that the flow channel reached the top surface at about 6 minutes after discharge began and became stable two minutes later.

The standard deviation of the above pressures is shown in Fig. 12.60c. It is quite uniform from the bottom to the level of Ring D with a value between 5 and 6kPa.
However, the pressure reduction is larger in the lower parts of the silo than that in the higher parts, so the standard deviation is relatively smaller in the lower parts (about 15% of the pressure reduction at the level of Ring A) than that in the higher parts (about 25% at the level of Ring D). It may be noted that the distribution of the standard deviation is also affected by the choice of load functions.

The variation of pressure reduction with height is difficult to characterise (Figs 12.60a and b). However, it becomes much simpler if the axisymmetric components are omitted: there is a large pressure reduction at each of the ring levels and a smaller reduction between them after the flow channel has been stabilised (Fig. 12.60d). This phenomenon may be related to the higher storing pressures at the ring levels. This pressure distribution has a nice trend in the form of the Janssen filling pressure distribution. It may be fitted to a value of about half (0.48) the Janssen storing pressure ($k=0.427$). The fitted curve is also shown in Fig. 12.60d for comparison.

The wall pressure during discharge can be obtained by superimposing the discharge pressure onto the storing pressure. The results on the meridian directly above the outlet at 15 minutes after the start of discharge are shown in Fig. 12.61. The storing pressure is also shown for comparison. The pressure during discharge is lower than the storing pressure in most places. At the level of Ring B, the pressure is effectively reduced to zero. However, the pressure between $z=3.8$ and $4.6m$ is slightly higher during discharge than that before discharge, which was very low.

The inferred pressure distributions at the level of Ring A at several instants after the start of discharge are shown in Fig. 12.62a. The first curve is almost unaffected by the discharge. A minute later, the pressure at the channel centre has reduced by 41kPa and by two minutes has come back to 34kPa. The low pressure region is fairly stable, and small peaks begin to develop at the sides of the pressure depression. Beyond these peaks, there are a pair of smaller pressure depressions. There is a very uniform pressure reduction of about 5kPa beyond $\theta=60^\circ$. However, this value has a high uncertainty with a standard deviation of the same order (Fig. 12.62a).
The corresponding pressure distributions at Ring B are shown in Fig. 12.62b. Similar phenomena occur, but the development of the pressure reduction is slower and slightly greater (45kPa). However, the final pressure drop (32kPa) is very similar to that at Ring A and is much smaller than the peak value. The same little wave at the edges of the channel and the small depression zones beyond them are also seen. The pressure reduction beyond θ=60° is significant here with an average of about 10kPa.

In Fig. 12.62c (Ring C), the pressures are slower to reach their final stable values. The peak and final pressure drops at the centre of the channel are only 16 and 12 kPa respectively, a significantly smaller change than that at Rings A and B. By contrast, the wave at the edge of the channel is very strong, with a 19kPa maximum pressure increase. The small depression zone beyond it is hardly seen here.

The pressure pattern in Ring D is shown in Fig. 12.62d. As discussed earlier, the pressure rises in the first few minutes and reaches a peak value of 21kPa at the centre. The centre of the higher pressure starts to depress and the pressures also drop a little elsewhere. This process continues and it takes several minutes to form the final pressure distribution, which again has the same shape as in the other rings. The maximum pressure reduction at the centre is about 16kPa, which is subjected to variation with time as shown in Figs 12.60a and b. The wave at the edge of the channel is also strong, with a 10kPa maximum pressure increase. The depression zone beyond it, seen in Rings A and B, is not seen here.

Figure 12.62d also shows the process of formation of the higher pressure peaks at the edges of the flow channel. Pressures rise before the flow channel reaches as high as the level. When the flow channel reaches the level, the pressure at the centre of the flow channel reduces quickly to a value lower than the storing pressure. However, the high pressure at the edges of the flow channel is retained for a long time (from 19kPa at t=3min to 10kPa) and remains positive (higher than the storing pressure) after the flow channel has become stable. This could well represent also the
formation of the peaks at the edges of the flow channel at other levels, but the
duration of this process may be much quicker in the lower parts of the silo, so that
the 1 minute sampling interval was too short to detect it.

The pressure distributions at each of the four ring levels at 10 minutes after the start
of discharge are compared in Fig. 12.63. The peaks at the edges of the flow channel
are clearly seen at all levels. The position of the peak varies from $\theta=29.5^\circ$ at the
lowest Ring A to $\theta=33.0^\circ$ at the highest Ring D, from which a fairly vertical flow
channel may be deduced.

The circumferential variation of the discharge pressure is something like the
vibration curve (e.g. displacement vs time) of a single degree of freedom system with
damping. The “coefficient of damping” seems to increase with the height. It is small
at lower levels, so more depression zones are seen outside the main reduction zone in
the flow channel. At higher levels, it is large so that the oscillation feature is
effectively prevented. The pressure reduction at the centre of flow channel decays
with height above the silo floor, but the relationship is not simply linear.
Furthermore, the pressure variation with height is much more complicated if the
variation in the barrels is included, which makes it difficult to use some simple forms
to characterise the discharge pressure on the 2D wall. While the characterisation of
the discharge pressure is important for engineering applications and standards, it is
beyond the scope of this thesis.

The determination of flow channel boundaries from the inferred discharge pressure
distribution is not straightforward. Although the block edge angles of the simple
inference in Section 12.7.3 matched astonishingly well with the results from
residence time measurements (Fig. 12.48), the shape of the pressure reduction in the
flow channel is clearly not rectangular but triangular or parabolic (Fig. 12.63).

The above results again demonstrated that there are peak pressures at the edges of the
flow channel as in some previous speculative suggestions (Jenike, 1967; Wood,
If the discharge pressure were always positive (increase) at the peak and negative (reduction) within the flow channel, it would be naturally expected that the flow channel boundary could be taken as the point where the pressure change is zero. However, the above results show that the edge peak pressures may also be negative, mainly because of the overall change of pressure (Harmonic 0). A simple adjusted proposition may be that the flow boundary lies where the pressure change is zero after the axisymmetric term (Harmonic 0) has been omitted.

Figure 12.64 shows the pressure distributions at the four ring levels with the components in Harmonics 0 and 1 omitted. The component of Harmonic 1 has been omitted because it has a relatively large standard deviation. A pair of positive peak pressures at the edges of the flow channel are seen at all the levels now. It may be noted that the value of the peak decreases in the height. The position of the zero pressure change varies from 17.5° to 20.3°. The results are shown in Fig. 12.65. This new position defines the channel as slightly smaller than the flow channel inferred from the residence time measurements at the lower levels of Rings A and B, but they match surprisingly well at the higher levels of Rings C and D.

The above process may be applied at any height. In this way the complete boundary of the flow channel, as it intersects the wall, may be defined (Fig. 12.66).

12.7.4.7  **Comparison with simple inference results**

To assess the validity of the simple method of inferring pressures from ring strains in Section 12.7.3, the results from the rigorous inference here (Figs 12.62 and 63) may be compared with those from the simple inference in Figs 12.40 and 41. At Ring A, the simple results (Fig. 12.40a) have a very similar pressure distribution to those of the rigorous inference (Fig. 12.62a). Both the pressure reduction at the centre and the position of the peaks at the edges of the flow channel show good agreement between the two methods. However, the value of the peaks at the edges of flow channel is much smaller in the simple method. The simple results also show a significant pressure drop on the opposite side of the silo from the flow channel. Similar
conclusions can be drawn for the discharge pressure distributions at the level of Ring B.

Unfortunately, such a good agreement cannot be found at the higher levels of Rings C and D. At Ring C, the simple method (Fig. 12.40c) produces a pressure reduction at the centre of the flow channel (33kPa) twice as large as the rigorous result 16kPa (Fig. 12.62c), and it also significantly underestimates the peak value at the edges of the flow channel (8kPa vs. 19kPa). The simple method also significantly overestimates the pressure reduction at the centre of flow channel at Ring D. Furthermore, the stage of formation of the edge peaks as seen in Fig. 12.62d is totally lost in the simple method (Fig. 12.40d).

The positions of edge peaks from the two methods are plotted in Fig. 12.65. The simple method results in very accurate assessments of the flow channel edges at the lower levels, but produces a much larger angle at higher levels.

In conclusion, the simple method produces fairly good results at the lower levels. At higher levels, it may result in errors in excess of 100%.

12.8 Correlations between discharge pressures and flow patterns

The pressure studies in this chapter have concentrated on fully eccentric discharge, because this is the least understood phenomenon of those explored here and produces the most practically damaging pressure distributions. This section therefore focuses particular attention on fully eccentric discharge.

The flow patterns shown in Chapter 7 indicated that the flow of iron ore was always in a rather narrow funnel flow mode. Concentric discharge produced a flow channel which penetrated to the top surface without intersecting the wall, and fully eccentric flow caused a narrow channel of flowing material to move in contact with the wall,
rising the full height to the surface. The flow channel in Tests PFA and PFB was of almost constant cross-section until it approached the surface. The circumferential range of its contact with the wall was ±21° from the outlet position.

The pressure distribution found in Test PFA also gives a strong indication of the dimensions of the flow channel, which is in itself a strong demonstration of the correlation of flow pattern with wall pressures. The boundary of the flow channel was estimated from the pressure distributions of Fig. 12.64, and also showed an almost vertical channel (Fig. 12.65). Of the alternative measures of the boundary of the moving solid, the estimates taken from the complete inferred pressure distribution are the best, and these indicate that the channel edge is at a circumferential coordinate of about ±18° at Rings A and B, increasing to about ±23° at Ring D. These measures closely match the estimates from the flow pattern measurement.

A further measure of the flow channel can be made from the inferred pressures. If the flow channel boundary intersection with the wall is known and the centre of the flow channel is assumed, together with a circular periphery (as seen in the flow pattern measurements), then the radius of the channel of flowing solid at each height can be estimated (see Eqs 7.4 and 7.5 in Chapter 7).

This could be a precise method of measuring flow channel geometry, and may be used to great advantage in future experiments because it does not require the seeding of markers which interfere with the filling process and could result in a significant change in the pressure distribution.

In conclusion, it is evident that there is a very high correlation between the measurements of flow pattern and the resulting induced wall pressures.
12.9 Conclusions

The rigorous technique developed in Chapters 8-10, together with a few simple techniques, have been used on a limited part of the experimental data set of the tests in the British Steel silo.

The finite element modelling of the silo has been studied comprehensively. In the strain pressure inference process, it is very important to correctly model ring stiffeners, especially the connection position between the barrel and the ring, the vertical and radial eccentricities and any axisymmetric imperfections. Any incorrect modelling of these aspects could result in large errors in the theoretical strain predictions and consequently big errors in the inferred pressure distributions.

The structural behaviour of the silo under both axisymmetric and harmonic pressure distributions has been studied. The magnitude of an axisymmetric pressure may be simply related to the hoop stress measured in barrel sections and the ring stains using the effective ring cross section analysis. However, these cannot be extended to cases of unsymmetrical pressures.

A very unsymmetrical filling pressure distribution has been found in Test PFA. The variation of the axisymmetric components with height is not smooth at all: higher pressures were found at the ring levels and lower pressures in the barrel sections in all four analysed experiments, probably caused by the axisymmetric inward imperfections at the ring locations. A large variation was found between different predictions from classical theories (over 100%) and design codes (up to 40%). If the mean pressure at each level is fitted to a Janssen pressure distribution, it is close to the largest theoretical prediction.

Attention has been principally focused on the pressures occurring during eccentric discharge in this chapter, as these are the tests in which the strongest correlation between flow pattern and wall stresses may be expected to be found, and where the record of silo failures is particularly serious.
The flow channel has been seen to take about 8 minutes to become stable and this early phase of discharge has been identified as the most damaging period to the structure. Above the outlet, higher pressures were seen at each level before the flow channel reached it. After the flow channel passes a level, the pressure at the centre of the flow channel drops quickly but those at the edges reduce slowly, to form a highly unsymmetrical pattern on the silo wall in a form similar to that speculated by several researchers in the past. The pressures were seen to be highly correlated with the flow pattern, and the detailed form of the pressure variation has been identified.

It is believed that this is the first time that the changing form of the eccentric discharge pressure distribution at the start of discharge has been observed, and indeed that the critical importance of the development of the flow channel to the silo's structural integrity has been identified.

From the viewpoint of wall strains, the worst condition for each part of the silo occurs at a different time. Moreover, because the peak structural response occurs during flow channel development and relates to a complex pattern of pressure at every level, the worst condition cannot be deduced as the envelope of the worst conditions at all measured times. Indeed if only pressures are measured, it is difficult to decide how the worst condition for the structure could be identified.

12.10 Notation

\[
\begin{align*}
&W \quad \text{weighting matrix} \\
&S \quad \text{sensitivity matrix} \\
\{a\} \quad \text{load parameters} \\
\{e\} \quad \text{strain residuals} \\
\{\epsilon\} \quad \text{strain observations} \\
A \quad \text{cross-sectional area} \\
\text{CoV} \quad \text{coefficient of variation}
\end{align*}
\]
E  Young’s modulus
I  second moment of area of the cross section
K_c  stiffness of rotational constraints on the effective ring cross section
M  bending moment
N  ring thrust
P  line loading
PoF  percentage of fit
R  radius
  e  eccentricity of cross section
  k  lateral pressure ratio
  l_e  effective length of wall which acts with a ring stiffener
m  line moment
p  distributed pressure
r  radius/radial coordinate
t  thickness of a wall/ring
x_c, z_c  coordinates of the centroid of an effective ring cross section
  φ  effective internal friction angle of solids
  φ_r  angle of repose of solids
  μ  wall friction coefficient
  v  Poisson’s ratio
  σ  stress/standard deviation

Subscripts
  a  load parameter
  e  effective cross section
  i  inner
  o  outer
  r  radial
  ε  strain
  θ  circumferential
Appendix 12A  Stresses in an annular plate under constant internal pressure

A stiffening annular plate may be subjected to both radial stress and bending moment at the inner edge, but the radial stress may be predominant. The stress solutions for a disc (the annular plate) with a unit thickness under constant internal pressure p are (Dugdale and Ruiz, 1971; Timoshenko and Goodier, 1970):

\[
\sigma_r = \frac{r^2_p}{r^2_0 - r^2_i} \left( 1 - \frac{r^2_i}{r^2} \right)
\]  
(12A.1a)

\[
\sigma_\theta = \frac{r^2_p}{r^2_0 - r^2_i} \left( 1 + \frac{r^2_0}{r^2} \right)
\]  
(12A.1b)

where \( r_i \) and \( r_o \) are the inner and outer radii of the disc respectively and \( r \) is the radial coordinate of a point in the disc. For the ring stiffener in the BS silo, \( r_o=2203\text{mm} \) and \( r_i=2103\text{mm} \). The stresses at the inner and outer edges of the disc are

\[
\sigma_n = \frac{r^2_p}{r^2_0 - r^2_i} \left( 1 - \frac{r^2_o}{r^2_i} \right)
\]  
(12A.2a)

\[
\sigma_\theta = \frac{r^2_p}{r^2_0 - r^2_i} \left( 1 + \frac{r^2_o}{r^2_i} \right)
\]  
(12A.2b)

\[ \sigma_m = 0 \]  
(12A.2c)

\[ \sigma_{m0} = 2 \frac{r^2_p}{r^2_0 - r^2_i} \]  
(12A.2d)
The circumferential strains at the inner and outer edges can be obtained using Hooke's law:

\[
\varepsilon_{ai} = \frac{r_1^2 \rho}{r_0 - r_1} \left(1 + \frac{r_0^2}{r_1} \right) \frac{1}{E} \left(1 - \frac{r_0^2}{r_1} \right) \frac{1}{E} = \frac{r_1^2 \rho}{r_0 - r_1} \frac{1}{E} \left(1 - \frac{r_0^2}{r_1} \right) \frac{1}{E} \left(1 + \frac{r_0^2}{r_1} \right) \frac{1}{E} 
\]  

(12A.3a)

\[
\varepsilon_{ao} = \frac{2 \pi r_1^2}{r_0^2 - r_1^2} \frac{1}{E_t}
\]  

(12A.3b)

in which E and \( v \) are Young's modulus and Poisson's ratio respectively. The relative difference between them is

\[
\frac{\varepsilon_{ai} - \varepsilon_{ao}}{\varepsilon_{ao}} = \frac{(1 + v) \frac{r_0^2}{r_1^2} - 1 - v}{1 - v + (1 + v) \frac{r_0^2}{r_1^2}}
\]  

(12A.4)

For the given values of \( r_i \) and \( r_o \) for the ring stiffener in the BS silo and \( v=0.3 \), Equation 12A.4 results in a difference of 5.95%. This agrees very well with the finite element prediction of 6.05%.

The circumferential stress in the ring cross-section given by thin ring theory is

\[
\sigma_{thin} = \frac{pR}{t} = \frac{p(r_o + r_i)}{2(r_o - r_i)}
\]  

(12A.5)

in which the mean radius \( R \) and the thickness \( t \) of the ring are defined by

\[
R = \frac{r_o + r_i}{2}
\]  

(12A.6a)

\[
t = r_o - r_i
\]  

(12A.6b)
Compared with the circumferential stress at the inner edge given by thick ring theory (Eq. 12A.2b), the relative difference is

\[
\frac{\sigma_{\text{thin}} - \sigma_{\text{th}}}{\sigma_{\text{th}}} = \frac{-(r_o - r_i)^2}{2(r_o^2 + r_i^2)} = \frac{-2t^2}{8R^2 + t^2} \approx \frac{-t^2}{4R^2} \tag{12A.7}
\]

Equation 12A.6 indicates that the circumferential stress predicted by thin ring theory (Eq. 12A.5) is a good approximation of that at the inner edge predicted by thick ring theory. For a very thick ring with \( R/t = 5 \), the error of this approximation is 1%. Because \( \sigma_o \) decreases when \( r \) increases (Eq. 12A.1b), the thin ring theory prediction of the circumferential stress is larger than the thick ring theory prediction in most parts of the cross section.
Fig. 12.1 Convergence of finite element mesh (unit = mm)
(a) Model 3a: Convention line elements  
(b) Model 5a: Joint elements

Fig. 12.2 Modelling of ring stiffeners: FE mesh

Fig. 12.3a Modelling of ring stiffeners: different FE mesh under harmonic pressures: strains in inner ring gauge
Fig. 12.3b Modelling of ring stiffeners: different FE mesh under harmonic pressures: strains in outer ring gauge

Fig. 12.3c Modelling of ring stiffeners: different FE mesh under harmonic pressures: errors in reference to axisymmetric elastic body analysis

Fig. 12.4 Axisymmetric imperfections in the BS silo adjacent to the rings
Fig. 12.5 Local imperfections in the BS silo adjacent to the rings

Fig. 12.6a Effect of a 4mm circumferential inwards imperfection on ring strains

Fig. 12.6b Effect of a 4mm circumferential inwards imperfection on ring strains: relative errors
Fig. 12.7 Finite element modelling of the BMHB silo
Fig. 12.8 Deformed silo section and angle ring

Fig. 12.9 Strains in the wall near the angle stiffener

Fig. 12.10 Circumferential strain distribution on the upper surface of the horizontal leg of the ring stiffener
a) Cross-section of effective ring and load  

b) Equivalent loads

c) Uniform radial line load  

d) Uniform line moment

Fig. 12.11 Effective ring
Fig. 12.12 Patch harmonic pressures on the silo centred on Ring C

Fig. 12.13 Circumferential strains on different rings in outer ring gauges at θ=0°
Fig. 12.14a Circumferential strain on Ring C in outer ring gauge at θ=0°

Fig. 12.14b Circumferential strain on Ring C at 9.5mm from outside edge at θ=0°: Effect of asymmetry of the ring

Fig. 12.15 Circumferential strain on a simple ring at θ=0°
Fig. 12.16 WFB: Strains on Barrel 1

Fig. 12.17a WFB: Membrane strains on Barrel 1

Fig. 12.17b WFB: Bending curvatures on Barrel 1
Fig. 12.17c WFB: Orientations of principle strains on Barrel 1

Fig. 12.18 Inferred pressures from strains measured in the water test WFB

Fig. 12.19a Strain history at position a on Barrel 1 for Test PFA
Fig. 12.19b Strain history at position b on Barrel 1 for Test PFA

Fig. 12.19c Strain history at position a-c on Ring A for Test PFA

Fig. 12.19d Strain history at position a-c on Ring C for Test PFA
Fig. 12.20a Storing membrane strain on Barrel 1 for Test PFA

Fig. 12.20b Storing membrane strain on Barrel 2 for Test PFA

Fig. 12.20c Storing strain on Ring A for Test PFA
Fig. 12.20d Storing strain on Ring C for Test PFA

Fig. 12.21a Mean Storing pressure from Simple interpretation: Test PFB

Fig. 12.21b Mean Storing pressure from Simple interpretation: Test PFA
Fig. 12.21c  Mean Storing pressure from Simple interpretation: Test PFC

Fig. 12.21d  Mean Storing pressure from Simple interpretation: Test PHA

Fig. 12.22  Load functions
Fig. 12.23a Storing pressure in Test PFA: Convergence of circumferential load functions: Percentage of fit

Fig. 12.23b Storing pressure in Test PFA: Convergence of circumferential load functions: Estimated standard deviation of strain observations

Fig. 12.24a Storing pressure in Test PFA: Convergence of vertical load functions: Percentage of fit
Fig. 12.24b Storing pressure in Test PFA: Convergence of vertical load functions: Estimated standard deviation of strain observations

Fig. 12.25a Observed vs inferred membrane strains on Barrel 2 Test PFA after filling

Fig. 12.25b Observed vs inferred membrane strains on Ring A: Test PFA after filling
Fig. 12.26 Test PFA: Axisymmetric pressure distribution after filling

Fig. 12.27a Test PFA: Normal wall pressure distribution after filling (3D surface)
Fig. 12.27b Test PFA: Normal wall pressures after filling (display with a different rotation angle as Fig. 12.27a)

Fig. 12.27c Test PFA: Normal wall pressures after filling (contours), kPa
Fig. 12.28a Test PFA: Standard deviation of normal wall pressures after filling

Fig. 12.28b Test PFA: Standard deviation of normal wall pressures after filling (contours), kPa
Fig. 12.29a Test PFA: Normal storing pressures on different vertical lines

Fig. 12.29b Test PFA: Standard deviation of pressures on different vertical lines

Fig. 12.30a Test PFA: Normal Storing pressures at different heights
Fig. 12.30b Test PFA: Standard deviation of pressures at different heights

Fig. 12.31 Test PFA: Axisymmetric pressure after filling: Fit to Janssen distribution

Fig. 12.32 Test PFA: Axisymmetric pressure after filling: Comparison with simple interpretation
Fig. 12.33a Test PFA: Axisymmetric pressure after filling: Comparison with predictions by classical theories

Fig. 12.33b Test PFA: Axisymmetric pressure after filling: Comparison with predictions by design codes

Fig. 12.34a Test PFA: Discharge strains on Ring A (variation with time)
Fig. 12.34b Test PFA: Discharge strains on Ring B (variation with time)

Fig. 12.34c Test PFA: Discharge strains on Ring C (variation with time)

Fig. 12.34d Test PFA: Discharge strains on Ring D (variation with time)
Fig. 12.34e Test PFA: Discharge strains on different rings

Fig. 12.35a Test PFA: Discharge strain in outer gauges on Ring B

Fig. 12.35b Test PFA: Discharge strain in outer gauges on Ring C
Fig. 12.36a Test PFA: Discharge strains on Barrel 1 (Variation with time)

Fig. 12.36b Test PFA: Discharge strains on Barrel 1 after 2 minutes since discharge

Fig. 12.37 Test PFB: Discharge strains on Ring A
Fig. 12.38 Test PFC: Discharge strains on Ring A

Fig. 12.39 Test PHA: Discharge strains on Ring A

Fig. 12.40a Test PFA: Incipient discharge pressure on Ring A
Fig. 12.40b Test PFA: Incipient discharge pressure on Ring B

Fig. 12.40c Test PFA: Incipient discharge pressure on Ring C

Fig. 12.40d Test PFA: Incipient discharge pressure on Ring D
Fig. 12.41 Test PFA: Incipient discharge pressure at 17/11/93 23:15

Fig. 12.42 Test PFA: Incipient discharge pressure on Ring A

Fig. 12.43 Test PFA: Development of incipient discharge pressure at θ=0°
Fig. 12.44 Test PFA: Pressure coefficient at 17/11/93 23:15

Fig. 12.45a Test PFA: Variation of harmonics with time: Harmonic 0

Fig. 12.45b Test PFA: Variation of harmonics with time: Harmonic 1
Fig. 12.45c Test PFA: Variation of harmonics with time: Harmonic 2

Fig. 12.45d Test PFA: Variation of harmonics with time: Harmonic 3

Fig. 12.45e Test PFA: Variation of harmonics with time: Harmonic 4
Fig. 12.45f Test PFA: Variation of harmonics with time: Harmonic 5

Fig. 12.45g Test PFA: Variation of harmonics with time: Harmonic 6

Fig. 12.45h Test PFA: Variation of harmonics with time: Harmonic 7
Fig. 12.45i Test PFA: Variation of harmonics with time: Harmonic 8

Fig. 12.45j Test PFA: Variation of harmonics with time: Harmonic 9

Fig. 12.45k Test PFA: Variation of harmonics with time: Harmonic 10
Fig. 12.45l Test PFA: Variation of harmonics with time: Harmonic 11

Fig. 12.45m Test PFA: Variation of harmonics with time: Harmonic 12

Fig. 12.46 Three-parameter model of discharge pressure distribution
Fig. 12.47 Test PFA: Discharge pressure at 17/11/93: 23:15: Blocks of pressure

Fig. 12.48 Test PFA: Incipient discharge pressure at 17/11/93: Blocks of pressure

Fig. 12.49a Test PFA: Simplified blocks of pressure: variation of load coefficient $p_a$ with time
Fig. 12.49b Test PFA: Simplified blocks of pressure: variation of load coefficient $p_b$ with time

Fig. 12.49c Test PFA: Simplified blocks of pressure: variation of block edge angle $\theta_0$ with time

Fig. 12.50a Test PFB: Incipient discharge pressure at Ring A
Fig. 12.50b Test PFB: Incipient discharge pressure at Ring B

Fig. 12.50c Test PFB: Incipient discharge pressure at Ring C

Fig. 12.50d Test PFB: Incipient discharge pressure at Ring D
Fig. 12.51a Test PFC: Incipient discharge pressure at Ring A

Fig. 12.51b Test PFC: Incipient discharge pressure at Ring B

Fig. 12.51c Test PFC: Incipient discharge pressure at Ring C
Fig. 12.51d Test PFC: Incipient discharge pressure at Ring D

Fig. 12.52 Block edge angle $\theta_0$ for PFA, PFB and PFC

Fig. 12.53a Incipient discharge pressure in Test PFA: Convergence of circumferential load functions: Percentage of fit
Fig. 12.53b Incipient discharge pressure in Test PFA: Convergence of circumferential load functions: Estimated standard deviation strain observations

Fig. 12.54a Test PFA: Observed vs inferred strains on Ring A at 10 minutes since discharge

Fig. 12.54b Test PFA: Observed vs inferred membrane strains on Barrel 4 at 10 minutes since discharge
Fig. 12.55a  Test PFA: Incipient discharge pressure at t=1 min (kPa)
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-0.48 Janssen storing pressure (k=0.427)

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Chapter 13

CONCLUSIONS

13.1 Introduction

Silo failures are very common in all countries in the world, and represent a source of major economic loss. The reasons for this high failure rate are believed mainly to be the poor understanding of solids flow patterns and pressures on the silo wall exerted by the stored solids.

Most previous experimental observations on solids flow patterns in silos are at model scale and have been inadequate to define the flowing channel geometry with certainty. Scientific measurements of flow patterns in full scale silos are almost unknown. Extrapolation of the observations made in laboratory models to full scale industrial silos is most uncertain.

Many classical theories and numerical methods are available for calculating wall pressure distributions in silos but their predictions are often a very poor match with experimental observations, especially during discharge. Large differences may be found in the codified design pressures from different countries. Moreover, many phenomena observed in experiments such as the random fluctuation of pressures with time cannot yet be described by existing theories and numerical methods.

The main body of this thesis has been concerned with the interpretation of solids flow patterns and wall pressures in full scale silos. The data processed in this thesis were obtained from full scale silo tests at the University of Edinburgh. This chapter
presents a review of the conclusions drawn in the previous chapters and sets out some topics relevant to the thesis for future study. Work under taken as part of the study but off the main stream of the thesis is presented in Appendices A to C and is not reviewed here.

13.2 Flow patterns in full scale silos

13.2.1 Interpretation techniques

The technique of residence time measurement was adopted to measure the solids flow patterns in the experiments analysed in this thesis. The chief difficulty in using this technique is the interpretation of the results. In this thesis, four techniques have been used to interpret solids flow patterns from residence time measurements. They are:

a) plotting residence time contours;
b) plotting mean velocity contours;
c) graphical visualisations; and
d) analysing the emerged mass from each seeded level.

All but the first technique have been newly developed in the thesis. The development of these techniques makes it possible to transform residence time measurements quantitatively into flow channel boundaries. Each of these techniques is briefly reviewed as follows.

13.2.1.1 Plotting residence time contours

Plotting contours of residence time in vertical and horizontal sections through the silo is the traditional method of presenting this information. Remaining mass (in Chapter 5) and remaining volume (in Chapter 7) were used instead of residence time in this thesis. They are effectively residence time measures, but are independent of solids
flow rate variations and a high value represents an early part of the discharge and low value a late part.

If a silo discharges solids under conditions of mass flow, the markers at any level emerge at substantially the same time. Contours of residence time at a given level are not closely spaced (Fig. 5.2). By contrast, if a silo exhibits funnel flow, the material immediately above the outlet emerges first, followed by solid from higher up the flow channel, then material from higher zones near the wall and finally solid from lower regions which has remained stationary throughout most of the discharge process. Contours of residence time at a given level are closely spaced across the flow channel boundary (Fig. 5.3).

However, apart from their use to distinguish mass flow from funnel flow, residence time contours can only be used as a qualitative indicator of the flow pattern. It is rather difficult to identify the flow channel boundary from them. Nevertheless, the residence time of a marker is the only information which can be easily obtained from a full scale experiment and it must be the starting point of all other interpretations.

13.2.1.2 Plotting mean velocity contours

Contours of mean velocity represent a considerable improvement on residence time readings (Figs 5.6 and 7). The mean velocity of a marker is given by the length of its shortest path from the initial position to the centre of the outlet divided by the residence time. The boundaries of the flow channel are qualitatively better defined, particularly in vertical sections through the silo, but still cannot be quantitatively defined. Because of the assumption that a marker takes the shortest path with a mean velocity, mean velocity contours cannot be used as a precise tool and cannot be used as the basis for more accurate interpretations.
13.2.1.3  *Graphical visualisations*

The technique of graphical visualisation described in Chapter 3 studies the flow pattern as a time-dependent phenomenon using residence times as data. There are two essential assumptions about the nature of the flow: the path taken by a marker and its speed towards the outlet. Many different options for these assumptions are available in the program suite FLOWVIS developed in this thesis. The user may improve the quality of the result by modifying the assumptions used in the interpretation.

The display of the solids flow pattern in a graphics display format has immediate appeal to industrial engineers, and can be used to gain a better understanding of many industrial materials handling problems. The program developed in this thesis is very flexible and is applicable to silos of different configurations and to any stored solid. It has great potential for industrial application. It is particularly useful for interpreting complex flow problems such as those in the British Gypsum silo (Chapter 5). However, addressing the flow as a time-dependent phenomenon means that it is difficult to translate the flow pattern into the form of commonly accepted simple flow images (Fig. 1.2). The flow channel boundary is also not clearly defined by this method.

13.2.1.4  *Analysis of emerged mass from each seeded level*

Because of the drawbacks in the above techniques, the analysis of emerged mass from each seeded level was developed to transform the flow pattern back into a classical image (Fig. 1.2). The method assumes that each marker represents the solid surrounding it at the given level. By integrating the emerged mass from the level over time, a normalised mass-time curve is obtained. A comprehensive discussion was presented in Chapter 5 of the characteristics of the normalised mass-time curve for mass and funnel flow.

Under conditions of mass flow, the normalised mass-time curve (Fig. 5.8a) has the following characteristics:
a) For a horizontal level at a given height, it may be a smooth curve. The starting point on the normalised time axis (the time at which the first part of the mass from the layer first reaches the outlet divided by the maximum residence time) is not small compared with unity, but depends on its distance from the initial top surface in the silo.

b) The curves for two or more horizontal layers at different heights remain almost parallel to each other and do not intersect. Their separation depends on the separation of their initial vertical coordinates.

Under conditions of funnel flow, the normalised mass-time curve (Fig. 5.8b) has the following characteristics which are different from those under mass flow:

a) For a horizontal level at a given height, there is a plateau in the curve. The length of the plateau is related to the duration of the stable channel at the level. However, this plateau may disappear if the level is close to the top surface. The starting point for the curve on the normalised time axis is small compared with unity.

b) The normalised mass-time curves for two or more horizontal layers at different heights intersect each other: some of the mass at the lower level reaches the outlet quite early but other parts do not appear until much later or not discharged at all.

It is thus possible to identify the flow pattern (mass or funnel flow) from a single normalised mass-time plot. For funnel flow, the plan cross-sectional area of the flow channel is defined from the plots. The flow channel boundary may be defined by assuming the shape and centre of the flow channel at each level. Alternatively, these latter parameters may be chosen from the residence time contours at each level, giving greater precision to the flow channel boundary.
The technique has been applied successfully in both the British Gypsum (Chapter 5) and British Steel tests (Chapter 7). This is believed to be the first time that the flow channel boundary has been quantitatively defined from residence time measurements.

13.2.2 Flow patterns in the gypsum silo at British Gypsum

The British Gypsum silo was recently constructed at British Gypsum's new East Leake plant (Chapter 4). It was fitted with an inverted cone bin discharge aid and air pads on the hopper section. It contained ground gypsum powder.

In Chapter 5, all the above interpretation techniques were used to interpret the experimental data from this silo. The tests revealed behaviour which was not expected in the silo. The conclusions are summarised as follows.

a) With injected air and the flow aid device in operation, a complex flow pattern occurred. Narrow pipe funnel flows with 3D features rose to the surface and resulted in rapid movement of a limited amount of material from the top surface to the outlet. This pattern was superimposed on a broad funnel flow. The location of the narrow pipe funnel flows was unpredictable and varied randomly from one experiment to another. It is also possible that the narrow pipe flow changed from time to time in a random manner even in a single experiment.

b) With the injected air and the flow aid device out of operation, the flow pattern in the silo was a broad-based funnel flow, occupying a zone as wide as the inverted cone and its surrounding gap, giving them the appearance of a single large outlet. The flow channel was fairly concentric in the lower parts of the silo but became unsymmetrical at higher levels. It touched one side of the silo wall half way up the cylinder and contained almost the entire cross section of the cylinder at the top surface, so that the surface appearance was mass flow.
13.2.3 Flow patterns in the iron ore pellets silo at British Steel

The new cylindrical steel silo at the British Steel Research Laboratories was specially designed for this research project (Chapters 6 and 11). It was 4.2m diameter and 9.5m high and designed for experiments under both concentric and eccentric discharge. Iron ore pellets were used in the experiments analysed in this thesis.

All the four methods described above were used to interpret the data (Chapter 7). The conclusions from the experiments are as follows.

a) Steep-sided funnel flow patterns occurred under both concentric and eccentric discharge. The plan cross-section of the flow channel was close to but not exactly circular.

b) Under concentric discharge, the axial symmetry of flow pattern was correlated to the filling height. If the filling height was small, the flow pattern was close to but not exactly axisymmetrical. It could be very unsymmetrical in the higher parts of the silo when the filling height was large. However, when the asymmetry of the flow pattern was expressed in terms of the angle of inclination of the centre of the flow channel, it was found to be relatively invariant with height. The magnitude and direction of this angle was found to vary randomly.

c) Under eccentric discharge (both fully eccentric and half eccentric), the flow pattern in the same silo was not simply related to the concentric discharge pattern as a horizontal translation of the flow channel. The flow channel tended to incline towards the nearest wall, but remained highly symmetrical relative to the vertical diametrical plane of symmetry through the outlets.

d) Under fully eccentric discharge, the flow pattern developed as a narrow channel against the silo wall when the discharge rate was low, and had a form similar to that proposed in earlier more speculative writings on the subject (e.g. Wood, 1983; Rotter, 1986b).
e) Under fully eccentric discharge, the size of the flow channel appeared to increase dramatically at the beginning of the second stage of discharge and formed a second wider stable flow channel. However, this increase in the size of the flow channel does not appear to be deleterious to the safety of the structure.

f) Under both concentric and eccentric discharge flows, the shape of the flow channel in the early stages of discharge may be described by a two parameter model (Eq. 7.6) in the lower parts of the silo. The flow channel angle $\theta_f$ and the maximum radius of the flow channel $R_f$ may describe the shape satisfactorily,

g) The value of $\theta_f$ may vary either linearly with the estimated mean vertical stress at the silo bottom (Eq. 7.10) or as a probability distribution (Eq. 7.13).

h) The maximum flow channel radius $R_f$ may be approximately described as linear in the square root of the estimated mean vertical stress at the silo bottom (Eq. 7.15). The effect of the eccentricity of discharge on $R_f$ may be approximately described by a power law model (Eq. 7.17).

13.3 Inference of wall pressures from wall strain measurements

13.3.1 Interpretation techniques

Three methods have been used in Chapter 12 to infer the wall pressures exerted by bulk solids from wall strain measurements. These were

a) simple inference of axisymmetrical components of pressures;

b) simple inference of local discharge pressures from ring strains; and

c) rigorous statistical inference of complete pressure distributions.
The quality of the outcome for each of these methods is very dependent on the accuracy of structural modelling of the silo and the quality of the strain measurements. Each of the methods is briefly reviewed separately as follows.

13.3.1.1  **Simple inference of axisymmetric components of pressures**
This traditional method simply relates the pressure $p$ at a barrel level to the mean observed circumferential stress $\bar{\sigma}_0$ ($\bar{\sigma}_0 = pR/t$) at that level. At ring levels, the pressure must be obtained with greater care by dividing the mean strain observations by the ring strain under unit pressure, which may be obtained by finite element calculation or an effective ring analysis.

The validity of this simple method is subjected severe limitations, including that
a)  the structure must be essentially axisymmetric;
b)  the pressure must be smoothly varying;
c)  all strain gauges in the barrel must be far from discontinuities or boundaries;
d)  either the pressure is essentially axisymmetric or there are many strain gauges equally located around the circumference of the silo;
e)  the method can only be used to determine the axisymmetrical components of pressures; and
f)  the pressure inferred from ring strain observations contains errors if the component due to frictional loadings cannot be identified separately.

In many previous studies, these conditions have not been honoured and consequently the method has been overused.

13.3.1.2  **Simple inference of discharge pressures from ring strains**
A simple and quick method was devised in Chapter 12 to infer the approximate changes in pressure at a ring level from the recorded ring strains. It follows the same procedure as the rigorous method (Chapter 8), but assumes that strains in a ring are caused by only the local pressure near it. The sensitivity matrix is obtained by
applying characteristic forms of local pressure variations around the ring. The whole silo structure is modelled in the structural analyses. This enables the pressure at each ring level to be inferred from the strain readings on that ring alone.

Compared with the rigorous method, this method uses three additional assumptions:

a) the vertical variation of pressure near the ring is known;
b) the influence of long-wave bending of the silo shell can be ignored; and
c) the effect of wall frictional tractions on ring strains can be ignored.

A comparison with the rigorous inference results in Chapter 12 showed that this method is able to capture the main features of the pressure distribution at all ring levels in the British Steel silo. The accuracy of this simple inference method was very good for some rings, but the errors were as big as 100% for others. It is probable that the validity of this method is highly related to the structural arrangement of the silo and the form of the pressure distribution.

13.3.1.3 Rigorous statistical inference of complete pressure distribution

A rigorous statistical method was developed in Chapter 8 to infer silo wall pressures from wall strain measurements. The theory was presented in its fullest form, in which all strain readings are used, and the loading parameters include normal pressure, vertical wall friction and circumferential wall friction.

The statistics of the method, including the statistics of the inferred load coefficients, the estimation of the variance of the strain observations, the analysis of strain residuals and the goodness-of-fit, were further discussed in Chapter 9. A lower bound to the variance of the load coefficients was identified. A dependence index between two sensitivity vectors was proposed, which is useful in selecting load functions when a large number of possible load functions are involved and in determining the strain gauge locations when planning an experiment.
Conclusions drawn from the discussions and example analyses in Chapters 8 and 9 are as follows.

a) When planning an experiment, it is important that a thorough structural analysis of the test silo should be made, followed by an investigation of the sensitivity of the loading parameters to the measuring station locations and data accuracy.

b) The coefficient of variation of the inferred pressures can be reduced linearly by reducing the level of noise.

c) The coefficient of variation of the inferred pressures can be reduced linearly by increasing the square root of the number of strain observations.

d) The use of two linearly dependent or nearly linearly dependent sensitivity vectors results in ill-conditioning in the normal equations, leading to failure of the inference.

e) The quality of the load functions, which may be represented by the dependence index, is dependent on the strain gauge placement pattern.

f) Errors in the inferred load coefficients are minimised by using orthogonal load functions.

The fundamental assumption for the method is that the silo structure can be accurately modelled. This requires that the structure is essentially free of defects. Provided that this assumption is satisfied, the method is general and can be applied to many other structures, not only axisymmetric silos. However, great care must be taken in modelling the structure. It is evident, as demonstrated in Chapter 12, that huge errors may arise from conventional simple structural modelling.
Chapter 10 briefly described the development of the program suite SIMA to implement the technique.

13.3.2 Wall pressures in the iron ore pellets silo at British Steel

The silo at British Steel had a 3mm thick cylindrical shell wall with five ring stiffeners (Chapter 11). A few sets of the experimental data were processed in Chapter 12 using the above methods. The conclusions from analyses of the as-filled condition are as follows.

a) The variation of the axisymmetric components of the filling pressures with height was not smooth: higher pressures were found at the ring levels and lower pressures in the barrel sections in all four processed experiments. Because the structure is particularly strong at the ring locations, the high filling pressures at these levels do not affect the silo design in any significant way. However, they may indicate that if a silo instrumentation system depends only on instrumented rings, the deduced pressures may be unrepresentative of the pressure on the remainder of the wall.

b) The filling pressure distribution was very unsymmetrical in Test PFA, which was the only experiment that had been analysed using the rigorous inference method.

c) A large discrepancy existed between different predictions by classical silo pressure theories (over 100%) and design codes (up to 40%).

d) For Test PFA, when the mean pressure was fitted to a Janssen pressure distribution, it was found to be close to the largest theoretical prediction.

The conclusions on the pressures at the beginning of eccentric discharge are as follows.
a) A slight increase in pressure develops at a level above the top of the rising flow channel.

b) After the developing flow channel reaches a level, the pressure at the centre of the flow channel drops quickly but that at the sides of the channel reduces slowly, to form lower pressures at the centre and pressure peaks at the edges in a highly unsymmetrical pattern on the silo wall similar in form to that speculated by several researchers in the past.

c) The flow channel took about 8 minutes to become stable in Test PFA.

d) The pressures occurring in all the flow regimes were highly correlated with the measured flow patterns: no predictive technique for pressures can afford to ignore the flow pattern.

e) The most critical instant under eccentric discharge appears to be during the development of the flow channel, rather than after the flow channel has developed: this appears to be the first observation of this fact, though with hindsight it is possible to see earlier evidence for this condition.

f) The many disasters previously caused by intentional or accidental eccentric discharge are not surprising when the results of these tests are considered.

g) The pressures occurring under fully eccentric discharge are extremely dangerous to the structural integrity of the silo, and deserve very careful further study.

h) The pressures occurring under half eccentric discharge are much less dangerous to the integrity of the silo, but remain significantly unsymmetrical and deserve further careful study to improve current ad-hoc design proposals.
i) It is believed that this is the first time that the changing form of the eccentric discharge pressure distribution at the start of discharge has been observed.

13.4 Future research

13.4.1 Interpretation techniques

One of the major tasks of this thesis has been the development of new techniques for the interpretation of solids flow and pressures in full scale silos. Although most of these new techniques have been proven to be useful and successful, much more work is needed.

For the interpretation of solids flow patterns from residence time measurements, much more work can be done to improve the graphical visualisation programs. The method of analysing the emerged mass from each seeded level may also be introduced into the program package. A good interpolation technique and a powerful computer contouring algorithm are also needed. Following these developments, the package could become a precise scientific interpretation tool for many applications.

For the method of analysing the emerged mass from each level, it is necessary to quantify the normalised mass-time curves for typical funnel flow and mass flow and the influence of the vertical coordinate of the level.

For the techniques in which silo wall pressures are inferred from wall strain measurements, more studies are needed to establish the conditions under which the method of simple inference of discharge pressures at ring levels from ring strains can be applied. It would be useful to study how the flow channel boundary on the silo walls can be accurately defined from discharge pressures inferred using the rigorous statistical method. It has been found that significant non-uniform pressure components exist during both filling and discharge. There are still major challenges in determining how to characterise these non-uniform pressures for standards codification and engineering applications.
13.4.2 Processing of the existing experimental data

A large number of experiments have been conducted in full scale silos using different bulk solids under the full scale silos research project at the University of Edinburgh. A huge database has been created of both solids flow and wall pressures in silos through these experiments. Most of the residence time data has been processed using several methods in this thesis to reveal the flow phenomena, but no analytical treatment has been developed to model these phenomena. It would be useful to make a comparison between different analytical flow predictions and these experimental observations, which may lead to a verified predictive method for silos design.

On wall pressure distributions, attention has been principally focused on the pressures occurring at the end of filling and during eccentric discharge in a few experiments. The total body of data is extremely large (about 1.5 million observations for each experiment), so its full processing and interpretation will take some years. However, a good start has been made, and new techniques for deducing reliable information have been developed in this thesis. Further interpretation of these tests will undoubtedly reveal more information and reach more reliable and more detailed conclusions.

13.4.3 Future experiments in full scale silos

As experiments in full scale silos have revealed many phenomena which differ from model scale silo experiments, many more experiments in full scale silos are clearly needed. The new cylindrical silo has great potential to produce conclusive new experiments on eccentric discharge over the next few years. A continuation of the project at the University of Edinburgh, funded by the Engineering and Science Research Council, is exploring the effect of particle size and particle size distributions in determining the flow patterns in funnel flow. It should lead to a much clearer picture of the causes of silo overpressure and the properties which are critical to safe and economic design.
Appendix A

LOCAL STIFFENING EFFECT OF "DOUBLE-DECK" BENDING STRAIN GAUGES

A.1 Introduction

The measurement of stresses on the internal surface of a closed structure, such as a box girder or containment structure (silo or tank), is often difficult and sometimes impossible. One remedy is to use a "double deck" (DD) or "sandwich" bending strain gauge which consists of a pair of foil gauges bonded onto the upper and lower surfaces of a plastic base or filling (Itoh, 1975; Kyowa 1977) (Fig. A.1). A DD bending gauge can be placed on the external surface of the structure to measure both the external surface strain and, by extrapolation, the internal surface strain in the test structure plate. These strains can then be translated into a membrane strain and a bending strain or curvature.

The bending curvature is deduced from the strain difference between the upper and lower constituent linear foil gauges in the DD gauge, so its sensitivity depends on the separation between the pair of foil gauges: the DD gauge plastic base thickness. Naturally the stiffness of the gauge increases as the gauge base thickness increases. If the stiffness of the DD gauge base is significant compared with that of the plate wall of the structure, its presence may cause a significant change in local stiffness, resulting in reduced inferred strains and possible modifications to the structural behaviour. Thus the measured strains may not give a good measure of the strains that
would occur if no gauge were present. The stiffness of the DD gauge base may be reduced by making the base from a material with a very low modulus. However, if the base or filling material is too flexible, shear lag may significantly affect the strain reading in the upper linear gauge.

The purpose of this study is to investigate the degree of local stiffening of the structure caused by the presence of a DD gauge. It is useful in considering the use of DD gauges on model structures where the plate thickness is small and the elastic modulus of the wall may also be low (e.g. thin model aluminium shells). The paper is not concerned with the problem of shear lag in the gauge, which has a smaller effect, but which remains important when strain observations are made on thicker high modulus plated structures.

An approximate analytical solution based on the classical theory of elasticity is presented in this appendix. The solution estimates the errors in the strain measurement. Three dimensional finite element calculations are used to verify the solution. The influence of several parameters, including the gauge and wall thicknesses, the elastic moduli, the Poisson's ratios and the length to thickness ratio of the DD gauge, are studied. The stiffening error is evaluated for typical situations. Where the stiffening effect is significant, it is advisable to correct the strain readings to account for local stiffening. A correction method is suggested that can be applied easily. The extrapolation error in determining the inside surface strain in the wall is also explored. This study should be beneficial to users in selecting appropriate gauges and interpreting the results and also to DD gauge suppliers in improving their product.

A.2 Assumptions

The following assumptions were made in analysing the problem.
1. The length and breadth of the DD gauge are small compared with the dimensions of the plate to which the DD gauge is attached: the plate can be treated as infinite.

2. The gauge is attached to a part of the plate in which the strain and curvature is essentially uniform.

3. The strain measured in the upper foil gauge is accurately represented by the strain at the centre of the upper surface of the DD gauge. The strain measured in the lower foil gauge is accurately represented by the strain at the centre of the lower DD gauge. This assumption requires the two constituent foil gauges to be short compared with the total length of the DD gauge and to be centrally located in the gauge (Fig. A.1).

4. Away from the ends of the DD gauge, the gauge deforms compatibly with the plate and plane sections remain plane in the cross-section of the plate-gauge composite system. This requires the DD gauge to be long compared with its thickness and the plate thickness.

5. Transverse normal strains in both the plate and the DD gauge are negligible.

6. For a curved specimen (shell), the radius of curvature of the specimen at the point of attachment of the DD gauge is so large that the wall can be treated as a flat plate.

A.3 Stress analysis of the gauge-plate composite system

A.3.1 Expected strains in gauges

A DD gauge of length $2L_g$ and width $2B_g$ is shown in Fig. A.1a, attached to the centre of a large plate subject to uniaxial stretching and bending in the direction of the gauges (Fig. A.1b), giving rise to a membrane strain $\varepsilon_\alpha$ and a bending curvature $\kappa_\alpha$ in the plate (Fig. A.1c). If the DD gauge does not cause any stiffening, the strains in the upper and lower linear gauges are (Fig. A.1c):
\[ \varepsilon_u = \varepsilon_0 + \frac{t_p + 2t_g}{2} \kappa_o \]  

\[ \varepsilon_l = \varepsilon_0 + \frac{t_p}{2} \kappa_o \]

where \( t_g \) and \( t_p \) are the thicknesses of the DD gauge and the plate respectively, the subscript \( u \) represents the upper gauge which is separated from the plate by the DD gauge base and the subscript \( l \) represents the lower gauge in close proximity to the plate surface.

### A.3.2 Stiffening effect of DD gauge

Because the filling or base of the DD sandwich gauge has a finite stiffness, finite forces are required to deform it into a strain state which is compatible with the plate. These forces are of the same sign as those in the plate itself, so the forces required to deform the gauge reduce the stress resultants in the plate, and consequently change the strains in the plate. The values and distributions of these forces can be obtained from a structural analysis and the actual strains in the gauges deduced. The assumption that plane sections remain plane for the combined plate and gauge means that this is the only phenomenon considered here (shear lag in the gauge is not considered).

A simple method of performing this analysis is to apply superposition, which also assist in the finite element verification. The analysis process is schematically presented in Fig. A.2. Figure A.2a shows the plate-gauge composite structure subjected to uniaxial tension and bending, which induces actual strains \( \varepsilon_u \) and \( \varepsilon_l \) in the upper and lower gauges respectively. The purpose of this analysis is to identify the differences between this actual strain state and that which would arise if the gauge caused no stiffening. The simplest method is therefore to identify the stiffening effect by superposition. The strain state without stiffening would occur if
additional fictitious forces (Fig. A.2c) were added to the actual forces (Fig. A.2b). Thus, these additional fictitious forces (Fig. A.2d) are those associated with the stiffening effect of the gauge. This is clarified in Fig. A.2e, where the assumed strain state exists, and Fig. A.2f which represents the stiffening error caused by the gauge. If the additional fictitious loads acting on the gauge ends are chosen so that they deform an unconnected DD gauge into a strain state that is compatible with the deformations of the plate when no gauge is present, then there are no forces between the DD gauge and the plate in Fig. A.2e, and the strains are those which would be found if the gauge does not stiffen the structure at all. These are here referred to as the "expected strains" (Eqs A.1). The local stiffening effect of the DD gauge may then be found as the stress (strain) state in the plate-gauge composite structure caused by the fictitious loads in Fig. A.2f. The stiffening errors in the upper and lower gauges $\varepsilon_{u}$ and $\varepsilon_{l}$ are the strains caused by these loads. The task of this paper is therefore to evaluate these fictitious forces, and to deduce their effect on the deformations of the gauge/plate composite system.

The plate without stiffening effect is deformed into a membrane strain $\varepsilon_{o}$ and a curvature $\kappa_{o}$ (Figs A.1c and 2e). The bending curvature in the gauge in Fig. A.2e is also $\kappa_{o}$, but its membrane strain is

$$\varepsilon_{og} = \varepsilon_{o} + \frac{t_{p} + t_{g}}{2} \kappa_{o} \quad (A.2)$$

The fictitious stress resultants applied to the gauge alone required to induce these strains (plane stress) are therefore

$$M_{g} = \frac{t_{g}^{3}}{12} E_{g} \kappa_{o} \quad (A.3a)$$

$$P_{g} = E_{g} t_{g} \varepsilon_{og} = E_{g} t_{g} \left[ \varepsilon_{o} + \frac{t_{p} + t_{g}}{2} \kappa_{o} \right] \quad (A.3b)$$
in which \( M_g \) and \( P_g \) are moment and axial tension force per unit width of gauge (stress resultants) acting on the ends of the gauge (Fig. A.2f) and \( E_g \) is Young's modulus for the DD gauge base.

From Fig. A.2, the observed strains \( \varepsilon_u \) and \( \varepsilon_l \), the expected strains \( \varepsilon_u \) and \( \varepsilon_l \) and the stiffening errors \( \varepsilon_{us} \) and \( \varepsilon_{ls} \) in the upper and lower gauges may be related as

\[
\bar{\varepsilon}_u = \varepsilon_u - \varepsilon_{us} \quad (A.4a)
\]

\[
\bar{\varepsilon}_l = \varepsilon_l - \varepsilon_{ls} \quad (A.4b)
\]

in which the minus signs before \( \varepsilon_{us} \) and \( \varepsilon_{ls} \) mean that a positive stiffening error indicates that the observed strain is smaller than the expected strain.

The evaluation of the DD gauge local stiffening effect is thus transformed into solving the static stress analysis problem of Fig. A.2f.

**A.3.3 Load transformation from gauge to plate**

The stress resultants on the gauge ends shown in Fig. A.3a (identical to Fig. A.2f) deform the plate-gauge composite structure. Because the gauge centreline is eccentric to the centreline of the plate by the distance of \( e = (t_g + t_p) / 2 \), the axial force applied to the gauge ends not only stretches the plate but also bends it. Even under purely axial loading, the plate is locally subject to bending due to the gauge stiffening effect.

To deform the gauge into a strain state which is compatible with the plate, the plate exerts some normal and shear stresses on it. The distributions of these stresses are represented schematically in Figs 3b and c. Although there are several analyses in
the literature for forces acting between an infinite plate and a bar of finite length which is placed at the middle surface of the plate (Lukasiewicz, 1979), the authors are not aware of any study which considers the bending stiffness and the eccentricity of the bar with respect to the plate, though this problem may easily be solved numerically. For simplicity and without a great loss of accuracy, the distribution of forces between the plate and the gauge is represented here by a pair of equal and opposite line forces and line moments acting close to the ends of the gauge (Figs 3d and e). Let these forces and moments be represented by \( p \) and \( m \) per unit width respectively and their distances from the centre of the gauge represented by

\[
L = L_g - \alpha_L t_g
\] (A.5)

where \( \alpha_L t_g \) is introduced to provide a calibrating variable \( \alpha_L \) which may be determined by comparing the analytical predictions with precise numerical results. An advantage of this arrangement is that it leads to a closed form solution with minimised errors.

From Assumption 2, the curvature of the plate is assumed to be uniform beneath the gauge, so a compatible deformation of the gauge base requires a uniform curvature too. This means that the bending moment within the gauge must be uniform throughout the gauge length, so the normal forces between the gauge and the plate, which introduce the bending moment into the gauge, must be strongly localised near the gauge ends. The representation of the interaction forces by line forces and moments near the ends (Fig. A.3d) is therefore not in serious error.

A.3.4 Equilibrium

The loads on the gauge and all the interaction forces between the gauge and the plate are here represented by pairs of equal and opposite forces in equilibrium. However, the shear forces between the gauge and the plate act at an eccentricity \( t_g/2 \) to the gauge centreline and \( t_p/2 \) to the plate centreline. In preparation for the stress analysis,
it is desirable to transform these forces to the respective centrelines. The line forces \( p \) can be transferred to the centrelines without changing their values, but they induce additional moments at the centrelines due to the eccentricities. The total moment on the gauge and plate centrelines due to the gauge-plate interaction are then (Fig. A.3f and h)

\[
m_g = \frac{p t_g}{2} - m \\
m_p = \frac{p t_p}{2} + m
\]  

(A.6a)

(A.6b)

**A.3.5 Deformation**

The deformations of the gauge are simple, and are deemed to be described by engineering bending theory. The strains at the centre of the upper and lower surface of the DD gauge base due to the loads shown in Fig. A.3f may then be found as

\[
\varepsilon_{gs} = \frac{P_g - p}{E_g t_g} \\
\kappa_{gs} = 12 \frac{M_g + m_g}{t_g^2 E_g}
\]  

(A.7a)

(A.7b)

in which the subscript \( s \) represents the stiffening effect.

Substituting Eqs A.3 and 6a into Eqs A.7 leads to

\[
\varepsilon_{gs} = \varepsilon_o + \frac{t_o + t_g}{2} \kappa_o - \frac{p}{t_g E_g}
\]  

(A.8a)
\[ \kappa_{gs} = \kappa_o + \frac{6 (p t_g - 2m)}{t_g^3 E_g} \]  

The deformations of the plate are more difficult to predict. Figure A.4a represents the middle surface of the plate subjected to a pair of equal and opposite in plane line forces, also shown in Fig. A.3g. The half length \( B \) over which the forces act is the gauge half width \( B_g \). Figure A.4b represents the same plate subjected to a pair of equal and opposite line moments, corresponding to Fig. A.3h. The plate is assumed to be much bigger than the gauge, so it can be treated as infinite. Analytical solutions for the stresses in an infinite plate subjected to these in plane forces (Fig. A.4a) and moments (Fig. A.4b) are presented in Appendices A1 and A2 respectively (Muskhelishvili, 1953; Godfrey, 1959; Lukasiewicz, 1979). The equal and opposite line forces \( p \) induce a membrane strain at the central point between the loads which is also the centre of the gauge of

\[ \varepsilon_s = \alpha_p \frac{p}{t_p E_p} \]  

where \( \alpha_p \) is the strain coefficient for an infinite plate under equal and opposite line forces (Appendix A1).

The value of \( \alpha_p \) depends on the ratio of the length \( 2B \) to the separation \( 2L \) of the pair of the line loads \( B/L \) and the Poisson's ratio of the plate material \( \nu_p \). For various values of \( B/L \) and \( \nu_p \), values of \( \alpha_p \) are given in Table A.1. The plate is under pure tension in this case, so the curvature is zero. A practical value for \( B/L \) might be 0.1 or 0.2. In this range, clearly \( \alpha_p \) is not very different from \( B/L \).

The equal and opposite line moments \( m_p \) (Fig. A.4b) induce a bending curvature beneath the centre of the gauge (centrally between the line loads) of
where \( \alpha_m \) is the bending curvature coefficient for an infinite plate under the equal and opposite moments (Appendix A2). Again, the value of \( \alpha_m \) depends on \( B/L \) and \( \nu_p \). Table A.2 presents the values of \( \alpha_m \) for various values of \( B/L \) and \( \nu_p \). The plate is under pure bending in this case, so the membrane strain is zero. In the practical range of geometries, \( \alpha_m \) is of the order of 0.3B/L.

### Table A.1 Coefficient \( \alpha_p \)

<table>
<thead>
<tr>
<th>B/L</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu_p=0.0 )</td>
<td>0.095</td>
<td>0.187</td>
<td>0.273</td>
<td>0.352</td>
<td>0.422</td>
<td>0.484</td>
<td>0.538</td>
<td>0.585</td>
<td>0.625</td>
<td>0.659</td>
</tr>
<tr>
<td>( \nu_p=0.1 )</td>
<td>0.101</td>
<td>0.198</td>
<td>0.289</td>
<td>0.371</td>
<td>0.443</td>
<td>0.505</td>
<td>0.558</td>
<td>0.602</td>
<td>0.640</td>
<td>0.671</td>
</tr>
<tr>
<td>( \nu_p=0.2 )</td>
<td>0.106</td>
<td>0.208</td>
<td>0.303</td>
<td>0.387</td>
<td>0.460</td>
<td>0.522</td>
<td>0.574</td>
<td>0.617</td>
<td>0.652</td>
<td>0.680</td>
</tr>
<tr>
<td>( \nu_p=0.3 )</td>
<td>0.111</td>
<td>0.217</td>
<td>0.315</td>
<td>0.401</td>
<td>0.475</td>
<td>0.537</td>
<td>0.588</td>
<td>0.628</td>
<td>0.661</td>
<td>0.686</td>
</tr>
<tr>
<td>( \nu_p=0.4 )</td>
<td>0.115</td>
<td>0.225</td>
<td>0.325</td>
<td>0.413</td>
<td>0.488</td>
<td>0.549</td>
<td>0.598</td>
<td>0.637</td>
<td>0.666</td>
<td>0.688</td>
</tr>
<tr>
<td>( \nu_p=0.5 )</td>
<td>0.118</td>
<td>0.231</td>
<td>0.334</td>
<td>0.423</td>
<td>0.498</td>
<td>0.558</td>
<td>0.606</td>
<td>0.642</td>
<td>0.669</td>
<td>0.688</td>
</tr>
</tbody>
</table>

### Table A.2 Coefficient \( \alpha_m \)

<table>
<thead>
<tr>
<th>B/L</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu_p=0.0 )</td>
<td>0.0319</td>
<td>0.0645</td>
<td>0.0979</td>
<td>0.132</td>
<td>0.168</td>
<td>0.204</td>
<td>0.239</td>
<td>0.274</td>
<td>0.308</td>
<td>0.341</td>
</tr>
<tr>
<td>( \nu_p=0.1 )</td>
<td>0.0316</td>
<td>0.0638</td>
<td>0.0970</td>
<td>0.131</td>
<td>0.166</td>
<td>0.202</td>
<td>0.237</td>
<td>0.272</td>
<td>0.305</td>
<td>0.337</td>
</tr>
<tr>
<td>( \nu_p=0.2 )</td>
<td>0.0307</td>
<td>0.0619</td>
<td>0.0940</td>
<td>0.127</td>
<td>0.161</td>
<td>0.195</td>
<td>0.230</td>
<td>0.263</td>
<td>0.296</td>
<td>0.327</td>
</tr>
<tr>
<td>( \nu_p=0.3 )</td>
<td>0.0291</td>
<td>0.0587</td>
<td>0.0891</td>
<td>0.121</td>
<td>0.153</td>
<td>0.185</td>
<td>0.218</td>
<td>0.250</td>
<td>0.281</td>
<td>0.310</td>
</tr>
<tr>
<td>( \nu_p=0.4 )</td>
<td>0.0268</td>
<td>0.0541</td>
<td>0.0823</td>
<td>0.111</td>
<td>0.141</td>
<td>0.171</td>
<td>0.201</td>
<td>0.230</td>
<td>0.259</td>
<td>0.286</td>
</tr>
<tr>
<td>( \nu_p=0.5 )</td>
<td>0.0240</td>
<td>0.0483</td>
<td>0.0735</td>
<td>0.0994</td>
<td>0.126</td>
<td>0.153</td>
<td>0.179</td>
<td>0.206</td>
<td>0.231</td>
<td>0.256</td>
</tr>
</tbody>
</table>
A.3.6 Compatibility

Two unknowns need to be found by applying compatibility conditions: the interaction stress resultants \( m \) and \( p \) between the gauge and the plate. The assumption that plane sections remain plane for the combined plate and gauge composite structure leads to two compatibility conditions: a) the bending curvature in the plate must equal that in the gauge and b) the strain at the gauged surface of the plate must equal the strain at the lower surface of the gauge base. These leads to

\[
\kappa_s = \kappa_{gs} \quad \text{(A.11a)}
\]

\[
\varepsilon_s + \frac{t_p}{2} \kappa_s = \varepsilon_{gs} - \frac{t_g}{2} \kappa_{gs} \quad \text{(A.11b)}
\]

Substituting Eqs A.6 to 10 into 11 and reorganising the equations leads to

\[
A_1 m + \frac{t_p}{2} A_2 p = m_0 \quad \text{(A.12a)}
\]

\[
\frac{6}{t_g} A_2 m + A_3 p = p_0 \quad \text{(A.12b)}
\]

where

\[
A_1 = 1 + \alpha_m \beta_1^3 \beta_E \quad \text{(A.13a)}
\]

\[
A_2 = \alpha_m \beta_1^2 \beta_E - 1 \quad \text{(A.13b)}
\]

\[
A_3 = 4 + \beta_1 \beta_E (\alpha_p + 3 \alpha_m) \quad \text{(A.13c)}
\]

\[
m_0 = \frac{E_p t_g^3 \kappa_0}{12} \quad \text{(A.13d)}
\]

580
\[ p_0 = t_g E_g \varepsilon_0 + \frac{t_h t_p}{2} E_g \kappa_0 \]  

(A.13e)

in which

\[ \beta_E = \frac{E_g}{E_p} \]  

(A.14a)

\[ \beta_t = \frac{t_g}{t_p} \]  

(A.14b)

are the modular ratio of gauge filling material to plate material and the thickness ratio of gauge to plate.

The interaction stress resultants between the gauge and the plate may then be obtained by solving Eqs A.12 to yield

\[ m = \frac{A_3 m_0 - t_g A_2 p_0/2}{A_1 A_3 - 3A_2^2} \]  

(A.15a)

\[ p = \frac{A_1 p_0 - 6A_2 m_0/t_g}{A_1 A_3 - 3A_2^2} \]  

(A.15b)

A.3.7 Stress and strain distributions in the plate

Equation A.15b gives the value of the pair of equal and opposite in plane line loads in the plate (Fig. A.4a). The value of the pair of equal and opposite line moments in the plate (Fig. A.4b) can be obtained by substituting Eqs A.15 into Eq. A.6b. Substituting these loads into the solutions in Appendices A1 and A2 respectively and superposing their results leads to the stress and strain distributions due to stiffening effect in the whole plate. Superposing these stress and strain states into the expected
stress and strain distributions, which involve no stiffening effect, leads to the actual stress and strain distributions.

The membrane strain and bending curvature due to the stiffening effect at the centre of the plate, which is the centre of the gauge, can be obtained by substituting Eqs A.6b and 15 into Eqs A.9 and 10 to obtain

\[
\varepsilon_s = \frac{\alpha_p}{t_p E_p} \frac{A_1 t_2 p_0 - 6A_2 m_0}{A_1 A_3 - 3A_2^2} \tag{A.16a}
\]

\[
\kappa_s = \frac{12\alpha_m (A_3 - 3A_2/\beta_p) m_0 - (t_2 A_2 - t_2 A_1) p_0/2}{t_p^3 E_p A_1 A_3 - 3A_2^2} \tag{A.16b}
\]

A.3.8 Strains in the gauges

The strain in the foil gauges due to stiffening effect (stiffening errors) can be calculated from the membrane strain and bending curvature in the gauge filling which can be obtained by substituting Eqs A.15 into Eqs A.8. Alternatively, they can be calculated from the membrane strain and bending curvature in the plate based on the assumption that plane sections remain plane. Substituting Eqs A.16 into Eqs A.1 leads to

\[
\varepsilon_u_s = \frac{6[\alpha_m(1+2\beta_p)(\beta_p A_3-3A_2) - \alpha_p A_2] m_0 + [\alpha_p A_1 + 3\alpha_m(1+2\beta_p)(1+\beta_p)] p_0 t_p}{t_p E_p (A_1 A_3 - 3A_2^2)} \tag{A.17a}
\]

\[
\varepsilon_l_s = \frac{6[\alpha_m(\beta_p A_3 - 3A_2) - \alpha_p A_2] m_0 + [\alpha_p A_1 + 3\alpha_m(1+\beta_p)] p_0 t_p}{t_p E_p (A_1 A_3 - 3A_2^2)} \tag{A.17b}
\]

Substituting Eqs A.17 into Eqs A.4 leads to the strains which will be observed in the foil gauges.
A.4 Relative stiffening errors

Equations A.17 present strain values (stiffening errors) in the foil gauges due to the stiffening effect. However, relative errors are often more important in interpreting measurements. Further, the stiffening errors $\varepsilon_{us}$ and $\varepsilon_{ls}$ are linear combinations of $m_0$ and $P_0$ (Eqs A.17) while $m_0$ and $P_0$ are linear combinations of $\varepsilon_0$ and $\kappa_0$ (Eqs A.13d and e), the stiffening errors $\varepsilon_{us}$ and $\varepsilon_{ls}$ are therefore also linear combinations of $\varepsilon_0$ and $\kappa_0$. It is then more convenient to look at pure tension and pure bending separately. Any other condition may be found by superposition.

A.4.1 Pure tension

For pure tension, there is no bending curvature in the ungauged plate. Substituting $\kappa_c=0$ into Eqs A.13d and 13e leads to

\[m_{t0} = 0\]  \hspace{1cm} (A.18a)

\[p_{t0} = t_E E_0 \varepsilon_0\]  \hspace{1cm} (A.18b)

in which the subscript $t$ indicates pure tension. Substituting Eqs A.18 into Eqs A.17 gives

\[\varepsilon_{uts} = \frac{\alpha m A_1 + 3 \alpha_m (1+2\beta_1)(1+\beta_2)}{A_1 A_3 - 3 A_2^2} \beta_1 \beta_2 E_0\]  \hspace{1cm} (A.19a)

\[\varepsilon_{lts} = \frac{\alpha m A_1 + 3 \alpha_m (1+\beta_2)}{A_1 A_3 - 3 A_2^2} \beta_1 \beta_2 E_0\]  \hspace{1cm} (A.19b)

In this case the expected strains in the upper and lower gauges are (Eqs A.1): \[\varepsilon_{ut} = \varepsilon_0\]  \hspace{1cm} (A.20a)
The relative stiffening errors (ratio of error to value for an ungauged plate) are then given by

\[
\varepsilon_{ult} = \frac{\varepsilon_{ult}}{\varepsilon_{ul}} = \frac{\alpha_p A_1 + 3\alpha_m (1+2\beta)(1+\beta_i)}{A_1 A_3 - 3A_2^2} \beta_i \beta_E \tag{A.21a}
\]

\[
\varepsilon_{ils} = \frac{\varepsilon_{ils}}{\varepsilon_{il}} = \frac{\alpha_p A_1 + 3\alpha_m (1+\beta_i)}{A_1 A_3 - 3A_2^2} \beta_i \beta_E \tag{A.21b}
\]

A.4.2 Pure bending

For pure bending, there is no membrane strain in the plate. Substituting \(\varepsilon_o = 0\) into Eqs A.13d and 13e leads to

\[
m_{bo} = \frac{E_p \kappa_0}{12} \tag{A.22a}
\]

\[
p_{bo} = \frac{t_{p} t_{p}}{2} E_g \kappa_0 \tag{A.22b}
\]

in which the subscript \(b\) indicates pure bending. Substituting Eqs A.22 into Eqs A.17 gives

\[
\varepsilon_{ubs} = \frac{\alpha_p (1+\beta_i) + \alpha_m (1+2\beta)(3+3\beta_i A_3 - 3A_2)^2 \beta_i}{A_1 A_3 - 3A_2^2} \frac{t_g}{2} \beta_E \kappa_0 \tag{A.23a}
\]

\[
\varepsilon_{lbs} = \frac{\alpha_p (1+\beta_i) + \alpha_m [3+(3+3\beta_i A_3 - 3A_2)^2 \beta_i]}{A_1 A_3 - 3A_2^2} \frac{t_g}{2} \beta_E \kappa_0 \tag{A.23b}
\]
In this case the expected strains in the upper and lower gauges are (Eqs A.1):

\[ \varepsilon_{ub} = t_p(1+2\beta_t)\kappa_v/2 \]  
\[ \varepsilon_{lb} = t_p\kappa_v/2 \]  

(A.24a)  

(A.24b)

The relative stiffening errors are then given by

\[ \varepsilon_{ubs} = \frac{\varepsilon_{ubs}}{\varepsilon_{ub}} = \frac{\alpha_p(1+\beta_t)+\alpha_n(1+2\beta_t)[3+(3+\beta_t A_3-3A_2)\beta_1]}{(1+2\beta_t)(A_1 A_3-3A_2^2)} \beta_4 \beta_E \]  
\[ \varepsilon_{lbs} = \frac{\varepsilon_{lbs}}{\varepsilon_{lb}} = \frac{\alpha_p(1+\beta_t)+\alpha_n[3+(3+\beta_t A_3-3A_2)\beta_1]}{A_1 A_3-3A_2^2} \beta_4 \beta_E \]  

(A.25a)  

(A.25b)

A.5 Finite element verification

The simplified analysis presented above was tested by using a three dimensional finite element analysis. The plate-DD gauge composite structure was modelled by using 20 node three dimensional elements in the commercial package ABAQUS (Hibbitt, Karlsson & Sorenson Inc., 1990). The plate was modelled as a 2Lp×2Lp square plate with a DD gauge attached at the centre (Fig. A.1a). Symmetry was used to analyse only a quarter of the structure (Fig. A.5). A typical mesh used one layer of 289 elements for the plate and one layer of 25 elements for the gauge.

The loads calculated from Eqs A.3 were transformed to equivalent distributed pressures and applied at the ends of the DD gauge (Fig. A.5). To provide clear comparisons with the analytical results, two load cases were used: pure tension and pure bending. For pure tension (\( \kappa_v=0 \)), the distributed pressures at the upper and lower surface of the gauge ends are:
\[ q_{ut} = E_t \varepsilon_o \] (A.26a)

\[ q_{lt} = E_t \varepsilon_o \] (A.26b)

For pure bending (\( \varepsilon_o = 0 \)) these loads become

\[ q_{ub} = E_t t_p \frac{(1+2\beta_p) \kappa_o}{2} \] (A.27a)

\[ q_{lb} = E_t t_p \frac{\kappa_o}{2} \] (A.27b)

For given values of \( \varepsilon_o \) and \( \kappa_o \), the values of loads in Eqs A.26 and 27 and the expected strains in the foil gauges (Eqs A.20 and 24) were calculated. These loads were then used to carry out the finite element analysis to find the strain in the foil gauges which were the strains at the centres of the upper and lower surfaces of the gauge filling. Comparing these strains with the expected strains give the relative stiffening errors. The value of 0.001 was used for \( \varepsilon_o \) and \( \kappa_o \) in this study.

The boundary conditions for the plate were carefully checked. Initially, all of four edges of the plate (Fig. A.1a) were either modelled as fixed (fixed boundary) or simple supported (simple boundary). For fixed boundaries, all the nodes at boundaries were restrained in all the three directions. For simple boundaries, only the nodes on the lower (ungauged) surface were restrained in the vertical direction (normal to the plate plane). The chief value of using these two limiting boundary conditions is, however, to identify the size of plate needed to achieve results which are independent of the plate size.

Figure A.6 shows the results of a study which explored the effect of the size of the plate to which the gauge is attached on the errors caused by gauge stiffening. Results for both fixed and simple boundaries are presented. The errors are essentially
independent of both plate size and boundary conditions as soon as the plate size exceeds about 10 gauge lengths. However, for the parameters used here ($\beta_\varepsilon=0.0147$, $\beta_t=2$, $L_g/t_g=7.5$, $B_g/L_g=0.156$ and $\nu_g=\nu_p=0.3$) derived from a commercial gauge and a steel plate, the errors due to stiffening can be seen to be very significant. Pure tension in the plate leads to a 2% error in the lower gauge and 8.5% error in the upper gauge, whilst pure bending leads to 6.8% error in the lower gauge and 5.5% error in the upper gauge. These significant errors arise from the use of a gauge twice as thick as the plate, which would be possibly used in model tests with very thin walls.

In all the following calculations, simple boundary conditions and a plate to gauge size ratio of $L_p/L_g=10$ were used.

The coefficient $\alpha_L$ in Eq. A.5 may be determined by comparing the analytical predictions with finite element predictions here. Figure A.7 presents a few finite element predictions for the stiffening error at the upper gauge under pure tension together with analytical predictions by using different $\alpha_L$ values ($E_g/E_p=0.0147$, $\nu_g=\nu_p=0.3$, $B_g/L_g=0.156$ and $L_g/t_g=7.5$). The analytical predictions with $\alpha_L=2$ fit the finite element calculations best. In all the following analytical calculations, the value of $\alpha_L=2$ was used.

Figure A.8a show the comparisons between analytical solutions and the finite element calculations for different thickness ratios ($L_g/t_g=7.5$ and $B_g/L_g=0.156$). For $E_g/E_p=0.0147$, the errors grow rapidly as the gauge thickness increases (Fig. A.8a): the lower gauge error under pure tension is almost linear with gauge thickness, but all other errors vary as a power greater than 2. Suppliers' advice$^3$ that the gauge thickness should be in the range of 1/3 to 1/2 of the plate thickness is therefore helpful in limiting these errors. The match between the present solution and the finite element calculations is very close.
The effect of the modular ratio on the stiffening errors is shown in Fig. A.8b \((t_g/t_p=2)\). A logarithmic scale is used for the modular ratio as this can vary over a wide range. A simple rule of thumb might be suggested that the modular ratio should be kept below 0.01. The finite element and analytical results are again in close agreement over a wide range. The figure also shows that the relative stiffening errors may exceed 100% if the gauge is used on a plate of low modulus. Here a 100% error means that the actual strain reading is 100% smaller than the expected strain with no stiffening (the strain reading is zero), while a greater than 100% stiffening error means that the actual strain reading is in the opposite sense to the applied load (Eqs A.4).

The sensitivity of the stiffening errors to change in the Poisson’s ratios of both the plate and the gauge were also investigated. The effects were found to be very small and the analytical predictions were in close agreement with the FE results.

### A.6 Parametric study and discussion

#### A.6.1 Effect of thickness ratio \(t_g/t_p\) on stiffening and extrapolation errors

##### A.6.1.1 Stiffening errors

From Eqs A.21 and 25, it is seen that all the relative stiffening errors will become zero if \(t_g \to 0\). This is the case for common single gauges and means that no stiffening error occurs, but the bending curvature cannot be deduced. If thickness ratio \(\beta=t_g/t_p\) increases to a huge value \((\beta \to \infty)\), all the relative stiffening errors will reach to 100%, i.e. no strain readings can be observed. This is the case for a thin plate attached to a massive gauge.

The influence of the thickness ratio \(t_g/t_p\) on the relative strain stiffening errors using typical values for a commercial gauge \((B_g/L_g=0.233, L_g/t_g=0.233, \nu_p=\nu_g=0.3)\) is shown in Fig. A.9a for a steel plate \((\beta_c=E_g/E_p=0.0147)\) and in Fig. A.9b for an
aluminium plate ($\beta_p=0.042$). These two figures are similar except the different y-axis scales. For this set of gauge parameters, the stiffening errors are less than 1% for the steel plate and less than 2.5% for the aluminium plate in the generally recommended range of $t_g/t_p \leq 1/2$ (Techni Measure, 1992). Stiffening errors increase rapidly as the thickness ratio increases. The relationships between thickness ratio and the stiffening errors may be estimated from Eqs A.21 and 25 within the practical range of parameters.

**A.6.1.2 Simplified solution**

It is known that $|\alpha_p| \leq 1$ (Table A.1) and $|\alpha_m| \leq 1$ (Table A.2). For many practical configurations, the following simplifications may be adopted: $\beta_p \beta_E << 1$, $\beta^2 \beta_E << 1$ and $\beta^3 \beta_E << 1$. The parameters in Eqs A.14 then become $A_1=1$, $A_2=-1$ and $A_3=4$ and Eqs A.21 and 25 reduce to

\[
e_{uts} = [\alpha_p + 3\alpha_m(1 + 2\beta_t)(1 + \beta_t)]\beta_t \beta_E \quad \text{(A.28a)}
\]

\[
e_{lts} = [\alpha_p + 3\alpha_m(1 + \beta_t)]\beta_t \beta_E \quad \text{(A.28b)}
\]

\[
e_{ubs} = \{[\alpha_p(1 + \beta_t)/(1 + 2\beta_t)] + \alpha_m(3 + 6\beta_t + 4\beta^2_t)\}\beta_t \beta_E \quad \text{(A.28c)}
\]

\[
e_{lbd} = [\alpha_p(1 + \beta_t) + \alpha_m(3 + 6\beta_t + 4\beta^2_t)]\beta_t \beta_E \quad \text{(A.28d)}
\]

The stiffening error of the lower gauge under pure tension (Eq. A.28b) is now quadratic in $\beta_t$ and that of the upper gauge (Eq. A.28a) is a cubic function of $\beta_t$. Under pure bending, the stiffening error for the lower gauge (Eq. A.28d) is a cubic function of $\beta_t$ while that of the upper gauge (Eq. A.28c) is a more complicated but approximately a cubic function. These relationships are evident in Figs 9.
A.6.1.3 Extrapolation error

The strain $e_i$ on the inside surface (ungauged surface) of the plate may be calculated by extrapolating the strains from the upper and the lower gauges:

$$e_i = e_1 - \frac{e_u - e_1}{\beta_i} \quad (A.29)$$

Any error in the upper and lower gauge observations may therefore transfer an error to the extrapolated opposite surface strain $e_i$. This error is here termed the extrapolation error. If there are relative errors $e_u$ and $e_1$ in the upper and lower gauges, the extrapolated strain and its relative error (extrapolation error) can be found as

$$
\bar{e}_i = e_i (1 - e_i) - \frac{e_u (1 - e_i) - e_i (1 - e_i)}{\beta_i} = e_i - \left[ e_i e_1 - \frac{e_u e_i - e_1 e_i}{\beta_i} \right] \quad (A.30a)
$$

$$e_i = \frac{e_i - \bar{e}_i}{\varepsilon_i} = \left[ e_i e_1 - \frac{e_u e_i - e_1 e_i}{\beta_i} \right] / \varepsilon_i \quad (A.30b)
$$

Again a positive relative error means a decrease in the strain. Substituting Eqs A.1 into Eq. A.30b and using appropriate conditions for pure tension ($\kappa_0 = 0$) and pure bending ($\kappa_0 = 0$), leads to

$$e_{it} = e_h - \frac{e_{ut} - e_{hi}}{\beta_i} \quad (A.31a)$$

$$e_{ib} = \frac{(1 + 2\beta_i) e_{ub} - e_{ib}}{\beta_i} - e_{ib} \quad (A.31b)$$
The extrapolation errors are thus linear in \(1/\beta_i\) and errors in the upper and lower gauges. Extrapolation errors due to both stiffening errors and measurement noise are explored here. Figure A.10a shows the extrapolation errors due to gauge stiffening using the same parameters as those in Fig. A.9a. These errors are evidently very small for gauge thickness less than the plate thickness, indicating that the ungauged surface is close to the point at which the two errors (upper and lower gauge) cancel each other.

Figure A.10b presents the extrapolation errors due to a 2% reading error (noise) in the upper gauge while the lower gauge is supposed to measure the strain accurately. This small error may evidently cause a huge extrapolation error when the thickness ratio is less than 1/5. The error is doubled (for pure tension) and quadrupled (for pure bending) even for \(\beta_i=t_g/t_p=0.5\): the upper limit of the generally recommended range of gauge thickness. The errors decrease slowly to about 1% (for pure tension) and 5% (for pure bending) when \(\beta_i=2\). It may therefore be concluded that a good choice of gauge thickness requires a careful compromise between the stiffening effect and the extrapolation error due to noise.

A.6.2 Effect of modular ratio \(E_g/E_p\) on stiffening errors and shear lag

A.6.2.1 Stiffening error

The stiffening errors naturally tend to zero as the elastic modulus of the gauge tends to zero, and become enormous as this modulus reaches the parent plate value. These features are evident in Eqs A.21 and 25.

Figure A.11 presents the influence of the modular ratio \(\beta_g=E_g/E_p\) on the relative stiffening errors for \(t_g/t_p=0.667\) using parameters for a commercial gauge (\(B_g/L_g=0.233, L_g/t_g=7.5\) and \(v_g=v_p=0.3\)). The range of \(\beta_g\) from 0 to 0.1 should be able to cover almost all practical cases. The stiffening errors increase almost linearly with \(\beta_g\).
A.6.2.2 Shear lag

The above analysis indicates that flexible materials with low Young's modulus should be used as the gauge filling to reduce the stiffening effect. However, shear lag may cause serious errors in the upper gauge if the material is too flexible. It is important to ensure that a flexible material does not cause other problems such as shear straining and creep.

A.6.3 Effect of the gauge aspect ratio on stiffening errors

The gauge aspect ratio $B_g/L_g$ appears to influence only the strain coefficients $\alpha_p$ and $\alpha_m$ in Eqs A.21 and 25. Figure A.12 presents the effect of $B_g/L_g$ on the relative strain stiffening errors for $t_g/t_p=0.667$ ($E_g/E_p=0.042$, $v_g=0.3$, and $L_g/t_g=7.5$). All the curves go through the origin because $\alpha_p=\alpha_m=0$ when $B_g/L_g=0$. The stiffening errors are quite linear in the range of $B_g/L_g$ from zero to 0.5, which covers all current gauge geometries. These linear relationships mean that either decreasing the width or increasing the length of a DD gauge will result in a reduction of the stiffening errors in a linear fashion: a long narrow gauge is the optimal form.

A.6.4 Effect of Poisson's ratios on stiffening errors

Figure A.13 shows the effect of the Poisson's ratio of the plate material on the relative strain stiffening errors for $t_g/t_p=0.667$ ($E_g/E_p=0.042$, $v_g=0.3$ and $L_g/t_g=7.5$). The stiffening errors decrease slightly with an increase in the value of the Poisson's ratio $v_p$. This effect is larger for the upper gauge than that for the lower gauge. In the practical range of values $v_p=1/6$ to $1/3$, the effect is less than 10%. Comparing with the strain coefficients $\alpha_p$ (Table A.1) and $\alpha_m$ (Table A.2), it is noted that when $v_p$ increases, $\alpha_p$ decreases but $\alpha_m$ increases. This indicates that bending dominates in the local stiffening effect.
Noting that the Poisson's ratio of the gauge filling does not appear in the solution, its effect on the stiffening errors can be ignored (Fig. A.8d).

### A.7 Correction of Strain Readings

If significant stiffening errors are unavoidable, the strain readings may need to be corrected. Given the relative stiffening errors $e_{uts}$, $e_{lts}$, $e_{ubs}$ and $e_{lbs}$ which can be calculated from Eqs A.21 and 25, and the strain readings of the upper and lower gauges $\bar{e}_u$ and $\bar{e}_l$, the following equations can be obtained from Eqs A.1, 4, 21 and 25:

\[ \bar{e}_u = (1-e_{ubs})e_o + t_p(1+2\beta_t)(1-e_{ubs})\kappa_o/2 \quad \text{(A.32a)} \]

\[ \bar{e}_l = (1-e_{lbs})e_o + t_p(1-e_{lbs})\kappa_o/2 \quad \text{(A.32b)} \]

Solving Eqs A.32 gives the corrected membrane strain $e_o$ and bending curvature $\kappa_o$:

\[ e_o = \frac{(1-e_{lbs})\bar{e}_u - (1+2\beta_t)(1-e_{ubs})\bar{e}_l}{(1-e_{uts})(1-e_{lbs}) - (1-e_{ubs})(1-e_{lbs})(1+2\beta_t)} \quad \text{(A.33a)} \]

\[ \kappa_o = \frac{(1-e_{uts})\bar{e}_l - (1-e_{lbs})\bar{e}_u}{(1-e_{uts})(1-e_{lbs}) - (1-e_{ubs})(1-e_{lbs})(1+2\beta_t)} \times \frac{2}{t_p} \quad \text{(A.33b)} \]

Substituting Eqs A.33 into Eqs A.1 leads to the corrected strain readings at the linear gauges

\[ \varepsilon_u = \frac{[e_{lts}-e_{lbs}-2\beta_t(1-e_{lts})]\bar{e}_u + (1+2\beta_t)(e_{ubs}-e_{uts})\bar{e}_l}{(1-e_{uts})(1-e_{lbs}) - (1-e_{ubs})(1-e_{lbs})(1+2\beta_t)} \quad \text{(A.34a)} \]
Eqs A.34 reduce to $\varepsilon_a = \varepsilon_u$ and $\varepsilon_f = \varepsilon_1$ if all the relative stiffening errors are zero. For a specific application, all the relative stiffening errors may be read from the graphs given in this study or calculated from the equations given.

A.8 Conclusions

An approximate analytical elastic solution for the local stiffening effect of a "Double Deck" (DD) bending strain gauge has been presented. The analytical solution has been verified using a 3D finite element analysis. The solution is applicable when the plate size is much larger than the DD gauge length (>10 times) and the constituent foil gauges are much shorter than the DD gauge length.

A correction method that can be easily applied has been proposed. The influences of several parameters including the ratio of the DD gauge thickness to wall thickness $\beta_t = t_\text{g}/t_\text{p}$, the ratio of the Young's modulus of the DD gauge filling to that of the wall material $\beta_E = E_\text{g}/E_\text{p}$, the ratio of width to the length of the DD gauge $B_\text{g}/L_\text{g}$ and the Poisson's ratios of both the DD gauge filling and the wall material have been studied. Extrapolation errors due to both gauge stiffening effect and gauge reading noises have also been discussed.

For most of the practical range of parameters (when the thickness ratio $\beta_t$ and modular ratio $\beta_E$ satisfy $\beta_t \beta_E << 1$, $\beta_t^2 \beta_E << 1$ and $\beta_t^3 \beta_E << 1$), the local stiffening effect errors are approximately quadratic and cubic functions of $\beta_t$ and linear in $\beta_E$ and the gauge aspect ratio $B_\text{g}/L_\text{g}$. The values of the Poisson's ratio for the plate and the gauge filling were found to be of minor significance. Random errors in gauge readings lead
to serious errors if the gauge is thin, but stiffening errors become important when the
gauge is thick. A compromise must therefore be found for each application.

The optimum thickness for the DD gauge depends on the elastic modulus of the wall,
the aspect ratio of the gauge, and the expected level of random errors in the gauge
readings from other sources. In particular, where random errors are expected to be a
problem, a gauge as thick as the plate itself may be desirable.

A.9 Notation

A  Coefficient
B  half length over which line a load/moment acts
B_g half width of a DD gauge
E  Young’s modulus
L  half separation of a pair of equal and opposite line loads/moments
L_g half length of a DD gauge
M_g fictitious moment per unit width of gauge acting on the ends of the gauge
P  point load
P_g fictitious tension force per unit width of gauge acting on the ends of the gauge
e relative error
e_uts relative stiffening error of upper gauge under pure tension
e_lts relative stiffening error of lower gauge under pure tension
e_ubs relative stiffening error of upper gauge under pure bending
e_lbs relative stiffening error of lower gauge under pure bending
e_i relative error of strain on the inside (ungauged) surface
e_it relative error of strain on the inside (ungauged) surface under pure tension
e_ib relative error of strain on the inside (ungauged) surface under pure bending
m line moment
p line load
q surface load
Appendix A1 An infinite plate subject to in plane line loads

A1.1 Under an in plane line load

Under a concentrated in plane force $P$, the membrane stresses in an infinite plate, expressed in a polar coordinate system (Fig. A1.1a), are given by (Muskhelishvili, 1953; Godfrey, 1959; Lukasiewicz, 1979)
\[
\sigma_z = -\frac{P(3+v_p)\cos\theta}{4\pi rt_p} \tag{A1.1a}
\]

\[
\sigma_\theta = \frac{P(1-v_p)\cos\theta}{4\pi rt_p} \tag{A1.1b}
\]

\[
\sigma_\rho = \frac{P(1-v_p)\sin\theta}{4\pi rt_p} \tag{A1.1c}
\]

in which \(v_p\) is the Poisson’s ratio of the plate material and \(t_p\) is the plate thickness.

Expressing the stresses in a Cartesian coordinate system with \(x-y\) axes as defined in Fig. A1.1a, Eqs A1.1 become

\[
\sigma_x = -\frac{P\cos\theta}{4\pi rt_p} [2+(1+v_p)\cos2\theta] \tag{A1.2a}
\]

\[
\sigma_y = -\frac{P\cos\theta}{4\pi rt_p} [2-(3-v_p)\cos2\theta] \tag{A1.2b}
\]

\[
\tau_{xy} = -\frac{P\sin\theta}{4\pi rt_p} [2+(1+v_p)\cos2\theta] \tag{A1.2c}
\]

When a line load \(p\) acts on the plate as shown in Fig. A1.1b, the force on a segment \(dy\) is given by

\[pdy = \frac{rd\theta}{\cos\theta} \tag{A1.3}\]

Substituting Eq. A1.3 into Eqs A1.2 and integrating from \(\theta_1\) to \(\theta_2\) as defined in Fig. A1.1b, leads to
\[
\sigma_x = \frac{p}{2\pi\rho}\left[\theta_1 - \theta_2 + \frac{1+\nu_p}{4}(\sin\theta_1 - \sin\theta_2)\right] \tag{A1.4a}
\]

\[
\sigma_y = \frac{p}{2\pi\rho}\left[\theta_1 - \theta_2 - \frac{3-\nu_p}{4}(\sin\theta_1 - \sin\theta_2)\right] \tag{A1.4b}
\]

\[
\tau_{xy} = \frac{p}{4\pi\rho}\left[(1-\nu_p)\ln\frac{r_1}{r_2} - (1+\nu_p)(\cos\theta_1 - \cos\theta_2)\right] \tag{A1.4c}
\]

A1.2 Under a pair of equal and opposite in plane line loads

The opposing pair of line loads due to the gauge stiffening is shown in Fig. A1.1c. The solution can be obtained from Eqs A1.4 by translating the origin of the coordinate system and superposing the results of both line loads:

\[
\sigma_x = \frac{p}{2\pi\rho}\left[\theta_1 - \theta_2 - \theta_3 + \theta_4 + \frac{1+\nu_p}{4}(\sin\theta_1 - \sin\theta_2 - \sin\theta_3 + \sin\theta_4)\right] \tag{A1.5a}
\]

\[
\sigma_y = \frac{p}{2\pi\rho}\left[(\theta_1 - \theta_2 - \theta_3 + \theta_4) - \frac{3-\nu_p}{4}(\sin\theta_1 - \sin\theta_2 - \sin\theta_3 + \sin\theta_4)\right] \tag{A1.5b}
\]

\[
\tau_{xy} = \frac{p}{4\pi\rho}\left[(1-\nu_p)\ln\frac{r_1 r_4}{r_2 r_3} - (1+\nu_p)(\cos\theta_1 - \cos\theta_2 - \cos\theta_3 + \cos\theta_4)\right] \tag{A1.5c}
\]

in which angles \(\theta_1, \theta_2, \theta_3\) and \(\theta_4\) and distances \(r_1, r_2, r_3\) and \(r_4\) are defined in Fig. A1.1c. It should be noted that angles \(\theta_1, \theta_2, \theta_3\) and \(\theta_4\) should be measured from \(-\pi/2\) to \(3\pi/2\).

The complete strain state in the plate can be calculated from the stresses given by Eqs A1.5 based on Hooke’s law. The membrane strain in the x direction at the origin is given by
Results

Under a pair of equal and opposite in-plane line loads, the direct membrane stresses (Eqs A1.5) and strains have finite values throughout the plate. The membrane shear stress and strain are singular at the point \( x=\pm L, \ y=\pm B \) as there (Fig. A1.1c). It should be noted that the membrane stresses (Eqs A1.5) are linear in the Poisson’s ratio but the direct strains (e.g. Eqs A1.6) are nonlinear in it. For the case of \( B/L=0.212 \) and \( v_p=0.3 \), Figures A1.2a and b present the direct membrane stress and strain distributions along the x and y axes respectively. The stresses and strains decrease to very small values for large values of \( x/L \) or \( y/L \).

Appendix A2  An infinite plate subject to line moments

A2.1 Under a line moment

Ignoring high order terms due to shear deformation, the moments, \( m_x, m_y \) and \( m_{xy} \), in an infinite plate under a line moment, \( m \) load (Fig. A2.1a) are given by (Lukasiewicz, 1979)

\[
m_x = \frac{m}{2\pi \ell} \left( \frac{1-v_p}{4} (\sin 2\theta_1 - \sin 2\theta_2) \right)
\]

(A2.1a)

\[
m_y = \frac{m}{2\pi \ell} \left( v_p (\theta_1 - \theta_2) + \frac{1-v_p}{4} (\sin 2\theta_1 - \sin 2\theta_2) \right)
\]

(A2.1b)

\[
m_{xy} = \frac{m}{4\pi \ell} \ln \frac{r_1}{r_2} + \frac{1}{2} (\cos 2\theta_1 - \cos 2\theta_2)
\]

(A2.1c)
A2.2 Under a pair of equal and opposite line moments

Performing transformations similar to those in Appendix A, the bending moments induced by a pair of opposing line moments (Fig. A2.1b) are given by

\[
m_x = \frac{m}{2\pi L} \left( \theta_1 - \theta_2 - \theta_3 + \theta_4 \right) - \frac{1 - \nu_p}{4} \left( \sin 2\theta_1 - \sin 2\theta_2 - \sin 2\theta_3 + \sin 2\theta_4 \right)
\]  
(A2.2a)

\[
m_y = \frac{m}{2\pi L} \left( \nu_p (\theta_1 - \theta_2 - \theta_3 + \theta_4) + \frac{1 - \nu_p}{4} (\sin 2\theta_1 - \sin 2\theta_2 - \sin 2\theta_3 + \sin 2\theta_4) \right)
\]  
(A2.2b)

\[
m_{xy} = \frac{m}{4\pi L} \left[ \ln \frac{r_1 r_2}{r_2 r_3} + \frac{1}{2} \left( \cos 2\theta_1 - \cos 2\theta_2 - \cos 2\theta_3 + \cos 2\theta_4 \right) \right]
\]  
(A2.2c)

The stresses on the upper surface of the plate are

\[
\sigma_x = \frac{6m_x}{t_p^2}
\]  
(A2.3a)

\[
\sigma_y = \frac{6m_y}{t_p^2}
\]  
(A2.3b)

\[
\tau_{xy} = \frac{6m_{xy}}{t_p^2}
\]  
(A2.3c)

The complete strain state in the plate can be calculated from these stresses based on Hooke's law. The bending curvature in the x direction at the origin is given by

\[
\kappa_x \bigg|_{y=0} = 2 \frac{\sigma_x - \nu_p \sigma_y}{E_p} \bigg|_{y=0} = \alpha_m \frac{12m}{t_p^3 E_p}
\]  
(A2.4)
A2.3 Results

Under a pair of equal and opposite line moments, the direct bending stresses (Eqs A2.3) and curvatures have finite values in the whole plate. The bending shear stress and strain are singular at the point \( x=\pm L, \ y=\pm B \) as there \( \lim_{r \to 0} \frac{r_1 r_4}{r_2 r_3} \to \pm \infty \) (Fig. A2.1b).

Again, the bending stresses (Eqs A2.3) are linear in the Poisson's ratio but the direct curvatures (e.g. Eq. A2.4) are nonlinear in it. For the case of \( B/L=0.212 \) and \( \nu_p=0.3 \), Figures A2.2a and b present the direct bending stress and curvature distributions along the x and y axes respectively. The stresses and curvatures decrease to very small values for large values of \( x/L \) or \( y/L \).
Fig. A.1 A DD strain gauge installed on a plate subjected to tension and bending
Plate-gauge subjected tension and bending

Plate-gauge subjected tension and bending

Plate-gauge subjected to external loads on the gauge

Plate-gauge subjected to external loads on the gauge

Forces which produce the ideal expected strains everywhere

Error forces which can be analysed as stiffening effect

Fig. A.2 Stiffening effect of DD gauge
Fig. A.3 Analysis of stiffening effect
a) Plate under a pair of equal and opposite in plane line loads

b) The plate under a pair of equal and opposite line moments

Fig. A.4 Forces on the plate transferred from the gauge
Fig. A.5 Finite element modelling of plate-gauge composite structure

Fig. A.6 Effect of plate size and boundary condition on stiffening errors (FE analysis)

Fig. A.7 Determination of coefficient $\alpha_L$
Fig. A.8 Comparison of analytical results with finite element calculations
Fig. A.9 Effect of thickness ratio on stiffening errors
a) Extrapolation error due to stiffening effect

b) Extrapolation error due to 2% noise in the upper gauge

Fig. A.10 Extrapolation error of the strain on the ungauged surface
Fig. A.11 Effect of modular ratio on stiffening errors

Fig. A.12 Effect of gauge aspect ratio on stiffening errors

Fig. A.13 Effect of Poisson's ratio of the plate on stiffening errors
a) A point load \( P \) at origin

b) A line load \( p \)

c) A pair of equal and opposite in plane line loads

Fig. A1.1 An infinite plate under in plane loads

Fig. A1.2a Membrane stress and strain distributions in an infinite plate under a pair of equal and opposite in plane line loads: Along x axis
Fig. A1.2b Membrane stress and strain distributions in an infinite plate under a pair of equal and opposite in plane line loads: Along y axis.

a) A line moment

b) A pair of equal and opposite line moments

Fig. A2.1 An infinite plate under line moments
Fig. A2.2 Moment and bending curvature distributions in an infinite plate under a pair of equal and opposite line moments
Appendix B

EFFECTIVE CROSS-SECTIONS OF ASYMMETRIC RINGBEAMS AND STIFFENERS FOR CYLINDRICAL SHELLS

B.1 Introduction

Ringbeams are often used to stiffen cylindrical tanks, silos and pressurised vessels. A ringbeam acts together with the axisymmetric shell to form a composite structure. The structural analysis of such a composite structure under general loadings is usually not simple, and has often to be numerical. However, the analysis under axisymmetric pressures may be simplified. Rotter (1983b) proposed an effective cross section of the ringbeam so that its membrane stress can be calculated from a simple ring analysis which is very easy for applications. Similar but less accurate methods have been proposed by Pippard and Baker (1957), Lambert (1968), API-620 (1970), Wozniak (1979), Ings (1981) and Wolf (1983). Gaylord and Gaylord (1984) suggested an alternative treatment of comparable complexity and accuracy.

However, a ringbeam is often placed at the end of a cylinder or where the wall thickness changes. Unsymmetrical cross sections are very often used for ring stiffeners in practice. Eccentric loadings and moments may also be applied on the ring stiffener. In these cases, the ring stiffener is subjected to not only tension, but also rotation which may cause significant deformations and stresses.

The purpose of this study is to investigate the deformation and stresses in a general unsymmetrical ring stiffener. It is found that a similar effective cross section as that proposed by Rotter (1983b) may be used to calculate the radial displacement and circumferential membrane stress, but additional rotational restraints provided by the shell should be included to calculate the rotation and bending stresses in the ringbeam. Moreover, the equation for calculating the length of the effective wall
section was different in Rotter's early work (1983b) when the ring is located at where the wall thickness changes or at the end of a cylinder. By including the rotation term for analysis, it can be simplified by using a unique equation regardless where a ringbeam is located.

B.2 Assumptions

The following assumptions were made in the analysis.

1. The cylindrical shell is thin so that it can be described by thin shell theory.
2. The ring stiffener is also thin and the difference between its radius and that of the shell can be ignored.
3. The ring stiffener is far from boundaries or discontinuities so that interactions among them can be ignored (long shell).

B.3 Deformation of a ringbeam stiffener attached to a shell

B.3.1 Internal forces of the ringbeam

For a given cylindrical shell of radius R, it is stiffened by a ring stiffener whose cross section may be of any shape (Fig. B.1a). A constant normal pressure p is applied on the cylindrical shell. The centroidal axis of the ring is also subjected to a uniform radial line loading P_r and a uniform line moment M_r (Fig. B.1b).

The cylindrical shell may be divided into two parts: one section above the ring stiffener (Section 1) and another below it (Section 2). Their thicknesses are t_1 and t_2 respectively. If the ringbeam is located at one end of the cylinder, either t_1 or t_2 should be set to zero. If the ring stiffener is attached to a uniform thickness wall, t_1=t_2. Note that the part of the wall to which the ring is attached is treated as part of the ring (Fig. B.1a).

The interaction forces between the shell sections and the ringbeam are shown in Fig. B.1b. The total radial line force P_r and line moment M_r acting on the centroidal axis of the ring are found as

$$P_r = ph + Q_1 + Q_2 + P$$  \hspace{1cm} (B.1a)
\[ m_r = M + M_1 - M_2 + phe + Q_2h_2 - Q_1h_1 \]  
(B.1b)

in which

\[ Q_1 = \text{meridional shear force between the ring and shell Section 1} \]
\[ Q_2 = \text{meridional shear force between the ring and shell Section 2} \]
\[ M_1 = \text{meridional bending moment between the ring and shell Section 1} \]
\[ M_2 = \text{meridional bending moment between the ring and shell Section 2} \]
\[ h = \text{height of the ringbeam which is attached on the wall} \]
\[ h_1 = \text{height of the ring above its centroidal axis} \]
\[ h_2 = \text{height of the ring below its centroidal axis} \]
\[ e = \text{eccentricity of the ring cross section (Fig. B.1).} \]

Parameters \( h, e, h_1 \) and \( h_2 \) may be found to have the following relationships

\[ h = h_1 + h_2 \]  
(B.2a)

\[ e = \frac{h_2 - h_1}{2} \]  
(B.2b)

The radial line force (Eq. B.1a) is resisted by a ring thrust \( N \) and the line moment (Eq. B.1b) is resisted by a bending moment \( M \) about \( r \) axis:

\[ N = p_r R^2 \]  
(B.3a)

\[ M = m_r R^2 \]  
(B.3b)

**B.3.2 Deformation of the ringbeam**

For the simple ring under radial load \( p_r \) and moment \( m_r \) (Eq. B.1), its radial displacement \( w_r \) and rotation angle \( \phi_r \) are given by (Flugge, 1973)

\[ w_r = \frac{p_r R^2}{EA} \]  
(B.4a)

\[ \phi_r = \frac{m_r R^2}{EI} \]  
(B.4b)
where $E$ is the Young's modulus for the ring material, and $A$ and $I$ are the area and the moment of inertia about $r$-$r$ axis for the ringbeam cross section respectively.

**B.3.3 Deformation of the shell sections**

Under the loadings as shown in Fig. B.1, the radial displacement $w_1$ and meridional rotation angle $\phi_1$ for the shell Section 1 at $z_1=0$ can be found as (Flugge, 1973; Calladine, 1983)

$$w_1 = \frac{R^2 p}{E t_1} - \frac{\lambda_1^2}{2\pi^2 D_1} \left( \frac{\pi}{\lambda_1} M_1 + Q_1 \right) \quad \text{(B.5a)}$$

$$\phi_1 = \frac{\lambda_1^2}{2\pi^2 D_1} \left( \frac{2\pi}{\lambda_1} M_1 + Q_1 \right) \quad \text{(B.5b)}$$

Similarly, the radial displacement $w_2$ and meridional rotation angle $\phi_2$ for the shell Section 2 at $z_2=0$ are given by

$$w_2 = \frac{R^2 p}{E t_2} - \frac{\lambda_2^2}{2\pi^2 D_2} \left( \frac{\pi}{\lambda_2} M_2 + Q_2 \right) \quad \text{(B.6a)}$$

$$\phi_2 = \frac{\lambda_2^2}{2\pi^2 D_2} \left( \frac{2\pi}{\lambda_2} M_2 + Q_2 \right) \quad \text{(B.6b)}$$

in which

$$D_i = \frac{E t_i^3}{12(1-\nu_i^2)} \quad \text{shell flexural rigidity}$$

$$\lambda_i = \frac{\pi}{\left[3(1-\nu_i^2)\right]^{1/4}\sqrt{R t_i}} \quad \text{meridional half bending wavelength for cylindrical shell}$$

which is reduced to $2.444\sqrt{R t_i}$ if $\nu_i=0.3$

$v_i = \text{Poisson's ratio of the material}$

$t_i = \text{the thickness of the shell and}$

$i = 1 \text{ and } 2 \text{ represents the upper and lower shell sections respectively.}$

From Eqs B.5 and B.6, the interaction forces between the ringbeam and the shell sections may be expressed as functions of deformations as follows.
\[ M_1 = \frac{2\pi^2 D_1}{\lambda_1^3} \left( \frac{\lambda_1}{\pi} \phi_1 + w_1 - \frac{R^2 p}{E t_1} \right) \]  \hspace{1cm} (B.7a)

\[ Q_1 = \frac{2\pi^2 D_1}{\lambda_1^3} \left( \frac{2R^2 p}{E t_1} - 2w_1 - \frac{\lambda_1}{\pi} \phi_1 \right) \]  \hspace{1cm} (B.7b)

\[ M_2 = \frac{2\pi^2 D_2}{\lambda_2^3} \left( \frac{\lambda_2}{\pi} \phi_2 + w_2 - \frac{R^2 p}{E t_2} \right) \]  \hspace{1cm} (B.7c)

\[ Q_2 = \frac{2\pi^2 D_2}{\lambda_2^3} \left( \frac{2R^2 p}{E t_2} - 2w_2 - \frac{\lambda_2}{\pi} \phi_2 \right) \]  \hspace{1cm} (B.7d)

**B.3.4 Compatibility**

The compatibility of deformation between the shell Section 1 and the ringbeam requires

\[ w_1 = w_r - \phi_1 h_1 \]  \hspace{1cm} (B.8a)

\[ \phi_1 = - \phi_r \]  \hspace{1cm} (B.8b)

Similarly, the compatibility between the shell Section 2 and ring beam requires

\[ w_2 = w_r + \phi_2 h_2 \]  \hspace{1cm} (B.9a)

\[ \phi_2 = \phi_r \]  \hspace{1cm} (B.9b)

The interaction forces between the ringbeam and the shell sections may be obtained by substituting Eqs B.8 and B.9 into Eqs B.7:

\[ M_1 = \frac{2\pi^2 D_1}{\lambda_1^3} \left[ w_r - \phi_1 h_1 - \frac{\lambda_1}{\pi} \phi_r - \frac{R^2 p}{E t_1} \right] \]  \hspace{1cm} (B.10a)

\[ Q_1 = \frac{2\pi^2 D_1}{\lambda_1^3} \left[ \frac{2R^2 p}{E t_1} - 2(w_r - \phi_1 h_1) + \frac{\lambda_1}{\pi} \phi_r \right] \]  \hspace{1cm} (B.10b)
\[ M_2 = \frac{2\pi^2 D_2}{\lambda_2} \left[ w_r + \frac{\lambda_2}{\pi} \frac{R^2 p}{E t_2} \right] \quad \text{(B.10c)} \]

\[ Q_2 = \frac{2\pi^2 D_2}{\lambda_2} \left[ 2R^2 - 2w_r + \frac{\lambda_2}{\pi} \phi_r \right] \quad \text{(B.10d)} \]

Substituting Eqs B.10 into B.1 gives

\[ p_r = p_0 - \frac{E A_1}{R^2} w_r - \frac{E A_2}{R^2} \phi_r \quad \text{(B.11a)} \]

\[ m_r = m_0 - \frac{E A_2}{R^2} w_r - \frac{E A_3}{R^2} \phi_r \quad \text{(B.11b)} \]

where

\[ p_0 = p \left( h + \frac{\lambda_1}{\pi} + \frac{\lambda_2}{\pi} \right) + P \quad \text{(B.12a)} \]

\[ m_0 = p \left( h e + \frac{\lambda_2^2}{2\pi^2} + \frac{\lambda_2 h_2}{\pi} - \frac{\lambda_1^2}{2\pi^2} - \frac{\lambda_1 h_1}{\pi} \right) + M \quad \text{(B.12b)} \]

\[ A_1 = \frac{\lambda_1 t_1}{\pi} + \frac{\lambda_2 t_2}{\pi} \quad \text{(B.12c)} \]

\[ A_2 = \frac{\lambda_2 t_2}{2\pi} \left( 2h_2 + \frac{\lambda_2}{\pi} \right) - \frac{\lambda_1 t_1}{2\pi} \left( 2h_1 + \frac{\lambda_1}{\pi} \right) \quad \text{(B.12d)} \]

\[ A_3 = \frac{\lambda_1 t_1}{2\pi} \left( 2h_1 + \frac{\lambda_1}{\pi} \right) + \frac{\lambda_1 t_1 h_1}{\pi} + \frac{\lambda_2 t_2}{2\pi} \left( 2h_2 + \frac{\lambda_2}{\pi} \right) + \frac{\lambda_2 t_2 h_2}{\pi} \quad \text{(B.12e)} \]

The deformations of the ringbeam can be obtained by substituting Eqs B.11 into B.4

\[ w_r = \frac{R^2 (I+A_3) p_0 - A_2 m_0}{E (A+A_1)(I+A_3) - A_3^2} \quad \text{(B.13a)} \]

\[ \phi_r = \frac{R^2 (A+A_1) m_0 - A_2 p_0}{E (A+A_1)(I+A_3) - A_3^2} \quad \text{(B.13b)} \]
B.4 Effective ring cross section

The purpose of this study is to investigate the possibility of calculating the deformations of and the stresses in the ring stiffener using simple ring analysis. It is assumed that effective shell segments of heights \( l_1 \) from each of the shell sections may act with the ring stiffener that may be treated as a thin ring. However, each of the shell segments may also provide an additional rotational restraint with stiffness \( k_i \) to the "effective ring" when the ring stiffener is subjected to rotation. The task here is to find the parameters \( l_1 \) and \( k_i \) so that the deformations of the ring centroidal axis based on "rotational restrained effective ring" analysis \( w_r \) and \( \phi_r \) are same as \( w_r \) and \( \phi_r \), which are obtained from rigorous shell analysis, i.e.

\[
\begin{align*}
  w_r &= w_{re} \\
  \phi_r &= \phi_{re}
\end{align*}
\]

(B.14a)  
(B.14b)

B.4.1 Geometrical properties of the effective ring cross section

For the proposed effective ring, its cross-sectional area is

\[
A_e = A + l_{c1}t_1 + l_{c2}t_2
\]

(B.15)

where \( l_{c1} \) and \( l_{c2} \) are the lengths of effective shell segments, which acts with the ring, from the shell Sections 1 and 2 respectively and subscript e represents effective ring cross section.

The centroid of the effective cross section \( C_e \) may differ from the centroid of the original ring stiffener \( C \) (Fig. B.2), and the distance between them can be found as

\[
\Delta e = \frac{l_{c2}t_2(h_2 + \frac{l_{c2}}{2}) - l_{c1}t_1(h_1 + \frac{l_{c1}}{2})}{A_e}
\]

(B.16)

The eccentricity and the moment of inertia of the effective ring cross section are therefore

\[
e_e = \frac{l_{c2} - l_{c1}}{2} + e - \Delta e
\]

(B.17a)
L_e = I + \frac{l_1^2 t_1}{12} + \frac{l_2^2 t_2}{12} + l_e t_1 \left( \frac{l_1}{2} + h_1 \right)^2 + l_e t_2 \left( \frac{l_2}{2} + h_2 \right)^2 - A_e \Delta e^2 \quad (B.17b)

### B.4.2 Loading on the effective ring centroidal axis

The loadings on the effective ring consist of two parts: those acting on the ring stiffener itself and those acting on the effective shell segments which act together with the ring. The equivalent radial line loading on the effective ring centroidal axis may then be found as

\[ p_{re} = p(h + l_{e1} + l_{e2}) + P_r \quad (B.18) \]

By taking the change of centroid by a distance of \( \Delta e \) into consideration, the moment on the centroidal axis of the effective ring is

\[ m_{re} = p(h + l_{e1} + l_{e2}) \left( \frac{l_{e2} - l_{e1}}{2} + e \right) + M - p_{re} \Delta e \quad (B.19) \]

in which the first two terms represent the moment about the original centroid of the ring stiffener and the last term represents the moment caused by the radial loading due to the change of position of the cross section centroid.

### B.4.3 Deformation of the ring stiffener based on effective ring analysis

The effective ring is analysed as a simple thin ring. The rotation angle of the effective cross section can be found as

\[ \phi_{re} = \frac{R^2}{E} \frac{m_{re}}{I_e + \frac{R^2}{E}(k_{e1} + k_{e2})} \quad (B.20) \]

What is of interests here is the deformation of the centroid of the original ring stiffener that differs from the centroid of the effective ring stiffener by \( \Delta e \). The radial displacement at the original centroid of the ring stiffener is

\[ w_{re} = \frac{R^2}{E} \left[ \frac{P_{re}}{A_e} - \frac{m_{re} \Delta e}{I_e + \frac{R^2}{E}(k_{e1} + k_{e2})} \right] \quad (B.21) \]
B.4.4 Effective ring cross section

Let the lengths of effective shell segments which act together with the ring stiffener

\[ l_{e1} = \frac{\lambda_1}{\pi} \]  
(B.22a)

\[ l_{e2} = \frac{\lambda_2}{\pi} \]  
(B.22b)

and the rotational restraint stiffnesses supplied by the shell sections

\[ k_{e1} = \frac{l_{e1}^3 t_1 E}{6R^2} \]  
(B.22c)

\[ k_{e2} = \frac{l_{e2}^3 t_2 E}{6R^2} \]  
(B.22d)

The parameters in Eqs B.12 can then be expressed as properties of the effective cross section (Eqs B.15 to B.19):

\[ A_1 = A_e \]  
(B.23a)

\[ A_2 = A_e \Delta e \]  
(B.23b)

\[ A_3 = I_e + A_e \Delta e^2 + \frac{k_{e1} R^2}{E} + \frac{k_{e2} R^2}{E} \]  
(B.23c)

\[ p_0 = p_{te} \]  
(B.23d)

\[ m_0 = m_{te} + p_0 \Delta e \]  
(B.23e)

With simple operations, the expressions on the right hand side of Eqs B.20 and 21 can be deduced by substituting Eqs B.23 into the right hand side of Eqs B.13. Thus, Eqs B.14 are satisfied and the parameters for an effective cross section are defined (Eqs B.22). The deformations of and stresses in the ring stiffener can therefore be obtained by using the simple “effective ring” analysis.
B.5 Deformations and stresses in shells nearby the ring stiffener

Once the deformations of the ring stiffener \( w_e \) and \( \phi_e \) are obtained using simple ring analysis, the deformations at the upper and lower edges of the ring can be obtained from Eqs B.8 and B.9. The interaction forces between the stiffener and the shell sections can then be obtained from Eqs B.7. The deformations and the meridional internal forces in the shell sections at a point \( z_i \) away from the ring-shell junction (Fig. B.1) are given by (Flugge, 1973; Calladine, 1983)

\[
\begin{align*}
    w_i(z_i) &= \frac{R^2_p}{E t_i} - \frac{\lambda_i^3}{2\pi D_i} \left[ \frac{\pi}{\lambda_i} M_i e^{-\pi z_i / \lambda_i} \left( \cos \frac{\pi}{\lambda_i} z_i - \sin \frac{\pi}{\lambda_i} z_i \right) + Q_i e^{-\pi z_i / \lambda_i} \cos \frac{\pi}{\lambda_i} z_i \right] \quad \text{(B.24a)} \\
    \frac{d w_i(z_i)}{dz_i} &= \frac{\lambda_i^2}{2\pi^2 D_i} \left[ \frac{2\pi}{\lambda_i} M_i e^{-\pi z_i / \lambda_i} \cos \frac{\pi}{\lambda_i} z_i + Q_i e^{-\pi z_i / \lambda_i} \left( \cos \frac{\pi}{\lambda_i} z_i + \sin \frac{\pi}{\lambda_i} z_i \right) \right] \quad \text{(B.24b)} \\
    M_i(z_i) &= M_i e^{-\pi z_i / \lambda_i} \left( \cos \frac{\pi}{\lambda_i} z_i + \sin \frac{\pi}{\lambda_i} z_i \right) + \frac{\lambda_i}{\pi} Q_i e^{-\pi z_i / \lambda_i} \sin \frac{\pi}{\lambda_i} z_i \quad \text{(B.24c)} \\
    Q_i(z_i) &= -\frac{2\pi}{\lambda_i} M_i e^{-\pi z_i / \lambda_i} \sin \frac{\pi}{\lambda_i} z_i + Q_i e^{-\pi z_i / \lambda_i} \left( \cos \frac{\pi}{\lambda_i} z_i - \sin \frac{\pi}{\lambda_i} z_i \right) \quad \text{(B.24d)}
\end{align*}
\]

where \( i = 1 \) or 2 represent the shell Sections 1 and 2 respectively.

B.6 Conclusions

The deformations and internal forces of a cylindrical shell stiffened by a general unsymmetrical ringbeam has been analysed using shell bending theory. Both radial forces and moments may be applied on the ring stiffeners. The results show that the deformation of and the stresses in the ring stiffener can be calculated using a simple ring analysis with an "effective ring cross section" which consists of the ring stiffener and segments of effective walls with additional terms of rotational restraints supplied by the walls. The length of effective segments of walls can be calculated using a very simple formulation (Eq. B.22), independent of the location of the ring stiffener. The deformation and stresses in the walls adjacent to the ring stiffener can also be found from the ring deformations.
B.7 Notation

A \quad \text{cross-sectional area/coefficient}
D \quad \text{shell flexural rigidity}
E \quad \text{Young's modulus}
I \quad \text{moment of inertia}
M \quad \text{bending moment}
M_r \quad \text{uniform line moment acting on the ring centroidal axis}
N \quad \text{ring thrust}
P_r \quad \text{uniform radial line load acting on the ring centroidal axis}
Q \quad \text{shear force}
R \quad \text{radius of the cylindrical shell}
e \quad \text{eccentricity of the ring cross section}
h \quad \text{height of ringbeam attaching on the shell}
l \quad \text{length of effective shell segment which acts with the ring}
m_r \quad \text{total line moment on the ring centroidal axis}
m_{re} \quad \text{total line moment on the effective ring centroidal axis}
p \quad \text{internal pressure}
p_r \quad \text{total radial line load on the ring centroidal axis}
p_{re} \quad \text{total radial line load on the effective ring centroidal axis}
t \quad \text{thickness}
w \quad \text{radial displacement}
z \quad \text{vertical coordinate}
\phi \quad \text{rotation angle}
\lambda \quad \text{meridional half bending wavelength}
v \quad \text{Poisson's ratio}

Subscript
1 \quad \text{shell section above the ringbeam}
2 \quad \text{shell section below the ringbeam}
e \quad \text{effective ring cross section}
r \quad \text{ring}
Fig. B.1 Notation for a ring stiffener in a cylindrical shell

Fig. B.2 Effective ring cross section
Appendix C

PREDICTION OF WALL PRESSURES IN COAL SILOS

C.1 Introduction

Coal silos have presented many problems in both flow and structural failures in recent years (Johnston, 1981) and continue to give rise to vigorous debates on how silo design procedures should be improved (ACI 313, 1989; ACI 313, 1992a). It has been argued that a better understanding of the properties of the ensiled materials and their interaction with the silo structure is one of the critical factors in improving design (Johnston, 1981).

This appendix describes part of a collaborative project undertaken to improve understanding of the mechanical behaviour of coal and its effect on solids arching and pressures acting on silo walls (Lohnes et al, 1994). The material tests and the constitutive modelling of samples of finely ground Shannon coal have been fully described elsewhere (Lohnes et al, 1994). This appendix presents finite element predictions of static silo wall pressures using the constitutive models derived from the material tests. The predictions are compared with those of existing classical theories and design codes from different countries.

Although the pressures during discharge of silos are often larger than those following filling, the design pressures are always related to the filling pressures. Agreement on the filling pressures is therefore the first requirement, and is the focus of this study.
It may also be noted that flow pressures do not exceed filling pressures in silos with internal flow.

Previous studies which used finite element method (FEM) to predict silo pressures mostly concentrated on complex empirically-based constitutive models (e.g. Bishara et al., 1983; Haussler and Eibl, 1984; Runesson and Nilsson, 1986), where the significance of many of the material parameters used in the computational model was not at all clear. In addition, few of these computations were related to the predictions of simple theories or codified rules, so it is not easy to see how standards for silo design can be improved as a result of the computations. Further, none of them appear to have studied the difficult and industrially important bulk solid coal. The present work attempts to avoid these shortcomings by describing the predictions in terms of well-established and widely recognised parameters, and by relating the results to those of current codified design methods.

C.2 Classical theories and design codes

Many classical theories exist for predicting the static pressures in silos after initial filling (Koenen, 1895; Janssen, 1896; Pieper and Wenzel, 1963; Walker, 1966; Homes, 1972; Walters, 1973a; Jenike et al., 1973; Reimbert and Reimbert, 1976). Most of these theories were derived from plastic equilibrium considerations on the basis of different simplifying assumptions. Discussions of these theories and fuller descriptions may be found elsewhere (Arnold et al., 1980; Ooi and Rotter, 1990b; Gaylord and Gaylord, 1984). A brief outline is given below.

Janssen (1896) was the first to derive the differential equation for the equilibrium of a slice of solid in a silo. He assumed that the ratio of horizontal pressure against the wall to the mean vertical stress in the stored solid (the lateral pressure ratio $k$) is invariant with depth in the silo, but did not define it. Considering vertical equilibrium and a horizontal top surface, he derived the pressure equation:
\[ p = p_0 \left[ 1 - e^{-2kz/R} \right] \]  

in which the maximum asymptotic pressure \( p_0 = \gamma R / 2\mu \), \( R \) is the radius of the silo, \( \mu \) is the wall friction coefficient, \( \gamma \) is the unit weight of the stored solid and \( z \) is the depth below the solid surface.

Many authors adopted the Janssen pressure distribution and attempted to find a good means of determining the lateral pressure ratio \( k \). Koenen (1895) proposed that the Rankine active pressure ratio should be used for this quantity. Jaky (1948) proposed that \( k \) should be given by \( k = 1 - \sin \phi \), a relationship which is widely used in the field of soil mechanics. The parameter \( \phi \) in this expression is the solid's internal friction angle. Pieper and Wenzel (1963) adopted this too. Walker (1966) deduced \( k \) by assuming that all the material is at active failure and that material adjacent to the wall is sliding down the wall. Walters (1973a) extended Walker's analysis to include non-uniformity of the vertical stresses and assumed that all the solid is at active failure. Jenike et al (1973) empirically proposed \( k = 0.40 \) and Homes (1972) similarly suggested \( k = 0.45 \) for most common solids. Ooi and Rotter (1990b) showed that under storing conditions, much of the solid is not at failure, so the lateral pressure ratio \( k \) can be approximated by the elastic value for a solid confined within an elastic shell: \( k = \nu / (1 - \nu + \alpha) \), in which \( \nu \) is the Poisson's ratio and \( \alpha \) is the relative stiffness parameter and is given by \( \alpha = E_t R / E_m t \) (Ooi, 1990).

Reimbert and Reimbert (1976) produced an alternative solution to the Janssen differential equation by curve fitting the experimental results subject to the equilibrium equations, thus effectively allowing \( k \) to vary between zero at the surface and the Rankine active pressure ratio at a great depth. Some comparisons of this solution with Janssen have been made (Briassoulis, 1986; Briassoulis, 1991; Reimberts and Reimberts, 1987). Much simpler design proposals for squat silos
were made by Lambert (1968) who proposed Rankine theory (1857) and Stewart (1972) who recommended Coulomb theory (1773).

Most design codes use Janssen theory, but different codes have adopted different values for the lateral pressure ratio k. A few examples are given here: ACI 313 (1992a) adopted $k=1-\sin \phi$ (Jaky, 1948); ISO (1991) and EC1 Part 4 (1992) adopted $k=1.1(1-\sin \phi)$; AS3774 (1990) adopted Walker's expression (1966).

C.3 Constitutive models for Shannon coal

Many constitutive models have been proposed for bulk solids and soils (e.g. Haussler and Eibl, 1984; Runesson and Nilsson, 1986). Most of those intended for computer applications involve a large number of parameters which must be known before any prediction is possible, so that very extensive solids testing is required. One important feature of the models used here is that only a limited number of parameters is required to describe the behaviour and these parameters can be easily extracted from a few simple tests. This allows the predictive method to be useful for practical applications involving other bulk solids.

The stiffness of a granular solid is known to vary significantly with confining stress. The chief aim in this project was to investigate the effect of the non-linearly varying bulk solid stiffness on the solid's behaviour in the silo and the pressures on silo walls. The purpose is to explore simple macroscopic changes, and to compare the outcomes with simple predictors, such as are used in codes and standards. For these reasons, two simple models (capable of modelling the non-linear elastic stiffness accurately) were chosen first, and a more complete model thereafter.

The derivations of the constitutive parameters for Shannon coal and the verification of the models are presented elsewhere (Lohnes et al, 1994). The parameters were derived from a series of isotropic compression and triaxial compression tests (Lohnes
et al, 1994), using a low stress triaxial apparatus developed at Iowa State University (Smith and Lohnes, 1981). A brief outline of the three chosen models, the hyperbolic model, the power law model and the modified Cam-clay model, is given here.

**C.3.1 Hyperbolic model**

The hyperbolic model (Duncan and Chang, 1970) characterises the nonlinear elastic part of the deviatoric stress-strain relation in the form of a hyperbola:

\[ \sigma_1 - \sigma_3 = \frac{E_1 \varepsilon}{1 - \varepsilon/E_0} \]  \hspace{1cm} (C.2)

Here, \( \sigma_1 - \sigma_3 \) is the deviatoric stress and \( \varepsilon \) is the axial strain. The two parameters which represent the nonlinear elastic part of this relationship are a reference strain \( \varepsilon_0 \) which describes the curvature of the hyperbola and the initial tangent modulus \( E_1 \). The initial tangent modulus was taken to vary with confining pressure \( \sigma_3 \) according to the Janbu equation (Janbu, 1963):

\[ E_i = K P_a \left( \frac{\sigma_3}{P_a} \right)^n \]  \hspace{1cm} (C.3)

where \( K \) is the modulus number and \( n \) is the modulus exponent. \( P_a \) is atmospheric pressure which is introduced to aid in conversion of one system of units to another. The model parameters are listed in Table C.1. They were obtained by regression analyses on the stress-strain data up to axial strains of 15%. The reference strain \( \varepsilon_0 \) was found to be relatively independent of confining stress and an average value was adopted.

Plastic yielding was modelled using the Mohr-Coulomb yield condition with a perfectly plastic plateau. The plastic deformations were represented by a non-associated flow with a dilation angle \( \psi \), which was taken to be a material constant.
and was determined from the volumetric response at large strains. In this case, Shannon coal displayed constant volume deformation at large strains, giving a zero dilation angle.

Table C.1 Parameters of various constitutive models for Shannon coal

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hyperbolic</td>
<td>$K=259$ kPa, $n=0.786$, $\varepsilon_c=0.420$</td>
</tr>
<tr>
<td>Power law</td>
<td>$K=109$ kPa, $n=1.00$, $\varepsilon_c=0.155$, $b=2.00$</td>
</tr>
<tr>
<td>Modified cam clay</td>
<td>$M=1.72$, $N=1.87$, $\lambda=0.0367$, $\kappa=0.0092$, $v=0.23$</td>
</tr>
</tbody>
</table>

C.3.2 Power law model

This model differs from the hyperbolic model only in the nonlinear elastic part of the deviatoric stress-strain relation. The power law gives an improved description (Ooi, 1990) because it increases the predicted stiffness of the solid at very low strains (below 1%) and permits both hardening and softening to be represented. This is particularly important in overconsolidated material, which is typically found in silos. This model therefore has better potential when applied to practical problems such as arching at the hopper outlet. The nonlinear elastic part of the deviatoric stress-strain relation is represented by:

$$\sigma_1 - \sigma_3 = E_i \varepsilon_b (\varepsilon/\varepsilon_i + 1)^b$$  \hspace{1cm} (C.4)

Again, the model parameters were determined by regression analyses on the test data and are given in Table C.1. The initial tangent modulus $E_i$ was expressed in the form of Janbu equation (Eq. C.3). The value of $b$ was taken as a mean value because it varied little between stress levels.
C.3.3 Modified Cam-clay model

In modified Cam-clay (Burland, 1965), the critical state line is defined in the stress-strain field by two equations (Schofield and Wroth, 1968):

\[ q = M p \]  \hspace{1cm} (C.5)

\[ v = \Gamma - \lambda \ln p \]  \hspace{1cm} (C.6)

where the stress parameters \( p \) and \( q \) are the mean effective stress and the deviatoric stress respectively, and \( v \) is the specific volume of the material (the volume of the material containing a unit volume of the solid particles).

For purely frictional material, the material constant \( M \) is given by

\[ M = \frac{6 \sin \phi'}{3 - \sin \phi'} \]  \hspace{1cm} (C.7)

In addition to the above, the volumetric changes on isotropic virgin loading are defined as

\[ v = N - \lambda \ln p \]  \hspace{1cm} (C.8)

while those during unloading and reloading from any point on the virgin line are represented by

\[ v = v_e - \kappa \ln p \]  \hspace{1cm} (C.9)

The material constant \( N \) is the value of the specific volume at a unit mean stress (\( p=1 \)), and it is related to the other material constants by

\[ N = \lambda - \kappa + \Gamma \]  \hspace{1cm} (C.10)
This critical state model, normally referred to as modified Cam-clay (Burland, 1965; Schofield and Wroth, 1968; Roscoe and Burland, 1968), has five parameters which are material constants: M, N, λ, κ and a further elastic constant which can either be a constant Poisson's ratio ν or a constant shear modulus G. The parameters derived from the test data are given in Table C.1.

C.3.4 Other relevant properties

Naturally the coal's bulk density increases under increasing consolidating stress. Figure C.1 shows how the bulk density of Shannon coal varies with the isotropic consolidating stress. The existing theories and codified rules normally use a constant value of bulk density in pressure calculations. The stress dependency of density can easily be modelled in the finite element program. However, to permit unbiased comparisons, a constant bulk density of γ=7.64 kN/m³ was used in all the analyses presented below. This value corresponds to a consolidating stress of 19.3 kPa, which is the mean stress in the silo as predicted by the Janssen equation.

In the experimental programme, wall friction tests on several different coals at two moisture contents were conducted using both mild and stainless steel (Lohnes et al, 1994). The average wall friction angles for stainless steel were 17.3°±1.2° and 17.1°±1.2° for moisture contents of 5% and 10% respectively. The corresponding values for mild steel were 19.0°±1.3° and 19.6°±1.6° respectively. There is very little variation between the wall friction angle at different moisture contents, or even for the different coals. In the following calculations, a wall friction angle of 19° was used throughout.

Finally for all the three constitutive models, the description of the elastic volumetric response is not complete without defining another elastic constant. This is given here by a constant Poisson's ratio ν, which was calculated from the volumetric strain data. An iterative procedure was used to deduce an appropriate single value from the test
data. First the approximate stress level in the silo is estimated and used to define a value of the principal stress ratio \( k' = \sigma_1/\sigma_3 \). The secant Poisson's ratio can then be calculated from the ratio of the volumetric strain \( \varepsilon_v \) to the axial strain \( \varepsilon_a \) at the appropriate deviator stress level using \( v = (1 - \varepsilon_v/\varepsilon_a)/2 \). From this first estimate, the stress ratio \( k' \) is calculated using \( k' = v/(1 - v) \), which is the equation for one-dimensional compression in a rigid smooth cylinder. This second estimate of \( k' \) is then used to find the second estimate of the Poisson's ratio. The process is repeated until the solution converged. The Poisson's ratio obtained is then the value which correspond to the stress ratio which might be expected to occur in a rigid silo. A mean value of \( v = 0.23 \) was obtained from all the tests and is used here.

### C.4 Finite element analyses of a squat silo

The comparison between the finite element and other predictions was undertaken by analysing a squat circular steel silo (Fig. C.2). The calculations were performed for the condition after filling with the coal. The silo is of height \( H = 10 \text{ m} \), radius \( R = 5 \text{ m} \) and wall thickness \( t = 10 \text{ mm} \). Shannon coal was characterised by \( \phi = 42.0^\circ \), \( v = 0.23 \), \( \phi_w = 19.0^\circ \) and an angle of repose of \( \phi_r = 25^\circ \).

The silo was modelled as a flexible axisymmetric structure, with a Mohr-Coulomb adhesion-frictional interface element between the bulk solid and the structure (Goodman et al, 1968; Ooi and Rotter, 1990b). Both the coal and the silo walls were modelled using the quadratic serendipity isoparametric element (Zienkiewicz and Taylor, 1989). Shannon coal was represented by the three constitutive models described above, as well as with a simple linear elastic treatment. In the linear elastic treatment, a constant Young's modulus of 1.8 MPa was estimated from test data. The filling process was not modelled and initial stresses were ignored. The loading from stored solid was applied as a "switched-on" gravity loading.
The calculations were performed using the finite element programs AFENA (Carter and Balaam, 1988) and ABAQUS (1995). AFENA was used for all four models described above, while ABAQUS was only used for the Modified Cam-clay model. The focus of this appendix is on the pressures which would occur after initial filling. Thus the solid is treated as normally consolidated. As a result, according to the modified Cam-clay model, the coal is at the yield surface from the start of loading.

C.5 Predictions for the example silo

The finite element predictions of silo wall pressures using each of the four above models are shown in Figs C.3 and 4. The normal pressure distributions using various classical theories and design codes are shown in Figs C.5 and 6 respectively. The Janssen pressures, using a lateral pressure ratio of \( k = 0.30 \) deduced from \( k = v/(1-v+\alpha) \) (Ooi, 1990) is shown as a reference case in all the plots.

It may be noted that the predictions from the linear elastic treatment and the non-linear elastic treatments (hyperbolic and power law models) are all very close to the Janssen distribution with \( k = 0.30 \). The slight differences are that the pressures fluctuate slightly and are slightly larger than the Janssen values near the bottom. For the storing condition, the nonlinearity of the solids stiffness seems to have little effect on the wall pressures.

However predictions using the modified Cam-clay model were found to be highly dependent on the shape of the assumed yield function. Figure C.7 shows how the parameter \( \beta \) is used to modify the shape of the yield function on the "wet" side of the critical state: the original modified Cam-clay model corresponds to a value of \( \beta = 1.0 \). A value between 0.5 and 1.0 is usually adopted. The predictions showed that much larger pressures are found when \( \beta = 1.0 \), with the maximum pressure some 40% larger than for \( \beta = 0.5 \). The prediction when \( \beta = 0.5 \) is very close to the finite element
predictions from the elastic and non-linear elastic-plastic models. Further studies are needed to explore the full significance of this parameter.

The maximum pressures predicted by the classical theories and codes vary from 14 kPa (0.25p) to 27 kPa (0.49p): the discrepancy between different predictions can reach 100%. The greater part of these differences arises from differences in the adopted value of the lateral pressure ratio k. Although these predictions depend on the values of the input parameters, it is clear that different theories and codes still give a wide scatter of pressure predictions.

The FEM calculations also indicate that the pressure variation near the bottom of the silo is much influenced by the silo base boundary condition: the restraint of the base naturally influences the pressures, but this is completely ignored in all classical theories, which are also used as the basis of code descriptions. The conical surcharge at the solid surface is traditionally accommodated by adjusting the origin of the depth coordinate to one third of the conical pile height when Janssen's theory or its modifications are used (e.g. Pieper and Wenzel, 1963; Walker, 1966; Homes, 1972; Walters, 1973a; Jenike et al, 1973). This results in finite pressures above the first wall contact, which are clearly wrong: the wall pressure at the highest wall contact must be zero (therefore k must be zero). This phenomenon is especially important in squat silos.

C.6 Conclusions

The appendix has described part of a collaborative project on the modelling of coals for silo applications. Three constitutive models have been used to represent the mechanical behaviour of Shannon coal using experimentally determined values of the model parameters based on triaxial tests. The models have been used to calculate the wall pressures in an example silo using the finite element method.
The finite element predictions from linear elastic, hyperbolic and power law models all gave almost the same pressure distribution, which is very close to the Janssen distribution with the lateral pressure ratio as proposed by Ooi (1990). The modified Cam-clay prediction was found to be sensitive to the shape of the yield function, giving much larger pressures when the value of the parameter $\beta$ is large. Further study is required to explore the significance of this parameter, and simple means of deducing its value from material tests.

The classical theories and design codes for silo loading have been briefly reviewed and compared with the finite element predictions. The finite element predictions lie close to the average of all the classical theory predictions, but they are at the lower bound of the design code values.

C.7 Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>Young's modulus</td>
</tr>
<tr>
<td>$H$</td>
<td>Height of the silo</td>
</tr>
<tr>
<td>$M$</td>
<td>critical state strength parameter (MCC model)</td>
</tr>
<tr>
<td>$N$</td>
<td>value of the specific volume at a unit mean stress (MCC model)</td>
</tr>
<tr>
<td>$P_a$</td>
<td>atmospheric pressure</td>
</tr>
<tr>
<td>$R$</td>
<td>radius of the silo</td>
</tr>
<tr>
<td>$k$</td>
<td>lateral pressure ratio</td>
</tr>
<tr>
<td>$k'$</td>
<td>principal stress ratio</td>
</tr>
<tr>
<td>$p$</td>
<td>normal wall pressure/mean effective stress</td>
</tr>
<tr>
<td>$q$</td>
<td>deviatoric stress</td>
</tr>
<tr>
<td>$t$</td>
<td>thickness of wall</td>
</tr>
<tr>
<td>$v$</td>
<td>specific volume</td>
</tr>
<tr>
<td>$\beta$</td>
<td>modified Cam-clay yield parameter</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>axial strain</td>
</tr>
<tr>
<td>$\phi$</td>
<td>friction angle</td>
</tr>
<tr>
<td>Symbol</td>
<td>Meaning</td>
</tr>
<tr>
<td>--------</td>
<td>---------</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>bulk density</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>consolidation parameter (MCC model)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>consolidation parameter (MCC model)</td>
</tr>
<tr>
<td>$\mu$</td>
<td>wall friction coefficient</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Modified Cam-clay parameter</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Poisson's ratio</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>principal stress</td>
</tr>
<tr>
<td>$\psi$</td>
<td>dilation angle</td>
</tr>
</tbody>
</table>

Subscripts
- $a$: axial
- $i$: initial
- $r$: repose
- $s$: solid
- $w$: wall
- $\infty$: asymptotic limit
Fig. C.1 Bulk density variation of Shannon coal

Fig. C.2 Example silo

a) geometry  
b) finite element mesh
Fig. C.3 Normal wall pressures predicted by finite element method

Fig. C.4 Normal wall pressures predicted by finite element method (Modified Cam-clay)
Fig. C.5 Normal wall pressures predicted by classical theories

Fig. C.6 Normal wall pressures predicted by design codes

Fig. C.7 Modified Cam-clay yield surface
REFERENCES


17. Askegaard, V. and Munch-Andersen, J. (1984) "Results from Tests with Normal and Shear Stress Cells in Medium Scale Silos" (in English), Report R191, Department of Structural Engineering, Technical University of Denmark, 10 pp.


125. Gram, J.P. (1883) "Ueber die Entwickelung reeler Functionen in Reihen Mittelst der Methode der kleinsten Quadrate", Journal fur de reine und angewandte Mathematik 94, pp.41-73


145. Ings, N.L. (1981) "Ring Beams", Structural Aspects of Steel Silos and Tanks, School of Civil and Mining Engineering, University of Sydney.


178. Koenen, M. (1895) "Berechnung des Seitenund Bodendrucks in Silos" (Calculation of Side and Floor Pressure in Silo Walls), Zentralblatt der Bauverwaltung, 16, 446-449.


185. Laplace, P.S. (1816) Premier Supplément to Laplace (1812)


224. Nielsen, J. (1979) "Opmaling af Silo i Karpalund" (in Danish), Nordic Group for Silo Research, Report No. 4, Technical University of Denmark, Department of Structural Engineering, 21pp (Measurements of the imperfect geometry of the Karpalund Silo)


229. Nielsen, J. and Kristiansen, N.O. (1979) "Pressure Measurements on a Silo in Karpalund" (in Danish), Nordic Group for Silo Research, Report No. 5, Technical University of Denmark, Department of Structural Engineering.


Stress and Strain Behaviour of Particulate Solids- Silo Stresses, Prague, August 1990, 10 pp.


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Rotter, J.M. (1985b) "Bending Theory of Shells for Bins and Silos", Design of Steel Bins for the Storage of Bulk Solids (ed. J.M. Rotter), University of Sydney, Sydney, Australia, pp71-81

Rotter, J.M. (1985c) "Analysis and Design of Ringbeams", Design of Steel Bins for the Storage of Bulk Solids (ed. J.M. Rotter), The University of Sydney, School of Civil and Mining Engineering and the Civil and Mining Engineering Foundation, Australia, pp.164-183.


