Splitting of the Hochschild cohomology of von Neumann algebras

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Dimosthenis Drivaliaris)
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Abstract

This thesis is concerned with the study of splitting for bounded and completely bounded Hochschild cohomology of von Neumann algebras.

Having as a starting point the notions of a split and a split exact complex, which are standard in homological algebra, we define five types of splitting for the (completely) bounded Hochschild cohomology group of $\mathcal{A}$, with coefficients in $X$, $\mathcal{H}_{cb}^n(\mathcal{A}, X)$. In general we could say that the study of splitting is the study of the invertibility of the coboundary map $\partial^n$. We show that all types of splitting are closely connected to geometric properties of the space of n-coboundaries $\mathcal{B}_n^\mathcal{A}(\mathcal{A}, X)$ and of the space of n-cocycles $\mathcal{Z}_n^\mathcal{A}(\mathcal{A}, X)$ and discuss the relation between the different types of splitting.

Then we define module actions on spaces of maps from, into and between $\mathcal{A}$-modules. Given an $\mathcal{A}$-module $X$ and a space $Y$ we make $L^1(Y, X)$ into an $\mathcal{A}$-module containing $X$. The modules $L^1(Y, X)$ inherit duality and normality from the module $X$; the completely bounded case is particularly interesting since we have to define a matricial norm structure on the tensor product of two matricially normed spaces $U$ and $V$ such that the tracial dual of $U \otimes V$ is completely isometrically isomorphic to the space of completely bounded maps from $U$ into the tracial dual of $V$. On the other hand we define a module structure on $L^1(X, Y)$ which generalises the notion of the dual $\mathcal{A}$-module of $X$. The completely bounded case is again non-trivial because we must consider a new matricial norm structure on $L^1_{cb}(X, Y)$ generalising the matricial norm structure of the tracial dual. Duality and normality of $L^1(X, Y)$ are also discussed.

We continue by studying the relation between splitting and the modules $L^1(Y, X)$. We show that whereas the vanishing of $\mathcal{H}_n^\mathcal{A}(\mathcal{A}, X)$ does not imply the vanishing of $\mathcal{H}_n^\mathcal{A}(\mathcal{A}, L^1(Y, X))$, for all spaces $Y$, the splitting of $\mathcal{H}_n^\mathcal{A}(\mathcal{A}, X)$ immediately implies the splitting of $\mathcal{H}_n^\mathcal{A}(\mathcal{A}, L^1(Y, X))$, for all spaces $Y$. We also prove that the third type of splitting of $\mathcal{H}_n^\mathcal{A}(\mathcal{A}, X)$ is equivalent to the vanishing of $\mathcal{H}_n^\mathcal{A}(\mathcal{A}, L^1(Y, X))$, for all spaces $Y$. The groups $\mathcal{H}_n^\mathcal{A}(\mathcal{A}, X^*)$ and $\mathcal{H}_n^\mathcal{A}(\mathcal{A}, L^1(X, Y^*))$ are related in a similar manner.

We finish with an investigation of splitting for the Hochschild cohomology of von Neumann algebras. We start by showing that the averaging results of Johnson, Kadison and Ringrose can be reformulated to similar results about splitting. This leads to splitting for both the bounded and the completely bounded
Hochschild cohomology complex of an amenable von Neumann algebra, with coefficients in a normal bimodule. We continue with the exceptional case of the cohomology into $B(H)$. There the splitting of $H^0_{\text{cb}}(\mathcal{M}, B(H))$ implies injectivity for $\mathcal{M}$ and in most of the cases the same holds for splitting of $H^1_c(\mathcal{M}, B(H))$. Similar results are obtained in the more general case of the cohomology into an injective von Neumann algebra containing $\mathcal{M}$. After that we prove that in all the cases where the Hochschild cohomology groups of a von Neumann algebra are known to vanish, they also split.
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Chapter 1

Introduction and preliminaries

1.1 Introduction

Hochschild cohomology of Banach, $C^*$- and von Neumann algebras was developed by Johnson, Kadison and Ringrose in their work in the late 1960's and early 1970's, which itself arises on the one hand from Hochschild's work on the cohomology groups of associative linear algebras and on the other hand from questions concerning derivations, which go back to Kaplansky's work in the early 1950's. The vanishing of Hochschild cohomology groups yields stability results about certain characteristics of the algebra -for example perturbations (see [J5] and [RaeTay] or [SSm1], Chapter 7) and extensions (see [Ri2], Section 2.3 or [Ri3], Section 2.2)- and results about derivations. More generally the vanishing of the cohomology groups, for all bimodules belonging in a particular family, offers characterisations for group algebras, $C^*$-algebras and von Neumann algebras. If the cohomology groups don't vanish, then they provide invariants which we can use to classify a class of algebras. Although such invariants have not been obtained using Hochschild cohomology groups, the "Elliott program" for the classification of $C^*$-algebras via $K$-theory offers an example of such a use of homological invariants in $C^*$-algebras theory (for more information on the $K$-theory of $C^*$-algebras see [Bl], [Da] and [We-O]).

Although we present the constructions and the results of Chapters 2, 3 and 4 for Banach algebras (in the bounded case) and for operator algebras (in the completely bounded case), the whole project started by trying to answer the following question concerning Hochschild cohomology groups of von Neumann algebras: for which von Neumann algebras $\mathcal{M}$ do the groups $H^n_\ast(\mathcal{M}, \mathcal{L}^1_\ast(\mathcal{M}, \mathcal{M}))$ and $H^n_\ast(\mathcal{M}, \mathcal{L}^1_\ast(\mathcal{M}, B(H)))$, where $\ast$ means either bounded or completely bounded, vanish?

In the fields of $C^*$- and von Neumann algebras investigations concerning Hochschild cohomology groups have proven to be very fruitful. First of all a cohomolog-
ical property, amenability, has been used by Connes to characterise injective von Neumann algebras and by Haagerup to characterise nuclear $C^*$-algebras. Moreover questions concerning cohomology groups were one of the main motivations behind the introduction of the Haagerup tensor product, by Effros and Kishimoto, and of completely bounded multilinear maps, by Christensen and Sinclair, in the 1980's; those two notions in their turn were central in the birth of the, now flourishing, field of operator spaces. Lately a connection between cohomological questions and decomposition properties of von Neumann algebras has started to emerge. And last, but not at all least, a still open cohomological problem -are all derivations from a $C^*$-algebra, acting on a Hilbert space $H$, into $B(H)$ inner?- has been proven to be equivalent to one of the most famous unanswered questions about $C^*$-algebras, the similarity problem.

Later in the thesis we will give the definitions of an amenable Banach/operator/von Neumann algebra and discuss in detail some of their properties. We would like to try to explain in a few words why the notion of amenability is so important. In general we can say that we call an algebra amenable if its cohomology groups vanish for all bimodules contained in a certain family. If we manage to choose the right family, then we are richly rewarded. We can construct a module and a cocycle from the algebra into the module and then use the vanishing of the cohomology group of the algebra, with coefficients in that module, to obtain an "invariant mean" for the algebra.

1.1.1 Overview of the thesis

All chapters, and many sections, of this thesis begin with a discussion which describes most of the results that they contain and a review of the existing results, where that applies, related to their topic. In the following few paragraphs we will try to point out the central theme of each chapter.

Section 1.2 contains most of the definitions and results about matricially normed spaces and Hochschild cohomology that the reader will need to go through the thesis. In the part about matricially normed spaces we present some new constructions related to the reversed tracial dual of a matricially normed space. We define a matricial norm structure on the space of completely bounded maps between two matricially normed spaces which generalises the matricial norm structure on the reversed tracial dual. We also define two matricial norm structures on the tensor product of two matricially normed spaces, which give duality results, similar to the well-known duality between the projective tensor product and the space of bounded linear maps, for the reversed tracial dual and the standard and the reversed tracial matricial norm structure on the space of completely bounded
maps.

In Chapter 2 we introduce the main idea of the thesis, the notion of splitting of Hochschild cohomology groups. We define five different types of splitting. The first four depend on the existence of a right inverse for the coboundary operator. The differences between them follow from choosing different domains for this inverse. The fifth type of splitting depends on the existence of a decomposition of the identity map on the space of cochains in terms of two consecutive coboundary operators and their right inverses. Whereas the study of Hochschild cohomology groups offers information about the relation between cocycles and coboundaries, the study of splitting provides mainly results about the complementation of the spaces of cocycles and coboundaries in the space of cochains. In the bounded case those results are proven to be equivalent to splitting. The same does not hold for the completely bounded case, because of the lack of an inverse mapping theorem for completely bounded maps. Many times when we study the splitting of cohomology groups the sum is more than the parts, i.e. the splitting of two consecutive cohomology groups gives us results that don't follow from either of the two splittings.

Chapter 3 is concerned with two ways of generating modules from a given module. Let $\mathcal{A}$ be a (Banach/operator) algebra, $X$ be a (Banach/completely bounded) $\mathcal{A}$-bimodule and $Y$ be a (Banach/matricially normed) space. Then we can make the spaces of (completely) bounded maps from $Y$ to $X$ and from $X$ to $Y$, $\mathcal{L}^1(Y, X)$ and $\mathcal{L}^1(X, Y)$, into $\mathcal{A}$-bimodules. The module actions on the first one are pointwise. The module actions on the second one generalise the module actions on dual modules. In the completely bounded case those two modules don't have the same matricial norm structure (the matricial norm structure on $\mathcal{L}^1_{cb}(Y, X)$ is the standard one and the matricial norm structure on $\mathcal{L}^1_{cb}(X, Y)$ is the reversed tracial one defined in Section 1.2). We show that the modules $\mathcal{L}^1(Y, X)$ and $\mathcal{L}^1(X, Y)$ are closely related, respectively, to the module $X$ and to its dual module $X^*$. In particular if the $n$th cohomology group of $\mathcal{A}$, with coefficients in $\mathcal{L}^1(Y, X)$, vanishes, then so does the $n$th cohomology group of $\mathcal{A}$, with coefficients in $X$, and if the $n$th cohomology group of $\mathcal{A}$, with coefficients in $\mathcal{L}^1(X, Y)$, vanishes, then the $n$th cohomology group of $\mathcal{A}$, with coefficients in $X^*$, also vanishes.

In Chapter 4 we investigate the relation between the topics of the two previous chapters. We start by proving that if $\mathcal{A}$ is a (Banach/operator) algebra, $X$ is a (Banach/operator completely bounded) $\mathcal{A}$-bimodule and $Y$ is a (Banach/matricially normed) space, then, for all five types of splitting, the splitting of a cohomology group of $\mathcal{A}$, with coefficients in $X$, is equivalent to the split-
ting of all the cohomology groups of $\mathcal{A}$, with coefficients in modules of the form $\mathcal{L}_c^1(Y, X)$, and if $\mathcal{A}$ is a (Banach/operator) algebra, $X$ is a (Banach$/L^1$ completely bounded) $\mathcal{A}$-bimodule and $Y$ is a (Banach/matricially normed) space, then the splitting of a cohomology group of $\mathcal{A}$, with coefficients in $X^*$, is equivalent to the splitting of all the cohomology groups of $\mathcal{A}$, with coefficients in modules of the form $\mathcal{L}_c^1(X, X^*)$. Then we show that the third type of splitting of a cohomology group of $\mathcal{A}$, with coefficients in $X$ or $X^*$, respectively, is characterised by the vanishing of all the cohomology groups of $\mathcal{A}$, with coefficients in modules of the form $\mathcal{L}_c^1(Y, X)$ or $\mathcal{L}_c^1(X, Y^*)$ respectively. This last result leads to the definition of a local version of amenability, $X$-amenability. The discussion about $X$-amenability shows in particular that the cohomology complex of an amenable algebra, with coefficients in a dual module, always splits.

In Chapter 5 we discuss the splitting of cohomology groups of von Neumann algebras. Using several results about conditional expectations, we prove that the splitting of the first completely bounded cohomology group of a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $H$, with coefficients in $\mathcal{B}(H)$, implies that $\mathcal{M}$ is injective and that in many cases the same is true in the bounded case. We also show that, in all the cases where we know that the cohomology group of a von Neumann algebra, with coefficients in itself, vanishes, this group splits.

1.1.2 Notation

Chapters 2, 3 and 4 are more or less self-contained. A knowledge of functional analysis together with the information about matricially normed spaces and Hochschild cohomology contained in Section 1.2 is enough to go through them (the terminology that we use when we refer to Banach algebras is the same with [BoD]). In Chapter 5 we use many notions and results concerning von Neumann algebras. We have tried to give the definitions for most of the terms and exact references for most of the results. Anything else can be found in any of the standard texts on $C^*$- and von Neumann algebras ([Di1], [Di2], [KR1], [Sa1], [StZ], [Ta]).

Most of the notation will be explained when we encounter a symbol for the first time. We would like to clarify some simple notational matters here. The set of natural numbers is always denoted by $\mathbb{N}$ (it will be clear from the context whether 0 is contained in $\mathbb{N}$ or not) and the set of complex numbers by $\mathbb{C}$. For any set $S$ the identity map from $S$ to $S$ will be denoted by $id_S$. We always consider vector spaces over $\mathbb{C}$. If $V$ and $U$ are vector spaces, then we will denote the space of linear maps from $V$ into $U$ by $\mathcal{L}^1(V, U)$. We will denote by $\mathcal{L}^n(V_1, \ldots, V_n; U)$ the space of $n$-linear maps from the Cartesian product of the vector spaces $V_1, \ldots, V_n$.
into the vector space $U$. If $V_1 = \ldots = V_n$, then we will write $L^n(V, U)$ instead of $L^n(V_1, \ldots, V_n; U)$. The tensor product of the vector spaces $V$ and $U$ will be denoted by $V \otimes U$. If $X$ is a normed space, then we will denote its dual by $X^*$. If $X$ is a normed space and $Y$ is a complemented subspace of $X$, then $X \oplus Y$ will denote a closed subspace of $X$ with $X = Y \oplus (X \oplus Y)$. If $X$ and $Y$ are normed spaces, then the space of bounded linear maps from $X$ into $Y$ will be denoted by $L_c(X, Y)$. We will write $L_c^1(X, Y)$ instead of $L_c^1(X, X)$. The space of bounded $n$-linear maps from $X_1 \times \ldots \times X_n$ into $Y$ will be denoted by $L_c^n(X_1, \ldots, X_n; Y)$. We will write $L_c^n(X, Y)$ instead of $L_c^n(X, \ldots, X; Y)$. We will denote the projective tensor product of two normed spaces $X$ and $Y$ by $X \hat{\otimes} Y$. If $S$ is a subset of a normed space $X$, then $\text{Span}(S)$ will denote the linear span of $S$ in $X$ and $\overline{\text{Span}(S)}$ will denote the norm closure of $\text{Span}(S)$. The inner product on a Hilbert space $H$ will be denoted by $\langle \cdot, \cdot \rangle$. We will denote the space of bounded linear operators from a Hilbert space $H$ into a Hilbert space $K$ by $B(H, K)$. We will denote $B(H, H)$ by $B(H)$.

1.2 Preliminaries

1.2.1 Matricially normed spaces

The notions of a matricially normed space and a completely bounded map are used in many places in this thesis. Since the subject is quite young there exists no book containing an account of the basics of operator space theory which we could use as a reference. Even if we refer the reader to Pisier's unpublished notes ([Pi7]) we would have to give the definitions of the tracial and the reversed tracial dual of a matricially normed space which are not discussed there. Thus we have decided to devote the first part of the preliminaries to a brief discussion of matricially normed spaces. We will give all the definitions and properties of matricially normed spaces which we will need in later parts of the thesis. In every definition and result we will give a reference to the paper or papers where, to the best of our knowledge, it first appeared. We will also introduce a new matricial norm structure on the space of completely bounded maps between two matricially normed spaces $V$ and $U$, $L_c^1(V, U)$, and two new matricial norm structures on $V \otimes U$.

The origins of the theory of operator spaces can be traced back to the study of completely positive and completely bounded maps between subspaces of $C^*$-algebras. Let $H$ be a Hilbert space. Then, for all $n \in \mathbb{N}$, the space of $n \times n$ matrices with entries from $B(H)$, $M_n(B(H))$, can be identified with the space of bounded linear operators on the direct sum of $n$ copies of $H$, $B(H^n)$. If $V$ is a subspace
of \( B(H) \), then, for all \( n \in \mathbb{N} \), \( \mathbb{M}_n(V) \) inherits a norm from \( \mathbb{M}_n(B(H)) = B(H^n) \). We will call \( V \) equipped with the sequence of norms \( \{ \| \cdot \|_{B(H^n)} \}_{n \in \mathbb{N}} \) an operator space. To keep track of all those norms we have to replace bounded maps with completely bounded maps. If \( V \) and \( U \) are operator spaces and \( \phi : V \to U \) is a bounded linear map, then, for each \( n \in \mathbb{N} \), we can define a bounded linear map \( \phi_n : \mathbb{M}_n(V) \to \mathbb{M}_n(U) \) with \( \phi_n((v_{ij})) = (\phi(v_{ij})) \), for all \( (v_{ij}) \in \mathbb{M}_n(V) \). We call \( \phi \) completely bounded if \( \| \phi \|_{cb} = \sup_{n \in \mathbb{N}} \| \phi_n \| < \infty \). Completely bounded maps were introduced by Arveson at 1969 ([Ar], Definition 1.2.1). They can be used to obtain information about some very interesting problems in operator algebras and operator theory, e.g. similarity problems (see [Pi1]), the derivation problem for \( C^* \)-algebras (see Section 5.3), the bounded projection problem for von Neumann algebras (see Sections 5.2 and 5.3) and the Hochschild cohomology problem for von Neumann algebras (see Section 5.4). Moreover completely bounded maps can be described in a very simple way. If \( H \) is a Hilbert space, \( A \) is a \( C^* \)-algebra and \( \phi : A \to B(H) \) is a completely bounded map, then there exist a \( * \)-representation \( \pi \) of \( A \) on a Hilbert space \( K \) and \( T,S \in B(H,K) \) with \( \| \phi \|_{cb} = \| T \| \| S \| \), such that \( \phi(a) = T^* \pi(a) S \), for all \( a \in A \) ([P1], Theorem 7.4). Completely bounded maps into \( B(H) \) can be decomposed into sums of completely positive ones. That yields a Hahn-Banach type theorem for completely bounded maps ([H5], [P2], [W]). The first and the third of the above mentioned results hold if we replace the \( C^* \)-algebra \( A \) with an operator space \( V \) (see [SSm1], Theorem 1.3.1 and Corollary 1.3.2). Paulsen's book ([P1]) and the review paper by Christensen and Sinclair ([CS2]) give very nice accounts of the theory of completely bounded maps between subspaces of \( C^* \)-algebras.

We can generalise the notion of completely bounded maps to normed spaces \( V \) equipped with a sequence \( \{ \| \cdot \|_n \}_{n \in \mathbb{N}} \) of norms on \( \mathbb{M}_n(V) \) which satisfy two simple conditions. Before we give the required definitions we fix some notation that we will use in the rest of the thesis. Let \( V \) be a vector space and \( n,m,n_1,n_2,m_1,m_2 \in \mathbb{N} \). We will denote the vector space of \( n \times m \) matrices with entries from \( V \) by \( \mathbb{M}_{n,m}(V) \). If \( (v_{ij}) \in \mathbb{M}_{n,m}(V) \), then \( (v_{ij})^T \) is the \( m \times n \) matrix \( (v_{ji}) \). If \( (v_{ij}) \in \mathbb{M}_{n_1,n_2}(V) \) and \( (v'_{st}) \in \mathbb{M}_{m_1,m_2}(V) \), then we will write \( (v_{ij}) \oplus (v'_{st}) \) for the \( (n_1 + m_1) \times (n_2 + m_2) \) matrix with \( (v_{ij}) \) in the upper left corner, \( (v'_{st}) \) in the lower right corner and zeros elsewhere. For each \( v \in V \) we will denote by \( v \otimes I_n \) the \( n \times n \) matrix with \( v \)'s on the diagonal and 0's elsewhere. We write 0 for any \( n \times m \) matrix with 0's everywhere (if \( n \) and \( m \) are not clear from the context, then we will write 0\(_{n,m}\)). We will denote the space of \( n \)-rows of elements of \( V \), \( \mathbb{M}_{1,n}(V) = \{ (v_1, \ldots, v_n) \mid v_1, \ldots, v_n \in V \} \), by \( \text{Row}_n(V) \) and the space of \( n \)-columns of elements of \( V \), \( \mathbb{M}_{n,1}(V) = \{ (v_1, \ldots, v_n)^T \mid v_1, \ldots, v_n \in V \} \), by \( \text{Col}_n(V) \). We
Definition 1.2.1. ([R1]) Let $V$ be a normed space and $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ be a sequence of norms, where, for each $n \in \mathbb{N}$, $\|\cdot\|_n$ is a norm on $M_n(V)$ (from now on we will call the norms $\|\cdot\|_n$ matrix norms). We will say that $V$ is a matricially normed space if the following two conditions hold:

1. For all $n,m \in \mathbb{N}$ and all $(v_{ij}) \in M_n(V)$,
   \[ M1. \| (v_{ij}) \oplus 0 \|_{n+m} = \|(v_{ij})\|_n \]

2. For all $n \in \mathbb{N}$, all $(v_{ij}) \in M_n(V)$ and all $(\alpha_{ij}), (\beta_{ij}) \in M_n$,
   \[ M2. \|(\alpha_{ij})(v_{ij})(\beta_{ij})\|_n \leq \|(\alpha_{ij})\|(v_{ij})\|_n \|(\beta_{ij})\| \]

Remark 1.2.1. Let $V$ be a matricially normed space.

(i) $M_n(V)$ is a matricially normed space, for all $n \in \mathbb{N}$.

(ii) $M_{nm}(V)$ is completely isometrically isomorphic to $M_n(M_m(V))$ and to $M_m(M_n(V))$, for all $n,m \in \mathbb{N}$ (see Definition 1.2.3 for the definition of a complete isometry). This allows us to write elements of $M_{nm}(V)$ in the form $(v_{ij}^st)$ with $1 \leq i,j \leq n$ and $1 \leq s,t \leq m$.

(iii) $\|v_{st}\|_1 \leq \|(v_{ij})\|_n \leq \sum_{1 \leq i,j \leq n} \|v_{ij}\|_1$, for all $1 \leq s,t \leq n$, all $n \in \mathbb{N}$ and all $(v_{ij}) \in M_n(V)$ ([R1], Proposition 2.1.(2)).

(iv) In general when we talk about matricially normed spaces we will not assume that they are Banach spaces. It follows immediately from (iii) that if a matricially normed space $V$ is a Banach space, then so is $M_n(V)$, for all $n \in \mathbb{N}$.

(v) If $V$ is a matricially normed space, then we can make $M_{nm}(V)$ into a normed space, for all $n,m \in \mathbb{N}$, if we embed it into $M_k(V)$, where $k = \max\{n,m\}$, by adding either $k - n$ zero $m$-rows, if $k = m$, or $k - m$ zero $n$-columns if $k = n$. We will denote the norm on $M_{nm}(V)$ obtained this way by $\|\cdot\|_{n,m}$. In particular $\text{Row}_n(V)$ and $\text{Col}_n(V)$ are normed spaces for all $n \in \mathbb{N}$.

Definition 1.2.2. ([R1]) Let $V$ be a matricially normed space.

(i) We will say that $V$ is an $L^\infty$ matricially normed space if
   \[ \|(v_{ij}) \oplus (v'_{st})\|_{n+m} = \max\{\|(v_{ij})\|_n, \|(v'_{st})\|_m\} \]
   for all $n,m \in \mathbb{N}$ and all $(v_{ij}) \in M_n(V)$, $(v'_{st}) \in M_m(V)$.

(ii) We will say that $V$ is an $L^p$ matricially normed space, for $1 \leq p < \infty$, if
   \[ \|(v_{ij}) \oplus (v'_{st})\|_{n+m} = \left(\|(v_{ij})\|_n^p + \|(v'_{st})\|_m^p\right)^{1/p} \]
   for all $n,m \in \mathbb{N}$ and all $(v_{ij}) \in M_n(V)$, $(v'_{st}) \in M_m(V)$. 

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Definition 1.2.3. Let $V$ and $U$ be matricially normed spaces and $\phi : V \to U$ be a linear map. For each $n \in \mathbb{N}$, we define $\phi_n : M_n(V) \to M_n(U)$ by $\phi_n((v_{ij})) = (\phi(v_{ij}))$, for all $(v_{ij}) \in M_n(V)$ ($\phi_n$ is called the $n$-amplification of $\phi$). We will say that $\phi$ is completely bounded if $\sup_{n \in \mathbb{N}} \|\phi_n\|_{L_1(M_n(V), M_n(U))} < \infty$. We will say that $\phi$ is a complete isomorphism if $\phi$ is a completely bounded invertible map with completely bounded inverse. We will say that $\phi$ is a complete isometry if $\phi_n$ is an isometry for all $n \in \mathbb{N}$. We will say that $\phi$ is a complete contraction if $\|\phi\|_{cb} \leq 1$.

Remark 1.2.2. Let $V$ and $U$ be matricially normed spaces.

(i) We will denote the space of completely bounded linear maps from $V$ into $U$ by $\mathcal{L}_{cb}(V, U)$. If we define $\|\phi\|_{cb} = \sup_{n \in \mathbb{N}} \|\phi_n\|_{L_1(M_n(V), M_n(U))}$, for all $\phi \in \mathcal{L}_{cb}(V, U)$, then $(\mathcal{L}_{cb}(V, U), \|\cdot\|_{cb})$ is a normed space.

(ii) If $V$ is an $L^p$ matricially normed space and $U$ is an $L^{p'}$ matricially normed space, with $1 \leq p < p' \leq \infty$, then $\mathcal{L}_{cb}^1(V, U) = \{0\}$ ([R1], Theorem 5.3).

(iii) If $V$ and $U$ are Banach spaces and $\phi : V \to U$ is an invertible bounded linear map, then, by the inverse mapping theorem, the inverse of $\phi$ is bounded. The same does not hold for completely bounded maps, i.e. the inverse of an invertible completely bounded linear map (between matricially normed spaces which are Banach spaces) is not automatically completely bounded as the following example shows: Let $l_2$ be the separable Hilbert space and $C$ and $R$ be the subspaces of $B(l_2)$ defined by $C = \overline{\text{span}}(\{e_{ij} \mid 1 \leq i\})$ and $R = \overline{\text{span}}(\{e_{ii} \mid 1 \leq i\})$. If $V$ is the subspace of $C \oplus \infty R$ (see Proposition 1.2.13(i) for the definition of the $\infty$ sum of matricially normed spaces) defined by $V = \{u \oplus u^T \mid u \in C\}$ and $\phi : V \to C$ is defined by $\phi(u \oplus u^T) = u$, for all $u \oplus u^T \in C \oplus \infty R$, then we can easily see that $\phi$ is a completely bounded linear map (with $\|\phi\|_{cb} \leq 1$). On the other hand, for each $n \in \mathbb{N}$, $d_{cb}(C_n, R_n) = n$, where $C_n = \overline{\text{span}}(\{e_{ii} \mid 1 \leq i \leq n\})$ and $R_n = \text{span}(\{e_{ii} \mid 1 \leq i \leq n\})$ and $d_{cb}(\cdot, \cdot)$ is the completely bounded analogue of the Banach-Mazur distance (see [Pi7], pp.17-18). Therefore $\phi^{-1}$ is not completely bounded.

Remark 1.2.3. Let $V$ and $U$ be matricially normed spaces and $\phi : V \to U$ be a linear map. For each $n \in \mathbb{N}$, we define $\phi_n : \text{Row}_n(V) \to \text{Row}_n(U)$ by $\phi_n((v_1, \ldots, v_n)) = (\phi(v_1), \ldots, \phi(v_n))$, for all $(v_1, \ldots, v_n) \in \text{Row}_n(V)$. We will say that $\phi$ is row bounded if $\sup_{n \in \mathbb{N}} \|\phi_n\|_{L_1(\text{Row}_n(V), \text{Row}_n(U))} < \infty$. We will denote the space of row bounded linear maps from $V$ into $U$ by $\mathcal{L}_{cb}^1(V, U)$. It is easy to see that $\mathcal{L}_{cb}^1(V, U)$ equipped with the norm $\|\phi\|_r = \sup_{n \in \mathbb{N}} \|\phi_n\|_{L_1(\text{Row}_n(V), \text{Row}_n(U))}$ becomes a normed space. Moreover every completely bounded map $\phi$ is row bounded and $\|\phi\| \leq \|\phi\|_r \leq \|\phi\|_{cb}$. Similarly we can define column bounded maps.

It is easy to see that an operator space is an $L^\infty$ matricially normed space.
Ruan's major achievement, which gave birth to operator space theory, was to prove that the converse is also true. If \( V \) is an \( L^\infty \) matricially normed space, then there exist a Hilbert space \( H \) and a complete isometry \( \theta : V \to B(H) \) ([R1], Theorem 3.1, see also [ER4] for a simple proof). From now on we will not distinguish between operator spaces and \( L^\infty \) matricially normed spaces.

Let \( V \) be an \( L^p \) matricially normed space and \( U \) be a closed subspace of \( V \). Then, by Remark 1.2.1(iii), \( M_n(U) \) is a closed subspace of \( M_n(V) \), for all \( n \in \mathbb{N} \). Hence \( U \) is an \( L^p \) matricially normed space with the sequence of matrix norms that it inherits from \( V \). Moreover \( V/U \) becomes an \( L^p \) matricially normed space if we identify \( M_n(V/U) \) with \( M_n(V)/M_n(U) \), for all \( n \in \mathbb{N} \).

**Proposition 1.2.1.** ([R1], Theorem 4.2) Let \( V \) be an \( L^p \) matricially normed space and \( U \) be a closed subspace of \( V \). If we define

\[
\|(v_{ij} + U)\|_{M_n(V/U)} = \|(v_{ij}) + M_n(U)\|_{M_n(V)/M_n(U)}
\]

for all \( n \in \mathbb{N} \) and all \( (v_{ij} + U) \in M_n(V/U) \), then \( V/U \) is an \( L^p \) matricially normed space. Moreover the quotient map \( q : V \to V/U \) is a complete contraction.

In the category of matricially normed spaces the notion of a complemented subspace is replaced by that of a completely complemented subspace. If \( V \) is a matricially normed space and \( U \) is a subspace of \( V \), then we will say that \( U \) is completely complemented in \( V \) if there exists a completely bounded projection \( \rho : V \to U \). If \( U \) is a completely complemented subspace of \( V \) and \( \rho : V \to V \ominus U \) is a completely bounded projection, then we can easily see that the map \( \tau : V/U \to V \) defined by \( \tau(v + U) = \rho(v) \), for all \( v \in V \), is completely bounded.

We move now to the dual of a matricially normed space \( V \). First of all we have to decide which is the correct "dual object" for the category of matricially normed spaces. It must be clear by now that it is the space of all completely bounded linear functionals on \( V \). It turns out that if \( f \) is a bounded linear functional on \( V \), then \( f \) is completely bounded with \( \|f\|_{cb} = \|f\| \) ([P1], Proposition 3.7). So the "dual object" of \( V \) is the dual Banach space \( V^* \) of \( V \). To finish the construction of the dual matricially normed space of \( V \) we must equip \( V^* \) with a sequence of matrix norms. This is where we encounter the first dichotomy in the theory of matricially normed spaces. There are two ways to define norms on \( M_n(V^*) \) (actually we will present three ways, but the second and the third are variations of each other). The first one is the right one when we are working in the category of operator spaces, but is not compatible with the notion of a dual completely bounded bimodule that we will use (see Section 1.2.2). On the other hand the second one makes us leave the category of operator spaces (but not the category
of matricially normed spaces), but works perfectly for dual completely bounded bimodules.

**Proposition 1.2.2.** Let $V$ be a matricially normed space. The dual $V^*$ of $V$ becomes a matricially normed space if we define

$$
\|(f_{ij})\|_n = \sup\{\|(f_{ij}(v_{st}))\| \mid \|(v_{st})\|_m \leq 1, m \in \mathbb{N}\}
$$

for all $n \in \mathbb{N}$ and all $(f_{ij}) \in \mathbb{M}_n(V^*)$. We will denote $V^*$ equipped with the sequence of matrix norms $\{\|\cdot\|_n\}$ by $V^*$ and call it the standard dual of $V$. The space $V^*$ is always an operator space. If $V$ is an operator space, then the canonical embedding $v \mapsto \hat{v} : V \to V^{*\ast}$ is a complete isometry. We will call a matricially normed space $V$ a dual matricially normed space if there exists a matricially normed space $U$ such that $V$ is completely isometrically isomorphic to $U^*$.

**Remark 1.2.4.** (i) The standard dual of a matricially normed space was defined independently in [BP], Example 2.10, and [ER2], Section 2. It was studied in detail in [B2]. All the three above mentioned papers are concerned with the case of an operator space $V$. The complete isometricity of $v \mapsto \hat{v}$, when $V$ is an operator space, was proved in [BP], Theorem 2.11 and in [ER2], Theorem 2.2.

(ii) We can easily see that $\|(f_{ij})\|_n$ coincides with $\|(f_{ij})\|_{c_b(V,\mathbb{M}_n)}$ if we define $(f_{ij}) : V \to \mathbb{M}_n$ by $(f_{ij})(v) = (f_{ij}(v))$, for all $v \in V$.

**Proposition 1.2.3.** Let $V$ be a matricially normed space. The dual $V^*$ of $V$ becomes a matricially normed space if we define

$$
\|(f_{ij})\|_n^t = \sup\{\sum_{1 \leq i,j \leq n} f_{ij}(v_{ij}) \mid \|(v_{ij})\|_n \leq 1\}
$$

for all $n \in \mathbb{N}$ and all $(f_{ij}) \in \mathbb{M}_n(V^*)$. We will denote $V^*$ equipped with the sequence of matrix norms $\{\|\cdot\|_n\}$ by $V^*_t$ and call it the tracial dual of $V$. If $V$ is an $L^p$ matricially normed space, $1 \leq p \leq \infty$, then $V^*_t$ is an $L^q$ matricially normed space where $\frac{1}{p} + \frac{1}{q} = 1$. The canonical embedding $v \mapsto \hat{v} : V \to (V^*_t)^t$ is a complete isometry. We will call a matricially normed space $V$ a tracial dual matricially normed space if there exists a matricially normed space $U$ such that $V$ is completely isometrically isomorphic to $U^*_t$.

**Remark 1.2.5.** (i) The tracial dual of a matricially normed space was defined in [ChE], p.161. Both the $L^p$ characterisations for $V^*_t$ and the complete isometricity of $v \mapsto \hat{v}$ were proved by Ruan ([R1], Theorem 5.1 and Corollary 2.4 respectively).

(ii) $\|(f_{ij})\|_n^t$ coincides with $\|(f_{ij})\|_{c_b(\mathbb{M}_n(V),\mathbb{C})}$ if we define $(f_{ij}) : \mathbb{M}_n(V) \to \mathbb{C}$ by $(f_{ij})(v_{ij}) = \sum_{1 \leq i,j \leq n} f_{ij}(v_{ij})$, for all $(v_{ij}) \in \mathbb{M}_n(V)$.
Proposition 1.2.4. Let $V$ be a matricially normed space. The dual $V^*$ of $V$ becomes a matricially normed space if we define

$$\|(f_{ij})\|_n^t = \sup\{|\sum_{1 \leq i,j \leq n} f_{ij}(v_{ji})| : \|(v_{ij})\|_n \leq 1\}$$

for all $n \in \mathbb{N}$ and all $(f_{ij}) \in M_n(V^*)$. We will denote $V^*$ equipped with the sequence of matrix norms $\{\|\cdot\|_n^t\}$ by $V^*_t$ and call it the reversed tracial dual of $V$. If $V$ is an $L^p$ matricially normed space, $1 \leq p \leq \infty$, then $V^*_t$ is an $L^q$ matricially normed space where $\frac{1}{p} + \frac{1}{q} = 1$. The canonical embedding $v \mapsto \hat{v} : V \to (V^*_t)_t$ is a complete isometry. We will call a matricially normed space $V$ a reversed tracial dual matricially normed space if there exists a matricially normed space $U$ such that $V$ is completely isometrically isomorphic to $U^*_t$.

Remark 1.2.6. (i) The reversed tracial dual of a matricially normed space was defined by Effros and Ruan ([ER1], p.145). The $L^p$ characterisations and the complete isometricity of $v \mapsto \hat{v}$ can be proved exactly in the same way as for the tracial dual.

(ii) It is obvious that $\|(f_{ij})\|_n^t$ coincides with $\|(f_{ij})\|_{L^1(M_n(V),C)}$ if we define $(f_{ij}) : M_n(V) \to \mathbb{C}$ by $(f_{ij})(v_{ij}) = \sum_{1 \leq i,j \leq n} f_{ij}(v_{ji})$, for all $(v_{ij}) \in M_n(V)$.

Let us mention here that Blecher proved that if $V$ is an operator space, then $V^{**}$ is completely isometrically isomorphic to $(V^*_t)^*_t$ ([B2], Theorem 2.5). It is easy to see that a similar result holds for $V^{**}$ and $(V^*_t)^*_t$.

From now on we will concentrate on the reversed tracial dual. The constructions presented in Propositions 1.2.4, 1.2.5, 1.2.8 and 1.2.9 have counterparts for the tracial dual.

As we mentioned in Remark 1.2.6(ii) the matrix norms associated with the standard dual of a matricially normed space coincide with $\|\cdot\|_{C^*_{cb}(V,M_n)}$ if we think of a matrix $(f_{ij}) \in M_n(V^*)$ as a map from $V$ into $M_n$ defined by $(f_{ij})(v) = (f_{ij}(v))$, for all $v \in V$. In a similar manner we can think of a matrix $(\phi_{ij}) \in M_n(L^1_{cb}(V,U))$ as a map from $V$ into $M_n(U)$ defined by $(\phi_{ij})(v) = (\phi_{ij}(v))$, for all $v \in V$, and get a matricial norm structure on $L^1_{cb}(V,U)$.

Proposition 1.2.5. Let $V$ and $U$ be matricially normed spaces. Then $L^1_{cb}(V,U)$ becomes a matricially normed space if we define

$$\|(\phi_{ij})\|_n = \sup\{|\|(\phi_{ij}(v_{st}))\|_{nm} : \|(v_{st})\|_m \leq 1, m \in \mathbb{N}\}$$

for all $n \in \mathbb{N}$ and all $(\phi_{ij}) \in M_n(L^1_{cb}(V,U))$. We will denote $L^1_{cb}(V,U)$ equipped with the sequence of matrix norms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ by $L^1_{cb}(V,U)$. We will call this matricial norm structure the standard matricial norm structure on $L^1_{cb}(V,U)$. If
U is an $L^p$ matricially normed space, then $L^1_{cb}(V, U)$ is an $L^p$ matricially normed space.

**Remark 1.2.7.** (i) The standard matricial norm structure on $L^1_{cb}(V, U)$ was defined independently in [BP], Example 2.10, and in [ER1], p.140. The $L^p$ characterisations are trivial.

(ii) If $V$ is a normed space and $U$ is a matricially normed space, then we can make $L^1_{cb}(V, U)$ into a matricially normed space if we define

$$
|||\phi_{ij}|||_n = \sup\{|||\phi_{ij}(v)|||_n \mid |||v||| \leq 1\}
$$

for all $n \in \mathbb{N}$ and all $(\phi_{ij}) \in \mathbb{M}_n(L^1_{cb}(V, U))$ ([ER1], p.140).

In a later part of the thesis (Section 3.2.2) we will need a matricial norm structure on $L^1_{cb}(V, U)$ generalising the matricial norm structure on $V^*$-$V$. To get the matrix norms $|||\cdot|||_n^r$ associated with $V^*$ we identified a matrix $(f_{ij}) \in \mathbb{M}_n(V^*)$ with a linear functional on $\mathbb{M}_n(V)$ defined by $(f_{ij})(v_{ij}) = \sum_{1 \leq i,j \leq n} f_{ij}(v_{ij})$, for all $(v_{ij}) \in \mathbb{M}_n(V)$, and defined $|||(f_{ij})|||_n^r$ to be the bounded norm of this functional. We have already mentioned that bounded functionals are completely bounded and their bounded and completely bounded norms coincide and so $|||f_{ij}|||_n^r = |||f_{ij}|||_{cb}(\mathbb{M}_n(V), \mathbb{C})$. To generalise this matricial norm structure to $L^1_{cb}(V, U)$ we have to identify $n \times n$ matrices of elements of $L^1_{cb}(V, U)$ with maps from $\mathbb{M}_n(V)$ into $U$ and take their completely bounded norms.

**Proposition 1.2.6.** Let $V$ and $U$ be matricially normed spaces. Then $L^1_{cb}(V, U)$ becomes a matricially normed space if we define

$$
|||\phi_{ij}|||_n^r = \sup\{\|\sum_{1 \leq i,j \leq n} \phi_{ij}(v_{ij}^{st})\|_m \mid \|v_{ij}^{st}\|_{nm} \leq 1, m \in \mathbb{N}\}
$$

for all $n \in \mathbb{N}$ and all $(\phi_{ij}) \in \mathbb{M}_n(L^1_{cb}(V, U))$. We will denote $L^1_{cb}(V, U)$ equipped with the sequence of matrix norms $\{|||\cdot|||_n^r\}_{n \in \mathbb{N}}$ by $L^1_{cb}(V, U)_rt$. We will call this matricial norm structure the reversed tracial matricial norm structure on $L^1_{cb}(V, U)$. If $V$ is an $L^1$ matricially normed space, then $L^1_{cb}(V, U)_rt$ is an operator space.

Before giving the proof of Proposition 1.2.6, we will prove a lemma, similar to [R1], Proposition 2.2.(2), which we will need for the proof. Both in the following lemma and in the proof of Proposition 1.2.6 we will denote $(\sum_{1 \leq i,j \leq n} \phi_{ij}(v_{ij}^{st}))$ by $(\phi_{ij})_m((v_{ij}^{st}))$, for all $n, m \in \mathbb{N}$, $(\phi_{ij}) \in \mathbb{M}_n(L^1_{cb}(V, U))$ and $(v_{ij}^{st}) \in \mathbb{M}_{nm}(V)$.

**Lemma 1.2.1.** Let $V$ and $U$ be matricially normed spaces, $n, r \in \mathbb{N}$, $(\phi_{ij}) \in \mathbb{M}_n(L^1_{cb}(V, U))$ and $(\psi_{pq}) \in \mathbb{M}_r(L^1_{cb}(V, U))$. Then

$$
|||\phi_{ij} \oplus (\psi_{pq})|||_{n+r}^r = \sup\{|||\phi_{ij} \oplus (\psi_{pq})|||_{nm+r}^r \mid \|v_{ij}^{st}\|_m \leq 1, m \in \mathbb{N}\}.
$$
Proof. Call the above defined supremum $\alpha$. It is easy to see that $\alpha \leq \| (\phi_{ij}) \oplus (\psi_{pq}) \|_{n+r}^{rt}$. On the other hand take $(z_{kl}^{st}) \in M_{nm+rm}(V)$ with $\| (z_{kl}^{st}) \|_{nm+rm} \leq 1$ and for all $1 \leq s, t \leq m$ let $(v_{ij}^{st})$ be the upper left $n \times n$ corner and $(w_{pq}^{st})$ be the lower right $r \times r$ corner of $(z_{kl}^{st})$. A straightforward calculation shows that

$$(\phi_{ij}) \oplus (\psi_{pq}) m ((z_{kl}^{st})) = (\phi_{ij}) m ((v_{ij}^{st})) + (\psi_{pq}) m ((w_{pq}^{st}))$$

Moreover $\| (v_{ij}^{st}) \oplus (w_{pq}^{st}) \|_{nm+rm} \leq 1$, by [R1], Proposition 2.2.(1). Thus $\| (\phi_{ij}) \oplus (\psi_{pq}) \|_{n+r}^{rt} = \alpha$. $\Box$

Proof of Proposition 1.2.6. $M1$ follows immediately from Lemma 1.2.1. If $n, m \in N$, $(\phi_{ij}) \in M_n (L^{1}_{cb}(V, U))$, $(\alpha_{ij})$, $(\beta_{ij}) \in M_n$ and $(v_{ij}^{st}) \in M_{nm}(V)$, then we can easily see that

$$( (\alpha_{ij})(\phi_{ij})(\beta_{ij}) )_m ((v_{ij}^{st})) = (\phi_{ij})_m ((\beta_{ij} \otimes I_m))(v_{ij}^{st})((\alpha_{ij} \otimes I_m)).$$

(1.1)

$M2$ follows immediately from (1.1) and $M2$ for $V$. To finish the proof suppose that $V$ is an $L^1$ matricially normed space. If $n, r, m \in N$, $(\phi_{ij}) \in M_n (L^{1}_{cb}(V, U))$, $(\psi_{pq}) \in M_r (L^{1}_{cb}(V, U))$, $(v_{ij}^{st}) \in M_{nm}(V)$ and $(w_{pq}^{st}) \in M_{rm}(V)$, then obviously

$$\| (\phi_{ij})_m ((v_{ij}^{st})) + (\psi_{pq})_m ((w_{pq}^{st})) \|_m$$

$$\leq \max \{ \| (\phi_{ij}) \|_n, \| (\psi_{pq}) \|_r \} (\| (v_{ij}^{st}) \|_{nm} + \| (w_{pq}^{st}) \|_{rm}).$$

(1.2)

But

$$\| (v_{ij}^{st}) \|_{nm} + \| (w_{pq}^{st}) \|_{rm} = \| (v_{ij}^{st}) \oplus (w_{pq}^{st}) \|_{nm+rm}$$

(1.3)

since $V$ is an $L^1$ matricially normed space. It follows immediately from (1.2), (1.3) and Lemma 1.2.1 that $L^{1}_{cb}(V, U)$ is an operator space. $\Box$

Remark 1.2.8. (i) $V^*_{rt}$ is completely isometrically isomorphic to $L^{1}_{cb}(V, C)_{rt}$, for all matricially normed spaces $V$.

(ii) If $V$ is a matricially normed space and $U$ is a normed space, then $L^{1}_{cb}(V, U)$ becomes a matricially normed space if we define

$$\| (\phi_{ij}) \|_{rt}^{st} = \sup \{ \| \sum_{1 \leq i, j \leq n} \phi_{ij} (v_{ij}) \| \| (v_{ij}) \|_n \leq 1 \}$$

for all $n \in N$ and all $(\phi_{ij}) \in M_n (L^{1}_{cb}(V, U))$. We will denote $L^{1}_{cb}(V, U)$ equipped with this matricial norm structure by $L^{1}_{cb}(V, U)_{rt}$.

If $V$ and $U$ are vector spaces, then $L^{1}(V, U^*)$ is isomorphic to $(V \otimes U)^*$. For Banach spaces $V$ and $U$ the same relation holds for bounded maps and the projective tensor product (see [BoD], Proposition 42.13 and the following Remark).
As we have seen so far there are two different matricial norm structures on $V^*$ and two different matricial norm structures on $\mathcal{L}_{cb}^1(V,U)$. Thus to extend the above mentioned result to matricially normed spaces we have to define a different tensor product for each combination of those structures. For the standard dual and the standard matricial norm structure on $\mathcal{L}_{cb}^1(V,U)$ such a tensor product has been constructed, the operator space projective tensor product $V \hat{\otimes}_{op} U$ of $V$ and $U$ ([BP], pp.286-287, and [ER2], Section 3). In the following three propositions we construct two tensor products giving similar results for $\mathcal{L}_{cb}^1(V,U_{rt}^*)$ and $\mathcal{L}_{cb}^1(V,U_{rt}^{*\rt})$ when one of the spaces $V$ and $U$ is an operator space and the other is an $L^1$ matricially normed space.

**Proposition 1.2.7.** Let $V$ and $U$ be matricially normed spaces, such that one of them is an operator space and the other is an $L^1$ matricially normed space, and $V \otimes U$ be their algebraic tensor product. If we define

$$
\|x\|_{rt} = \inf \{ \| (u_{ij}) \|_m \| (u_{ij}) \|_m \mid x = \sum_{1 \leq i,j \leq m} u_{ij} \otimes u_{ji}, m \in \mathbb{N} \}
$$

for all $x \in V \otimes U$, then $\| \cdot \|_{rt}$ is a norm on $V \otimes U$. We will denote $V \otimes U$ equipped with the norm $\| \cdot \|_{rt}$ by $V \hat{\otimes}_r U$. Then $(V \hat{\otimes}_r U)^*$ is isometrically isomorphic to $\mathcal{L}_{cb}^1(V,U_{rt}^*)$.

To prove Proposition 1.2.7 we will need the following lemma.

**Lemma 1.2.2.** Let $U$ be an $L^1$ matricially normed space, $m \in \mathbb{N}$, $(u_{ij}) \in M_m(U)$ and $f \in U^*$. Then

$$
tr(|(f(u_{ij}))|) \leq \|f\| \|\ (u_{ij})\|_m
$$

where $tr : M_m \rightarrow \mathbb{C}$ is the canonical trace on $M_m$ with $tr(I_m) = m$.

**Proof.** From the polar decomposition of the matrix $(f(u_{ij}))$ there exists a unitary matrix $(\alpha_{ij}) \in M_m$ such that $|(f(u_{ij}))| = (\alpha_{ij})(f(u_{ij}))$. Then

$$
tr(|(f(u_{ij}))|) = tr((\alpha_{ij})(f(u_{ij})))
$$

$$
= \sum_{1 \leq i,j \leq m} \alpha_{ij} f(u_{ji})
$$

$$
= \sum_{1 \leq i,j \leq m} (\alpha_{ij} f)(u_{ji})
$$

$$
\leq \|((\alpha_{ij} f))\|_m \|\ (u_{ij})\|_m
$$

$$
= \|((\alpha_{ij})(f \otimes I_m))\|_m \|\ (u_{ij})\|_m
$$

$$
\leq \|f \otimes I_m\|_m \|\ (u_{ij})\|_m
$$

$$
= \|f\| \|\ (u_{ij})\|_m
$$

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where the fourth step follows from the definition of the matrix norms on $U^*_{rt}$, the sixth from $M2$ for $U$ and the seventh from $U^*_{rt}$ being an operator space (since $U$ is an $L^1$ matricially normed space).

Proof of Proposition 1.2.7. Throughout the proof we will denote $\sum_{1 \leq i,j \leq m} v_{ij} \otimes u_{ji}$ by $(v_{ij}) \otimes (u_{ij})$. Without loss of generality we assume that $V$ is an operator space and $U$ is an $L^1$ matricially normed space. To show that $\| \cdot \|_{rt}$ is well-defined let $x = \sum_{1 \leq k \leq m} v_k \otimes u_k \in V \otimes U$. Then $x = (v_{ij}) \otimes (u_{ij})$, where $(v_{ij}) \in \mathcal{M}_m(V)$, with $v_1, \ldots, v_m$ in the diagonal and zeros elsewhere, and $(u_{ij}) \in \mathcal{M}_m(U)$, with $u_1, \ldots, u_m$ in the diagonal and zeros elsewhere, and hence $\|x\|_{rt} \leq \|(v_{ij})\|_m \|(u_{ij})\|_m < \infty$.

The next step is to prove that if $\|x\|_{rt} = 0$, for some $x \in V \otimes U$, then $x = 0$. We will do that by proving that $\|x\|_\lambda \leq \|x\|_{rt}$, for all $x \in V \otimes U$, where $\| \cdot \|_\lambda$ is the injective normed space tensor product norm on $V \otimes U$ (for the definition of $\| \cdot \|_\lambda$ see [BoD], Definition 42.7; there $\| \cdot \|_\lambda$ is denoted by $w(.)$ and called the weak tensor norm). Let $x = (v_{ij}) \otimes (u_{ij}) \in V \otimes U$ and $f \in V^*$, $g \in U^*$, with $\|f\|, \|g\| \leq 1$. Using the previous lemma and a well-known inequality about the canonical trace $\text{tr} : \mathcal{M}_m \rightarrow \mathbb{C}$ we get

$$\left| \sum_{1 \leq i,j \leq m} f(v_{ij})g(u_{ji}) \right| = \left| \text{tr}((f(v_{ij}))(g(u_{ij}))) \right| \leq \|(f(v_{ij}))\|_m \|\text{tr}(g(u_{ij}))\|_m \\
\leq \|(f(v_{ij}))\|_m \|(u_{ij})\|_m \\
\leq \|(v_{ij})\|_m \|(u_{ij})\|_m.$$ 

Taking the infimum over $f \in V^*$ and $g \in U^*$, with $\|f\|, \|g\| \leq 1$, on the left hand side of the previous inequality, we get $\|x\|_\lambda \leq \|(v_{ij})\|_m \|(u_{ij})\|_m$. But the infimum of $\|(v_{ij})\|_m \|(u_{ij})\|_m$ over $m \in \mathbb{N}$ and $(v_{ij}) \in \mathcal{M}_m(V)$ and $(u_{ij}) \in \mathcal{M}_m(U)$ with $x = (v_{ij}) \otimes (u_{ij})$ is equal to $\|x\|_{rt}$ and hence $\|x\|_\lambda \leq \|x\|_{rt}$.

To prove the triangle inequality take non-zero $x, y \in V \otimes U$ and $(v_{ij}) \in \mathcal{M}_m(V)$, $(v'_{st}) \in \mathcal{M}_m(V)$, $(u_{ij}) \in \mathcal{M}_n(U)$ and $(u'_{st}) \in \mathcal{M}_m(U)$ with $x = (v_{ij}) \otimes (u_{ij})$ and $y = (v'_{st}) \otimes (u'_{st})$. If $\alpha = \|(v_{ij})\|_n$, $\beta = \|(v'_{st})\|_m$, $(apq) = (\alpha v_{ij}) \oplus (\beta v'_{st})$ and $(bpq) = (\alpha u_{ij}) \oplus (\beta u'_{st})$, then a straightforward calculation shows that $x + y = (apq) \otimes (bpq)$ and thus

$$\|x + y\|_{rt} \leq \|(apq)\|_{n+m} \|(bpq)\|_{n+m}.$$ 

(1.4)

Since $V$ is an operator space, $\|(apq)\|_{n+m} = 1$. Moreover $\|(bpq)\|_{n+m}$ is equal to $\|(v_{ij})\|_n \|(u_{ij})\|_n + \|(v'_{st})\|_m \|(u'_{st})\|_m$. Combining those two observations with (1.4) we get $\|x + y\|_{rt} \leq \|x\|_{rt} + \|y\|_{rt}$. It follows immediately from the definition of $\| \cdot \|_{rt}$ and the matricial norm structure on $U^*_{rt}$ that the map $\phi \mapsto \Psi_\phi : \mathcal{L}^1_{cb}(V, U^*_{rt}) \rightarrow$
(V ⊗ U)* defined by Ψφ(v ⊗ u) = ϕ(v)(u), for all ϕ ∈ L1cb(V, U∗ r ), all v ∈ V and all u ∈ U is an isometric isomorphism.

Remark 1.2.9. (i) The tensor product V ⊗t U has been defined in [Itt1], Section 2, in the special case where V is a C*-algebra A and U is the space of trace class operators on H, C1(H). A similar tensor product was defined in [EKi], pp.263-4, for V = B(H) and U = C1(H).

(ii) If V is an operator space and U is an Lp matricially normed space, with 1 < p ≤ ∞, (or the other way round), then Ker(∥.∥rt) = V ⊗ U. To see that take non-zero v ∈ V and u ∈ U. Then, for each m ∈ N, v ⊗ u = ∑1≤k≤m(1/m)v ⊗ u = (1/m)v ⊗ I_m ⊗ (u ⊗ I_m) and hence ∥v ⊗ u∥rt ≤ 1/mv ⊗ I_m ∥u ⊗ I_m∥ = 1/mv1/p∥u∥.

Proposition 1.2.8. Let V be an operator space, U be an L1 matricially normed space and V ⊗ U be their algebraic tensor product. The vector space V ⊗t U equipped with the sequence of matrix norms {∥∥∥mnrt} defined by

\[ ∥(x_{ij})∥_{mnrt} = \text{inf}\{∥(v_{ij})∥m∥(u_{ij})∥_{m} | (x_{ij}) = \left( \sum_{1≤s,t≤m} v_{st} ⊗ u_{st}^{*} \right), m ∈ N\} \]

for all n ∈ N and all (x_{ij}) ∈ Mn(V ⊗ U) is an L1 matricially normed space. We will denote it by V ⊗t U and call it the right reversed tracial tensor product of V and U. Moreover (V ⊗t U)* is completely isometrically isomorphic to Lcb(V, U∗ r).

Proof. Throughout the proof we will denote ∑1≤s,t≤m v_{st} ⊗ u_{st}^{*} by (v_{st}) ⊗ (u_{st}^{*}). It is obvious that ∥∥∥_1rt coincides with ∥∥∥_rt. Following in the steps of the proof of the previous proposition we can show that ∥∥∥_nrt is a norm on Mn(V ⊗ U) for all n ∈ N. M1 for V ⊗r U can be proved using M1 for U and the following relation (which can be proved in a similar manner to the proof of Lemma 1.2.1):

If n, r ∈ N, (x_{ij}) ∈ Mn(V ⊗ U) and (y_{pq}) ∈ Mr(V ⊗ U), then

\[ ∥(x_{ij}) ⊙ (y_{pq})∥_{nr} = \text{inf}\{∥(v_{ij})∥m∥(w_{pq})∥_{(m+r)m} | (x_{ij}) = (v_{ij}) ⊙ (w_{pq}), (y_{pq}) = (v_{ij}) ⊙ (w_{pq}), m ∈ N\}. \] (1.5)

M2 for V ⊗r U follows from M2 for U and a straightforward calculation. To prove that V ⊗ U is an L1 matricially normed space let (x_{ij}) ∈ Mn(V ⊗ U) and (y_{pq}) ∈ Mr(V ⊗ U). If (v_{ij}) ∈ Mn(V), (v_{ij}^{*}) ∈ Mn(U) and (w_{pq}^{*}) ∈ Mn(U) with (x_{ij}) = (v_{ij}) ⊙ (w_{pq}^{*}) and (y_{pq}) = (v_{ij}) ⊙ (w_{pq}^{*}), then

\[ ∥(x_{ij})∥n + ∥(y_{pq})∥r ≤ ∥(v_{ij})∥m(∥(u_{ij}^{*})∥_{nm} + ∥(w_{pq}^{*})∥_{rm}). \] (1.6)

Since U is an L1 matricially normed space ∥(u_{ij}^{*})∥_{nm} + ∥(w_{pq}^{*})∥_{rm} = ∥(u_{ij}^{*}) ⊙ (w_{pq}^{*})∥_{nm+rm}. Combining (1.5), (1.6) and the last relation we get that ∥(x_{ij})∥n +
It is obvious that \( \| (x_{ij})\|_n + \| (y_{pq})\|_r \geq \| (x_{ij}) \oplus (y_{pq})\|_{n+r} \) and therefore \( V \otimes_{rt} U \) is an \( L^1 \) matricially normed space. It is trivial to show that the map \( \phi \mapsto \Psi_\phi : \mathcal{L}_{cb}^r(V,U^*_{rt}) \to (V \otimes_{rt} U)^*_{rt} \) defined in the proof of Proposition 1.2.7 is a complete isometry.

**Proposition 1.2.9.** Let \( V \) be an \( L^1 \) matricially normed space, \( U \) be an operator space and \( V \otimes U \) be their algebraic tensor product. The vector space \( V \otimes U \) equipped with the sequence of matrix norms \( \{\| \cdot \|_{n}^{rt}\} \) defined by

\[
\| (x_{ij}) \|_{n}^{rt} = \inf \left\{ \| (v_{ij}^{st})\|_m \| (u_{st})\|_n \mid (x_{ij}) = \left( \sum_{1 \leq s,t \leq m} v_{ij}^{st} \otimes u_{ts}, m \in \mathbb{N} \right) \right\}
\]

for all \( n \in \mathbb{N} \) and all \( (x_{ij}) \in M_n(V \otimes U) \) is an \( L^1 \) matricially normed space. We will denote it by \( V \otimes_{rt} U \) and call it the left reversed tracial tensor product of \( V \) and \( U \). Moreover \((V \otimes_{rt} U)^*_{rt}\) is completely isometrically isomorphic to \( L_{cb}^r(V,U^*_{rt}) \).

**Proof.** It is similar to the proof of Proposition 1.2.8. \( \Box \)

We can easily see that \( \{\| \cdot \|_{n}^{rt}\} \) and \( \{\| \cdot \|_{n}^{rt}\} \) also define matricial norm structures on the completion of \( V \otimes U \) with respect to the norm \( \| \cdot \|_{rt} \).

If \( V \) and \( U \) are Banach spaces, then \( V \hat{\otimes} U \) is isometrically isomorphic to \( U \hat{\otimes} V \) (and thus \( L_b^1(V,U^*) \) is isometrically isomorphic to \( L_b^1(U,V^*) \)). Similar relations hold for the operator space projective tensor product, the standard dual and the standard matricial norm structure on \( \mathcal{L}_{cb}^b(V,U) \) ([BP], Proposition 5.4). The following proposition describes an analogue of those results for the reversed tracial constructions.

**Proposition 1.2.10.** Let \( V \) be an operator space and \( U \) be an \( L^1 \) matricially normed space. Then the following hold:

(i) \( V \otimes_{rt} U \) is completely isometrically isomorphic to \( U \otimes_{rt} V \).

(ii) \( L_{cb}^r(V,U^*_{rt}) \) is completely isometrically isomorphic to \( L_{cb}^r(U,V^*_{rt}) \).

**Proof.** Let \( J : V \otimes_{rt} U \to U \otimes_{rt} V \) be defined by \( J(v \otimes u) = u \otimes v \), for all \( v \in V \) and all \( u \in U \). If \( n \in \mathbb{N} \) and \( (x_{ij}) \in M_n(V \otimes U) \), then

\[
\| (J(x_{ij})) \|_{n}^{rt} = \inf \left\{ \| (v_{ij}^{st})\|_m \| (v_{st})\|_n \mid (J(x_{ij})) = \left( \sum_{1 \leq s,t \leq m} v_{ij}^{st} \otimes u_{ts}, m \in \mathbb{N} \right) \right\}
\]

\[
= \inf \left\{ \| (v_{st})\|_m \| (u_{ij}^{st})\|_n \mid (x_{ij}) = \left( \sum_{1 \leq s,t \leq m} v_{st} \otimes u_{ij}^{st}, m \in \mathbb{N} \right) \right\}
\]

\[
= \| (x_{ij}) \|_{n}^{rt}
\]

which shows that \( J \) is a complete isometry and thus proves (i). (ii) follows from (i) and Propositions 1.2.8 and 1.2.9. \( \Box \)
Remark 1.2.10. We can show directly that the second part of the previous proposition holds for all matricially normed spaces V and U.

We finish our discussion of tensor products with a tensor product of matricially normed spaces that has no analogue in the category of Banach spaces, the Haagerup tensor product. It would not be an exaggeration to say that the Haagerup tensor product and its twin notion of completely bounded multilinear maps (see Definition 1.2.4) are the elements of the theory of operator spaces that make it such a powerful tool in the study of operator algebras (we will see many examples of that throughout the thesis; one that we will not refer directly to, but is hidden behind some of the results in Section 5.4, is the non-commutative Grothendieck inequality proved by Haagerup and Pisier in [H2] and [Pi2]).

Proposition 1.2.11. Let V and U be matricially normed spaces and V ⊗ U be their algebraic tensor product. If \((v_{ij}) \in M_{n,m}(V)\) and \((u_{ij}) \in M_{m,n}(U)\), then we will denote the \(n \times n\) matrix \((\sum_{1 \leq k \leq m} v_{ik} \otimes u_{kj})\) by \((v_{ij}) \otimes (u_{ij})\). For all \(n \in \mathbb{N}\) and all \((x_{ij}) \in M_n(V \otimes U)\), we define

\[
\| (x_{ij}) \|_n^h = \inf \left\{ \sum_{1 \leq k \leq l} \| (u^k_{ij}) \|_{n,m_k} \| (u^k_{ij}) \|_{m_k,n} \mid (x_{ij}) = \sum_{1 \leq k \leq l} (u^k_{ij}) \otimes (v^k_{ij}), l, m_1, ..., m_l \in \mathbb{N} \right\}.
\]

Then \(\| . \|_n^h\) is a norm on \(M_n(V \otimes U)\), for all \(n \in \mathbb{N}\). The completion of \(V \otimes U\), with respect to \(\| . \|_n^h\), equipped with the sequence of matrix norms \(\{\| . \|_n^h\}_{n \in \mathbb{N}}\) is a matricially normed space. We will denote it by \(V \otimes_h U\) and call it the Haagerup tensor product of V and U. The vector space \(V \otimes U\) equipped with \(\{\| . \|_n^h\}_{n \in \mathbb{N}}\) is also a matricially normed space denoted by \(V \otimes^h U\). If V and U are operator spaces, then

\[
\| (x_{ij}) \|_n^h = \inf \left\{ \| (v_{ij}) \|_{n,m} \| (u_{ij}) \|_{m,n} \mid (x_{ij}) = (v_{ij}) \otimes (u_{ij}), m \in \mathbb{N} \right\}
\]

for all \(n \in \mathbb{N}\) and all \((x_{ij}) \in M_n(V \otimes U)\) and \(V \otimes_h U\) and \(V \otimes^h U\) are operator spaces.

The norm \(\| . \|_n^h\), when V and U are operator spaces, was introduced by Effros and Kishimoto ([EKi], Section 2). The matrix norms \(\| . \|_n^h, n > 1\), for operator spaces were introduced by Paulsen and Smith in [PSm]. The Haagerup tensor product of matricially normed spaces was studied by Blecher and Paulsen ([BP], Section 3). It is called the Haagerup tensor product because of the previous use of \(\| . \|_n^h\) in [H4]. There are very nice accounts of the Haagerup tensor product in [Pi7], Chapter 3, and in [SSm1], Section 1.4.

Another notion that we will need is that of a completely bounded multilinear map.
Definition 1.2.4. Let $V_1, \ldots, V_n$ and $U$ be matricially normed spaces and $\phi : V_1 \times \ldots \times V_n \to U$ be an $n$-linear map. For each $m \in \mathbb{N}$, we define $\phi_m : M_m(V_1) \times \ldots \times M_m(V_n) \to M_m(U)$ by

$$
\phi_m((v_{ij}^1), \ldots, (v_{ij}^n)) = \left( \sum_{1 \leq k_1, k_2, \ldots, k_{n-1} \leq m} \phi(v_{i k_1 k_2}^1, v_{k_2 k_3}^2, \ldots, v_{k_{n-1} i}^n) \right)
$$

for all $(v_{ij}^1) \in M_m(V_1), \ldots, (v_{ij}^n) \in M_m(V_n)$. We will say that $\phi$ is completely bounded if $\sup_{m \in \mathbb{N}} \|\phi_m\|_{cb} < \infty$. We will say that $\phi$ is completely contractive if $\|\phi\|_{cb} \leq 1$. We will denote the space of completely bounded $n$-linear maps from $V_1 \times \ldots \times V_n$ into $U$ by $\mathcal{L}^n_{cb}(V_1, \ldots, V_n; U)$.

As in Proposition 1.2.5 we can make $\mathcal{L}^n_{cb}(V_1, \ldots, V_n; U)$ into a matricially normed space by identifying $(\phi_{ij}) \in M_m(\mathcal{L}^n_{cb}(V_1, \ldots, V_n; U))$ with an $n$-linear map from $V_1 \times \ldots \times V_n$ into $M_m(U)$.

Proposition 1.2.12. Let $V_1, \ldots, V_n$ and $U$ be matricially normed spaces. Then $\mathcal{L}^n_{cb}(V_1, \ldots, V_n; U)$ becomes a matricially normed space if we define

$$
\|\phi_{ij}\|_m = \sup\{\|((\phi_{ij})_1((v_{i1}^1), \ldots, (v_{i1}^n)))\|_{mr} \mid \|v_{i1}^1\|_r, \ldots, \|v_{i1}^n\|_r \leq 1, r \in \mathbb{N}\}
$$

for all $m \in \mathbb{N}$ and all $(\phi_{ij}) \in M_m(\mathcal{L}^n_{cb}(V_1, \ldots, V_n; U))$. If $U$ is an $L^p$ matricially normed space, then so is $\mathcal{L}^n_{cb}(V_1, \ldots, V_n; U)$.

Remark 1.2.11. (i) Completely bounded multilinear maps were defined in [CS1].

(ii) The representation theorem for completely bounded linear maps can be extended to completely bounded multilinear maps. If $H$ is a Hilbert space, $A_1, \ldots, A_n$ are C*-algebras and $\phi : A_1 \times \ldots \times A_n \to B(H)$ is a completely bounded $n$-linear map, then there exist $*$-representations $\pi_i$ of $A_i$ on a Hilbert space $K_i$, $1 \leq i \leq n$, and $T_i \in B(H_{i+1}, H_i)$, $0 \leq i \leq n$, where $H_0 = H_{n+1} = H$, with $\|\phi\|_{cb} = \|T_0\| \ldots \|T_n\|$ and $\phi(a_1, \ldots, a_n) = T_0 \pi_1(a_1) T_1 \ldots \pi_n(a_n) T_n$, for all $a_1 \in A_1, \ldots, a_n \in A_n$ ([CS1], Theorem 5.2, see also [PSm], Section 3, for a generalisation to operator spaces).

(iii) $\mathcal{L}^2_{cb}(V_1, V_2; \mathbb{C})$ is isometrically isomorphic to $(V_1 \otimes_{h} V_2)^*$ ([PSm], Propositions 1.3 and 1.4). For results on the standard dual of the Haagerup tensor product see [BSm] and [ER3]. We don't know of any results describing the reversed tracial dual of $V_1 \otimes_{h} V_2$.

(iv) It is easy to see that the space $\mathcal{L}^n_{cb}(V_1, \ldots, V_n; U)$ is not isomorphic to the space $\mathcal{L}^1_{cb}(V_1, \mathcal{L}^{n-1}_{cb}(V_2, \ldots, V_n; U))$. In particular $\mathcal{L}^2_{cb}(V_1, V_2; \mathbb{C})$ and $\mathcal{L}^1_{cb}(V_1, V_2^*)$ are not isomorphic.
There are two other notions of complete boundedness for multilinear maps. The first one is that of matricially bounded or jointly completely bounded multilinear maps introduced in [BP], Definition 5.3, and [ER2], Section 2. If $L^2_{nb}(V, U; \mathbb{C})$ is the space of matricially bounded bilinear forms on $V \times U$, then $L^2_{nb}(V, U; \mathbb{C})$ is completely isometrically isomorphic to $L^1_{cb}(V, U^*)$ and to $(V \hat{\otimes}_{os} U)^*$. The second is that of reversed tracially completely bounded multilinear maps (see [B1], [H1t] and [It2] for the normalised version of those maps). Matrix norms can be constructed on the space of those maps giving results similar to the previous one for the reversed tracial dual, the reversed tracial matricial norm structure on $L^1_{cb}(V, U)$ and the two reversed tracial tensor products.

We finish our account of matricially normed spaces with the definition of $l^\infty$ and $l^1$ direct sums of a family of matricially normed spaces $\{V_\lambda \mid \lambda \in \Lambda\}$. The $l^\infty$ direct sum is obtained by identifying $M_n(l^\infty(V_\lambda \mid \lambda \in \Lambda))$ with $l^\infty(M_n(V_\lambda) \mid \lambda \in \Lambda)$, for all $n \in \mathbb{N}$. The $l^1$ direct sum is obtained by identifying $l^1(V_\lambda \mid \lambda \in \Lambda)$ with the predual of $l^\infty(V_\lambda^* \mid \lambda \in \Lambda)$. Both definitions are due to Blecher ([B2], p.23). He defined $l^\infty$ and $l^1$ direct sums for families $\{V_\lambda \mid \lambda \in \Lambda\}$ of operator spaces, but it is easy to see that the definitions also hold for families of matricially normed spaces.

**Proposition 1.2.13.** Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be a family of matricially normed spaces.

(i) $l^\infty(V_\lambda \mid \lambda \in \Lambda)$ will denote the vector space $l^\infty(V_\lambda \mid \lambda \in \Lambda)$ equipped with the matrix norms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ defined by

$$\|(\sum_{\lambda \in \Lambda} \Theta v^{ij}_\lambda)\|_n = \sup_{\lambda \in \Lambda} \|(v^{ij}_\lambda)\|_n$$

for all $(\sum_{\lambda \in \Lambda} \Theta v^{ij}_\lambda) \in M_n(l^\infty(V_\lambda \mid \lambda \in \Lambda))$ and all $n \in \mathbb{N}$.

(ii) $l^1(V_\lambda \mid \lambda \in \Lambda)$ will denote the vector space $l^1(V_\lambda \mid \lambda \in \Lambda)$ equipped with the matrix norms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ defined by

$$\|(\sum_{\lambda \in \Lambda} \Theta v^{ij}_\lambda)\|_n = \sup_{\lambda \in \Lambda} \|(f_{pq}(\sum_{\lambda \in \Lambda} \Theta v^{ij}_\lambda))\| \in BALL(M_m(l^\infty(V_\lambda^* \mid \lambda \in \Lambda))), m \in \mathbb{N}$$

for all $(\sum_{\lambda \in \Lambda} \Theta v^{ij}_\lambda) \in M_n(l^1(V_\lambda \mid \lambda \in \Lambda))$ and all $n \in \mathbb{N}$, where $BALL(M_m(l^\infty(V_\lambda^* \mid \lambda \in \Lambda)))$ is the unit ball of $M_m(l^\infty(V_\lambda^* \mid \lambda \in \Lambda))$. The canonical maps $i_{\lambda_0} : V_{\lambda_0} \rightarrow l^1(V_\lambda \mid \lambda \in \Lambda)$ and $q_{\lambda_0} : l^1(V_\lambda \mid \lambda \in \Lambda) \rightarrow V_{\lambda_0}$ are respectively a complete isometry and a completely contractive map, for all $\lambda_0 \in \Lambda$.

Similar definitions hold for finite families of matricially normed spaces.

**Remark 1.2.12.** For a different definition of $l^1$ direct sums of operator spaces see [Pi7], Section 2.5. Pisier has also defined $l^p$ direct sums of operator spaces, for $1 < p < \infty$, using interpolation (see [Pi7], Section 2.8). We will not use those direct sums.
1.2.2 Hochschild cohomology

Cohomology groups of associative linear algebras were introduced by Hochschild in a series of three papers ([Ho1], [Ho2], [Ho3]) in the 1940's. During the 1950's and 1960's there was much interest in questions concerning derivations on Banach algebras (see [BoD], Section 18) and on $C^*$- and von Neumann algebras (see the introduction to Section 5.4 for more information). That lead to the introduction of bounded Hochschild cohomology groups in [Kam] and in [Gu] in the early 1960's.

What really put the study of bounded Hochschild cohomology on the map was the pioneering work of Johnson, Kadison and Ringrose in the late 1960's and early 1970's ([J1], [J2], [JKRi], [KRi4], [KRi5]). In the mid 1980's Christensen, Effros and Sinclair defined completely bounded Hochschild cohomology groups in [CES].

In this section we are going to give the definitions concerning modules and the Hochschild cohomology complex which we will use in the thesis. Almost everything we will say (and much more about $\mathcal{A}$-modules and Hochschild cohomology of Banach, $C^*$- and von Neumann algebras) can be found in [BoD], Sections 9, 43 and 44, [J2], [J4], [J6], [Pa1], [Pie2], [Ri2], [Ri3] and [SSm1].

We start by recalling the definition of an $\mathcal{A}$-module if $\mathcal{A}$ is an associative linear algebra.

**Definition 1.2.5.** Let $\mathcal{A}$ and $\mathcal{B}$ be associative linear algebras over $\mathbb{F}$ and $X$ be a vector space over $\mathbb{F}$. We will say that $X$ is a left $\mathcal{A}$-module if there exists a bilinear map $(a, x) \mapsto ax : A \times X \to X$ with $a_1(a_2x) = (a_1a_2)x$, for all $a_1, a_2 \in \mathcal{A}$ and all $x \in X$. Similarly we can define a right $\mathcal{A}$-module. We will say that $X$ is an $(\mathcal{A}, \mathcal{B})$-bimodule if $X$ is both a left $\mathcal{A}$-module and a right $\mathcal{B}$-module and $(ax)b = a(xb)$, for all $a \in \mathcal{A}$, all $b \in \mathcal{B}$ and all $x \in X$. We will call an $(\mathcal{A}, \mathcal{A})$-bimodule an $\mathcal{A}$-bimodule.

From now on we will deal only with $\mathcal{A}$-bimodules. It is easy to see that most of the definitions and constructions concerning $\mathcal{A}$-bimodules that we will discuss have analogues for one-sided $\mathcal{A}$-modules and $(\mathcal{A}, \mathcal{B})$-bimodules.

1.2.2.1 Bounded Hochschild cohomology

We start with four definitions concerning Banach $\mathcal{A}$-bimodules, a small discussion of ways to construct Banach $\mathcal{A}$-bimodules from given Banach $\mathcal{A}$-bimodules and the definition of dual $\mathcal{A}$-bimodules.

**Definition 1.2.6.** Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach space. We will say that $X$ is a Banach $\mathcal{A}$-bimodule if $X$ is an $\mathcal{A}$-bimodule and the maps
(a, x) \mapsto ax : A \times X \to X and (a, x) \mapsto xa : A \times X \to X are bounded bilinear maps, i.e. if there exists K > 0 such that \|ax\| \leq K\|a\|\|x\| and \|xa\| \leq K\|a\|\|x\|, for all a \in A and all x \in X. If K \leq 1, then we will say that X is contractive.

Obviously we can define normed A-bimodules which are not Banach spaces. We will only consider Banach A-bimodules.

**Definition 1.2.7.** Let A be a Banach algebra and X be a Banach A-bimodule.

(i) If A is unital with unit element e, then we will say that X is unital if ex = x = xe, for all x \in X.

(ii) If A has a bounded approximate identity \{e_\lambda\}_{\lambda \in \Lambda}, then we will say that X is essential, or nonunital, if \lim_{\lambda \in \Lambda}(e_\lambda x) = x = \lim_{\lambda \in \Lambda}(xe_\lambda), for all x \in X.

(iii) We will denote the set of elements of X that commute with A, \{x \in X \mid ax = xa, for all a \in A\}, by \mathcal{Z}(A, X). If \mathcal{Z}(A, X) = X, then we will say that X is abelian.

(iv) The annihilator of X in A is the set \text{Ann}_A(X) = \{a \in A \mid ax = 0 = xa, for all x \in X\}. If \text{Ann}_A(X) = A, then we will say that X is annihilating. If \text{Ann}_A(X) = \{0\}, then we will say that X is faithful.

**Definition 1.2.8.** Let A be a Banach algebra and X and Y be Banach A-bimodules. If \phi : X \to Y is a bounded linear map, then we will say that \phi is a bounded A-module map, or A-modular map, or A-module homomorphism, if \phi(ax) = a\phi(x) and \phi(xa) = \phi(x)a, for all a \in A and all x \in X. We will say that X and Y are A-module isomorphic if there exists an invertible bounded A-module map \phi : X \to Y. We will denote the set of bounded A-module maps between X and Y by \mathcal{L}_c(X, Y : /A).

**Definition 1.2.9.** Let A be a Banach algebra, X be a Banach A-bimodule and Y be a closed subspace of X. We will say that Y is a closed A-submodule of X if ay \in Y and ya \in Y, for all a \in A and all y \in Y. We will say that Y is a complemented A-submodule of X if there exists a bounded projection \rho : X \to Y which is an A-module map.

Let A be a Banach algebra. It is easy to see that if X is a Banach A-bimodule and Y is a closed A-submodule of X, then X/Y becomes a Banach A-bimodule if we define the module actions of A on X/Y by \alpha(x + Y) = \alpha x + Y and (x + Y)a = xa + Y, for all \alpha \in A and all x \in X. If \{X_\lambda \mid \lambda \in \Lambda\} is a family of uniformly bounded Banach A-bimodules (i.e. if there exists K > 0 with \|ax_\lambda\| \leq K\|a\|\|x_\lambda\| and \|x_\lambda a\| \leq K\|a\|\|x_\lambda\|, for all a \in A, all x_\lambda \in X_\lambda and all \lambda \in \Lambda), then \mathcal{L}^p(X_\lambda \mid \lambda \in \Lambda), 1 \leq p \leq \infty, and c_0(X_\lambda \mid \lambda \in \Lambda) are Banach A-bimodules with module actions defined coefficientwise. If \tilde{A} is the unitisation of
\( \mathcal{A} \) and \( X \) is a Banach \( \mathcal{A} \)-bimodule, then \( X \) becomes a unital Banach \( \mathcal{A} \)-bimodule, denoted by \( \hat{X} \), with \((a \oplus t)x = ax + tx \) and \( x(a \oplus t) = xa + tx \), for all \( a \oplus t \in \mathcal{A} \) and all \( x \in X \). Obviously if \( B \) is a Banach algebra and \( J : B \to \mathcal{A} \) is an algebra homomorphism, then any Banach \( \mathcal{A} \)-bimodule \( X \) becomes a Banach \( B \)-bimodule with the module actions defined by \( bx = J(b)x \) and \( xb = xJ(b) \), for all \( b \in B \) and all \( x \in X \).

A notion that is in the centre of the study of bounded Hochschild cohomology is the notion of a dual \( \mathcal{A} \)-bimodule. If \( \mathcal{A} \) is a Banach algebra and \( X \) is a Banach \( \mathcal{A} \)-bimodule, then it is easy to see that the dual of \( X \), \( X^* \), becomes a Banach \( \mathcal{A} \)-bimodule if we define

\[
(af)(x) = f(xa) \quad \text{and} \quad (fa)(x) = f(ax)
\]

for all \( a \in \mathcal{A} \), \( f \in X^* \) and all \( x \in X \). We will call modules of that form dual \( \mathcal{A} \)-bimodules. We can easily see that a Banach \( \mathcal{A} \)-bimodule \( X \) is a dual \( \mathcal{A} \)-bimodule if and only if there exists a Banach space \( X_* \) such that \( X \) is isometrically isomorphic to \((X_*)^* \) and the maps \( x \mapsto ax : X \to X \) and \( x \mapsto xa : X \to X \) are weak* continuous for all \( a \in \mathcal{A} \). Obviously a weak* closed \( \mathcal{A} \)-submodule of a dual \( \mathcal{A} \)-bimodule is also a dual \( \mathcal{A} \)-bimodule.

We can now define the bounded Hochschild cohomology complex of a Banach algebra \( \mathcal{A} \), with coefficients in a Banach \( \mathcal{A} \)-bimodule \( X \). For each \( n \geq 1 \) let \( \mathcal{L}_c^n(\mathcal{A}, X) \) be the space of bounded \( n \)-linear maps (which we will also call (bounded) \( n \)-cochains) from \( \mathcal{A}^n \) into \( X \) and \( \mathcal{L}_c^0(\mathcal{A}, X) = X \). The coboundary map \( \partial^n : \mathcal{L}_c^n(\mathcal{A}, X) \to \mathcal{L}_c^{n+1}(\mathcal{A}, X) \) is defined by

\[
\partial^n(x)(a) = ax - xa \tag{1.8}
\]

for all \( x \in X \) and all \( a \in \mathcal{A} \) and by

\[
\partial^n(\phi)(a_1, \ldots, a_{n+1}) = a_1\phi(a_2, \ldots, a_{n+1}) + \sum_{1 \leq k \leq n} (-1)^k \phi(a_1, \ldots, a_k a_{k+1}, \ldots, a_{n+1}) + (-1)^{n+1} \phi(a_1, \ldots, a_n) a_{n+1} \tag{1.9}
\]

for all \( n \geq 1 \), all \( \phi \in \mathcal{L}_c^n(\mathcal{A}, X) \) and all \( a_1, \ldots, a_{n+1} \in \mathcal{A} \). It is obvious that, for all \( n \geq 0 \), \( \partial^n \) is a well-defined bounded linear map, with \( \| \partial^n \| \leq n + 2K \), where \( K \) is the bound of the module actions. A tedious, but straightforward, calculation shows that \( \partial^n \partial^{n-1} = 0 \), for all \( n \geq 1 \) ([Ho1], p.60, or [SSm1], pp.2-3). Hence we can form the bounded Hochschild cohomology complex of \( \mathcal{A} \), with coefficients in \( X \),

\[
\mathcal{L}_c^0(\mathcal{A}, X) \xrightarrow{\partial^0} \mathcal{L}_c^1(\mathcal{A}, X) \xrightarrow{\partial^1} \mathcal{L}_c^2(\mathcal{A}, X) \xrightarrow{\partial^2} \ldots
\]
For all \( n \geq 1 \) we will denote \( \text{Ker}(\partial^n) \) by \( \mathcal{Z}^n_c(\mathcal{A}, X) \) and \( \text{Im}(\partial^{n-1}) \) by \( \mathcal{B}^n_c(\mathcal{A}, X) \). We will call the elements of \( \mathcal{Z}^n_c(\mathcal{A}, X) \) (bounded) \( n \)-cocycles and the elements of \( \mathcal{B}^n_c(\mathcal{A}, X) \) (bounded) \( n \)-coboundaries. Since \( \partial^n \partial^{n-1} = 0 \), \( \mathcal{B}^n_c(\mathcal{A}, X) \) is a linear subspace of \( \mathcal{Z}^n_c(\mathcal{A}, X) \). Thus we can take the quotient \( \mathcal{Z}^n_c(\mathcal{A}, X)/\mathcal{B}^n_c(\mathcal{A}, X) \). This quotient is called the \( n \)th bounded Hochschild cohomology group of \( \mathcal{A} \), with coefficients in \( X \), and is denoted by \( \mathcal{H}^n_c(\mathcal{A}, X) \). Note that \( \mathcal{Z}^1_c(\mathcal{A}, X) \) coincides with the set of bounded derivations from \( \mathcal{A} \) into \( X \) (i.e. bounded linear maps \( D : \mathcal{A} \rightarrow X \) with \( D(a_1a_2) = a_1D(a_2) + D(a_1)a_2 \), for all \( a_1, a_2 \in \mathcal{A} \)) and \( \mathcal{B}^1_c(\mathcal{A}, X) \) coincides with the set of inner derivations from \( \mathcal{A} \) into \( X \) (i.e. maps of the form \( \partial(x) : \mathcal{A} \rightarrow X \) with \( \partial(x)(a) = ax - xa \), for all \( a \in \mathcal{A} \), where \( x \in X \) and thus \( \mathcal{H}^1_c(\mathcal{A}, X) = \{0\} \) if and only if all the bounded derivations from \( \mathcal{A} \) into \( X \) are inner. (Obviously \( \mathcal{Z}^0_c(\mathcal{A}, X) = \text{Ker}(\partial^0) = \mathcal{Z}(\mathcal{A}, X) \). We can define \( \mathcal{B}^0_c(\mathcal{A}, X) \) to be equal to \( \{0\} \) and take the 0th bounded Hochschild cohomology group \( \mathcal{H}^0_c(\mathcal{A}, X) = \mathcal{Z}(\mathcal{A}, X) \). We will consider cohomology groups only for \( n \geq 1 \). We will say that \( \mathcal{H}^n_c(\mathcal{A}, X) \) vanishes if \( \mathcal{H}^n_c(\mathcal{A}, X) = \{0\} \). We must mention here that \( \mathcal{B}^n_c(\mathcal{A}, X) \) is not always a norm closed subspace of \( \mathcal{Z}^n_c(\mathcal{A}, X) \) (see [KLRi], Example 6.2) and therefore \( \mathcal{H}^n_c(\mathcal{A}, X) \) is not always a normed space.

A very useful technique in the study of Hochschild cohomology groups is the reduction of dimension trick discovered by Hochschild ([Ho1], Section 3) for the purely algebraic case and extended to the bounded case by Johnson ([J2], Section 1.1.a). If \( \mathcal{A} \) is a Banach algebra and \( X \) is a Banach \( \mathcal{A} \)-bimodule, then, for all \( n \geq 1 \), \( \mathcal{L}^n_c(\mathcal{A}, X) \) becomes a Banach \( \mathcal{A} \)-bimodule with the module actions of \( \mathcal{A} \) on \( \mathcal{L}^n_c(\mathcal{A}, X) \) defined by

\[
(a\phi)(a_1, \ldots, a_n) = a\phi(a_1, \ldots, a_n)
\]

and

\[
(\phi a)(a_1, \ldots, a_n) = \phi(aa_1, \ldots, a_n)
+ \sum_{1 \leq k \leq n-1} (-1)^k \phi(a, a_1, \ldots, a_ka_{k+1}, \ldots, a_n)
+ (-1)^n \phi(a, a_1, \ldots, a_{n-1})a_n
\]

Then the natural isometric isomorphism \( J : \mathcal{L}^{n+1}_c(\mathcal{A}, X) \rightarrow \mathcal{L}^1_c(\mathcal{A}, \mathcal{L}^n_c(\mathcal{A}, X)) \) defined by \( J(\Phi)(a)(a_1, \ldots, a_n) = \Phi(a, a_1, \ldots, a_n) \), for all \( \Phi \in \mathcal{L}^{n+1}_c(\mathcal{A}, X) \) and all \( a, a_1, \ldots, a_n \in \mathcal{A} \), induces an isomorphism between the cohomology groups \( \mathcal{H}_c^{n+1}(\mathcal{A}, X) \) and \( \mathcal{H}_c^1(\mathcal{A}, \mathcal{L}^n_c(\mathcal{A}, X)) \).

We finish our discussion of bounded Hochschild cohomology with the definition of an amenable and an \( n \)-amenable Banach algebra.

**Definition 1.2.10.** Let \( \mathcal{A} \) be a Banach algebra.
We will say that $A$ is amenable if $\mathcal{H}_c^n(A, X) = \{0\}$, for all dual $A$-bimodules $X$ and all $n \in \mathbb{N}$.

(ii) For each $n \in \mathbb{N}$, we will say that $A$ is $n$-amenable if $\mathcal{H}_c^n(A, X) = \{0\}$, for all dual $A$-bimodules $X$.

**Remark 1.2.13.** (i) Amenable Banach algebras were defined by Johnson ([J2], Section 5). One of the earliest results about bounded Hochschild cohomology was Johnson’s characterisation of amenable groups (see Section 5.1 for the definition of amenable groups) in terms of amenability of their group algebras: if $G$ is a locally compact group, then $L^1(G)$ is an amenable Banach algebra if and only if $G$ is an amenable group ([J2], Theorem 2.5).

(ii) The study of $n$-amenability for $n > 1$ was initiated by Effros and Kishimoto ([EKi], Section 3). The term $n$-amenable is due to Paterson ([Pa2]).

(iii) If $A$ is a Banach algebra and $X$ is a dual $A$-bimodule, then, for all $n \geq 1$, the reduction of dimension module $\mathcal{L}_c^n(A, X)$ is also dual. Thus if $A$ is $m$-amenable, then $A$ is $(m + n)$-amenable, for all $n \geq 1$. In particular if $A$ is 1-amenable, then $A$ is amenable.

(iv) A Banach algebra $A$ is amenable if and only if there exists a virtual diagonal for $A$, i.e. an element $M$ of $(A \hat{\otimes} A)^{**}$ with $aM = Ma$ and $\pi^{**}(M)a = \hat{a}$, for all $a \in A$, where the module actions of $A$ on $A \hat{\otimes} A$ are defined by $a(a_1 \otimes a_2) = (aa_1) \otimes a_2$ and $(a_1 \otimes a_2)a = a_1 \otimes (a_2a)$, for all $a, a_1, a_2 \in A$, $\hat{a}$ is the element of $A^{**}$ corresponding to $a$ and $\pi : A \hat{\otimes} A \to A$ is defined by $\pi(a_1 \otimes a_2) = a_1a_2$, for all $a_1, a_2 \in A$ (see [J1], Theorem 1.3 or [BoD], Theorem 43.9). A similar characterisation of $n$-amenability of $A$, in terms of $n$-virtual diagonals, which are generalisations of virtual diagonals, has been obtained, if $A$ is unital, by Effros and Kishimoto ([EKi], Theorem 3.1, see also [Pa2], Theorem 3.2).

**Remark 1.2.14.** We can also view bounded Hochschild cohomology of Banach algebras as a relative homology theory. This is the approach developed by Helemskii ([He]). We will not use this approach in this thesis.

### 1.2.2.2 Completely bounded Hochschild cohomology

If $A$ is a $C^*$-algebra and $V$ is an operator space, then the Christensen-Sinclair representation theorem for completely bounded multilinear maps (Remark 1.2.11(ii)) gives a nice description of the spaces $\mathcal{L}_c^n(A, V)$, for which no analogue exists in the bounded case. This was the main motivation behind the introduction of completely bounded Hochschild cohomology groups (and vice versa). For those groups to be well-defined the coboundary map $\partial^n$ must map $\mathcal{L}_c^n(A, V)$ into $\mathcal{L}_c^{n+1}(A, V)$ and be completely bounded. That happens if we consider algebras which are
operator spaces and have completely bounded multiplication and modules which are operator spaces and have completely bounded module actions. We start with the definitions of those algebras and modules.

The correct category of algebras to consider when we study completely bounded Hochschild cohomology are operator algebras. We will call a normed algebra \( A \) an operator algebra if it is a subalgebra of \( \mathcal{B}(H) \) for some Hilbert space \( H \) and is equipped with the natural sequence of matrix norms \( \{\|\cdot\|_{\mathcal{B}(H^n)}\}_{n\in\mathbb{N}} \). If \( A \) is an operator algebra, then the multiplication \( (a_1, a_2) \mapsto a_1 a_2 : A \times A \to A \) is a completely contractive bilinear map. On the other hand if \( A \) is a normed algebra which is an operator space and has completely bounded multiplication, then \( A \) is completely isomorphic to an operator algebra (completely isometrically isomorphic if \( A \) has a unit element of norm 1 or a contractive approximate identity and the multiplication is completely contractive) as it was proved in [BRS], Theorem 3.1, [R2], Theorem 2.2, and [133], Theorem 2.2 (there is a very nice account of those results in Chapter 4 of [Pi7]).

**Definition 1.2.11.** Let \( A \) be an operator algebra and \( X \) be a matricially normed space. We will say that \( X \) is a completely bounded \( A \)-bimodule if \( X \) is an \( A \)-bimodule and the maps \( (a, x) \mapsto ax : A \times X \to X \) and \( (a, x) \mapsto xa : A \times X \to X \) are completely bounded bilinear maps, i.e. if there exists \( K > 0 \) such that \( \|(\sum_{1 \leq k \leq n} a_{ik} x_{kj})\|_n \leq K \|(a_{ij})\|_n \|(x_{ij})\|_n \) and \( \|(\sum_{1 \leq k \leq n} x_{ik} a_{kj})\|_n \leq K \|(a_{ij})\|_n \|(x_{ij})\|_n \), for all \( n \in \mathbb{N} \), all \( (a_{ij}) \in \mathcal{M}_n(A) \) and all \( (x_{ij}) \in \mathcal{M}_n(X) \). If \( K \leq 1 \), then we will say that \( X \) is completely contractive. We will call \( X \) an operator completely bounded \( A \)-bimodule if \( X \) is an operator space. We will call \( X \) an \( L^p \) completely bounded \( A \)-bimodule, for \( 1 \leq p < \infty \), if \( X \) is an \( L^p \) matricially normed space.

**Remark 1.2.15.** If \( A \subseteq \mathcal{B}(H) \) is an operator algebra and \( X \) is a subspace of \( \mathcal{B}(H) \) such that \( axb \in X \), for all \( a, b \in A \) and all \( x \in X \), then \( X \) is an operator completely contractive \( A \)-bimodule. All operator completely bounded \( A \)-bimodules are of that form up to complete isomorphism, i.e. if \( X \) is an operator completely bounded \( A \)-bimodule, then there exist a Hilbert space \( K \) and complete isomorphisms \( \theta : X \to \mathcal{B}(K) \) and \( \pi : A \to \mathcal{B}(K) \), such that \( \pi \) is an algebra homomorphism, with \( \theta(axb) = \pi(a)\theta(x)\pi(b) \), for all \( a, b \in A \) and all \( x \in X \) ([B4], Theorem 2.2). If moreover \( A \) has a contractive approximate identity and \( X \) is an essential operator completely contractive \( A \)-bimodule, then \( \theta \) and \( \pi \) are complete isometries ([B4], Theorem 2.4). If \( A \) is a \( C^* \)-algebra, then \( \pi \) is a \( * \)-representation ([CES], Corollary 3.3, [ER1], Theorem 2.1).

It is easy to see that Definitions 1.2.7, 1.2.8 and 1.2.9 and the constructions
described after Definition 1.2.9 have completely bounded counterparts.

Suppose that $A$ is an operator algebra and $X$ is a completely bounded $A$-bimodule. Which of the three matricial norm structures that we have defined on $X^*$ makes it a completely bounded $A$-bimodule with the module actions defined as in (1.7)? A straightforward calculation shows that the answer to this question is the reversed tracial dual $X^*_{rt}$ of $X$ ([ER1], pp.148-9). We will call a completely bounded $A$-bimodule $X$ a dual completely bounded $A$-bimodule if there exists a completely bounded $A$-bimodule $Y$ with $X = Y^*_{rt}$ and the module actions of $A$ on $X$ are defined as in (1.7) (with $Y$ in the place of $X$).

If $A$ is an operator algebra and $X$ is an operator completely bounded $A$-bimodule, then, for all $n \in \mathbb{N}$, the coboundary map $\partial^n$ defined as in (1.8) and (1.9) maps $L^+_cb(A, X)$ into $L^{n+1}_cb(A, X)$ and is completely bounded with $\|\partial^n\|_{cb} \leq n+2K$ (the $L^\infty$ property of both $A$ and $X$ is essential in proving that). Thus we can form the completely bounded Hochschild cohomology complex of $A$, with coefficients in $X$,

$$L^0_{cb}(A, X) \xrightarrow{\partial^0} L^1_{cb}(A, X) \xrightarrow{\partial^1} L^2_{cb}(A, X) \xrightarrow{\partial^2} \ldots$$

As in the bounded case we define, for all $n \geq 1$, $Z^n_{cb}(A, X)$ to be the kernel of $\partial^n$, $B^n_{cb}(A, X)$ to be the image of $\partial^{n-1}$ and $H^n_{cb}(A, X)$ to be the quotient of $Z^n_{cb}(A, X)$ by $B^n_{cb}(A, X)$. We will call $H^n_{cb}(A, X)$ the $n$th completely bounded Hochschild cohomology group of $A$, with coefficients in $X$. The elements of $L^n_{cb}(A, X)$ will be called (completely bounded) $n$-cochains, the elements of $Z^n_{cb}(A, X)$ (completely bounded) $n$-coboundaries. Obviously $\partial^n(x)$ is completely bounded for all $x \in X$ and thus the spaces $B^1_{cb}(A, X)$ and $B^1_c(A, X)$ coincide.

As we mentioned in Remark 1.2.11(iv), the space $L^{n+1}_{cb}(A, X)$ is not isomorphic to the space $L^1_{cb}(A, L^n_{cb}(A, X))$. Thus the reduction of dimension trick that we discussed in the bounded case does not apply here.

The notions of an amenable and an $n$-amenable Banach algebra are replaced in the completely bounded case by the notions of a completely amenable and a completely $n$-amenable operator algebra.

**Definition 1.2.12.** Let $A$ be an operator algebra.

(i) We will say that $A$ is completely amenable if $H^n_{cb}(A, X) = \{0\}$, for all dual operator completely bounded $A$-bimodules $X$ and all $n \in \mathbb{N}$.

(ii) For each $n \in \mathbb{N}$, we will say that $A$ is completely $n$-amenable if $H^n_{cb}(A, X) = \{0\}$, for all dual operator completely bounded $A$-bimodules $X$.

**Remark 1.2.16.** Due to the lack of a reduction of dimension trick an analogue of Remark 1.2.13(iii) does not hold here, i.e. complete $m$-amenability of $A$ does
not imply complete \((m+n)\)-amenability of \(A\). In particular completely 1-amenable does not imply completely amenable. We don’t know of any characterisations of complete \(n\)-amenability in terms of \(n\)-virtual diagonals. An argument involving the minimal operator space structure on a Banach space \(X\), \(MIN(X)\), and the automatic complete boundedness of all bounded maps from an operator space \(Y\) into \(MIN(X)\) ([Pi7], p.16) shows that complete \(n\)-amenability implies \(n\)-amenability, for all \(n \in \mathbb{N}\). On the other hand 1-amenability implies complete 1-amenability since \(B^1_{cb}(A, X) = B^1_c(A, X)\).

Remark 1.2.17. Ruan has developed a different kind of completely bounded cohomology ([R3], [R4]). He considered algebras with matricially bounded multiplication and modules with matricially bounded module actions and constructed the complex \((\mathcal{L}^*_mb(A, X), \partial^n)\). Moreover he introduced a notion of amenability, operator amenability, for that cohomology complex. We will not deal with that kind of completely bounded cohomology. We believe that the constructions and the results of Chapters 2, 3 and 4 hold there as well.

1.2.2.3 Normal Hochschild cohomology

If \(A\) is an operator algebra, then it inherits the ultraweak topology from \(B(H)\). The existence of this topology on \(A\) leads to the definition of normal \(A\)-bimodules and normal cohomology groups.

Definition 1.2.13. Let \(A\) be an operator algebra and \(X\) be a dual \(A\)-bimodule. We will say that \(X\) is a normal \(A\)-bimodule if the maps \(a \mapsto ax : A \to X\) and \(a \mapsto xa : A \to X\) are ultraweak-weak\(^*\) continuous for all \(x \in X\).

It is easy to see that a weak\(^*\) closed \(A\)-submodule of a normal \(A\)-bimodule is a normal \(A\)-bimodule.

If \(A\) is an operator algebra and \(X\) is a normal \(A\)-bimodule, then the coboundary map \(\partial^n\) maps the space of \(n\)-linear separately ultraweak-weak\(^*\) continuous maps \(\mathcal{L}^n_w(A, X)\) into the space of \((n+1)\)-linear separately ultraweak-weak\(^*\) continuous maps \(\mathcal{L}^{n+1}_w(A, X)\). That allows us to define the normal Hochschild cohomology complex of \(A\), with coefficients in \(X\),

\[
\mathcal{L}^0_w(A, X) \xrightarrow{\partial^0} \mathcal{L}^1_w(A, X) \xrightarrow{\partial^1} \mathcal{L}^2_w(A, X) \xrightarrow{\partial^2} \ldots
\]

The \(n\)th normal Hochschild cohomology group of \(A\), with coefficients in \(X\), \(\mathcal{H}^n_w(A, X)\), is the quotient of \(\mathcal{Z}^n_w(A, X) = \text{Ker}(\partial^n)\) by \(\mathcal{B}^n_w(A, X) = \text{Im}(\partial^{n-1})\).

In a similar manner we can define normal completely bounded \(A\)-bimodules (see [ER1], Theorem 4.1, for a representation theorem for normal completely
bounded $\mathcal{A}$-bimodules $X$, when $\mathcal{A}$ is a von Neumann algebra and $X$ has a predual which is a Banach space), the normal completely bounded Hochschild cohomology complex of $\mathcal{A}$, with coefficients in $X$, and the normal completely bounded cohomology groups $\mathcal{H}^n_{\text{wcb}}(\mathcal{A}, X)$.

For von Neumann algebras we have the following notion of amenability (see the opening discussion in Section 5.2 for more information on amenable von Neumann algebras).

**Definition 1.2.14.** Let $\mathcal{M}$ be a von Neumann algebra. We will say that $\mathcal{M}$ is amenable if $\mathcal{H}^n_{\text{w}}(\mathcal{M}, X) = \{0\}$, for all normal $\mathcal{M}$-bimodules $X$ and all $n \in \mathbb{N}$.

Using reduction of dimension, we can easily see that $\mathcal{M}$ is amenable if and only if $\mathcal{H}^1_{\text{w}}(\mathcal{M}, X) = \{0\}$, for all normal $\mathcal{M}$-bimodules $X$.

### 1.2.2.4 $\mathcal{B}$-relative Hochschild cohomology

The notion of module maps defined in Definition 1.2.8 can be extended to multi-linear maps.

**Definition 1.2.15.** Let $\mathcal{A}$ be a Banach algebra, $X$ and $Y$ be Banach $\mathcal{A}$-bimodules and $\phi : X^n \to Y$ be a bounded $n$-linear map, $n > 1$. We will say that $\phi$ is a bounded $\mathcal{A}$-module, or $\mathcal{A}$-modular map, if

$$a\phi(x_1, \ldots, x_n) = \phi(ax_1, \ldots, x_n)$$

$$\phi(x_1, \ldots, x_k a, x_{k+1}, \ldots, x_n) = \phi(x_1, \ldots, x_k, ax_{k+1}, \ldots, x_n)$$

$$\phi(x_1, \ldots, x_n)a = \phi(x_1, \ldots, x_n a)$$

for all $a \in \mathcal{A}$, all $x_1, \ldots, x_k, x_{k+1}, \ldots, x_n \in X$ and all $1 \leq k \leq n - 1$. We will denote the set of all bounded $n$-linear $\mathcal{A}$-module maps between $X^n$ and $Y$ by $\mathcal{L}^n_c(X, Y : /\mathcal{A})$.

If $\mathcal{A}$ is a Banach algebra, $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and $X$ is a Banach $\mathcal{A}$-bimodule, then $\partial^n$ maps $\mathcal{L}^n_c(\mathcal{A}, X : /\mathcal{B})$ into $\mathcal{L}^{n+1}_c(\mathcal{A}, X : /\mathcal{B})$. That allows us to define the bounded $\mathcal{B}$-relative Hochschild cohomology complex of $\mathcal{A}$, with coefficients in $X$,

$$\mathcal{L}^0_c(\mathcal{A}, X : /\mathcal{B}) \xrightarrow{\partial^0} \mathcal{L}^1_c(\mathcal{A}, X : /\mathcal{B}) \xrightarrow{\partial^1} \mathcal{L}^2_c(\mathcal{A}, X : /\mathcal{B}) \xrightarrow{\partial^2} \ldots$$

where $\mathcal{L}^n_c(\mathcal{A}, X : /\mathcal{B}) = \{x \in X \mid bx = xb, \text{ for all } b \in \mathcal{B}\}$. The $n$th bounded $\mathcal{B}$-relative Hochschild cohomology group of $\mathcal{A}$, with coefficients in $X$, $\mathcal{H}^n_{\text{r}}(\mathcal{A}, X : /\mathcal{B})$ is defined in a similar manner to the ones defined previously. We can also define the groups $\mathcal{H}^n_{\text{cb}}(\mathcal{A}, X : /\mathcal{B})$, $\mathcal{H}^n_{\text{w}}(\mathcal{A}, X : /\mathcal{B})$ and $\mathcal{H}^n_{\text{wcb}}(\mathcal{A}, X : /\mathcal{B})$.
A slightly different version of $\mathcal{B}$-relative Hochschild cohomology can be defined if we assume that the elements of $\mathcal{L}_c^n(\mathcal{A}, X : /\mathcal{B})$ have the properties of Definition 1.2.15 together with the property that $\phi(a_1, ..., a_n) = 0$ if any of the $a_k$'s, $1 \leq k \leq n$, is an element of $\mathcal{B}$. We will not consider this version.
Chapter 2

The splitting of the cohomology

In this chapter we will discuss the notion of splitting for Hochschild cohomology groups and the Hochschild cohomology complex. For a minute we go back to the purely algebraic case. Let us consider an associative linear algebra $\mathcal{A}$ and an $\mathcal{A}$-bimodule $X$ and form the Hochschild cohomology complex

$$
\cdots \xrightarrow{\partial_{n-2}} \mathcal{L}^{n-1}(\mathcal{A}, X) \xrightarrow{\partial_{n-1}} \mathcal{L}^{n}(\mathcal{A}, X) \xrightarrow{\partial_{n}} \mathcal{L}^{n+1}(\mathcal{A}, X) \xrightarrow{\partial_{n+1}} \cdots
$$

The vector space $\mathcal{L}^{n}(\mathcal{A}, X)$ can be decomposed into the direct sum of its subspaces $\mathcal{B}^{n}(\mathcal{A}, X)$ and $\mathcal{L}^{n}(\mathcal{A}, X) \oplus \mathcal{B}^{n}(\mathcal{A}, X)$. On the other hand $\mathcal{L}^{n-1}(\mathcal{A}, X)$ can be decomposed into the direct sum of $\mathcal{Z}^{n-1}(\mathcal{A}, X)$ and $\mathcal{L}^{n-1}(\mathcal{A}, X) \oplus \mathcal{Z}^{n-1}(\mathcal{A}, X)$. Since $\mathcal{Z}^{n-1}(\mathcal{A}, X)$ and $\mathcal{B}^{n}(\mathcal{A}, X)$ are, respectively, the kernel and the image of the coboundary map $\partial_{n-1}$, those decompositions give rise to a map

$$s_{n} : \mathcal{L}^{n}(\mathcal{A}, X) \to \mathcal{L}^{n-1}(\mathcal{A}, X)$$

with $\partial_{n-1} s_{n} \partial_{n-1} = \partial_{n-1}$. We can see that both $\partial_{n-1} s_{n}$ and $s_{n} \partial_{n-1}$ are projections, with $\text{Im}(\partial_{n-1} s_{n}) = \mathcal{B}^{n}(\mathcal{A}, X)$ and $\text{Im}(s_{n} \partial_{n-1}) = \mathcal{L}^{n-1}(\mathcal{A}, X) \oplus \mathcal{Z}^{n-1}(\mathcal{A}, X)$. Thus if we take two consecutive such maps $s_{n}$ and $s_{n+1}$, the map $\partial_{n-1} s_{n} + s_{n+1} \partial_{n}$ is a projection that maps $\mathcal{L}^{n}(\mathcal{A}, X)$ onto $\mathcal{L}^{n}(\mathcal{A}, X) \oplus (\mathcal{Z}^{n}(\mathcal{A}, X) \oplus \mathcal{B}^{n}(\mathcal{A}, X))$, which is isomorphic to $\mathcal{L}^{n}(\mathcal{A}, X) \oplus \mathcal{H}^{n}(\mathcal{A}, X)$. So, if $\mathcal{H}^{n}(\mathcal{A}, X)$ vanishes, then $\partial_{n-1} s_{n} + s_{n+1} \partial_{n} = \text{id}_{\mathcal{L}^{n}(\mathcal{A}, X)}$. (For a more general discussion on the same lines see [We], pp.15-16). Those ideas lead to the definition of a split complex which is standard in homological algebra.

**Definition 2.0.1 ([We], Definition 1.4.1)** A complex

$$C = \cdots \xrightarrow{\partial_{n-1}} C_{n-1} \xrightarrow{\partial_{n}} C_{n} \xrightarrow{\partial_{n+1}} C_{n+1} \xrightarrow{\partial_{n+2}} \cdots$$

is called split if, for all $n \in \mathbb{N}$, there exist maps $s_{n} : C_{n} \to C_{n+1}$ such that $\partial_{n+1} s_{n} \partial_{n+1} = \partial_{n+1}$. The maps $s_{n}$ are called splitting maps. If in addition $C$ is
acyclic (exact as a sequence), we say that $C$ is split exact.

A natural question that arises in the study of the cohomology theory of Banach algebras, which is related to the ideas discussed in the previous paragraph, is the following: Suppose that $A$ is a Banach algebra and $X$ is a Banach $A$-bimodule. If $\mathcal{H}_c(A, X)$ vanishes, i.e. if any bounded derivation from $A$ into $X$ is an inner derivation, can we recover, in a "good" way, for any derivation $D \in \mathcal{Z}^1_c(A, X)$, an element $x \in X$, with $D = \partial^0(x)$? This question can be reformulated in the following way: Does there exist a bounded linear map $s_1 : \mathcal{Z}^1_c(A, X) \to X$, with $\partial^0 s_1(D) = D$, for all $D \in \mathcal{Z}^1_c(A, X)$? A careful glance at the map $s_1$, will show that it has the same defining property as a splitting exact map of Definition 2.0.1, but instead of being defined on $\mathcal{L}^1_c(A, X)$ it is defined on $\mathcal{Z}^1_c(A, X)$. Obviously the previous question can also be asked for $n > 1$ and for the completely bounded cohomology.

The discussion so far gives us five ways to define splitting for bounded and completely bounded cohomology groups. The first two depend on the existence of a map $s_n$ from $\mathcal{B}^n_c(A, X)$ or $\mathcal{L}^n_c(A, X)$ into $\mathcal{L}^{n-1}_c(A, X)$ with $\partial^{n-1} s_n \partial^{n-1} = \partial^n$ and correspond to the homological notion of a split complex. The third and the fourth depend on the existence of a map $s_n$ from $\mathcal{Z}^n(A, X)$ or $\mathcal{L}^n(A, X)$ into $\mathcal{L}^{n-1}_c(A, X)$, with $\partial^{n-1} s_n(\phi) = \phi$, for all $\phi \in \mathcal{Z}^n(A, X)$ and correspond to the homological notion of a split exact complex. The fifth depends on the existence of a pair of maps $s_n : \mathcal{L}^n_c(A, X) \to \mathcal{L}^{n-1}_c(A, X)$ and $s_{n+1} : \mathcal{L}^{n+1}_c(A, X) \to \mathcal{L}^n(A, X)$, decomposing the identity on $\mathcal{L}^n_c(A, X)$ into the sum $\partial^{n-1} s_n + s_{n+1} \partial^{n-1}$. It is obvious that in all five cases the corresponding continuity conditions must hold for the maps $s_n$. Although defining types of splitting with the splitting maps $s_n$ defined on $\mathcal{B}^n_c(A, X)$ or on $\mathcal{Z}^n(A, X)$ instead of $\mathcal{L}^n(A, X)$ seems to be an unrequired complication those types of splitting turn out to be quite interesting. The first one ($s_n$ from $\mathcal{B}^n_c(A, X)$ into $\mathcal{L}^{n-1}_c(A, X)$ with $\partial^{n-1} s_n \partial^{n-1} = \partial^n$), although being very weak, is related to injectivity for von Neumann algebras as we will show in Section 5.3. The second one ($s_n$ from $\mathcal{Z}^n(A, X)$ into $\mathcal{L}^{n-1}_c(A, X)$ with $\partial^{n-1} s_n(\phi) = \phi$, for all $\phi \in \mathcal{Z}^n(A, X)$) is the only one on which the averaging and lifting results for the cohomology of von Neumann algebras can be applied (see Section 5.1).

There are two central themes in the study of splitting. The first is the relation between the existence of splitting maps and the complementation of $\mathcal{Z}^{n-1}_c(A, X)$ in $\mathcal{L}^n(A, X)$ and of $\mathcal{B}^n_c(A, X)$ or $\mathcal{Z}^n(A, X)$ in $\mathcal{L}^n(A, X)$ (in the purely algebraic case this complementation is automatic and hence splitting maps always exist, as we mentioned in the first paragraph). In general we tend to think of the
study of Hochschild cohomology as the study of the relation between cocycles and coboundaries. The above mentioned connection shows that studying the Hochschild cohomology complex we can also obtain information about the way $B^n_c(A, X)$ and $Z^n_c(A, X)$ "sit" inside $L^n(A, X)$. The second theme is the interaction between the splitting of consecutive cohomology groups.

We will study first the bounded case, then the completely bounded one and finish with the case of the bounded relative cohomology.

We could have defined similar types of splitting for normal cohomology groups. There the categorically correct type of continuity for the splitting maps would have been weak$^*$ continuity. We don’t have enough space to study this normal version of splitting. Moreover in the cases where we use splitting arguments for the normal cohomology groups in Chapter 5 we only have to consider bounded splitting maps.

### 2.1 Splitting of the bounded Hochschild cohomology

In this section we study the notion of splitting for the bounded Hochschild cohomology. In the first part we will study the notion of splitting for bounded Hochschuld cohomology groups. We will define five different types of splitting for the group $H^n_c(A, X)$. As we will see all five types of splitting are equivalent to geometric properties of the spaces $Z^{n-1}_c(A, X)$, $B^n_c(A, X)$ and $Z^n_c(A, X)$. After defining each type of splitting we will give its geometric characterisation. The splitting of $H^n_c(A, X)$ depends on the existence of certain maps, which we call splitting maps. For each type of splitting we will give the properties of the set of splitting maps, which show that the individual splitting maps are not important. After doing that we will give the relation between the type of splitting we are studying and the ones previously defined. When we have defined all five types of splitting, given their geometric characterisations and the properties of their respective sets of splitting maps and studied the relation between them, we will give some algebraic properties of splitting (i.e. a reduction of dimension result and some remarks about how splitting behaves with respect to isomorphic modules, submodules, quotients and direct sums of modules, unital modules and the unitisation of the algebra). We will finish with a remark about the relation between the splitting of the first bounded cohomology group of an algebra $A$, with coefficients in an $A$-bimodule $X$, and the complementation of $Z(A, X)$ in $X$.

In the second part of the section we will study the splitting of the bounded Hochschild cohomology complex. The main result is that the third, fourth and
fifth type of splitting coincide on the complex and so we don’t have to distinguish between them on that level.

2.1.1 Splitting of the bounded Hochschild cohomology groups

**Definition 2.1.1.** Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. We say that the $n$th bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (I) if there exists a bounded linear map

$$s_n : \mathcal{B}_c^n(\mathcal{A}, X) \to \mathcal{L}_c^{n-1}(\mathcal{A}, X)$$

with

$$\partial^{n-1}s_n = id_{\mathcal{B}_c^n(\mathcal{A}, X)}$$

The map $s_n$ is called a splitting map of the first kind.

The following proposition gives a geometric characterisation of split (I). In particular it shows that if $\mathcal{H}_c^n(\mathcal{A}, X)$ splits (I), then $\mathcal{B}_c^n(\mathcal{A}, X)$ is a closed subspace of $\mathcal{L}_c^n(\mathcal{A}, X)$ and so $\mathcal{H}_c^n(\mathcal{A}, X)$ is a Banach space.

**Proposition 2.1.1.** Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) The $n$th bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (I).

(ii) $\mathcal{Z}_c^{n-1}(\mathcal{A}, X)$ is complemented in $\mathcal{L}_c^{n-1}(\mathcal{A}, X)$ and $\mathcal{B}_c^n(\mathcal{A}, X)$ is a closed subspace of $\mathcal{L}_c^n(\mathcal{A}, X)$.

**Proof.** If $\mathcal{H}_c^n(\mathcal{A}, X)$ splits (I), then there exists a bounded linear map

$$s_n : \mathcal{B}_c^n(\mathcal{A}, X) \to \mathcal{L}_c^{n-1}(\mathcal{A}, X)$$

with $\partial^{n-1}s_n = id_{\mathcal{B}_c^n(\mathcal{A}, X)}$. Given $\phi \in \mathcal{L}_c^{n-1}(\mathcal{A}, X)$, the condition for split (I) implies that

$$(s_n\partial^{n-1})^2(\phi) = s_n(\partial^{n-1}s_n(\partial^{n-1}(\phi))) = s_n\partial^{n-1}(\phi)$$

and thus $s_n\partial^{n-1}$ and $id_{\mathcal{L}_c^{n-1}(\mathcal{A}, X)} - s_n\partial^{n-1}$ are bounded projections. Obviously $id_{\mathcal{L}_c^{n-1}(\mathcal{A}, X)} - s_n\partial^{n-1}$ maps $\mathcal{L}_c^{n-1}(\mathcal{A}, X)$ into $\mathcal{Z}_c^{n-1}(\mathcal{A}, X)$. Moreover, if $\phi$ is an $(n-1)$-cocycle, then

$$(id_{\mathcal{L}_c^{n-1}(\mathcal{A}, X)} - s_n\partial^{n-1})(\phi) = \phi - s_n(\partial^{n-1}(\phi)) = \phi$$

and hence $\text{Im}(id_{\mathcal{L}_c^{n-1}(\mathcal{A}, X)} - s_n\partial^{n-1}) = \mathcal{Z}_c^{n-1}(\mathcal{A}, X)$. Therefore $\mathcal{Z}_c^{n-1}(\mathcal{A}, X)$ is complemented in $\mathcal{L}_c^{n-1}(\mathcal{A}, X)$. To show that $\mathcal{B}_c^n(\mathcal{A}, X)$ is closed, take a sequence $\{\partial^{n-1}(\phi_m)\}_{m \in \mathbb{N}}$ in $\mathcal{B}_c^n(\mathcal{A}, X)$ converging to some $\psi \in \mathcal{L}_c^n(\mathcal{A}, X)$. Then
\{\partial^{n-1}(\phi_m)\}_{m \in \mathbb{N}} \text{ is Cauchy and hence } \{s_n\partial^{n-1}(\phi_m)\}_{m \in \mathbb{N}} \text{ converges to some } \phi \in \mathcal{L}^{n-1}_{c}(A, X). \text{ Therefore}

\[
\partial^{n-1}(\phi) = \lim_{m \to \infty} \partial^{n-1}s_n(\partial^{n-1}(\phi_m)) \\
= \lim_{m \to \infty} \partial^{n-1}(\phi_m) \\
= \psi
\]

and so \(\psi \in \mathcal{B}^n_c(A, X)\), which proves that \(\mathcal{B}^n_c(A, X)\) is closed.

Conversely suppose that \(\mathcal{Z}^{n-1}_{c}(A, X)\) is complemented in \(\mathcal{L}^{n-1}_{c}(A, X)\) and \(\mathcal{B}^n_c(A, X)\) is a closed subspace of \(\mathcal{L}^{n}_{c}(A, X)\). Since \(\mathcal{Z}^{n-1}_{c}(A, X) = \text{Ker}(\partial^{n-1})\) is complemented in \(\mathcal{L}^{n-1}_{c}(A, X)\) and \(\mathcal{B}^n_c(A, X) = \text{Im}(\partial^{n-1})\) is closed, from the inverse mapping theorem, \(\mathcal{B}^n_c(A, X)\) is isomorphic to \(\mathcal{L}^{n-1}_{c}(A, X) \oplus \mathcal{Z}^{n-1}_{c}(A, X)\) and so there exists a bounded linear map

\[
\pi : \mathcal{B}^n_c(A, X) \to \mathcal{L}^{n-1}_{c}(A, X)
\]

with \((\partial^{n-1}\pi)(\psi) = \psi\), for all \(\psi \in \mathcal{B}^n_c(A, X)\), i.e. \(\pi\) is a splitting map of the first kind. \(\square\)

**Remark 2.1.1.** In [J6], pp.253-4, Johnson gave the following definition: If \(A\) is a Banach algebra, \(X\) and \(Y\) are Banach \(A\)-modules and \(\phi : X \to Y\) is a bounded \(A\)-module map, then \(\phi\) is called admissible if \(\text{Ker}(\phi)\) is a complemented subspace of \(X\) and \(\text{Im}(\phi)\) is a closed subspace of \(Y\). This definition can be generalised to bounded linear maps between normed spaces. We will call a bounded linear map \(\phi : X \to Y\) between the normed spaces \(X\) and \(Y\) admissible if \(\text{Ker}(\phi)\) is complemented in \(X\) and \(\text{Im}(\phi)\) is closed in \(Y\). If \(X\) and \(Y\) are Banach spaces and \(\phi : X \to Y\) is an admissible map, then there exists a bounded map \(\psi : \text{Im}(\phi) \to X\) with \(\phi \psi = \text{id}_{\text{Im}(\phi)}\). Using this terminology the result of the previous proposition can be rephrased in the following way: \(\mathcal{H}^n_c(A, X)\) splits (I) if and only if \(\partial^{n-1}\) is admissible.

We must mention that the complementation of \(\mathcal{Z}^{n-1}_{c}(A, X)\) in \(\mathcal{L}^{n-1}_{c}(A, X)\) does not imply by itself that \(\mathcal{H}^n_c(A, X)\) splits (I) (see the remarks following Proposition 2.1.23).

In the following proposition we show that the set of splitting maps of the first kind is a closed convex set which is stable under addition with maps mapping \(\mathcal{B}^n_c(A, X)\) into \(\mathcal{Z}^{n-1}_{c}(A, X)\).

**Proposition 2.1.2.** Let \(A\) be a Banach algebra and \(X\) be a Banach \(A\)-bimodule. If the \(n\)th bounded cohomology group of \(A\), with coefficients in \(X\), splits (I), then the following hold:
(i) The set of splitting maps of the first kind is a convex closed subset of $\mathcal{L}_c^1(\mathcal{B}_c^n(\mathcal{A}, X), \mathcal{L}_c^{n-1}(\mathcal{A}, X))$.

(ii) If $s_n$ is a splitting map of the first kind and

$$\mathcal{T} = \{ \Phi \in \mathcal{L}_c^1(\mathcal{B}_c^n(\mathcal{A}, X), \mathcal{L}_c^{n-1}(\mathcal{A}, X)) \mid \Phi(\mathcal{B}_c^n(\mathcal{A}, X)) \subseteq \mathcal{Z}_c^{n-1}(\mathcal{A}, X) \}$$

then the set of splitting maps of the first kind is the equivalence class of $s_n$ in $\mathcal{L}_c^1(\mathcal{B}_c^n(\mathcal{A}, X), \mathcal{L}_c^{n-1}(\mathcal{A}, X))/\mathcal{T}$.

Proof. (i) It is obvious that the set of splitting maps of the first kind is convex. Moreover if $\{s_n^\lambda\}_{\lambda \in \Lambda}$ is a net of splitting maps of the first kind converging, pointwise on $\mathcal{B}_c^n(\mathcal{A}, X)$, to a map $s_n$, then it is easy to see that $s_n$ is a splitting map of the first kind.

(ii) It is easy to see that if $\Phi \in \mathcal{T}$, then $\partial^{n-1}\Phi \partial^{n-1} = 0$ and therefore

$$\partial^{n-1}(s_n + \Phi)(\partial^{n-1}(\phi)) = \partial^{n-1}(\phi)$$

for all $\partial^{n-1}(\phi) \in \mathcal{B}_c^n(\mathcal{A}, X)$, i.e. $s_n + \Phi$ is a splitting map of the first kind. On the other hand if $s_n$ and $s_n'$ are splitting maps of the first kind, then $\partial^{n-1}(s_n - s_n')(\partial^{n-1}(\phi)) = 0$, for all $\partial^{n-1}(\phi) \in \mathcal{B}_c^n(\mathcal{A}, X)$ and so $s_n - s_n' \in \mathcal{T}$. \qed

We move now to the second type of splitting. Its only difference from the first type is that the splitting maps are defined on the whole of $\mathcal{L}_c^n(\mathcal{A}, X)$ and not just on $\mathcal{B}_c^n(\mathcal{A}, X)$.

Definition 2.1.2. Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. We say that the $n$th bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (II) if there exists a bounded linear map

$$s_n : \mathcal{L}_c^n(\mathcal{A}, X) \to \mathcal{L}_c^{n-1}(\mathcal{A}, X)$$

with

$$\partial^{n-1}s_n\partial^{n-1} = \partial^{n-1}$$

The map $s_n$ is called a splitting map of the second kind.

The following proposition gives a geometric characterisation of split (II).

Proposition 2.1.3. Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) The $n$th bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (II).

(ii) $\mathcal{Z}_c^{n-1}(\mathcal{A}, X)$ is complemented in $\mathcal{L}_c^{n-1}(\mathcal{A}, X)$ and $\mathcal{B}_c^n(\mathcal{A}, X)$ is complemented in $\mathcal{L}_c^n(\mathcal{A}, X)$.
Proof. If the \( n \)th bounded cohomology group of \( A \), with coefficients in \( X \), splits (II), then there exists a bounded linear map

\[
s_n : \mathcal{L}_c^n(A, X) \to \mathcal{L}_c^{n-1}(A, X)
\]

with \( \partial^{n-1}s_n\partial^{n-1} = \partial^{n-1} \). As in Proposition 2.1.1, we can prove that \( \text{id}_{\mathcal{L}_c^{n-1}(A, X)} - s_n\partial^{n-1} \) is a bounded projection that maps \( \mathcal{L}_c^{n-1}(A, X) \) onto \( Z_c^{n-1}(A, X) \). Moreover since

\[
(\partial^{n-1}s_n)^2 = (\partial^{n-1}s_n\partial^{n-1})s_n = \partial^{n-1}s_n
\]

\( \partial^{n-1}s_n \) is a bounded projection. It is easy to see that \( \text{Im}(\partial^{n-1}s_n) = B_c^n(A, X) \) and therefore \( B_c^n(A, X) \) is complemented in \( \mathcal{L}_c^n(A, X) \).

Conversely suppose that \( Z_c^{n-1}(A, X) \) is complemented in \( \mathcal{L}_c^{n-1}(A, X) \) and \( B_c^n(A, X) \) is complemented in \( \mathcal{L}_c^n(A, X) \). As in Proposition 2.1.1, there exists a bounded linear map

\[
\pi : B_c^n(A, X) \to \mathcal{L}_c^{n-1}(A, X)
\]

with \( (\partial^{n-1}\pi)(\psi) = \psi \), for all \( \psi \in B_c^n(A, X) \). Moreover, since \( B_c^n(A, X) \) is complemented in \( \mathcal{L}_c^n(A, X) \), there exists a projection

\[
\rho : \mathcal{L}_c^n(A, X) \to \mathcal{L}_c^n(A, X)
\]

with \( \text{Im}(\rho) = B_c^n(A, X) \). Now if

\[
s_n : \mathcal{L}_c^n(A, X) \to \mathcal{L}_c^{n-1}(A, X)
\]

is defined by \( s_n = \pi\rho \), then given \( \phi \in \mathcal{L}_c^{n-1}(A, X) \) we get

\[
\partial^{n-1}s_n\partial^{n-1}(\phi) = \partial^{n-1}\pi(\rho(\partial^{n-1}(\phi)) = \partial^{n-1}\pi(\partial^{n-1}(\phi)) = \partial^{n-1}(\phi)
\]

and so \( s_n \) is a splitting map of the second kind. \( \square \)

In particular the previous proposition implies that if \( \mathcal{H}_c^n(A, X) \) splits (II), then \( \mathcal{H}_c^n(A, X) \) is a Banach space isomorphic to \( Z_c^n(A, X) \oplus B_c^n(A, X) \).

Using the previous proposition we can see that the simplest case of splitting of the second kind arises when \( B_c^n(A, X) = \{0\} \) (since then \( Z_c^{n-1}(A, X) = \mathcal{L}_c^{n-1}(A, X) \) and so \( Z_c^{n-1}(A, X) \) is complemented in \( \mathcal{L}_c^{n-1}(A, X) \) and \( B_c^n(A, X) \) is complemented in \( \mathcal{L}_c^n(A, X) \)). For example, if \( A \) is a Banach algebra and \( X \) is an abelian Banach \( A \)-bimodule, then the first bounded cohomology group of \( A \), with coefficients in \( X \), splits (II).
The existence of a splitting map of the second kind \( s_n : \mathcal{L}^n_c(A, X) \to \mathcal{L}^{n-1}_c(A, X) \) does not imply that \( \mathcal{H}^n_c(A, X) = \{0\} \). To see that let \( A = A(\mathbb{D}) \) be the disc algebra and \( X = \mathbb{C} \), with the module actions of \( A \) on \( X \) defined by \( f \lambda = f(z_0)\lambda = \lambda f \), for all \( f \in A(\mathbb{D}) \) and all \( \lambda \in \mathbb{C} \), where \( z_0 \in \text{int}(\mathbb{D}) \). Then \( \mathcal{H}^1_c(A, X) \) does not vanish ([J2], p.88). On the other hand using the previous observation we can see that \( \mathcal{H}^1_c(A, X) \) splits (II), since \( X \) is an abelian \( A \)-bimodule.

The following proposition gives the properties of the set of splitting maps of the second kind.

**Proposition 2.1.4.** Let \( A \) be a Banach algebra and \( X \) be a Banach \( A \)-bimodule. If the \( n \)th bounded cohomology group of \( A \), with coefficients in \( X \), splits (II), then the following hold:

(i) The set of splitting maps of the second kind is a convex closed subset of \( \mathcal{L}^n_c(\mathcal{L}^n_c(A, X), \mathcal{L}^{n-1}_c(A, X)) \).

(ii) If \( s_n \) is a splitting map of the second kind and

\[ \mathcal{T} = \{ \Phi \in \mathcal{L}^1_c(\mathcal{L}^n_c(A, X), \mathcal{L}^{n-1}_c(A, X)) \mid \Phi(\mathcal{B}^n(A, X)) \subseteq \mathcal{Z}^{n-1}_c(A, X) \} \]

then the set of splitting maps of the second kind is the equivalence class of \( s_n \) in \( \mathcal{L}^n_c(\mathcal{L}^n_c(A, X), \mathcal{L}^{n-1}_c(A, X))/\mathcal{T} \).

**Proof.** It is similar to the proof of Proposition 2.1.2. \( \square \)

For the following three types of splitting we will give the properties of the sets of splitting maps right after their geometric characterisation without any further comments. In all three cases the proofs are similar to that of Proposition 2.1.2.

We can see from Proposition 2.1.3 that the splitting of the second kind of the \( n \)th bounded cohomology group of \( A \), with coefficients in \( X \), gives us information about both \( \mathcal{L}^{n-1}_c(A, X) \) and \( \mathcal{L}^n_c(A, X) \). So if two consecutive cohomology groups \( \mathcal{H}^n_c(A, X) \) and \( \mathcal{H}^{n+1}_c(A, X) \) split (II) we have information about \( \mathcal{L}^n_c(A, X) \) coming from both those splittings. In the following proposition we show how this information can be combined.

**Proposition 2.1.5.** Let \( A \) be a Banach algebra and \( X \) be a Banach \( A \)-bimodule. If both the \( n \)th and the \((n+1)\)th bounded cohomology groups of \( A \), with coefficients in \( X \), split (II) and \( s_n \) and \( s_{n+1} \) are splitting maps of the second kind, then the map

\[ \partial^{n-1}s_n + s_{n+1}\partial^n : \mathcal{L}^n_c(A, X) \to \mathcal{L}^n_c(A, X) \]

is a bounded projection, with

\[ \text{Im}(\partial^{n-1}s_n + s_{n+1}\partial^n) = \mathcal{L}^n_c(A, X) \subseteq (\mathcal{Z}^n_c(A, X) \ominus \mathcal{B}^n_c(A, X)). \]
Proof. As we showed in the proof of Proposition 2.1.3, $\partial^{n-1}s_n$ is a bounded projection that maps $\mathcal{L}^n_c(A, X)$ onto $\mathcal{B}^n_c(A, X)$ and $s_{n+1}\partial^n$ is a bounded projection mapping $\mathcal{L}^n_c(A, X)$ onto $\mathcal{L}^n_c(A, X) \oplus Z^n_c(A, X)$. Now the result follows immediately, since $\mathcal{B}^n_c(A, X) \subseteq Z^n_c(A, X)$.

It is obvious from the definitions of splits (I) and (II) that the second type of splitting is stronger than the first one. The following proposition gives the exact relation between them.

**Proposition 2.1.6.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) The $n$-th bounded cohomology group of $A$, with coefficients in $X$, splits (I) and $\mathcal{B}^n_c(A, X)$ is complemented in $\mathcal{L}^n_c(A, X)$.

(ii) The $n$-th bounded cohomology group of $A$, with coefficients in $X$, splits (II).

**Proof.** It follows directly from the geometric characterisations of the first and the second type of splitting (Propositions 2.1.1 and 2.1.3).

**Remark 2.1.2.** We could have defined an intermediate type of splitting between the first and the second one, by demanding that the splitting maps are defined on $Z^n_c(A, X)$. Obviously the splitting of that type is equivalent to the complementation of $Z^{n-1}_c(A, X)$ in $\mathcal{L}^{n-1}_c(A, X)$ and of $\mathcal{B}^n_c(A, X)$ in $Z^n_c(A, X)$. Moreover it is stronger than the first one and weaker than the second. As it will not appear in any of the cases we will discuss in Chapter 5, we decided not to deal with that type of splitting.

We continue with the definition of the third type of splitting.

**Definition 2.1.3.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. We say that the $n$th bounded cohomology group of $A$, with coefficients in $X$, splits (III) if there exists a bounded linear map

$$s_n : Z^n_c(A, X) \to \mathcal{L}^{n-1}_c(A, X)$$

with

$$\partial^{n-1}s_n = id_{Z^n_c(A, X)}$$

The map $s_n$ is called a splitting map of the third kind.

In the following proposition we give a geometric characterisation of split (III). Whereas the two previous types of splitting do not imply the vanishing of the cohomology group, the third one does.
Proposition 2.1.7. Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) The $n$th bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (III).
(ii) $Z^n(A,X)$ is complemented in $L^{n-1}_c(A,X)$ and $\mathcal{H}^n_c(A,X) = \{0\}$.

Proof. Suppose that $\mathcal{H}^n_c(A,X)$ splits (III). Then there exists a bounded linear map

$$s_n : Z^n_c(A,X) \rightarrow L^{n-1}_c(A,X)$$

with $\partial^{n-1}s_n = id_{Z^n_c(A,X)}$. Obviously that implies the vanishing of $\mathcal{H}^n_c(A,X)$. Moreover, as in Proposition 2.1.1, we can prove that $id_{Z^n_c(A,X)} - s_n\partial^{n-1}$ is a bounded projection, with $\text{Im}(id_{Z^n_c(A,X)} - s_n\partial^{n-1}) = Z^{n-1}_c(A,X)$. Therefore $Z^{n-1}_c(A,X)$ is complemented in $L^{n-1}_c(A,X)$.

Conversely suppose that $\mathcal{H}^n_c(A,X) = \{0\}$ and $Z^{n-1}_c(A,X)$ is complemented in $L^{n-1}_c(A,X)$. Since $Z^n_c(A,X)$ is complemented in $L^{n-1}_c(A,X)$ and $\mathcal{B}_c^n(A,X)(= Z^n_c(A,X)$, since $\mathcal{H}^n_c(A,X) = \{0\}$) is closed, there exists, as in the proof of Proposition 2.1.1, a bounded linear map

$$\pi : Z^n_c(A,X) \rightarrow L^{n-1}_c(A,X)$$

with $\partial^{n-1}\pi(\psi) = \psi$, for all $\psi \in Z^n_c(A,X)$. Hence $\pi$ is a splitting map of the third kind. \qed

Although the condition "$Z^{n-1}_c(A,X)$ is complemented in $L^{n-1}_c(A,X)$" in (ii) of the previous proposition appears to be quite weak it is not. In Section 5.3 we will show that there are many cases where $\mathcal{H}^n_c(A,X)$ does not split (III) although it vanishes.

Proposition 2.1.8. Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. If the $n$th bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (III), then the following hold:

(i) The set of splitting maps of the third kind is a convex closed subset of $L^1_c(Z^n_c(A,X), L^{n-1}_c(A,X))$.
(ii) If $s_n$ is a splitting map of the third kind and

$$\mathcal{T} = \{ \Phi \in L^1_c(Z^n_c(A,X), L^{n-1}_c(A,X)) \mid \Phi(Z^n_c(A,X)) \subseteq Z^{n-1}_c(A,X) \}$$

then the set of splitting maps of the third kind is the equivalence class of $s_n$ in $L^1_c(Z^n_c(A,X), L^{n-1}_c(A,X))/\mathcal{T}$.

The example of the disc algebra and the module $\mathcal{C}$ that we discussed after Proposition 2.1.3 gives an example of a cohomology group which splits (II), and
thus (I), but does not vanish, which implies that it does not split (III). On the other hand it is easy to see that the third type of splitting implies the first one. We have not been able to find an example of a bounded Hochschild cohomology group that splits (III), but not (II) (to get such an example we need to find a Banach algebra $A$ and a Banach $A$-bimodule $X$ such that $Z_c^{n-1}(A, X)$ is complemented in $L_c^{n-1}(A, X)$, $H_c^n(A, X) = \{0\}$ and $Z_c^n(A, X)$ is not complemented in $L_c^n(A, X)$). We will discuss the relation between the first three types of splitting, together with their relationship to the fourth one, after we have defined the fourth type of splitting and discussed its properties.

**Definition 2.1.4.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. We say that the $n$th bounded cohomology group of $A$, with coefficients in $X$, splits (IV) if there exists a bounded linear map

$$s_n : L_c^n(A, X) \to L_c^{n-1}(A, X)$$

with

$$\partial^{n-1}s_n(\phi) = \phi$$

for all $\phi \in Z_c^n(A, X)$. We call $s_n$ a splitting map of the fourth kind.

In the following proposition we give a geometric characterisation of split (IV).

**Proposition 2.1.9.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) The $n$th bounded cohomology group of $A$, with coefficients in $X$, splits (IV).

(ii) $Z_c^{n-1}(A, X)$ is complemented in $L_c^{n-1}(A, X)$, $Z_c^n(A, X)$ is complemented in $L_c^n(A, X)$ and $H_c^n(A, X) = \{0\}$.

**Proof.** (i) $\Rightarrow$ (ii) Let

$$s_n : L_c^n(A, X) \to L_c^{n-1}(A, X)$$

be a splitting map of the fourth kind. Obviously $H_c^n(A, X) = \{0\}$. Moreover, as in Proposition 2.1.1, we can prove that $id_{L_c^{n-1}(A, X)} - s_n\partial^{n-1}$ is a bounded projection with image $Z_c^{n-1}(A, X)$. On the other hand if $\phi \in L_c^n(A, X)$, then

$$(\partial^{n-1}s_n)^2(\phi) = \partial^{n-1}s_n(\partial^{n-1}(s_n(\phi))) = \partial^{n-1}(s_n(\phi))$$

since $B_c^n(A, X) \subseteq Z_c^n(A, X)$, and so $\partial^{n-1}s_n$ is a bounded projection. It is easy to see that $\text{Im}(\partial^{n-1}s_n) = Z_c^n(A, X)$ and therefore $Z_c^n(A, X)$ is complemented in $L_c^n(A, X)$. 

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(ii) ⇒ (i) As in Proposition 2.1.7, we can prove that there exists a map
\[ \pi : \mathcal{Z}_c^n(A, X) \to \mathcal{L}^{n-1}_c(A, X) \]
with \( \partial^{n-1}\pi = \text{id}_{\mathcal{Z}_c^n(A, X)} \). Since \( \mathcal{Z}_c^n(A, X) \) is complemented in \( \mathcal{L}_c^n(A, X) \), there exists a bounded projection
\[ \rho : \mathcal{L}_c^n(A, X) \to \mathcal{Z}_c^n(A, X) \]
If \( \phi \in \mathcal{Z}_c^{n-1}(A, X) \), then \( \partial^{n-1}\pi(\rho(\phi)) = \partial^{n-1}\pi(\phi) = \phi \) and thus \( s_n = \pi \rho \) is a splitting map of the fourth kind.

An immediate consequence of the previous proposition is that if \( \mathcal{Z}_c^n(A, X) = \{0\} \), then the nth bounded cohomology group of \( A \), with coefficients in \( X \), splits (IV).

**Proposition 2.1.10.** Let \( A \) be a Banach algebra and \( X \) be a Banach \( A \)-bimodule. If the nth bounded cohomology group of \( A \), with coefficients in \( X \), splits (IV), then the following hold:

(i) The set of splitting maps of the fourth kind is a convex closed subset of \( \mathcal{L}_c^1(\mathcal{L}_c^n(A, X), \mathcal{L}_c^{n-1}(A, X)) \).

(ii) If \( s_n \) is a splitting map of the fourth kind and
\[ T = \{ \Phi \in \mathcal{L}_c^1(\mathcal{L}_c^n(A, X), \mathcal{L}_c^{n-1}(A, X)) \mid \Phi(\mathcal{Z}_c^n(A, X)) \subseteq \mathcal{Z}_c^{n-1}(A, X) \} \]
then the set of splitting maps of the fourth kind is the equivalence class of \( s_n \) in \( \mathcal{L}_c^1(\mathcal{L}_c^n(A, X), \mathcal{L}_c^{n-1}(A, X))/T \).

Using again the example of the disc algebra and its module \( C \) we can see that the first and the second type of splitting do not imply the fourth one. On the other hand, as we show in the following proposition, the fourth type of splitting implies the other three.

**Proposition 2.1.11.** Let \( A \) be a Banach algebra and \( X \) be a Banach \( A \)-bimodule. Then, for all \( n \in \mathbb{N} \), the following are equivalent:

(i) The nth bounded cohomology group of \( A \), with coefficients in \( X \), splits (I), \( \mathcal{B}_c^n(A, X) \) is complemented in \( \mathcal{L}_c^n(A, X) \) and \( \mathcal{H}_c^n(A, X) = \{0\} \).

(ii) The nth bounded cohomology group of \( A \), with coefficients in \( X \), splits (II) and \( \mathcal{H}_c^n(A, X) = \{0\} \).

(iii) The nth bounded cohomology group of \( A \), with coefficients in \( X \), splits (III) and \( \mathcal{Z}_c^n(A, X) \) is complemented in \( \mathcal{L}_c^n(A, X) \).

(iv) The nth bounded cohomology group of \( A \), with coefficients in \( X \), splits both (II) and (III).

(v) The nth bounded cohomology group of \( A \), with coefficients in \( X \), splits (IV).
Proof. (i) $\Leftrightarrow$ (ii) It follows immediately from Proposition 2.1.6.

(ii) $\Rightarrow$ (iii) Since $\mathcal{H}_c^n(A, X)$ splits (II), $\mathcal{L}_c^{n-1}(A, X)$ is complemented in $\mathcal{L}_c^{n-1}(A, X)$ and $\mathcal{B}_c^n(A, X)$ is complemented in $\mathcal{L}_c^{n}(A, X)$, by Proposition 2.1.3. The first one together with the vanishing of $\mathcal{H}_c^n(A, X)$ implies that $\mathcal{H}_c^n(A, X)$ splits (III), by Proposition 2.1.7.

(iii) $\Rightarrow$ (iv) Since the $n$th bounded cohomology group of $A$, with coefficients in $X$, splits (III), $\mathcal{L}_c^{n-1}(A, X)$ is complemented in $\mathcal{L}_c^{n-1}(A, X)$ and $\mathcal{H}_c^n(A, X) = \{0\}$, i.e. $\mathcal{Z}_c^n(A, X) = \mathcal{B}_c^n(A, X)$, by Proposition 2.1.7. Thus, by Proposition 2.1.3, $\mathcal{H}_c^n(A, X)$ splits (II).

(iv) $\Rightarrow$ (ii) Since $\mathcal{H}_c^n(A, X)$ splits (III), $\mathcal{H}_c^n(A, X) = \{0\}$, by Proposition 2.1.7.

(iii) $\Leftrightarrow$ (v) It follows from Propositions 2.1.7 and 2.1.9.

It is easy to see that the equivalence between (i) and (iii) holds if we omit "$\mathcal{B}_c^n(A, X)$ is complemented in $\mathcal{L}_c^n(A, X)$" in (i) and "$\mathcal{Z}_c^n(A, X)$ is complemented in $\mathcal{L}_c^n(A, X)$" in (iii). That gives us the relation between the first and the third type of splitting.

Although (iii) and (v) of the previous proposition show that the third type of splitting is weaker than the fourth one, we have not been able to find an example of a bounded Hochschild cohomology group that splits (III), but does not split (IV). A bounded Hochschild cohomology group that splits (III), but does not split (II), will provide us with such an example.

In a manner similar to Proposition 2.1.5, we can see that if both $\mathcal{H}_c^{n-1}(A, X)$ and $\mathcal{H}_c^n(A, X)$ split (IV), then $\partial^{n-1} s_n$ and $s_{n+1} \partial^n$ are bounded projections that map $\mathcal{L}_c^n(A, X)$ onto $\mathcal{L}_c^n(A, X)$ and $\mathcal{L}_c^n(A, X) \ominus \mathcal{Z}_c^n(A, X)$ respectively and so $\partial^{n-1} s_n + s_{n+1} \partial^n = id_{\mathcal{L}_c^n(A, X)}$. That leads us to the definition of the fifth type of splitting.

Definition 2.1.5. Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. We say that the $n$th bounded cohomology group of $A$, with coefficients in $X$, splits (V), if there exist bounded linear maps

$$s_n : \mathcal{L}_c^n(A, X) \rightarrow \mathcal{L}_c^{n-1}(A, X)$$

and

$$s_{n+1} : \mathcal{L}_c^{n+1}(A, X) \rightarrow \mathcal{L}_c^n(A, X)$$

with

$$\partial^{n-1} s_n + s_{n+1} \partial^n = id_{\mathcal{L}_c^n(A, X)}$$

We call the pair $(s_n, s_{n+1})$ a pair of splitting maps of the fifth kind.
In the following proposition we characterise the fifth type of splitting geometrically.

**Proposition 2.1.12.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) The $n$th bounded cohomology group of $A$, with coefficients in $X$, splits (V).

(ii) $\mathcal{Z}^{-1}_c(A, X)$ is complemented in $\mathcal{L}^{-1}_c(A, X)$, $\mathcal{Z}^n_c(A, X)$ is complemented in $\mathcal{L}^n_c(A, X)$, $\mathcal{B}^{n+1}_c(A, X)$ is complemented in $\mathcal{L}^{n+1}_c(A, X)$ and $\mathcal{H}^n_c(A, X) = \{0\}$.

**Proof.** Let $(s_n, s_{n+1})$ be a pair of splitting maps of the fifth kind. Then the coboundary condition $\partial^n \partial^{n-1} = 0$ implies that both $s_n$ and $s_{n+1}$ are splitting maps of the second kind. Thus, by Proposition 2.1.3, $\mathcal{Z}^{-1}_c(A, X)$, $\mathcal{Z}^n_c(A, X)$ and $\mathcal{B}^{n+1}_c(A, X)$ are complemented in $\mathcal{L}^{-1}_c(A, X)$, $\mathcal{L}^n_c(A, X)$ and $\mathcal{L}^{n+1}_c(A, X)$ respectively. Moreover if $\phi \in \mathcal{Z}^n_c(A, X)$, then

$$\phi = \partial^n s_n(\phi) + s_{n+1} \partial^n(\phi) = \partial^n s_n(\phi)$$

and therefore $\mathcal{H}^n_c(A, X) = \{0\}$.

Conversely, as in Propositions 2.1.3 and 2.1.5, we can prove that since $\mathcal{Z}^{-1}_c(A, X)$ is complemented in $\mathcal{L}^{-1}_c(A, X)$, $\mathcal{B}^n_c(A, X)$ (since $\mathcal{H}^n_c(A, X) = \{0\}$ and so $\mathcal{Z}^n_c(A, X) = \mathcal{B}^n_c(A, X)$) and $\mathcal{Z}^n_c(A, X)$ are complemented in $\mathcal{L}^n_c(A, X)$ and $\mathcal{B}^{n+1}_c(A, X)$ is complemented in $\mathcal{L}^{n+1}_c(A, X)$, there exist bounded linear maps

$$s_n : \mathcal{L}^n_c(A, X) \rightarrow \mathcal{L}^{-1}_c(A, X)$$

and

$$s_{n+1} : \mathcal{L}^{n+1}_c(A, X) \rightarrow \mathcal{L}^n_c(A, X)$$

such that $\partial^n s_n + s_{n+1} \partial^n$ is a projection with

$$\text{Im}(\partial^n s_n + s_{n+1} \partial^n) = \mathcal{L}^n_c(A, X) \ominus (\mathcal{Z}^n_c(A, X) \ominus \mathcal{B}^n_c(A, X))$$

But $\mathcal{H}^n_c(A, X) = \{0\}$ and so $\mathcal{Z}^n_c(A, X) = \mathcal{B}^n_c(A, X)$, i.e. $\partial^n s_n + s_{n+1} \partial^n = \text{id}_{\mathcal{L}_c(A, X)}$ and thus $\mathcal{Z}^n_c(A, X)$ splits (V).

**Proposition 2.1.13.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. If the $n$th bounded cohomology group of $A$, with coefficients in $X$, splits (V), then the following hold:

(i) The set of pairs of splitting maps of the fifth kind is a closed convex subset of $\mathcal{L}^1_c(\mathcal{L}^n_c(A, X), \mathcal{L}^{n-1}_c(A, X)) \times \mathcal{L}^1_c(\mathcal{L}^{n+1}_c(A, X), \mathcal{L}^n_c(A, X))$.

(ii) If $(s_n, s_{n+1})$ is a pair of splitting maps of the fifth kind and $T$ is the set of pairs $(\Phi, \Psi)$ in $\mathcal{L}^1_c(\mathcal{L}^n_c(A, X), \mathcal{L}^{n-1}_c(A, X)) \times \mathcal{L}^1_c(\mathcal{L}^{n+1}_c(A, X), \mathcal{L}^n_c(A, X))$ which
satisfy the condition $\partial^{n-1}\Phi = -\Psi \partial^n$, then the set of pairs of splitting maps of the fifth kind is the equivalence class of $(s_n, s_{n+1})$ in $\mathcal{L}^1_{\mathcal{C}}(\mathcal{L}^{n}_c(A, X), \mathcal{L}^{n-1}_c(A, X)) \times \mathcal{L}^1_{\mathcal{C}}(\mathcal{L}^{n+1}_c(A, X), \mathcal{L}^{n}_c(A, X))/T$.

As we said after Proposition 2.1.3, if $A = A(D)$ and $X = \mathbb{C}$, with the module actions of $A$ on $X$ defined by $\lambda f = \lambda f(z_0) = f\lambda$, for all $f \in A$ and all $\lambda \in X$, where $z_0 \in int(D)$, then the first bounded cohomology group of $A$, with coefficients in $X$, splits (II). Moreover if we define

$$\pi : \mathcal{L}^2_c(A, X) \to \mathcal{L}^1_c(A, X)$$

by $\pi(\phi)(f) = \phi(a, f)$, for all $\phi \in \mathcal{L}^2_c(A, X)$ and all $f \in A$, where $a(z) = z - z_0$, for all $z \in D$ and $\tilde{f}(z) = (z - z_0)^{-1}(f(z) - f(z_0))$, if $z \neq z_0$, and $\tilde{f}(z_0) = f'(z_0)$,

$$\sigma : \mathcal{L}^2_c(A, X) \to \mathcal{L}^2_c(A, X)$$

by $\sigma = id_{\mathcal{L}^2_c(A, X)} + \partial^1 \pi$, and

$$\tau : \mathcal{L}^2_c(A, X) \to \mathcal{L}^1_c(A, X)$$

by $\tau(\phi)(f) = f(z_0)\sigma(\phi)(1, 1)$, for all $\phi \in \mathcal{L}^2_c(A, X)$ and all $f \in A$, where 1 is the identity $1(z) = 1$ of $A$ then $s_2 = \tau - \pi$ is a splitting map of the fourth kind.

Before we show that let us make two remarks that we will need in the proof. If $I_0 = \{f \in A \mid f(z_0) = 0\}$, then it is easy to see that

$$a\tilde{f} = f \quad (2.1)$$

for all $f \in I_0$ and that

$$(f_1f_2)^* = \tilde{f}_1f_2 \quad (2.2)$$

for all $f_1 \in I_0$ and all $f_2 \in A$. Consider $\phi \in \mathcal{Z}^2_c(A, X)$. Then (2.1), (2.2) and $\partial^2(\phi) = 0$ imply that

$$-\phi(f_1, f_2) = \partial^1(\pi(\phi))(f_1, f_2) \quad (2.3)$$

for all $f_1 \in I_0$ and all $f_2 \in A$. On the other hand since $\phi \in \mathcal{Z}^2_c(A, X)$, $\sigma(\phi) \in \mathcal{Z}^2_c(A, X)$ and thus $\sigma(\phi)(1, f) = f(z_0)\sigma(\phi)(1, 1)$, for all $f \in A$. From this it is easy to see that

$$f_1(z_0)\sigma(\phi)(1, f_2) = \partial^1(\tau(\phi))(f_1, f_2) \quad (2.4)$$

for all $f_1, f_2 \in A$. Every $f \in A$ can be written in the form $(f - f(z_0)1) + f(z_0)1$, where $f - f(z_0)1 \in I_0$. Hence, by (2.3), for all $f_1, f_2 \in A$,

$$\partial^1(\pi(\phi))(f_1, f_2) = -\phi(f_1, f_2) + f_1(z_0)\phi(1, f_2)$$

$$+ f_1(z_0)\partial^1(\pi(\phi))(1, f_2)$$

$$= -\phi(f_1, f_2) + f_1(z_0)\sigma(\phi)(1, f_2)$$

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Using (2.4) we get that $\partial^1((\tau - \pi)(\phi)) = \phi$. Therefore $\mathcal{H}_c^n(A, X)$ splits (IV). (The proof that $s_2$ is a splitting map of the fourth kind is a rephrasing of the proof of [J2], Proposition 9.1.(i)). But as we mentioned before $\mathcal{H}_c^1(A, X)$ does not vanish and so by Proposition 2.1.12, it does not split (V). So split (II) of $\mathcal{H}_c^n(A, X)$ and split (IV) of $\mathcal{H}_c^{n+1}(A, X)$ do not imply split (V) for $\mathcal{H}_c^n(A, X)$. As we will prove in the following proposition, for $\mathcal{H}_c^n(A, X)$ to split (V) we need $\mathcal{H}_c^n(A, X)$ to split (III) or (IV) and $\mathcal{H}_c^{n+1}(A, X)$ to split (II).

**Proposition 2.1.14.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) The $n$th and the $(n+1)$th bounded cohomology groups of $A$, with coefficients in $X$, split (II) and $\mathcal{H}_c^n(A, X) = \{0\}$.

(ii) The $n$th bounded cohomology group of $A$, with coefficients in $X$, splits (III) and the $(n+1)$th splits (II).

(iii) The $n$th bounded cohomology group of $A$, with coefficients in $X$, splits (IV) and the $(n+1)$th splits (II).

(iv) The $n$th bounded cohomology group of $A$, with coefficients in $X$, splits (V).

**Proof.** (i) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (iii) follow from the geometric properties of the second, the fourth and the fifth type of splitting (Propositions 2.1.3, 2.1.5, 2.1.9 and 2.1.12). On the other hand (iii) $\Rightarrow$ (ii) follows from Proposition 2.1.11 and (ii) $\Rightarrow$ (i) follows from Propositions 2.1.7 and 2.1.11. 

The previous proposition shows that if $\mathcal{H}_c^n(A, X)$ splits (V), then $\mathcal{H}_c^n(A, X)$ splits (IV) (and hence (III)) and $\mathcal{H}_c^{n+1}(A, X)$ splits (II). Does the fifth type of splitting of $\mathcal{H}_c^n(A, X)$ imply that $\mathcal{H}_c^{n+1}(A, X)$ splits (III) or (IV)? The next proposition shows that for that to happen we also need $\mathcal{H}_c^n(A, X) = \{0\}$. We have not been able to find an example of a bounded Hochschild cohomology group $\mathcal{H}_c^n(A, X)$ which splits (V), but $\mathcal{H}_c^{n+1}(A, X)$ does not split (III) or (IV).

**Proposition 2.1.15.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) The $n$th and the $(n+1)$th bounded cohomology groups of $A$, with coefficients in $X$, split (II), $\mathcal{H}_c^n(A, X) = \{0\}$ and $\mathcal{H}_c^{n+1}(A, X) = \{0\}$.

(ii) The $n$th and the $(n+1)$th bounded cohomology groups of $A$, with coefficients in $X$, split (III) and $\mathcal{Z}_c^{n+1}(A, X)$ is complemented in $\mathcal{L}_c^{n+1}(A, X)$.

(iii) The $n$th bounded cohomology group of $A$, with coefficients in $X$, splits (III) and the $(n+1)$th splits (IV).

(iv) The $n$th and the $(n+1)$th bounded cohomology groups of $A$, with coefficients in $X$, split (IV).
(v) The $n$th bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (V) and $\mathcal{H}^{n+1}_c(\mathcal{A}, X) = \{0\}$.

**Proof.** It follows directly from Propositions 2.1.11 and 2.1.14. \qed

In the following seven propositions we give some algebraic properties of splitting. Since all of them hold for all five types of splitting and the proofs are similar we give the proof only for split (IV).

As we mentioned in Section 1.2.2, $\mathcal{H}^n_c(\mathcal{A}, X)$ is isomorphic to $\mathcal{H}^1_c(\mathcal{A}, \mathcal{L}^{n-1}_c(\mathcal{A}, X))$. Using this isomorphism we will show that the splitting of $\mathcal{H}^n_c(\mathcal{A}, X)$ is equivalent to the splitting of $\mathcal{H}^1_c(\mathcal{A}, \mathcal{L}^{n-1}_c(\mathcal{A}, X))$. That allows us to consider only the case of the first bounded cohomology group when we study general properties of splitting.

**Proposition 2.1.16.** Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. Then, for all $n > 1$, $\mathcal{H}^n_c(\mathcal{A}, X)$ splits (I), (II), (III), (IV) or (V) respectively if and only if $\mathcal{H}^1_c(\mathcal{A}, \mathcal{L}^{n-1}_c(\mathcal{A}, X))$ splits (I), (II), (III), (IV) or (V) respectively.

**Proof.** If

$$J : \mathcal{L}^n_c(\mathcal{A}, X) \to \mathcal{L}^1_c(\mathcal{A}, \mathcal{L}^{n-1}_c(\mathcal{A}, X))$$

is the canonical isomorphism defined by $J(\Phi)(a)(a_1, ..., a_{n-1}) = \Phi(a, a_1, ..., a_{n-1})$, for all $\Phi \in \mathcal{L}^n_c(\mathcal{A}, X)$ and all $a, a_1, ..., a_{n-1} \in \mathcal{A}$, then

$$J \partial^{n-1} = \Delta^0 \quad (2.5)$$

where

$$\partial^{n-1} : \mathcal{L}^{n-1}_c(\mathcal{A}, X) \to \mathcal{L}^n_c(\mathcal{A}, X)$$

and

$$\Delta^0 : \mathcal{L}^{n-1}_c(\mathcal{A}, X) \to \mathcal{L}^1_c(\mathcal{A}, \mathcal{L}^{n-1}_c(\mathcal{A}, X))$$

are the coboundary maps. Using (2.5) we can prove that if

$$s_n : \mathcal{L}^n_c(\mathcal{A}, X) \to \mathcal{L}^{n-1}_c(\mathcal{A}, X)$$

is a splitting map of the fourth kind then so is $S_1 = s_n J^{-1}$ and if

$$S_1 : \mathcal{L}^1_c(\mathcal{A}, \mathcal{L}^{n-1}_c(\mathcal{A}, X)) \to \mathcal{L}^{n-1}_c(\mathcal{A}, X)$$

is a splitting map of the fourth kind, then $s_n = S_1 J$ is also a splitting map of the fourth kind. \qed
It is obvious that if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic Banach algebras and $X$ is a Banach $\mathcal{A}$- (and thus $\mathcal{B}$-) bimodule, then, for all $n \in \mathbb{N}$, $\mathcal{H}_c^n(\mathcal{A}, X)$ splits (I), (II), (III), (IV) or (V) respectively if and only $\mathcal{H}_c^n(\mathcal{B}, X)$ splits (I), (II), (III), (IV) or (V) respectively.

The following two propositions show that splitting is a property that respects module isomorphisms and complemented submodules.

**Proposition 2.1.17.** Let $\mathcal{A}$ be a Banach algebra and $X$ and $Y$ be two isomorphic Banach $\mathcal{A}$-bimodules. If the $n$th bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V) respectively, then the $n$th bounded cohomology group of $\mathcal{A}$, with coefficients in $Y$, splits (I), (II), (III), (IV) or (V) respectively.

**Proof.** If $J : X \to Y$ is an $\mathcal{A}$-module isomorphism, then it is easy to see that, for all $n \in \mathbb{N}$, the map

$$J_n : \mathcal{L}_c^n(\mathcal{A}, X) \to \mathcal{L}_c^n(\mathcal{A}, Y)$$

defined by $J_n(\Phi)(a_1, \ldots, a_n) = J(\Phi(a_1, \ldots, a_n))$, for all $\Phi \in \mathcal{L}_c^n(\mathcal{A}, X)$ and all $a_1, \ldots, a_n \in \mathcal{A}$ is an isomorphism, with $J_{n+1}\partial^n = \Delta^n J_n$, where

$$\partial^n : \mathcal{L}_c^n(\mathcal{A}, X) \to \mathcal{L}_c^{n+1}(\mathcal{A}, X)$$

and

$$\Delta^n : \mathcal{L}_c^n(\mathcal{A}, Y) \to \mathcal{L}_c^{n+1}(\mathcal{A}, Y)$$

are the coboundary maps. If now $s_n : \mathcal{L}_c^n(\mathcal{A}, X) \to \mathcal{L}_c^{n-1}(\mathcal{A}, X)$ is a splitting map of the fourth kind, then $S_n : \mathcal{L}_c^n(\mathcal{A}, Y) \to \mathcal{L}_c^{n-1}(\mathcal{A}, Y)$ defined by $S_n = J_{n-1}s_nJ_n^{-1}$ is also a splitting map of the fourth kind. \qed

**Proposition 2.1.18.** Let $\mathcal{A}$ be a Banach algebra, $X$ be a Banach $\mathcal{A}$-bimodule and $Z$ be a complemented $\mathcal{A}$-submodule of $X$. If the $n$th bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V) respectively then:

(i) The $n$th bounded cohomology group of $\mathcal{A}$, with coefficients in $Z$, splits (I), (II), (III), (IV) or (V) respectively.

(ii) The $n$th bounded cohomology group of $\mathcal{A}$, with coefficients in $X/Z$, splits (I), (II), (III), (IV) or (V) respectively.

**Proof.** (i) Since $Z$ is a complemented $\mathcal{A}$-submodule of $X$, there exists a bounded projection $\rho : X \to Z$, which is an $\mathcal{A}$-module homomorphism. Now, for each $n \in \mathbb{N}$, define

$$\rho_n : \mathcal{L}_c^n(\mathcal{A}, X) \to \mathcal{L}_c^n(\mathcal{A}, Z)$$

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with \( \rho_n(\Phi)(a_1, \ldots, a_n) = \rho(\Phi(a_1, \ldots, a_n)) \), for all \( \Phi \in \mathcal{L}_c^n(A, X) \) and all \( a_1, \ldots, a_n \in A \). Then, since \( \rho \) is an \( A \)-module homomorphism, \( \partial^n \rho_n = \rho_{n+1} \partial^n \) and hence if \( s_n : \mathcal{L}_c^n(A, X) \to \mathcal{L}_c^{n-1}(A, X) \) is a splitting map of the fourth kind, then so is \( S_n = \rho_{n-1} s_n \mid_{\mathcal{L}_c^2(A, X)} \).

(ii) Since \( Z \) is complemented in \( X \), there exists a closed submodule \( Z' \) of \( X \) which is isomorphic to \( X/Z \). Obviously \( Z' \) is also complemented in \( X \) and thus the result follows from (i) and Proposition 2.1.17.

**Remark 2.1.3.** The converse of (i) of the previous proposition does not hold. In Section 5.3 we will give an example of a Banach algebra \( A \), a Banach \( A \)-bimodule \( X \) and a complemented \( A \)-submodule \( Z \) of \( X \) where \( \mathcal{H}_c^1(A, Z) \) splits (III), but \( \mathcal{H}_c^n(A, X) \) does not split (III).

Part (i) of the previous proposition shows in particular that if the cohomology group \( \mathcal{H}_c^n(A, X_1 \oplus X_2) \) splits, then so do \( \mathcal{H}_c^n(A, X_1) \) and \( \mathcal{H}_c^n(A, X_2) \) and, more generally, that if for a family \( \{ X_\lambda \mid \lambda \in \Lambda \} \) of uniformly bounded Banach \( A \)-bimodules and for some \( 1 \leq p \leq \infty \), \( \mathcal{H}_c^p(A, p(X_\lambda \mid \lambda \in \Lambda)) \) splits, then \( \mathcal{H}_c^n(A, X_\lambda) \) splits, for all \( \lambda \in \Lambda \). In the following proposition we show that the converse holds in the finite case and for \( p = \infty \) in the infinite case.

**Proposition 2.1.19.** Let \( A \) be a Banach algebra. Then the following hold:

(i) If \( X_1 \) and \( X_2 \) are Banach \( A \)-bimodules and \( \mathcal{H}_c^n(A, X_1) \) and \( \mathcal{H}_c^n(A, X_2) \) split (I), (II), (III), (IV) or (V) respectively, then \( \mathcal{H}_c^n(A, X_1 \oplus_p X_2) \) splits (I), (II), (III), (IV) or (V) respectively, for all \( 1 \leq p \leq \infty \).

(ii) If \( \{ X_\lambda \mid \lambda \in \Lambda \} \) is a family of uniformly bounded Banach \( A \)-bimodules and \( \mathcal{H}_c^n(A, X_\lambda) \) splits (I), (II), (III), (IV) or (V) respectively, for all \( \lambda \in \Lambda \), and there exists a family \( \{ s^n_\lambda \mid \lambda \in \Lambda \} \) of splitting maps of the first, second, third or fourth kind, with \( \sup_{\lambda \in \Lambda} ||s^n_\lambda|| < \infty \), or a family \( \{ (s^n_\lambda, s^{n+1}_\lambda) \mid \lambda \in \Lambda \} \) of pairs of splitting maps of the fifth kind, with \( \sup_{\lambda \in \Lambda} ||s^n_\lambda|| < \infty \) and \( \sup_{\lambda \in \Lambda} ||s^{n+1}_\lambda|| < \infty \), then \( \mathcal{H}_c^n(A, l^\infty(X_\lambda \mid \lambda \in \Lambda)) \) splits (I), (II), (III), (IV) or (V) respectively.

**Proof.** (i) If, for each \( n \in \mathbb{N} \), the map

\[
J_n : \mathcal{L}_c^n(A, X_1) \oplus_p \mathcal{L}_c^n(A, X_2) \to \mathcal{L}_c^n(A, X_1 \oplus_p X_2)
\]

is the canonical isomorphism defined by

\[
J_n(\phi \oplus \psi)(a_1, \ldots, a_n) = \phi(a_1, \ldots, a_n) \oplus \psi(a_1, \ldots, a_n)
\]

for all \( \phi \oplus \psi \in \mathcal{L}_c^n(A, X_1) \oplus \mathcal{L}_c^n(A, X_2) \) and all \( a_1, \ldots, a_n \in A \), then it is easy to see that

\[
\Delta^n J_n(\phi \oplus \psi) = J_{n+1}(\partial^n_1(\phi) \oplus \partial^n_2(\psi))
\]
for all \( \phi \oplus \psi \in \mathcal{L}^n_c(A, X_1) \oplus \mathcal{L}^n_c(A, X_2) \), where

\[
\Delta^n : \mathcal{L}^n_c(A, X_1 \oplus_p X_2) \to \mathcal{L}^{n+1}_c(A, X_1 \oplus_p X_2)
\]

\[
\partial_1^n : \mathcal{L}^n_c(A, X_1) \to \mathcal{L}^{n+1}_c(A, X_1)
\]

\[
\partial_2^n : \mathcal{L}^n_c(A, X_2) \to \mathcal{L}^{n+1}_c(A, X_2)
\]

are the coboundary maps. If now

\[
s_1^n : \mathcal{L}^n_c(A, X_1) \to \mathcal{L}^{n-1}_c(A, X_1)
\]

and

\[
s_2^n : \mathcal{L}^n_c(A, X_2) \to \mathcal{L}^{n-1}_c(A, X_2)
\]

are splitting maps of the fourth kind, then (2.6) implies that the map

\[
S_n : \mathcal{L}^n_c(A, X_1 \oplus X_2) \to \mathcal{L}^{n-1}_c(A, X_1 \oplus X_2)
\]

defined by

\[
S_n(\Phi) = J_{n-1}(s_1^n(J_{n-1}^{-1}(\Phi)_1) \oplus s_2^n(J_{n-1}^{-1}(\Phi)_2))
\]

for all \( \Phi \in \mathcal{L}^n_c(A, X_1 \oplus X_2) \), where \( J_{n-1}^{-1}(\Phi)_1 \) and \( J_{n-1}^{-1}(\Phi)_2 \) are the \( \mathcal{L}^n_c(A, X_1) \) and \( \mathcal{L}^n_c(A, X_2) \) direct summands of \( J_{n-1}^{-1}(\Phi) \) respectively, is a splitting map of the fourth kind.

(ii) It is similar to the proof of (i). Let us just remark that we need \( \sup_{\lambda \in \Lambda} \|s_\lambda^n\| < \infty \), for \( \sum_{\lambda \in \Lambda} \oplus s_\lambda^n(J_{n-1}^{-1}(\Phi)_\lambda) \) to belong in \( \mathcal{L}^{n-1}_c(A, l^\infty(X_\lambda | \lambda \in \Lambda)) \).

When we study the Hochschild cohomology of unital algebras we only need to consider unital modules. To prove that, we first show that the cohomology groups vanish when the algebra acts trivially on the one side of the module and then decompose any module \( X \) into a direct sum of the unital one \( eXe \) and \( eX(1-e), (1-e)Xe, (1-e)X(1-e) \) which have trivial actions on one or both sides of them ([J2], pp.9-10 and p.12 or [Pie2], Proposition 0.1, for the bounded case, and [Ho1], p.61, for the purely algebraic case) and thus get that \( \mathcal{H}^n_c(A, X) \) and \( \mathcal{H}^n_c(A, eXe) \) are isomorphic. In the following two propositions we shall prove that the same thing holds for splitting.

**Proposition 2.1.20.** Let \( A \) be a unital Banach algebra and \( X \) be a Banach \( A \)-bimodule. If \( A \) acts trivially on the one side of \( X \), then \( \mathcal{H}^n_c(A, X) \) splits (I), (II), (III), (IV) and (V), for all \( n \in \mathbb{N} \).
Proof. We can, without loss of generality, assume that \( A \) acts trivially on the left of \( X \) (i.e. that \( ax = 0 \), for all \( a \in A \) and all \( x \in X \)). If \( e \) is the unit element of \( A \), then define

\[
s_1 : L^1_e(A, X) \rightarrow X
\]

by \( s_1(\phi) = -\phi(e) \), for all \( \phi \in L^1_e(A, X) \). Now if \( \phi \in L^1_e(A, X) \) and \( a \in A \), then

\[
\partial^0(s_1(\phi))(a) = -a\phi(e) + \phi(e)a = \phi(e)a - e\phi(a) = \phi(ea) = \phi(a)
\]

where the second and the fourth equality follow from the trivial action of \( A \) on the left of \( X \) and the third from \( \phi \) being a cocycle. Hence \( H^1_e(A, X) \) splits (IV). Now if \( A \) acts trivially on the left of \( X \), then it also acts trivially on the left of \( L^{n-1}_e(A, X) \), for any \( n > 1 \). Thus for \( n > 1 \) the result follows using the reduction of dimension technique established in Proposition 2.1.16.

Proposition 2.1.21. Let \( A \) be a unital Banach algebra, with unit element \( e \), and \( X \) be a Banach \( A \)-bimodule. Then, for all \( n \in \mathbb{N} \), the following are equivalent:

(i) \( H^n_e(A, X) \) splits (I), (II), (III), (IV) or (V) respectively.

(ii) \( H^n(A, eXe) \) splits (I), (II), (III), (IV) or (V) respectively.

Proof. (i) \( \Rightarrow \) (ii) It follows from Proposition 2.1.18(i), since \( eXe \) is a complemented submodule of \( X \).

(ii) \( \Rightarrow \) (i) We can write \( X \) as the direct sum of the modules \( eXe, (1 - e)Xe, eX(1 - e) \) and \( (1 - e)X(1 - e) \). Since \( A \) acts trivially at least on one side of the last three we get the result from Propositions 2.1.19 and 2.1.20. \( \Box \)

In his second paper on the cohomology of associative linear algebras Hochschild showed that if \( A \) is an associative linear algebra with unitisation \( \tilde{A} \) and \( X \) is an \( A \)-bimodule, then \( H^n(A, X) \simeq H^n(\tilde{A}, \tilde{X}) \), for all \( n \in \mathbb{N} \) ([Ho2], Theorem 2). This result also holds in the bounded case ([Pie2], Proposition 0.2). We will briefly discuss how that can be proved and then show that a similar result holds for splitting. For each \( n \in \mathbb{N} \), let \( \partial^n : L^n_e(A, X) \rightarrow L^{n+1}_e(A, X) \) and \( \Delta^n : L^n_e(\tilde{A}, \tilde{X}) \rightarrow L^{n+1}_e(\tilde{A}, \tilde{X}) \) be the coboundary maps. If \( J_n : L^n_e(\tilde{A}, \tilde{X}) \rightarrow L^n_e(A, X) \) is the restriction map, then it is easy to see that

\[
J_n\Delta^{n-1} = \partial^{n-1}J_{n-1}
\]  

On the other hand if \( I_n : L^n_e(A, X) \rightarrow L^n_e(\tilde{A}, \tilde{X}) \) is defined by \( I_n(\phi)(a_1 \oplus t_1, ..., a_n \oplus t_n) = \phi(a_1, ..., a_n) \), for all \( \phi \in L^n_e(A, X) \) and all \( a_1 \oplus t_1, ..., a_n \oplus t_n \in \tilde{A} \), then

\[
\Delta^{n-1}I_{n-1} = I_n\partial^{n-1}
\]  

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Obviously
\[ J_nI_n = id_{L_c^n(A,X)} \]  
(2.9)

Moreover if \( \Phi \in L_c^n(\tilde{A}, \tilde{X}) \), with \( \Phi(\tilde{a}_1, ..., \tilde{a}_n) = 0 \), if \( \tilde{a}_k = 0 \oplus 1 \), for some \( 1 \leq k \leq n \), then
\[ I_nJ_n(\Phi) = \Phi \]  
(2.10)

(2.7), (2.8), (2.9), (2.10) together with the following lemma, which is due to Hochschild ([Ho2], Lemma 1), give the isomorphism of \( \mathcal{H}_c^n(A, X) \) and \( \mathcal{H}_c^n(\tilde{A}, \tilde{X}) \).

We give a slightly modified version of this lemma.

**Lemma 2.1.1.** Let \( A \) be a unital Banach algebra, with unit element \( e \), and \( X \) be a unital Banach \( A \)-bimodule. Then, for all \( n \in \mathbb{N} \), there exists a bounded linear map
\[ K_n : L_c^n(A, X) \rightarrow L_c^{n-1}(A, X) \]

such that, for all \( \phi \in Z_c^n(A, X) \), \( (\phi - \partial^{n-1}K_n(\phi))(a_1, ..., a_n) = 0 \), if \( a_k = e \), for some \( 1 \leq k \leq n \).

**Proposition 2.1.22.** Let \( A \) be a Banach algebra, \( \tilde{A} \) be the unitisation of \( A \) and \( X \) be a Banach \( A \)-bimodule. Then, for all \( n \in \mathbb{N} \), the following are equivalent:

(i) The \( n \)th bounded cohomology group of \( A \), with coefficients in \( X \), splits (I), (II), (III), (IV) or (V) respectively.

(ii) The \( n \)th bounded cohomology group of \( \tilde{A} \), with coefficients in \( \tilde{X} \), splits (I), (II), (III), (IV) or (V) respectively.

**Proof.** The notation is as in the preceding discussion. To prove (i) \( \Rightarrow \) (ii) suppose that \( s_n : L_c^n(A, X) \rightarrow L_c^{n-1}(A, X) \) is a splitting map of the fourth kind and define \( S_n : L_c^n(\tilde{A}, \tilde{X}) \rightarrow L_c^{n-1}(\tilde{A}, \tilde{X}) \) by \( S_n = I_{n-1}s_nJ_n(id_{L_c^2(\tilde{A}, \tilde{X})} - \Delta^{n-1}K_n) \). If \( \Phi \in Z_c^n(\tilde{A}, \tilde{X}) \), then, by (2.8),
\[ \Delta^{n-1}S_n(\Phi) = I_n\partial^{n-1}s_nJ_n((id_{L_c^2(\tilde{A}, \tilde{X})} - \Delta^{n-1}K_n)(\Phi)). \]

(2.7) implies that \( J_n \) maps \( Z_c^n(\tilde{A}, \tilde{X}) \) into \( Z_c^n(A, X) \). Moreover it is obvious that \((id_{L_c^2(\tilde{A}, \tilde{X})} - \Delta^{n-1}K_n)(\Phi) \in Z_c^n(\tilde{A}, \tilde{X}) \). Hence, since \( s_n \) is a splitting map of the fourth kind,
\[ \Delta^{n-1}S_n(\Phi) = I_nJ_n((id_{L_c^2(\tilde{A}, \tilde{X})} - \Delta^{n-1}K_n)(\Phi)). \]

Now using Lemma 2.1.1 and (2.10) we get that \( \Delta^{n-1}S_n(\Phi) = \Phi \) which shows that \( S_n \) is a splitting map of the fourth kind.

Conversely if \( S_n : L_c^n(\tilde{A}, \tilde{X}) \rightarrow L_c^{n-1}(\tilde{A}, \tilde{X}) \) is a splitting map of the fourth kind, then we can prove that \( s_n = J_{n-1}S_nI_n \) is a splitting map of the fourth kind using (2.7), (2.8) and (2.9).
Remark 2.1.4. The previous result can be generalised to direct sums of algebras.

We finish with a proposition relating the splitting of the first bounded cohomology group $H^1_c(A, X)$ with the complementation of $Z(A, X)$ in $X$. This proposition will be used in Section 5.3 to provide a link between splitting and injectivity for von Neumann algebras.

**Proposition 2.1.23.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. If the first bounded cohomology group of $A$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V) and

$$s_1 : L^1_c(A, X) \to X$$

is a splitting map of the first, second, third or fourth kind or belongs in a pair of splitting maps of the fifth kind, then

$$id_X - s_1 \partial^0 : X \to X$$

is a bounded projection, with $\text{Im}(id_X - s_1 \partial^0) = Z(A, X)$.

**Proof.** As we have seen in Propositions 2.1.1, 2.1.3, 2.1.7, 2.1.9 and 2.1.12, if the $n$th bounded cohomology group of $A$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V), then $id_{C^n(A, X)} - s_n \partial^{n-1}$ is a bounded projection with image $Z^{n-1}_c(A, X)$. So in that particular case $id_X - s_1 \partial^0$ is a projection with image $Z^0_c(A, X)$ and the result follows since $Z^0_c(A, X) = Z(A, X)$. \qed

Does the converse hold, i.e. does the complementation of $Z(A, X)$ in $X$ imply the splitting of $H^1_c(A, X)$? Unfortunately the answer is negative even for the first type of splitting as the following example shows. Take the $C^*$-algebra $A$ constructed in [KLRi], Example 6.2. The center $Z(A)$ of $A$ is equal to $Ce$, where $e$ is the unit element of $A$, and therefore it is complemented in $A$. On the other hand $B^1_c(A, A)$ is not a closed subspace of $L^1_c(A, A)$ and therefore, by Proposition 2.1.1, $H^1_c(A, A)$ does not split (I).

### 2.1.2 Splitting of the bounded Hochschild cohomology complex

So far we discussed splitting for individual cohomology groups. As we have seen the information obtained from the splitting of two consecutive cohomology groups can be combined to give results stronger than the ones that follow from the splitting of the two individual groups (see Propositions 2.1.5, 2.1.14 and 2.1.15). That will become more clear in the following discussion about splitting of the cohomology complex.
Definition 2.1.6. Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. We say that the bounded Hochschild cohomology complex of $\mathcal{A}$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V) respectively if the $n$th bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V) respectively, for all $n \in \mathbb{N}$.

The geometric characterisations of the first and the second type of splitting of bounded Hochschild cohomology groups (Propositions 2.1.1 and 2.1.3) yield the following geometric characterisations for the first and the second type of splitting of the bounded Hochschild cohomology complex.

Corollary 2.1.1. Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. Then the following are equivalent:

(i) The bounded Hochschild cohomology complex of $\mathcal{A}$, with coefficients in $X$, splits (I).

(ii) $\mathcal{B}^n_{\mathcal{C}}(\mathcal{A}, X)$ is a closed subspace of $\mathcal{L}^n_{\mathcal{C}}(\mathcal{A}, X)$ and $\mathcal{Z}^n_{\mathcal{C}}(\mathcal{A}, X)$ is complemented in $\mathcal{L}^n_{\mathcal{C}}(\mathcal{A}, X)$, for all $n \in \mathbb{N}$.

Corollary 2.1.2. Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. Then the following are equivalent:

(i) The bounded Hochschild cohomology complex of $\mathcal{A}$, with coefficients in $X$, splits (II).

(ii) $\mathcal{B}^n_{\mathcal{C}}(\mathcal{A}, X)$ and $\mathcal{Z}^n_{\mathcal{C}}(\mathcal{A}, X)$ are both complemented in $\mathcal{L}^n_{\mathcal{C}}(\mathcal{A}, X)$, for all $n \in \mathbb{N}$.

The two previous corollaries show that the first and the second type of splitting do not coincide on the complex level. We mention here that the type of splitting of Remark 2.1.2 (same defining property with splits (I) and (II) but the splitting maps are defined on $\mathcal{Z}^n_{\mathcal{C}}(\mathcal{A}, X)$) would coincide with split (II) for the whole complex.

In the following proposition we show that splits (III), (IV) and (V) are equivalent when we talk about the whole complex.

Proposition 2.1.24. Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. Then the following are equivalent:

(i) The bounded Hochschild cohomology complex of $\mathcal{A}$, with coefficients in $X$, splits (III).

(ii) The bounded Hochschild cohomology complex of $\mathcal{A}$, with coefficients in $X$, splits (IV).

(iii) The bounded Hochschild cohomology complex of $\mathcal{A}$, with coefficients in $X$, splits (V).
(iv) $Z^n_c(A, X)$ is complemented in $L^n_c(A, X)$ and $H^n_c(A, X) = \{0\}$, for all $n \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (iv) It follows from Proposition 2.1.7.

(iv) $\Rightarrow$ (iii) Let $n \in \mathbb{N}$. By the hypothesis, $Z^{n-1}_c(A, X)$, $Z^n_c(A, X)$ and $Z^{n+1}_c(A, X)$ are complemented in $L^{n-1}_c(A, X)$, $L^n_c(A, X)$ and $L^{n+1}_c(A, X)$ respectively and $H^{n-1}_c(A, X)$, $H^n_c(A, X)$ and $H^{n+1}_c(A, X)$ vanish. Since $H^{n+1}_c(A, X)$ vanishes, $E^{n+1}_c(A, X) = Z^{n+1}_c(A, X)$ and hence the result follows from Proposition 2.1.12.

(iii) $\Rightarrow$ (ii) It follows from Proposition 2.1.14.

(ii) $\Rightarrow$ (i) It follows from Proposition 2.1.11. □

Since splits (III), (IV) and (V) are equivalent on the complex level we will refer only to the first, the second and the third type of splitting when we are talking about the complex. In the following proposition we give the relation between them.

**Proposition 2.1.25.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. Then the following are equivalent:

(i) The bounded Hochschild cohomology complex of $A$, with coefficients in $X$, splits (I) and $H^n_c(A, X) = \{0\}$, for all $n \in \mathbb{N}$.

(ii) The bounded Hochschild cohomology complex of $A$, with coefficients in $X$, splits (II) and $H^n_c(A, X) = \{0\}$, for all $n \in \mathbb{N}$.

(iii) The bounded Hochschild cohomology complex of $A$, with coefficients in $X$, splits (III).

Proof. (i) $\Rightarrow$ (iii) follows from the remark after Proposition 2.1.11. (iii) $\Rightarrow$ (ii) follows from Propositions 2.1.24 and 2.1.11. (ii) $\Rightarrow$ (i) follows from Proposition 2.1.6. □

### 2.2 Splitting of the completely bounded Hochschild cohomology

We move now to the splitting of the completely bounded Hochschild cohomology. In the first part of the section we study the splitting of completely bounded Hochschild cohomology groups. As both the definitions and the properties of all five types of splitting are similar to the ones in Section 2.1, we will present them in a different order. We will start by giving the definitions. Then we will give the geometric properties of the five types of splitting and the relation between them. We will then show some algebraic properties and finish with a result about the
relation between the splitting of $H_{cb}^1(A, X)$ and the complete complementation of $Z(A, X)$ in $X$ and some remarks on how splitting of completely bounded cohomology groups and bounded cohomology groups are related.

We used the expression geometric properties instead of geometric characterisations, because whereas the splitting of completely bounded cohomology groups implies complementation results similar to the ones in the bounded case, the converse does not hold, i.e. those complementations do not imply the splitting of completely bounded cohomology groups. That happens because of the use in the proofs in the bounded case of the inverse mapping theorem which does not hold for completely bounded maps.

In the second part we will define splitting for the completely bounded Hochschild cohomology complex and prove that splits (III), (IV) and (V) are equivalent for the complex as in the bounded case.

2.2.1 Splitting of the completely bounded Hochschild cohomology groups

We start by defining the five types of splitting for completely bounded Hochschild cohomology groups. The defining properties of the splitting maps are exactly the same with the ones in the bounded case. The only difference is that here we demand that the splitting maps are completely bounded instead of bounded.

Definition 2.2.1. Let $A$ be an operator algebra and $X$ be an operator completely bounded $A$-bimodule.

(i) We say that the $n$th completely bounded cohomology group of $A$, with coefficients in $X$, splits (I) if there exists a completely bounded linear map

$$s_n : B_{cb}^n(A, X) \rightarrow L_{cb}^{n-1}(A, X)$$

with

$$\partial^{n-1}s_n = id_{B_{cb}^n(A, X)}.$$ 

The map $s_n$ is called a splitting map of the first kind.

(ii) We say that the $n$th completely bounded cohomology group of $A$, with coefficients in $X$, splits (II) if there exists a completely bounded linear map

$$s_n : L_{cb}^n(A, X) \rightarrow L_{cb}^{n-1}(A, X)$$

with

$$\partial^{n-1}s_n\partial^{n-1} = \partial^{n-1}.$$ 

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The map \( s_n \) is called a splitting map of the second kind.

(iii) We say that the \( n \)th completely bounded cohomology group of \( A \), with coefficients in \( X \), splits (III) if there exists a completely bounded linear map
\[
s_n : Z^n_{cb}(A, X) \to \mathcal{L}^{n-1}_{cb}(A, X)
\]
with
\[
\partial^{n-1}s_n = id_{Z^n_{cb}(A, X)}.
\]
The map \( s_n \) is called a splitting map of the third kind.

(iv) We say that the \( n \)th completely bounded cohomology group of \( A \), with coefficients in \( X \), splits (IV) if there exists a completely bounded linear map
\[
s_n : \mathcal{L}^n_{cb}(A, X) \to \mathcal{L}^{n-1}_{cb}(A, X)
\]
with
\[
\partial^{n-1}s_n(\phi) = \phi
\]
for all \( \phi \in Z^n_{cb}(A, X) \). We call \( s_n \) a splitting map of the fourth kind.

(v) We say that the \( n \)th completely bounded cohomology group of \( A \), with coefficients in \( X \), splits (V), if there exist completely bounded linear maps
\[
s_n : \mathcal{L}^n_{cb}(A, X) \to \mathcal{L}^{n-1}_{cb}(A, X)
\]
and
\[
s_{n+1} : \mathcal{L}^{n+1}_{cb}(A, X) \to \mathcal{L}^n_{cb}(A, X)
\]
with
\[
\partial^{n-1}s_n + s_{n+1}\partial^n = id_{\mathcal{L}^n_{cb}(A, X)}.
\]
We call the pair \((s_n, s_{n+1})\) a pair of splitting maps of the fifth kind.

The following six propositions describe the geometric properties of the five types of splitting. Since the proofs are similar to the ones in the first section, we will only give a detailed proof for the first one. We will also explain there why in the completely bounded case the geometric properties do not imply splitting.

**Proposition 2.2.1.** Let \( A \) be an operator algebra, \( X \) be an operator completely bounded \( A \)-bimodule, which is a Banach space, and \( n \in \mathbb{N} \). If the \( n \)th bounded cohomology group of \( A \), with coefficients in \( X \), splits (I), then \( Z^n_{cb}(A, X) \) is completely complemented in \( \mathcal{L}^{n-1}_{cb}(A, X) \) and \( B^n_{cb}(A, X) \) is a closed subspace of \( \mathcal{L}^n_{cb}(A, X) \).
Proof. Let \( s_n : B^n_{cb}(A, X) \to L_{cb}^{n-1}(A, X) \) be a splitting map of the first kind. As in Proposition 2.1.1, we can prove that \( id_{L_{cb}^{n-1}(A, X)} - s_n\partial^{n-1} \) is a projection mapping \( L_{cb}^{n-1}(A, X) \) onto \( Z_{cb}^{n-1}(A, X) \). Its complete boundedness follows from the complete boundedness of \( id_{L_{cb}^{n-1}(A, X)} \), \( s_n \) and \( \partial^{n-1} \). Hence \( Z_{cb}^{n-1}(A, X) \) is completely complemented in \( L_{cb}^{n-1}(A, X) \). Since \( X \) is a Banach space, we can prove that \( B^n_{cb}(A, X) \) is closed as in the proof of Proposition 2.1.1.

On the other hand suppose that \( B^n_{cb}(A, X) = \text{Im}(\partial^{n-1}) \) is closed and \( Z_{cb}^{n-1}(A, X) = \ker(\partial^{n-1}) \) is completely complemented in \( L_{cb}^{n-1}(A, X) \). Then, as in Proposition 2.1.1, we can construct a map \( \pi : B^n_{cb}(A, X) \to L_{cb}^{n-1}(A, X) \) with \( (\partial^{n-1}\pi)(\psi) = \psi \), for all \( \psi \in B^n_{cb}(A, X) \). Is \( \pi \) completely bounded? In the bounded case we used the inverse mapping theorem to prove the boundedness of \( \pi \). Unfortunately a similar result does not hold for completely bounded maps (see Remark 1.2.2(iii)). \( \Box \)

Remark 2.2.1. In Remark 2.1.1 we discussed the notion of an admissible map. For the completely bounded case we can define the similar notion of a completely admissible map. If \( X \) and \( Y \) are matricially normed spaces and \( \phi : X \to Y \) is a completely bounded map, then we will call \( \phi \) completely admissible if \( \ker(\phi) \) is completely complemented in \( X \) and \( \text{Im}(\phi) \) is closed in \( Y \). We mentioned in Remark 2.1.1 that if \( X \) and \( Y \) are Banach spaces, then there exists a bounded right inverse \( \psi : \text{Im}(\phi) \to X \) of \( \phi \). A question that seems to be of some interest is to determine when \( \psi \) is completely bounded.

Proposition 2.2.2. Let \( A \) be an operator algebra, \( X \) be an operator completely bounded \( A \)-bimodule and \( n \in \mathbb{N} \). If the \( n \)-th completely bounded cohomology group of \( A \), with coefficients in \( X \), splits \( II \), then \( Z_{cb}^{n-1}(A, X) \) is completely complemented in \( L_{cb}^{n-1}(A, X) \) and \( B^n_{cb}(A, X) \) is completely complemented in \( L_{cb}^{n}(A, X) \).

Proof. It is similar to the proof of \( (i) \Rightarrow (ii) \) in Proposition 2.1.3. \( \Box \)

Remark 2.2.2. The second observation following Proposition 2.1.3 holds here as well, i.e. if \( B^n_{cb}(A, X) = \{ 0 \} \), then \( H^n_{cb}(A, X) \) splits \( II \). The converse part of the geometric characterisation not holding here, we must construct the splitting map. But that is elementary: just take \( s_n : L_{cb}^{n}(A, X) \to L_{cb}^{n-1}(A, X) \), with \( s_n(\phi) = 0 \), for all \( \phi \in L_{cb}^{n}(A, X) \).

Proposition 2.2.3. Let \( A \) be an operator algebra, \( X \) be an operator completely bounded \( A \)-bimodule and \( n \in \mathbb{N} \). If both the \( n \)-th and the \((n+1)\)-th completely bounded cohomology groups of \( A \), with coefficients in \( X \), split \( II \) and \( s_n \) and \( s_{n+1} \) are splitting maps of the second kind, then the map

\[
\partial^{n-1}s_n + s_{n+1}\partial^n : L_{cb}^{n}(A, X) \to L_{cb}^{n}(A, X)
\]
is a completely bounded projection, with

\[ \text{Im}(\partial^{n-1}s_n + s_{n+1}\partial^n) = \mathcal{L}_{cb}^n(A, X) \odot (\mathcal{Z}_{cb}^n(A, X) \oplus \mathcal{B}_{cb}^n(A, X)) \]

\textbf{Proof.} It is similar to the proof of Proposition 2.1.5. \qed

\textbf{Proposition 2.2.4.} Let \( A \) be an operator algebra, \( X \) be an operator completely bounded \( A \)-bimodule and \( n \in \mathbb{N} \). If the \( n \)-th completely bounded cohomology group of \( A \), with coefficients in \( X \), splits (III), then \( Z_{cb}^{n-1}(A, X) \) is completely complemented in \( \mathcal{L}_{cb}^{n-1}(A, X) \) and \( \mathcal{H}_{cb}^n(A, X) = \{0\} \).

\textbf{Proof.} It is similar to the proof of \((i) \Rightarrow (ii)\) in Proposition 2.1.7. \qed

\textbf{Proposition 2.2.5.} Let \( A \) be an operator algebra, \( X \) be an operator completely bounded \( A \)-bimodule and \( n \in \mathbb{N} \). If the \( n \)-th completely bounded cohomology group of \( A \), with coefficients in \( X \), splits (IV), then \( Z_{cb}^{n-1}(A, X) \) is completely complemented in \( \mathcal{L}_{cb}^{n-1}(A, X) \), \( Z_{cb}^n(A, X) \) is completely complemented in \( \mathcal{L}_{cb}^n(A, X) \) and \( \mathcal{H}_{cb}^n(A, X) = \{0\} \).

\textbf{Proof.} It is the same with the proof of \((i) \Rightarrow (ii)\) in Proposition 2.1.9. \qed

\textbf{Remark 2.2.3.} If \( Z_{cb}^n(A, X) = \{0\} \), then \( \mathcal{H}_{cb}^n(A, X) \) splits (IV).

\textbf{Proposition 2.2.6.} Let \( A \) be an operator algebra, \( X \) be an operator completely bounded \( A \)-bimodule and \( n \in \mathbb{N} \). If the \( n \)-th completely bounded cohomology group of \( A \), with coefficients in \( X \), splits (V), then \( Z_{cb}^{n-1}(A, X) \) is completely complemented in \( \mathcal{L}_{cb}^{n-1}(A, X) \), \( Z_{cb}^n(A, X) \) is completely complemented in \( \mathcal{L}_{cb}^n(A, X) \), \( \mathcal{B}_{cb}^{n+1}(A, X) \) is completely complemented in \( \mathcal{L}_{cb}^{n+1}(A, X) \) and \( \mathcal{H}_{cb}^n(A, X) = \{0\} \).

\textbf{Proof.} It is similar to the proof of \((i) \Rightarrow (ii)\) in Proposition 2.1.12. \qed

The sets of splitting maps have properties similar to the ones described in Propositions 2.1.2, 2.1.4, 2.1.8, 2.1.10 and 2.1.13.

The following three propositions describe the relation between the five types of splitting. Due to the lack of geometric characterisations we can't use the proofs of the first section.

\textbf{Proposition 2.2.7.} Let \( A \) be an operator algebra and \( X \) be an operator completely bounded \( A \)-bimodule. Then, for all \( n \in \mathbb{N} \), the following are equivalent:

(i) The \( n \)-th completely bounded cohomology group of \( A \), with coefficients in \( X \), splits (I), \( \mathcal{B}_{cb}^n(A, X) \) is completely complemented in \( \mathcal{L}_{cb}^n(A, X) \) and \( \mathcal{H}_{cb}^n(A, X) = \{0\} \).

(ii) The \( n \)-th completely bounded cohomology group of \( A \), with coefficients in \( X \), splits (II) and \( \mathcal{H}_{cb}^n(A, X) = \{0\} \).
(iii) The $n$th completely bounded cohomology group of $A$, with coefficients in $X$, splits (III) and $Z^n_{cb}(A, X)$ is completely complemented in $L^n_{cb}(A, X)$.

(iv) The $n$th completely bounded cohomology group of $A$, with coefficients in $X$, splits both (II) and (III).

(v) The $n$th completely bounded cohomology group of $A$, with coefficients in $X$, splits (IV).

Proof. To prove (i) $\Rightarrow$ (ii) let $s_n : B^n_{cb}(A, X) \to L^{n-1}_{cb}(A, X)$ be a splitting map of the first kind. Since $B^n_{cb}(A, X)$ is completely complemented in $L^n_{cb}(A, X)$, there exists a completely bounded projection $\rho : L^n_{cb}(A, X) \to B^n_{cb}(A, X)$. It is obvious that the map $s_n \rho$ is a splitting map of the second kind. (ii) $\Rightarrow$ (iii) follows immediately from the definition of the second type of splitting and Proposition 2.2.2. (iii) $\Rightarrow$ (iv): As in (i) $\Rightarrow$ (ii) we can prove that the complete complementation of $Z^n_{cb}(A, X)$ in $L^n_{cb}(A, X)$ implies that $H^n_{cb}(A, X)$ splits (II). (iv) $\Rightarrow$ (v) follows from the definition of split (II) and Proposition 2.2.4. (v) $\Rightarrow$ (i) is an immediate consequence of the definition of the fourth type of splitting and Proposition 2.2.5.  

Remark 2.2.4. (i) The equivalence between (i) and (ii) in the previous proposition is true if we omit "$H^n_{cb}(A, X) = \{0\}$" from both of them.

(ii) The equivalence between (i) and (iii) remains true if we omit "$B^n_{cb}(A, X)$ is completely complemented in $L^n_{cb}(A, X)$" from (i) and "$Z^n_{cb}(A, X)$ is completely complemented in $L^n_{cb}(A, X)$" from (iii).

Proposition 2.2.8. Let $A$ be an operator algebra and $X$ be an operator completely bounded $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) The $n$th and the $(n+1)$th completely bounded cohomology groups of $A$, with coefficients in $X$, split (II) and $H^n_{cb}(A, X) = \{0\}$.

(ii) The $n$th completely bounded cohomology group of $A$, with coefficients in $X$, splits (III) and the $(n+1)$th splits (II).

(iii) The $n$th completely bounded cohomology group of $A$, with coefficients in $X$, splits (IV) and the $(n+1)$th splits (II).

(iv) The $n$th completely bounded cohomology group of $A$, with coefficients in $X$, splits (V).

Proof. For (iv) $\Rightarrow$ (iii) let $(s_n, s_{n+1})$ be a pair of splitting maps of the fifth kind. Their defining property $\partial^{n-1}s_n + s_{n+1}\partial^n = id_{L^n_{cb}(A, X)}$ and the coboundary condition $\partial^n\partial^{n-1} = 0$ imply that both of them are splitting maps of the second kind. Moreover, by Proposition 2.2.6, $H^n_{cb}(A, X) = \{0\}$. Hence $H^n_{cb}(A, X)$ splits (IV), by Proposition 2.2.7. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) follows from Propositions 2.2.2 and 2.2.7 and (i) $\Rightarrow$ (iv) follows from Proposition 2.2.3.  

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Proposition 2.2.9. Let $\mathcal{A}$ be an operator algebra and $X$ be an operator completely bounded $\mathcal{A}$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) The $n$th and the $(n+1)$th completely bounded cohomology groups of $\mathcal{A}$, with coefficients in $X$, split (II), $\mathcal{H}^n_{\text{cb}}(\mathcal{A}, X) = \{0\}$ and $\mathcal{H}^{n+1}_{\text{cb}}(\mathcal{A}, X) = \{0\}$.

(ii) The $n$th and the $(n+1)$th completely bounded cohomology groups of $\mathcal{A}$, with coefficients in $X$, split (III) and $\mathcal{Z}^n_{\text{cb}}(\mathcal{A}, X)$ is completely complemented in $\mathcal{L}^{n+1}_{\text{cb}}(\mathcal{A}, X)$.

(iii) The $n$th completely bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (III) and the $(n+1)$th splits (IV).

(iv) The $n$th and the $(n+1)$th completely bounded cohomology groups of $\mathcal{A}$, with coefficients in $X$, split (IV).

(v) The $n$th completely bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (V) and $\mathcal{H}^{n+1}_{\text{cb}}(\mathcal{A}, X) = \{0\}$.

Proof. It follows from Propositions 2.2.7 and 2.2.8. □

As we mentioned in Section 1.2.2 a reduction of dimension argument does not hold for the completely bounded cohomology. Thus here we don’t have a result similar to that of Proposition 2.1.16. As in the bounded case, if $\mathcal{A}$ and $\mathcal{B}$ are completely isomorphic operator algebras and $X$ is a completely bounded $\mathcal{A}$-bimodule, then the splitting, of any type, of $\mathcal{H}^n_{\text{cb}}(\mathcal{A}, X)$ and of $\mathcal{H}^n_{\text{cb}}(\mathcal{B}, X)$ are equivalent. The following two propositions describe the situation with respect to isomorphic modules, submodules and quotients. We omit the proofs since they are exactly the same with those of Propositions 2.1.17 and 2.1.18 (the map $J_n$ defined in the proof of 2.1.17 and the map $\rho_n$ defined in the proof of 2.1.18 are obviously completely bounded).

Proposition 2.2.10. Let $\mathcal{A}$ be an operator algebra and $X$ and $Y$ be two completely isomorphic operator completely bounded $\mathcal{A}$-bimodules. If the $n$th completely bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V) respectively, then the $n$th completely bounded cohomology group of $\mathcal{A}$, with coefficients in $Y$, splits (I), (II), (III), (IV) or (V) respectively.

Proposition 2.2.11. Let $\mathcal{A}$ be an operator algebra, $X$ be an operator completely bounded $\mathcal{A}$-bimodule and $Z$ be a completely complemented $\mathcal{A}$-submodule of $X$. If the $n$th completely bounded cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V) respectively then:

(i) The $n$th completely bounded cohomology group of $\mathcal{A}$, with coefficients in $Z$, splits (I), (II), (III), (IV) or (V) respectively.

(ii) The $n$th completely bounded cohomology group of $\mathcal{A}$, with coefficients in $X/Z$, splits (I), (II), (III), (IV) or (V) respectively.
Results similar to those of Propositions 2.1.19, 2.1.20, 2.1.21 and 2.1.22 also hold here.

As in Proposition 2.1.23 the splitting of $H^1_{	ext{cb}}(A, X)$ implies the complete complementation of $Z(A, X)$ in $X$.

**Proposition 2.2.12.** Let $A$ be an operator algebra and $X$ be an operator completely bounded $A$-bimodule. If the first completely bounded cohomology group of $A$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V) and

$$s_1 : \mathcal{L}^1_{\text{cb}}(A, X) \to X$$

is a splitting map of the first, second, third or fourth kind or belongs to a pair of splitting maps of the fifth kind, then

$$\text{id}_X - s_1 \varrho^0 : X \to X$$

is a completely bounded projection, with $\text{Im}(\text{id}_X - s_1 \varrho^0) = Z(A, X)$.

**Proof.** It is similar to the proof of Proposition 2.1.23. \qed

In general we don’t know whether the splitting of $H^n_{\text{cb}}(A, X)$ implies the splitting of $H^n_c(A, X)$, since most of the times the spaces $\mathcal{L}^n_{\text{cb}}(A, X)$, $Z^n_{\text{cb}}(A, X)$ and $B^n_{\text{cb}}(A, X)$ are smaller than the spaces $\mathcal{L}^n_c(A, X)$, $Z^n_c(A, X)$ and $B^n_c(A, X)$. There is one exceptional case which we describe in the following proposition.

**Proposition 2.2.13.** Let $A$ be an operator algebra and $X$ be an operator completely bounded $A$-bimodule. If $H^1_{\text{cb}}(A, X)$ splits (I), then $H^1_c(A, X)$ splits (I).

**Proof.** Since $H^1_{\text{cb}}(A, X)$ splits (I) there exists a completely bounded map $s_1 : B^1_{\text{cb}}(A, X) \to X$ with $\varrho^0 s_1 = \text{id}_{B^1_{\text{cb}}(A, X)}$. But, as we said in Section 1.2.2, $B^1_{\text{cb}}(A, X) = B^1_c(A, X)$ and therefore $H^1_c(A, X)$ splits (I). \qed

Since $B^1_{\text{cb}}(A, X) = B^1_c(A, X)$ and $Z^1_{\text{cb}}(A, X) \subseteq Z^1_c(A, X)$ the vanishing of $H^1_c(A, X)$ always implies the vanishing of $H^1_{\text{cb}}(A, X)$. We don’t know whether the same holds for splitting because we demanded that the splitting maps for completely bounded cohomology groups are completely bounded.

### 2.2.2 Splitting of the completely bounded Hochschild cohomology complex

**Definition 2.2.2.** Let $A$ be an operator algebra and $X$ be an operator completely bounded $A$-bimodule. We say that the completely bounded Hochschild cohomology complex of $A$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V) if the $n$th completely bounded cohomology group of $A$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V) respectively, for all $n \in \mathbb{N}$. 64
The geometric properties of the first and the second type of splitting of completely bounded cohomology groups (Propositions 2.2.1 and 2.2.2) imply geometric properties of splits (I) and (II) of the completely bounded cohomology complex similar to (i) \(\Rightarrow\) (ii) of Corollaries 2.1.1 and 2.1.2.

As in the bounded case splits (III), (IV) and (V) of the complex are equivalent.

**Proposition 2.2.14.** Let \(A\) be an operator algebra and \(X\) be an operator completely bounded \(A\)-bimodule. Then the following are equivalent:

(i) The completely bounded Hochschild cohomology complex of \(A\), with coefficients in \(X\), splits (III).

(ii) The completely bounded Hochschild cohomology complex of \(A\), with coefficients in \(X\), splits (IV).

(iii) The completely bounded Hochschild cohomology complex of \(A\), with coefficients in \(X\), splits (V).

Proof. (iii) \(\Rightarrow\) (ii) and (ii) \(\Rightarrow\) (i) follow immediately from Propositions 2.2.8 and 2.2.7 respectively. (i) \(\Rightarrow\) (iii) follows from Propositions 2.2.4 and 2.2.9. 

The connection between splits (I), (II) and (III) of the complex is the same as the one described in Proposition 2.1.25.

### 2.3 Splitting of the \(B\)-relative Hochschild cohomology

In this section we will study the notion of splitting for \(B\)-relative Hochschild cohomology. We phrase our results for the bounded case and give indications about the required modifications for the completely bounded case.

#### 2.3.1 Splitting of the \(B\)-relative Hochschild cohomology groups

Since the coboundary map \(\partial^n\) maps \(L^c_n(\mathcal{A}, \mathcal{X} : :B)\) into \(L^{n-1}_c(\mathcal{A}, \mathcal{X} : :B)\) we can give the following definitions for the five types of splitting of \(\mathcal{H}^n_c(\mathcal{A}, \mathcal{X} : :B)\).

**Definition 2.3.1.** Let \(A\) be a Banach algebra, \(B\) be a subalgebra of \(A\) and \(X\) be a Banach \(A\)-bimodule.

(i) We say that the \(n\)th bounded \(B\)-relative cohomology group of \(A\), with coefficients in \(X\), splits (I) if there exists a bounded linear map

\[s_n : B^c_n(\mathcal{A}, \mathcal{X} : :B) \rightarrow L^{n-1}_c(\mathcal{A}, \mathcal{X} : :B)\]
with

$$\partial^{n-1}s_n = id_{\mathcal{L}^n_c(A, X : /B)}.$$ 

The map $s_n$ is called a splitting map of the first kind.

(ii) We say that the $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (II) if there exists a bounded linear map

$$s_n : \mathcal{L}^n_c(A, X : /B) \to \mathcal{L}^{n-1}_c(A, X : /B)$$

with

$$\partial^{n-1}s_n\partial^{n-1} = \partial^{n-1}.$$ 

The map $s_n$ is called a splitting map of the second kind.

(iii) We say that the $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (III) if there exists a bounded linear map

$$s_n : \mathcal{Z}^n_c(A, X : /B) \to \mathcal{L}^{n-1}_c(A, X : /B)$$

with

$$\partial^{n-1}s_n = id_{\mathcal{Z}^n_c(A, X : /B)}.$$ 

The map $s_n$ is called a splitting map of the third kind.

(iv) We say that the $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (IV) if there exists a bounded linear map

$$s_n : \mathcal{L}^n_c(A, X : /B) \to \mathcal{L}^{n-1}_c(A, X : /B)$$

with

$$\partial^{n-1}s_n(\phi) = \phi$$

for all $\phi \in \mathcal{Z}^n_c(A, X : /B)$. We call $s_n$ a splitting map of the fourth kind.

(v) We say that the $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (V), if there exist bounded linear maps

$$s_n : \mathcal{L}^n_c(A, X : /B) \to \mathcal{L}^{n-1}_c(A, X : /B)$$

and

$$s_{n+1} : \mathcal{L}^{n+1}_c(A, X : /B) \to \mathcal{L}^n_c(A, X : /B)$$

with

$$\partial^{n-1}s_n + s_{n+1}\partial^n = id_{\mathcal{L}^n_c(A, X : /B)}.$$ 

We call the pair $(s_n, s_{n+1})$ a pair of splitting maps of the fifth kind.
In a similar way we can define splitting for completely bounded \( B \)-relative cohomology groups.

The geometric characterisations of the five types of splitting are exactly the same with the ones discussed in Section 2.1.1. We describe them in the following proposition. The proofs of (a), (b), (c), (d) and (e) are similar to the proofs of Propositions 2.1.1, 2.1.3, 2.1.7, 2.1.9 and 2.1.12 respectively.

**Proposition 2.3.1.** Let \( A \) be a Banach algebra, \( B \) be a subalgebra of \( A \), \( X \) be a Banach \( A \)-bimodule and \( n \in \mathbb{N} \).

(a) The following are equivalent:

(i) The \( n \)th bounded \( B \)-relative cohomology group of \( A \), with coefficients in \( X \), splits (I).

(ii) \( Z_{c}^{n-1}(A, X : /B) \) is complemented in \( \mathcal{L}_{c}^{n-1}(A, X : /B) \) and \( \mathcal{B}_{c}^{n}(A, X : /B) \) is a closed subspace of \( \mathcal{L}_{c}^{n}(A, X : /B) \).

(b) The following are equivalent:

(i) The \( n \)th bounded \( B \)-relative cohomology group of \( A \), with coefficients in \( X \), splits (II).

(ii) \( Z_{c}^{n-1}(A, X : /B) \) is complemented in \( \mathcal{L}_{c}^{n-1}(A, X : /B) \) and \( \mathcal{B}_{c}^{n}(A, X : /B) \) is complemented in \( \mathcal{L}_{c}^{n}(A, X : /B) \).

(c) The following are equivalent:

(i) The \( n \)th bounded \( B \)-relative cohomology group of \( A \), with coefficients in \( X \), splits (III).

(ii) \( Z_{c}^{n-1}(A, X : /B) \) is complemented in \( \mathcal{L}_{c}^{n-1}(A, X : /B) \) and \( \mathcal{H}_{c}^{n}(A, X : /B) = \{0\} \).

(d) The following are equivalent:

(i) The \( n \)th bounded \( B \)-relative cohomology group of \( A \), with coefficients in \( X \), splits (IV).

(ii) \( Z_{c}^{n-1}(A, X : /B) \) is complemented in \( \mathcal{L}_{c}^{n-1}(A, X : /B) \), \( Z_{c}^{n}(A, X : /B) \) is complemented in \( \mathcal{L}_{c}^{n}(A, X : /B) \) and \( \mathcal{H}_{c}^{n}(A, X : /B) = \{0\} \).

(e) The following are equivalent:

(i) The \( n \)th bounded \( B \)-relative cohomology group of \( A \), with coefficients in \( X \), splits (V).

(ii) \( Z_{c}^{n-1}(A, X : /B) \) is complemented in \( \mathcal{L}_{c}^{n-1}(A, X : /B) \), \( Z_{c}^{n}(A, X : /B) \) is complemented in \( \mathcal{L}_{c}^{n}(A, X : /B) \), \( \mathcal{B}_{c}^{n+1}(A, X : /B) \) is complemented in \( \mathcal{L}_{c}^{n+1}(A, X : /B) \) and \( \mathcal{H}_{c}^{n}(A, X : /B) = \{0\} \).

**Remark 2.3.1.** Parts (b) and (d) imply in particular that if \( \mathcal{B}_{c}^{n}(A, X : /B) = \{0\} \) or \( Z_{c}^{n}(A, X : /B) = \{0\} \) respectively, then \( \mathcal{H}_{c}^{n}(A, X : /B) \) splits (II) or (IV) respectively.
Remark 2.3.2. If $A$ is a unital algebra with unit element $e$ and $X$ is a unital $A$-bimodule, then it is easy to see that $\mathcal{L}_C^a(A, X) = \mathcal{L}_C^a(A, X : /Ce)$, for all $n \in \mathbb{N}$. So in that case the bounded cohomology group $\mathcal{H}_C^a(A, X)$ coincides with the bounded $\mathbb{C}e$-relative cohomology group $\mathcal{H}_C^a(A, X : /Ce)$. Therefore we can use the example of the disc algebra $A(\mathbb{D})$ and its module $\mathbb{C}$ to obtain an example of a relative cohomology group that splits (II), but does not vanish.

Propositions 2.2.1, 2.2.2, 2.2.4, 2.2.5 and 2.2.6 can be rephrased to give the geometric properties of the five types of splitting of $\mathcal{H}_C^a(A, X : /B)$.

The sets of splitting maps have properties similar to the ones described in Propositions 2.1.2, 2.1.4, 2.1.8, 2.1.10 and 2.1.13.

The relation between the five types of splitting is described in the following proposition. The proofs of (a), (b) and (c) are similar to the proofs of Propositions 2.1.11, 2.1.14 and 2.1.15 respectively.

**Proposition 2.3.2.** Let $A$ be a Banach algebra, $B$ be a subalgebra of $A$, $X$ be a Banach $A$-bimodule and $n \in \mathbb{N}$.

(a) The following are equivalent:

(i) The $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (I), $\mathcal{B}_C^a(A, X : /B)$ is complemented in $\mathcal{L}_C^a(A, X : /B)$ and $\mathcal{H}_C^a(A, X : /B) = \{0\}$.

(ii) The $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (II) and $\mathcal{H}_C^a(A, X : /B) = \{0\}$.

(iii) The $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (III) and $\mathcal{Z}_C^a(A, X : /B)$ is complemented in $\mathcal{L}_C^a(A, X : /B)$.

(iv) The $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits both (II) and (III).

(v) The $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (IV).

(b) The following are equivalent:

(i) The $n$th and the $(n+1)$th bounded $B$-relative cohomology groups of $A$, with coefficients in $X$, split (II) and $\mathcal{H}_C^a(A, X : /B) = \{0\}$.

(ii) The $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (III) and the $(n+1)$th splits (II).

(iii) The $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (IV) and the $(n+1)$th splits (II).

(iv) The $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (V).

(c) The following are equivalent:
(i) The $n$th and the $(n+1)$th bounded $B$-relative cohomology groups of $A$, with coefficients in $X$, split (II), $\mathcal{H}^n_c(A, X : /B) = \{0\}$ and $\mathcal{H}^{n+1}_c(A, X : /B) = \{0\}$.

(ii) The $n$th and the $(n+1)$th bounded $B$-relative cohomology groups of $A$, with coefficients in $X$, split (III) and $Z^n(A, X : /B)$ is complemented in $L^{n+1}_c(A, X)$.

(iii) The $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (III) and the $(n+1)$th (IV).

(iv) The $n$th and the $(n+1)$th bounded $B$-relative cohomology groups of $A$, with coefficients in $X$, split (IV).

(v) The $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (V) and $\mathcal{H}^{n+1}_c(A, X : /B) = \{0\}$.

**Remark 2.3.3.** (i) The equivalence between (i) and (ii) in (a) holds if we omit "$\mathcal{H}^n(A, X : /B) = \{0\}$" from both of them.

(ii) The equivalence between (i) and (iii) in (a) remains true if we omit "$B^n(A, X : /B)$ is complemented in $L^n_c(A, X : /B)$" from (i) and "$Z^n_c(A, X : /B)$ is complemented in $L^n_c(A, X : /B)$" from (iii).

Similar results hold in the completely bounded case.

A reduction of dimension result does not hold here, since the canonical isomorphism $J : L^n_c(A, X) \to L^1_c(A, L^{n-1}_c(A, X))$ does not map $L^n_c(A, X : /B)$ into $L^1_c(A, L^{n-1}_c(A, X) : /B)$. To see that consider the case $n = 2$ and take $\phi \in L^2_c(A, X : /B)$, $a_1, a_2 \in A$ and $b \in B$. Then

\[
(J(\phi)(a_1)b)(a_2) = J(\phi)(a_1)(ba_2) - J(\phi)(a_1)(b)a_2 = \phi(a_1, ba_2) - \phi(a_1, b)a_2
\]

and

\[
J(\phi)(a_1b)(a_2) = \phi(a_1b, a_2) = \phi(a_1, ba_2)
\]

and hence $J(\phi)$ is not right $B$-modular. (If we consider the kind of $B$-relative cohomology groups that we discussed at the end of Section 1.2.2, then a reduction of dimension result does hold).

The results of the two previous sections about isomorphic modules and complemented submodules hold here as well. We repeat them since we will need them in Chapter 4.

**Proposition 2.3.3.** Let $A$ be a Banach algebra, $B$ be a subalgebra of $A$ and $X$ and $Y$ two isomorphic Banach $A$-bimodules. If the $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V) respectively, then the $n$th bounded $B$-relative cohomology group of $A$, with coefficients in $Y$, splits (I), (II), (III), (IV) or (V) respectively.
Proof. It is similar to the proof of Proposition 2.1.17. We just have to observe that the restriction to $\mathcal{L}_c^n(\mathcal{A}, X : /B)$ of the map $J_n$ defined there gives us the required isomorphism.

Proposition 2.3.4. Let $\mathcal{A}$ be a Banach algebra, $\mathcal{B}$ be a subalgebra of $\mathcal{A}$, $X$ be a Banach $\mathcal{A}$-bimodule and $Z$ be a complemented $\mathcal{A}$-submodule of $X$. If the $n$th bounded $\mathcal{B}$-relative cohomology group of $\mathcal{A}$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V) respectively then:

(i) The $n$th bounded $\mathcal{B}$-relative cohomology group of $\mathcal{A}$, with coefficients in $Z$, splits (I), (II), (III), (IV) or (V) respectively.

(ii) The $n$th bounded $\mathcal{B}$-relative cohomology group of $\mathcal{A}$, with coefficients in $X/Z$, splits (I), (II), (III), (IV) or (V) respectively.

Proof. It is similar to the proof of Proposition 2.1.18. □

In Propositions 2.1.23 and 2.2.12 we proved that the splitting of any type of $\mathcal{H}_c^1(\mathcal{A}, X)$ implies the (complete) complementation of $\mathcal{Z}(\mathcal{A}, X)$ in $X$. Without mentioning it we used there the identification of $\mathcal{L}_c^0(\mathcal{A}, X)$ with $X$. In the case of the $\mathcal{B}$-relative cohomology $\mathcal{L}_c^0(\mathcal{A}, X : /B)$ is not equal to $X$ but to $\{x \in X \mid bx = xb, \text{ for all } b \in \mathcal{B}\}$. Therefore the splitting of any type of $\mathcal{H}_c^1(\mathcal{A}, X : /B)$ implies the (complete) complementation of $\mathcal{Z}(\mathcal{A}, X)$ in $\{x \in X \mid bx = xb, \text{ for all } b \in \mathcal{B}\}$. In particular the splitting of $\mathcal{H}_c^1(\mathcal{A}, X : /\mathcal{A})$ does not imply any complementation result for $\mathcal{Z}(\mathcal{A}, X)$.

In the following proposition we show that $\mathcal{A}$-relative cohomology groups of an algebra $\mathcal{A}$, with coefficients in any $\mathcal{A}$-bimodule $X$, are especially well-behaved with respect to splitting.

Proposition 2.3.5. Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. Then the following hold:

(i) $\mathcal{H}_c^{2n+1}(\mathcal{A}, X : /\mathcal{A})$ splits (II), for all $n \in \mathbb{N}$.

(ii) If $\mathcal{A}$ has a bounded approximate identity and $X$ is neounital, then $\mathcal{H}_c^{2n+1}(\mathcal{A}, X : /\mathcal{A})$ splits (IV), for all $n \in \mathbb{N}$.

(iii) If $\mathcal{A}$ is unital and $X$ is unital, then $\mathcal{H}_c^n(\mathcal{A}, X : /\mathcal{A})$ splits (IV), for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$ and consider $\phi \in \mathcal{L}_c^n(\mathcal{A}, X : /\mathcal{A})$ and $a_1, \ldots, a_{2n+1} \in \mathcal{A}$. Then the $\mathcal{A}$-modularity of $\phi$ implies that
\[ \partial^{2n}(\phi)(a_1, \ldots, a_{2n+1}) = a_1 \phi(a_2, \ldots, a_{2n+1}) - \phi(a_1 a_2, \ldots, a_{2n+1}) \\
+ \sum_{1 \leq i \leq n-1} (-1)^i \phi(a_1, \ldots, a_{2i} a_{2i+1}, \ldots, a_{2n+1}) \\
+ \sum_{1 \leq i \leq n-1} (-1)^{2i+1} \phi(a_1, \ldots, a_{2i+1} a_{2i+2}, \ldots, a_{2n+1}) \\
+ (-1)^{2n} \phi(a_1, \ldots, a_{2n} a_{2n+1}) + (-1)^{2n+1} \phi(a_1, \ldots, a_{2n} a_{2n+1}) = 0. \]

Therefore

\[ \partial^{2n}(\phi) = 0 \tag{2.11} \]

for all \( \phi \in \mathcal{L}_c^{2n}(\mathcal{A}, X : /\mathcal{A}) \). Similarly we can prove that

\[ \partial^{2n+1}(\phi)(a_1, \ldots, a_{2n+2}) = a_1 \phi(a_2, \ldots, a_{2n+2}) \tag{2.12} \]

for all \( \phi \in \mathcal{L}_c^{2n+1}(\mathcal{A}, X : /\mathcal{A}) \) and all \( a_1, \ldots, a_{2n+2} \in \mathcal{A} \). (2.11) implies that \( \mathcal{B}_c^{2n+1}(\mathcal{A}, X : /\mathcal{A}) = \{0\} \) and therefore, by Remark 2.3.1, \( \mathcal{H}_c^{2n+1}(\mathcal{A}, X : /\mathcal{A}) \) splits (II).

Now to prove (ii), let \( \{e_\lambda\}_{\lambda \in \Lambda} \) be a bounded approximate identity in \( \mathcal{A} \) and \( \phi \in \mathcal{Z}_c^{2n+1}(\mathcal{A}, X : /\mathcal{A}) \). Then, from (2.12), \( \partial^{2n+1}(\phi)(e_\lambda, a_1, \ldots, a_{2n+1}) = e_\lambda \phi(a_1, \ldots, a_{2n+1}) \), for all \( \lambda \in \Lambda \) and all \( a_1, \ldots, a_{2n+1} \in \mathcal{A} \). But \( \partial^{2n+1}(\phi) = 0 \) and thus \( e_\lambda \phi(a_1, \ldots, a_{2n+1}) \) is equal to 0, for all \( \lambda \in \Lambda \) and all \( a_1, \ldots, a_{2n+1} \in \mathcal{A} \), which taking the limit over \( \Lambda \) gives us, since \( X \) is neo-unital, \( \phi = 0 \). Hence \( \mathcal{Z}_c^{2n+1}(\mathcal{A}, X : /\mathcal{A}) = \{0\} \) and thus, by Remark 2.3.1, \( \mathcal{H}_c^{2n+1}(\mathcal{A}, X : /\mathcal{A}) \) splits (IV).

For (iii) let \( e \) be the unit element of \( \mathcal{A} \) and for all \( n \in \mathbb{N} \) define

\[ s_n : \mathcal{L}_c^n(\mathcal{A}, X : /\mathcal{A}) \to \mathcal{L}_c^{n-1}(\mathcal{A}, X : /\mathcal{A}) \]

by

\[ s_n(\phi)(a_1, \ldots, a_{n-1}) = \phi(e, a_1, \ldots, a_{n-1}) \]

for all \( \phi \in \mathcal{L}_c^n(\mathcal{A}, X : /\mathcal{A}) \) and all \( a_1, \ldots, a_{n-1} \in \mathcal{A} \). If \( n = 2k \), \( \phi \in \mathcal{Z}_c^{2k}(\mathcal{A}, X : /\mathcal{A})(= \mathcal{L}_c^{2k}(\mathcal{A}, X : /\mathcal{A})) \) and \( a_1, \ldots, a_{2k+1} \in \mathcal{A} \), then, from (2.12),

\[ \partial^{2k-1}s_{2k}(\phi)(a_1, \ldots, a_{2k+1}) = a_1 s_{2k}(\phi)(a_2, \ldots, a_{2k+1}) = a_1 \phi(e, a_2, \ldots, a_{2k+1}) = \phi(a_1, \ldots, a_{2k+1}) \]

and thus \( \mathcal{H}_c^{2k}(\mathcal{A}, X : /\mathcal{A}) \) splits (IV). For \( n = 2k+1 \) we have as in (ii) that \( \mathcal{Z}_c^{2k+1}(\mathcal{A}, X : /\mathcal{A}) = \{0\} \) and so it is immediate that \( s_{2k+1} \) is a splitting map of the fourth kind. \( \square \)
2.3.2 Splitting of the $B$-relative Hochschild cohomology complex

As in the two previous sections, we will say that the (completely) bounded $B$-relative Hochschild cohomology complex of $A$, with coefficients in $X$, splits (I), (II), (III), (IV) or (V) respectively if $\mathcal{H}_{c(b)}^n(A, X : /B)$ splits (I), (II), (III), (IV) or (V) respectively, for all $n \in \mathbb{N}$.

The equivalence between the third, fourth and fifth type of splitting of the complex holds here as well.

**Proposition 2.3.6.** Let $A$ be a Banach algebra, $B$ be a subalgebra of $A$ and $X$ be a Banach $A$-bimodule. Then the following are equivalent:

(i) The bounded $B$-relative Hochschild cohomology complex of $A$, with coefficients in $X$, splits (III).

(ii) The bounded $B$-relative Hochschild cohomology complex of $A$, with coefficients in $X$, splits (IV).

(iii) The bounded $B$-relative Hochschild cohomology complex of $A$, with coefficients in $X$, splits (V).

(iv) $Z^n_c(A, X : /B)$ is complemented in $L^n_c(A, X : /B)$ and $\mathcal{H}^n_c(A, X : /B) = \{0\}$, for all $n \in \mathbb{N}$.

The previous proposition also holds for the completely bounded case if we omit (iv).
Chapter 3

Modules of maps from, into and between $\mathcal{A}$-modules

In this chapter we shall study modules of maps from, into and between $\mathcal{A}$-modules. We start, in the first section, by giving a module structure to the space $\mathcal{L}_1^1(Y, X)$ of maps from a space $Y$ into an $\mathcal{A}$-module $X$, with pointwise module actions of $\mathcal{A}$ on $\mathcal{L}_1^1(Y, X)$. As we shall see $X$ can be embedded, as a complemented submodule, into $\mathcal{L}_1^1(Y, X)$, for all spaces $Y$, and therefore the modules $\mathcal{L}_1^1(Y, X)$ are closely related to $X$ in terms of cohomology. We will also show that the modules $\mathcal{L}_1^1(Y, X)$ inherit many algebraic and topological properties from the module $X$ and are "well-behaved", i.e. module maps between different $X$’s and linear maps between different $Y$’s can be lifted to module maps on $\mathcal{L}_1^1(Y, X)$, submodules of $X$ and subspaces of $Y$ give rise to submodules of $\mathcal{L}_1^1(Y, X)$ and most of the standard module constructions involving modules of the form $\mathcal{L}_1^1(Y, X)$ yield modules of the form $\mathcal{L}_1^1(Y, X)$. We will discuss first the bounded case and then the completely bounded one (where $\mathcal{L}_{cb}^1(Y, X)$ has the standard matricial norm structure).

In the second section we will study modules $\mathcal{L}_1^1(X, Y)$ of maps from an $\mathcal{A}$-module $X$ into a space $Y$. Here the module actions generalise the module actions on the dual module $X^*$ of $X$. Whereas the modules $\mathcal{L}_1^1(Y, X)$ of the first section can be thought of as "extensions" of $X$, the modules $\mathcal{L}_1^1(X, Y)$ of this section are "extensions" of $X^*$. Again we will show that the modules $\mathcal{L}_1^1(X, Y)$ inherit many properties of $X^*$ and are "well-behaved". As in the first section, we will deal first with the bounded and then with the completely bounded case. In the bounded case things are similar to the first section, with the exception of duality and normality for the module $\mathcal{L}_c^1(X, Y)$, which are connected with duality of $Y$ as a Banach space and normality of the dual $\mathcal{A}$-module $X^*$ of $X$. In the completely bounded case things become a bit more complicated. For $\mathcal{L}_{cb}^1(X, Y)$ to become a completely bounded $\mathcal{A}$-module we will have to use the reversed tracial matricial norm structure on $\mathcal{L}_{cb}^1(X, Y)$ which we defined in Section 1.2.1. This is something
we should have expected, since the modules $\mathcal{L}_{cb}^1(X, Y)$ generalise the notion of the dual module and if $X$ is a completely bounded $\mathcal{A}$-module, then the dual completely bounded $\mathcal{A}$-module $X^*$ of $X$ is equipped with the reversed tracial dual matricial norm structure and not with the standard dual one.

In the third section we deal with the relation between the classes of modules defined in the first and the second section. In particular we will show that the modules $\mathcal{L}_1^1(X, Y^*)$ and $\mathcal{L}_1^1(Y, X^*)$ are $\mathcal{A}$-module isomorphic. We will also see that if both $X$ and $Y$ are $\mathcal{A}$-modules, then the module actions defined in the two previous sections are related for $\mathcal{A}$-module maps.

In the fourth section we combine the module actions defined in the first and the second section to make spaces $\mathcal{L}_1^1(X_1, X_2)$ of bounded maps between left or right $\mathcal{A}$-modules $X_1$ and $X_2$ into $\mathcal{A}$-bimodules. Using the results of the first two sections we then prove certain properties of modules of that form. Because of the use of different matricial norm structures in the definitions of $\mathcal{L}_{cb}^1(Y, X)$ and $\mathcal{L}_{cb}^1(X, Y)$ the definition and the results of this section can not be extended to the completely bounded case.

The proofs of most of the results in Sections 3.1, 3.2 and 3.3 are exactly the same for the left and the right module action. When that happens we will only give the proof for the left module action.

### 3.1 Modules of maps into $\mathcal{A}$-modules

#### 3.1.1 The modules $\mathcal{L}_c^1(Y, X)$

As we mentioned in the opening discussion, in this part we study modules of bounded maps from a Banach space $Y$ into a Banach $\mathcal{A}$-module $X$. Although we state our results for $\mathcal{A}$-bimodules, they also hold for left and right $\mathcal{A}$-modules and for $(\mathcal{A}, \mathcal{B})$-bimodules. The plan is the following: We start by defining the modules $\mathcal{L}_c^1(Y, X)$. Then we discuss how the modules $\mathcal{L}_c^1(Y, X)$ are related to the module $X$. We continue by showing that certain algebraic and topological properties of $X$ are inherited by $\mathcal{L}_c^1(Y, X)$. After that we prove that the modules $\mathcal{L}_c^1(Y, X)$ are "well-behaved". Then using the relation between $\mathcal{L}_c^1(Y, X)$ and $X$ that we established in the beginning, we show how the cohomology groups of $\mathcal{A}$, with coefficients in $X$ and in $\mathcal{L}_c^1(Y, X)$, are related. We finish with a result about the relation between the multiplication on $\mathcal{L}_c^1(X)$ and the module actions of $\mathcal{A}$ on $\mathcal{L}_c^1(X)$ and some remarks about submodules of $\mathcal{L}_c^1(Y, X)$.

**Proposition 3.1.1.** Let $\mathcal{A}$ be a Banach algebra, $X$ be a Banach $\mathcal{A}$-bimodule and $Y$ be a Banach space. Then the space of bounded linear maps from $Y$ into
$X$, $\mathcal{L}^1_c(Y, X)$, becomes a Banach $\mathcal{A}$-bimodule, with the module actions of $\mathcal{A}$ on $\mathcal{L}^1_c(Y, X)$ defined by

$$(a\phi)(y) = a \phi(y)$$

and

$$(\phi a)(y) = \phi(y)a$$

for all $a \in \mathcal{A}$, all $\phi \in \mathcal{L}^1_c(Y, X)$ and all $y \in Y$.

Proof. The algebraic part follows directly from the pointwise definition of the module actions. Moreover if $K > 0$ is such that $\|ax\| \leq K\|a\|\|x\|$ and $\|xa\| \leq K\|a\|\|x\|$, for all $a \in \mathcal{A}$ and all $x \in X$, then $\|a\phi\| \leq K\|a\|\|\phi\|$ and $\|\phi a\| \leq K\|a\|\|\phi\|$, for all $a \in \mathcal{A}$ and all $\phi \in \mathcal{L}^1_c(Y, X)$. Therefore $\mathcal{L}^1_c(Y, X)$ is a Banach $\mathcal{A}$-bimodule. We should observe here that the constant $K$ does not depend on $Y$; in particular if $X$ is contractive, then so is $\mathcal{L}^1_c(Y, X)$.

By letting $Y = \mathbb{C}$ we can take the module $X$ itself as a module of the form $\mathcal{L}^1_c(Y, X)$, the simplest of this family of modules.

If $X$ is a Banach $\mathcal{A}$-bimodule and $Y_1, \ldots, Y_n$ are Banach spaces, then, in a similar manner to Proposition 3.1.1, we can make $\mathcal{L}^n_c(Y_1, \ldots, Y_n; X)$ into a Banach $\mathcal{A}$-bimodule. It is easy to see that the module $\mathcal{L}^n_c(Y_1, \ldots, Y_n; X)$ is isometrically $\mathcal{A}$-module isomorphic to the module $\mathcal{L}^1_c(Y_1, \mathcal{L}^{n-1}_c(Y_2, \ldots, Y_n; X))$. Hence when we discuss the properties of the modules $\mathcal{L}^{n}_c(Y_1, \ldots, Y_n; X)$ we need to consider only the case $n = 1$.

Given a non-zero $f \in Y^*$ the module $X$ can be embedded into the module $\mathcal{L}^1_c(Y, X)$ via the map

$$x \mapsto x_f : X \to \mathcal{L}^1_c(Y, X)$$

defined by $x_f(y) = f(y)x$, for all $x \in X$ and all $y \in Y$. It is easy to see that this map is an $\mathcal{A}$-module homomorphism because of the pointwise definition of the module actions. Moreover $\|x_f\| = \|f\|\|x\|$, for all $x \in X$. So $X$ is (isometrically if $\|f\| = 1$) $\mathcal{A}$-module isomorphic to the closed submodule $\{x_f | x \in X\}$ of $\mathcal{L}^1_c(Y, X)$.

On the other hand given a non-zero $y \in Y$ take the "estimation" map

$$\phi \mapsto \phi(y) : \mathcal{L}^1_c(Y, X) \to X.$$ 

We can easily see, using the Hahn-Banach theorem, that $\|\phi \mapsto \phi(y)\| = \|y\|$ and that $\phi \mapsto \phi(y)$ maps $\mathcal{L}^1_c(Y, X)$ onto $X$. Moreover the pointwise nature of the module actions shows that $\phi \mapsto \phi(y)$ is an $\mathcal{A}$-module homomorphism.
Now given a non-zero $f \in Y^*$ and $y \in Y$ with $f(y) = 1$, the composition of those two maps gives us a bounded projection (with norm equal to 1, if $\|f\| \leq 1$ and $\|y\| \leq 1$) that maps $L_c^1(Y, X)$ onto $\{x| x \in X \}$ and is an $A$-module homomorphism. Thus $X$ is (isomorphic to) a complemented $A$-submodule of $L_c^1(Y, X)$.

The pointwise definition of the module actions of $A$ on $L_c^1(Y, X)$ together with the previous observations show that $L_c^1(Y, X)$ possesses certain algebraic properties if and only if $X$ does, i.e. $L_c^1(Y, X)$ is unital if and only if $X$ is, $L_c^1(Y, X)$ is abelian if and only if $X$ is and $Ann_A(L_c^1(Y, X)) = Ann_A(X)$ and thus $L_c^1(Y, X)$ is respectively faithful/annihilating if and only if $X$ is respectively faithful/annihilating. It is obvious that if $L_c^1(Y, X)$ is neounital, then so is $X$. We don’t know under which conditions neounitallity of $X$ implies that $L_c^1(Y, X)$ is neounital.

Moreover duality and normality of $X$ are automatically inherited by $L_c^1(Y, X)$ as we will show in the two following propositions.

**Proposition 3.1.2.** Let $A$ be a Banach algebra, $X$ be a dual $A$-bimodule and $Y$ be a Banach space. Then $L_c^1(Y, X)$ is a dual $A$-bimodule.

**Proof.** Since $X$ is a dual $A$-bimodule, there exists a Banach space $X_*$ such that $X = (X_*)^*$ and the maps $x \mapsto ax$ and $x \mapsto xa$ are weak* continuous for all $a \in A$. Now, since $X = (X_*)^*$, $L_c^1(Y, X)$ is isometrically isomorphic to $(Y \hat{\otimes} X_*)^*$ via the map

$$\phi \mapsto \Psi_\phi : L_c^1(Y, X) \rightarrow (Y \hat{\otimes} X_*)^*$$

defined by $\Psi_\phi(y \otimes z) = \phi(y)(z)$, for all $\phi \in L_c^1(Y, X)$, all $y \in Y$ and all $z \in X_*$, and extended to $Y \hat{\otimes} X_*$ ([BoD], Proposition 42.13 and following Remark). In the rest of the proof we will identify $L_c^1(Y, X)$ with $(Y \hat{\otimes} X_*)^*$.

Before we move on with the proof we would like to remind the reader of two properties of the weak* topology on the dual of a Banach space. Let $V$ and $U$ be Banach spaces. (1) A linear map $T : V^* \rightarrow U^*$ is weak* continuous if and only if $\{T(f_\lambda)\}_{\lambda \in \Lambda}$ converges weak* to $T(f)$, for all $f \in V^*$ and all bounded nets $\{f_\lambda\}_{\lambda \in \Lambda} \subseteq V^*$ converging weak* to $f$. The proof of this property can be obtained combining the following three facts: (i) $T$ is weak* continuous if and only if $T_u : V^* \rightarrow C$, with $T_u(f) = T(f)(u)$, for all $f \in V^*$, is weak* continuous for all $u \in U$. (ii) A linear functional $F : V^* \rightarrow C$ is weak* continuous if and only if $F$ is continuous with respect to the bounded weak* topology on $V^*$ (see [DuSc], Theorem V.5.6). (iii) A linear functional $F : V^* \rightarrow C$ is continuous with respect to the bounded weak* topology on $V^*$ if and only if $\lim_{\lambda \in \Lambda} F(f_\lambda) = F(f)$, for all $f \in V^*$ and all bounded nets $\{f_\lambda\}_{\lambda \in \Lambda} \subseteq V^*$ converging weak* to $f$ (which follows
immediately from the definition of the bounded weak* topology on \( V^* \) which can be found in [DuSc], Definition V.5.3). (2) Let \( W \) be a dense subset of \( V \), \( \{f_\lambda\}_{\lambda \in \Lambda} \) be a bounded net in \( V^* \) and \( f \in V^* \). If \( \lim_{\lambda \in \Lambda} f_\lambda(w) = f(w) \), for all \( w \in W \), then \( \{f_\lambda\}_{\lambda \in \Lambda} \) converges weak* to \( f \).

Let \( a \in A \). To prove the weak* continuity of the map \( \phi \mapsto a\phi : \mathcal{L}_c^1(Y, X) \to \mathcal{L}_c^1(Y, X) \) consider a bounded net \( \{\phi_\lambda\}_{\lambda \in \Lambda} \) in \( \mathcal{L}_c^1(Y, X) \) converging weak* to some \( \phi \in \mathcal{L}_c^1(Y, X) \). Then \( \lim_{\lambda \in \Lambda} \phi_\lambda(y)(z) = \phi(y)(z) \), for all \( y \in Y \) and all \( z \in X_* \), and therefore the net \( \{\phi_\lambda(y)\}_{\lambda \in \Lambda} \) converges weak* in \( X \) to \( \phi(y) \), for all \( y \in Y \). Thus, by the weak* continuity of the map \( x \mapsto ax \) and the pointwise definition of the left module action of \( A \) on \( \mathcal{L}_c^1(Y, X) \), the net \( \{(a\phi_\lambda)(y)\}_{\lambda \in \Lambda} \) converges weak* in \( X \) to \( (a\phi)(y) \), for all \( y \in Y \). So \( \lim_{\lambda \in \Lambda} (a_\lambda)(y)(z) = (a\phi)(y)(z) \), for all \( y \in Y \) and all \( z \in X_* \). Combining that with \( Y \otimes X_* \) being dense in \( Y \otimes X \), and the boundedness of the net \( \{a\phi_\lambda\}_{\lambda \in \Lambda} \) (which follows from the boundedness of \( \{\phi_\lambda\}_{\lambda \in \Lambda} \)), we get that the net \( \{a\phi_\lambda\}_{\lambda \in \Lambda} \) converges weak* to \( a\phi \) and the weak* continuity of the map \( \phi \mapsto a\phi : \mathcal{L}_c^1(Y, X) \to \mathcal{L}_c^1(Y, X) \) has been proved. Therefore \( \mathcal{L}_c^1(Y, X) \) is a dual \( A \)-bimodule.

Alternatively, since \( X \) is a dual \( A \)-bimodule, \( X_* \) is also an \( A \)-bimodule. We can make \( Y \otimes X_* \) into an \( A \)-bimodule with \( A \) acting on \( X_* \) and prove that \( \mathcal{L}_c^1(Y, X) \) is the dual \( A \)-bimodule of \( Y \otimes X_* \).

We can see from the alternative proof that the previous proposition holds in the more general case where \( X \) is the dual of a normed \( A \)-bimodule \( X_* \) which is not a Banach space.

**Proposition 3.1.3.** Let \( M \) be a von Neumann algebra, \( X \) be a normal \( M \)-bimodule and \( Y \) be a Banach space. Then \( \mathcal{L}_c^1(Y, X) \) is a normal \( M \)-bimodule.

**Proof.** By the previous proposition, \( \mathcal{L}_c^1(Y, X) \) is a dual \( M \)-bimodule. Since \( X \) is a normal \( M \)-bimodule, the map \( a \mapsto ax : M \to X \) is ultraweak-weak* continuous, for all \( x \in X \). Take \( \phi \in \mathcal{L}_c^1(Y, X) \). To prove the ultraweak-weak* continuity of the map \( a \mapsto a\phi : M \to \mathcal{L}_c^1(Y, X) \), consider a net \( \{a_\lambda\}_{\lambda \in \Lambda} \) in \( M \) converging ultraweakly to some \( a \in M \). Since \( M \) is a von Neumann algebra, the ultraweak topology on \( M \) is the weak* topology on \( M \) ([StZ], pp.15-19). Hence, using the property that we discussed in the second paragraph of the proof of Proposition 3.1.2, we may assume that \( \{a_\lambda\}_{\lambda \in \Lambda} \) is bounded. Let \( X_* \) be as in the proof of the previous proposition. Then, for all \( z \in X_* \) and all \( y \in Y \), we have, using the identification of \( \mathcal{L}_c^1(Y, X) \) with \( (Y \otimes X_*)^* \) and the ultraweak-weak* continuity of
\[
\lim_{\lambda \in \Lambda} (a_\lambda \phi) (y \otimes z) = \lim_{\lambda \in \Lambda} a_\lambda \phi (y \otimes z) \\
= (\lim_{\lambda \in \Lambda} a_\lambda) \phi (y \otimes z) \\
= a \phi (y \otimes z) \\
= (a \phi) (y \otimes z).
\]

Since \( Y \otimes X_\ast \) is dense in \( Y \hat{\otimes} X_\ast \) and \( \{a_\lambda \phi\}_{\lambda \in \Lambda} \) is bounded (since \( \{a_\lambda\}\) is), \( \{a_\lambda \phi\}_{\lambda \in \Lambda} \) converges weak* to \( a \phi \). Hence \( \mathcal{L}_c^1 (Y, X) \) is a normal \( \mathcal{M} \)-bimodule. \( \Box \)

The following two propositions show that bounded module homomorphisms between \( X \)'s and bounded linear maps between \( Y \)'s can be lifted, via composition, to bounded module homomorphisms between \( \mathcal{L}_c^1 (Y, X) \)'s and that submodules of \( X \) and subspaces of \( Y \) give rise to submodules of \( \mathcal{L}_c^1 (Y, X) \).

**Proposition 3.1.4.** Let \( \mathcal{A} \) be a Banach algebra, \( X_1 \) and \( X_2 \) be Banach \( \mathcal{A} \)-bimodules and \( Y_1 \) and \( Y_2 \) be Banach spaces.

(i) If \( \pi : X_1 \rightarrow X_2 \) is a bounded \( \mathcal{A} \)-module homomorphism, then

\[
\pi_{Y_1} : \mathcal{L}_c^1 (Y_1, X_1) \rightarrow \mathcal{L}_c^1 (Y_1, X_2)
\]

defined by \( \pi_{Y_1} (\phi) = \pi \phi \), for all \( \phi \in \mathcal{L}_c^1 (Y_1, X_1) \), is a bounded \( \mathcal{A} \)-module homomorphism, with \( \| \pi_{Y_1} \| = \| \pi \| \).

(ii) If \( \tau : Y_1 \rightarrow Y_2 \) is a bounded linear map, then

\[
\tau_{X_1} : \mathcal{L}_c^1 (Y_2, X_1) \rightarrow \mathcal{L}_c^1 (Y_1, X_1)
\]

defined by \( \tau_{X_1} (\phi) = \phi \tau \), for all \( \phi \in \mathcal{L}_c^1 (Y_2, X_1) \), is a bounded \( \mathcal{A} \)-module homomorphism, with \( \| \tau_{X_1} \| = \| \tau \| \).

**Proof.** It is easy to see by a straightforward calculation and the Hahn-Banach theorem that both \( \pi_{Y_1} \) and \( \tau_{X_1} \) are bounded maps with \( \| \pi_{Y_1} \| = \| \pi \| \) and \( \| \tau_{X_1} \| = \| \tau \| \).

If \( \phi \in \mathcal{L}_c^1 (Y_1, X_1) \), \( a \in \mathcal{A} \) and \( y \in Y_1 \), then

\[
(a \pi_{Y_1} (\phi))(y) = a \pi_{Y_1} (\phi)(y) = a \pi (\phi(y)) = \pi (a \phi(y))
\]

where the first and the fourth step follow from the pointwise definition of the module action and the third step follows from \( \pi \) being an \( \mathcal{A} \)-module homomorphism. Hence \( \pi_{Y_1} \) is an \( \mathcal{A} \)-module homomorphism.

On the other hand if \( \phi \in \mathcal{L}_c^1 (Y_2, X_1) \), \( a \in \mathcal{A} \) and \( y \in Y_1 \), then

\[
(a \tau_{X_1} (\phi))(y) = a \tau_{X_1} (\phi)(y) = a \phi (\tau (y))
\]

where the first and the fourth step follow from the pointwise definition of the module action and the third step follows from \( \pi \) being an \( \mathcal{A} \)-module homomorphism. Hence \( \tau_{X_1} \) is an \( \mathcal{A} \)-module homomorphism.
where the first and the third step follow from the definition of the left module action.

In particular the previous proposition shows that if we start with (isometrically) isomorphic modules $X_1$ and $X_2$ and (isometrically) isomorphic spaces $Y_1$ and $Y_2$, then the modules $L_c^1(Y_1, X_1)$ and $L_c^1(Y_2, X_2)$ are (isometrically) isomorphic.

**Proposition 3.1.5.** Let $A$ be a Banach algebra, $X$ be a Banach $A$-bimodule and $Y$ be a Banach space.

(i) If $X_1$ is a closed $A$-submodule of $X$, then $L_c^1(Y, X_1)$ is a closed $A$-submodule of $L_c^1(Y, X)$. Moreover if $X_1$ is a complemented $A$-submodule of $X$, then $L_c^1(Y, X_1)$ is a complemented $A$-submodule of $L_c^1(Y, X)$.

(ii) If $Y_1$ is a complemented subspace of $Y$, then $L_c^1(Y_1, X)$ is a complemented $A$-submodule of $L_c^1(Y, X)$.

(iii) If $X_1$ is a closed $A$-submodule of $X$ and $Y_1$ is a complemented subspace of $Y$, then $L_c^1(Y_1, X_1)$ is a closed $A$-submodule of $L_c^1(Y, X)$. Moreover if $X_1$ is a complemented $A$-submodule of $X$, then $L_c^1(Y_1, X_1)$ is a complemented $A$-submodule of $L_c^1(Y, X)$.

Proof. (i) Obviously $L_c^1(Y, X_1)$ is a closed subspace of $L_c^1(Y, X)$ (or more formally it is isometrically isomorphic to the closed subspace $\{\phi \in L_c^1(Y, X) \mid \phi(y) \in X_1, \text{for all } y \in Y\}$ of $L_c^1(Y, X)$). Now consider $a \in A$, $\phi \in L_c^1(Y, X_1)$ and $y \in Y$. Then $\phi(y) \in X_1$ and so $a\phi(y) \in X_1$, since $X_1$ is an $A$-submodule of $X$, which proves that $a\phi \in L_c^1(Y, X_1)$. Thus $L_c^1(Y_1, X_1)$ is an $A$-submodule of $L_c^1(Y, X)$.

If $X_1$ is complemented in $X$, then there exists a bounded projection $\rho : X \to X$, with $Im(\rho) = X_1$, which is an $A$-module homomorphism. Then, by (i) of the previous proposition, the map $\rho_Y : L_c^1(Y, X) \to L_c^1(Y, X)$ is a bounded $A$-module homomorphism. Moreover it is easy to see that $\rho_Y$ is a projection, with $Im(\rho_Y) = L_c^1(Y, X_1)$, and therefore $L_c^1(Y, X_1)$ is complemented in $L_c^1(Y, X)$.

(ii) Since $Y_1$ is complemented in $Y$, there exists a bounded projection $\rho : Y \to Y$, with $Im(\rho) = Y_1$. Using the second part of the previous proposition we see that $\rho_X : L_c^1(Y, X) \to L_c^1(Y, X)$ is a bounded $A$-module homomorphism. Moreover $\rho_X$ is a projection mapping $L_c^1(Y, X)$ onto $S = \{\phi \in L_c^1(Y, X) \mid \phi(y) = 0, \text{for all } y \in Y \ominus Y_1\}$. But $S$ is isomorphic to $L_c^1(Y_1, X)$ via the restriction map $\phi \mapsto \phi|_{Y_1} : S \to L_c^1(Y_1, X)$ and therefore $L_c^1(Y_1, X)$ is a complemented $A$-submodule of $L_c^1(Y, X)$.

(iii) follows immediately from (i) and (ii).

In the following two propositions we study the relation between direct sums of $X$’s and $Y$’s and direct sums of $L_c^1(Y, X)$’s and between quotients of $X$ by
Proposition 3.1.6. If \( \mathcal{A} \) is a Banach algebra, then the following hold:

(i) If \( X_1 \) and \( X_2 \) are Banach \( \mathcal{A} \)-bimodules and \( Y \) is a Banach space, then for all \( 1 \leq p \leq \infty \), \( \mathcal{L}_c^1(Y, X_1) \oplus_p \mathcal{L}_c^1(Y, X_2) \) is \( \mathcal{A} \)-module isomorphic to \( \mathcal{L}_c^1(Y, X_1 \oplus_p X_2) \).

(ii) If \( \{X_\lambda|\lambda \in \Lambda\} \) is a family of uniformly bounded Banach \( \mathcal{A} \)-bimodules and \( Y \) is a Banach space, then:

(a) \( l^\infty(\mathcal{L}_c^1(Y, X_\lambda)|\lambda \in \Lambda) \) is isometrically \( \mathcal{A} \)-module isomorphic to \( \mathcal{L}_c^1(Y, l^\infty(X_\lambda)|\lambda \in \Lambda) \).

(b) \( c_0(\mathcal{L}_c^1(Y, X_\lambda)|\lambda \in \Lambda) \) can be isometrically embedded as an \( \mathcal{A} \)-submodule into \( \mathcal{L}_c^1(Y, c_0(X_\lambda)|\lambda \in \Lambda) \).

(c) For all \( 1 \leq p < \infty \), \( l^p(\mathcal{L}_c^1(Y, X_\lambda)|\lambda \in \Lambda) \) can be contractively embedded as an \( \mathcal{A} \)-submodule into \( \mathcal{L}_c^1(Y, l^p(X_\lambda)|\lambda \in \Lambda) \).

(iii) If \( X \) is a Banach \( \mathcal{A} \)-bimodule and \( Y_1 \) and \( Y_2 \) are Banach spaces, then, for all conjugate \( 1 \leq p, q \leq \infty \), \( \mathcal{L}_c^1(Y_1, X) \oplus_p \mathcal{L}_c^1(Y_2, X) \) is \( \mathcal{A} \)-module isomorphic to \( \mathcal{L}_c^1(Y_1 \oplus_q Y_2, X) \). Moreover if \( p = \infty \) and \( q = 1 \), then they are isometrically isomorphic.

(iv) If \( X \) is a Banach \( \mathcal{A} \)-bimodule and \( \{Y_\lambda|\lambda \in \Lambda\} \) is a family of Banach spaces, then:

(a) \( l^\infty(\mathcal{L}_c^1(Y_\lambda, X)|\lambda \in \Lambda) \) is isometrically \( \mathcal{A} \)-module isomorphic to \( \mathcal{L}_c^1(l^1(Y_\lambda|\lambda \in \Lambda), X) \).

(b) \( l^1(\mathcal{L}_c^1(Y_\lambda, X)|\lambda \in \Lambda) \) can be contractively embedded as an \( \mathcal{A} \)-submodule into \( \mathcal{L}_c^1(c_0(Y_\lambda|\lambda \in \Lambda), X) \).

(c) For all conjugate \( 1 < p, q \leq \infty \), \( l^p(\mathcal{L}_c^1(Y_\lambda, X)|\lambda \in \Lambda) \) can be contractively embedded as an \( \mathcal{A} \)-submodule into \( \mathcal{L}_c^1(l^q(Y_\lambda|\lambda \in \Lambda), X) \).

Proof. (i) Let

\[
J : \mathcal{L}_c^1(Y, X_1) \oplus_p \mathcal{L}_c^1(Y, X_2) \to \mathcal{L}_c^1(Y, X_1 \oplus_p X_2)
\]

be defined by

\[
J(\phi_1 \oplus \phi_2)(y) = \phi_1(y) + \phi_2(y)
\]

for all \( \phi_1 \oplus \phi_2 \in \mathcal{L}_c^1(Y, X_1) \oplus \mathcal{L}_c^1(Y, X_2) \) and all \( y \in Y \). It is easy to see that \( J \) is an (isometric if \( p = \infty \)) isomorphism. To see that it is not an isometry if \( 1 \leq p < \infty \) just consider a Banach space \( X \) and the projections

\[
P_1 : X \oplus_p X \to X
\]
and

\[ P_2 : X \oplus_p X \rightarrow X \]

with \( P_1(x_1 \oplus x_2) = x_1 \) and \( P_2(x_1 \oplus x_2) = x_2 \), for all \( x_1 \oplus x_2 \in X \oplus X \); then \( \|P_1 \oplus P_2\|_p = 2^{1/p} \) and \( \|J(P_1 \oplus P_2)\| = 1 \). To show that \( J \) is an \( \mathcal{A} \)-module homomorphism let us consider \( a \in \mathcal{A} \), \( \phi_1 \oplus \phi_2 \in \mathcal{L}_c^1(Y, X_1) \oplus \mathcal{L}_c^1(Y, X_2) \) and \( y \in Y \). Then

\[
J(a(\phi_1 \oplus \phi_2))(y) = (a(\phi_1))(y) + (a(\phi_2))(y) = a(\phi_1(y)) + a(\phi_2(y))
\]

which proves that \( J \) is a left \( \mathcal{A} \)-module homomorphism.

(ii) Since \( \{X_{\lambda}\mid \lambda \in \Lambda\} \) is a uniformly bounded family of Banach \( \mathcal{A} \)-bimodules, \( \{\mathcal{L}_c^1(Y, X_{\lambda})\mid \lambda \in \Lambda\} \) is also a family of uniformly bounded Banach \( \mathcal{A} \)-bimodules, because, as we mentioned at the end of the proof of Proposition 3.1.1, the constant \( K \) does not depend on \( Y \). Now consider the map

\[
J : l^\infty(\mathcal{L}_c^1(Y, X_{\lambda})\mid \lambda \in \Lambda) \rightarrow \mathcal{L}_c^1(Y, l^\infty(X_{\lambda}\mid \lambda \in \Lambda))
\]

defined by

\[
J(\sum_{\lambda \in \Lambda} \phi_{\lambda})(y) = \sum_{\lambda \in \Lambda} \phi_{\lambda}(y)
\]

for all \( \sum_{\lambda \in \Lambda} \phi_{\lambda} \in l^\infty(\mathcal{L}_c^1(Y, X_{\lambda})\mid \lambda \in \Lambda) \) and all \( y \in Y \). It is easy to see that it is an isometric isomorphism. Moreover we can see that its restriction to \( c_0(\mathcal{L}_c^1(Y, X_{\lambda})\mid \lambda \in \Lambda) \) is an isometry. Considering a family of Banach spaces \( \{Y_n\mid n \in \mathbb{N}\} \) and the identity map on \( c_0(Y_n\mid n \in \mathbb{N}) \) we can see that \( J \) is not onto. As for its restriction to \( l^p(\mathcal{L}_c^1(Y, X_{\lambda})\mid \lambda \in \Lambda), 1 \leq p < \infty \), it is also easy to see that it is a contractive embedding; we can see that it is not onto using an argument similar to that for \( c_0(\mathcal{L}_c^1(Y, X_{\lambda})\mid \lambda \in \Lambda) \) and that it is not an isometry using an argument similar to that of part (i). We can also prove that \( J \) is an \( \mathcal{A} \)-module homomorphism as in part (i).

(iii) Let

\[
J : \mathcal{L}_c^1(Y_1, X) \oplus_p \mathcal{L}_c^1(Y_2, X) \rightarrow \mathcal{L}_c^1(Y_1 \oplus_q Y_2, X)
\]

be defined by

\[
J(\phi_1 \oplus \phi_2)(y_1 \oplus y_2) = \phi_1(y_1) + \phi_2(y_2)
\]

for all \( \phi_1 \oplus \phi_2 \in \mathcal{L}_c^1(Y_1, X) \oplus \mathcal{L}_c^1(Y_2, X) \) and all \( y_1 \oplus y_2 \in Y_1 \oplus Y_2 \). It is easy to prove that \( J \) is an isomorphism. Moreover it is easy to see that it is an isometry.
if \( p = \infty \) and \( q = 1 \). To see that it is not an isometry for the other values of \( p \) and \( q \) consider a Banach space \( X \) and the maps

\[
\tau_1 : X \to X \oplus_q X
\]

and

\[
\tau_2 : X \to X \oplus_q X
\]
defined by \( \tau_1(x) = x \oplus 0 \) and \( \tau_2(x) = 0 \oplus x \), for all \( x \in X \); then \( \| \tau_1 \oplus \tau_2 \|_p = 2^{1/p} \) and \( \|J(\tau_1 \oplus \tau_2)\| = 1 \). To show that \( J \) is an \( \mathcal{A} \)-module homomorphism let \( a \in \mathcal{A} \), \( \phi_1 \oplus \phi_2 \in \mathcal{L}^1_c(Y_1, X) \oplus \mathcal{L}^1_c(Y_2, X) \) and \( y_1 \oplus y_2 \in Y_1 \oplus Y_2 \). Then

\[
J(a(\phi_1 \oplus \phi_2))(y_1 \oplus y_2) = (a\phi_1)(y_1) \oplus (a\phi_2)(y_2)
= aJ(\phi_1 \oplus \phi_2)(y_1 \oplus y_2)
= (aJ(\phi_1 \oplus \phi_2))(y_1 \oplus y_2)
\]

which proves that \( J \) is a left \( \mathcal{A} \)-module homomorphism.

(iv) It is easy to see that \( \{ \mathcal{L}^1_c(Y_\lambda, X)\mid \lambda \in \Lambda \} \) is a family of uniformly bounded \( \mathcal{A} \)-bimodules. Let

\[
J : l^\infty(\mathcal{L}^1_c(Y_\lambda, X)\mid \lambda \in \Lambda) \to \mathcal{L}^1_c(l^1(Y_\lambda\mid \lambda \in \Lambda), X)
\]

be defined by

\[
J(\sum_{\lambda \in \Lambda} \phi_\lambda)(\sum_{\lambda \in \Lambda} y_\lambda) = \sum_{\lambda \in \Lambda} \phi_\lambda(y_\lambda)
\]

for all \( \sum_{\lambda \in \Lambda} \phi_\lambda \in l^\infty(\mathcal{L}^1_c(Y_\lambda, X)\mid \lambda \in \Lambda) \) and all \( \sum_{\lambda \in \Lambda} y_\lambda \in l^1(Y_\lambda\mid \lambda \in \Lambda) \). We can easily see that \( J \) is an isometric isomorphism and that its restrictions to \( l^p(\mathcal{L}^1_c(Y_\lambda, X)\mid \lambda \in \Lambda) \), for \( 1 \leq p < \infty \), are contractive embeddings. As in part (iii) we can prove that those restrictions are not isometries. Moreover we can see that they are not in general onto if we consider a family \( \{ Y_n\mid n \in \mathbb{N} \} \) of Banach spaces and the identity map on \( c_0(Y_n\mid n \in \mathbb{N}) \) or \( l^q(Y_n\mid n \in \mathbb{N}) \), for \( 1 < q \leq \infty \). We can also show that \( J \) is an \( \mathcal{A} \)-module homomorphism as in part (iii).

\[
\Box.
\]

**Proposition 3.1.7.** Let \( \mathcal{A} \) be a Banach algebra, \( X \) be a Banach \( \mathcal{A} \)-bimodule and \( Y \) be a Banach space.

(i) If \( X_1 \) is a closed \( \mathcal{A} \)-submodule of \( X \), then \( \mathcal{L}^1_c(Y, X)/\mathcal{L}^1_c(Y, X_1) \) can be isometrically embedded as an \( \mathcal{A} \)-submodule into \( \mathcal{L}^1_c(Y, X/X_1) \). Moreover if \( X_1 \) is complemented in \( X \) as a subspace, then \( \mathcal{L}^1_c(Y, X)/\mathcal{L}^1_c(Y, X_1) \) is isometrically \( \mathcal{A} \)-module isomorphic to \( \mathcal{L}^1_c(Y, X/X_1) \).

(ii) If \( Y_1 \) is a complemented subspace of \( Y \), then \( \mathcal{L}^1_c(Y, X)/\mathcal{L}^1_c(Y_1, X) \) is \( \mathcal{A} \)-module isomorphic to \( \mathcal{L}^1_c(Y/Y_1, X) \).
Proof. (i) It follows from Proposition 3.1.5(i) that $L^1_c(Y, X_1)$ is a closed $\mathcal{A}$-submodule of $L^1_c(Y, X)$ and hence $L^1_c(Y, X)/L^1_c(Y, X_1)$ is an $\mathcal{A}$-bimodule. With the help of the Hahn-Banach theorem we can see that the map

$$J : L^1_c(Y, X)/L^1_c(Y, X_1) \rightarrow L^1_c(Y, X/X_1)$$

defined by

$$J(\phi + L^1_c(Y, X_1))(y) = \phi(y) + X_1$$

for all $\phi + L^1_c(Y, X_1) \in L^1_c(Y, X)/L^1_c(Y, X_1)$ and all $y \in Y$, is an isometry. Moreover if $\phi \in L^1_c(Y, X)$, $a \in \mathcal{A}$ and $y \in Y$, then

$$J(a(\phi + L^1_c(Y, X_1)))(y) = J(a\phi + L^1_c(Y, X_1))(y)$$

$$= (a\phi)(y) + X_1$$

$$= a\phi(y) + X_1$$

$$= aJ(\phi + L^1_c(Y, X_1))(y)$$

$$= (aJ(\phi + L^1_c(Y, X_1)))(y)$$

and thus $J$ is an $\mathcal{A}$-module homomorphism.

If $X_1$ is a complemented subspace of $X$ and $\rho : X \rightarrow X \ominus X_1$ is a bounded projection, then given $\Phi \in L^1_c(Y, X/X_1)$, let $\phi : Y \rightarrow X$, be defined by $\phi = \tau\Phi$, where $\tau(x + X_1) = \rho(x)$, for all $x \in X$. Then it is easy to see that $\Phi = J(\phi + L^1_c(Y, X_1))$ and therefore $J$ is onto.

(ii) It follows from Proposition 3.1.5(ii) that $L^1_c(Y_1, X)$ is a closed $\mathcal{A}$-submodule of $L^1_c(Y, X)$ and hence $L^1_c(Y, X)/L^1_c(Y_1, X)$ is an $\mathcal{A}$-bimodule. If $\rho : Y \rightarrow Y \ominus Y_1$ is a bounded projection, then

$$J : L^1_c(Y, X)/L^1_c(Y_1, X) \rightarrow L^1_c(Y/Y_1, X)$$

which is defined by

$$J(\phi + L^1_c(Y_1, X))(y + Y_1) = \phi(\rho(y))$$

for all $\phi + L^1_c(Y, X) \in L^1_c(Y, X)/L^1_c(Y_1, X)$ and all $y \in Y$, is an isomorphism. Moreover if $\phi \in L^1_c(Y, X)$, $a \in \mathcal{A}$ and $y \in Y$, then

$$J(a(\phi + L^1_c(Y_1, X)))(y + Y_1) = J(a\phi + L^1_c(Y_1, X))(y + Y_1)$$

$$= (a\phi)(\rho(y))$$

$$= a\phi(\rho(y))$$

$$= aJ(\phi + L^1_c(Y_1, X))(y + Y_1)$$

$$= (aJ(\phi + L^1_c(Y_1, X)))(y + Y_1)$$

and therefore $J$ is an $\mathcal{A}$-module homomorphism. \qed
It is easy to see that if $\mathcal{A}$ is a Banach algebra with unitisation $\tilde{\mathcal{A}}$, $X$ is a Banach $\mathcal{A}$-bimodule and $Y$ is a Banach space, then the $\tilde{\mathcal{A}}$-bimodules $\mathcal{L}_c^1(Y, \tilde{X})$ and $\mathcal{L}_c^1(Y, X)^\sim$ coincide.

In the following proposition we study the relation between the modules $\mathcal{L}_c^1(Y, X)$ and the reduction of dimension modules $\mathcal{L}_c^n(\mathcal{A}, X)$.

**Proposition 3.1.8.** Let $\mathcal{A}$ be a Banach algebra, $X$ be a Banach $\mathcal{A}$-bimodule and $Y$ be a Banach space. Then $\mathcal{L}_c^n(\mathcal{A}, \mathcal{L}_c^1(Y, X))$ and $\mathcal{L}_c^1(Y, \mathcal{L}_c^n(\mathcal{A}, X))$ are isometrically $\mathcal{A}$-module isomorphic.

**Proof.** Let

$$J : \mathcal{L}_c^n(\mathcal{A}, \mathcal{L}_c^1(Y, X)) \to \mathcal{L}_c^1(Y, \mathcal{L}_c^n(\mathcal{A}, X))$$

be the canonical isometric isomorphism defined by

$$J(\Phi)(y)(a_1, \ldots, a_n) = \Phi(a_1, \ldots, a_n)(y)$$

for all $\Phi \in \mathcal{L}_c^n(\mathcal{A}, \mathcal{L}_c^1(Y, X))$, all $y \in Y$ and all $a_1, \ldots, a_n \in \mathcal{A}$. It is easy to see that $J$ is a left $\mathcal{A}$-module homomorphism. To show that it is a right $\mathcal{A}$-module homomorphism consider $\Phi \in \mathcal{L}_c^n(\mathcal{A}, \mathcal{L}_c^1(Y, X))$, $a \in \mathcal{A}$, $y \in Y$ and $a_1, \ldots, a_n \in \mathcal{A}$. Then

$$J(\Phi a)(y)(a_1, \ldots, a_n) = (\Phi a)(a_1, \ldots, a_n)(y)$$

$$= \Phi(aa_1, \ldots, a_n)(y)$$

$$+ \sum_{1 \leq k \leq n-1} (-1)^k \Phi(a, a_1, \ldots, akak+1, \ldots, a_n)(y)$$

$$+ (-1)^n \Phi(a, a_1, \ldots, an-1)(y)a_n$$

$$= J(\Phi)(y)(aa_1, \ldots, a_n)$$

$$+ \sum_{1 \leq k \leq n-1} (-1)^k J(\Phi)(y)(a, a_1, \ldots, akak+1, \ldots, a_n)$$

$$+ (-1)^n J(\Phi)(y)(a, a_1, \ldots, an-1)a_n$$

$$= (J(\Phi)(y)a)(a_1, \ldots, a_n)$$

$$= (J(\Phi)a)(y)(a_1, \ldots, a_n)$$

$$\square$$

In the following proposition we show how the cohomology groups of $\mathcal{A}$, with coefficients in $X$ and $\mathcal{L}_c^1(Y, X)$, are related.

**Proposition 3.1.9.** Let $\mathcal{A}$ be a Banach algebra, $\mathcal{B}$ be a subalgebra of $\mathcal{A}$, $X$ be a Banach $\mathcal{A}$-bimodule, $Y$ be a Banach space and $n \in \mathbb{N}$. Then the following hold:
(i) For all non-zero \( f \in Y^* \), the map
\[
F \mapsto F_1 : \mathcal{L}_c^n(A, X) \rightarrow \mathcal{L}_c^n(A, \mathcal{L}_c^1(Y, X))
\]
defined by \( F_1(a_1, \ldots, a_n)(y) = f(y)F(a_1, \ldots, a_n) \), for all \( F \in \mathcal{L}_c^n(A, X) \), all \( a_1, \ldots, a_n \in A \) and all \( y \in Y \) is a one-to-one bounded map, with \( \|F_1\| = \|f\| \|F\| \). Moreover
\[
F(tF + t'F') = tF_1 + t'F'_1, \quad \text{for all } F \in \mathcal{L}_c^n(A, X), \text{ all } f, f' \in Y^* \text{ and all } t, t' \in \mathbb{C}.
\]

(ii) For all non-zero \( y \in Y \), the map
\[
\Phi \mapsto \Phi_y : \mathcal{L}_c^n(A, \mathcal{L}_c^1(Y, X)) \rightarrow \mathcal{L}_c^n(A, X)
\]
defined by \( \Phi_y(a_1, \ldots, a_n) = \Phi(a_1, \ldots, a_n)(y) \), for all \( \Phi \in \mathcal{L}_c^n(A, \mathcal{L}_c^1(Y, X)) \) and all \( a_1, \ldots, a_n \in A \) is a bounded map, with \( \|\Phi \mapsto \Phi_y\| = \|y\| \), which maps \( \mathcal{L}_c^n(A, \mathcal{L}_c^1(Y, X)) \) onto \( \mathcal{L}_c^n(A, X) \). Moreover \( \Phi_{ty + ty'} = t\Phi_y + t'\Phi_{y'} \), for all \( \Phi \in \mathcal{L}_c^n(A, \mathcal{L}_c^1(Y, X)) \), all \( y, y' \in Y \) and all \( t, t' \in \mathbb{C} \).

(iii) For all non-zero \( f \in Y^* \), \( F \mapsto F_1 \) maps \( \mathcal{L}_c^n(A, X : /B) \) into \( \mathcal{L}_c^n(A, \mathcal{L}_c^1(Y, X) : /B) \).

(iv) For all non-zero \( y \in Y \), \( \Phi \mapsto \Phi_y \) maps \( \mathcal{L}_c^n(A, \mathcal{L}_c^1(Y, X) : /B) \) onto \( \mathcal{L}_c^n(A, X : /B) \).

(v) If
\[
\partial^n : \mathcal{L}_c^n(A, X) \rightarrow \mathcal{L}_c^{n+1}(A, X)
\]
and
\[
\Delta^n : \mathcal{L}_c^n(A, \mathcal{L}_c^1(Y, X)) \rightarrow \mathcal{L}_c^{n+1}(A, \mathcal{L}_c^1(Y, X))
\]
are the coboundary maps, then, for all \( f \in Y^* \) and all \( F \in \mathcal{L}_c^n(A, X) \),
\[
\Delta^n(F_f) = (\partial^n(F))_f
\]
and, for all \( y \in Y \) and all \( \Phi \in \mathcal{L}_c^n(A, \mathcal{L}_c^1(Y, X)) \),
\[
\partial^n(\Phi_y) = (\Delta^n(\Phi))_y
\]

(vi) For all non-zero \( f \in Y^* \), \( F \mapsto F_1 \) maps \( \mathcal{Z}_c^n(A, X) \) into \( \mathcal{Z}_c^n(A, \mathcal{L}_c^1(Y, X)) \), \( \mathcal{B}_c^n(A, X) \) into \( \mathcal{B}_c^n(A, \mathcal{L}_c^1(Y, X)) \) and \( \mathcal{H}_c^n(A, X) \) into \( \mathcal{H}_c^n(A, \mathcal{L}_c^1(Y, X)) \).

(vii) For all non-zero \( y \in Y \), \( \Phi \mapsto \Phi_y \) maps \( \mathcal{Z}_c^n(A, \mathcal{L}_c^1(Y, X)) \) onto \( \mathcal{Z}_c^n(A, X) \), \( \mathcal{B}_c^n(A, \mathcal{L}_c^1(Y, X)) \) onto \( \mathcal{B}_c^n(A, X) \) and \( \mathcal{H}_c^n(A, \mathcal{L}_c^1(Y, X)) \) onto \( \mathcal{H}_c^n(A, X) \).

(viii) For all non-zero \( f \in Y^* \), \( F \mapsto F_1 \) maps \( \mathcal{Z}_c^n(A, X : /B) \) into \( \mathcal{Z}_c^n(A, \mathcal{L}_c^1(Y, X) : /B) \), \( \mathcal{B}_c^n(A, X : /B) \) into \( \mathcal{B}_c^n(A, \mathcal{L}_c^1(Y, X) : /B) \) and \( \mathcal{H}_c^n(A, X : /B) \) into \( \mathcal{H}_c^n(A, \mathcal{L}_c^1(Y, X) : /B) \).

(ix) For all non-zero \( y \in Y \), \( \Phi \mapsto \Phi_y \) maps \( \mathcal{Z}_c^n(A, \mathcal{L}_c^1(Y, X) : /B) \) onto \( \mathcal{Z}_c^n(A, X : /B) \), \( \mathcal{B}_c^n(A, \mathcal{L}_c^1(Y, X) : /B) \) onto \( \mathcal{B}_c^n(A, X : /B) \) and \( \mathcal{H}_c^n(A, \mathcal{L}_c^1(Y, X) : /B) \) onto \( \mathcal{H}_c^n(A, X : /B) \).
Proof. Parts (i) and (ii) follow from a straightforward calculation and the Hahn-Banach theorem.

A straightforward calculation shows that if $A$ is an algebra, $X_1$ and $X_2$ are $A$-bimodules and $I : X_1 \to X_2$ is an $A$-module homomorphism, then, for all $n \in \mathbb{N}$, the map

$$I_n : \mathcal{L}^n(A, X_1) \to \mathcal{L}^n(A, X_2)$$

defined by

$$I_n(\phi)(a_1, \ldots, a_n) = I(\phi(a_1, \ldots, a_n))$$

for all $\phi \in \mathcal{L}^n(A, X_1)$ and all $a_1, \ldots, a_n \in A$, maps $Z^n(A, X_1)$, $B^n(A, X_1)$ and $H^n(A, X_1)$ into $Z^n(A, X_2)$, $B^n(A, X_2)$ and $H^n(A, X_2)$ respectively and $\mathcal{L}^n(A, X_1 : /B)$, $Z^n(A, X_1 : /B)$, $B^n(A, X_1 : /B)$ and $H^n(A, X_1 : /B)$ into $\mathcal{L}^n(A, X_2 : /B)$, $Z^n(A, X_2 : /B)$, $B^n(A, X_2 : /B)$ and $H^n(A, X_2 : /B)$ respectively, for any subalgebra $B$ of $A$. Parts (iii), (vi) and (viii) follow from the previous observation with $X_1 = X, X_2 = \mathcal{L}^1_c(Y, X)$ and $I = x \mapsto x_f$ and parts (iv), (vii) and (ix) follow with $X_1 = \mathcal{L}^1_c(Y, X), X_2 = X$ and $I = \phi \mapsto \phi(y)$. The onto in (iv), (vii) and (ix) follows from (iii), (vi) and (viii) respectively and the Hahn-Banach theorem.

In particular the previous proposition implies that if $H^n_c(A, \mathcal{L}^1_c(Y, X))$ or $H^n_c(A, \mathcal{L}^1_c(Y, X) : /B)$ vanishes for some Banach space $Y$, then so does $H^n_c(A, X)$ or $H^n_c(A, X : /B)$ (the previous result follows immediately from the complementation of $X$ in $\mathcal{L}^1_c(Y, X)$ established in the remarks following Proposition 3.1.1; we discuss the map $\Phi \mapsto \Phi_y$ because it will be used in Chapter 4). The converse does not hold (see Section 5.3 for an example of a Banach algebra $A$, a Banach $A$-bimodule $X$ and a Banach space $Y$ with $H^n_c(A, X) = \{0\}$ and $H^n_c(A, \mathcal{L}^1_c(Y, X)) \neq \{0\}$). As we shall see in Chapter 4, the vanishing of the groups $H^n_c(A, \mathcal{L}^1_c(Y, X))$ or $H^n_c(A, \mathcal{L}^1_c(Y, X) : /B)$, for all Banach spaces $Y$, implies more than just the vanishing of the groups $H^n_c(A, X)$ and $H^n_c(A, X : /B)$.

In the following proposition we give the relation between the multiplication on $\mathcal{L}^1_c(X)$ and the module actions of $A$ on $\mathcal{L}^1_c(X)$.

**Proposition 3.1.10.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. Then the following hold:

(i) For all $a \in A$ and all $\phi_1, \phi_2 \in \mathcal{L}^1_c(X)$,

$$a(\phi_1 \phi_2) = (a\phi_1)\phi_2$$

and

$$(\phi_1 a)\phi_2 = (\phi_1 \phi_2)a.$$
(ii) If $\phi \in L^1_c(X)$, then $\phi$ is an $A$-module homomorphism if and only if

$$\phi(a\psi) = a(\phi \psi)$$

and

$$\phi(\psi a) = (\phi \psi)a$$

for all $a \in A$ and all $\psi \in L^1_c(X)$.

Proof. (i) follows immediately from the definition of the module actions of $A$ on $L^1_c(X)$. For the first part of (ii) let $\phi$ be a left $A$-module homomorphism, $a \in A$, $\psi \in L^1_c(X)$ and $x \in X$. Then

$$(\phi(a\psi))(x) = \phi(a\psi(x)) = a\phi(\psi(x)) = (a(\phi \psi))(x).$$

On the other hand, if $\phi(a\psi) = a(\phi \psi)$, for all $a \in A$ and all $\psi \in L^1_c(X)$, take $\psi = id_X$. Then, for all $x \in X$,

$$\phi(ax) = \phi((a id_X)(x)) = (\phi(a id_X))(x) = a(\phi id_X)(x) = a\phi(x)$$

and so $\phi$ is a left $A$-module homomorphism. \qed

We should mention here that in the situation described in the previous proposition the left and the right module action of $A$ on $L^1_c(X)$ coincide with composition with the left and the right multiplication maps $L_a$ and $R_a$ respectively.

A question that arises if $Y$ is also an $A$-bimodule and $B$ is a subalgebra of $A$ is whether $L^1_c(Y, X : B)$ is also an $A$-bimodule. In the following proposition we show that this happens if $B$ is contained in the centre of $A$.

Proposition 3.1.11. Let $A$ be a Banach algebra, $B$ be a subalgebra of $A$ and $X$ and $Y$ be Banach $A$-bimodules. Then the following hold:

(i) For all $b \in B$, all $\phi \in L^1_c(Y, X : B)$ and all $y \in Y$,

$$(b\phi)(y) = \phi(by)$$

and

$$(\phi b)(y) = \phi(yb).$$

(ii) If $B$ is contained in the centre $Z(A)$ of $A$, then $L^1_c(Y, X : B)$ is a Banach $A$-bimodule.
Proof. The first part follows from the pointwise definition of the module actions of $\mathcal{A}$ on $\mathcal{L}_c^1(Y, X)$. To prove the second part let $\phi \in \mathcal{L}_c^1(Y, X : /\mathcal{B})$, $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $y \in Y$. Then

$$(a\phi)(by) = ab\phi(y) = b(a\phi)(y)$$

since $\mathcal{B} \subseteq \mathcal{Z}(\mathcal{A})$. Moreover it is easy to see that $(a\phi)(yb) = (a\phi)(y)b$. Hence $a\phi \in \mathcal{L}_c^1(Y, X : /\mathcal{B})$. □

Remark 3.1.1. As we mentioned after Proposition 3.1.1 we can also give $\mathcal{L}_c^n(Y, X)$ an $\mathcal{A}$-bimodule structure for $n > 1$. So $\mathcal{L}_c^n(\mathcal{A}, X)$ is an $\mathcal{A}$-bimodule for all $n \geq 1$.

Is the coboundary map $\partial^n : \mathcal{L}_c^n(\mathcal{A}, X) \to \mathcal{L}_c^{n+1}(\mathcal{A}, X)$ an $\mathcal{A}$-module homomorphism? A straightforward calculation shows that

$$\partial^n(a\phi)(a_1, \ldots, a_{n+1}) = (a\partial^n(\phi))(a_1, \ldots, a_{n+1}) + ((a_1a - aa_1)\phi)(a_2, \ldots, a_{n+1})$$

and

$$\partial^n(a\phi)(a_1, \ldots, a_{n+1}) = (\partial^n(\phi)a)(a_1, \ldots, a_{n+1}) + (-1)^{n+1}(\phi(aa_{n+1} - a_{n+1}a))(a_1, \ldots, a_n)$$

for all $\phi \in \mathcal{L}_c^n(\mathcal{A}, X)$ and all $a, a_1, \ldots, a_{n+1} \in \mathcal{A}$ and thus the answer in general is no. But if $\mathcal{A}$ is abelian, then $\partial^n$ is an $\mathcal{A}$-module homomorphism and $\mathcal{L}_c^n(\mathcal{A}, X)$ and $\mathcal{B}_c^n(\mathcal{A}, X)$ are $\mathcal{A}$-bimodules.

3.1.2 The modules $\mathcal{L}_{cb}^1(Y, X)$

As we mentioned in Remark 1.2.7(ii), if $X$ is a matricially normed space and $Y$ is a normed space, then $\mathcal{L}_c^1(Y, X)$ becomes a matricially normed space, with matrix norms defined by

$$\|(\phi_{ij})\|_m = \sup\{\|(\phi_{ij}(y))\|_m \mid \|y\| \leq 1\}$$

for all $m \in \mathbb{N}$ and all $(\phi_{ij}) \in \mathbb{M}_m(\mathcal{L}_c^1(Y, X))$. Moreover if $Y$ is also a matricially normed space, then the standard matricial norm structure on $\mathcal{L}_{cb}^1(Y, X)$ is defined by

$$\|(\phi_{ij})\|_m = \sup\{\|(\phi_{ij}(y_{st}))\|_{mt} \mid \|(y_{st})\|_t \leq 1, t \in \mathbb{N}\}$$

(3.1)

for all $m \in \mathbb{N}$ and all $(\phi_{ij}) \in \mathbb{M}_m(\mathcal{L}_{cb}^1(Y, X))$ (Proposition 1.2.5).

Using those two matricial norm structures, we can make $\mathcal{L}_c^1(Y, X)$ and $\mathcal{L}_{cb}^1(Y, X)$ into completely bounded $\mathcal{A}$-modules, when $X$ is a completely bounded $\mathcal{A}$-module. In their work on completely bounded $\mathcal{A}$-modules Effros and Ruan mention those modules when $X$ is a $C^*$-algebra $\mathcal{A}$; they call them range modules and use them.
as examples of completely bounded $\mathcal{A}$-modules ([ER1], p.140). In this part we will define the modules $\mathcal{L}^1_{\text{cb}}(Y, X)$ and discuss some of their properties. We will not show again that the natural embeddings and isomorphisms that appear in the proofs are $\mathcal{A}$-module homomorphisms, since the calculations are exactly the same with those in the first part. The order of the presentation of the results is the same with the first part. Similar results hold for the matricially normed space $\mathcal{L}^1_{\text{cb}}(Y, X)$, when $X$ is a completely bounded $\mathcal{A}$-bimodule and $Y$ is a normed space.

**Proposition 3.1.12.** Let $\mathcal{A}$ be an operator algebra, $X$ be a completely bounded $\mathcal{A}$-bimodule and $Y$ be a matricially normed space. Then the space of completely bounded linear maps from $Y$ into $X$, $\mathcal{L}^1_{\text{cb}}(Y, X)$, becomes a completely bounded $\mathcal{A}$-bimodule, with the module actions of $\mathcal{A}$ on $\mathcal{L}^1_{\text{cb}}(Y, X)$ defined by

$$(a \phi)(y) = a \phi(y)$$

and

$$(\phi a)(y) = \phi(y)a$$

for all $a \in \mathcal{A}$, all $\phi \in \mathcal{L}^1_{\text{cb}}(Y, X)$ and all $y \in Y$. Moreover if $X$ is an operator completely bounded $\mathcal{A}$-bimodule, then $\mathcal{L}^1_{\text{cb}}(Y, X)$ is also an operator completely bounded $\mathcal{A}$-bimodule.

**Proof.** Since $X$ is a completely bounded $\mathcal{A}$-bimodule, it is a matricially normed space. Hence $\mathcal{L}^1_{\text{cb}}(Y, X)$ becomes a matricially normed space, with the matricial norms defined as in (3.1). Moreover, since $X$ is a completely bounded $\mathcal{A}$-bimodule, the map $(a, x) \mapsto ax : \mathcal{A} \times X \to X$ is completely bounded, i.e. there exists $K > 0$ such that

$$\| (a_{ij})(x_{ij}) \|_m \leq K \| (a_{ij}) \|_m \| (x_{ij}) \|_m$$  \hspace{1cm} (3.2)

for all $m \in \mathbb{N}$, all $(x_{ij}) \in \mathbb{M}_m(X)$ and all $(a_{ij}) \in \mathbb{M}_m(\mathcal{A})$ (see Definition 1.2.11). Using Proposition 3.1.1 we get that $\mathcal{L}^1_{\text{cb}}(Y, X)$ is an $\mathcal{A}$-bimodule. If we consider $\phi \in \mathcal{L}^1_{\text{cb}}(Y, X)$, $a \in \mathcal{A}$, $m \in \mathbb{N}$ and $(y_{ij}) \in \mathbb{M}_m(Y)$, we have

$$\| ((a \phi)(y_{ij})) \|_m = \| (a \phi)(y_{ij}) \|_m$$

$$= \| (a \otimes I_m)(\phi(y_{ij})) \|_m$$

$$\leq K \| (a \otimes I_m) \|_m \| (\phi(y_{ij})) \|_m$$

$$= K \| a \| \| (\phi(y_{ij})) \|_m$$

$$\leq K \| a \| \| \phi \|_{\text{cb}} \| (y_{ij}) \|_m$$

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the third step following from (3.2), the fourth from the $L^\infty$ property of $\mathcal{A}$ and the fifth from the complete boundedness of $\phi$. Thus $a\phi$ is completely bounded, which proves that $\mathcal{L}_cb^1(Y, X)$ is an $\mathcal{A}$-submodule of $\mathcal{L}_c^1(Y, X)$ and therefore an $\mathcal{A}$-bimodule. Now to prove that $\mathcal{L}_cb^l(Y, X)$ is a completely bounded $\mathcal{A}$-bimodule consider $m \in \mathbb{N}$, $(\phi_{ij}) \in \mathbb{M}_m(\mathcal{L}_cb^1(Y, X))$ and $(a_{ij}) \in \mathbb{M}_m(\mathcal{A})$. If $l \in \mathbb{N}$ and $(y_{st}) \in \mathbb{M}_l(Y)$, then

$$\|((\sum_{1 \leq k \leq n} (a_{ik}\phi_{kj})(y_{st}))\|_{ml} = \|((\sum_{1 \leq k \leq n} a_{ik}\phi_{kj}(y_{st}))\|_{ml}$$

$$= \|((a_{ij}) \otimes I_l)(\phi_{ij}(y_{st}))\|_{ml}$$

$$\leq K\|(a_{ij}) \otimes I_l\|\|\phi_{ij}(y_{st})\|_{ml}$$

$$= K\|(a_{ij})\|_m\|\phi_{ij}(y_{st})\|_{ml}$$

$$\leq K\|(a_{ij})\|_m\|\phi_{ij}\|_m\|y_{st}\|_{l}$$

the third step following from (3.2), the fourth from the $L^\infty$ property of $\mathcal{A}$ and the fifth from (3.1). Since the previous inequality holds for all $l \in \mathbb{N}$ and all $(y_{st}) \in \mathbb{M}_l(Y)$, we get, using (3.1), that

$$\|((\sum_{1 \leq k \leq n} a_{ik}\phi_{kj})\|_m \leq K\|(a_{ij})\|_m\|\phi_{ij}\|_m$$

which proves that $\mathcal{L}_cb^l(Y, X)$ is a completely bounded $\mathcal{A}$-bimodule. Since the constant $K$ has not changed, if $X$ is completely contractive, then $\mathcal{L}_cb^1(Y, X)$ is also completely contractive.

To finish the proof suppose that $X$ is an operator completely bounded $\mathcal{A}$-bimodule. Then $X$ is an operator space and hence, by Proposition 1.2.5, $\mathcal{L}_cb^1(Y, X)$ is also an operator space. $\square$

The simplest case arises again when $X$ is an operator completely bounded $\mathcal{A}$-bimodule and $Y = \mathbb{C}$, where we have that $\mathcal{L}_cb^1(Y, X) = X$.

As in the bounded case, we can consider $\mathcal{A}$-bimodules of completely bounded $n$-linear maps, for $n > 1$, from matricially normed spaces $Y_1, \ldots, Y_n$ into a completely bounded $\mathcal{A}$-bimodule $X$, $\mathcal{L}_cb^n(Y_1, \ldots, Y_n; X)$. We remind the reader that $\mathcal{L}_cb^n(Y_1, \ldots, Y_n; X)$ is not isomorphic to $\mathcal{L}_cb^1(Y_1, \mathcal{L}_cb^{n-1}(Y_2, \ldots, Y_n; X))$ (see Remark 1.2.11(iv)).

In the first part we established the relation between the modules $X$ and $\mathcal{L}_c^1(Y, X)$ using the maps $x \mapsto x_f$ and $\phi \mapsto \phi(y)$. Those maps work in the completely bounded case as long as $X$ is an operator completely bounded $\mathcal{A}$-bimodule. Let $f$ be a non-zero element of $Y^*$. To see that the map $x \mapsto x_f$ is
well-defined consider $x \in X$, $f \in Y^*$, $m \in \mathbb{N}$ and $(y_{ij}) \in \mathbb{M}_m(Y)$. Then
\[
\| (x_f(y_{ij})) \|_m = \| (f(y_{ij})x) \|_m \\
\leq \| (f(y_{ij})) \|_m \| x \otimes I_m \|_m \\
= \| (f(y_{ij})) \|_m \| x \| \\
\leq \| f \| \| (y_{ij}) \|_m \| x \|
\]
where the third step follows from the $L^\infty$ property of $X$ and the fourth from the complete boundedness of $f$. Moreover if $m, l \in \mathbb{N}$, $(x_{ij}) \in \mathbb{M}_m(X)$ and $(y_{st}) \in \mathbb{M}_l(Y)$, then
\[
\| ((x_{ij})_f(y_{st})) \|_{ml} = \| (f(y_{st})x_{ij}) \|_{ml} \\
\leq \| (f(y_{st})) \otimes I_m \| \| (x_{ij} \otimes I_l) \|_{ml} \\
= \| (f(y_{st})) \| \| (x_{ij} \otimes I_l) \|_{ml} \\
= \| (f(y_{st})) \| \| (x_{ij}) \|_m \\
\leq \| f \| \| (y_{st}) \|_l \| (x_{ij}) \|_m
\]
the fourth step following from $(x_{ij}) \otimes I_l$ being equal to a product of the form $(\alpha_{pq}) (x_{ij} \otimes I_l) (\beta_{pq})$ where $(\alpha_{pq}), (\beta_{pq}) \in \mathbb{M}_{ml}$ are permutation matrices. Since the previous inequality holds for all $l \in \mathbb{N}$ and all $(y_{st}) \in \mathbb{M}_l(Y)$, we get, using the definition of the norm on $\mathbb{M}_m(L^1_b(Y, X))$, that
\[
\| ((x_{ij})_f) \|_m \leq \| f \| \| (x_{ij}) \|_m
\]
which proves that $x \mapsto x_f$ is completely bounded. By the remarks following Proposition 3.1.1,
\[
\| f \| \| (x_{ij}) \|_m = \| (x_{ij})_f \|
\]
where $(x_{ij})_f$ is viewed as a bounded map from $Y$ into $\mathbb{M}_m(X)$. It is easy to see that $(x_{ij})_f$ as a completely bounded map from $Y$ into $\mathbb{M}_m(X)$ coincides with $((x_{ij})_f)$ as an element of $\mathbb{M}_m(L^1_b(Y, X))$ (see the discussion before Proposition 1.2.5) and hence
\[
\| (x_{ij})_f \|_{cb} = \| ((x_{ij})_f) \|_m.
\]
Using (3.3), (3.4) and (3.5) we get that $X$ is completely (isometrically if $\| f \| = 1$) isomorphic to $\{ x_f \mid x \in X \}$. On the other hand a similar calculation shows that
\[
\| (x_{f_{ij}}) \|_m \leq \| x \| \| (f_{ij}) \|_m
\]
for all $x \in X$, all $m \in \mathbb{N}$ and all $(f_{ij}) \in \mathbb{M}_m(Y^*)$. 

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Passing to the map \( \phi \mapsto \phi(y) \) let us consider \( y \in Y \) and \( (\phi_{ij}) \in M_m(\mathcal{L}_{cb}^1(Y, X)) \). Then it follows immediately from the definition of the norm on \( M_m(\mathcal{L}_{cb}^1(Y, X)) \) that \( \| (\phi_{ij}(y)) \|_m \leq \| (\phi_{ij}) \|_m \| y \|_m \), which proves that \( \phi \mapsto \phi(y) \) is completely bounded, with \( \| \phi \mapsto \phi(y) \|_{cb} \leq \| y \|_m \). Combining that with the results in the bounded case we can easily see that \( \| \phi \mapsto \phi(y) \|_{cb} = \| y \|_m \).

As in the first part combining those two maps we get a completely bounded projection of \( \mathcal{L}_{cb}^1(Y, X) \) onto \( \{ x_f \mid x \in X \} \), which shows that \( \{ x_f \mid x \in X \} \) is a completely complemented \( \mathcal{A} \)-submodule of \( \mathcal{L}_{cb}^1(Y, X) \).

It is easy to see that the relation between certain algebraic properties of \( X \) and \( L_c(Y, X) \) that we stated in p.76 holds for \( L_{cb}^1(Y, X) \) as well.

As in the bounded case, duality and normality of \( X \) are inherited by \( L_{cb}^1(Y, X) \) when \( X \) is an operator completely bounded \( \mathcal{A} \)-bimodule and \( Y \) is an operator space, as we shall see in the two following propositions.

**Proposition 3.1.13.** Let \( \mathcal{A} \) be an operator algebra, \( X \) be a dual operator completely bounded \( \mathcal{A} \)-bimodule and \( Y \) be an operator space. Then \( \mathcal{L}_{cb}^1(Y, X) \) is a dual operator completely bounded \( \mathcal{A} \)-bimodule.

**Proof.** Since \( X \) is a dual operator completely bounded \( \mathcal{A} \)-bimodule, there exists an \( L^1 \) completely bounded \( \mathcal{A} \)-bimodule \( X_* \) such that \( X = (X_*)_r \). By Proposition 1.2.8, \( Y \otimes_{rt} X_* \) is an \( L^1 \) matricially normed space, with \( (Y \otimes_{rt} X_*)_r \cong \mathcal{L}_{cb}^1(Y, X) \). Moreover it is easy to see that \( Y \otimes_{rt} X_* \) becomes a completely bounded \( \mathcal{A} \)-bimodule, with the module actions of \( \mathcal{A} \) on \( Y \otimes_{rt} X_* \) defined by

\[
a(y \otimes z) = y \otimes (az)
\]

and

\[
(y \otimes z)a = y \otimes (za)
\]

for all \( a \in \mathcal{A} \) and all \( y \otimes z \in Y \otimes_{rt} X_* \). Therefore \( \mathcal{L}_{cb}^1(Y, X) \) is a dual operator completely bounded \( \mathcal{A} \)-bimodule. \( \square \)

**Proposition 3.1.14.** Let \( \mathcal{A} \) be an operator algebra, \( X \) be a normal operator completely bounded \( \mathcal{A} \)-bimodule and \( Y \) be an operator space. Then \( \mathcal{L}_{cb}^1(Y, X) \) is a normal operator completely bounded \( \mathcal{A} \)-bimodule.

**Proof.** It follows from Proposition 3.1.13, that \( \mathcal{L}_{cb}^1(Y, X) \) is a dual operator completely bounded \( \mathcal{A} \)-bimodule. The ultraweak-weak* continuity of the maps \( a \mapsto a\phi : \mathcal{A} \rightarrow \mathcal{L}_{cb}^1(Y, X) \) and \( a \mapsto \phi a : \mathcal{A} \rightarrow \mathcal{L}_{cb}^1(Y, X) \) can be proved as in Proposition 3.1.3. \( \square \)
In the following four propositions we discuss the situation with respect to module homomorphisms, submodules, direct sums and quotients.

**Proposition 3.1.15.** Let \( A \) be an operator algebra, \( X_1 \) and \( X_2 \) be completely bounded \( A \)-bimodules and \( Y_1 \) and \( Y_2 \) be matricially normed spaces.

(i) If \( \pi : X_1 \to X_2 \) is a completely bounded \( A \)-module homomorphism, then 
\[
\pi_{Y_1} : \mathcal{L}_{cb}^1(Y_1, X_1) \to \mathcal{L}_{cb}^1(Y_1, X_2)
\]
defined by \( \pi_{Y_1}(\phi) = \pi \phi \), for all \( \phi \in \mathcal{L}_{cb}^1(Y_1, X_1) \), is a completely bounded \( A \)-module homomorphism, with \( \|\pi_{Y_1}\|_{cb} \leq \|\pi\|_{cb} \).

(ii) If \( \tau : Y_1 \to Y_2 \) is a completely bounded linear map, then 
\[
\tau_{X_1} : \mathcal{L}_{cb}^1(Y_2, X_1) \to \mathcal{L}_{cb}^1(Y_1, X_1)
\]
defined by \( \tau_{X_1}(\phi) = \phi \tau \), for all \( \phi \in \mathcal{L}_{cb}^1(Y_2, X_1) \), is a completely bounded \( A \)-module homomorphism, with \( \|\tau_{X_1}\|_{cb} \leq \|\tau\|_{cb} \).

**Proof.** (i) To prove that \( \pi_{Y_1} \) is completely bounded let us consider \( m \in \mathbb{N} \) and \( (\phi_{ij}) \in \mathcal{M}_m(\mathcal{L}_{cb}^1(Y_1, X_1)) \). If \( l \in \mathbb{N} \) and \( (y_{st}) \in \mathcal{M}_l(Y_1) \), then 
\[
\|(\pi_{Y_1}(\phi_{ij})(y_{st}))\|_{ml} = \|(\pi(\phi_{ij}(y_{st})))\|_{ml} \\
\leq \|\pi\|_{cb} \|(\phi_{ij}(y_{st}))\|_{ml} \\
\leq \|\pi\|_{cb} \|(\phi_{ij})\|_m \|(y_{st})\|_l
\]
the second step following from the complete boundedness of \( \pi \) and the third from the definition of the norm on \( \mathcal{M}_m(\mathcal{L}_{cb}^1(Y_1, X_1)) \). Using the definition of the norm on \( \mathcal{M}_m(\mathcal{L}_{cb}^1(Y_1, X_2)) \), we get that 
\[
\|(\pi_{Y_1}(\phi_{ij}))\|_m \leq \|\pi\|_{cb} \|(\phi_{ij})\|_m
\]
which proves that \( \pi_{Y_1} \) is completely bounded, with \( \|\pi_{Y_1}\|_{cb} \leq \|\pi\|_{cb} \).

(ii) To prove that \( \tau_{X_1} \) is completely bounded consider \( m \in \mathbb{N} \) and \( (\phi_{ij}) \in \mathcal{M}_m(\mathcal{L}_{cb}^1(Y_2, X_1)) \). Taking \( l \in \mathbb{N} \) and \( (y_{st}) \in \mathcal{M}_l(Y_1) \), we get 
\[
\|(\tau_{X_1}(\phi_{ij}(y_{st})))\|_{ml} = \|(\phi_{ij}(\tau(y_{st})))\|_{ml} \\
\leq \|\phi_{ij}\|_m \|\tau(y_{st})\|_l \\
\leq \|\phi_{ij}\|_m \|\tau\|_{cb} \|y_{st}\|_l
\]
the second step following from the definition of the norm on \( \mathcal{M}_m(\mathcal{L}_{cb}^1(Y_2, X_1)) \) and the third from the complete boundedness of \( \tau \). Now taking on both sides of the previous inequality the supremum over \( l \in \mathbb{N} \) and \( (y_{st}) \in \mathcal{M}_l(Y_1) \), with \( \|(y_{st})\|_l \leq 1 \), we get 
\[
\|(\tau_{X_1}(\phi_{ij}))\|_m \leq \|\tau\|_{cb} \|(\phi_{ij})\|_m
\]
which proves that \( \tau_{X_1} \) is completely bounded, with \( \|\tau_{X_1}\|_{cb} \leq \|\tau\|_{cb} \).
Proposition 3.1.16. Let $A$ be an operator algebra, $X$ be a completely bounded $A$-bimodule and $Y$ be a matricially normed space.

(i) If $X_1$ is a closed $A$-submodule of $X$, then $L^{1}_{cb}(Y, X_1)$ is a closed $A$-submodule of $L^{1}_{cb}(Y, X)$. Moreover if $X_1$ is a completely complemented $A$-submodule of $X$, then $L^{1}_{cb}(Y, X_1)$ is a completely complemented $A$-submodule of $L^{1}_{cb}(Y, X)$.

(ii) If $Y_1$ is a completely complemented subspace of $Y$, then $L^{1}_{cb}(Y, X_1)$ is a completely complemented $A$-submodule of $L^{1}_{cb}(Y, X)$.

(iii) If $X_1$ is a closed $A$-submodule of $X$ and $Y_1$ is a completely complemented subspace of $Y$, then $L^{1}_{cb}(Y_1, X_1)$ is a closed $A$-submodule of $L^{1}_{cb}(Y, X)$. Moreover if $X_1$ is a completely complemented $A$-submodule of $X$, then $L^{1}_{cb}(Y_1, X_1)$ is a completely complemented $A$-submodule of $L^{1}_{cb}(Y, X)$.

Proof. (i) The first part is obvious and the second part can be proved, using Proposition 3.1.15, in a manner similar to the proof of Proposition 3.1.5(ii).

(ii) It is similar to the proof of Proposition 3.1.5(ii) (using part (ii) of the previous proposition in the place of Proposition 3.1.4(ii)).

Proposition 3.1.17. If $A$ is an operator algebra then the following hold:

(i) If $\{X_{\lambda} | \lambda \in \Lambda\}$ is a family of uniformly bounded completely bounded $A$-bimodules and $Y$ is a matricially normed space, then:

(a) $l^{\infty}(L^{1}_{cb}(Y, X_{\lambda}) | \lambda \in \Lambda)$ is completely isometrically $A$-module isomorphic to $L^{1}_{cb}(Y, l^{\infty}(X_{\lambda} | \lambda \in \Lambda))$.

(b) $l^{1}(L^{1}_{cb}(Y, X_{\lambda}) | \lambda \in \Lambda)$ can be completely contractively embedded as an $A$-submodule into $L^{1}_{cb}(Y, l^{1}(X_{\lambda} | \lambda \in \Lambda))$.

(ii) If $X$ is a completely bounded $A$-bimodule and $\{Y_{\lambda} | \lambda \in \Lambda\}$ is a family of matricially normed spaces, then:

(a) $l^{\infty}(L^{1}_{cb}(Y_{\lambda}, X) | \lambda \in \Lambda)$ can be completely contractively embedded as an $A$-submodule into $L^{1}_{cb}(l^{1}(Y_{\lambda} | \lambda \in \Lambda), X)$.

(b) $l^{1}(L^{1}_{cb}(Y_{\lambda}, X) | \lambda \in \Lambda)$ can be completely contractively embedded as an $A$-submodule into $L^{1}_{cb}(l^{\infty}(Y_{\lambda} | \lambda \in \Lambda), X)$.

Similar results hold in the finite case.

Proof. We recall from Proposition 1.2.13 that if $\{V_{\lambda} | \lambda \in \Lambda\}$ is a family of matricially normed spaces, then $l^{\infty}(V_{\lambda} | \lambda \in \Lambda)$ becomes a matricially normed space with matrix norms $\{\| \cdot \|_n^{\infty}\}_{n \in \mathbb{N}}$ defined by

$$\| \sum_{\lambda \in \Lambda} \Theta v^{ij}_{\lambda} \|_{n}^{\infty} = \sup_{\lambda \in \Lambda} \| v^{ij}_{\lambda} \|_{n}$$

for all $(\sum_{\lambda \in \Lambda} \Theta v^{ij}_{\lambda}) \in M_{n}(l^{\infty}(V_{\lambda} | \lambda \in \Lambda))$ and all $n \in \mathbb{N}$, and $l^{1}(V_{\lambda} | \lambda \in \Lambda)$
becomes a matricially normed space with matrix norms \( \{ \| \cdot \|_n \}_{n \in \mathbb{N}} \) defined by

\[
\left\| \left( \sum_{\lambda \in \Lambda} \phi_{ij} \right) \right\|_n = \sup \{ \left\| (f_{pq} \left( \sum_{\lambda \in \Lambda} \phi_{ij} \right)) \right\| \mid (f_{pq}) \in \text{BALL}(\mathbb{M}_m(l^\infty(V^*_\lambda \mid \lambda \in \Lambda))), m \in \mathbb{N} \}
\]

for all \( \left( \sum_{\lambda \in \Lambda} \phi_{ij} \right) \in \mathbb{M}_n(l^1(V^*_\lambda \mid \lambda \in \Lambda)) \) and all \( n \in \mathbb{N} \), where \( \text{BALL}(\mathbb{M}_m(l^\infty(V^*_\lambda \mid \lambda \in \Lambda))) \) is the unit ball of \( \mathbb{M}_m(l^\infty(V^*_\lambda \mid \lambda \in \Lambda)) \).

(i) Let

\[
J : l^p(\mathcal{L}_{cb}^1(Y, X_\lambda) \mid \lambda \in \Lambda) \longrightarrow \mathcal{L}_{cb}^1(Y, l^p(X_\lambda \mid \lambda \in \Lambda))
\]

be defined, as in Proposition 3.1.6, by \( J(\sum_{\lambda \in \Lambda} \phi_{ij})(y) = \sum_{\lambda \in \Lambda} \phi_{ij}(y) \), for all \( \sum_{\lambda \in \Lambda} \phi_{ij} \in l^p(\mathcal{L}_{cb}^1(Y, X_\lambda) \mid \lambda \in \Lambda) \) and all \( y \in Y \), where \( p = 1, \infty \).

For \( p = \infty \) it follows immediately from the definition of the \( l^\infty \) matrix norms that \( J \) is a complete isometry.

To prove that, for \( p = 1 \), \( J \) is a complete contraction we will need the following observation: If \( V \) and \( U \) are matricially normed spaces, \( f \in U^* \) and \( v \in V \), then it is easy to see that \( (f, v) : \mathcal{L}_{cb}^1(V, U) \to \mathbb{C} \) defined by \( (f, v)(\phi) = f(\phi(v)) \), for all \( \phi \in \mathcal{L}_{cb}^1(V, U) \), is a bounded linear functional with \( \| (f, v) \| \leq \| f \| \| v \| \). Moreover if \( (f_{ij}) \in \mathbb{M}_n(U^*) \) and \( (v_{st}) \in \mathbb{M}_m(V) \), then \( \| ((f_{ij}, v_{st})) \|_{nm} \leq \| (f_{ij}) \|_n \| (v_{st}) \|_m \).

Now if \( f = \sum_{\lambda \in \Lambda} \phi_{ij} \in l^\infty(X^*_\lambda \mid \lambda \in \Lambda) \) and \( y \in Y \), then from the previous remark \( \sum_{\lambda \in \Lambda} \phi_{ij}(y) \in l^\infty(\mathcal{L}_{cb}^1(Y, X^*_\lambda) \mid \lambda \in \Lambda) \). We will denote \( \sum_{\lambda \in \Lambda} \phi_{ij}(y) \) by \( (f, y) \). It is obvious that for all \( n, m \in \mathbb{N} \), \( (f_{ij}) \in \mathbb{M}_n(l^\infty(X^*_\lambda \mid \lambda \in \Lambda)) \) and \( (y_{st}) \in \mathbb{M}_m(Y) \), \( \| ((f_{ij}, y_{st})) \|_{nm} \leq \| (f_{ij}) \|_n \| (y_{st}) \|_m \). Let \( n, m \in \mathbb{N} \), \( \sum_{\lambda \in \Lambda} \phi_{ij}(y) \in \mathbb{M}_n(l^\infty(\mathcal{L}_{cb}^1(Y, X^*_\lambda) \mid \lambda \in \Lambda)) \) and \( (y_{st}) \in \mathbb{M}_m(Y) \), with \( \| (y_{st}) \|_m \leq 1 \). Then

\[
\| (J(\sum_{\lambda \in \Lambda} \phi_{ij})(y_{st})) \|_m = \| (\sum_{\lambda \in \Lambda} \phi_{ij}(y_{st})) \|_m
\]

\[
\leq \sup \{ \| (f_{pq} \left( \sum_{\lambda \in \Lambda} \phi_{ij} \right)(y_{st})) \| \mid (f_{pq}) \in \text{BALL}(\mathbb{M}_r(l^\infty(X^*_\lambda \mid \lambda \in \Lambda))), r \in \mathbb{N} \}
\]

\[
\leq \sup \{ \| (f_{pq}(\sum_{\lambda \in \Lambda} \phi_{ij}))(y_{st}) \| \mid (f_{pq}) \in \text{BALL}(\mathbb{M}_r(l^\infty(X^*_\lambda \mid \lambda \in \Lambda))), r \in \mathbb{N} \}
\]

\[
\leq \sup \{ \| (F_{kl}(\sum_{\lambda \in \Lambda} \phi_{ij})) \| \mid (F_{kl}) \in \text{BALL}(\mathbb{M}_N(l^\infty(\mathcal{L}_{cb}^1(Y, X^*_\lambda) \mid \lambda \in \Lambda))), N \in \mathbb{N} \}
\]

\[
= \| (\sum_{\lambda \in \Lambda} \phi_{ij}) \|_n
\]

from the definition of the matrix norms on \( l^1(X^*_\lambda \mid \lambda \in \Lambda) \) and on \( l^1(\mathcal{L}_{cb}^1(Y, X^*_\lambda) \mid \lambda \in \Lambda) \) and the previous discussion. Therefore \( J \) is a complete contraction (a similar calculation with \( n = 1 \) shows that \( J \) is well-defined).
Let \( J : \mathcal{P}(L^1_{cb}(Y, X)|\lambda \in \Lambda) \to L^1_{cb}(l^p(Y, X)|\lambda \in \Lambda), X) \) be defined by \( J(\sum_{\lambda \in \Lambda} \Theta \phi_\lambda)(\sum_{\lambda \in \Lambda} \Theta y_\lambda) = \sum_{\lambda \in \Lambda} \phi_\lambda(y_\lambda) \), for all \( \sum_{\lambda \in \Lambda} \Theta \phi_\lambda \in \mathcal{P}(L^1_{cb}(Y, X)|\lambda \in \Lambda) \) and all \( \sum_{\lambda \in \Lambda} \Theta y_\lambda \in l^p(Y, X)|\lambda \in \Lambda) \), where \( p = 1, \infty \) and \( q = \infty, 1 \).

To prove that, for \( p = \infty \) and \( q = 1 \), \( J \) is a complete contraction we will need the following observation: Let \( V \) and \( U \) be matricially normed spaces, \( \phi \in L^1_{cb}(V, U) \) and \( g \in U^* \). If \( \phi_g : V \to \mathbb{C} \) is defined by \( \phi_g(v) = g(\phi(v)) \), for all \( v \in V \), then \( \phi_g \) is a bounded linear functional with \( ||\phi_g|| \leq ||\phi|| ||g|| \). Moreover if \( (\phi_{ij}) \in M_n(L^1_{cb}(V, U)) \) and \( (g_{st}) \in M_m(U^*) \), then \( ||((\phi_{ij})_{g_{st}})||_{nm} \leq ||(\phi_{ij})||_n ||(g_{st})||_m \) (see the remarks after Proposition 3.2.11 for more on those maps). Now if \( \sum_{\lambda \in \Lambda} \Theta \phi_\lambda \in l^\infty(L^1_{cb}(Y, X)|\lambda \in \Lambda) \) and \( g \in Y^* \), then \( (\sum_{\lambda \in \Lambda} \Theta \phi_\lambda)_g = \sum_{\lambda \in \Lambda} \Theta (\phi_\lambda)_g \in l^\infty(Y, \lambda \in \Lambda) \). For all \( n, m \in \mathbb{N} \), \( (\sum_{\lambda \in \Lambda} \Theta \phi_\lambda) \in M_n(l^\infty(L^1_{cb}(Y, X)|\lambda \in \Lambda)) \) and \( (g_{st}) \in M_m(X^*) \). \( ||((\sum_{\lambda \in \Lambda} \Theta \phi_\lambda)_{g_{st}})||_{nm} \leq ||(\sum_{\lambda \in \Lambda} \Theta \phi_\lambda)||_n ||(g_{st})||_m \). If \( (\sum_{\lambda \in \Lambda} \Theta \phi_\lambda) \in M_n(l^\infty(L^1_{cb}(Y, X)|\lambda \in \Lambda)), with \( ||(\sum_{\lambda \in \Lambda} \Theta \phi_\lambda)||_n \leq 1 \), and \( (\sum_{\lambda \in \Lambda} \Theta y_\lambda) \in M_m(l^1(Y, X)|\lambda \in \Lambda)) \), then, from the preceding discussion and the definition of the norm on \( M_m(l^1(Y, X)|\lambda \in \Lambda)) \), we get

\[
||((\sum_{\lambda \in \Lambda} \Theta \phi_\lambda)(\sum_{\lambda \in \Lambda} \Theta y_\lambda))||_{nm} = ||(\sum_{\lambda \in \Lambda} \phi_{ij}(y_{st}))||_{nm}
\leq \sup\{||g_{pq}(\sum_{\lambda \in \Lambda} \phi_{ij}(y_{st}))|| | (g_{pq}) \in BALL(M_r(X^*)), r \in \mathbb{N}\}
= \sup\{||((\sum_{\lambda \in \Lambda} \Theta \phi_\lambda)_{g_{pq}})(\sum_{\lambda \in \Lambda} \Theta y_\lambda)_{g_{pq}})|| | (g_{pq}) \in BALL(M_r(X^*)), r \in \mathbb{N}\}
= \sup\{||G_{kl}(\sum_{\lambda \in \Lambda} \Theta y_{st}^{ij}))|| | (G_{kl}) \in BALL(M_N(l^\infty(Y, \lambda \in \Lambda))), N \in \mathbb{N}\}
= ||(\sum_{\lambda \in \Lambda} \Theta y_{st}^{ij})||_n^{1(n)}
\]

which shows that \( J \) is a complete contraction.

From the discussion before the proof of (i) for \( p = 1 \) we get that if \( \sum_{\lambda \in \Lambda} \Theta y_\lambda \in l^\infty(Y, \lambda \in \Lambda) \) and \( f \in X^* \), then \( (\sum_{\lambda \in \Lambda} \Theta y_\lambda)(f, y_\lambda) = \sum_{\lambda \in \Lambda} \Theta (f, y_\lambda) \in l^\infty(L^1_{cb}(X, Y, \lambda)^*, \lambda \in \Lambda) \), and, for all \( m, n \in \mathbb{N} \), \( (\sum_{\lambda \in \Lambda} \Theta y_{st}^{ij}) \in M_n(l^\infty(Y, \lambda \in \Lambda)) \) and \( (f_{st}) \in M_m(X^*), ||(f_{st}, \sum_{\lambda \in \Lambda} \Theta y_{st}^{ij})||_{nm} \leq ||(f_{st})||_m ||(\sum_{\lambda \in \Lambda} \Theta y_{st}^{ij})||_n^{\infty}. \) Using that we can prove that \( J \) is a complete contraction if \( p = 1 \) and \( q = \infty \).

**Proposition 3.1.18.** Let \( A \) be an operator algebra, \( X \) be a completely bounded \( A \)-bimodule and \( Y \) be a matricially normed space.
(i) If $X_1$ is a closed $A$-submodule of $X$, then $L^1_{cb}(Y, X)/L^1_{cb}(Y, X_1)$ can be completely contractively embedded as an $A$-submodule into $L^1_{cb}(Y, X/X_1)$. Moreover if $X_1$ is completely complemented in $X$ as a subspace, then $L^1_{cb}(Y, X)/L^1_{cb}(Y, X_1)$ is completely $A$-module isomorphic to $L^1_{cb}(Y, X/X_1)$.

(ii) If $Y_1$ is a completely complemented subspace of $Y$, then $L^1_{cb}(Y, X)/L^1_{cb}(Y_1, X)$ is completely $A$-module isomorphic to $L^1_{cb}(Y/Y_1, X)$.

**Proof.** As we said in Proposition 1.2.1, if $V$ is a matricially normed space and $U$ is a closed subspace of $V$, then $V/U$ becomes a matricially normed space by identifying $M_m(V/U)$ with $M_m(V)/M_m(U)$, for all $m \in \mathbb{N}$.

(i) Let us define, as in Proposition 3.1.7(i),

$$J : L^1_{cb}(Y, X)/L^1_{cb}(Y, X_1) \to L^1_{cb}(Y, X/X_1)$$

by

$$J(\phi + L^1_{cb}(Y, X_1))(y) = \phi(y) + X_1$$

for all $\phi \in L^1_{cb}(Y, X)$ and all $y \in Y$. To prove that $J$ is well-defined consider $\phi \in L^1_{cb}(Y, X)$, $m \in \mathbb{N}$ and $(y_{ij}) \in \mathbb{M}_m(Y)$. Then, by the definition of the quotient matricial norms and the complete boundedness of $\phi$,

$$\|(J(\phi + L^1_{cb}(Y, X_1))(y_{ij}))\|_m = \|(\phi(y_{ij}) + X_1)\|_m$$

$$\leq \|\phi(y_{ij})\|_m$$

$$\leq \|\phi\|_m \|(y_{ij})\|_m$$

which proves that $J(\phi + L^1_{cb}(Y, X_1))$ is completely bounded. Moreover if $m \in \mathbb{N}$, $(\phi_{ij}) \in \mathbb{M}_m(L^1_{cb}(Y, X))$, $(\psi_{ij}) \in (\phi_{ij}) + \mathbb{M}_m(L^1_{cb}(Y, X_1))$, $l \in \mathbb{N}$ and $(y_{st}) \in \mathbb{M}_l(Y)$, then

$$\|(J(\phi_{ij} + L^1_{cb}(Y, X_1))(y_{st}))\|_m = \|(J(\psi_{ij} + L^1_{cb}(Y, X_1))(y_{st}))\|_m$$

$$= \|(\psi_{ij}(y_{st}) + X_1)\|_m$$

$$\leq \|\psi_{ij}(y_{st})\|_m$$

$$\leq \|\psi_{ij}\|_m \|(y_{st})\|_l.$$ 

The infimum of $\|\psi_{ij}\|_m$ over $(\psi_{ij}) \in (\phi_{ij}) + \mathbb{M}_m(L^1_{cb}(Y, X_1))$ is equal to $\|\phi_{ij} + L^1_{cb}(Y, X_1)\|_m$ and therefore $J$ is a complete contraction.

If $X_1$ is completely complemented in $X$, then there exists a completely bounded projection $\rho : X \to X$, with $Im(\rho) = X \ominus X_1$. As we said after Proposition 1.2.1, the map $\tau : X/X_1 \to X$ defined by $\tau(x + X_1) = \rho(x)$, for all $x \in X$, is completely bounded. Now we can prove, as in Proposition 3.1.7(i), that $J$ is onto. To see
that $J^{-1}$ is completely bounded take $m \in \mathbb{N}$, $(\Phi_{ij}) \in \mathbb{M}_m(\mathcal{L}_{cb}^1(Y, X/X_1)), l \in \mathbb{N}$ and $(y_{st}) \in \mathbb{M}_l(Y)$. Then

$$\|((\Phi_{ij}\tau)(y_{st}))\|_{ml} \leq \|((\Phi_{ij})\|_m \|\tau\|_{cb} \|y_{st}\|_l$$

which implies that

$$\|((\Phi_{ij}\tau))\|_m \leq \|((\Phi_{ij})\|_m \|\tau\|_{cb}$$

Hence

$$\|((J^{-1}(\Phi_{ij}))\|_m = \|((\Phi_{ij}\tau + \mathcal{L}_{cb}^1(Y, X_1))\|_m$$

$$\leq \|((\Phi_{ij}\tau))\|_m$$

$$\leq \|((\Phi_{ij})\|_m \|\tau\|_{cb}$$

which shows that $J^{-1}$ is completely bounded.

(ii) Since $Y_1$ is completely complemented in $Y$, we have a completely bounded projection $\rho : Y \to Y$, with $Im(\rho) = Y \ominus Y_1$. As in Proposition 3.1.7(ii), we define

$$J : \mathcal{L}_{cb}^1(Y, X)/\mathcal{L}_{cb}^1(Y_1, X) \to \mathcal{L}_{cb}^1(Y/Y_1, X)$$

by

$$J(\phi + \mathcal{L}_{cb}^1(Y_1, X))(y + Y_1) = \phi(\rho(y))$$

for all $\phi \in \mathcal{L}_{cb}^1(Y, X)$ and all $y \in Y$. To prove that $J$ is well-defined consider $\phi \in \mathcal{L}_{cb}^1(Y, X), m \in \mathbb{N}, (y_{ij}) \in \mathbb{M}_m(Y)$ and $(z_{ij}) \in \mathbb{M}_m(Y_1)$. Then

$$\|((J(\phi + \mathcal{L}_{cb}^1(Y_1, X))(y_{ij} + Y_1))\|_m = \|((\phi(\rho(y_{ij})))\|_m$$

$$= \|((\phi(\rho(y_{ij} + z_{ij})))\|_m$$

$$\leq \|\phi\|_{cb}\|\rho\|_{cb}\|(y_{ij} + Y_1)\| + \|z_{ij}\|_m.$$
and then taking the infimum over \((\psi_{ij}) \in (\phi_{ij}) + \mathbb{M}_m(\mathcal{L}_{cb}^1(Y_1, X))\) on the right hand side we conclude that

\[
\|(J(\phi_{ij} + \mathcal{L}_{cb}^1(Y_1, X)))\|_m \leq \|\rho\|_{cb}\|(J(\phi_{ij} + (\mathcal{L}_{cb}^1(Y_1, X)))\|_m
\]

which proves that \(J\) is completely bounded.

From Proposition 1.2.1 the quotient map \(q : Y \to Y/Y_1\) is completely bounded. Hence if \(\Phi \in \mathcal{L}_{cb}^1(Y/Y_1, X)\), then \(\Phi q \in \mathcal{L}_{cb}^1(Y, X)\). It is easy to prove that \(\Phi = J(\Phi q + \mathcal{L}_{cb}^1(Y_1, X))\). Thus \(J\) is onto. Moreover \(J^{-1}\) is completely bounded as the following calculation shows. Take \(m \in \mathbb{N}\), \((\Phi_{ij}) \in \mathbb{M}_m(\mathcal{L}_{cb}^1(Y/Y_1, X)), l \in \mathbb{N}\) and 

\[
((y_{st})) \in \mathbb{M}_l(Y).
\]

Then

\[
\|(\Phi_{ij}(q(y_{st})))\|_{ml} \leq \|(\Phi_{ij})\|_m\|q\|_{cb}\|(y_{st})\|_l
\]

which shows that

\[
\|(\Phi_{ij}q)\|_m \leq \|(\Phi_{ij})\|_m\|q\|_{cb}.
\]

Thus

\[
\|(J^{-1}(\Phi_{ij}))\|_m = \|(\Phi_{ij}q + \mathcal{L}_{cb}^1(Y_1, X))\|_m
\]

\[
\leq \|(\Phi_{ij}q)\|_{ml}
\]

\[
\leq \|(\Phi_{ij})\|_m\|q\|_{cb}.
\]

\(\square\)

A reduction of dimension result does not hold for completely bounded cohomology (see p.29). So a discussion of the connection between \(\mathcal{L}_{cb}^1(Y, X)\) and \(\mathcal{L}_{cb}^n(A, X)\) similar to Proposition 3.1.8 would be out of place here.

As in the first part, we can relate the completely bounded cohomology, with coefficients in \(X\), to the completely bounded cohomology, with coefficients in \(\mathcal{L}_{cb}^1(Y, X)\). We must mention here that although part \((vii)\) of the following proposition shows that the vanishing of \(\mathcal{H}_{cb}^n(A, \mathcal{L}_{cb}^1(Y, X))\) implies the vanishing of \(\mathcal{H}_{cb}^n(A, X)\), the converse does not hold (see Section 5.3). On the other hand the vanishing of \(\mathcal{H}_{cb}^n(A, \mathcal{L}_{cb}^1(Y, X))\), for all matricially normed spaces \(Y\), implies more than the vanishing of \(\mathcal{H}_{cb}^n(A, X)\) (see Section 4.1).

**Proposition 3.1.19.** Let \(A\) be an operator algebra, \(B\) be a subalgebra of \(A\), \(X\) be an operator completely bounded \(A\)-bimodule, \(Y\) be a matricially normed space and \(n \in \mathbb{N}\). Then the following hold:

\(i\) For all non-zero \(f \in Y^*\), the map

\[
F \mapsto F_f : \mathcal{L}_{cb}^n(A, X) \to \mathcal{L}_{cb}^n(A, \mathcal{L}_{cb}^1(Y, X))
\]

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defined by \( F_f(a_1, ..., a_n)(y) = f(y)F(a_1, ..., a_n) \), for all \( F \in L^\infty_{cb}(A, X) \), all \( a_1, ..., a_n \in A \) and all \( y \in Y \) is a one-to-one completely bounded map, with \( \|F_f\|_{cb} \leq \|f\| \|F\|_{cb} \).

Moreover \( F_{tf+t'f'} = tf_f + t'f' \), for all \( F \in L^\infty_{cb}(A, X) \), all \( f, f' \in Y^* \) and all \( t, t' \in \mathbb{C} \).

(ii) For all non-zero \( y \in Y \), the map

\[ \Phi \mapsto \Phi_y : L^\infty_{cb}(A, L^1_{cb}(Y, X)) \to L^\infty_{cb}(A, X) \]

defined by \( \Phi_y(a_1, ..., a_n) = \Phi(a_1, ..., a_n)(y) \), for all \( \Phi \in L^\infty_{cb}(A, L^1_{cb}(Y, X)) \) and all \( a_1, ..., a_n \in A \) is a completely bounded map, with \( \|\Phi \mapsto \Phi_y\|_{cb} \leq \|y\|_{cb} \), which maps \( L^\infty_{cb}(A, L^1_{cb}(Y, X)) \) onto \( L^\infty_{cb}(A, X) \). Moreover \( \Phi_{ty+t'y'} = t\Phi_y + t'\Phi_{y'} \), for all \( \Phi \in L^\infty_{cb}(A, L^1_{cb}(Y, X)) \), all \( y, y' \in Y \) and all \( t, t' \in \mathbb{C} \). If \( m, l \in \mathbb{N}, (\Phi_{ij}) \in M_m(L^\infty_{cb}(A, L^1_{cb}(Y, X))) \) and \( (y_{st}) \in M_l(Y) \), then

\[ \|((\Phi_{ij})_{ys})\|_{ml} \leq \|((\Phi_{ij})_{ms})\| \|y_{st}\|_{l} \]

(iii) For all non-zero \( f \in Y^* \), \( F \mapsto F_f \) maps \( L^\infty_{cb}(A, X : /B) \) into \( L^\infty_{cb}(A, L^1_{cb}(Y, X)) : /B \).

(iv) For all non-zero \( y \in Y \), \( \Phi \mapsto \Phi_y \) maps \( L^\infty_{cb}(A, L^1_{cb}(Y, X)) : /B \) onto \( L^\infty_{cb}(A, X : /B) \).

(v) If

\[ \partial^n : L^\infty_{cb}(A, X) \to L^{n+1}_{cb}(A, X) \]

and

\[ \Delta^n : L^\infty_{cb}(A, L^1_{cb}(Y, X)) \to L^{n+1}_{cb}(A, L^1_{cb}(Y, X)) \]

are the coboundary maps, then, for all \( f \in Y^* \) and all \( F \in L^\infty_{cb}(A, X) \),

\[ \Delta^n(F_f) = (\partial^n(F))_f \]

and, for all \( y \in Y \) and all \( \Phi \in L^\infty_{cb}(A, L^1_{cb}(Y, X)) \),

\[ \partial^n(\Phi_y) = (\Delta^n(\Phi))_y. \]

(vi) For all non-zero \( f \in Y^* \), \( F \mapsto F_f \) maps \( Z^n_{cb}(A, X) \) into \( Z^n_{cb}(A, L^1_{cb}(Y, X)) \), \( B^n_{cb}(A, X) \) into \( B^n_{cb}(A, L^1_{cb}(Y, X)) \) and \( H^n_{cb}(A, X) \) into \( H^n_{cb}(A, L^1_{cb}(Y, X)) \).

(vii) For all non-zero \( y \in Y \), \( \Phi \mapsto \Phi_y \) maps \( Z^n_{cb}(A, L^1_{cb}(Y, X)) \) onto \( Z^n_{cb}(A, X) \), \( B^n_{cb}(A, L^1_{cb}(Y, X)) \) onto \( B^n_{cb}(A, X) \) and \( H^n_{cb}(A, L^1_{cb}(Y, X)) \) onto \( H^n_{cb}(A, X) \).

(viii) For all non-zero \( f \in Y^* \), \( F \mapsto F_f \) maps \( Z^n_{cb}(A, X : /B) \) into \( Z^n_{cb}(A, L^1_{cb}(Y, X)) : /B \), \( B^n_{cb}(A, X : /B) \) into \( B^n_{cb}(A, L^1_{cb}(Y, X)) : /B \) and \( H^n_{cb}(A, X : /B) \) into \( H^n_{cb}(A, L^1_{cb}(Y, X)) : /B \).
(ix) For all non-zero \( y \in Y \), \( \Phi \mapsto \Phi_y \) maps \( Z^n_{cb}(A, L^1_{cb}(Y, X)) : /B \) onto \( Z^n_{cb}(A, X : /B) \), \( B^n_{cb}(A, L^1_{cb}(Y, X)) : /B \) onto \( B^n_{cb}(A, X : /B) \) and \( H^n_{cb}(A, L^1_{cb}(Y, X)) : /B \) onto \( H^n_{cb}(A, X : /B) \).

**Proof.** (i) If \( y \in Y^* \), \( F \in L^n_{cb}(A, X) \), \( a_1, \ldots, a_n \in A \), \( m \in \mathbb{N} \) and \( (y_{ij}) \in M_m(Y) \), then

\[
\|(F_f(a_1, \ldots, a_n)(y_{ij}))\|_m = \|(f(y_{ij})F(a_1, \ldots, a_n))\|_m
\]

\[
\leq \|(f(y_{ij}))\|_m \|F(a_1, \ldots, a_n)\|_m
\]

\[
= \|(f(y_{ij}))\|_m \|F(a_1, \ldots, a_n)\|
\]

\[
\leq \|f\| \|(y_{ij})\|_m \|F(a_1, \ldots, a_n)\|
\]

which proves that \( F_f(a_1, \ldots, a_n) \in L^1_{cb}(Y, X) \). Moreover if \( m \in \mathbb{N} \), \( (a^1_{ij}), \ldots, (a^n_{ij}) \in M_m(A) \), \( l \in \mathbb{N} \) and \( (y_{st}) \in M_l(Y) \) we can prove as in the previous step that

\[
\|(\sum_{1 \leq k_1, \ldots, k_{n-1} \leq m} F_f(a^1_{ik_1}, \ldots, a^n_{ik_n-1})(y_{st}))\|_m
\]

\[
\leq \|f\| \|(\sum_{1 \leq k_1, \ldots, k_{n-1} \leq m} F(a^1_{ik_1}, \ldots, a^n_{ik_n-1}))\|_m \|y_{st}\|_l
\]

and thus

\[
\|(\sum_{1 \leq k_1, \ldots, k_{n-1} \leq m} F_f(a^1_{ik_1}, \ldots, a^n_{ik_n-1}))\|_m \leq \|f\| \|(\sum_{1 \leq k_1, \ldots, k_{n-1} \leq m} F(a^1_{ik_1}, \ldots, a^n_{ik_n-1}))\|_m
\]

\[
\leq \|f\| \|F_{cb}(a^1_{ij})\|_m \|y_{st}\|_l \|a^n_{ij}\|_m.
\]

Therefore \( F_f \) is completely bounded, i.e. \( F \mapsto F_f \) is well-defined. Now if \( m \in \mathbb{N} \), \( (F_{ij}) \in M_m(L^n_{cb}(A, X)) \), \( l \in \mathbb{N} \) and \( (a^1_{st}), \ldots, (a^n_{st}) \in M_l(A) \) we get in a similar manner that

\[
\|(\sum_{1 \leq k_1, \ldots, k_{n-1} \leq m} (F_{ij})(a^1_{sk_1}, \ldots, a^n_{sk_n-1}))\|_m \leq \|f\| \|(F_{ij})\|_m \|(a^1_{st})\|_l \|a^n_{st}\|_l
\]

and hence by the definition of the norm on \( M_m(L^n_{cb}(A, X)) \), \( \|(F_{ij})\|_m \leq \|f\| \|(F_{ij})\|_m \), which implies that \( F \mapsto F_f \) is completely bounded, with \( \|F \mapsto F_f\|_{cb} \leq \|f\| \).

(ii) By the way that \( \Phi \mapsto \Phi_y \) is defined, it is easy to see that it is completely bounded, with \( \|\Phi \mapsto \Phi_y\|_{cb} \leq \|y\| \). Moreover, using the Hahn-Banach theorem, we can show that it is onto. If \( m, l \in \mathbb{N} \), \( (\Phi_{ij}) \in M_m(L^n_{cb}(A, L^1_{cb}(Y, X))) \) and
(y_{st}) \in M_q(Y)$, then, for all $r \in \mathbb{N}$ and all $(a_{pq}^1, \ldots, a_{pq}^n) \in M_r(A)$,
\[
\|((\Phi_{ij})_{y_{st}}(a_{pq}^1, \ldots, a_{pq}^n))\|_{m|r}
\leq \|((\Phi_{ij})_{y_{st}}(a_{pq}^1, \ldots, a_{pq}^n))\|_{m|r}(y_{st})\|_t
\]
which shows that
\[
\|((\Phi_{ij})_{y_{st}})\|_{m|t} \leq \|((\Phi_{ij})_{y_{st}})\|_{m|(y_{st})}\|_t.
\]
The rest is similar to the proof of Proposition 3.1.9.

Results similar to those of Propositions 3.1.10 and 3.1.11 and Remark 3.1.1 also hold for the completely bounded case.

### 3.2 Modules of maps from $A$-modules

#### 3.2.1 The modules $L_c^1(X, Y)$

In this section we study modules of bounded maps from a Banach $A$-module $X$ into a Banach space $Y$. (We said in the opening discussion for Section 3.1.2 that Effros and Ruan called modules of the form $L_c^{1b}(Y, X)$ range modules; in a similar manner we could have called the modules $L_c^{1b}(X, Y)$ domain modules).

As we mentioned in the introduction, the module actions of $A$ on $L_c^1(X, Y)$ are similar to the module actions of $A$ on the dual module $X^*$ of $X$. Although we present our results for $A$-bimodules, they also hold for one-sided $A$-modules and for $(A, B)$-bimodules. Let us just remark that, because of the way the module actions of $A$ on $L_c^1(X, Y)$ are defined, if $X$ is respectively a left/right $A$-module, then $L_c^1(X, Y)$ is respectively a right/left $A$-module. Moreover $(A, B)$-bimodules yield $(B, A)$-bimodules. We will present our results in the same order as in Section 3.1.1. We will start by defining the modules $L_c^1(X, Y)$. Then we will show how the modules $L_c^1(X, Y)$ are related to the dual module $X^*$ of $X$. We will then talk about the relation between algebraic and topological properties of $X$ and of $L_c^1(X, Y)$. As we shall see duality and normality of $L_c^1(X, Y)$ are not related to the duality and the normality of $X$, but to the duality of $Y$ (as a Banach space) and the normality of $X^*$. We will continue by showing that the modules $L_c^1(X, Y)$ are "well-behaved". Then we will discuss how the cohomology groups of $A$, with coefficients in $X^*$ and in $L_c^1(X, Y)$, are related and finish with a result
about the relation between the multiplication and the module actions of \( A \) on \( \mathcal{L}_c^1(X) \) and a result about the spaces \( \mathcal{L}_c^1(X, Y : /B) \) of module maps when \( Y \) is also an \( A \)-module.

**Proposition 3.2.1.** Let \( A \) be a Banach algebra, \( X \) be a Banach \( A \)-bimodule and \( Y \) be a Banach space. Then the space of bounded linear maps from \( X \) into \( Y \), \( \mathcal{L}_c^1(X, Y) \), becomes a Banach \( A \)-bimodule, with the right and the left module actions of \( A \) on \( \mathcal{L}_c^1(X, Y) \) defined respectively by

\[
(\phi \circ a)(x) = \phi(ax)
\]

and

\[
(a \circ \phi)(x) = \phi(xa)
\]

for all \( a \in A \), all \( \phi \in \mathcal{L}_c^1(X, Y) \) and all \( x \in X \).

**Proof.** The algebraic part is similar to proving that the dual module \( X^* \) of \( X \) is an \( A \)-bimodule. We can easily see that if \( K > 0 \), with \( ||ax|| \leq K||a|| ||x|| \) and \( ||xa|| \leq K||a|| ||x|| \), for all \( a \in A \) and all \( x \in X \), then \( ||a \circ \phi|| \leq K||a|| ||\phi|| \) and \( ||\phi \circ a|| \leq K||a|| ||\phi|| \), for all \( a \in A \) and all \( \phi \in \mathcal{L}_c^1(X, Y) \), i.e. \( \mathcal{L}_c^1(X, Y) \) is a Banach \( A \)-bimodule and moreover \( K \) does not depend on \( Y \).

If \( Y = \mathbb{C} \), then \( \mathcal{L}_c^1(X, Y) \) is the dual module \( X^* \) of \( X \).

Now if \( X_1, \ldots, X_n \) are Banach \( A \)-bimodules and \( Y \) is a Banach space, then for all \( 1 \leq k_1, k_2 \leq n \) we can make \( \mathcal{L}^n_c(X_1, \ldots, X_n; Y) \) into a Banach \( A \)-bimodule, with the right and left module actions of \( A \) on \( \mathcal{L}^n_c(X_1, \ldots, X_n; Y) \) defined by

\[
(\phi \circ_{k_1} a)(x_1, \ldots, x_n) = \phi(x_1, \ldots, ax_{k_1}, \ldots, x_n)
\]

and

\[
(a \circ_{k_2} \phi)(x_1, \ldots, x_n) = \phi(x_1, \ldots, x_{k_2} a, \ldots, x_n)
\]

for all \( a \in A \), all \( \phi \in \mathcal{L}^n_c(X_1, \ldots, X_n; Y) \) and all \( x_1 \in X_1, \ldots, x_{k_1} \in X_{k_1}, x_{k_2} \in X_{k_2}, \ldots, x_n \in X_n \). If \( k_1 = k_2 = k \), then we can easily see that \( \mathcal{L}^n_c(X_1, \ldots, X_n; Y) \) is isometrically \( A \)-module isomorphic to \( \mathcal{L}_{c}^1(X_k, \mathcal{L}_{c}^{n-1}(X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n; Y)) \).

It is easy to see that for any non-zero \( y \in Y \) the map

\[
f \mapsto f_y : X^* \to \mathcal{L}_c^1(X, Y)
\]

defined by \( f_y(x) = f(x)y \), for all \( f \in X^* \) and all \( x \in X \), is an embedding of \( X^* \) into \( \mathcal{L}_c^1(X, Y) \), with \( ||f_y|| = ||y|| ||f|| \), for all \( f \in X^* \). Moreover it is an \( A \)-module.
homomorphism and so \(X^*\) is (isometrically if \(\|y\| = 1\)) \(A\)-module isomorphic to the closed \(A\)-submodule \(\{f_y|f \in X^*\}\) of \(\mathcal{L}_c^1(X, Y)\).

Conversely, we can see that for any non-zero \(g \in Y^*\) the map

\[
\phi \mapsto \phi_g : \mathcal{L}_c^1(X, Y) \to X^*
\]

defined by \(\phi_g(x) = g(\phi(x))\), for all \(\phi \in \mathcal{L}_c^1(X, Y)\) and all \(x \in X\), is a bounded map with \(\|\phi \mapsto \phi_g\| = \|g\|\), which maps \(\mathcal{L}_c^1(X, Y)\) onto \(X^*\). Moreover we can easily see that it is an \(A\)-module homomorphism.

Now if \(g \in Y^*\) and \(y \in Y\), with \(g(y) = 1\), then the composition of those two maps is a projection (with norm 1 if \(\|g\| \leq 1\) and \(\|y\| \leq 1\)) that maps \(\mathcal{L}_c^1(X, Y)\) onto \(\{f_y|f \in X^*\}\) and is an \(A\)-module homomorphism. Thus \(X^*\) is a complemented \(A\)-submodule of \(\mathcal{L}_c^1(X, Y)\).

The way the module actions of \(A\) on \(\mathcal{L}_c^1(X, Y)\) are defined together with the previous remarks and the Hahn-Banach theorem show that \(\mathcal{L}_c^1(X, Y)\) has certain algebraic properties if and only if \(X\) does, i.e. \(\mathcal{L}_c^1(X, Y)\) is unital if and only if \(X\) is, \(\mathcal{L}_c^1(X, Y)\) is abelian if and only if \(X\) is and \(Ann_A(\mathcal{L}_c^1(X, Y)) = Ann_A(X)\) and thus \(\mathcal{L}_c^1(X, Y)\) is respectively annihilating/faithful if and only if \(X\) is respectively annihilating/faithful. We haven’t been able to discover any connection between neounitality of \(X\) and \(\mathcal{L}_c^1(X, Y)\) (obviously if \(\mathcal{L}_c^1(X, Y)\) is neounital, then \(X^*\) is also neounital).

**Proposition 3.2.2.** Let \(A\) be a Banach algebra, \(X\) be a Banach \(A\)-bimodule and \(Y\) be a Banach space. If \(Y\) is a dual Banach space, then \(\mathcal{L}_c^1(X, Y)\) is a dual \(A\)-bimodule.

**Proof.** Since \(Y\) is a dual Banach space, there exists a normed space \(Y_*\) such that \(Y = (Y_*)^*\). Hence, as in Proposition 3.1.2, \(\mathcal{L}_c^1(X, Y)\) is isometrically isomorphic to \((X \hat{\otimes} Y_*)^*\) via the map

\[
\phi \mapsto \Psi_\phi : \mathcal{L}_c^1(X, Y) \to (X \hat{\otimes} Y_*)^*
\]

defined by \(\Psi_\phi(x \otimes z) = \phi(x)(z)\), for all \(\phi \in \mathcal{L}_c^1(X, Y)\), all \(x \in X\) and all \(z \in Y_*\) and extended to \(X \hat{\otimes} Y_*\).

To prove that \(\mathcal{L}_c^1(X, Y)\) is a dual \(A\)-bimodule we have to prove that, for all \(a \in A\), the maps \(\phi \mapsto a \circ \phi\) and \(\phi \mapsto \phi \circ a\) are weak* continuous. To prove that \(\phi \mapsto a \circ \phi\) is weak* continuous consider \(a \in A\) and a bounded net \(\{\phi_\lambda\}_{\lambda \in \Lambda}\) converging weak* to some \(\phi \in \mathcal{L}_c^1(X, Y)\) (the discussion in the second paragraph of the proof of Proposition 3.1.2 allows us to consider only bounded nets). Then \(\lim_{\lambda \in \Lambda} \phi_\lambda(x)(z) = \phi(x)(z)\), for all \(x \in X\) and all \(z \in Y_*\). Thus \(\lim_{\lambda \in \Lambda} \phi_\lambda(xa)(z) = \phi(xa)(z)\), for all \(x \in X\) and all \(z \in Y_*\), which implies that, for
all \( x \in X \), \( \{\phi_\lambda(xa)\}_{\lambda \in \Lambda} \) converges weak* in \( Y \) to \( \phi(xa) \). By the definition of the left module action of \( \mathcal{A} \) on \( \mathcal{L}^1_c(X, Y) \) that means that, for all \( x \in X \), \( \{(a \circ \phi_\lambda)(x)\}_{\lambda \in \Lambda} \) converges weak* in \( Y \) to \( (a \circ \phi)(x) \). Hence \( \lim_{\lambda \in \Lambda} (a \circ \phi_\lambda)(x)(z) = (a \circ \phi)(x)(z) \), for all \( x \in X \) and all \( z \in Y \). The convergence of \( \{a \circ \phi_\lambda\}_{\lambda \in \Lambda} \) to \( a \circ \phi \) on \( X \otimes Y \) together with \( X \otimes Y \) being dense in \( X \otimes Y \) and the boundedness of \( \{a \circ \phi_\lambda\}_{\lambda \in \Lambda} \) imply that the net \( \{a \circ \phi_\lambda\}_{\lambda \in \Lambda} \) converges to \( a \circ \phi \) weak* and so the map \( \phi \mapsto a \circ \phi \) is weak* continuous. Thus \( \mathcal{L}^1_c(X, Y) \) is a dual \( \mathcal{A} \)-bimodule.

Alternatively we can make \( X \otimes Y \) into an \( \mathcal{A} \)-bimodule, with \( \mathcal{A} \) acting on \( X \) and show that \( \mathcal{L}^1_c(X, Y) \) is the dual \( \mathcal{A} \)-bimodule of \( X \otimes Y \).

**Proposition 3.2.3.** Let \( \mathcal{M} \) be a von Neumann algebra, \( X \) be a Banach \( \mathcal{M} \)-bimodule and \( Y \) be a Banach space. If \( Y \) is a dual Banach space and the dual \( \mathcal{M} \)-bimodule \( X^* \) of \( X \) is a normal \( \mathcal{M} \)-bimodule, then \( \mathcal{L}^1_c(X, Y) \) is a normal \( \mathcal{M} \)-bimodule.

**Proof.** As in the previous proposition, since \( Y \) is a dual Banach space, there exists a normed space \( Y_\ast \) such that \( Y = (Y_\ast)^* \) and \( \mathcal{L}^1_c(X, Y) \) is isometrically isomorphic to \( (X \otimes Y_\ast)^* \).

To prove that \( \mathcal{L}^1_c(X, Y) \) is a normal \( \mathcal{M} \)-bimodule, we need to prove that, for all \( \phi \in \mathcal{L}^1_c(X, Y) \), the maps \( \phi \mapsto \phi \circ a \) and \( \phi \mapsto a \circ \phi \) are ultraweak-weak* continuous. Let \( \phi \in \mathcal{L}^1_c(X, Y) \) and consider a net \( \{a_\lambda\}_{\lambda \in \Lambda} \) in \( \mathcal{M} \), which converges ultraweakly to some \( a \in \mathcal{M} \). As in the proof of Proposition 3.1.3 we may assume that \( \{a_\lambda\}_{\lambda \in \Lambda} \) is bounded. Consider also \( x \in X \) and \( z \in Y \). Since \( X^* \) is a normal \( \mathcal{M} \)-bimodule, the map \( \phi \mapsto \phi \circ a \) is ultraweak-weak* continuous, for all \( f \in X^* \). Now if \( \tilde{z} \) is the element of \((Y_\ast)^{**} = Y^* \) corresponding to \( z \), then, by the remarks after Proposition 3.2.1, \( \phi \circ a \mapsto a \circ \phi \) is ultraweak-weak* continuous. That implies that \( \{a_\lambda \circ \phi_\lambda\}_{\lambda \in \Lambda} \) converges weak* to \( a \circ \phi \) and so \( \lim_{\lambda \in \Lambda} (a_\lambda \circ \phi_\lambda)(x) = (a \circ \phi)(x) \). Using the definition of the module action of \( \mathcal{M} \) on \( X^* \) we can see that \( (a_\lambda \circ \phi_\lambda)(x) = \phi_\lambda(xa_\lambda) \), for all \( \lambda \in \Lambda \), and \( (a_\lambda \circ \phi_\lambda)(x) = \phi_\lambda(xa) \).

Moreover by the way \( \phi_\lambda \) and \( \tilde{z} \) are defined we get \( \phi_\lambda(xa_\lambda) = \phi(xa_\lambda)(z) \), for all \( \lambda \in \Lambda \), and \( \phi_\lambda(xa) = \phi(xa)(z) \). By the way the left module action of \( \mathcal{M} \) on \( \mathcal{L}^1_c(X, Y) \) is defined \( \phi(xa_\lambda) = (a_\lambda \circ \phi)(x) \), for all \( \lambda \in \Lambda \) and \( \phi(xa) = (a \circ \phi)(x) \). Hence \( \lim_{\lambda \in \Lambda} (a_\lambda \circ \phi)(x)(z) = (a \circ \phi)(x)(z) \), for all \( x \in X \) and all \( z \in Y \). As before this implies that \( \{a_\lambda \circ \phi\}_{\lambda \in \Lambda} \) converges weak* to \( a \circ \phi \), which proves that the map \( a \mapsto a \circ \phi \) is ultraweak-weak* continuous. Therefore \( \mathcal{L}^1_c(X, Y) \) is a normal \( \mathcal{M} \)-bimodule.

In the following two propositions we show that module homomorphisms between \( X \)'s and bounded maps between \( Y \)'s can be lifted as composition maps to
module maps on $L^1_c(X, Y)$ and that submodules of $X$ and subspaces of $Y$ give rise to submodules of $L^1_c(X, Y)$.

**Proposition 3.2.4.** Let $A$ be a Banach algebra, $X_1$ and $X_2$ be Banach $A$-bimodules and $Y_1$ and $Y_2$ be Banach spaces.

(i) If $\tau : X_1 \to X_2$ is a bounded $A$-module homomorphism, then the map

$$\tau Y_1 : L^1_c(X_2, Y_1) \to L^1_c(X_1, Y_1)$$

defined by $\tau Y_1(\phi) = \phi \tau$, for all $\phi \in L^1_c(X_2, Y_1)$, is a bounded $A$-module homomorphism, with $\|\tau Y_1\| = \|\tau\|$.

(ii) If $\pi : Y_1 \to Y_2$ is a bounded linear map, then the map

$$\pi X_1 : L^1_c(X_1, Y_1) \to L^1_c(X_2, Y_2)$$

defined by $\pi X_1(\phi) = \pi \phi$, for all $\phi \in L^1_c(X_1, Y_1)$, is a bounded $A$-module homomorphism, with $\|\pi X_1\| = \|\pi\|$.

**Proof.** As in Proposition 3.1.4, we can see that $\tau Y_1$ and $\pi X_1$ are bounded maps, with $\|\tau Y_1\| = \|\tau\|$ and $\|\pi X_1\| = \|\pi\|$.

To prove that $\tau Y_1$ is an $A$-module homomorphism take $a \in A$ and $\phi \in L^1_c(X_2, Y_1)$. Then, for all $x_1 \in X_1$,

$$\tau Y_1(a \circ \phi)(x_1) = (a \circ \phi)(\tau(x_1)) = \phi(\tau(x_1)a) = \phi(\tau(x_1)a)$$

$$= \tau Y_1(\phi)(x_1a) = (a \circ \tau Y_1(\phi))(x_1)$$

since $\tau$ is an $A$-module homomorphism.

To prove that $\pi X_1$ is an $A$-module homomorphism consider $a \in A$ and $\phi \in L^1_c(X_1, Y_1)$. Then, for all $x_1 \in X_1$,

$$\pi X_1(a \circ \phi)(x_1) = \pi((a \circ \phi)(x_1)) = \pi(\phi(x_1a))$$

$$= \pi X_1(\phi)(x_1a) = (a \circ \pi X_1(\phi))(x_1).$$

$\square$

In particular the previous proposition implies that (isometrically) isomorphic $A$-bimodules $X_1$ and $X_2$ and (isometrically) isomorphic Banach spaces $Y_1$ and $Y_2$ yield (isometrically) isomorphic $A$-bimodules $L^1_c(X_1, Y_1)$ and $L^1_c(X_2, Y_2)$.

**Proposition 3.2.5.** Let $A$ be a Banach algebra, $X$ be a Banach $A$-bimodule and $Y$ be a Banach space.

(i) If $X_1$ is a complemented $A$-submodule of $X$, then $L^1_c(X_1, Y)$ is a complemented $A$-submodule of $L^1_c(X, Y)$.
(ii) If \( Y_1 \) is a closed subspace of \( Y \), then \( \mathcal{L}_c^1(\mathcal{L}_c^1, Y_1) \) is a closed \( A \)-submodule of \( \mathcal{L}_c^1(\mathcal{L}_c^1, Y) \). Moreover if \( Y_1 \) is a complemented subspace of \( Y \), then \( \mathcal{L}_c^1(\mathcal{L}_c^1, Y_1) \) is a complemented \( A \)-submodule of \( \mathcal{L}_c^1(\mathcal{L}_c^1, Y) \).

(iii) If \( X_1 \) is a complemented \( A \)-submodule of \( X \) and \( Y_1 \) is a closed subspace of \( Y \), then \( \mathcal{L}_c^1(X_1, Y_1) \) is a closed \( A \)-submodule of \( \mathcal{L}_c^1(X, Y) \). Moreover if \( Y_1 \) is a complemented subspace of \( Y \), then \( \mathcal{L}_c^1(X_1, Y_1) \) is a complemented \( A \)-submodule of \( \mathcal{L}_c^1(X, Y) \).

Proof. (i) Since \( X_1 \) is a complemented \( A \)-submodule of \( X \), there exists a bounded projection \( \rho : X \to X \) mapping \( X \) onto \( X_1 \), which is an \( A \)-module homomorphism. By the previous proposition, the map

\[
\rho_Y : \mathcal{L}_c^1(X, Y) \to \mathcal{L}_c^1(X, Y)
\]

is a bounded \( A \)-module homomorphism. Moreover it is a projection, with \( \text{Im}(\rho_Y) = \mathcal{L}_c^1(X_1, Y) \), which implies that \( \mathcal{L}_c^1(X_1, Y) \) is a complemented \( A \)-submodule of \( \mathcal{L}_c^1(X, Y) \).

(ii) If \( a \in A \) and \( \phi \in \mathcal{L}_c^1(X, Y_1) \), then \( \phi(xa) \in Y_1 \), for all \( x \in X \), and so \( (a \circ \phi)(x) \in Y_1 \), for all \( x \in X \), which implies that \( a \circ \phi \in \mathcal{L}_c^1(X, Y_1) \). Thus \( \mathcal{L}_c^1(X, Y_1) \) is an \( A \)-submodule of \( \mathcal{L}_c^1(X, Y) \). Now if \( Y_1 \) is complemented in \( Y \), then there exists a projection \( \rho : Y \to Y \) that maps \( Y \) onto \( Y_1 \). By the previous proposition, the map

\[
\rho_X : \mathcal{L}_c^1(X, Y) \to \mathcal{L}_c^1(X, Y)
\]

is a bounded \( A \)-module homomorphism. Moreover it is a projection, with \( \text{Im}(\rho_X) = \mathcal{L}_c^1(X_1, Y) \) and thus \( \mathcal{L}_c^1(X_1, Y) \) is complemented in \( \mathcal{L}_c^1(X, Y) \).

(iii) follows directly from (i) and (ii). \( \square \)

The following two propositions describe what happens with direct sums and quotients.

**Proposition 3.2.6.** Let \( A \) be a Banach algebra. Then the following hold:

(i) If \( X_1 \) and \( X_2 \) are Banach \( A \)-bimodules and \( Y \) is a Banach space, then, for all conjugate \( 1 \leq p, q \leq \infty \), \( \mathcal{L}_c^1(X_1, Y) \oplus_p \mathcal{L}_c^1(X_2, Y) \) and \( \mathcal{L}_c^1(X_1 \oplus_q X_2, Y) \) are \( A \)-module isomorphic. Moreover if \( p = \infty \) and \( q = 1 \), then they are isometrically \( A \)-module isomorphic.

(ii) If \( \{X_\lambda|\lambda \in \Lambda\} \) is a family of uniformly bounded Banach \( A \)-bimodules and \( Y \) is a Banach space, then:

(a) \( l^\infty(\mathcal{L}_c^1(X_\lambda, Y)|\lambda \in \Lambda) \) is isometrically \( A \)-module isomorphic to \( \mathcal{L}_c^1(l^1(X_\lambda|\lambda \in \Lambda), Y) \).
(b) \( l^1(\mathcal{L}_c^1(X, Y)|\lambda \in \Lambda) \) can be contractively embedded as an \( \mathcal{A} \)-submodule into \( \mathcal{L}_c^1(c_0(X|\lambda \in \Lambda), Y) \).

(c) \( l^p(\mathcal{L}_c^1(X, Y)|\lambda \in \Lambda) \) can be contractively embedded as an \( \mathcal{A} \)-submodule into \( \mathcal{L}_c^1(l^q(X|\lambda \in \Lambda), Y) \), for all conjugate \( 1 < p, q < \infty \).

(iii) If \( X \) is a Banach \( \mathcal{A} \)-bimodule and \( Y_1 \) and \( Y_2 \) are Banach spaces, then \( \mathcal{L}_c^1(X, Y_1) \oplus_p \mathcal{L}_c^1(X, Y_2) \) and \( \mathcal{L}_c^1(X, Y_1 \oplus_p Y_2) \) are \( \mathcal{A} \)-module isomorphic, for all \( 1 \leq p \leq \infty \). Moreover they are isometrically \( \mathcal{A} \)-module isomorphic if \( p = \infty \).

(iv) If \( X \) is a Banach \( \mathcal{A} \)-bimodule and \( \{Y_\lambda|\lambda \in \Lambda\} \) is a family of Banach spaces, then:

(a) \( l^\infty(\mathcal{L}_c^1(X, Y_\lambda)|\lambda \in \Lambda) \) is isometrically \( \mathcal{A} \)-module isomorphic to \( \mathcal{L}_c^1(X, l^\infty(Y_\lambda|\lambda \in \Lambda)) \).

(b) \( c_0(\mathcal{L}_c(X, Y_\lambda)|\lambda \in \Lambda) \) can be isometrically embedded as an \( \mathcal{A} \)-submodule into \( \mathcal{L}_c^1(X, c_0(Y_\lambda|\lambda \in \Lambda)) \).

(c) \( l^p(\mathcal{L}_c(X, Y_\lambda)|\lambda \in \Lambda) \) can be contractively embedded as an \( \mathcal{A} \)-submodule into \( \mathcal{L}_c^1(X, l^p(Y_\lambda|\lambda \in \Lambda)) \), for all \( 1 < p < \infty \).

Proof. (i) Let

\[
J : \mathcal{L}_c^1(X_1, Y) \oplus_p \mathcal{L}_c^1(X_2, Y) \to \mathcal{L}_c^1(X_1 \oplus_q X_2, Y)
\]

be the (isometric if \( p = \infty \)) isomorphism defined, as in Proposition 3.1.6(iii), by

\[
J(\phi_1 \oplus \phi_2)(x_1 \oplus x_2) = \phi_1(x_1) + \phi_2(x_2)
\]

for all \( \phi_1 \oplus \phi_2 \in \mathcal{L}_c^1(X_1, Y) \oplus \mathcal{L}_c^1(X_2, Y) \) and all \( x_1 \oplus x_2 \in X_1 \oplus X_2 \). To prove that \( J \) is an \( \mathcal{A} \)-module homomorphism consider \( a \in \mathcal{A}, \phi_1 \oplus \phi_2 \in \mathcal{L}_c^1(X_1, Y) \oplus \mathcal{L}_c^1(X_2, Y) \) and \( x_1 \oplus x_2 \in X_1 \oplus X_2 \). Then

\[
J(a(\phi_1 \oplus \phi_2))(x_1 \oplus x_2) = (a \circ \phi_1)(x_1) + (a \circ \phi_2)(x_2)
\]

\[
= \phi_1(x_1 a) + \phi_2(x_2 a)
\]

\[
= J(\phi_1 \oplus \phi_2)((x_1 \oplus x_2)a)
\]

\[
= (a \circ J(\phi_1 \oplus \phi_2))(x_1 \oplus x_2)
\]

(ii) Since \( \{X_\lambda|\lambda \in \Lambda\} \) is a family of uniformly bounded Banach \( \mathcal{A} \)-modules, \( \{\mathcal{L}_c^1(X_\lambda, Y)|\lambda \in \Lambda\} \) is also a family of uniformly bounded Banach \( \mathcal{A} \)-modules, because, as we mentioned in Proposition 3.2.1, the constant \( K \) does not depend on \( Y \). Let

\[
J : l^\infty(\mathcal{L}_c^1(X_\lambda, Y)|\lambda \in \Lambda) \to \mathcal{L}_c^1(l^1(X|\lambda \in \Lambda), Y)
\]

be the isometric isomorphism defined, as in Proposition 3.1.6(iv), by

\[
J(\sum_{\lambda \in \Lambda} \phi_\lambda)(\sum_{\lambda \in \Lambda} x_\lambda) = \sum_{\lambda \in \Lambda} \phi_\lambda(x_\lambda)
\]
for all \( \sum_{\lambda \in \Lambda} \Theta \phi_\lambda \in l^\infty(\mathcal{L}_c^1(X, Y), |\lambda \in \Lambda) \) and all \( \sum_{\lambda \in \Lambda} \Theta x_\lambda \in l^1(X, Y, |\lambda \in \Lambda) \).

We can prove, as in part (i), that \( J \) and its restrictions to \( l^p(\mathcal{L}_c^1(X, Y), |\lambda \in \Lambda), \)
\( 1 \leq p < \infty \), are \( \mathcal{A} \)-module homomorphisms.

(iii) Let
\[
J : \mathcal{L}_c^1(X, Y_1 \oplus_p \mathcal{L}_c^1(X, Y_2) \rightarrow \mathcal{L}_c^1(X, Y_1 \oplus_p Y_2)
\]
be the (isometric if \( p = \infty \)) isomorphism defined, as in Proposition 3.1.6(i), by
\[
J(\Theta \phi_1 \Theta \phi_2)(x) = \phi_1(x) + \phi_2(x)
\]
for all \( \phi_1 \Theta \phi_2 \in \mathcal{L}_c^1(X, Y_1) \oplus \mathcal{L}_c^1(X, Y_2) \) and for all \( x \in X \). If \( a \in \mathcal{A}, \phi_1 \Theta \phi_2 \in \mathcal{L}_c^1(X, Y_1) \oplus \mathcal{L}_c^1(X, Y_2) \) and \( x \in X \), then
\[
J(a(\phi_1 \Theta \phi_2))(x) = (a \circ \phi_1)(x) + (a \circ \phi_2)(x)
\]
\[
= \phi_1(xa) + \phi_2(xa)
\]
\[
= J(\phi_1 \Theta \phi_2)(xa)
\]
\[
= (a \circ J(\phi_1 \Theta \phi_2))(x)
\]
which proves that \( J \) is an \( \mathcal{A} \)-module homomorphism.

(iv) Since the constant \( K \) does not depend on \( Y \), the family \( \{\mathcal{L}_c^1(X, Y_1) | \lambda \in \Lambda\} \)
is a family of uniformly bounded \( \mathcal{A} \)-modules. Let
\[
J : l^\infty(\mathcal{L}_c^1(X, Y_1), |\lambda \in \Lambda) \rightarrow \mathcal{L}_c^1(X, l^\infty(Y, Y, |\lambda \in \Lambda))
\]
be the isometric isomorphism defined, as in Proposition 3.1.6(ii), by
\[
J(\sum_{\lambda \in \Lambda} \Theta \phi_\lambda)(x) = \sum_{\lambda \in \Lambda} \Theta \phi_\lambda(x)
\]
for all \( \sum_{\lambda \in \Lambda} \Theta \phi_\lambda \in l^\infty(\mathcal{L}_c^1(X, Y_1), |\lambda \in \Lambda) \) and all \( x \in X \). As in part (iii) we can prove that \( J \) and its restrictions to \( c_0(\mathcal{L}_c^1(X, Y_1), |\lambda \in \Lambda) \) and \( l^p(\mathcal{L}_c^1(X, Y_1), |\lambda \in \Lambda) \), for \( 1 \leq p < \infty \), are \( \mathcal{A} \)-module homomorphisms.

**Proposition 3.2.7.** Let \( \mathcal{A} \) be a Banach algebra, \( X \) be a Banach \( \mathcal{A} \)-bimodule and \( Y \) be a normed space. Then the following hold:

(i) If \( X_1 \) is a complemented \( \mathcal{A} \)-submodule of \( X \), then \( \mathcal{L}_c^1(X, Y)/\mathcal{L}_c^1(X_1, Y) \) is \( \mathcal{A} \)-module isomorphic to \( \mathcal{L}_c^1(X/X_1, Y) \).

(ii) If \( Y_1 \) is a closed subspace of \( Y \), then \( \mathcal{L}_c^1(X, Y)/\mathcal{L}_c^1(X, Y_1) \) can be isometrically embedded as an \( \mathcal{A} \)-submodule into \( \mathcal{L}_c^1(X, Y/Y_1) \). Moreover if \( Y_1 \) is a complemented subspace of \( Y \), then \( \mathcal{L}_c^1(X, Y)/\mathcal{L}_c^1(X, Y_1) \) is isometrically \( \mathcal{A} \)-module isomorphic to \( \mathcal{L}_c^1(X, Y/Y_1) \).
Proof. (i) It follows from Proposition 3.2.5(i) that $L_c^1(X, Y)$ is a closed $A$-submodule of $L_c^1(Y, X)$ and so $L_c^1(X, Y)/L_c^1(X_1, Y)$ is an $A$-bimodule. Since $X_1$ is a complemented submodule of $X$, there exists a bounded projection $\rho : X \to X$, with $\text{Im}(\rho) = X \ominus X_1$, which is an $A$-module homomorphism. Now to see that the isomorphism

$$J : L_c^1(X, Y)/L_c^1(X_1, Y) \to L_c^1(X/X_1, Y)$$

defined, as in Proposition 3.1.7(ii), by

$$J(\phi + L_c^1(X, Y))(x + X_1) = \phi(\rho(x))$$

for all $\phi + L_c^1(X_1, Y) \in L_c^1(X, Y)/L_c^1(X_1, Y)$ and all $x \in X$, is an $A$-module homomorphism, consider $a \in A$, $\phi \in L_c^1(X, Y)$ and $x + X_1 \in X/X_1$. Then

$$J(a(\phi + L_c^1(X_1, Y)))(x + X_1) = (a \circ \phi)(\rho(x))$$

$$= \phi(\rho(x)a)$$

$$= \phi(\rho(xa))$$

$$= J(\phi + L_c^1(X_1, Y))(xa + X_1)$$

$$= (a \circ J(\phi + L_c^1(X_1, Y)))(x + X_1).$$

(ii) By Proposition 3.2.5(ii), $L_c^1(X, Y_1)$ is a closed $A$-submodule of $L_c^1(X, Y)$ and thus $L_c^1(X, Y)/L_c^1(X, Y_1)$ is an $A$-bimodule. Let

$$J : L_c^1(X, Y)/L_c^1(X, Y_1) \to L_c^1(X, Y/Y_1)$$

be the isometric embedding (isomorphism if $Y_1$ is complemented in $Y$) defined, as in Proposition 3.1.7(i), by

$$J(\phi + L_c^1(X, Y_1))(x) = \phi(x) + Y_1$$

for all $\phi \in L_c^1(X, Y)$ and all $x \in X$. To show that $J$ is an $A$-module homomorphism consider $a \in A$, $\phi + L_c^1(X, Y_1) \in L_c^1(X, Y)/L_c^1(X, Y_1)$ and $x \in X$. Then

$$J(a(\phi + L_c^1(X, Y_1)))(x) = J((a \circ \phi) + L_c^1(X, Y_1))(x)$$

$$= (a \circ \phi)(x) + Y_1$$

$$= \phi(xa) + Y_1$$

$$= J(\phi + L_c^1(X, Y_1))(xa)$$

$$= (a \circ J(\phi + L_c^1(X, Y_1)))(x).$$

\qed
As in the first section it is easy to see that if $\mathcal{A}$ is a Banach algebra with unitisation $\tilde{\mathcal{A}}$, $X$ is a Banach $\mathcal{A}$-bimodule and $Y$ is a Banach space, then the $\tilde{\mathcal{A}}$-bimodules $\mathcal{L}_c^1(\tilde{X}, Y)$ and $\mathcal{L}_c^1(X, Y)$ coincide.

A result similar to that of Proposition 3.1.8, does not hold, i.e. the modules $\mathcal{L}_c^1(X, Y)$ do not "commute" with the reduction of dimension modules $\mathcal{L}_c^n(\mathcal{A}, X)$. To see that consider $a \in \mathcal{A}$, $\Phi \in \mathcal{L}_c^n(\mathcal{A}, \mathcal{L}_c^1(X, Y))$, $a_1, ..., a_n \in \mathcal{A}$, $x \in X$ and let $J : \mathcal{L}_c^n(\mathcal{A}, \mathcal{L}_c^1(X, Y)) \to \mathcal{L}_c^1(X, \mathcal{L}_c^n(\mathcal{A}, Y))$ be the canonical isomorphism. Then

$$J(\Phi a)(x)(a_1, ..., a_n) = (\Phi a)(a_1, ..., a_n)(x)$$

$$= \Phi(aa_1, ..., a_n)(x)$$

$$+ \sum_{1 \leq k \leq n-1} (-1)^k \Phi(a, a_1, ..., a_k a_{k+1}, ..., a_n)(x)$$

$$+ (-1)^n \Phi(a, a_1, ..., a_{n-1}) \circ a_n)(x)$$

$$= \Phi(aa_1, ..., a_n)(x)$$

$$+ \sum_{1 \leq k \leq n-1} (-1)^k \Phi(a, a_1, ..., a_k a_{k+1}, ..., a_n)(x)$$

$$+ (-1)^n \Phi(a, a_1, ..., a_{n-1})(a_n x)$$

wheras

$$(J(\Phi) \circ a)(x)(a_1, ..., a_n) = J(\Phi)(ax)(a_1, ..., a_n)$$

$$= \Phi(a_1, ..., a_n)(ax)$$

In the following proposition we discuss the relation between the cohomology, with coefficients in $X^*$, and the cohomology, with coefficients in $\mathcal{L}_c^1(X, Y)$.

**Proposition 3.2.8.** Let $\mathcal{A}$ be a Banach algebra, $X$ be a Banach $\mathcal{A}$-bimodule and $Y$ be a Banach space. Then the following hold:

(i) For all non-zero $y \in Y$, the map

$$F \mapsto F_y : \mathcal{L}_c^n(\mathcal{A}, X^*) \to \mathcal{L}_c^n(\mathcal{A}, \mathcal{L}_c^1(X, Y))$$

defined by $F_y(a_1, ..., a_n)(x) = F(a_1, ..., a_n)(x)y$, for all $F \in \mathcal{L}_c^n(\mathcal{A}, X^*)$, all $a_1, ..., a_n \in \mathcal{A}$ and all $x \in X$ is a one-to-one bounded map, with $\|F_y\| = \|y\|\|F\|$, for all $F \in \mathcal{L}_c^n(\mathcal{A}, X^*)$. Moreover $F_{ty + t'y'} = tF_y + t'F_{y'}$, for all $F \in \mathcal{L}_c^n(\mathcal{A}, X^*)$, all $y, y' \in Y$ and all $t, t' \in \mathbb{C}$.

(ii) For all non-zero $g \in Y^*$, the map

$$\Phi \mapsto \Phi_g : \mathcal{L}_c^n(\mathcal{A}, \mathcal{L}_c^1(X, Y)) \to \mathcal{L}_c^n(\mathcal{A}, X^*)$$

defined by $\Phi_g(a_1, ..., a_n)(x) = g(\Phi(a_1, ..., a_n)(x))$, for all $\Phi \in \mathcal{L}_c^n(\mathcal{A}, \mathcal{L}_c^1(X, Y))$, all $a_1, ..., a_n \in \mathcal{A}$ and all $x \in X$ is a bounded map, with $\|\Phi \mapsto \Phi_g\| = \|g\|$, that
maps $\mathcal{L}_c^n(A, \mathcal{L}_c^1(X, Y))$ onto $\mathcal{L}_c^n(A, X^*)$. Moreover $\Phi_{tg + t'g'} = t\Phi_g + t'\Phi_{g'}$, for all $\Phi \in \mathcal{L}_c^n(A, \mathcal{L}_c^1(X, Y))$, all $g, g' \in Y^*$ and all $t, t' \in \mathbb{C}$.

(iii) For all non-zero $y \in Y$, $F \mapsto F_y$ maps $\mathcal{L}_c^n(A, X^*: /B)$ into $\mathcal{L}_c^n(A, \mathcal{L}_c^1(X, Y): /B)$.

(iv) For all non-zero $g \in Y^*$, $\Phi \mapsto \Phi_g$ maps $\mathcal{L}_c^n(A, \mathcal{L}_c^1(X, Y): /B)$ onto $\mathcal{L}_c^n(A, X^*: /B)$.

(v) If

$$\partial^n : \mathcal{L}_c^n(A, X^*) \to \mathcal{L}_c^{n+1}(A, X^*)$$

and

$$\Delta^n : \mathcal{L}_c^n(A, \mathcal{L}_c^1(X, Y)) \to \mathcal{L}_c^{n+1}(A, \mathcal{L}_c^1(X, Y))$$

are the coboundary maps, then, for all $y \in Y$ and all $F \in \mathcal{L}_c^n(A, X^*)$,

$$\Delta^n(F_y) = (\partial^n(F))_y$$

and, for all $g \in Y^*$ and all $\Phi \in \mathcal{L}_c^n(A, \mathcal{L}_c^1(X, Y))$,

$$\partial^n(\Phi_g) = (\Delta^n(\Phi))_g.$$
Proposition 3.2.9. Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. Then the following hold:

(i) For all $a \in A$ and all $\phi_1, \phi_2 \in \mathcal{L}^1_c(X)$,

$$\phi_1(\phi_2 \circ a) = (\phi_1 \phi_2) \circ a$$

and

$$\phi_1(a \circ \phi_2) = a \circ (\phi_1 \phi_2)$$

(ii) If $\phi \in \mathcal{L}^1_c(X)$, then $\phi$ is an $A$-module homomorphism if and only if

$$(\psi \circ a)\phi = (\psi \phi) \circ a$$

and

$$(a \circ \psi)\phi = a \circ (\psi \phi)$$

for all $a \in A$ and all $\psi \in \mathcal{L}^1_c(X)$.

Proposition 3.2.10. Let $A$ be a Banach algebra, $B$ be a subalgebra of $A$ and $X$ and $Y$ be Banach $A$-bimodules. Then the following hold:

(i) For all $\phi \in \mathcal{L}^1_c(X,Y:B)$, $b \in B$ and $x \in X$,

$$(\phi \circ b)(x) = b\phi(x)$$

and

$$(b \circ \phi)(x) = \phi(x)b$$

(ii) If $B \subseteq Z(A)$, then $\mathcal{L}^1_c(X,Y:B)$ is a Banach $A$-bimodule.

3.2.2 The modules $\mathcal{L}^1_{cb}(X,Y)$

We mentioned after Proposition 3.2.1 that if $Y = \mathbb{C}$, then the module $\mathcal{L}^1_c(X,Y)$ coincides with the dual module of $X$. As we said in p.29, in the case of a completely bounded $A$-module $X$, the dual module of $X$ does not have the standard dual matricial norm structure, but the reversed tracial dual matricial norm structure. That shows that for the space $\mathcal{L}^1_{cb}(X,Y)$ to become a completely bounded $A$-module, with the module actions of Proposition 3.2.1, we have to use the reversed tracial matricial norm structure on $\mathcal{L}^1_{cb}(X,Y)$.

We recall from Remark 1.2.8(ii), that if $X$ is a matricially normed space and $Y$ is a normed space, then $\mathcal{L}^1_c(X,Y)$ becomes a matricially normed space, with

$$\| (\phi_{ij})^{r}_{m} \| = \sup \{ \| \sum_{1 \leq i,j \leq m} \phi_{ij}(x_{ji}) \| : \| (x_{ij}) \|_{m} \leq 1 \}$$
for all $m \in \mathbb{N}$ and all $(\phi_{ij}) \in \mathbb{M}_m(L^1_{cb}(X,Y))$. Moreover, if $Y$ is also a matricially normed space, then the reversed tracial matrix norms on $L^1_{cb}(X,Y)$ are defined by

$$
\|\phi_{ij}\|_{m}^{rt} = \sup\{\|\left( \sum_{1 \leq i,j \leq m} \phi_{ij}(x_{ij}^{st}) \right)\|_l : \|x_{ij}^{st}\|_{ml} \leq 1, l \in \mathbb{N}\}
$$

for all $m \in \mathbb{N}$ and all $(\phi_{ij}) \in \mathbb{M}_m(L^1_{cb}(X,Y))$ (Proposition 1.2.6).

We will show that $L^1_{cb}(X,Y)_{rt}$ is a completely bounded $\mathcal{A}$-bimodule if $X$ is a completely bounded $\mathcal{A}$-bimodule and $Y$ is a matricially normed space and discuss some of its properties. Similar results hold for $L^1_c(X,Y)_{rt}$ if $Y$ is a normed space.

**Proposition 3.2.11.** Let $\mathcal{A}$ be an operator algebra, $X$ be a completely bounded $\mathcal{A}$-bimodule and $Y$ be a matricially normed space. Then the space of completely bounded linear maps from $X$ into $Y$, with the reversed tracial matricial norm structure, $L^1_{cb}(X,Y)_{rt}$, becomes a completely bounded $\mathcal{A}$-bimodule, with the right and the left module actions of $\mathcal{A}$ on $L^1_{cb}(X,Y)$ defined by

$$(\phi \circ a)(x) = \phi(ax)$$

and

$$(a \circ \phi)(x) = \phi(xa)$$

for all $a \in \mathcal{A}$, all $\phi \in L^1_{cb}(X,Y)$ and all $x \in X$. Moreover if $X$ is an $L^1$ completely bounded $\mathcal{A}$-bimodule, then $L^1_{cb}(X,Y)_{rt}$ is an operator completely bounded $\mathcal{A}$-bimodule.

**Proof.** Since $X$ is a completely bounded $\mathcal{A}$-bimodule, there exists $K > 0$ with

$$\|(a_{ij})(x_{ij})\|_m \leq K\|(a_{ij})\|_m\|(x_{ij})\|_m$$

for all $m \in \mathbb{N}$, all $(x_{ij}) \in X$ and all $(a_{ij}) \in \mathbb{M}_m(\mathcal{A})$. By Proposition 3.2.1, $L^1_c(X,Y)$ is an $\mathcal{A}$-bimodule. Now if $\phi \in L^1_{cb}(X,Y)$, $a \in \mathcal{A}$, $m \in \mathbb{N}$ and $(x_{ij}) \in \mathbb{M}_m(X)$, then

$$\|((a \circ \phi)(x_{ij}))\|_m = \|(\phi(x_{ij}a))\|_m$$

$$\leq \|\phi\|_{cb}\|(x_{ij}a)\|_m$$

$$= \|\phi\|_{cb}\|(x_{ij})(a \otimes I_m)\|_m$$

$$\leq \|\phi\|_{cb}K\|(x_{ij})\|_m\|(a \otimes I_m)\|_m$$

$$= \|\phi\|_{cb}K\|(x_{ij})\|_m\|a\|$$

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which proves that \( a \circ \phi \) is completely bounded. Therefore \( \mathcal{L}_{cb}^1(X, Y) \) is an \( \mathcal{A} \)-bimodule. Moreover if \( m \in \mathbb{N} \), \((\phi_{ij}) \in \mathbb{M}_m(\mathcal{L}_{cb}^1(X, Y))\), \((a_{ij}) \in \mathbb{M}_m(\mathcal{A})\), \( l \in \mathbb{N} \) and \((x_{ij}^t) \in \mathbb{M}_{ml}(X)\), then

\[
\|( \sum_{1 \leq i,j \leq m} \sum_{1 \leq k \leq m} (a_{ik} \circ \phi_{kj})(x_{ji}^t) )\|_l = \|( \sum_{1 \leq i,j \leq m} \sum_{1 \leq k \leq m} \phi_{kj}(x_{ji}^t a_{ik}) )\|_l = \|( \sum_{1 \leq i,j \leq m} \phi_{ij}(\sum_{1 \leq k \leq m} x_{jk}^t a_{ki}) )\|_l = \|(\phi_{ij})\|_{m}^r \|\sum_{1 \leq k \leq m} x_{ik}^t a_{kj}\|_{ml} = \|(\phi_{ij})\|_{m}^r \|\sum_{1 \leq i,j \leq m} (a_{ij}) \otimes I_l\|_{ml} = K \|(\phi_{ij})\|_{m}^r \|\sum_{1 \leq i,j \leq m} (a_{ij})\|_{m}.
\]

Since the previous inequality holds for all \( l \in \mathbb{N} \) and all \((x_{ij}^t) \in \mathbb{M}_l(X)\), we get, by the definition of \( \|(\cdot)\|_{m}^r \), that

\[
\|( \sum_{1 \leq k \leq m} a_{ik} \circ \phi_{kj})\|_{m}^r \leq K \|(a_{ij})\|_{m} \|(\phi_{ij})\|_{m}^r
\]

which shows that \( \mathcal{L}_{cb}^1(X, Y)_{rt} \) is a completely bounded \( \mathcal{A} \)-bimodule.

From Proposition 1.2.6, if \( X \) is an \( L^1 \) matricially normed space, then \( \mathcal{L}_{cb}^1(X, Y)_{rt} \) is an operator space. Thus if \( X \) is an \( L^1 \) completely bounded \( \mathcal{A} \)-bimodule, then \( \mathcal{L}_{cb}^1(X, Y)_{rt} \) is an operator completely bounded \( \mathcal{A} \)-bimodule.

From Remark 1.2.8(i), if \( Y = \mathbb{C} \), then \( \mathcal{L}_{cb}^1(X, Y)_{rt} \) coincides with the dual \( \mathcal{A} \)-bimodule of \( X \).

To extend the map \( f \mapsto f_y \) that we defined in Section 3.2.1 to the completely bounded case we need \( Y \) to be an operator space. As in Section 3.2.1, for each \( y \in Y \), we define

\[
f \mapsto f_y : X^*_{rt} \rightarrow \mathcal{L}_{cb}^1(X, Y)_{rt}
\]

by \( f_y(x) = f(x)y \), for all \( f \in X^* \) and all \( x \in X \). To see that this map is well-defined take \( f \in X^* \), \( m \in \mathbb{N} \) and \((x_{ij}) \in \mathbb{M}_m(X)\). Then, from the \( L^\infty \) property of \( Y \) and the complete boundedness of \( f \), we have

\[
\|(f_y(x_{ij}))\|_m = \|(f(x_{ij})y)\|_m \\
\leq \|(f(x_{ij}))\|_m \|y \otimes I_m\|_m \\
= \|(f(x_{ij}))\|_m \|y\| \\
\leq \|f\| \|(x_{ij})\|_m \|y\|.
\]
which shows that $f_y$ is completely bounded.

Moreover if $m, l \in \mathbb{N}$, $(f_{ij}) \in \mathbb{M}_m(X^*)$ and $(x_{ij}^*) \in \mathbb{M}_{ml}(X)$, then

$$
\|(\sum_{1 \leq i,j \leq m} (f_{ij})_y(x_{ij}^*))\|_t = \|(\sum_{1 \leq i,j \leq m} f_{ij}(x_{ij}^*))y\|_t
\leq \|(\sum_{1 \leq i,j \leq m} f_{ij}(x_{ij}^*))\|\|y \otimes I_t\|_t
= \|(\sum_{1 \leq i,j \leq m} f_{ij}(x_{ij}^*))\|\|y\|
$$

from the $L^\infty$ property of $Y$. Now, by the definition of the matrix norms on $\mathcal{L}_{cb}^1(X,Y)_t$ and on $X^*_t$,

$$
\|(f_{ij})_y\|_{rt}^t \leq \|y\|\|(f_{ij})\|_{mt}^t
$$

and thus $f \mapsto f_y$ is completely bounded, with $\|f \mapsto f_y\|_{cb} \leq \|y\|$. From the results following Proposition 3.2.1, we have that $\|f_y\| = \|y\|\|f\|$, for all $f \in X^*$ and hence $\|f \mapsto f_y\|_{cb} = \|y\|$, which shows that $X^*_t$ is completely isomorphic to $\{f_y \mid f \in X^*\}$.

Again as in Section 3.2.1, for each $g \in Y^*$, we define

$$
\phi \mapsto \phi_g : \mathcal{L}_{cb}^1(X,Y)_t \to X^*_t
$$

with $\phi_g(x) = g(\phi(x))$, for all $\phi \in \mathcal{L}_{cb}^1(X,Y)$ and all $x \in X$. If $m \in \mathbb{N}$, $(\phi_{ij}) \in \mathbb{M}_m(\mathcal{L}_{cb}^1(X,Y))$ and $(x_{ij}) \in \mathbb{M}_m(X)$, then

$$
| \sum_{1 \leq i,j \leq m} (\phi_{ij})_g(x_{ji}) | = | g(\sum_{1 \leq i,j \leq m} \phi_{ij}(x_{ji})) |
\leq \|g\|\|\sum_{1 \leq i,j \leq m} \phi_{ij}(x_{ji})\|
\leq \|g\|\|(\phi_{ij})\|_{mt}^t\|(x_{ij})\|_m
$$

and hence

$$
\|((\phi_{ij})_g)\|_{mt}^t \leq \|g\|\|(\phi_{ij})\|_{mt}^t
$$

which shows that $\phi \mapsto \phi_g$ is completely bounded, with $\|\phi \mapsto \phi_g\|_{cb} \leq \|g\|$. Moreover, using the Hahn-Banach theorem, we get that $\|\phi \mapsto \phi_g\|_{cb} = \|g\|$. A calculation similar to the previous one shows that if $m, l \in \mathbb{N}$, $(\phi_{ij}) \in \mathbb{M}_m(\mathcal{L}_{cb}^1(X,Y)_t)$ and $(g_{st}) \in \mathbb{M}_t(Y^*_t)$, then $\|((\phi_{ij})_{g_{st}})\|_{mt}^t \leq \|g_{st}\|\|(\phi_{ij})\|_{mt}^t$.

Combining the above two maps we get a completely bounded projection mapping $\mathcal{L}_{cb}^1(X,Y)_t$ onto $\{f_y \mid f \in X^*\}$, which shows that $\{f_y \mid f \in X^*\}$ is a completely complemented $\mathcal{A}$-submodule of $\mathcal{L}_{cb}^1(X,Y)_t$, if $Y$ is an operator space.

Let's see now what happens with duality and normality.
Proposition 3.2.12. Let $A$ be an operator algebra, $X$ be an $L^1$ completely bounded $A$-bimodule and $Y$ be a reversed tracial dual $L^1$ matricially normed space. Then $\mathcal{L}_{cb}(X,Y)_{rt}$ is a dual operator completely bounded $A$-bimodule.

Proof. Since $Y$ is a reversed tracial dual $L^1$ matricially normed space, there exists an operator space $Y_*$ such that $Y = (Y_*)_{rt}$. Therefore, by Proposition 1.2.9, $X_{rt} \otimes Y_*$ is an $L^1$ matricially normed space, with $\mathcal{L}_{cb}(X,Y)_{rt} \simeq (X_{rt} \otimes Y_*)_{rt}$. Since $X_{rt} \otimes Y_*$ becomes a completely bounded $A$-bimodule if we define $a(x \otimes z) = (ax) \otimes z$ and $(x \otimes z)a = (xa) \otimes z$, for all $a \in A$ and all $x \otimes z \in X \otimes Y_*$, $\mathcal{L}_{cb}(X,Y)_{rt}$ is a dual operator completely bounded $A$-bimodule. \qed

Proposition 3.2.13. Let $A$ be an operator algebra, $X$ be an $L^1$ completely bounded $A$-bimodule and $Y$ be a reversed tracial dual $L^1$ matricially normed space. If $X^*$ is a normal $A$-bimodule, then $\mathcal{L}_{cb}^1(X,Y)_{rt}$ is a normal operator completely bounded $A$-bimodule.

Proof. It follows from Propositions 3.2.12 and 3.2.3. \qed

As in the previous part we examine now what happens with module homomorphisms, submodules, direct sums and quotients.

Proposition 3.2.14. Let $A$ be an operator algebra, $X_1$ and $X_2$ be completely bounded $A$-bimodules and $Y_1$ and $Y_2$ be matricially normed spaces.

(i) If $\tau : X_1 \rightarrow X_2$ is a completely bounded $A$-module homomorphism, then the map

$$\tau_{Y_1} : \mathcal{L}_{cb}^1(X_2,Y_1)_{rt} \rightarrow \mathcal{L}_{cb}^1(X_1,Y_1)_{rt}$$

defined by $\tau_{Y_1}(\phi) = \phi \tau$, for all $\phi \in \mathcal{L}_{cb}^1(X_2,Y_1)$, is a completely bounded $A$-module homomorphism, with $\|\tau_{Y_1}\|_cb \leq \|\tau\|_cb$.

(ii) If $\pi : Y_1 \rightarrow Y_2$ is a completely bounded linear map, then the map

$$\pi_{X_1} : \mathcal{L}_{cb}^1(X_1,Y_1)_{rt} \rightarrow \mathcal{L}_{cb}^1(X_1,Y_2)_{rt}$$

defined by $\pi_{X_1}(\phi) = \pi \phi$, for all $\phi \in \mathcal{L}_{cb}^1(X_1,Y_1)$, is a completely bounded $A$-module homomorphism, with $\|\pi_{X_1}\|_cb \leq \|\pi\|_cb$.

Proof. It is exactly the same with the proof of Proposition 3.1.15 with the standard matrix norms replaced by the reversed tracial ones. \qed

Proposition 3.2.15. Let $A$ be an operator algebra, $X$ be a completely bounded $A$-bimodule and $Y$ be a matricially normed space.

(i) If $X_1$ is a completely complemented $A$-submodule of $X$, then $\mathcal{L}_{cb}^1(X_1,Y)_{rt}$ is a completely complemented $A$-submodule of $\mathcal{L}_{cb}^1(X,Y)_{rt}$.
(ii) If $Y_1$ is a closed subspace of $Y$, then $\mathcal{L}_{cb}^1(X, Y_1)_{rt}$ is a closed $A$-submodule of $\mathcal{L}_{cb}^1(X, Y)_{rt}$. Moreover if $Y_1$ is a completely complemented subspace of $Y$, then $\mathcal{L}_{cb}^1(X, Y_1)_{rt}$ is a completely complemented $A$-submodule of $\mathcal{L}_{cb}^1(Y, X)_{rt}$.

(iii) If $X_1$ is a completely complemented $A$-submodule of $X$ and $Y_1$ is a closed subspace of $Y$, then $\mathcal{L}_{cb}^1(X_1, Y_1)_{rt}$ is a closed $A$-submodule of $\mathcal{L}_{cb}^1(X, Y)_{rt}$. Moreover if $Y_1$ is a completely complemented subspace of $Y$, then $\mathcal{L}_{cb}^1(X_1, Y_1)_{rt}$ is a completely complemented $A$-submodule of $\mathcal{L}_{cb}^1(X, Y)_{rt}$.

Proof. It follows from Proposition 3.2.14 in a similar manner to the proof of Proposition 3.2.5. □

Proposition 3.2.16. Let $A$ be an operator algebra. Then the following hold:

(i) If $\{X_\lambda | \lambda \in \Lambda\}$ is a family of uniformly bounded completely bounded $A$-bimodules and $Y$ is a matricially normed space, then:

(a) $l^{\infty}(\mathcal{L}_{cb}^1(X_\lambda, Y) | \lambda \in \Lambda)$ can be completely contractively embedded as an $A$-submodule into $\mathcal{L}_{cb}^1(l^1(X_\lambda | \lambda \in \Lambda), Y)$.

(b) $l^1(\mathcal{L}_{cb}^1(X_\lambda, Y) | \lambda \in \Lambda)$ can be completely contractively embedded as an $A$-submodule into $\mathcal{L}_{cb}^1(l^{\infty}(X_\lambda | \lambda \in \Lambda), Y)$.

(iv) If $X$ is a completely bounded $A$-bimodule and $\{Y_\lambda | \lambda \in \Lambda\}$ is a family of matricially normed spaces, then:

(a) $l^{\infty}(\mathcal{L}_{cb}^1(X, Y_\lambda) | \lambda \in \Lambda)$ is completely isometrically $A$-module isomorphic to $\mathcal{L}_{cb}^1(X, l^{\infty}(Y_\lambda | \lambda \in \Lambda))$.

(b) $l^1(\mathcal{L}_{cb}^1(X, Y_\lambda) | \lambda \in \Lambda)$ can be completely contractively embedded as an $A$-submodule into $\mathcal{L}_{cb}^1(X, l^1(Y_\lambda | \lambda \in \Lambda))$.

Similar results hold in the finite case.

Proof. It is exactly the same with the proof of Proposition 3.1.17 with the standard matrix norms replaced by the reversed tracial ones. □

Proposition 3.2.17. Let $A$ be an operator algebra, $X$ be a completely bounded $A$-bimodule and $Y$ be a matricially normed space. Then the following hold:

(i) If $X_1$ is a completely complemented $A$-submodule of $X$, then $\mathcal{L}_{cb}^1(X, Y)_{rt}/\mathcal{L}_{cb}^1(X_1, Y)_{rt}$ is completely $A$-module isomorphic to $\mathcal{L}_{cb}^1(X/X_1, Y)_{rt}$.

(ii) If $Y_1$ is a closed subspace of $Y$, then $\mathcal{L}_{cb}^1(X, Y)_{rt}/\mathcal{L}_{cb}^1(X, Y_1)_{rt}$ can be completely embedded as an $A$-submodule into $\mathcal{L}_{cb}^1(X, Y/Y_1)_{rt}$. Moreover if $Y_1$ is a completely complemented subspace of $Y$, then $\mathcal{L}_{cb}^1(X, Y)_{rt}/\mathcal{L}_{cb}^1(X, Y_1)_{rt}$ is completely $A$-module isomorphic to $\mathcal{L}_{cb}^1(X, Y/Y_1)_{rt}$.

Proof. It is exactly the same with the proof of Proposition 3.1.18 with the standard matrix norms replaced by the reversed tracial ones. □
The following proposition describes the relation between the cohomology, with coefficients in $X_{rt}^*$, and the cohomology, with coefficients in $L_{cb}^1(X,Y)_{rt}$.

**Proposition 3.2.18.** Let $A$ be an operator algebra, $X$ be an $L^1$ completely bounded $A$-bimodule and $Y$ be an operator space. Then the following hold:

(i) For all non-zero $y \in Y$ the map,

$$F \mapsto F_y : L_{cb}^n(A,X_{rt}^*) \to L_{cb}^n(A,L_{cb}^1(X,Y)_{rt})$$

defined by $F_y(a_1,...,a_n)(x) = F(a_1,...,a_n)(x)y$, for all $F \in L_{cb}^n(A,X_{rt}^*)$, all $a_1,...,a_n \in A$ and all $x \in X$, is a one-to-one completely bounded map, with $\|F_y\|_{cb} \leq \|y\|\|F\|_{cb}$, for all $F \in L_{cb}^n(A,X_{rt}^*)$. Moreover $F_{ty+t'y'} = tF_y + t'F_{y'}$, for all $F \in L_{cb}^n(A,X_{rt}^*)$, all $y,y' \in Y$ and all $t,t' \in \mathbb{C}$.

(ii) For all non-zero $g \in Y^*$, the map

$$F \mapsto \Phi_g : L_{cb}^n(A,L_{cb}^1(X,Y)_{rt}) \to L_{cb}^n(A,X_{rt}^*)$$

defined by $\Phi_g(a_1,...,a_n)(x) = g(\Phi(a_1,...,a_n))(x)$, for all $\Phi \in L_{cb}^n(A,L_{cb}^1(X,Y)_{rt})$, all $a_1,...,a_n \in A$ and all $x \in X$, is a completely bounded map, with $\|\Phi \mapsto \Phi_g\|_{cb} \leq \|g\|$, that maps $L_{cb}^n(A,L_{cb}^1(X,Y)_{rt})$ onto $L_{cb}^n(A,X_{rt}^*)$. Moreover $\Phi_{tg + t'g'} = t\Phi_g + t'\Phi_{g'}$, for all $\Phi \in L_{cb}^n(A,L_{cb}^1(X,Y)_{rt})$, all $g,g' \in Y^*$ and all $t,t' \in \mathbb{C}$. If $m,l \in \mathbb{N}$, $(\Phi_{ij}) \in M_m(L_{cb}^n(A,L_{cb}^1(X,Y)_{rt}))$ and $(g_{st}) \in M_l(Y_{rt}^*)$, then $\|((\Phi_{ij})_{g_{st}})\|_{mt} \leq \|(g_{st})\|t\|\{(\Phi_{ij})\}\|_m$.

(iii) For all non-zero $y \in Y$, $F \mapsto F_y$ maps $L_{cb}^n(A,X_{rt}^*) : /B$ into $L_{cb}^n(A,L_{cb}^1(X,Y)_{rt}) : /B$.

(iv) For all non-zero $g \in Y^*$, $\Phi \mapsto \Phi_g$ maps $L_{cb}^n(A,L_{cb}^1(X,Y)_{rt}) : /B$ onto $L_{cb}^n(A,X_{rt}^*) : /B$.

(v) If

$$\partial^n : L_{cb}^n(A,X_{rt}^*) \to L_{cb}^{n+1}(A,X_{rt}^*)$$

and

$$\Delta^n : L_{cb}^n(A,L_{cb}^1(X,Y)_{rt}) \to L_{cb}^{n+1}(A,L_{cb}^1(X,Y)_{rt})$$

are the coboundary maps, then, for all $y \in Y$ and all $F \in L_{cb}^n(A,X_{rt}^*)$,

$$\Delta^n(F_y) = (\partial^n(F))_y$$

and for all $g \in Y^*$ and all $\Phi \in L_{cb}^n(A,L_{cb}^1(X,Y)_{rt})$,

$$\partial^n(\Phi_g) = (\Delta^n(\Phi))_g.$$
(vi) For all non-zero \( y \in Y \), \( F \mapsto F_y \) maps \( Z_{cb}^n(A, X_r^*) \) into \( Z_{cb}^n(A, L_{cb}^1(X, Y)_{rt}) \), \( B_{cb}^n(A, X_r^*) \) into \( B_{cb}^n(A, L_{cb}^1(X, Y)_{rt}) \) and \( H_{cb}^n(A, X_r^*) \) into \( H_{cb}^n(A, L_{cb}^1(X, Y)_{rt}) \).

(vii) For all non-zero \( g \in Y^* \), \( \Phi \mapsto \Phi_g \) maps \( Z_{cb}^n(A, L_{cb}^1(X, Y)_{rt}) \) onto \( Z_{cb}^n(A, X_r^*) \), \( B_{cb}^n(A, L_{cb}^1(X, Y)_{rt}) \) onto \( B_{cb}^n(A, X_r^*) \) and \( H_{cb}^n(A, L_{cb}^1(X, Y)_{rt}) \) onto \( H_{cb}^n(A, X_r^*) \).

(viii) For all non-zero \( y \in Y \), \( F \mapsto F_y \) maps \( Z_{cb}^n(A, L_{cb}^1(X, Y)_{rt}) \) onto \( Z_{cb}^n(A, L_{cb}^1(X, Y)_{rt} : /B) \) and \( H_{cb}^n(A, L_{cb}^1(X, Y)_{rt}) \) onto \( H_{cb}^n(A, L_{cb}^1(X, Y)_{rt} : /B) \).

(ix) For all non-zero \( g \in Y^* \), \( \Phi \mapsto \Phi_g \) maps \( Z_{cb}^n(A, L_{cb}^1(X, Y)_{rt} : /B) \) onto \( Z_{cb}^n(A, L_{cb}^1(X, Y)_{rt} : /B) \) and \( H_{cb}^n(A, L_{cb}^1(X, Y)_{rt} : /B) \) onto \( H_{cb}^n(A, L_{cb}^1(X, Y)_{rt} : /B) \).

Parts (ii), (iv), (vii) and (ix) hold if \( Y \) is just a matricially normed space.

Proof. The proofs of (i) and (ii) are generalisations of the ones in pp.112-114. The rest is similar to the proof of Proposition 3.1.9.

Results similar to the ones in Propositions 3.2.9 and 3.2.10 also hold here.

### 3.3 Relations between the modules \( L_{cb}^1(Y, X) \) and the modules \( L_{cb}^1(X, Y) \)

Let \( X \) be a Banach \( A \)-bimodule and \( Y \) be a Banach space. Then as in the second section we can make the space of bounded linear maps from \( X \) into \( Y^* \), \( L_{cb}^1(X, Y^*) \), into an \( A \)-bimodule. On the other hand as in the first section we can make \( L_{cb}^1(Y, X^*) \) into an \( A \)-bimodule. So the isomorphic spaces \( L_{cb}^1(X, Y^*) \) and \( L_{cb}^1(Y, X^*) \) are both \( A \)-bimodules. We will prove that those two \( A \)-bimodules are \( A \)-module isomorphic.

**Proposition 3.3.1.** Let \( A \) be a Banach algebra, \( X \) be a Banach \( A \)-bimodule and \( Y \) be a Banach space. Then \( L_{cb}^1(X, Y^*) \) and \( L_{cb}^1(Y, X^*) \) are isometrically \( A \)-module isomorphic.

**Proof.** Let

\[
J : L_{cb}^1(X, Y^*) \to L_{cb}^1(Y, X^*)
\]

be the isometric isomorphism defined by \( J(\phi)(y)(x) = \phi(x)(y) \), for all \( \phi \in L_{cb}^1(X, Y^*) \), all \( x \in X \) and all \( y \in Y \). To prove that \( J \) is an \( A \)-module homomorphism just consider \( a \in A \), \( \phi \in L_{cb}^1(X, Y^*) \), \( x \in X \) and \( y \in Y \). Then

\[
J(\phi \circ a)(y)(x) = (\phi \circ a)(x)(y) = \phi(ax)(y) = J(\phi)(y)(ax)
\]

\[
= (J(\phi)(y)a)(x) = (J(\phi)a)(y)(x)
\]

\[\square\]
As we proved in Proposition 1.2.10(ii) (see also Remark 1.2.10), \( L_{cb}^1(V, U_{ri}) \) is completely isometrically isomorphic to \( L_{cb}^1(U, V_{ri})_{rt} \), for all matricially normed spaces \( V \) and \( U \). That leads to a result similar to that of the previous proposition for the completely bounded case.

**Proposition 3.3.2.** Let \( \mathcal{A} \) be an operator algebra, \( X \) be a completely bounded \( \mathcal{A} \)-bimodule and \( Y \) be a matricially normed space. Then \( L_{cb}^1(X, Y_{ri})_{rt} \) and \( L_{cb}^1(Y, X_{ri}) \) are completely isometrically \( \mathcal{A} \)-module isomorphic.

**Remark 3.3.1.** Combining the results of the two previous propositions with the results of Propositions 3.1.2, 3.1.3, 3.1.13 and 3.1.14 we get alternative proofs of Propositions 3.2.2, 3.2.3, 3.2.12 and 3.2.13 (and vice versa).

If both \( X_1 \) and \( X_2 \) are \( \mathcal{A} \)-bimodules, then we can define both the module structure of Section 3.1 and that of Section 3.2 on \( L_c^1(X_1, X_2) \). Propositions 3.1.11(i) and 3.2.10(i) give the following relation between those module actions.

**Corollary 3.3.1.** Let \( \mathcal{A} \) be a Banach algebra, \( \mathcal{B} \) be a subalgebra of \( \mathcal{A} \) and \( X_1 \) and \( X_2 \) be Banach \( \mathcal{A} \)-bimodules. Then

\[
b\phi = \phi \circ b
\]

and

\[
\phi b = b \circ \phi
\]

for all \( b \in \mathcal{B} \) and all \( \phi \in L_c^1(X_1, X_2 : /\mathcal{B}) \).

A similar result holds in the completely bounded case.

### 3.4 Modules of maps between \( \mathcal{A} \)-modules

In the first section we defined module actions of \( \mathcal{A} \) on the space of maps \( L_c^1(Y, X) \) from a space \( Y \) into an \( \mathcal{A} \)-module \( X \) and in the second section we defined module actions of \( \mathcal{A} \) on the space \( L_c^1(X, Y) \) of maps from an \( \mathcal{A} \)-module \( X \) into a space \( Y \).

If \( X_1 \) and \( X_2 \) are both left Banach \( \mathcal{A} \)-modules, then we can define, as in Section 3.1, a left module action of \( \mathcal{A} \) on \( L_c^1(X_1, X_2) \) and, as in Section 3.2, a right module action of \( \mathcal{A} \) on \( L_c^1(X_1, X_2) \). In this section we are going to prove that those module actions make \( L_c^1(X_1, X_2) \) into an \( \mathcal{A} \)-bimodule and then using some of the results obtained in the first two sections (or to be more precise their counterparts for left \( \mathcal{A} \)-modules) show how certain properties of \( X_1 \) and \( X_2 \) are related to properties of \( L_c^1(X_1, X_2) \). We present the results when \( X_1 \) and \( X_2 \) are left \( \mathcal{A} \)-modules; they also hold for right \( \mathcal{A} \)-modules.
Unfortunately in the completely bounded case the use of different matricial norm structures in the definition of the modules \( \mathcal{L}^1_{cb}(Y,X) \) and \( \mathcal{L}^1_{cb}(X,Y) \) does not allow us to make a similar construction.

We start with the definition of the modules \( \mathcal{L}^1_c(X_1, X_2) \)

**Proposition 3.4.1.** Let \( A \) be a Banach algebra and \( X_1 \) and \( X_2 \) be Banach left \( A \)-modules. Then \( \mathcal{L}^1_c(X_1, X_2) \) becomes a Banach \( A \)-bimodule, with the left and the right module actions of \( A \) on \( \mathcal{L}^1_c(X_1, X_2) \) defined respectively by

\[
(a \phi)(x_1) = a \phi(x_1)
\]

and

\[
(\phi \circ a)(x_1) = \phi(ax_1)
\]

for all \( a \in A \), all \( \phi \in \mathcal{L}^1_c(X_1, X_2) \) and all \( x_1 \in X_1 \).

**Proof.** From Propositions 3.1.1 and 3.2.1 respectively, \( \mathcal{L}^1_c(X_1, X_2) \) is a Banach left and right \( A \)-module. Moreover it is easy to see that \((a_1 \phi) \circ a_2 = a_1 (\phi \circ a_2)\), for all \( a_1, a_2 \in A \) and all \( \phi \in \mathcal{L}^1_c(X_1, X_2) \), and therefore \( \mathcal{L}^1_c(X_1, X_2) \) is a Banach \( A \)-bimodule. \( \square \)

The remarks following Propositions 3.1.1 and 3.2.1 respectively show that \( \mathcal{L}^1_c(X_1, X_2) \) is related to \( X_2 \) as a left \( A \)-module and to \( X_1^* \) as a right \( A \)-module.

The two following propositions discuss duality and normality for \( \mathcal{L}^1_c(X_1, X_2) \)

**Proposition 3.4.2.** Let \( A \) be a Banach algebra and \( X_1 \) and \( X_2 \) be Banach left \( A \)-modules. If \( X_2 \) is a dual left \( A \)-module, then \( \mathcal{L}^1_c(X_1, X_2) \) is a dual \( A \)-bimodule.

**Proof.** Since \( X_2 \) is a dual left \( A \)-module, \( \mathcal{L}^1_c(X_1, X_2) \) is a dual left \( A \)-module from Proposition 3.1.2. Moreover since \( X_2 \) is a dual left \( A \)-module it is a dual Banach space. Thus, by Proposition 3.2.2, \( \mathcal{L}^1_c(X_1, X_2) \) is a dual right \( A \)-module.

Alternatively we can give a constructive proof. First we make \( X_1 \otimes (X_2)^* \) into an \( A \)-bimodule with module actions defined by \( a(x_1 \otimes z) = (ax_1) \otimes z \) and \((x_1 \otimes z)a = x_1 \otimes (za)\), for all \( a \in A \), all \( x_1 \in X_1 \) and all \( z \in (X_2)^* \), and then show that \( \mathcal{L}^1_c(X_1, X_2) \) is the dual \( A \)-bimodule of \( X_1 \otimes (X_2)^* \). \( \square \)

**Remark 3.4.1.** (i) The module actions of \( A \) on \( X_1 \otimes (X_2)^* \), defined in the alternative proof of the previous proposition have been used in the proof of the equivalence between \( n \)-amenability and the existence of an \( n \)-virtual diagonal.

(ii) If \( A \) is an operator algebra and \( X_1 \) and \( X_2 \) are completely bounded left \( A \)-modules, then we can easily see that neither \( X_1 \otimes \mathbb{B}(X_2) \) nor \( X_1 \otimes \mathbb{B}(X_2) \) becomes
a completely bounded $A$-bimodule with the module actions defined as in the alternative proof of the previous proposition. On the other hand a straightforward calculation shows that $X_1 \otimes_h X_2$ is a completely bounded $A$-bimodule with respect to those module actions. Hence $(X_1 \otimes_h X_2)^*_t$ is a dual completely bounded $A$-bimodule. Unfortunately there are no results describing the reversed tracial dual of the Haagerup tensor product. Such a description would give a completely bounded analogue of the construction discussed in this section.

**Proposition 3.4.3.** Let $A$ be an operator algebra and $X_1$ and $X_2$ be Banach left $A$-modules. If the dual right $A$-module $X_1^*$ of $X_1$ is a normal right $A$-module and $X_2$ is a normal left $A$-module, then $\mathcal{L}^1_c(X_1, X_2)$ is a normal $A$-bimodule.

**Proof.** It follows from Propositions 3.1.3 and 3.2.3. \qed

The situation with respect to module homomorphisms, submodules, direct sums and quotients is described in the following four propositions.

**Proposition 3.4.4.** Let $A$ be a Banach algebra and $X_1$, $X$, $X_2$ and $X_4$ be Banach left $A$-modules.

(i) If $\tau : X_1 \to X_4^\#$ is a bounded left $A$-module homomorphism, then the map

$$\tau_{X_2} : \mathcal{L}^1_c(X_1, X_2) \to \mathcal{L}^1_c(X_1, X_2)$$

defined by $\tau_{X_2}(\phi) = \phi \tau$, for all $\phi \in \mathcal{L}^1_c(X_1, X_2)$, is a bounded two-sided $A$-module homomorphism, with $\|\tau_{X_2}\| = \|\tau\|$.

(ii) If $\pi : X_2 \to X_2^\#$ is a bounded left $A$-module homomorphism, then the map

$$\pi_{X_1} : \mathcal{L}^1_c(X_1, X_2) \to \mathcal{L}^1_c(X_1, X_2)$$

defined by $\pi_{X_1}(\phi) = \pi \phi$, for all $\phi \in \mathcal{L}^1_c(X_1, X_2)$, is a bounded two-sided $A$-module homomorphism, with $\|\pi_{X_1}\| = \|\pi\|$.

**Proof.** It follows from Propositions 3.1.4 and 3.2.4. \qed

**Proposition 3.4.5.** Let $A$ be a Banach algebra and $X_1$ and $X_2$ be Banach left $A$-modules.

(i) If $X_1^\#$ is a complemented left $A$-submodule of $X_1$, then $\mathcal{L}^1_c(X_1, X_2)$ is a complemented two-sided $A$-submodule of $\mathcal{L}^1_c(X_1, X_2)$.

(ii) If $X_2^\#$ is a closed left $A$-submodule of $X_2$, then $\mathcal{L}^1_c(X_1, X_2)$ is a closed two-sided $A$-submodule of $\mathcal{L}^1_c(X_1, X_2)$. Moreover if $X_2^\#$ is a complemented left $A$-submodule of $X_2$, then $\mathcal{L}^1_c(X_1, X_2^\#)$ is a complemented two-sided $A$-submodule of $\mathcal{L}^1_c(X_1, X_2)$.
(iii) If $X_1^\dagger$ is a complemented left $A$-submodule of $X_1$ and $X_2^\dagger$ is a closed left $A$-submodule of $X_2$, then $\mathcal{L}^1_c(X_1^\dagger, X_2^\dagger)$ is a closed two-sided $A$-submodule of $\mathcal{L}^1_c(X_1, X_2)$. Moreover if $X_2^\dagger$ is a complemented left $A$-submodule of $X_2$, then $\mathcal{L}^1_c(X_1^\dagger, X_2^\dagger)$ is a complemented two-sided $A$-submodule of $\mathcal{L}^1_c(X_1, X_2)$.

Proof. It follows from Propositions 3.1.5 and 3.2.5.

Proposition 3.4.6. Let $A$ be a Banach algebra. Then the following hold:

(i) If $X_1$, $X_1^\dagger$ and $X_2$ are Banach left $A$-modules, then, for all conjugate $1 \leq p, q \leq \infty$, $\mathcal{L}^1_c(X_1, X_2)\oplus \mathcal{L}^1_c(X_1^\dagger, X_2)$ and $\mathcal{L}^1_c(X_1\oplus X_1^\dagger, X_2)$ are two-sided $A$-module isomorphic. Moreover if $p = \infty$ and $q = 1$, then they are isometrically two-sided $A$-module isomorphic.

(ii) If $\{X_1^\lambda|\lambda \in \Lambda\}$ is a family of uniformly bounded Banach left $A$-modules and $X_2$ is a Banach left $A$-module, then:

(a) $l^\infty(\mathcal{L}^1_c(X_1^\lambda, X_2)|\lambda \in \Lambda)$ is isometrically two-sided $A$-module isomorphic to $\mathcal{L}^1_c(l^1(X_1^\lambda|\lambda \in \Lambda), X_2)$.

(b) $l^1(\mathcal{L}^1_c(X_1^\lambda, X_2)|\lambda \in \Lambda)$ can be contractively embedded as a two-sided $A$-submodule into $\mathcal{L}^1_c(c_0(X_1^\lambda|\lambda \in \Lambda), X_2)$.

(c) $l^p(\mathcal{L}^1_c(X_1^\lambda, Y)|\lambda \in \Lambda)$ can be contractively embedded as a two-sided $A$-submodule into $\mathcal{L}^1_c(l^q(X_1^\lambda|\lambda \in \Lambda), X_2)$, for all conjugate $1 < p, q < \infty$.

(iii) If $X_1$, $X_2$ and $X_2^\dagger$ are Banach left $A$-modules, then $\mathcal{L}^1_c(X_1, X_2)\oplus \mathcal{L}^1_c(X_1, X_2^\dagger)$ and $\mathcal{L}^1_c(X_1, X_2\oplus X_2^\dagger)$ are two-sided $A$-module isomorphic, for all $1 \leq p \leq \infty$. Moreover they are isometrically two-sided $A$-module isomorphic if $p = \infty$.

(iv) If $X_1$ is a Banach left $A$-module and $\{X_2^\lambda|\lambda \in \Lambda\}$ is a family of uniformly bounded Banach left $A$-modules, then:

(a) $l^\infty(\mathcal{L}^1_c(X_1, X_2^\lambda)|\lambda \in \Lambda)$ is isometrically two-sided $A$-module isomorphic to $\mathcal{L}^1_c(X_1, l^\infty(X_2^\lambda|\lambda \in \Lambda)).$

(b) $c_0(\mathcal{L}^1_c(X_1, X_2^\lambda)|\lambda \in \Lambda)$ can be isometrically embedded as a two-sided $A$-submodule into $\mathcal{L}^1_c(X_1, c_0(X_2^\lambda|\lambda \in \Lambda)).$

(c) $l^p(\mathcal{L}^1_c(X_1, X_2^\lambda)|\lambda \in \Lambda)$ can be contractively embedded as a two-sided $A$-submodule into $\mathcal{L}^1_c(X_1, l^p(X_2^\lambda|\lambda \in \Lambda))$, for all $1 < p < \infty$.

Proof. It follows from Propositions 3.1.6 and 3.2.6.

Proposition 3.4.7. Let $A$ be a Banach algebra and $X_1$ and $X_2$ be Banach left $A$-modules. Then the following hold:

(i) If $X_1^\dagger$ is a complemented left $A$-submodule of $X_1$, then $\mathcal{L}^1_c(X_1, X_2)/\mathcal{L}^1_c(X_1^\dagger, X_2)$ is isometrically two-sided $A$-module isomorphic to $\mathcal{L}^1_c(X_1/X_1^\dagger, X_2)$.

(ii) If $X_2^\dagger$ is a closed left $A$-submodule of $X_2$, then $\mathcal{L}^1_c(X_1, X_2/X_2^\dagger)$ can be isometrically embedded as a two-sided $A$-submodule into $\mathcal{L}^1_c(X_1, X_2/X_2^\dagger)$. Moreover
if $X_2^\ddagger$ is a complemented left $A$-submodule of $X_2$, then $L^1_c(X_1, X_2/X_2^\ddagger)$ is isometrically two-sided $A$-module isomorphic to $L^1_c(X_1, X_2/X_2^\ddagger)$.

**Proof.** It follows from Propositions 3.1.7 and 3.2.7.

As we mentioned after Proposition 3.4.1, $L^1_c(X_1, X_2)$ is related to $X_2$ as a left $A$-module and to $X_1^\ddagger$ as a right $A$-module. Hence we don't have any straightforward way of relating the cohomology, with coefficients in $L^1_c(X_1, X_2)$, either to the cohomology, with coefficients in $X_1$, or to the cohomology, with coefficients in $X_2$. 

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Chapter 4

The relation between the splitting of the cohomology with coefficients in a module $X$ and the modules of maps from and into $X$

In the remarks following Proposition 3.1.9 and those preceding Proposition 3.1.19 we said that the vanishing of $\mathcal{H}_n^*(\mathcal{A}, \mathcal{L}_1^*(Y, X))$ or of $\mathcal{H}_n^*(\mathcal{A}, \mathcal{L}_1^*(Y, X) : \mathcal{B})$, for some space $Y$, implies the vanishing of $\mathcal{H}_n^*(\mathcal{A}, X)$ or of $\mathcal{H}_n^*(\mathcal{A}, X : \mathcal{B})$ respectively. We also promised that in Section 5.3 we will give an example showing that the converse does not hold. We shall prove in this chapter that the modules $\mathcal{L}_1^*(Y, X)$ behave much better with respect to splitting. More precisely we will prove in the first part of the first section that, for all five types of splitting, splitting of the cohomology, with coefficients in $X$, is equivalent to splitting of the cohomology, with coefficients in $\mathcal{L}_1^*(Y, X)$, for all spaces $Y$.

We have already mentioned that in Section 5.3 we will show that the vanishing of $\mathcal{H}_n^*(\mathcal{A}, X)$ does not imply that $\mathcal{H}_n^*(\mathcal{A}, X)$ splits (III). On the other hand in the above mentioned remarks we hinted that the vanishing of $\mathcal{H}_n^*(\mathcal{A}, \mathcal{L}_1^*(Y, X))$, for all spaces $Y$, implies more than the vanishing of $\mathcal{H}_n^*(\mathcal{A}, X)$. We will prove in the second part of the first section that it implies split (III) of $\mathcal{H}_n^*(\mathcal{A}, X)$. That will be done by constructing a cocycle $\Phi \in Z_n^*(\mathcal{A}, \mathcal{L}_1^*(Z_n^*(\mathcal{A}, X), X))$ the cobounding of which gives rise to a splitting map of the third kind $s_n : Z_n^*(\mathcal{A}, X) \to \mathcal{L}_n^*(\mathcal{A}, X)$.

In the second section we combine the results of the first section with the results of Section 3.3 to obtain results about the relation between splitting and the modules $\mathcal{L}_1^*(X, Y)$ in the special case of a dual $\mathcal{A}$-bimodule $X$.

In the third section we introduce the notion of $X$-amenability and discuss its relation to splitting. In particular we prove that if $\mathcal{A}$ is a (completely) amenable
Banach (operator) algebra, then the (completely) bounded Hochschild cohomology complex of $\mathcal{A}$, with coefficients in a dual (operator completely bounded) $\mathcal{A}$-bimodule, splits (III).

4.1 The relation between splitting and the modules $\mathcal{L}^1_*(Y, X)$

4.1.1 The relation between the splitting of the cohomology with coefficients in $X$ and the splitting of the cohomology with coefficients in $\mathcal{L}^1_*(Y, X)$

As we mentioned in the introduction, in this part of the chapter we are going to prove that the splitting of both the cohomology groups and the cohomology complex, with coefficients in a module $X$, is equivalent to the splitting of the cohomology, with coefficients in $\mathcal{L}^1_*(Y, X)$, for any space $Y$. As the results hold for all five types of splitting and the proofs are similar, we will only give the proof for the third type of splitting. Throughout the module actions of $\mathcal{A}$ on $\mathcal{L}^1_*(Y, X)$ are the ones defined in Section 3.1. Moreover the coboundary map between $\mathcal{L}^n_*(\mathcal{A}, X)$ and $\mathcal{L}^{n+1}_*(\mathcal{A}, X)$ is denoted by $\partial^n$ and the coboundary map between $\mathcal{L}^n_*(\mathcal{A}, \mathcal{L}^1_*(Y, X))$ and $\mathcal{L}^{n+1}_*(\mathcal{A}, \mathcal{L}^1_*(Y, X))$ is denoted by $\Delta^n$.

Proposition 4.1.1. Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $\mathcal{H}^n_*(\mathcal{A}, X)$ splits (I), (II), (III), (IV) or (V) respectively.

(ii) $\mathcal{H}^n_*(\mathcal{A}, \mathcal{L}^1_*(Y, X))$ splits (I), (II), (III), (IV) or (V) respectively, for some Banach space $Y$.

(iii) $\mathcal{H}^n_*(\mathcal{A}, \mathcal{L}^1_*(Y, X))$ splits (I), (II), (III), (IV) or (V) respectively, for all Banach spaces $Y$.

Proof. $(i) \Rightarrow (iii)$ Let us start with the case $n = 1$. Since $\mathcal{H}^1_*(\mathcal{A}, X)$ splits (III), there exists a bounded linear map

$$s_1: \mathcal{Z}^1_*(\mathcal{A}, X) \to X$$

with $\partial_0 s_1 = id_{\mathcal{Z}^1_*(\mathcal{A}, X)}$. Now consider a Banach space $Y$ and define

$$S_1: \mathcal{Z}^1_*(\mathcal{A}, \mathcal{L}^1_*(Y, X)) \to \mathcal{L}^1_*(Y, X)$$

by $S_1(\Phi)(y) = s_1(\Phi y)$, for all $\Phi \in \mathcal{Z}^1_*(\mathcal{A}, \mathcal{L}^1_*(Y, X))$ and all $y \in Y$, where $\Phi y(a) = \Phi(a)(y)$, for all $a \in \mathcal{A}$, as in Proposition 3.1.9(ii). Using Proposition 3.1.9(ii) it is
easy to see that $S_1$ is a bounded linear map. Now consider $\Phi \in Z_c^1(A, X)$, $a \in A$ and $y \in Y$. Then

$$\Delta^0(S_1(\Phi))(a)(y) = aS_1(\Phi)(y) - S_1(\Phi)(y)a$$

$$= as_1(\Phi y) - s_1(\Phi y)a$$

$$= \phi(1)(s_1(\Phi y))(a)$$

$$= \phi_1(a)$$

$$= \phi(a)(y)$$

since, by Proposition 3.1.9(vii), $\phi_y \in Z_c^1(A, X)$. Hence $\Delta^0S_1 = id_{Z_c^1(A, L_c^1(Y, X))}$ and so $H_c^1(A, L_c^1(Y, X))$ splits (III). Now suppose that $n > 1$ and let $Y$ be a Banach space. Since $H_c^n(A, X)$ splits (III), we get, using reduction of dimension, that $H_c^1(A, L_c^{-1}(A, X))$ also splits (III). From the first part of the proof, we can conclude that $H_c^1(A, L_c(Y, L_c^{-1}(A, X)))$ splits (III). As we proved in Proposition 3.1.8, $L_c(Y, L_c^{-1}(A, X))$ and $L_c^{-1}(A, L_c(Y, X))$ are $A$-module isomorphic and hence $H_c^1(A, L_c^{-1}(A, L_c(Y, X)))$ also splits (III). Now using reduction of dimension again we get that $H_c^n(A, L_c^1(Y, X))$ splits (III).

It is obvious that (iii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i) As we observed in Section 3.1 (remarks after Proposition 3.1.1), given a non-zero functional $f$ on $Y$, $X$ is $A$-module isomorphic to the complemented $A$-submodule $\{x_f \mid x \in X\}$ of $L_c(Y, X)$. Therefore if $H_c^n(A, L_c^1(Y, X))$ splits (III), for some Banach space $Y$, then so does $H_c^1(A, X)$, by Propositions 2.1.17 and 2.1.18(i).

Applying the previous proposition to the bounded Hochschild cohomology complex we get the following corollary.

**Corollary 4.1.1.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. Then the following are equivalent:

(i) The bounded Hochschild cohomology complex of $A$, with coefficients in $X$, splits (I), (II) or (III) respectively.

(ii) The bounded Hochschild cohomology complex of $A$, with coefficients in $L_c^1(Y, X)$, splits (I), (II) or (III) respectively, for some Banach space $Y$.

(iii) The bounded Hochschild cohomology complex of $A$, with coefficients in $L_c^1(Y, X)$, splits (I), (II) or (III) respectively, for all Banach spaces $Y$.

We now move to the completely bounded case. The proof is similar to that of Proposition 4.1.1. Due to the lack of a reduction of dimension result we have to give a direct proof of the case $n > 1$ in (i) $\Rightarrow$ (iii).
Proposition 4.1.2. Let $A$ be an operator algebra and $X$ be an operator completely bounded $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $\mathcal{H}^n_{cb}(A,X)$ splits (I), (II), (III), (IV) or (V) respectively.

(ii) $\mathcal{H}^n_{cb}(A,L^1_{cb}(Y,X))$ splits (I), (II), (III), (IV) or (V) respectively, for some matricially normed space $Y$.

(iii) $\mathcal{H}^n_{cb}(A,L^1_{cb}(Y,X))$ splits (I), (II), (III), (IV) or (V) respectively, for all matricially normed spaces $Y$.

Proof. (ii) $\Rightarrow$ (i) follows in a manner similar to the proof of (ii) $\Rightarrow$ (i) in Proposition 4.1.1 using the remarks following Proposition 3.1.12 and Propositions 2.2.10 and 2.2.11(i). To prove (i) $\Rightarrow$ (iii) suppose that $\mathcal{H}^n_{cb}(A,X)$ splits (III) and let $s_n : Z^n_{cb}(A,X) \to L^{n-1}_{cb}(A,X)$ be a splitting map of the third kind and $Y$ be a matricially normed space. For $n = 1$ define $S_1 : Z^1_{cb}(A,L^1_{cb}(Y,X)) \to L^1_{cb}(Y,X)$ as in the proof of Proposition 4.1.1. For $n > 1$ define

$$S_n : Z^n_{cb}(A,L^1_{cb}(Y,X)) \to L^{n-1}_{cb}(A,L^1_{cb}(Y,X))$$

by $S_n(\Phi)(a_1,\ldots,a_{n-1})(y) = s_n(\Phi_y)(a_1,\ldots,a_{n-1})$, for all $\Phi \in Z^n_{cb}(A,L^1_{cb}(Y,X))$, all $a_1,\ldots,a_{n-1} \in A$ and all $y \in Y$, where $\Phi_y$ is defined by $\Phi_y(a_1,\ldots,a_{n-1}) = \Phi(a_1,\ldots,a_{n-1})(y)$ as in Proposition 3.1.19(ii). We showed in the proof of Proposition 4.1.1 that $S_1$ has the defining property of a splitting map of the third kind. A similar calculation shows that the same holds for $S_n$, if $n > 1$. To finish the proof we must prove that $S_n$ maps $Z^n_{cb}(A,L^1_{cb}(Y,X))$ into $L^{n-1}_{cb}(A,L^1_{cb}(Y,X))$ and is completely bounded. We will only prove the complete boundedness of $S_n$ for $n > 1$ (the case $n = 1$ follows in a similar manner, the well-definedness of $S_n$ follows from a similar calculation first with $m = l = 1$ and then with $m = 1$).

Consider $r,l,m \in \mathbb{N}$, $(\Phi_{ij}) \in \mathcal{M}_{nm}(Z^n_{cb}(A,L^1_{cb}(Y,X)))$, $(a^1_{st}),\ldots,(a^n_{st}) \in \mathcal{M}_l(A)$ and $(y_{pq}) \in \mathcal{M}_r(Y)$. Then

\[
\|((\sum_{1 \leq k_1,\ldots,k_{n-1} \leq l} S_n(\Phi_{ij})(a^1_{sk_1},\ldots,a^n_{k_{n-1}l})(y_{pq})))\|_{\mathcal{M}_{nm}} \\
= \|((\sum_{1 \leq k_1,\ldots,k_{n-1} \leq l} s_n(\Phi_{ij})(y_{pq})(a^1_{sk_1},\ldots,a^n_{k_{n-1}l})))\|_{\mathcal{M}_{mr}} \\
\leq \|s_n((\Phi_{ij})(y_{pq}))\|_{\mathcal{M}_{mr}} \|a^1_{st}\| \cdots \|a^n_{st}\| \\
\leq \|s_n\|_{cb} \|((\Phi_{ij})(y_{pq}))\|_{\mathcal{M}_{mr}} \|a^1_{st}\| \cdots \|a^n_{st}\| \\
\leq \|s_n\|_{cb} \|((\Phi_{ij})(y_{pq}))\|_{\mathcal{M}_{mr}} \|a^1_{st}\| \cdots \|a^n_{st}\| \\
\]

where the second step follows from $(s_n((\Phi_{ij})(y_{pq})) \in \mathcal{M}_{mr}(Z^{n-1}_{cb}(A,X))$, the third from the complete boundedness of $s_n$ and the fourth from Proposition 3.1.19(ii). \qed
A similar result holds for the completely bounded Hochschild cohomology complex.

We finish with $B$-relative Hochschild cohomology.

**Proposition 4.1.3.** Let $A$ be a Banach algebra, $B$ be a subalgebra of $A$ and $X$ be a Banach $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $\mathcal{H}^n_c(A, X : /B)$ splits (I), (II), (III), (IV) or (V) respectively.

(ii) $\mathcal{H}^n_c(A, L^1_c(Y, X) : /B)$ splits (I), (II), (III), (IV) or (V) respectively, for some Banach space $Y$.

(iii) $\mathcal{H}^n_c(A, L^1_c(Y, X) : /B)$ splits (I), (II), (III), (IV) or (V) respectively, for all Banach spaces $Y$.

**Proof.** Suppose that $\mathcal{H}^n_c(A, X : /B)$ splits (III) and let

$$s_n : Z^n_c(A, X : /B) \rightarrow L^{n-1}_c(A, X : /B)$$

be a splitting map of the third kind and $Y$ be a Banach space. If we define

$$S_n : Z^n_c(A, L^1_c(Y, X) : /B) \rightarrow L^{n-1}_c(A, L^1_c(Y, X) : /B)$$

as in the proof of Proposition 4.1.1, if $n = 1$, and as in the proof of Proposition 4.1.2, if $n > 1$, then the only thing we have to prove is that $S_n$ maps $Z^n_c(A, L^1_c(Y, X) : /B)$ into $L^{n-1}_c(A, L^1_c(Y, X) : /B)$. Consider $\Phi \in Z^n_c(A, L^1_c(Y, X) : /B)$, $b \in B$, $a_1, ..., a_{n-1} \in A$ and $y \in Y$. Then

$$S_n(\Phi)(ba_1, ..., a_{n-1})(y) = s_n(\Phi)(ba_1, ..., a_{n-1}) = bs_n(\Phi_y)(a_1, ..., a_{n-1})$$

$$= bS_n(\Phi)(a_1, ..., a_{n-1})(y) = (bS_n(\Phi)(a_1, ..., a_{n-1}))(y)$$

since $\Phi_y \in Z^n_c(A, X : /B)$ (Proposition 3.1.9(ix)) and $s_n$ maps $Z^n_c(A, X : /B)$ into $L^{n-1}_c(A, X : /B)$. Thus $S_n(\Phi)(ba_1, ..., a_{n-1}) = bS_n(\Phi)(a_1, ..., a_{n-1})$. Similarly we can prove that

$$S_n(\Phi)(a_1, ..., a_k b, a_{k+1}, ..., a_{n-1}) = S_n(\Phi)(a_1, ..., a_k b a_{k+1}, ..., a_{n-1})$$

for all $1 \leq k \leq n-2$, and that $S_n(\Phi)(a_1, ..., a_{n-1} b)$ is equal to $S_n(\Phi)(a_1, ..., a_{n-1}) b$. The proof of (ii) $\Rightarrow$ (i) follows in the steps of the proof of (ii) $\Rightarrow$ (i) in Proposition 4.1.1. Instead of Propositions 2.1.17 and 2.1.18(i) we use Propositions 2.3.3 and 2.3.4(i)

The results of the previous proposition also hold for the bounded $B$-relative Hochschild cohomology complex. Moreover we can prove a completely bounded version of the same results using Proposition 3.1.19(ix) in the place of Proposition 3.1.9(ix).
4.1.2 The relation between the splitting of the cohomology with coefficients in \( X \) and the vanishing of the cohomology with coefficients in \( L^1_c(Y, X) \)

In this part we will show that the third type of splitting of the cohomology, with coefficients in \( X \), is characterised by the vanishing of the cohomology, with coefficients in \( L^1_c(Y, X) \), for all spaces \( Y \). We will do that by constructing a test cocycle the cobounding of which gives rise to a splitting map of the third kind. The module actions of \( A \) on \( L^1_c(Y, X) \) are the ones defined in Section 3.1. The coboundary maps are denoted as in the first part.

**Proposition 4.1.4.** Let \( A \) be a Banach algebra and \( X \) be a Banach \( A \)-bimodule. Then, for all \( n \in \mathbb{N} \), the following are equivalent:

(i) \( \mathcal{H}_c^n(A, X) \) splits (III).
(ii) \( \mathcal{H}_c^n(A, L^1_c(Y, X)) \) splits (III), for some Banach space \( Y \).
(iii) \( \mathcal{H}_c^n(A, L^1_c(Y, X)) \) splits (III), for all Banach spaces \( Y \).
(iv) \( \mathcal{H}_c^n(A, L^1(Y, X)) = \{0\} \), for all Banach spaces \( Y \).

**Proof.** (i) \( \Leftrightarrow \) (ii) \( \Leftrightarrow \) (iii) follows from Proposition 4.1.1. (iii) \( \Rightarrow \) (iv) follows from Proposition 2.1.7. To prove (iv) \( \Rightarrow \) (i) we start with the case \( n = 1 \). Define

\[
\chi : A \to L^1_c(Z(A, X), X)
\]

with \( \chi(a)(\phi) = \phi(a) \), for all \( a \in A \) and all \( \phi \in Z^1_c(A, X) \). It is obvious that \( \chi \) is a bounded linear map. Moreover if \( a_1, a_2 \in A \) and \( \phi \in Z^1_c(A, X) \), then

\[
\Delta^1(\chi)(a_1, a_2)(\phi) = a_1\chi(a_2)(\phi) - \chi(a_1 a_2)(\phi) + \chi(a_1)(\phi)a_2
\]

\[
= a_1\phi(a_2) - \phi(a_1 a_2) + \phi(a_1)a_2 = 0
\]

since \( \phi \in Z^1_c(A, X) \) and so \( \chi \in Z^1_c(A, L^1_c(Z^1_c(A, X), X)) \). Obviously \( Z^1_c(A, X) \) is a Banach space and so, by our hypothesis, \( \mathcal{H}^1_c(A, L^1_c(Z^1_c(A, X), X)) \) vanishes. Thus there exists \( s_1 \in L^1_c(Z^1_c(A, X), X) \), with \( \chi = \Delta^0(s_1) \). Now if \( \phi \in Z^1_c(A, X) \) and \( a \in A \), then

\[
\partial^0(s_1(\phi))(a) = a(s_1(\phi)) - (s_1(\phi))a = (as_1 - s_1a)(\phi)
\]

\[
= \Delta^0(s_1)(\phi) = \chi(a)(\phi) = \phi(a)
\]

and therefore \( \partial^0 s_1 = id_{Z^1_c(A, X)} \). Hence \( \mathcal{H}_c^1(A, X) \) splits (III). For \( n > 1 \) we can prove it using reduction of dimension, the isomorphism between \( L^1_c(A, L^1_c(Y, X)) \) and \( L^1_c(Y, L^1_c(A, X)) \) and the first part of the proof. \( \square \)

A similar result holds for the bounded Hochschild cohomology complex.

We move now to the completely bounded case. Again we have to give a direct proof of the case \( n > 1 \) in (iv) \( \Rightarrow \) (i).

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Proposition 4.1.5. Let $A$ be an operator algebra and $X$ be an operator completely bounded $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $\mathcal{H}_{cb}^{n}(A, X)$ splits (III).

(ii) $\mathcal{H}_{cb}^{n}(A, L_{cb}^{1}(Y, X))$ splits (III), for some matricially normed space $Y$.

(iii) $\mathcal{H}_{cb}^{n}(A, L_{cb}^{1}(Y, X))$ splits (III), for all matricially normed spaces $Y$.

(iv) $\mathcal{H}_{cb}^{n}(A, L_{cb}^{1}(Y, X)) = \{0\}$, for all matricially normed spaces $Y$.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ follows from Proposition 4.1.2 and $(iii) \Rightarrow (iv)$ follows from Proposition 2.2.4. The proof of $(iv) \Rightarrow (i)$ for $n = 1$ is similar to the one in Proposition 4.1.4. It is easy to see that the cocycle $\chi$ maps $A$ into $L_{cb}^{1}(Z_{cb}^{1}(A, X), X)$ and is completely bounded. Thus $s_1$ is also completely bounded. Suppose that $n > 1$ and define

$$\Phi : A^n \rightarrow L_{cb}^{1}(Z_{cb}^{n}(A, X), X)$$

by $\Phi(a_1, \ldots, a_n)(\phi) = \phi(a_1, \ldots, a_n)$, for all $a_1, \ldots, a_n \in A$ and all $\phi \in Z_{cb}^{n}(A, X)$. Take $m, r \in \mathbb{N}$, $(a_{ij}^{1})$, $(a_{ij}^{n}) \in M_{m}(A)$ and $(\phi_{st}) \in M_{r}(Z_{cb}^{n}(A, X))$. Then

$$|| \left( \sum_{1 \leq k_1, \ldots, k_{n-1} \leq m} \Phi(a_{ik_1}^{1}, \ldots, a_{k_{n-1}}^{n})(\phi_{st}) \right) ||_{mr}$$

$$= || \left( \sum_{1 \leq k_1, \ldots, k_{n-1} \leq m} \phi_{st}(a_{ik_1}^{1}, \ldots, a_{k_{n-1}}^{n}) \right) ||_{mr}$$

$$\leq ||(\phi_{st})||_{r}|| (a_{ij}^{1}) ||_{m} \ldots || (a_{ij}^{n}) ||_{m}.$$

Hence $\Phi$ is completely bounded (the same calculation with $m = 1$ shows that $\Phi$ is well-defined). A calculation similar to the one for $\chi$ shows that $\Phi$ is a cocycle. Thus there exists $\Psi \in L_{cb}^{-1}(A, L_{cb}^{1}(Z_{cb}^{n}(A, X), X))$ with $\Phi = \Delta^{-1}(\Psi)$. If we define $s_{n} : Z_{cb}^{n}(A, X) \rightarrow L_{cb}^{n-1}(A, X)$ by $s_{n}(\phi)(a_1, \ldots, a_{n-1}) = \Psi(a_1, \ldots, a_{n-1})(\phi)$, for all $\phi \in Z_{cb}^{n}(A, X)$ and all $a_1, \ldots, a_{n-1} \in A$, then $s_{n}$ is a splitting map of the third kind.

The previous proposition is also true for the completely bounded Hochschild cohomology complex.

Proposition 4.1.6. Let $A$ be a Banach algebra, $B$ be a subalgebra of $A$ and $X$ be a Banach $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $\mathcal{H}_{cb}^{n}(A, X : /B)$ splits (III).

(ii) $\mathcal{H}_{cb}^{n}(A, L_{cb}^{1}(Y, X) : /B)$ splits (III), for some Banach space $Y$.

(iii) $\mathcal{H}_{cb}^{n}(A, L_{cb}^{1}(Y, X) : /B)$ splits (III), for all Banach spaces $Y$.

(iv) $\mathcal{H}_{cb}^{n}(A, L_{cb}^{1}(Y, X) : /B) = \{0\}$, for all Banach spaces $Y$. 

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Proof. The proof follows the steps of the two previous proofs. The test cochain is defined as in the proof of Proposition 4.1.4, if $n = 1$, and as in the proof of Proposition 4.1.5, if $n > 1$. The only change that we have to make is that $Y$ must be $Z^n_c(A, X : /B)$. Then it is easy to show that $s_n$, which is defined as in the two previous proofs, will map $Z^n_c(A, X : /B)$ into $L^{n-1}_c(A, X : /B)$. \hfill $\square$

Similar results hold for the bounded $B$-relative Hochschild cohomology complex, for completely bounded $B$-relative cohomology groups and for the completely bounded $B$-relative Hochschild cohomology complex.

**Remark 4.1.1.** The proofs of Propositions 4.1.4, 4.1.5 and 4.1.6 show that the following hold:

(a) If $A$ is a Banach algebra and $X$ is a Banach $A$-bimodule, then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $H^n_c(A, X)$ splits (III).

(ii) $H^n_c(A, L^1_c(Z^n_c(A, X), X)) = \{0\}$.

(b) If $A$ is an operator algebra and $X$ is an operator completely bounded $A$-bimodule, then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $H^n_{cb}(A, X)$ splits (III).

(ii) $H^n_{cb}(A, L^1_{cb}(Z^n_{cb}(A, X), X)) = \{0\}$.

(c) If $A$ is a Banach algebra, $B$ is a subalgebra of $A$ and $X$ is a Banach $A$-bimodule, then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $H^n_c(A, X : /B)$ splits (III).

(ii) $H^n_c(A, L^1_c(Z^n_c(A, X : /B), X : /B)) = \{0\}$.

Results similar to (ii) $\Rightarrow$ (i) of (a), (b) and (c) of the previous remark also hold for the first type of splitting. For example if $B^n_c(A, X)$ is closed and $H^n_c(A, L^1_c(B^n_c(A, X), X)) = \{0\}$, then $H^n_c(A, X)$ splits (I).

### 4.2 The relation between splitting and the modules $L^1_*(X, Y)$

We proved in Section 3.3 that if $X$ is an $A$-bimodule, then the modules $L^1_*(Y, X^*)$ and $L^1_*(X, Y^*)$ are $A$-module isomorphic, for all spaces $Y$. Combining that result with the results of the previous section we can relate the splitting of the cohomology, with coefficients in $X^*$, with the cohomology, with coefficients in $L^1_*(X, Y)$. We start by proving that splitting of the cohomology, with coefficients in $X^*$, is equivalent to splitting of the cohomology, with coefficients in $L^1_*(X, Y)$, for all dual $Y$, and then we show that the third type of splitting of $H^n_*(A, X^*)$ is equivalent to the vanishing of $H^n_c(A, L^1_c(X, Y))$, for all dual $Y$. We don't know
whether the splitting of \( \mathcal{H}_n^*(A, X^*) \) implies the splitting of \( \mathcal{H}_n^*(A, \mathcal{L}_1^1(X, Y)) \) if \( Y \) is not dual. We don’t even know if the third type of splitting of \( \mathcal{H}_n^*(A, X^*) \) implies the vanishing of \( \mathcal{H}_n^*(A, \mathcal{L}_1^1(X, Y)) \) for a non-dual \( Y \). Throughout this section the module actions of \( A \) on \( \mathcal{L}_1^1(X, Y) \) are the ones defined in Section 3.2.

The coboundary map between \( \mathcal{L}_n^*(A, X^*) \) and \( \mathcal{L}_n^{n+1}(A, X^*) \) is denoted by \( \partial^n \) and the one between \( \mathcal{L}_n^*(A, \mathcal{L}_1^1(X, Y)) \) and \( \mathcal{L}_n^{n+1}(A, \mathcal{L}_1^1(X, Y)) \) by \( \Delta^n \). We will sketch direct proofs of the results which "show" why we need \( Y \) to be dual.

**Proposition 4.2.1.** Let \( A \) be a Banach algebra and \( X \) be a Banach \( A \)-bimodule. Then, for all \( n \in \mathbb{N} \), the following are equivalent:

1. \( \mathcal{H}_n^*(A, X^*) \) splits (I), (II), (III), (IV) or (V) respectively.
2. \( \mathcal{H}_n^*(A, \mathcal{L}_1^1(X, Y^*)) \) splits (I), (II), (III), (IV) or (V) respectively, for some Banach space \( Y \).
3. \( \mathcal{H}_n^*(A, \mathcal{L}_1^1(X, Y^*)) \) splits (I), (II), (III), (IV) or (V) respectively, for all Banach spaces \( Y \).

**Proof.** Without loss of generality suppose that \( \mathcal{H}_n^*(A, \mathcal{L}_1^1(X, Y)) \) splits (III) for some Banach space \( Y \) (not necessarily dual). By the remarks following Proposition 3.2.1, \( X^* \) is \( A \)-module isomorphic to the complemented \( A \)-submodule \( \{ f_y \mid f \in X^* \} \) of \( \mathcal{L}_1^1(X, Y) \), for any non-zero \( y \in Y \). Using Propositions 2.1.17 and 2.1.18(i) and the previous observation we get that \( \mathcal{H}_n^*(A, X^*) \) splits (III). So (ii) implies (i) for all Banach spaces \( Y \) and not just for dual ones.

On the other hand suppose that \( \mathcal{H}_n^*(A, X^*) \) splits (III) and let \( Y \) be a Banach space. By Proposition 4.1.1, \( \mathcal{H}_n^*(A, \mathcal{L}_1^1(Y, X^*)) \) splits (III). But \( \mathcal{L}_1^1(Y, X^*) \) is \( A \)-module isomorphic to \( \mathcal{L}_1^1(X, Y^*) \), by Proposition 3.3.1, and therefore \( \mathcal{H}_n^*(A, \mathcal{L}_1^1(X, Y^*)) \) splits (III), by Proposition 2.1.17.

To get a direct proof of (i) \( \Rightarrow \) (iii) suppose that \( \mathcal{H}_n^*(A, X^*) \) splits (III), let \( s_n : \mathcal{Z}_c^n(A, X^*) \to \mathcal{L}_c^{n-1}(A, X^*) \) be a splitting map of the third kind and \( Y \) be a Banach space and define

\[
S_n : \mathcal{Z}_c^n(A, \mathcal{L}_c^1(X, Y^*)) \to \mathcal{L}_c^{n-1}(A, \mathcal{L}_1^1(X, Y^*))
\]

by

\[
S_n(\Phi)(a_1, ..., a_{n-1})(x)(y) = s_n(\Phi_\hat{y})(a_1, ..., a_{n-1})(x)
\]

for all \( \Phi \in \mathcal{Z}_c^n(A, \mathcal{L}_c^1(X, Y^*)) \), all \( a_1, ..., a_{n-1} \in A \), all \( x \in X \) and all \( y \in Y \), where \( \hat{y} \) is the element of \( Y^{**} \) corresponding to \( y \) and \( \Phi_\hat{y} \) is defined, as in Proposition 3.2.8(ii), by \( \Phi_\hat{y}(a_1, ..., a_n)(x) = \hat{y}(\Phi(a_1, ..., a_n)(x)) \), for all \( a_1, ..., a_n \in A \) and all \( x \in X \). If \( \Phi \in \mathcal{Z}_c^n(A, \mathcal{L}_c^1(X, Y^*)) \), \( a_1, ..., a_n \in A \), \( x \in X \) and \( y \in Y \), then

\[
\Delta^{n-1}(S_n(\Phi))(a_1, ..., a_n)(x)(y) = \partial^{n-1}(s_n(\Phi_\hat{y}))(a_1, ..., a_n)(x)
\]

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and so $\Delta^{n-1}(S_n(\Phi)) = \Phi$.

Using Propositions 4.1.2 and 3.3.2 we can prove the following similar result for completely bounded cohomology groups.

**Proposition 4.2.2.** Let $A$ be an operator algebra and $X$ be an $L^1$ completely bounded $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

1. $H^*_cb(A, X^*)$ splits (I), (II), (III), (IV) or (V) respectively.
2. $H^*_cb(A, L^1_c(X, Y^*))$ splits (I), (II), (III), (IV) or (V) respectively, for some matricially normed space $Y$.
3. $H^*_cb(A, L^1_c(X, Y^*))$ splits (I), (II), (III), (IV) or (V) respectively, for all matricially normed spaces $Y$.

The following two propositions follow immediately from Propositions 4.1.4, 4.1.5, 3.3.1 and 3.3.2.

**Proposition 4.2.3.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

1. $H^c_n(A, X^*)$ splits (III).
2. $H^c_n(A, L^1_c(X, Y^*))$ splits (III), for some Banach space $Y$.
3. $H^c_n(A, L^1_c(X, Y^*))$ splits (III), for all Banach spaces $Y$.
4. $H^c_n(A, L^1_c(X, Y^*)) = \{0\}$, for all Banach spaces $Y$.

**Sketch of a direct proof of (iv) $\Rightarrow$ (i).** Let

$$\Phi : A^n \rightarrow L^1_c(X, Z^*_c(A, X^*))$$

be defined by $\Phi(a_1, \ldots, a_n)(x)(\phi) = \phi(a_1, \ldots, a_n)(x)$, for all $a_1, \ldots, a_n \in A$, all $x \in X$ and all $\phi \in Z^*_c(A, X^*)$. It is easy to show that

$$\Delta^n(\Phi)(a_1, \ldots, a_{n+1})(x)(\phi) = \partial^n(\phi)(a_1, \ldots, a_{n+1})(x)$$

for all $a_1, \ldots, a_{n+1} \in A$, all $x \in X$ and all $\phi \in Z^*_c(A, X^*)$ and thus $\Phi$ is an $n$-cocycle. Therefore there exists $\Psi \in L^{-1}_c(A, L^1_c(X, Z^*_c(A, X^*))$, with $\Phi = \Delta^{n-1}(\Psi)$. If we define

$$s_n : Z^*_c(A, X^*) \rightarrow L^{-1}_c(A, X^*)$$

by $s_n(\phi)(a_1, \ldots, a_{n-1})(x) = \Psi(a_1, \ldots, a_{n-1})(x)(\phi)$, for all $\phi \in Z^*_c(A, X^*)$, all $a_1, \ldots, a_{n-1} \in A$ and all $x \in X$, then $s_n$ is a splitting map of the third kind.

**Proposition 4.2.4.** Let $A$ be an operator algebra and $X$ be an $L^1$ completely bounded $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:
(i) $H^0_{cb}(A, X^*)$ splits (III).
(ii) $H^1_{cb}(A, L^1_{cb}(X, Y^*_{rt}))$ splits (III), for some matricially normed space $Y$.
(iii) $H^0_{cb}(A, L^1_{cb}(X, Y^*_{rt}))$ splits (III), for all matricially normed spaces $Y$.
(iv) $H^0_{cb}(A, L^1_{cb}(X, Y^*)) = \{0\}$, for all matricially normed spaces $Y$.

Results similar to those of Propositions 4.2.1, 4.2.2, 4.2.3 and 4.2.4 hold for the bounded and the completely bounded Hochschild cohomology complex, for $B$-relative cohomology groups and for the $B$-relative Hochschild cohomology complex.

### 4.3 $X$-amenability and splitting

If $A$ is a Banach (operator) algebra and $X$ is a Banach ($L^1$ completely bounded) $A$-bimodule, then the results of Chapter 3 show that the class

$$\mathcal{F} = \{ L^1_{cb}(X, Y^*_{rt}) | Y \text{ Banach (operator) space} \}$$

is a class of dual (operator completely bounded) $A$-bimodules which contain $X^*_{rt}$ as a (completely) complemented $A$-submodule. Thus we can think of $\mathcal{F}$ as a class of "$X$-dual (operator completely bounded) $A$-bimodules". (By Propositions 3.3.1 and 3.3.2, $\mathcal{F}$ can be identified with the class of $A$-bimodules $L^1_{cb}(Y, X^*_{rt})$ where $Y$ is a Banach (operator space). We recall from Definitions 1.2.10 and 1.2.12 that a Banach (operator) algebra $A$ is called (completely) $n$-amenable if $H^0_{cb}(A, X)$ vanishes, for all dual (operator completely bounded) $A$-bimodules. Putting those two observations together we get the following definition.

**Definition 4.3.1.** (i) Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. If $n \in \mathbb{N}$, then we will say that $A$ is $(X, n)$-amenable if $H^0_{cb}(A, L^1_{cb}(X, Y^*))$ vanishes, for all Banach spaces $Y$. If $A$ is $(X, n)$-amenable, for all $n \in \mathbb{N}$, then we will say that $A$ is $X$-amenable.

(ii) Let $A$ be an operator algebra and $X$ be an $L^1$ completely bounded $A$-bimodule. If $n \in \mathbb{N}$, then we will say that $A$ is completely $(X, n)$-amenable if $H^0_{cb}(A, L^1_{cb}(X, Y^*_{rt}))$ vanishes, for all operator spaces $Y$. If $A$ is completely $(X, n)$-amenable, for all $n \in \mathbb{N}$, then we will say that $A$ is completely $X$-amenable.

**Remark 4.3.1.** Reduction of dimension arguments do not work for (complete) $(X, n)$-amenability. (Complete) $(X, n)$-amenability does not imply (complete) $(X, n+m)$-amenability. Moreover (completely) $(X, 1)$-amenable and (completely) $X$-amenable are not the same.

The following corollary is an immediate consequence of the definitions. It shows that we can think of (complete) $X$-amenability as a local version of (complete) amenability.
Corollary 4.3.1. (i) Let $A$ be a Banach algebra. Then, for all $n \in \mathbb{N}$, $A$ is $n$-amenable if and only if $A$ is $(X, n)$-amenable, for all Banach $A$-bimodules $X$. Moreover $A$ is amenable if and only if $A$ is $X$-amenable, for all Banach $A$-bimodules $X$.

(ii) Let $A$ be an operator algebra. Then, for all $n \in \mathbb{N}$, $A$ is completely $n$-amenable if and only if $A$ is completely $(X, n)$-amenable, for all $L^1$ completely bounded $A$-bimodules $X$. Moreover $A$ is completely amenable if and only if $A$ is completely $X$-amenable, for all $L^1$ completely bounded $A$-bimodules $X$.

We recall that a Banach algebra is $n$-amenable if and only if there exists an $n$-virtual diagonal of $A$ (Remark 1.2.13(iv)). The following two corollaries show that the analogue of an $n$-virtual diagonal for $(X, n)$-amenability is a splitting map of the third kind $s_n : Z^n(A, X) \to L^{n-1}(A, X)$. The first one follows immediately from Proposition 4.2.3. We must be a bit more careful for (ii) $\Rightarrow$ (i) of the second one, because the definition of complete $(X, n)$-amenability depends on the vanishing of $H^n_{cb}(A, L^1_{cb}(X, Y^*_{rt}))$, and thus of $H^n_{cb}(A, L^1_{cb}(X^*_r))$, only for operator spaces $Y$. Since $X$ is an $L^1$ matricially normed space, $X^*_r$ is an operator space. Hence $Z^n_{cb}(A, X^*_r)$ is also an operator space and the result follows from Remark 4.1.1(b).

Corollary 4.3.2. Let $A$ be a Banach algebra and be $X$ a Banach $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $H^n_{cb}(A, X^*)$ splits (III).

(ii) $A$ is $(X, n)$-amenable.

Corollary 4.3.3. Let $A$ be an operator algebra and $X$ be an $L^1$ completely bounded $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $H^n_{cb}(A, X^*_r)$ splits (III).

(ii) $A$ is completely $(X, n)$-amenable.

Applying the results of the two previous corollaries on the (completely) bounded Hochschild cohomology complex we get parts (a)(i) and (b)(i) of the following corollary. Parts (a)(ii) and (b)(ii) follow from parts (a)(i) and (b)(i) and Corollary 4.3.1.

Corollary 4.3.4. (a) Let $A$ be a Banach algebra.

(i) If $X$ is a Banach $A$-bimodule, then $A$ is $X$-amenable if and only if the bounded Hochschild cohomology complex of $A$, with coefficients in $X^*$, splits (III).

(ii) $A$ is amenable if and only if the bounded Hochschild cohomology complex of $A$, with coefficients in $X$, splits (III), for all dual $A$-bimodules $X$.

(b) Let $A$ be an operator algebra.
(i) If $X$ is an $L^1$ completely bounded $A$-bimodule, then $A$ is completely $X$-amenable if and only if the completely bounded Hochschild cohomology complex of $A$, with coefficients in $X^*$, splits (III).

(ii) $A$ is completely amenable if and only if the completely bounded Hochschild cohomology complex of $A$, with coefficients in $X$, splits (III), for all dual operator completely bounded $A$-bimodules $X$.

Using reduction of dimension and Proposition 2.1.14 we can easily see that if a Banach algebra $A$ is $n$-amenable, then $H^m_c(A, X)$ splits (V), for all dual $A$-bimodules $X$ and all $m \geq n$. 

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Chapter 5

Splitting of the cohomology of von Neumann algebras

In this chapter we are going to study the splitting of bounded and completely bounded Hochschild cohomology groups of von Neumann algebras. In the first section we will show that averaging and lifting results similar to the ones obtained by Johnson, Kadison and Ringrose hold for the third type of splitting. In the second section we will show that results similar to the ones proved in Section 4.3 for amenable Banach algebras and completely amenable operator algebras also hold for amenable von Neumann algebras, i.e. a von Neumann algebra $\mathcal{M}$ is amenable if and only if the (completely) bounded Hochschild cohomology complex of $\mathcal{M}$, with coefficients in $X$, splits (III), for all normal (operator completely bounded) $\mathcal{M}$-bimodules $X$. In the third section we will prove that there is a close relation between splitting of the first bounded and completely bounded cohomology group of a von Neumann algebra $\mathcal{M}$, with coefficients in an injective von Neumann algebra which contains $\mathcal{M}$, and injectivity. In the fourth section we will show that, in all the cases where they are known to vanish, the bounded and completely bounded cohomology groups of a von Neumann algebra $\mathcal{M}$, with coefficients in $\mathcal{M}$, split (III).

We don't have enough space to discuss the splitting of the cohomology groups of a von Neumann algebra, with coefficients in the compacts (see [JPar], [Po4] and [Ra] for results concerning those groups), and the splitting of the cohomology groups of a $C^*$-algebra, with coefficients in its dual, and of a von Neumann algebra, with coefficients in its predual (for existing results see [BuPas2], [H1], Section 4 and [CS3]). We believe that in all the cases where we know that those groups vanish, they split (III).

Throughout the chapter the unit element of a unital algebra $\mathcal{A}$ will be denoted by $1_{\mathcal{A}}$. All inclusions $\mathcal{A}_1 \subseteq \mathcal{A}_2$ of unital algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital, i.e. $1_{\mathcal{A}_1} = 1_{\mathcal{A}_2}$. Moreover we will assume that all modules over a unital algebra are
unital (we are allowed to do that because of Proposition 2.1.21). If $\mathcal{M}$ is a von Neumann algebra, then we will denote the commutant of $\mathcal{M}$ by $\mathcal{M}'$.

Two notions that will be central in this chapter are those of a hyperfinite and an injective von Neumann algebra. We recall their definitions.

**Definition 5.0.1.** Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $H$.

1. We will say that $\mathcal{M}$ is injective if there exists a bounded projection $\rho : B(H) \to \mathcal{M}$ with $\|\rho\| = 1$.
2. We will say that $\mathcal{M}$ is hyperfinite if there exists an upwards directed family $\{\mathcal{M}_\lambda\}_{\lambda \in \Lambda}$ (i.e. $\Lambda$ is a directed set and for each $\lambda \in \Lambda$ and $\lambda' \in \Lambda$ such that $\lambda \leq \lambda'$, $\mathcal{M}_\lambda \subseteq \mathcal{M}_{\lambda'}$) of finite dimensional subalgebras of $\mathcal{M}$, with $1_{\mathcal{M}} = 1_{\mathcal{M}_\lambda}$, for all $\lambda \in \Lambda$, such that $\bigcup_{\lambda \in \Lambda} \mathcal{M}_\lambda$ is weakly dense in $\mathcal{M}$.

### 5.1 Averaging and lifting results

Averaging and lifting results have played a very important role in the study of the cohomology of von Neumann algebras. By averaging results we mean results which relate the Hochschild cohomology groups of an algebra $\mathcal{A}$, with coefficients in an $\mathcal{A}$-bimodule $X$, to the $\mathcal{B}$-relative Hochschild cohomology groups of $\mathcal{A}$, with coefficients in $X$, for some subalgebra $\mathcal{B}$ of $\mathcal{A}$. By lifting results we mean results relating cohomology groups with different continuity assumptions.

In the bounded case averaging and lifting results were established in the series of the three papers by Johnson, Kadison and Ringrose ([KRi4], [KRi5], [JKRi]) which initiated the study of the cohomology of operator algebras. In the first paper Kadison and Ringrose proved that if $\mathcal{A}$ is a unital $C^*$-algebra with centre $Z$, $X$ is a dual $\mathcal{A}$-bimodule and $n \in \mathbb{N}$, then, for each $\phi \in \mathcal{L}^n_c(\mathcal{A}, X)$, there exists $\psi \in \mathcal{L}^{n-1}_c(\mathcal{A}, X)$ with $(\phi - \partial^{n-1}(\psi))(a_1, ..., a_n) = 0$, if $a_k \in Z$, for some $1 \leq k \leq n$ (Theorem 3.4). In the third paper the same was shown to be true with any amenable subalgebra $\mathcal{B}$ of $\mathcal{A}$ in the place of $Z$ (Theorem 4.1, see also the proof of Theorem 3.3 in the second paper). In other words they proved that for any amenable subalgebra $\mathcal{B}$ of a unital $C^*$-algebra $\mathcal{A}$ and for any dual $\mathcal{A}$-bimodule $X$, $\mathcal{L}^n_c(\mathcal{A}, X) = \mathcal{B}^n_c(\mathcal{A}, X) + \mathcal{E}^n_c(\mathcal{A}, X : /\mathcal{B})$. The main lifting result of those three papers is Theorem 5.6 of [JKRi], which says that if $\mathcal{A}$ is a unital $C^*$-algebra acting on a Hilbert space $H$, $\mathcal{A}$ is the weak closure of $\mathcal{A}$ and $X$ is a normal $\mathcal{A}$-bimodule, then $\mathcal{H}^n_c(\mathcal{A}, X)$ and $\mathcal{H}^n_\mathcal{B}(\mathcal{A}, X)$ are isomorphic, for all $n \in \mathbb{N}$ (thus a von Neumann algebra $\mathcal{M}$ is amenable if and only if $\mathcal{H}^n_c(\mathcal{M}, X) = \{0\}$, for all
normal $\mathcal{M}$-bimodules $X$ and all $n \in \mathbb{N})$. A clear account of the above mentioned results can be found in [Ri2], Chapter 4 (for the averaging results) and Chapters 5 and 6 (for the lifting results). We should mention here that Craw gave a proof of the previously mentioned lifting result using homological methods (see [Cr]).

In their paper which introduced the notion of completely bounded cohomology Christensen, Effros and Sinclair showed that similar results hold in the completely bounded case ([CES], pp.293-296, see also [CS7], Theorem 3.4). Both the bounded and the completely bounded case are discussed in detail in Chapter 3 of [SSm1].

We will prove that the above mentioned lifting and averaging results yield similar results for the third type of splitting. The following proposition summarises the results of this section.

**Proposition 5.1.1.** Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $H$, $A$ be a $C^*$-subalgebra of $\mathcal{M}$ which is weakly dense in $\mathcal{M}$, $\mathcal{N}$ be a hyperfinite subalgebra of $\mathcal{M}$ and $X$ be a normal $\mathcal{M}$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $\mathcal{H}_n^\beta(\mathcal{M}, X)$ splits (III).

(ii) There exists a bounded linear map

$$s_n : Z_w^n(\mathcal{M}, X) \to \mathcal{L}_w^{n-1}(\mathcal{M}, X)$$

with $\partial^{n-1}s_n = \text{id}_{Z_w^n(\mathcal{M}, X)}$.

(iii) There exists a bounded linear map

$$s_n : Z_w^n(\mathcal{M}, X : /\mathcal{N}) \to \mathcal{L}_w^{n-1}(\mathcal{M}, X : /\mathcal{N})$$

with $\partial^{n-1}s_n = \text{id}_{Z_w^n(\mathcal{M}, X : /\mathcal{N})}$.

(iv) There exists a bounded linear map

$$s_n : Z_w^n(\mathcal{A}, X) \to \mathcal{L}_w^{n-1}(\mathcal{A}, X)$$

with $\partial^{n-1}s_n = \text{id}_{Z_w^n(\mathcal{A}, X)}$.

Moreover if $X$ is a normal operator completely bounded $\mathcal{M}$-bimodule, then the previous equivalence holds with the subscripts "c" and "w" replaced by "cb" and "wcb" respectively and "completely bounded" instead of "bounded" in parts (ii), (iii) and (iv).

To prove Proposition 5.1.1 we need to prove that the isomorphisms between $\mathcal{H}_{c(b)}^n(\mathcal{M}, X)$, $\mathcal{H}_{w(cb)}^n(\mathcal{M}, X)$, $\mathcal{H}_{w(cb)}^n(\mathcal{M}, X : /\mathcal{N})$ and $\mathcal{H}_{w(cb)}^n(\mathcal{A}, X)$ constructed in [Kri4], [Kri5], [JKRi] and [CES] can be described via (completely) bounded maps between the corresponding spaces of $n$-cochains or $n$-cocycles. Fortunately most of that has already been done in Chapter 3 of [SSm1]. We will describe how
those maps are defined, prove that they are completely bounded in the completely bounded case (in [SSm1] it is proved that they are well-defined in the completely bounded case, but not that they are completely bounded) and use them to show the equivalence between the splitting of different cohomology groups.

The first step is to show that if \( \mathcal{A} \) is a unital \( C^* \)-algebra acting on a Hilbert space \( H \), \( \mathcal{U} \) is an amenable subgroup of the group of unitaries of \( \mathcal{A} \), \( \mathcal{B} \) is the \( C^* \)-subalgebra of \( \mathcal{A} \) generated by \( \mathcal{U} \), \( X \) is a dual (operator completely bounded) \( \mathcal{A} \)-bimodule and \( n \in \mathbb{N} \), then \( \mathcal{H}^n_{cb}(\mathcal{A}, X) \) splits (III) if and only if \( \mathcal{H}^n_{cb}(\mathcal{A}, X : /\mathcal{B}) \) splits (III).

We start by recalling the definition of an invariant mean. If \( \mathcal{U} \) is a topological group, then a bounded linear functional \( \mu \) on the space of bounded continuous complex-valued functions on \( \mathcal{U} \), \( BC(\mathcal{U}) \), is called a right invariant mean on \( \mathcal{U} \) if \( \mu(f) \geq 0 \), for all \( f \in BC(\mathcal{U}) \) with \( f \geq 0 \), \( \mu(1) = 1 \), where \( 1(u) = 1 \), for all \( u \in \mathcal{U} \), and \( \mu(f_u) = \mu(f) \), for all \( f \in BC(\mathcal{U}) \) and all \( u \in \mathcal{U} \), where \( f_u(u) = f(uw) \), for all \( u \in \mathcal{U} \). We will denote \( \mu(f) \) by \( \int f(u)d\mu(u) \), for all \( f \in BC(\mathcal{U}) \). We must mention here that if \( \mu \) is a right invariant mean, then \( ||\mu|| = 1 \). A topological group \( \mathcal{U} \) is called amenable if there exists a right invariant mean on \( \mathcal{U} \) (for more information on amenable groups see [Gr], [Pa1] and [Pie1]).

If \( \mathcal{U} \) is an amenable group, \( \mu \) is a right invariant mean on \( \mathcal{U} \), \( X = (X_*)^* \) is a dual Banach space and \( BC(\mathcal{U}, X) \) is the space of bounded continuous (with respect to the norm topology on \( X \)) functions from \( \mathcal{U} \) to \( X \), then we can define a generalised right invariant mean \( \bar{\mu} : BC(\mathcal{U}, X) \to X \) by \( \bar{\mu}(F)(z) = \mu(u \mapsto F(u)(z)) \), for all \( F \in BC(\mathcal{U}, X) \) and all \( z \in X_* \). We call \( \bar{\mu} \) a generalised right invariant mean because, by [Ri2], Lemma 4.2, \( \bar{\mu} \) is a bounded linear map, with \( ||\bar{\mu}|| = 1 \) and has the following properties:

(i) If \( F \in BC(\mathcal{U}, X) \), \( w \in \mathcal{U} \) and \( F_w(u) = F(uw) \), for all \( u \in \mathcal{U} \), then \( \bar{\mu}(F_w) = \bar{\mu}(F) \).

(ii) If \( F(u) = x \), for all \( u \in \mathcal{U} \), then \( \bar{\mu}(F) = x \).

(iii) If \( \mathcal{A} \) is a Banach algebra, \( \mathcal{U} \) is a subgroup of \( \mathcal{A} \), \( X \) is a dual \( \mathcal{A} \)-bimodule, \( F \in BC(\mathcal{U}, X) \), \( a_1, a_2 \in \mathcal{A} \) and \( F'(u) = a_1 F(u) a_2 \), for all \( u \in \mathcal{U} \), then \( \bar{\mu}(F') = a_1 \bar{\mu}(F) a_2 \).

As in [SSm1] we will denote \( \bar{\mu}(F) \) by \( \int F(u)d\bar{\mu}(u) \), for all \( F \in BC(\mathcal{U}, X) \).

Using this notation we have that \( \int (F(u)d\bar{\mu}(u))(z) = \int F(u)(z)d\mu(u) \), for all \( F \in BC(\mathcal{U}, X) \) and all \( z \in X_* \).

To establish the equivalence between the splitting of the groups \( \mathcal{H}^n_{cb}(\mathcal{A}, X) \) and \( \mathcal{H}^n_{cb}(\mathcal{A}, X : /\mathcal{B}) \) we have to define certain averaging maps from \( \mathcal{L}^n_{cb}(\mathcal{A}, X) \) into either \( \mathcal{L}^n_{cb}(\mathcal{A}, X) \) or \( \mathcal{L}^n_{cb}(\mathcal{A}, X) \).

**Lemma 5.1.1.** ([SSm1], Lemma 3.2.3) Let \( \mathcal{A} \) be a unital \( C^* \)-algebra acting on...
a Hilbert space $H$, $U$ be an amenable subgroup of the group of unitaries of $A$, $\mu$ be a right invariant mean on $U$, $X$ be a dual $A$-bimodule and $n \in \mathbb{N}$. Then the maps

$$F_k : \mathcal{L}_c^n(A, X) \to \mathcal{L}_c^{n-1}(A, X), \ 0 \leq k \leq n - 1$$

defined by

$$F_0(\phi)(a_1, ..., a_{n-1}) = \int u^* \phi(u, a_1, ..., a_{n-1}) d\mu(u)$$

and by

$$F_k(\phi)(a_1, ..., a_{n-1}) = \int \phi(a_1, ..., a_{k-1}, a_k u^*, u, a_{k+1}, ..., a_{n-1}) d\mu(u), \ 1 \leq k \leq n - 1$$

for all $\phi \in \mathcal{L}_c^n(A, X)$ and all $a_1, ..., a_{n-1} \in A$, and the maps

$$G_k : \mathcal{L}_c^n(A, X) \to \mathcal{L}_c^n(A, X), \ 0 \leq k \leq n$$

defined by

$$G_0(\phi)(a_1, ..., a_n) = \int u^* \phi(ua_1, ..., a_n) d\mu(u)$$

$$G_k(\phi)(a_1, ..., a_n) = \int \phi(a_1, ..., a_k u^*, ua_{k+1}, ..., a_n) d\mu(u), \ 1 \leq k \leq n - 1$$

$$G_n(\phi)(a_1, ..., a_n) = \int \phi(a_1, ..., a_n u^*) ud\mu(u)$$

for all $\phi \in \mathcal{L}_c^n(A, X)$ and all $a_1, ..., a_n \in A$, where $\mu$ is the generalised mean on $BC(U, X)$, are bounded linear maps with norm less than or equal to 1.

If $X$ is a dual operator completely bounded $A$-bimodule, then the subscript "c" can be replaced by "cb" and the maps defined above are completely bounded, with completely bounded norm less than or equal to 1.

Proof. The only part that is not proved in [SSm1] is the complete boundedness of the maps $F_k$ and $G_k$. We will prove it for $G_k, 1 \leq k \leq n - 1$. In proving that $G_k$ maps $\mathcal{L}_c^n(A, X)$ into $\mathcal{L}_c^n(A, X)$ Sinclair and Smith observe that this follows from the commuting of $m$-amplifications and averages. We will try to make that more clear. Since $X$ is a dual operator completely bounded $A$-bimodule, there exists an $L^1$ completely bounded $A$-bimodule $X_*$ with $X = (X_*)^*$. For all $m \in \mathbb{N}$, $\mathbb{M}_m(X)$ can be identified with $\mathbb{M}_m(X_*)^*$ via $(x_{ij})(z_{ij}) = \sum_{1 \leq i, j \leq m} x_{ij}(z_{ji})$, for all $(x_{ij}) \in$
$M_m(X)$ and all $(z_{ij}) \in M_m(X_*)$ (see Remark 1.2.6(ii)). Since $M_m(X)$ is a dual Banach space there exists a generalised right invariant mean $\bar{\mu}_m : BC(U, M_m(X)) \to M_m(X)$. If $\phi \in L_{cb}^m(A, X)$ and $(a_{ij}^1), \ldots, (a_{ij}^n) \in M_m(A)$, then the map $u \mapsto \phi_m((a_{ij}^1), \ldots, (a_{ij}^k)(u^* \otimes I_m), (u \otimes I_m)(a_{ij}^{k+1}), \ldots, (a_{ij}^n))$ is in $BC(U, M_m(X))$ and its norm is not greater than $\|\phi\|_{cb}\|(a_{ij}^1)\|_m \ldots\|(a_{ij}^n)\|_m$. Therefore, since $\|\bar{\mu}_m\| \leq 1,$

$$\|\int \phi_m((a_{ij}^1), \ldots, (a_{ij}^k)(u^* \otimes I_m), (u \otimes I_m)(a_{ij}^{k+1}), \ldots, (a_{ij}^n))d\bar{\mu}_m(u)\|_m \leq \|\phi\|_{cb}\|(a_{ij}^1)\|_m \ldots\|(a_{ij}^n)\|_m. \quad (5.1)$$

If $(z_{ij}) \in M_m(X_*)$, then, by the linearity of the right invariant mean $\mu$, we get

$$(G_k(\phi))_m((a_{ij}^1), \ldots, (a_{ij}^n))(z_{ij})$$

$$= \sum_{1 \leq i,j \leq m} \sum_{1 \leq l_1, \ldots, l_{n-1} \leq m} \left( \int \phi(a_{ij}^{l_1}, \ldots, a_{ij}^{k+l_{l_1-1}}u^*, ua_{ij}^{k+l_{l_1}}, \ldots, a_{ij}^l)d\mu(u)(z_{ij}) \right)$$

$$= \sum_{1 \leq i,j \leq m} \sum_{1 \leq l_1, \ldots, l_{n-1} \leq m} \left( \int \phi(a_{ij}^{l_1}, \ldots, a_{ij}^{k+l_{l_1-1}}u^*, ua_{ij}^{k+l_{l_1}}, \ldots, a_{ij}^l)(z_{ij})d\mu(u) \right)$$

$$= \int \sum_{1 \leq i,j \leq m} \sum_{1 \leq l_1, \ldots, l_{n-1} \leq m} \phi(a_{ij}^{l_1}, \ldots, a_{ij}^{k+l_{l_1-1}}u^*, ua_{ij}^{k+l_{l_1}}, \ldots, a_{ij}^l)(z_{ij})d\mu(u)$$

$$= \int \phi_m((a_{ij}^1), \ldots, (a_{ij}^k)(u^* \otimes I_m), (u \otimes I_m)(a_{ij}^{k+1}), \ldots, (a_{ij}^n))(z_{ij})d\mu(u)$$

$$= \int \phi_m((a_{ij}^1), \ldots, (a_{ij}^k)(u^* \otimes I_m), (u \otimes I_m)(a_{ij}^{k+1}), \ldots, (a_{ij}^n))d\bar{\mu}_m(u)((z_{ij}))$$

and so

$$(G_k(\phi))_m((a_{ij}^1), \ldots, (a_{ij}^n))$$

$$= \int \phi_m((a_{ij}^1), \ldots, (a_{ij}^k)(u^* \otimes I_m), (u \otimes I_m)(a_{ij}^{k+1}), \ldots, (a_{ij}^n))d\bar{\mu}_m(u)$$

which demonstrates the commuting of $m$-amplifications and averages indicated by Sinclair and Smith. Therefore, by (5.1), we have

$$\|(G_k(\phi))_m((a_{ij}^1), \ldots, (a_{ij}^n))\|_m \leq \|\phi\|_{cb}\|(a_{ij}^1)\|_m \ldots\|(a_{ij}^n)\|_m.$$

which shows that $G_k(\phi) \in L_{cb}^m(A, X)$. To prove that $G_k$ is completely bounded take $m, r \in \mathbb{N}$, $(\phi_{ij}) \in M_m(L_{cb}^m(A, X))$ and $(a_{st}^1), \ldots, (a_{st}^n) \in M_r(A)$. Then, as before, we have

$$\|(G_k(\phi_{ij}))_r((a_{st}^1), \ldots, (a_{st}^n))\|_mr$$

$$= \int ((\phi_{ij})_r((a_{st}^1), \ldots, (a_{st}^k)(u^* \otimes I_r), (u \otimes I_r)(a_{st}^{k+1}), \ldots, (a_{st}^n))d\bar{\mu}_{mr}(u)\|_mr$$

$$\leq \|(\phi_{ij})_m\|(a_{st}^1)\|_r \ldots\|(a_{st}^n)\|_r$$

which shows that $G_k$ is completely bounded with $\|G_k\|_{cb} \leq 1$. \hfill \square
Using the maps $F_k$ and $G_k$ we can now define the maps which we will need to prove the equivalence between the splitting of $\mathcal{H}_{\varepsilon(b)}^n(A, X)$ and of $\mathcal{H}_{\varepsilon(b)}^n(A, X : /B)$.

**Lemma 5.1.2.** ([SSm1], Lemma 3.2.6) Let $A$ be a unital $C^*$-algebra acting on a Hilbert space $H$, $\mathcal{U}$ be an amenable group of unitaries in $A$, $B$ be the $C^*$-subalgebra of $A$ generated by $\mathcal{U}$ and $X$ be a dual $A$-bimodule. Then, for all $n \in \mathbb{N}$, there exists a bounded projection $Q_n : \mathcal{L}_c^n(A, X) \to \mathcal{L}_c^n(A, X : /B)$ with $\|Q_n\| \leq 1$, such that $\partial^{n-1}Q_{n-1} = Q_n\partial^{n-1}$.

If $X$ is a dual operator completely bounded $A$-bimodule, then $Q_n : \mathcal{L}_{cb}^n(A, X) \to \mathcal{L}_{cb}^n(A, X : /B)$ is completely bounded with $\|Q_n\|_{cb} \leq 1$.

The projection $Q_n$ is defined by $Q_n = G_n\ldots G_0$, where the maps $G_k$, $0 \leq k \leq n$, are defined as in Lemma 5.1.1. Therefore $Q_n$ is completely bounded with $\|Q_n\|_{cb} \leq 1$.

**Lemma 5.1.3.** ([SSm1], Lemma 3.2.4) Let $A$ be a unital $C^*$-algebra acting on a Hilbert space $H$, $\mathcal{U}$ be an amenable group of unitaries in $A$, $B$ be the $C^*$-subalgebra of $A$ generated by $\mathcal{U}$ and $X$ be a dual $A$-bimodule. Then, for all $n \in \mathbb{N}$, there exists a bounded linear map $K_n : \mathcal{L}_c^n(A, X) \to \mathcal{L}_c^{n+1}(A, X)$ with $\|K_n\| \leq ((n+2)^{n-1} - 1)/(n+1)$, such that $\phi - \partial^{n-1}K_n(\phi) \in \mathcal{Z}_c^n(A, X : /B)$, for all $\phi \in \mathcal{Z}_c^n(A, X)$.

If $X$ is a dual operator completely bounded $A$-bimodule, then we can replace the subscript "c" with the subscript "cb" and $K_n$ is completely bounded with $\|K_n\|_{cb} \leq ((n+2)^{n-1} - 1)/(n+1)$.

If $\mathcal{U}$ is finite dimensional and $X$ is a normal (operator completely bounded) $A$-bimodule, then the subscripts "c" and "cb" can be replaced by "w" and "wcb" respectively.

The map $K_n$ is defined inductively in the following manner: maps $J_1, \ldots, J_n$ are constructed with $\|J_k\|_{cb} \leq ((n+2)^{k-1} - 1)/(n+1)$ such that $\phi - \partial^{n-1}J_k(\phi)(a_1, \ldots, a_n) = 0$ if any of $a_1, \ldots, a_k$ is in $B$. Then $K_n = J_n$. For $k = 1$, $J_1$ is defined to be equal to $F_0$, where $F_0$ is as in Lemma 5.1.1. Therefore $J_1$ is completely bounded with $\|J_1\|_{cb} \leq 1$. If $J_k$ with $\|J_k\|_{cb} \leq ((n+2)^{k-1} - 1)/(n+1)$ has been defined, then $J_{k+1}$ is defined by $J_{k+1} = J_k - (-1)^kF_k(id_{\mathcal{L}_{cb}^n(A, X)} - \partial^{n-1}J_k)$. We can see that $J_{k+1}$ is completely bounded with $\|J_{k+1}\|_{cb} \leq ((n+2)^{k+1} - 1)/(n+1)$, using Lemma 5.1.1, the inductive hypothesis and $\|\partial^{n-1}\|_{cb} \leq n + 1$.

Now we can show the equivalence between split (III) of $\mathcal{H}_{\varepsilon(b)}^n(A, X)$ and of $\mathcal{H}_{\varepsilon(b)}^n(A, X : /B)$.

**Proposition 5.1.2.** Let $A$ be a unital $C^*$-algebra acting on a Hilbert space $H$, $\mathcal{U}$ be an amenable group of unitaries in $A$, $B$ be the $C^*$-subalgebra of $A$ generated by $\mathcal{U}$ and $X$ be a dual $A$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:
If $X$ is a dual operator completely bounded $A$-bimodule, then the subscript "c" can be replaced with the subscript "cb".

Proof. (i) $\Rightarrow$ (ii) Let $s_n : Z^n_c(A, X) \to L^{n-1}_c(A, X)$ be a splitting map of the third kind. If $s'_n : Z^n_c(A, X : /B) \to L_c^{n-1}(A, X : /B)$ is defined by $s'_n = Q_{n-1}s_n$, then, by Lemma 5.1.2, $s'_n$ is a well-defined bounded linear map. Consider $\phi \in Z^n_c(A, X : /B)$. Then

$$(\partial^{n-1}s'_n)(\phi) = (\partial^{n-1}Q_{n-1}s_n)(\phi) = (Q_n\partial^{n-1}s_n)(\phi) = Q_n(\phi) = \phi$$

since $\partial^{n-1}Q_{n-1} = Q_n\partial^{n-1}$ (Lemma 5.1.2), $s_n$ is a splitting map of the third kind and $Q_n$ is a projection onto $L^n_c(A, X : /B)$. Therefore $\partial^{n-1}s'_n = id_{Z^n_c(A, X : /B)}$ and so $H^n_c(A, X : /B)$ splits (III).

(ii) $\Rightarrow$ (i) Let $s_n : Z^n_c(A, X : /B) \to L^{n-1}_c(A, X : /B)$ be a splitting map of the third kind and define $s'_n : Z^n_c(A, X) \to L_c^{n-1}(A, X)$ by $s'_n = s_n(id_{Z^n_c(A, X)} - \partial^{n-1}K_n) + K_n$. From Lemma 5.1.3, $s'_n$ is a well-defined bounded linear map. Moreover if $\phi \in Z^n_c(A, X)$, then it follows from Lemma 5.1.3 and $s_n$ being a splitting map of the third kind that

$$(\partial^{n-1}s'_n)(\phi) = \partial^{n-1}s_n(\phi - (\partial^{n-1}K_n)(\phi)) + (\partial^{n-1}K_n)(\phi) = \phi - (\partial^{n-1}K_n)(\phi) + (\partial^{n-1}K_n)(\phi) = \phi$$

and thus $H^n_c(A, X)$ splits (III).

The completely bounded case follows in the same way, since the maps $Q_n$ and $K_n$ are completely bounded.

It is easy to see that the direction (i) $\Rightarrow$ (ii) in the previous proposition holds for the other types of splitting as well. The same is not true for the direction (ii) $\Rightarrow$ (i), since the map $id_{Z^n_c(A, X)} - \partial^{n-1}K_n$ maps only the elements of $Z^n_c(A, X)$ into $L^n_c(A, X : /B)$.

The second step is to prove the equivalence between the "splitting" of the groups $H^n_w(\mathcal{A}, X)$ and $H^n_{w(cb)}(\mathcal{A}, X)$ if $\mathcal{A}$ is a $C^*$-algebra acting on a Hilbert space $H$, $\mathcal{A}$ is the weak closure of $\mathcal{A}$ and $X$ is a normal (operator completely bounded) $\mathcal{A}$-bimodule.

Proposition 5.1.3. Let $\mathcal{A}$ be a $C^*$-algebra acting on a Hilbert space $H$, $\mathcal{A}$ be the weak closure of $\mathcal{A}$ and $X$ be a normal $\mathcal{A}$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

1. $H^n_w(\mathcal{A}, X)$ splits (III).
2. $H^n_{w(cb)}(\mathcal{A}, X)$ splits (III).
(i) There exists a bounded linear map

$$s_n : Z_w^n(A, X) \to L_w^{n-1}(A, X)$$

with $\partial^{n-1}s_n = id_{Z_w^n(A, X)}$.

(ii) There exists a bounded linear map

$$s_n : Z_w^n(\tilde{A}, X) \to L_w^{n-1}(\tilde{A}, X)$$

with $\partial^{n-1}s_n = id_{Z_w^n(\tilde{A}, X)}$.

Moreover if $X$ is a normal operator completely bounded $\tilde{A}$-bimodule, then the previous equivalence holds with the subscript "w" replaced by "wcb" and "completely bounded" instead of "bounded".

A similar result holds for normal (completely bounded) $B$-relative cohomology groups if $B$ is a subalgebra of $A$.

Proof. For each $n \in \mathbb{N}$, we define the restriction map $\psi \mapsto \psi |_{A^n} : L_w^n(\tilde{A}, X) \to L_w^n(A, X)$. By [SSm1], Lemma 3.3.3, for each $\phi \in L_w^n(A, X)$, there exists a unique $\tilde{\phi} \in L_w^n(\tilde{A}, X)$, with $\|\phi\| = \|\tilde{\phi}\|$, such that $\tilde{\phi} |_{A^n} = \phi$. Thus $\psi \mapsto \psi |_{A^n}$ is an isometric isomorphism. Moreover, since $X$ is a normal $\tilde{A}$-bimodule, $\psi \mapsto \psi |_{A^n}$ commutes with the coboundary operator, i.e. $\partial^n(\psi) |_{A^{n+1}} = \partial^n(\psi |_{A^n})$, for all $\psi \in L_w^n(\tilde{A}, X)$. The equivalence of (i) and (ii) follows now easily.

In the completely bounded case we take again the restriction map $\psi \mapsto \psi |_{A^n} : L_{wcb}^n(\tilde{A}, X) \to L_{wcb}^n(A, X)$. To prove that it is an isometric isomorphism take $\phi \in L_{wcb}^n(A, X)$. By what we said in the bounded case, there exists unique $\Phi \in L_{wcb}^n(M_m(A), M_m(X))$ with $\Phi |_{M_m(A)^n} = \phi_m$ and $\|\Phi\| = \|\phi_m\|$. On the other hand $(\tilde{\phi})_m \in L_{wcb}^n(M_m(\tilde{A}), M_m(X)) = L_{wcb}^n(M_m(\tilde{A}), M_m(X))$ and $(\tilde{\phi})_m |_{M_m(A)^n} = \phi_m$. Thus, by the uniqueness of $\Phi$, $(\tilde{\phi})_m = \Phi$ and so $\|(\tilde{\phi})_m\| = \|\Phi\| = \|\phi_m\|$ which shows that $\tilde{\phi}$ is completely bounded with $\|\tilde{\phi}\|_{cb} = \|\phi\|_{cb}$. (Alternatively we can prove that using Kaplansky's density theorem, as in [SSm1]). Therefore the restriction map is an isometric isomorphism. To prove that it is completely isometric let $m \in \mathbb{N}$ and $(\psi_{ij}) \in M_m(L_{wcb}^n(\tilde{A}, X))$. Then $(\psi_{ij} |_{A^n}) \in L_{wcb}^n(M_m(A), M_m(X))$ and, from what we said in the first part, there exists unique $\Psi \in L_{wcb}^n(M_m(A)^-, M_m(X))$ with $\Psi |_{M_m(A)} = (\psi_{ij} |_{A^n})$ and $\|\Psi\|_{cb} = \|(\psi_{ij} |_{A^n})\|_{L_{wcb}^n(M_m(A), M_m(X))} = \|(\psi_{ij} |_{A^n})\|_{cb}$. Using the uniqueness of $\Psi$ we get that $\Psi = (\psi_{ij})$ and so $\|(\psi_{ij})\| = \|(\psi_{ij} |_{A^n})\|_{cb}$, i.e. the restriction map is a complete isometry. The rest follows as in the bounded case.

Obviously the result also holds if $s_n$ is a map with the defining property of a splitting map of the first, second or fourth kind and if $(s_n, s_{n+1})$ is a pair of maps with the defining property of a pair of splitting maps of the fifth kind.

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The third step is to prove that the splitting of $\mathcal{H}^n_{(b)}(A, X)$ is equivalent to the "splitting" of $\mathcal{H}^n_{w(cb)}(A, X)$ if $A$ is a C*-algebra acting on a Hilbert space $H$ and $X$ is a normal (operator completely bounded) $\bar{A}$-bimodule.

**Lemma 5.1.4.** ([SSm1], Lemma 3.3.4) Let $A$ be a C*-algebra acting on a Hilbert space $H$, $\bar{A}$ be the weak closure of $A$, $\pi$ be the universal representation of $A$, $p$ be the minimal central projection in $\pi(A)^{-}$ with $p\pi(A)^{-} = \bar{A}$ and $X$ be a normal $\bar{A}$-bimodule. Then, for all $n \in \mathbb{N}$, there exist bounded linear maps $T_n : L^n_{c}(A, X) \to L^n_{w}(\pi(A)^{-}, X)$, $S_n : L^n_{w}(\pi(A)^{-}, X) \to L^n_{w}(\bar{A}, X)$ and $W_n : L^n_{w}(\pi(A)^{-}, X) \to L^n_{cb}(A, X)$ with $\|T_n\|, \|S_n\|, \|W_n\| \leq 1$, such that $\partial^n T_n = T_{n+1} \partial^n$ and $\partial^n S_n = S_{n+1} \partial^n$. Moreover $S_n T_n$ is a projection mapping $L^n_{c}(A, X)$ onto $L^n_{w}(\bar{A}, X)$.

If $X$ is a normal operator completely bounded $\bar{A}$-bimodule, then the subscripts "c" and "w" can be replaced by "cb" and "wcb" respectively and $T_n, S_n$ and $W_n$ are completely bounded with $\|T_n\|_{cb}, \|S_n\|_{cb}, \|W_n\|_{cb} \leq 1$.

Take $\theta$ to be an isomorphism from $p\pi(A)^{-}$ onto $\bar{A}$ with $\theta(p\pi(a)) = a$, for all $a \in A$ and $\theta(pb) = \pi^{-1}(b)$, for all $b \in \pi(A)$. The map $T_n$ is defined in two steps. First we take the map $\phi : L^n_{cb}(A, X) \to L^n_{wcb}(\pi(A), X)$ defined by $\phi(b_1, ..., b_n) = \theta(\phi_1(pb_1), ..., \phi_1(pb_n))$, for all $\phi \in L^n_{cb}(A, X)$ and all $b_1, ..., b_n \in \pi(A)$, and then define $T_n(\phi)$ to be equal to $\bar{\phi}_1$, where $\bar{\phi}_1$ is the unique extension of $\phi_1$ on $(\pi(A)^{-})^n$. The automatic complete boundedness of $\theta$ (since it is a *-isomorphism) and the complete boundedness of the map $b \mapsto \phi b$ imply that $\phi \mapsto \bar{\phi}_1$ is completely bounded. From what we said in the proof of Proposition 5.1.3, $\phi_1 \mapsto \bar{\phi}_1$ is also completely bounded and hence $T_n$ is completely bounded.

The map $S_n$ is defined by $S_n(\psi)(a_1, ..., a_n) = \psi(\theta^{-1}(a_1), ..., \theta^{-1}(a_n))$, for all $\psi \in L^n_{wcb}(\pi(A)^{-}, X)$ and all $a_1, ..., a_n \in \bar{A}$. Thus it is completely bounded, since $\theta^{-1}$ is.

The map $W_n$ is defined by $W_n(\psi)(a_1, ..., a_n) = \psi(\pi(a_1), ..., \pi(a_n))$, for all $\psi \in L^n_{wcb}(\pi(A)^{-}, X)$ and all $a_1, ..., a_n \in A$. Its complete boundedness follows from the complete boundedness of $\pi$.

**Lemma 5.1.5.** ([SSm1], Lemma 3.3.5) Let $A$ be a C*-algebra acting on a Hilbert space $H$, $\bar{A}$ be the weak closure of $A$ and $X$ be a normal $\bar{A}$-bimodule. Then, for all $n \in \mathbb{N}$, there exists a bounded linear map $J_n : L^n_{c}(A, X) \to L^{n-1}_{c}(A, X)$ with $\|J_n\| \leq ((n + 2)^n - 1)/(n + 1)$, such that $\phi - \partial^{n-1} J_n(\phi) \in Z^n_{w}(A, X)$, for all $\phi \in Z^n_{w}(A, X)$.

If $X$ is a normal operator completely bounded $\bar{A}$-bimodule, then we can replace the subscripts "c" and "w" with the subscripts "cb" and "wcb" respectively and $J_n$ is completely bounded with $\|J_n\|_{cb} \leq ((n + 2)^n - 1)/(n + 1)$.

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Let $p$ be the minimal central projection in $\pi(A)$ such that $p\pi(A)$ is isomorphic to $\tilde{A}$, $U$ be the (finite dimensional) subgroup $\{1, \pi(A), 2p\}$ of the unitary group of $\pi(A)$ and $B$ be the $C^*$-subalgebra of $\pi(A)$ generated by $U$. By Lemma 5.1.3, there exists a completely bounded linear map $K_n : \mathcal{L}_{wcb}^n(\pi(A)^{-}, X) \to \mathcal{L}_{wcb}^{n-1}(\pi(A)^{-}, X)$ with $\|K_n\|_{cb} \leq ((n+2)^n - 1)/(n+1)$, such that $\phi - \partial^{n-1}K_n(\phi) \in \mathcal{Z}_{wcb}^n(\pi(A)^{-}, X : /B)$, for all $\phi \in \mathcal{Z}_{wcb}^n(\pi(A)^{-}, X)$. The map $J_n$ is defined by $J_n = W_{n-1}K_nT_n$ and therefore it is completely bounded with $\|J_n\|_{cb} \leq ((n+2)^n - 1)/(n+1)$.

Now we can prove the equivalence between the splitting of $\mathcal{H}_{c}\text{cb}(A, X)$ and the "splitting" of $\mathcal{H}_{wcb}(A, X)$.

**Proposition 5.1.4.** Let $A$ be a $C^*$-algebra acting on a Hilbert space $H$, $\tilde{A}$ be the weak closure of $A$ and $X$ be a normal $\tilde{A}$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $\mathcal{H}_c^n(A, X)$ splits (III).

(ii) There exists a bounded linear map

$$s_n : \mathcal{Z}_w^n(A, X) \to \mathcal{L}_w^{n-1}(A, X)$$

with $\partial^{n-1}s_n = id_{\mathcal{Z}_w(A,X)}$.

Moreover if $X$ is a normal operator completely bounded $\tilde{A}$-bimodule, then the previous equivalence holds with the subscripts "c" and "w" replaced by "cb" and "wcb" respectively and "bounded" in (ii) replaced by "completely bounded".

**Proof.** It is similar to the proof of Proposition 5.1.2 with $Q_n$ replaced by $S_nT_n$ and $K_n$ replaced by $J_n$.

We note that, as in Proposition 5.1.2, whereas (i) $\Rightarrow$ (ii) holds for the other types of splitting, (ii) $\Rightarrow$ (i) does not.

The last step is to show that an averaging result similar to that of Proposition 5.1.2 holds for the normal cohomology in the case of a von Neumann algebra $M$ and a hyperfinite subalgebra $N$ of $M$.

**Lemma 5.1.6.** ([SSm1], Lemma 3.4.2) Let $M$ be a von Neumann algebra acting on a Hilbert space $H$, $N$ be a hyperfinite subalgebra of $M$ and $X$ be a normal $M$-bimodule. Then, for all $n \in \mathbb{N}$, there exists a bounded linear map $L_n : \mathcal{L}_w^n(M, X) \to \mathcal{L}_w^{n-1}(M, X)$ with $\|L_n\| \leq 2(((n+2)^n - 1)/(n+1))^2 + (n+2)^n - 1$, such that $\phi - \partial^{n-1}L_n(\phi) \in \mathcal{Z}_w^n(M, X : /N)$, for all $\phi \in \mathcal{Z}_w^n(M, X)$.

If $X$ is a normal operator completely bounded $M$-bimodule, then the subscript "w" can be replaced by "wcb" and $L_n$ is completely bounded, with $\|L_n\|_{cb} \leq 2(((n+2)^n - 1)/(n+1))^2 + (n+2)^n - 1$.
The map \( L_n \) is defined by \( L_n = J_n + K_n - J_n \delta^{n-1} K_n \) and hence, by Lemmas 5.1.3 and 5.1.5, it is completely bounded with \( \| L_n \|_{cb} \leq 2(((n+2)^n - 1)/(n + 1))^2 + (n+2)^n - 1 \).

**Lemma 5.1.7.** ([SSm1], Lemma 3.4.3) Let \( \mathcal{M} \) be a von Neumann algebra acting on a Hilbert space \( H \), \( \mathcal{N} \) be a hyperfinite subalgebra of \( \mathcal{M} \) and \( X \) be a normal \( \mathcal{M} \)-bimodule. Then, for all \( n \in \mathbb{N} \), there exists a bounded projection \( P_n : \mathcal{L}_w^n(\mathcal{M}, X) \to \mathcal{L}_w^n(\mathcal{M}, X : /\mathcal{N}) \) with \( \| P_n \| \leq 1 \), such that \( \vartheta^{n-1} P_{n-1} = P_n \vartheta^{n-1} \).

If \( X \) is a normal operator completely bounded \( \mathcal{M} \)-bimodule, then the subscript "\( w \)" can be replaced by "\( wcb \)" and \( P_n \) is completely bounded with \( \| P_n \|_{cb} \leq 1 \).

The projection \( P_n \) is defined by \( P_n = S_n T_n Q_n |_{\mathcal{L}_{wcb}(\mathcal{M}, X)} \) and therefore by Lemmas 5.1.2 and 5.1.4 it is completely bounded with \( \| P_n \|_{cb} \leq 1 \).

Arguing along the lines of the proof of Proposition 5.1.2 with \( Q_n \) replaced by \( P_n \) and \( K_n \) replaced by \( L_n \) we get the following proposition.

**Proposition 5.1.5.** Let \( \mathcal{M} \) be a von Neumann algebra acting on a Hilbert space \( H \), \( \mathcal{N} \) be a hyperfinite subalgebra of \( \mathcal{M} \) and \( X \) be a normal \( \mathcal{M} \)-bimodule. Then, for all \( n \in \mathbb{N} \), the following are equivalent:

(i) There exists a bounded linear map

\[
\begin{align*}
 s_n : \mathcal{Z}_w^n(\mathcal{M}, X) &\to \mathcal{L}_w^{n-1}(\mathcal{M}, X),
\end{align*}
\]

with \( \vartheta^{n-1} s_n = id_{\mathcal{Z}_w^n(\mathcal{M}, X)} \).

(ii) There exists a bounded linear map

\[
\begin{align*}
 s_n : \mathcal{Z}_w^n(\mathcal{M}, X : /\mathcal{N}) &\to \mathcal{L}_w^{n-1}(\mathcal{M}, X : /\mathcal{N}),
\end{align*}
\]

with \( \vartheta^{n-1} s_n = id_{\mathcal{Z}_w^n(\mathcal{M}, X : /\mathcal{N})} \).

If \( X \) is a normal operator completely bounded \( \mathcal{M} \)-bimodule, then the previous equivalence holds with the subscript "\( w \)" replaced by "\( wcb \)" and "completely bounded" replaced by "bounded".

Now we can prove Proposition 5.1.1.

**Proof of Proposition 5.1.1.** (i) \(\Leftrightarrow\) (ii) follows from Proposition 5.1.4, (ii) \(\Leftrightarrow\) (iii) from Proposition 5.1.5 and (ii) \(\Leftrightarrow\) (iv) from Proposition 5.1.3.

**Remark 5.1.1.** In the last paragraph of the introduction to Chapter 2 we said that we could have defined a notion of normal splitting for normal cohomology groups where the splitting maps are weak* continuous. It would be interesting to know whether Proposition 5.1.1 holds with "bounded" replaced by "weak* continuous" in parts (ii), (iii) and (iv).
It can be proved, using the averaging results that we discussed, that if $\mathcal{M}$ is a von Neumann algebra, $\mathcal{M}_I$, $\mathcal{M}_{II_1}$, $\mathcal{M}_{II_\infty}$ and $\mathcal{M}_{III}$ are respectively the type $I$, $II_1$, $II_\infty$ and $III$ central direct summands of $\mathcal{M}$ and $X$ is a normal (operator completely bounded) $\mathcal{M}$-bimodule, then, for all $n \in \mathbb{N}$, $\mathcal{H}_{c(b)}^n(\mathcal{M}, X)$ is isomorphic to $\mathcal{H}_{c(b)}^n(\mathcal{M}_I, X) \oplus \mathcal{H}_{c(b)}^n(\mathcal{M}_{II_1}, X) \oplus \mathcal{H}_{c(b)}^n(\mathcal{M}_{II_\infty}, X) \oplus \mathcal{H}_{c(b)}^n(\mathcal{M}_{III}, X)$ ([SSm1], Theorem 3.3.7 and Corollary 3.3.8). We finish the discussion of averaging and lifting results with a similar result about splitting.

**Proposition 5.1.6.** Let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ be a direct sum decomposition of a von Neumann algebra $\mathcal{M}$ into von Neumann algebras $\mathcal{M}_1$ and $\mathcal{M}_2$ with $\mathcal{M} = e_1 \oplus e_2$ and $X$ be a normal $\mathcal{M}$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $\mathcal{H}_c^n(\mathcal{M}, X)$ splits (III).

(ii) $\mathcal{H}_c^n(\mathcal{M}_1, X)$ and $\mathcal{H}_c^n(\mathcal{M}_2, X)$ split (III).

(iii) $\mathcal{H}_c^n(\mathcal{M}_1, e_1 X e_1)$ and $\mathcal{H}_c^n(\mathcal{M}_2, e_2 X e_2)$ split (III).

If $X$ is a normal operator completely bounded $\mathcal{M}$-bimodule, then the same equivalence holds with "cb" in the place of "c".

A similar result holds for normal (completely bounded) cohomology groups.

**Proof.** We will prove it for the bounded case. The rest follows in a similar manner. Let $\mathcal{B}$ be the $C^*$-subalgebra of $\mathcal{M}$ generated by $e_1$ and $e_2$. It is easy to see that $\mathcal{L}_c^n(\mathcal{M}, X : /\mathcal{B})$ is isomorphic to $\mathcal{L}_c^n(\mathcal{M}_1, X : /\mathcal{B}) \oplus \mathcal{L}_c^n(\mathcal{M}_2, X : /\mathcal{B})$ via the map $\phi \mapsto \phi |_{\mathcal{M}_1} \oplus \phi |_{\mathcal{M}_2} : \mathcal{L}_c^n(\mathcal{M}_1, X : /\mathcal{B}) \to \mathcal{L}_c^n(\mathcal{M}_1, X : /\mathcal{B}) \oplus \mathcal{L}_c^n(\mathcal{M}_2, X : /\mathcal{B})$ and $\mathcal{L}_c^n(\mathcal{M}_k, X : /\mathcal{B})$ is isomorphic to $\mathcal{L}_c^n(\mathcal{M}_k, e_k X e_k : /\mathcal{B})$, $k = 1, 2$, via the map $\psi \mapsto \tilde{\psi} : \mathcal{L}_c^n(\mathcal{M}_k, X : /\mathcal{B}) \to \mathcal{L}_c^n(\mathcal{M}_k, e_k X e_k : /\mathcal{B})$, defined by $\tilde{\psi}(m) = e_k \psi(m) e_k$, for all $\psi \in \mathcal{L}_c^n(\mathcal{M}_k, X : /\mathcal{B})$. Since both $e_1$ and $e_2$ are central projections, $\mathcal{B}$ is contained in the centre of $\mathcal{M}$ and thus it is generated by an amenable group of unitaries. The equivalence between (i), (ii) and (iii) follows from Proposition 5.1.2 and the two above mentioned isomorphisms. □

**Corollary 5.1.1.** Let $\mathcal{M}$ be a von Neumann algebra, $\mathcal{M}_I = e_I \mathcal{M}$, $\mathcal{M}_{II_1} = e_{II_1} \mathcal{M}$, $\mathcal{M}_{II_\infty} = e_{II_\infty} \mathcal{M}$ and $\mathcal{M}_{III} = e_{III} \mathcal{M}$ be the type $I$, $II_1$, $II_\infty$ and $III$ central direct summands of $\mathcal{M}$ and $X$ be a normal $\mathcal{M}$-bimodule. Then, for all $n \in \mathbb{N}$, the following are equivalent:

(i) $\mathcal{H}_c^n(\mathcal{M}, X)$ splits (III).

(ii) $\mathcal{H}_c^n(\mathcal{M}_1, X)$, $\mathcal{H}_c^n(\mathcal{M}_{II_1}, X)$, $\mathcal{H}_c^n(\mathcal{M}_{II_\infty}, X)$ and $\mathcal{H}_c^n(\mathcal{M}_{III}, X)$ split (III).

(iii) $\mathcal{H}_c^n(\mathcal{M}_1, e_1 X e_1)$, $\mathcal{H}_c^n(\mathcal{M}_{II_1}, e_{II_1} X e_{II_1})$, $\mathcal{H}_c^n(\mathcal{M}_{II_\infty}, e_{II_\infty} X e_{II_\infty})$ and $\mathcal{H}_c^n(\mathcal{M}_{III}, e_{III} X e_{III})$ split (III).

If $X$ is a normal operator completely bounded $\mathcal{M}$-bimodule, then the same holds with "cb" in the place of "c".

A similar result holds for normal (completely bounded) cohomology groups.
Remark 5.1.2. The use of Proposition 5.1.2 in the proof of Proposition 5.1.6 does not allow us to obtain results similar to those of Proposition 5.1.6 and Corollary 5.1.1 for the other types of splitting.

5.2 Amenable von Neumann algebras

One of the most important applications of the cohomology theory of operator algebras has been the characterisation, by Connes, of hyperfinite von Neumann algebras as the amenable ones. We start by reminding the reader how that was achieved and by discussing some other related results and questions. Accounts of the relation between amenable, injective and hyperfinite von Neumann algebras can be found in [Pal], [Pie2] and [Th].

Using the averaging results of the previous section, Kadison and Ringrose proved that if a C*-algebra $A$ is the closed linear span of an amenable subgroup of its unitary group, then $\mathcal{H}_c^n(A, X) = \{0\}$, for all dual $A$-bimodules $X$ and all $n \in \mathbb{N}$ ([KRi5], Theorem 3.3), i.e. $A$ is amenable in the sense of Definition 1.2.10(i). By applying the lifting result that we referred to in the previous section, they showed, jointly with Johnson, that if $M$ is a hyperfinite von Neumann algebra, then $\mathcal{H}_c^n(M, X)$ vanishes, for all normal $M$-bimodules $X$ and all $n \in \mathbb{N}$ ([JKRi], Corollary 6.4), namely $M$ is an amenable von Neumann algebra (see Definition 1.2.14). Since all the averaging and lifting techniques used in the proof of that result also hold in the completely bounded case the result holds in the completely bounded case as well, i.e. if $M$ is a hyperfinite von Neumann algebra, then $\mathcal{H}_c^{cb}(M, X) = \{0\}$, for all normal operator completely bounded $M$-bimodules $X$ and all $n \in \mathbb{N}$ ([SSm1], Corollary 3.4.6).

In his celebrated paper on the classification of injective factors, Connes proved that if a factor with separable predual is injective, then it is hyperfinite ([Co1], Theorem 6, p.74). That result was later generalised to von Neumann algebras with non-separable predual by Elliott ([El], Corollary 5). (For simpler proofs see [H3] and [Po3]).

The last step in establishing the equivalence between hyperfiniteness, amenability and injectivity was Connes proof that every amenable von Neumann algebra $M$ is injective ([Co2], Theorem 1, see also [Co1], Remark 5.33 for the first appearance of the ideas that lead to the proof of that theorem). He proved that by constructing a normal $M$-bimodule and a derivation into that bimodule, the cobounding of which implies that $M$ is injective.

Bunce and Paschke gave a simpler proof of that last result, which we will discuss briefly, because it shows an application of the modules $L_c^1(Y, X)$ that
we defined in Section 3.1.1. They defined a quasi-expectation to be a bounded projection (not necessarily of norm one) \( Q \) from a unital \( C^* \)-algebra \( A \) onto a unital \( C^* \)-subalgebra \( B \) of \( A \), with \( Q(b_1 ab_2) = b_1 Q(a) b_2 \), for all \( a \in A \) and all \( b_1, b_2 \in B \). In other words a quasi-expectation is a bounded projection, which is a \( B \)-module homomorphism. The existence of a quasi-expectation from \( \mathcal{B}(H) \) onto a von Neumann algebra \( \mathcal{M} \) implies the injectivity of \( \mathcal{M} \) ([BuPas1], Theorem 2). Moreover, if \( \mathcal{N} \) is a von Neumann algebra and \( \mathcal{M} \) is an amenable von Neumann subalgebra of \( \mathcal{N} \), then there exists a quasi-expectation from \( \mathcal{N} \) onto \( \mathcal{M}' \cap \mathcal{N} \) ([BuPas1], Theorem 3). Combining Theorems 2 and 3 (with \( \mathcal{N} = \mathcal{B}(H) \)) of [BuPas1] we get Connes' result that amenability implies injectivity. The proof of Theorem 3 of [BuPas1] in the case \( \mathcal{N} = \mathcal{B}(H) \) goes as follows: Take the \( \mathcal{M} \)-bimodule \( L^1_c(\mathcal{B}(H), \mathcal{B}(H)) \) defined as in Proposition 3.1.1. Then, by Proposition 3.1.3, \( L^1_c(\mathcal{B}(H), \mathcal{B}(H)) \) is a normal \( \mathcal{M} \)-bimodule. It is easy to see that \( Y = L^1_c(\mathcal{B}(H), \mathcal{B}(H) : /\mathcal{M}') \) is a weak* closed \( \mathcal{M} \)-submodule of \( L^1_c(\mathcal{B}(H), \mathcal{B}(H)) \) and thus a normal \( \mathcal{M} \)-bimodule. Moreover \( X = \{ \phi \in Y \mid \phi(m') = 0, \text{for all } m' \in \mathcal{M}' \} \) is a weak* closed \( \mathcal{M} \)-submodule of \( Y \) and hence a normal \( \mathcal{M} \)-bimodule.

If we define \( D : \mathcal{M} \to X \), by \( D(m)(a) = ma - am \), for all \( m \in \mathcal{M} \) and all \( a \in \mathcal{B}(H) \), then obviously \( D \in Z^1_c(\mathcal{M}, X) \). Therefore the amenability of \( \mathcal{M} \) implies the existence of \( \phi \in X \), with \( D = \partial^0(\phi) \). If we define \( Q : \mathcal{B}(H) \to \mathcal{B}(H) \), by \( Q = id_{\mathcal{B}(H)} - \phi \), then the vanishing of \( \phi \) on \( \mathcal{M}' \) and \( D = \partial^0(\phi) \) imply that \( Q \) is a bounded projection, with \( Im(Q) = \mathcal{M}' \). Since \( \phi \) is an \( \mathcal{M}' \)-module homomorphism, \( Q \) is also an \( \mathcal{M}' \)-module homomorphism and so it is a quasi-expectation.

Lately it was proved -in [CS4], Theorem 3.1 (see also Corollary 3.4 there for a simple proof of amenability \( \Rightarrow \) injectivity) and in [Pi6], Theorem 2.9 (see also [Pi4], Corollaire 5)- that the existence of a completely bounded projection \( \rho : \mathcal{B}(H) \to \mathcal{M} \) implies the injectivity of \( \mathcal{M} \). It is easy to see, using the minimal operator space structure \( MIN(X) \) on a Banach space \( X \) and the automatic complete boundedness of bounded maps \( \phi : Y \to MIN(X) \) for any operator space \( Y \) (see [Pi7], p.16), that the vanishing of \( \mathcal{H}^1_{cb}(\mathcal{M}, X) \), for all normal operator completely bounded \( \mathcal{M} \)-bimodules, implies the vanishing of \( \mathcal{H}^1_c(\mathcal{M}, X) \), for all normal \( \mathcal{M} \)-bimodules \( X \), and hence, from [Co2], Theorem 1, the injectivity of \( \mathcal{M} \). We will give a straightforward proof of that result using the above mentioned result of Christensen-Sinclair and Pisier. Although this approach has not appeared in the literature, we believe that many people must have known about it.

**Proposition 5.2.1.** Let \( \mathcal{M} \) be a von Neumann algebra acting on a Hilbert space \( H \). Then \( \mathcal{H}^n_{cb}(\mathcal{M}, X) \) vanishes, for all normal operator completely bounded \( \mathcal{M} \)-bimodules \( X \) and all \( n \in \mathbb{N} \), if and only if \( \mathcal{M} \) is injective.
Proof. From [CS4], Theorem 3.1 or [Pi6], Theorem 2.9, to prove that a von Neumann algebra $\mathcal{N}$ is injective, we need to show the existence of a completely bounded projection $\rho : \mathcal{B}(H) \to \mathcal{N}$. Obviously $\mathcal{B}(H)$ is a normal operator completely bounded $\mathcal{M}$-bimodule. Hence, by Proposition 3.1.14, $\mathcal{L}^{1}_{cb}(\mathcal{B}(H), \mathcal{B}(H))$ is also a normal operator completely bounded $\mathcal{M}$-bimodule. Moreover

$$X = \{ \phi \in \mathcal{L}^{1}_{cb}(\mathcal{B}(H), \mathcal{B}(H)) \mid \phi(m') = 0, \text{for all } m' \in \mathcal{M}' \}$$

is also a normal operator completely bounded $\mathcal{M}$-bimodule, since it is a weak* closed submodule of $\mathcal{L}^{1}_{cb}(\mathcal{B}(H), \mathcal{B}(H))$. As in the proof of [BuPasl], Theorem 3, we define $D : \mathcal{M} \to X$, by $D(m)(a) = ma - am$, for all $m \in \mathcal{M}$ and all $a \in \mathcal{B}(H)$. The $L^\infty$ property of $\mathcal{M}$ and $X$ implies that $D \in \mathcal{Z}^{1}_{cb}(\mathcal{M}, X)$. So, by the hypothesis, there exists $\phi \in X$, with $D = \partial^{0}(\phi)$. Then $\rho = \text{id}_{\mathcal{B}(H)} - \phi$ is a completely bounded projection from $\mathcal{B}(H)$ onto $\mathcal{M}'$ and therefore $\mathcal{M}'$ is injective. Thus, by [Co1], Proposition 6.4.(a), $\mathcal{M}$ is injective.

On the other hand, if $\mathcal{M}$ is injective, then it is hyperfinite ([Co1], Theorem 6) and thus $\mathcal{H}^{n}_{cb}(\mathcal{M}, X)$ vanishes, for all normal operator completely bounded $\mathcal{M}$-bimodules $X$ and all $n \in \mathbb{N}$ ([SSm1], Corollary 3.4.6). We finish this discussion with a, still open, question related to what we said so far. If $\mathcal{M}$ is a von Neumann algebra for which $\mathcal{H}^{n}_{cb}(\mathcal{M}, X)$ vanishes, for all normal $\mathcal{M}$-bimodules $X$, is $\mathcal{M}$ injective (see [SSm1], Problem 8.4.5)? Christensen and Sinclair constructed an $\mathcal{M}$-bimodule and a 2-cochain into that bimodule, the cobounding of which implies the injectivity of $\mathcal{M}'$ and thus of $\mathcal{M}$ ([CS6]). Unfortunately their module is not dual.

In the two following propositions we will show that if $\mathcal{M}$ is a hyperfinite von Neumann algebra, then both the bounded and the completely bounded cohomology complex of $\mathcal{M}$, with coefficients in a normal (operator completely bounded) $\mathcal{M}$-bimodule $X$, splits (III). Moreover we will show that the splitting of the cohomology complex, with coefficients in $X$, for all normal (operator completely bounded) $\mathcal{M}$-bimodules $X$, implies nothing more than the amenability of the algebra.

We start with the bounded case.

**Proposition 5.2.2.** If $\mathcal{M}$ is a von Neumann algebra, then the following are equivalent:

(i) $\mathcal{M}$ is injective.

(ii) $\mathcal{M}$ is hyperfinite.

(iii) $\mathcal{M}$ is amenable.

(iv) The bounded Hochschild cohomology complex of $\mathcal{M}$, with coefficients in $X$, splits (III), for all normal $\mathcal{M}$-bimodules $X$.
Proof. (i) $\iff$ (ii) $\iff$ (iii) It follows from [Co2], Theorem 1, [Co1], Theorem 6 and [El], Corollary 5.

(ii) $\Rightarrow$ (iv) Let $n \in \mathbb{N}$ and $X$ be a normal $\mathcal{M}$-bimodule. Since $\mathcal{M}$ is hyperfinite, by Proposition 5.1.1, to prove that $\mathcal{H}_c^n(\mathcal{M}, X)$ splits (III), we need to prove that there exists a bounded linear map

$$s_n : \mathcal{Z}_w^n(\mathcal{M}, X : /\mathcal{M}) \to \mathcal{L}_w^{n-1}(\mathcal{M}, X : /\mathcal{M})$$

with $\partial^{n-1} s_n = \text{id}_{\mathcal{Z}_w^n(\mathcal{M}, X : /\mathcal{M})}$. As in the proof of Proposition 2.3.5, we can define $s_n : \mathcal{Z}_w^n(\mathcal{M}, X : /\mathcal{M}) \to \mathcal{L}_w^{n-1}(\mathcal{M}, X : /\mathcal{M})$ by

$$s_n(\phi)(m_1, \ldots, m_{n-1}) = \phi(1\mathcal{M}, m_1, \ldots, m_{n-1})$$

for all $\phi \in \mathcal{Z}_w^n(\mathcal{M}, X : /\mathcal{M})$ and all $m_1, \ldots, m_{n-1} \in \mathcal{M}$. It is easy to see that $s_n$ preserves separate ultraweak-weak* continuity. Moreover $\partial^{n-1} s_n = \text{id}_{\mathcal{Z}_w^n(\mathcal{M}, X : /\mathcal{M})}$. Hence $\mathcal{H}_c^n(\mathcal{M}, X)$ splits (III).

Alternatively we can duplicate the arguments that lead to Corollary 4.3.4(a)(ii) and prove (ii) $\Rightarrow$ (iv) using Proposition 3.1.3, [JKRi], Corollary 6.4 (which shows that all hyperfinite von Neumann algebras are amenable) and Corollary 4.1.1.

(iv) $\Rightarrow$ (iii) From Proposition 2.1.24, since the bounded Hochschild cohomology complex of $\mathcal{M}$, with coefficients in $X$, splits (III), for all normal $\mathcal{M}$-bimodules $X$, $\mathcal{H}_c^n(\mathcal{M}, X) = \{0\}$, for all normal $\mathcal{M}$-bimodules $X$ and all $n \in \mathbb{N}$. \qed

Before we move to the completely bounded case, we would like to discuss briefly what the geometric characterisation of split (III) and the results of Sections 4.1 and 4.2 about the relation between the splitting of $\mathcal{H}_c^n(\mathcal{M}, X)$ and the splitting of $\mathcal{H}_c^n(\mathcal{M}, \mathcal{L}^1_c(Y, X))$ and of $\mathcal{H}_c^n(\mathcal{M}, \mathcal{L}^1_c(X, Y^*))$ ($X$ is normal and so it has a predual $X_\ast$) imply here. The geometric characterisation (Proposition 2.1.24) immediately implies the following corollary.

**Corollary 5.2.1.** Let $\mathcal{M}$ be a von Neumann algebra. If $\mathcal{M}$ is hyperfinite, then $\mathcal{Z}_c^n(\mathcal{M}, X)$ is complemented in $\mathcal{L}_c^n(\mathcal{M}, X)$, for all $n \in \mathbb{N}$ and all normal $\mathcal{M}$-bimodules $X$.

We will see in the following section that in many cases the converse of Corollary 5.2.1 is also true.

We recall from Corollary 4.1.1, that the third type of splitting of the bounded Hochschild cohomology complex of a Banach algebra $\mathcal{A}$, with coefficients in a Banach $\mathcal{A}$-bimodule $X$, is equivalent to the splitting of the cohomology complex of $\mathcal{A}$, with coefficients in $\mathcal{L}^1_c(Y, X)$, for all Banach spaces $Y$. Combining that with the result of Proposition 5.2.2, we get that if $\mathcal{M}$ is a hyperfinite von Neumann algebra and $X$ is a normal $\mathcal{M}$-bimodule, then the bounded Hochschild cohomology
complex of $\mathcal{M}$, with coefficients in $\mathcal{L}_c^1(Y, X)$, splits (III), for all Banach spaces $Y$. But all modules of the form $\mathcal{L}_c^1(Y, X)$ are normal, since $X$ is, and so what we just said follows immediately from Proposition 5.2.2, without any use of Corollary 4.1.1. Exactly the same holds for the modules $\mathcal{L}_c^1(X^*, Y^*)$.

A question that arises here is whether the splitting of $\mathcal{H}_c^1(\mathcal{M}, X)$, for all normal $\mathcal{M}$-bimodules $X$, implies that $\mathcal{M}$ is injective. For the third type of splitting this question is equivalent, from Proposition 3.1.3 and Proposition 4.1.4, to the question we mentioned in the opening discussion about the relation between the vanishing of $\mathcal{H}_c^1(\mathcal{M}, X)$, for all normal $\mathcal{M}$-bimodules $X$, and injectivity. Things become more interesting if we consider this question for the fourth and the fifth type of splitting. Split (IV) and (V) of $\mathcal{H}_c^2(\mathcal{M}, X)$, for all normal $\mathcal{M}$-bimodules $X$, seem to be stronger conditions than the vanishing of $\mathcal{H}_c^2(\mathcal{M}, X)$, for all normal $\mathcal{M}$-bimodules $X$.

We move now to the completely bounded case.

**Proposition 5.2.3.** If $\mathcal{M}$ is a von Neumann algebra, then the following are equivalent:

1. $\mathcal{M}$ is injective.
2. $\mathcal{M}$ is hyperfinite.
3. $\mathcal{M}$ is amenable.
4. $\mathcal{H}_{cb}^n(\mathcal{M}, X)$ vanishes, for all normal operator completely bounded $\mathcal{M}$-bimodules $X$ and all $n \in \mathbb{N}$.

5. The completely bounded Hochschild cohomology complex of $\mathcal{M}$, with coefficients in $X$, splits (III), for all normal operator completely bounded $\mathcal{M}$-bimodules $X$.

**Proof.** $(i) \iff (ii) \iff (iii) \iff (iv)$ follows from Proposition 5.2.1, [Co1], Theorem 6, [El], Corollary 5 and [Co2], Theorem 1. $(ii) \Rightarrow (v)$ can be proved as $(ii) \Rightarrow (iv)$ in Proposition 5.2.2 (both proofs can be used). $(v)$ implies $(iv)$ from Proposition 2.2.4.

A corollary similar to Corollary 5.2.1 holds in the completely bounded case. The remarks following Corollary 5.2.1 also make sense here.

**Remark 5.2.1.** (i) If a $C^*$-algebra is amenable as a Banach algebra, then it is nuclear ([Co2], Corollary 2). Haagerup showed that the converse is also true ([H1], Theorem 3.1). It follows immediately from those results and Corollary 4.3.4(a)(ii) that a $C^*$-algebra $\mathcal{A}$ is nuclear if and only if the bounded Hochschild cohomology complex of $\mathcal{A}$, with coefficients in $X$, splits (III), for all dual $\mathcal{A}$-bimodules $X$. 

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(ii) If \( \mathcal{M} \) and \( \mathcal{N} \) are von Neumann algebras, then the binormal projective tensor product \( \mathcal{M} \hat{\otimes}^\sigma \mathcal{N} \) of \( \mathcal{M} \) and \( \mathcal{N} \) is defined as the dual of the space of normal bilinear forms on \( \mathcal{M} \times \mathcal{N} \) ([E], p.139). A normal virtual diagonal for \( \mathcal{M} \) is an element \( \delta \) of \( \mathcal{M} \hat{\otimes}^\sigma \mathcal{M} \) with \( \pi(M) = Mm \), for all \( m \in \mathcal{M} \) and \( \pi(M) = 1_{\mathcal{M}} \) (where \( \pi(m_1 \otimes m_2) = m_1 m_2 \), for all \( m_1, m_2 \in \mathcal{M} \) ([E], p.147)). Haagerup's proof of the amenability of nuclear \( C^* \)-algebras depends on proving that if \( \mathcal{M} \) is an amenable von Neumann algebra, then there exists a normal virtual diagonal for \( \mathcal{M} \) ([H1], proof of Theorem 3.1, see also [E], Theorem 3.1 for a different approach). An analogue of this result in terms of splitting would be that if \( \mathcal{M} \) is an amenable von Neumann algebra, then the normal Hochschild cohomology complex of \( \mathcal{M} \), with coefficients in \( X \), splits normally (i.e. the splitting maps are weak* continuous), for all normal \( \mathcal{M} \)-bimodules \( X \). If the result of Remark 5.1.1 holds, then that would follow immediately from Proposition 5.2.2.

(iii) Following Definitions 1.2.10(ii) and 1.2.12(ii) we can call a von Neumann algebra \( \mathcal{M} \) (completely) \( n \)-amenable if \( \mathcal{H}^n_{c(b)}(\mathcal{M}, X) = \{0\} \), for all normal (operator completely bounded) \( \mathcal{M} \)-bimodules \( X \). Arguments similar to the ones leading to Corollary 4.3.4(a)(i) and (b)(i) show that a von Neumann algebra \( \mathcal{M} \) is (completely) \( n \)-amenable if and only if \( \mathcal{H}^n_{c(b)}(\mathcal{M}, X) \) splits (III), for all normal (operator completely bounded) \( \mathcal{M} \)-bimodules \( X \). It would be very interesting to obtain a characterisation of \( n \)-amenability for von Neumann algebras.

Type I von Neumann algebras are injective ([To1], Corollary 7.2.1 and Theorem 7.2). Thus combining Corollary 5.1.1 and Propositions 5.2.2 and 5.2.3 we get the following corollary.

**Corollary 5.2.2.** Let \( \mathcal{M} \) be a von Neumann algebra, \( \mathcal{M}_{II_1} = e_{II_1} \mathcal{M}, \mathcal{M}_{II_\infty} = e_{II_\infty} \mathcal{M} \) and \( \mathcal{M}_{III} = e_{III} \mathcal{M} \) be the type \( II_1, II_\infty \) and \( III \) central direct summands of \( \mathcal{M} \) and \( X \) be a normal (operator completely bounded) \( \mathcal{M} \)-bimodule. Then, for all \( n \in \mathbb{N} \), the following are equivalent:

(i) \( \mathcal{H}^n_{c(b)}(\mathcal{M}, X) \) splits (III).

(ii) \( \mathcal{H}^n_{c(b)}(\mathcal{M}_{II_1}, X) \), \( \mathcal{H}^n_{c(b)}(\mathcal{M}_{II_\infty}, X) \) and \( \mathcal{H}^n_{c(b)}(\mathcal{M}_{III}, X) \) split (III).

(iii) \( \mathcal{H}^n_{c(b)}(\mathcal{M}_{II_1}, e_{II_1} X e_{II_1}) \), \( \mathcal{H}^n_{c(b)}(\mathcal{M}_{II_\infty}, e_{II_\infty} X e_{II_\infty}) \) and \( \mathcal{H}^n_{c(b)}(\mathcal{M}_{III}, e_{III} X e_{III}) \) split (III).

A similar result holds for normal (completely bounded) cohomology groups.

**5.3 The cohomology into \( B(H) \) and injective von Neumann algebras**

Maybe the most interesting open problem in the Hochschild cohomology of \( C^* \)- and von Neumann algebras is the derivation problem which asks whether all
derivations from a C*-subalgebra of \( B(H) \) into \( B(H) \) are inner, i.e. whether \( \mathcal{H}_c^1(\mathcal{A}, B(H)) = \{0\} \), for all C*-algebras \( \mathcal{A} \) acting on a Hilbert space \( H \). In this section we will discuss the splitting of the groups \( \mathcal{H}_{c(b)}^1(\mathcal{M}, B(H)) \) for a von Neumann algebra \( \mathcal{M} \) acting on \( H \). We will start with a review of the existing results about the groups \( \mathcal{H}_{c(b)}^n(\mathcal{A}, B(H)) \) and a brief discussion of conditional expectations. Then we will show that the splitting of the groups \( \mathcal{H}_c^1(\mathcal{M}, B(H)) \) and \( \mathcal{H}_{c(b)}^1(\mathcal{M}, B(H)) \) is closely related to injectivity of \( \mathcal{M} \).

All the existing results about derivations from a C*-algebra acting on a Hilbert space \( H \) into \( B(H) \) are contained in two papers by Christensen from the late 1970's and the early 1980's ([C1] and [C2], see also Section 2 of [C3] for a review of those results). Using some of the techniques that he developed in his study of perturbations of von Neumann algebras (see the references in [C3]) he obtained several conditions equivalent to the vanishing of \( \mathcal{H}_c^1(\mathcal{A}, B(H)) \) ([C2], Theorem 3.1) and gave an affirmative answer to the derivation problem for properly infinite von Neumann algebras ([C1], Theorem 3.2), for type \( II_1 \) von Neumann algebras which are stable under tensoring with the hyperfinite type \( II_1 \) factor ([C2], Corollary 3.3)(2) combined with Theorem 3.1) and for C*-algebras which have a cyclic vector ([C2], Corollary 5.4). Moreover he observed that for those three classes of C*-algebras \( \mathcal{H}_c^1(\mathcal{A}, \mathcal{N}) \) vanishes if \( \mathcal{N} \) is an injective von Neumann algebra containing \( \mathcal{A} \) ([C3], Theorem 2.3(ii)).

If all the derivations from \( \mathcal{A} \) into \( B(H) \) are completely bounded, then they are inner ([C2], Theorem 3.1(4)). In other words \( \mathcal{H}_{cb}^1(\mathcal{A}, B(H)) \) vanishes for all C*-algebras \( \mathcal{A} \) acting on \( H \). More generally \( \mathcal{H}_{cb}^1(\mathcal{A}, \mathcal{N}) = \{0\} \) if \( \mathcal{N} \) is an injective von Neumann algebra containing \( \mathcal{A} \). Christensen, Effros and Sinclair proved, using the Christensen-Sinclair representation theorem for completely bounded multilinear maps, that the same holds for \( n > 1 \) ([CES], Theorem 4.3). Combining that with an automatic complete boundedness result, which we will discuss in the second part of Section 5.4, they showed that \( \mathcal{H}_{cb}^n(\mathcal{M}, \mathcal{N}) \) vanishes, for all \( n \in \mathbb{N} \), if \( \mathcal{M} \) is either a properly infinite von Neumann algebra or a type \( II_1 \) von Neumann algebra stable under tensoring with the hyperfinite type \( II_1 \) factor and \( \mathcal{N} \) is an injective von Neumann algebra containing \( \mathcal{M} \) ([CES], Corollary 5.5).

The derivation problem is central in the study of C*-algebras because of its connection to Kadison's similarity problem (is every representation of a C*-algebra \( \mathcal{A} \) on a Hilbert space \( H \) similar to a \( * \)-representation of \( \mathcal{A} \) on \( H \) ?). It was known from the beginning of the study of the derivation problem that an affirmative answer to the similarity problem implies an affirmative answer to the derivation problem (see [C3], p.266 and [P1], pp.130-132). Recently Kirchberg proved that the converse is also true and so the similarity problem and the derivation.
problem are equivalent ([Ki], Corollary 1). A nice account of the relation between
the similarity problem, the derivation problem and complete boundedness can be
found in [Pi1], pp.73-75 and pp.128-130.

Let \( \mathcal{A} \) be a \( C^* \)-algebra and \( \mathcal{B} \) be a \( C^* \)-subalgebra of \( \mathcal{A} \). A bounded linear map
\( \mathcal{E} : \mathcal{A} \rightarrow \mathcal{B} \) is called a conditional expectation if: (i) \( \mathcal{E} \) is a projection mapping
\( \mathcal{A} \) onto \( \mathcal{B} \), (ii) \( \mathcal{E} \) is a \( \mathcal{B} \)-module map, i.e. \( \mathcal{E}(ab) = \mathcal{E}(a)b \) and \( \mathcal{E}(ba) = b\mathcal{E}(a) \), for
all \( a \in \mathcal{A} \) and all \( b \in \mathcal{B} \) and (iii) \( \mathcal{E} \) is positive, i.e. \( \mathcal{E}(a) \in \mathcal{B}_+ \), for all \( a \in \mathcal{A}_+ \)
([St], p.116). Conditional expectations were introduced in the 1950's. One of the
problems related to them, which has attracted much attention through the years,
is whether there are algebraic or norm conditions on a projection \( \rho : \mathcal{A} \rightarrow \mathcal{B} \) which
will imply that either \( \rho \) itself is a conditional expectation or, more generally, that
there exists a conditional expectation \( \mathcal{E} : \mathcal{A} \rightarrow \mathcal{B} \). The first result of this type
was proved by Tomiyama in the late 1950's. If \( \mathcal{A} \) is a \( C^* \)-algebra and \( \mathcal{B} \) is a
\( C^* \)-subalgebra of \( \mathcal{A} \), then a bounded projection \( \rho : \mathcal{A} \rightarrow \mathcal{B} \) with \( \|\rho\| = 1 \) is a
conditional expectation ([To2] or [To1], Theorem 3.1, see also [St], Theorem 9.1
for a simple proof). In particular if \( \mathcal{M} \) is an injective von Neumann algebra, then
there exists a conditional expectation \( \mathcal{E} : \mathcal{B}(H) \rightarrow \mathcal{M} \). In the previous section we
discussed some results about quasi-expectations (projections \( Q : \mathcal{A} \rightarrow \mathcal{B} \) which
are \( \mathcal{B} \)-module maps) proved by Bunce and Pascke in [BuPas1]. In the same paper
they showed that if \( \mathcal{A} \) is a unital \( C^* \)-algebra and \( \mathcal{M} \subseteq \mathcal{A} \) is a finite, countably
decomposable von Neumann algebra with \( 1_{\mathcal{A}} = 1_{\mathcal{M}} \), then the existence of a
quasi-expectation \( Q : \mathcal{A} \rightarrow \mathcal{M} \) implies the existence of a conditional expectation
\( \mathcal{E} : \mathcal{A} \rightarrow \mathcal{M} \) (although this result is not stated in that form in [BuPas1], it
follows from the proof of (iii) \( \Rightarrow \) (i) in Proposition 1 of that paper; it appears in
that form in [St], Theorem 10.27). Every conditional expectation is completely
positive ([St], Proposition 9.3) and thus completely bounded. Does the existence
of a completely bounded projection \( \rho : \mathcal{A} \rightarrow \mathcal{B} \) imply the existence of a conditional
expectation \( \mathcal{E} : \mathcal{A} \rightarrow \mathcal{B} \)? We saw in Section 5.2 that Christensen, Sinclair and
Pisier showed that if there exists a completely bounded projection from \( \mathcal{B}(H) \)
on to a von Neumann algebra \( \mathcal{M} \) acting on \( H \), then \( \mathcal{M} \) is injective and so there
exists a conditional expectation \( \mathcal{E} : \mathcal{B}(H) \rightarrow \mathcal{M} \). The same three authors proved
that more general results hold in this context. Pisier proved in [Pi5], Theorem
1.1 that if \( \mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{B}(H) \) are von Neumann algebras such that \( \mathcal{M} \) is semi-
finite, then the existence of a completely bounded projection \( \rho : \mathcal{N} \rightarrow \mathcal{M} \) implies
the existence of a bounded projection \( \tilde{\rho} : \mathcal{N} \rightarrow \mathcal{M} \) with \( \|\tilde{\rho}\| = 1 \) (and thus of
a conditional expectation \( \mathcal{E} : \mathcal{N} \rightarrow \mathcal{M} \)). Christensen and Sinclair proved an
even more general result. If \( \mathcal{M} \) is a von Neumann algebra, \( \mathcal{A} \) is a \( C^* \)-algebra,
\( \theta : \mathcal{M} \rightarrow \mathcal{A} \) is a *-homomorphism and \( \rho : \mathcal{A} \rightarrow \mathcal{M} \) is a completely bounded
map with $\rho \theta = id_M$, then there exists a completely positive $\mathcal{M}$-module map $\tilde{\rho} : \mathcal{A} \to \mathcal{M}$ which maps $\mathcal{A}$ onto $\mathcal{M}$, with $\tilde{\rho} \theta = id_M$ and $\|\tilde{\rho}\| = 1$ ([CS5], Theorem 5.1).

A question related to the one we discussed in the previous paragraph is the following (called sometimes the bounded projection problem): Does the existence of a bounded projection from $\mathcal{B}(H)$ onto a von Neumann algebra $\mathcal{M}$ acting on $H$ imply that $\mathcal{M}$ is injective? Pisier showed that the answer is yes if $\mathcal{M}$ is isomorphic to $\mathcal{M} \hat{\otimes} \mathcal{M}$ ([Pi3], Theorem 1.4). Christensen and Sinclair generalised this result to properly infinite von Neumann algebras and type $II_1$ von Neumann algebras which are equal to the tensor product $S \hat{\otimes} T$ of two type $II_1$ von Neumann algebras $S$ and $T$ ([CS4], Corollary 3.2). In a recent preprint Pop, Sinclair and Smith introduced the notion of a norming subalgebra (see Definition 5.4.1) and proved that if $\mathcal{M}$ is a type $II_1$ von Neumann algebra with an injective norming subalgebra, then the existence of a bounded projection $\rho : \mathcal{B}(H) \to \mathcal{M}$ implies that $\mathcal{M}$ is injective ([PopSSm], Theorem 6.5).

We observed in Chapter 2 (Propositions 2.1.23 and 2.2.12) that if $\mathcal{A}$ is a Banach (operator) algebra and $X$ is a Banach (operator completely bounded) $\mathcal{A}$-bimodule, then the splitting, of any type, of $\mathcal{H}_{c(b)}^1(\mathcal{A}, X)$ implies the (complete) complementation of $\mathcal{Z}(\mathcal{A}, X)$ in $X$. Obviously if $\mathcal{M} \subseteq \mathcal{N}$ are von Neumann algebras, then $\mathcal{Z}(\mathcal{M}, \mathcal{N}) = \mathcal{M}' \cap \mathcal{N}$. Thus Proposition 2.1.23 and Proposition 2.2.12 combined with [CS5], Theorem 5.1 give, respectively, parts (i) and (ii) of the following corollary.

**Corollary 5.3.1.** Let $H$ be a Hilbert space and $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{B}(H)$ be an inclusion of von Neumann algebras.

(i) If $\mathcal{H}_{c}^1(\mathcal{M}, \mathcal{N})$ splits (I) or (II) or (III) or (IV) or (V), then $\mathcal{M}' \cap \mathcal{N}$ is complemented in $\mathcal{N}$.

(ii) If $\mathcal{H}_{cb}^1(\mathcal{M}, \mathcal{N})$ splits (I) or (II) or (III) or (IV) or (V), then there exists a conditional expectation $E : \mathcal{N} \to \mathcal{M}' \cap \mathcal{N}$.

**Remark 5.3.1.** We mentioned in Section 5.2 that Bunce and Paschke proved that if $\mathcal{M}$ is an amenable von Neumann algebra and $\mathcal{N}$ is a von Neumann algebra containing $\mathcal{M}$, then there exists a quasi-expectation $Q : \mathcal{N} \to \mathcal{M}' \cap \mathcal{N}$ ([BuPas1], Theorem 3). As in Proposition 5.2.1 we can prove, using [CS5], Theorem 5.1, that if $\mathcal{M} \subseteq \mathcal{N}$ are von Neumann algebras such that $\mathcal{M}$ is amenable, then there exists a conditional expectation $E : \mathcal{N} \to \mathcal{M}' \cap \mathcal{N}$. The proof of this result depends on constructing a normal operator completely bounded $\mathcal{M}$-bimodule $X$, using the vanishing of $\mathcal{H}_{cb}^1(\mathcal{M}, X)$ to get a completely bounded projection $\rho : \mathcal{N} \to \mathcal{M}' \cap \mathcal{N}$ and then using [CS5], Theorem 5.1. We can give a proof of that result that
does not use the amenability of \( M \), but its hyperfiniteness. Let \( M \subseteq N \) be von Neumann algebras such that \( M \) is hyperfinite. By Proposition 5.2.3, (ii) \( \Rightarrow \) (iv), \( \mathcal{H}^1_{cb}(M, N) \) splits (III) (note that the proof of (ii) \( \Rightarrow \) (iv) in Proposition 5.2.3, which is exactly the same with the proof of (ii) \( \Rightarrow \) (iv) in Proposition 5.2.2, uses only the hyperfiniteness of \( M \) and the results of Section 5.1). Thus, by Corollary 5.3.1(ii) (which follows from the geometric properties of split (III) and [CS5], Theorem 5.1), there exists a conditional expectation \( E : N \to M' \cap N \).

We will use Corollary 5.3.1 to obtain results about the splitting of the groups \( \mathcal{H}^1_{cb}(M, N) \), where \( N \) is an injective von Neumann algebra containing \( M \). We start with the completely bounded case. We recall that if \( N \) is injective and \( M \) is contained in \( N \), then \( \mathcal{H}^1_{cb}(M, N) \) vanishes. Thus \( \mathcal{H}^1_{cb}(M, N) \) splits (I) or (II) respectively if and only if \( \mathcal{H}^1_{cb}(M, N) \) splits (III) or (IV) respectively (Remark 2.2.4(ii) and Proposition 2.2.7). Hence in the following proposition we refer only to splits (III), (IV) and (V).

**Proposition 5.3.1.** Let \( M \) be a von Neumann algebra acting on a Hilbert space \( H \).

(i) If \( N \subseteq B(H) \) is an injective von Neumann algebra containing \( M \) and \( \mathcal{H}^1_{cb}(M, N) \) splits (III) or (IV) or (V), then \( M' \cap N \) is injective.

(ii) If \( \mathcal{H}^1_{cb}(M, B(H)) \) splits (III) or (IV) or (V), then \( M \) is injective.

_Proof._ (i) By Corollary 5.3.1(ii) there exists a conditional expectation \( E_1 : N \to M' \cap N \). Since \( N \) is injective, there exists a conditional expectation \( E_2 : B(H) \to N \). Obviously \( E_1 E_2 : B(H) \to M' \cap N \) is a conditional expectation and thus \( M' \cap N \) is injective. If \( N = B(H) \), then it follows immediately from (i) that the splitting of \( \mathcal{H}^1_{cb}(M, B(H)) \) implies that \( M' \) is injective. Thus \( M \) is injective, by [Co1], Proposition 6.4.(a), and so we have a proof of (ii).

**Remark 5.3.2.** The converse of part (ii) of the previous proposition follows by Proposition 5.2.3. Thus (ii) gives a characterisation of injective von Neumann algebras.

Combining [C2], Theorem 3.1 and Proposition 5.3.1 we can see that if \( M \) is a non-injective von Neumann algebra acting on a Hilbert space \( H \), then the group \( \mathcal{H}^1_{cb}(M, B(H)) \) vanishes, but does not split (III). Combining Proposition 5.3.1, Remark 5.3.2, Remark 4.1.1(b) and Proposition 3.3.2 we get the following characterisation of injectivity: A von Neumann algebra \( M \) acting on a Hilbert space \( H \) is injective if and only if either \( \mathcal{H}^1_{cb}(M, L^1_{cb}(Z^1_{cb}(M, B(H)), B(H))) = \{0\} \) or \( \mathcal{H}^1_{cb}(M, L^1_{cb}(C_1(H), (Z^1_{cb}(M, B(H))))_{rt}) = \{0\} \) (where \( C_1(H) \) denotes the space of trace class operators on \( H \) with the reversed tracial predual matricial structure).
norm structure). In particular if we consider a non-injective von Neumann algebra $\mathcal{M}$ we can see that the vanishing of $\mathcal{H}_c^1(\mathcal{M}, \mathcal{B}(H))$ does not imply the vanishing either of $\mathcal{H}_c^1(\mathcal{M}, \mathcal{L}_c^1(Y, \mathcal{B}(H)))$ or of $\mathcal{H}_c^1(\mathcal{M}, \mathcal{L}_c^1(\mathcal{C}_1(H), Y))$ for all matricially normed spaces $Y$.

We continue with the bounded case. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $H$. If $\mathcal{B}_c^1(\mathcal{M}, \mathcal{B}(H))$ is closed, then $\mathcal{H}_c^1(\mathcal{M}, \mathcal{B}(H))$ vanishes ([C2], Theorem 3.1). Thus $\mathcal{H}_c^1(\mathcal{M}, \mathcal{B}(H))$ splits (I) or (II) respectively if and only if $\mathcal{H}_c^1(\mathcal{M}, \mathcal{B}(H))$ splits (III) or (IV) respectively (Proposition 2.1.11 and the following remark). Therefore in part (ii) of the following proposition we talk only about splits (III), (IV) and (V).

**Proposition 5.3.2.** Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $H$.

(i) If $\mathcal{N} \subseteq \mathcal{B}(H)$ is a von Neumann algebra which is complemented in $\mathcal{B}(H)$ as a subspace and contains $\mathcal{M}$ and $\mathcal{H}_c^1(\mathcal{M}, \mathcal{N})$ splits (I) or (II) or (III) or (IV) or (V), then $\mathcal{M} \cap \mathcal{N}$ is injective if: (a) $\mathcal{M} \cap \mathcal{N}$ is properly infinite or (b) $\mathcal{M} \cap \mathcal{N}$ is type $II_1$ and (1) $\mathcal{M} \cap \mathcal{N} = S \otimes T$, where $S$ and $T$ are type $II_1$ von Neumann algebras, or (2) $\mathcal{M} \cap \mathcal{N}$ has an injective norming subalgebra.

(ii) If $\mathcal{H}_c^1(\mathcal{M}, \mathcal{B}(H))$ splits (III) or (IV) or (V), then $\mathcal{M}$ is injective if (a) $\mathcal{M}$ is type $III$, (b) $\mathcal{M}$ is type $II_\infty$ with properly infinite commutant, (c) $\mathcal{M}$ is type $II_\infty$ with finite commutant and $\mathcal{M}'$ is either of the form $S \otimes T$, where $S$ and $T$ are type $II_1$ von Neumann algebras, or has an injective norming subalgebra, (d) $\mathcal{M}$ is type $II_1$ with properly infinite commutant, (e) $\mathcal{M}$ is type $II_1$ with finite commutant and $\mathcal{M}'$ is either of the form $S \otimes T$, where $S$ and $T$ are type $II_1$ von Neumann algebras, or has an injective norming subalgebra.

**Proof.** (i) By Corollary 5.3.1 there exists a bounded projection $\rho_1 : \mathcal{N} \rightarrow \mathcal{M}' \cap \mathcal{N}$. Since $\mathcal{N}$ is complemented in $\mathcal{B}(H)$, there exists a bounded projection $\rho_2 : \mathcal{B}(H) \rightarrow \mathcal{N}$. The bounded projection $\rho_1 \rho_2 : \mathcal{B}(H) \rightarrow \mathcal{M}' \cap \mathcal{N}$ together with [CS4], Corollary 3.2 (for (a) and (b)(1)) and [PopSSm], Theorem 6.5 (for (b)(2)) show that $\mathcal{M}$ is injective. (ii) follows immediately from (i), since the commutant of a type $III$ von Neumann algebra is a type $III$ von Neumann algebra and the commutant of a type $II$ von Neumann algebra is a type $II$ von Neumann algebra ([StZ], Theorem 6.4).

**Remark 5.3.3.** (i) For the definition of the term "norming subalgebra" see Definition 5.4.1.

(ii) It follows immediately from Proposition 5.2.2, that $\mathcal{H}_c^1(\mathcal{M}, \mathcal{B}(H))$ splits (V) (and thus (III) and (IV)), for all injective von Neumann algebras $\mathcal{M}$. It is obvious that an affirmative answer to the bounded projection problem will imply
that the splitting of $\mathcal{H}_c^1(\mathcal{M}, \mathcal{B}(H))$ always implies the injectivity of $\mathcal{M}$. Combining those two observations we can see that an affirmative answer to the bounded projection problem leads to a characterisation of injective von Neumann algebras as the ones for which the first bounded cohomology group, with coefficients in $\mathcal{B}(H)$, splits (III) (or (IV) or (V)).

Using [C1], Theorem 3.2 and Proposition 5.3.2(ii), we can see that if $\mathcal{M}$ is a type III von Neumann algebra which is not injective, then $\mathcal{H}_c^1(\mathcal{M}, \mathcal{B}(H))$ vanishes, but does not split (III). Moreover, by Remark 4.1.1(a) and Proposition 3.3.1, $\mathcal{H}_c^1(\mathcal{M}, \mathcal{L}_c^1(\mathcal{Z}_c(\mathcal{M}, \mathcal{B}(H)), \mathcal{B}(H)))$ and $\mathcal{H}_c^1(\mathcal{M}, \mathcal{L}_c^1(\mathcal{C}_1(H), \mathcal{Z}_c^1(\mathcal{M}, \mathcal{B}(H))^*))$ do not vanish for such an algebra (although $\mathcal{H}_c^1(\mathcal{M}, \mathcal{B}(H)) = \{0\}$).

An affirmative answer to the derivation problem will imply that if $\mathcal{M}$ is a von Neumann algebra acting on a Hilbert space $H$ and $\mathcal{N} \subseteq \mathcal{B}(H)$ is an injective von Neumann algebra containing $\mathcal{M}$, then $\mathcal{H}_c^1(\mathcal{M}, \mathcal{N})$ vanishes (because of the existence of a conditional expectation $E : \mathcal{B}(H) \to \mathcal{N}$, $\mathcal{N}$ is a complemented $\mathcal{M}$-submodule of $\mathcal{B}(H)$). We don't know whether a negative answer to the derivation problem would imply the existence of a von Neumann algebra $\mathcal{M}$ with non-vanishing $\mathcal{H}_c^1(\mathcal{M}, \mathcal{N})$, for all injective von Neumann algebras $\mathcal{N} \subseteq \mathcal{B}(H)$ which contain $\mathcal{M}$. What we can show is that if $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{B}(H)$ is an inclusion of von Neumann algebras where $\mathcal{N}$ is injective, then the splitting of $\mathcal{H}_c^1(\mathcal{M}, \mathcal{N})$ does not imply the splitting of $\mathcal{H}_c^1(\mathcal{M}, \mathcal{B}(H))$. To obtain a counterexample to that we will need the following proposition, which is due to Popa ([Po1], Corollary 4.1) in the factor case and follows as a corollary of a result of Sinclair and Smith ([SSm4], Theorem 8) in the general case.

**Proposition 5.3.3.** Let $\mathcal{M}$ be a type II$_1$ von Neumann algebra with separable predual and centre $\mathcal{Z}$. Then there exists an injective subalgebra $\mathcal{N}$ of $\mathcal{M}$ with $\mathcal{N} \cap \mathcal{M} = \mathcal{Z}$. If $\mathcal{M}$ is a factor, then $\mathcal{N}$ is a factor.

**Counterexample 5.3.1.** There exists a von Neumann algebra $\mathcal{M}$ acting on a separable Hilbert space $H$ and an injective von Neumann algebra $\mathcal{N} \subseteq \mathcal{B}(H)$ which contains $\mathcal{M}$, such that $\mathcal{H}_c^1(\mathcal{M}, \mathcal{N})$ splits (III) and $\mathcal{H}_c^1(\mathcal{M}, \mathcal{B}(H))$ vanishes, but does not split (III).

**Proof.** Let $\mathcal{M}_1$ be a non-injective type II$_1$ von Neumann algebra acting on a separable Hilbert space $H$, with properly infinite commutant $\mathcal{M}_1'$, and let $\mathcal{Z}$ be the centre of $\mathcal{M}_1$. By Proposition 5.3.3, there exists an injective von Neumann algebra $\mathcal{N}_1 \subseteq \mathcal{M}_1$ such that $\mathcal{N}_1' \cap \mathcal{M}_1 = \mathcal{Z}$. Let $\mathcal{M} = \mathcal{M}_1'$ and $\mathcal{N} = \mathcal{N}_1'$. Obviously $\mathcal{M} \subseteq \mathcal{N}$. Since $\mathcal{M}' \cap \mathcal{N} = \mathcal{Z}$, $\mathcal{M}' \cap \mathcal{N}$ is complemented in $\mathcal{N}$. Since $\mathcal{M}$ is properly infinite and $\mathcal{N}$ is injective (since $\mathcal{N}_1 = \mathcal{N}'$ is), $\mathcal{H}_c^1(\mathcal{M}, \mathcal{N}) = \{0\}$,
by [C3], Theorem 2.2(ii) and Theorem 2.3(ii). Hence $\mathcal{H}_c^1(\mathcal{M}, \mathcal{N})$ splits (III), by Proposition 2.1.7. If $\mathcal{H}_c^1(\mathcal{M}, \mathcal{B}(H))$ splits (III), then, by Proposition 5.3.2(ii), $\mathcal{M}$ is injective and thus $\mathcal{M}_1 = \mathcal{M}'$ is injective, which leads to a contradiction. \( \Box \)

Let $\mathcal{M} \subseteq \mathcal{N}$ be von Neumann algebras. It follows immediately from Proposition 3.1.1 that $\mathcal{L}_c^1(\mathcal{M}, \mathcal{N})$ is a Banach $\mathcal{N}$-, and thus $\mathcal{M}' \cap \mathcal{N}$-, bimodule. A straightforward calculation shows that $\partial^1(a_1 \phi a_2) = a_1 \partial^1(\phi) a_2$, for all $a_1, a_2 \in \mathcal{M}' \cap \mathcal{N}$ and all $\phi \in \mathcal{L}_c^1(\mathcal{M}, \mathcal{N})$, and therefore $\mathcal{Z}_c^1(\mathcal{M}, \mathcal{N})$ is an $\mathcal{M}' \cap \mathcal{N}$-bimodule. Now suppose that $\mathcal{H}_c^1(\mathcal{M}, \mathcal{N})$ splits (III) and let $s_1 : \mathcal{Z}_c^1(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{N}$ be a splitting map of the third kind. If $s_1$ is an $\mathcal{M}' \cap \mathcal{N}$-module map, then so is $s_1 \partial^0$. Therefore $\text{id}_\mathcal{N} - s_1 \partial^0 : \mathcal{N} \rightarrow \mathcal{M}' \cap \mathcal{N}$ is a quasi-expectation. It follows immediately from [St], Theorem 10.27 that if the above hold and $\mathcal{M}' \cap \mathcal{N}$ is finite and countably decomposable, then there exists a conditional expectation $E : \mathcal{N} \rightarrow \mathcal{M}' \cap \mathcal{N}$. Moreover we have the following result about injectivity.

**Proposition 5.3.4.** Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $H$.

(i) Suppose that $\mathcal{N} \subseteq \mathcal{B}(H)$ is an injective von Neumann algebra which contains $\mathcal{M}$. If $\mathcal{H}_c^1(\mathcal{M}, \mathcal{N})$ splits (III) and there exists a splitting map of the third kind $s_1 : \mathcal{Z}_c^1(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{N}$ which is an $\mathcal{M}' \cap \mathcal{N}$-module map, then $\mathcal{M}' \cap \mathcal{N}$ is injective.

(ii) If $\mathcal{H}_c^1(\mathcal{M}, \mathcal{B}(H))$ splits (III) and there exists a splitting map of the third kind $s_1 : \mathcal{Z}_c^1(\mathcal{M}, \mathcal{B}(H)) \rightarrow \mathcal{B}(H)$ which is an $\mathcal{M}'$-module map, then $\mathcal{M}$ is injective.

**Proof.** (i) It follows from the preceding discussion that there exists a quasi-expectation $Q : \mathcal{N} \rightarrow \mathcal{M}' \cap \mathcal{N}$. Since $\mathcal{N}$ is injective, there exists a conditional expectation $E : \mathcal{B}(H) \rightarrow \mathcal{N}$. Then $QE : \mathcal{B}(H) \rightarrow \mathcal{M}' \cap \mathcal{N}$ is also a quasi-expectation and hence, by [BuPas1], Theorem 2, $\mathcal{M}' \cap \mathcal{N}$ is injective. (ii) follows immediately from (i) with $\mathcal{N} = \mathcal{B}(H)$. \( \Box \)

We can easily see that the converse of part (ii) of the previous proposition is true: Let $\mathcal{M}$ be an injective von Neumann algebra and $E : \mathcal{B}(H) \rightarrow \mathcal{M}'$ be a conditional expectation. Then $s_1 : \mathcal{Z}_c^1(\mathcal{M}, \mathcal{B}(H)) \rightarrow \mathcal{B}(H)$ defined by $s_1(\partial^0(x)) = x - E(x)$, for all $\partial^0(x) \in \mathcal{Z}_c^1(\mathcal{M}, \mathcal{B}(H))$ (since $\mathcal{M}$ is injective, $\mathcal{Z}_c^1(\mathcal{M}, \mathcal{B}(H)) = \mathcal{B}_c^1(\mathcal{M}, \mathcal{B}(H))$) is a splitting map of the third kind, which is an $\mathcal{M}'$-module map. Therefore part (ii) gives us a characterisation of injective von Neumann algebras.

We have not been able to prove any results about the splitting of the groups $\mathcal{H}_{c(b)}^n(\mathcal{M}, \mathcal{B}(H))$ if $n > 1$. Ideally the splitting of $\mathcal{H}_{c(b)}^n(\mathcal{M}, \mathcal{B}(H))$ must imply that $\mathcal{M}$ is (completely) $n$-amenable (see Remark 5.2.1(iii)).
5.4 The cohomology into the algebra

The study of the cohomology theory of operator algebras has its origins in the study of derivations on von Neumann algebras. Derivations have been studied extensively during the fifties and sixties, mainly because of their connection to automorphisms. This work culminated in the Kadison-Sakai theorem, which states that all the derivations on a von Neumann algebra $\mathcal{M}$ are inner, i.e. in cohomological terms $\mathcal{H}_c(\mathcal{M}, \mathcal{M})$ vanishes. We give a quick review of the results that lead to the Kadison-Sakai theorem. As we will see there are two recurring themes in the proofs of all those results, the existence of extensions of derivations on the weak closure of some $C^*$-algebra and the behaviour of derivations on the centre or, more generally, on the commutant of the algebra. Both techniques have been inherited by cohomology theory and appear there as the lifting and averaging results of Section 5.1 and the extended cobounding result of Kadison and Ringrose ([Ri2], Theorem 8.2).

In 1952 Kaplansky showed, using Singer’s result that all derivations on a commutative $C^*$-algebra vanish ([Ka], Lemma 15), that all derivations of a type I von Neumann algebra are inner ([Ka], Theorem 9); actually he proved it for AW*-algebras, something that shows the central role of projections when dealing with derivations. In the same paper he conjectured that all derivations on a $C^*$-algebra are continuous (Remark 2). That was proved by Sakai in [Sa2] (as Ringrose showed later in [Ri1] that is true for all derivations from a $C^*$-algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $X$). The next step was Miles’ proof that every derivation on a $C^*$-algebra is implemented by an element in the weak closure of a faithful representation of the algebra ([Mi], Theorem). He achieved that by showing that a derivation can be extended to a derivation on the universal representation of the algebra ([Mi], Lemma 1). Kadison then proved, using the annihilation of the centre of the algebra by any derivation and the existence of a unique ultraweakly continuous extension of a derivation to a derivation on the weak closure of the algebra ([K1], Theorem 2 and Lemma 3 respectively), that any derivation on a $C^*$ subalgebra of $B(H)$ is spatially implemented, i.e. it is implemented by an element of $B(H)$ ([K1], Theorem 4; for a generalisation of that result to higher cohomology groups see [KRi5], Theorem 2.4 and [Ri2], Theorem 8.2). As a corollary of that he showed that all derivations on a hyperfinite von Neumann algebra are inner (Theorem 7). Just before Sakai’s proof, Kadison and Ringrose proved that all derivations on the von Neumann group algebra of a discrete group are inner ([KRi2], Theorem 1.1). Finally Sakai proved in [Sa3], using Kadison’s spatial implementation result, that all derivations on a von Neumann algebra are inner.
Different proofs of that result appeared later in [K1], Theorem 15, [KRi3], pp.36-7, [JRi], Theorem 3, [AEIPeTo], Theorem 2.7 and [K2], Lemma 4 (the last two also contain some estimates on the norm of the element implementing the derivation). Accounts of the Kadison-Sakai theorem can be found in [Sa1], Section 4.1, [Ri2], Chapter 3 and [SSm1], Chapter 2.

In their first paper on the cohomology of operator algebras Kadison and Ringrose proved that $H^n_c(M, M)$ vanishes, for all $n \in \mathbb{N}$, if $M$ is a type I von Neumann algebra ([(KRi4), Theorem 4.4). They extended that to hyperfinite algebras in their second paper ([(KRi5), Theorem 3.1). Three years later Johnson gave an example of a non-hyperfinite von Neumann algebra $M$, with $H^2_c(M, M) = \{0\}$ in [J3], Theorem 2.

That is where things stood till the introduction of completely bounded cohomology in [CES]. Some time after the definition of completely bounded cohomology Christensen and Sinclair showed that, for all von Neumann algebras $M$ and all $n \in \mathbb{N}, H^n_{cb}(M, M) = \{0\}$ ([CS7], Theorem 4.2). A different proof of that result was obtained in [CS5], Theorem 7.1.

With that result in hand the next step was, by using automatic complete boundedness results or otherwise, to prove the vanishing of $H^n_c(M, M)$. In [CS7], Theorem 5.1 Christensen and Sinclair proved that if the type $II_1$ central direct summand of $M$ is isomorphic to its tensor product with the type $II_1$ hyperfinite factor $\mathcal{R}$, then all the bounded cohomology groups of $M$, with coefficients in $M$, vanish. In Theorem 5.2 of the same paper it was proved that if $M$ is a type $II_1$ factor, with property $\Gamma$, then $H^2_c(M, M) = \{0\}$ (a result due to Christensen). The next case that was settled was that of a type $II_1$ von Neumann algebra with separable predual and a Cartan subalgebra. Pop and Smith showed that $H^2_c(M, M)$ vanishes for a type $II_1$ factor, with a Cartan subalgebra ([PopSm], Theorem 3.1). This result was generalised to a type $II_1$ von Neumann algebra with a Cartan subalgebra in [CPopSSm], Theorem 5.4. It was also shown in Theorem 5.5 of the same paper that if $M$ is a type $II_1$ von Neumann algebra with separable predual and a Cartan subalgebra, then $H^2_c(M, M)$ vanishes. Both results depend on automatic complete boundedness for certain maps and have been obtained in a situation more general than the Cartan subalgebra one ([CPopSSm], Propositions 5.1 and 5.3). Using those more general results Ge and Popa proved that if for a type $II_1$ factor $M$ there exist a hyperfinite subalgebra $\mathcal{R}$ of $M$, an abelian *-subalgebra $\mathcal{A}$ of $M$ and $\xi_1, ..., \xi_n \in L^2(M, tr)$, with $\text{Span}(\mathcal{A}\{\xi_1, ..., \xi_n\}\mathcal{R}) = L^2(M, tr)$, then $H^2_c(M, M) = \{0\}$ ([GPo], Corollary 3.5). The case of a type $II_1$ von Neumann algebra with separable predual and a Cartan subalgebra was finally settled for all $n \in \mathbb{N}$ in [SSm2], Theorem 5.1.
(see also [SSm3], Theorem 4.1). In a recent preprint Pop, Sinclair and Smith introduced the notion of a norming algebra and showed that the existence of a norming subalgebra of $M$ together with some other conditions implies the vanishing of $\mathcal{H}_c^\pi(M, M)$ ([PopSSm], Theorem 6.1). The cases of a type $II_1$ algebra stable under tensoring with the hyperfinite type $II_1$ factor, of a type $II_1$ algebra with a Cartan subalgebra and of some of the algebras studied in [GPo] can be obtained as corollaries of that result.

Most of the results about the cohomology of $M$, with coefficients in $M$, can be found in [SSm1]. The completely bounded case is discussed in Section 4.3, the case of algebras with type $II_1$ central direct summand stable under tensoring with the hyperfinite type $II_1$ factor in Section 6.2, the case of the second and the third bounded cohomology group of a type $II_1$ von Neumann algebra with a Cartan subalgebra in Section 6.3 and the case of a type $II_1$ factor with property $\Gamma$ in Section 6.4.

In this section we will prove that in all the cases, except the case of a type $II_1$ factor with property $\Gamma$, where the cohomology groups of $M$, with coefficients in $M$, are known to vanish, they also split (III). We believe that the same holds for type $II_1$ factors with property $\Gamma$.

### 5.4.1 The completely bounded case

As we said in the introduction, the vanishing of the completely bounded cohomology groups of a von Neumann algebra $M$, with coefficients in $M$, has been proved in two different ways. Here we will use the approach in [CS5] to show that the completely bounded cohomology complex of $M$, with coefficients in $M$, splits (III).

It was proved in [CS5], Theorem 3.3 (see also [SSm1], Section 1.7), that if $A$ is a $C^*$-algebra and $N$ is a von Neumann algebra contained in $A$, then there exists a projection of norm less than or equal to $1$ mapping $L_c^b(A, A)$ onto the space of completely bounded right $N$-module maps from $A$ into $A$.

$$L_c^b(A, A) = \{ \phi \in L_c^b(A, N) \mid \phi(an) = \phi(a)n, \text{ for all } a \in A \text{ and all } n \in N \}.$$ 

To avoid any confusion we must say here that in [CS5] the previous set is denoted by $L_c^1(A, N)$.

For completeness we state here the parts of that theorem that we will need, in the case $A = N = M$. (1) of the following proposition is not contained in the original phrasing of the theorem, but it can be found in [CS5], pp.627-8. The same rephrasing together with some more results about the projection $\rho$ can be found in [SSm2], Theorem 3.1.
Proposition 5.4.1. ([CS5], Theorem 3.3) Let \( \mathcal{M} \) be a von Neumann algebra acting on a Hilbert space \( H \). Then there exists a bounded projection

\[ \rho : \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \to \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \]

with \( \|\rho\| \leq 1 \), which has the following property:

1. There exists a net \( \{\alpha_{\lambda}\}_{\lambda \in \Lambda} \) of maps

\[ \alpha_{\lambda} : \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \to \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \]

defined by

\[ \alpha_{\lambda}(\phi)(m) = \sum_{1 \leq k < \infty} \phi(mv_{k,\lambda}^*)v_{k,\lambda} \]

for all \( \phi \in \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \) and all \( m \in \mathcal{M} \), where, for all \( \lambda \in \Lambda \), \( \{v_{k,\lambda}\}_{k \in \mathbb{N}} \) is a sequence in \( \mathcal{M} \) with \( \sum_{1 \leq k \leq N} v_{k,\lambda}^*v_{k,\lambda} = 1_M \) strongly, such that the net \( \{\alpha_{\lambda}(\phi)(m)\}_{\lambda \in \Lambda} \) converges ultraweakly to \( \rho(\phi)(m) \), for all \( \phi \in \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \) and all \( m \in \mathcal{M} \).

The projection \( \rho \) will be used to define the splitting maps for the groups \( \mathcal{H}^n_{cb}(\mathcal{M}, \mathcal{M}) \). Hence we have to prove that it is completely bounded.

Proposition 5.4.2. Let \( \mathcal{M} \) be a von Neumann algebra acting on a Hilbert space \( H \). Then the projection

\[ \rho : \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \to \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \]

of Proposition 5.4.1 is completely bounded with \( \|\rho\|_{cb} = 1 \).

Proof. The proof will be done in three steps. First we will prove that, for all \( N \in \mathbb{N} \) and all \( \vartheta = (v_1, ..., v_N) \in \mathcal{M}^N \) with \( \sum_{1 \leq k \leq N} v_k^*v_k \leq 1_M \), the map

\[ \alpha_{\vartheta} : \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \to \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \]

defined by

\[ \alpha_{\vartheta}(\phi)(m) = \sum_{1 \leq k \leq N} \phi(mv_k^*)v_k \]

for all \( \phi \in \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \) and all \( m \in \mathcal{M} \), is completely bounded, with \( \|\alpha_{\vartheta}\|_{cb} \leq 1 \). Then, using the first step, we will prove that, for all sequences \( \{v_k\}_{k \in \mathbb{N}} \) in \( \mathcal{M} \) with \( \sum_{1 \leq k < \infty} v_k^*v_k = 1_M \) strongly, the map

\[ \alpha : \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \to \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \]
defined by

\[ \alpha(\phi)(m) = \sum_{1 \leq k < \infty} \phi(mv_k^*)v_k \]

for all \( \phi \in \mathcal{L}_{cb}^1(M, M) \) and all \( m \in M \), is completely bounded with \( \|\alpha\|_{cb} = 1 \). The complete boundedness of \( \rho \) will then follow from the second step and Proposition 5.4.1(1).

To prove the first step take \( v_1, \ldots, v_N \in M \), with \( \sum_{1 \leq k \leq N} v_k^*v_k \leq 1_M \). If \( l, r \in \mathbb{N} \), \( (\phi_{ij}) \in \mathbb{M}_l(\mathcal{L}_{cb}^1(M, M)) \) and \( (m_{st}) \in \mathbb{M}_r(M) \), then

\[
(\alpha_0(\phi_{ij})(m_{st})) = \left( \sum_{1 \leq k < N} \phi_{ij}(m_{st}v_k^*)v_k \right)
\]

can be written as the product of the \( lr \times lrN \) matrix

\[
A = (\phi_{ij}(m_{st}v_k^*))
\]

and the \( lrN \times lr \) matrix

\[
B = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \otimes I_{lr}.
\]

It is easy to see that by multiplying \( (a_{lr(N-1),lrN})^T \) by a suitable permutation matrix we get the matrix

\[
(\phi_{ij})_{rN} \begin{pmatrix} (m_{st}v_k^*) \\ 0_{r(N-1),rN} \end{pmatrix}.
\]

Hence, using the definition of the standard norm on \( \mathbb{M}_l(\mathcal{L}_{cb}^1(M, M)) \), we have

\[
\|A\|_{lr,lrN} = \| (a_{lr(N-1),lrN}) \|_{lrN} = \| (\phi_{ij})_{rN} \begin{pmatrix} (m_{st}v_k^*) \\ 0_{r(N-1),rN} \end{pmatrix} \|_{lrN} \leq \| (\phi_{ij})_{rN} \|_{r} \| (m_{st}v_k^*) \|_{r,rN}.
\]

Now multiplying \( (m_{st}v_k^*) \) by a permutation matrix we get \( (m_{st})(v_k^* \otimes I_r) \), the norm of which can be seen, using \( \sum_{1 \leq k \leq N} v_k^*v_k \leq 1_M \) and the C*-property of \( \mathbb{M}_{rN}(M) \), to be less than or equal to \( \| (m_{st}) \|_r \). Thus

\[
\|A\|_{lr,lrN} \leq \| (\phi_{ij})_{rN} \|_{r} \| (m_{st}) \|_r.
\]
On the other hand multiplying $B$ by a permutation matrix gives us $(v_k \otimes I_t)$, the norm of which can be easily seen to be less than or equal to 1. Therefore $\alpha_0$ is completely bounded with $\|\alpha_0\|_{cb} \leq 1$.

For the second step consider a sequence $\{v_k\}$ in $\mathcal{M}$ with $\sum_{1 \leq k < \infty} v_k v_k^* = 1_{\mathcal{M}}$ strongly and let $l, r \in \mathbb{N}$, $(\phi_{ij}) \in \mathcal{M}_l (\mathcal{L}_{cb}(\mathcal{M}, \mathcal{M}))$ and $\mathcal{(m_{st})} \in \mathcal{M}_r (\mathcal{M})$. By [CS5], Lemma 3.2, the series $\sum_{1 \leq k < \infty} \phi(m_{st} v_k) v_k$ is weakly convergent for all $\phi \in \mathcal{L}_{cb}^1(\mathcal{M}, \mathcal{M})$ and all $m \in \mathcal{M}$. Therefore, given $\varepsilon > 0$, for all $(i, j) \in \{1, \ldots, l\} \times \{1, \ldots, l\}$, $(s, t) \in \{1, \ldots, r\} \times \{1, \ldots, r\}$ and all $\bar{\xi} = \sum_{(i, s) \in \{1, \ldots, l\} \times \{1, \ldots, r\}} \otimes \xi_i^s$, $\bar{\eta} = \sum_{(i, t) \in \{1, \ldots, l\} \times \{1, \ldots, r\}} \otimes \eta_i^t \in H^{tr}$, there exists $N^{(s,t)}_{(i,j)} \in \mathbb{N}$ with

$$\left| \sum_{N \leq k < \infty} \left| \phi_{ij}(m_{st} v_k) v_k(\xi_i^j) \right| \eta_i^t \right| \leq \frac{1}{l^2 r^2} \varepsilon \left\| (\phi_{ij}) \right\|_l \left\| (m_{st}) \right\|_r \left\| \bar{\xi} \right\| \left\| \bar{\eta} \right\|$$

(5.2)

for all $N \geq N^{(s,t)}_{(i,j)}$. Take $N_0 = \max\{N^{(s,t)}_{(i,j)} \mid 1 \leq i, j \leq l, 1 \leq s, t \leq r\}$. Then

$$\left| \langle (\alpha(\phi_{ij})(m_{st}))(\bar{\xi}) \mid \bar{\eta} \right| \leq \sum_{1 \leq i, j \leq 1} \sum_{1 \leq s, t \leq r} \sum_{1 \leq k \leq N_0 - 1} \left| \phi_{ij}(m_{st} v_k^*) v_k(\xi_i^j) \right| \eta_i^t \right| \left| 
\sum_{1 \leq i, j \leq 1} \sum_{1 \leq s, t \leq r} \sum_{N_0 \leq k < \infty} \left| \phi_{ij}(m_{st} v_k^*) v_k(\xi_i^j) \right| \eta_i^t \right| \right|.$$

(5.3)

If $v = (v_1, \ldots, v_{N_0 - 1})$, then it is easy to see that the first part of the right hand side in (5.3) is equal to

$$\left| \langle (\alpha_0(\phi_{ij})(m_{st}))(\bar{\xi}) \mid \bar{\eta} \right|$$

which, by the first part of the proof, is less than or equal to

$$\left\| (\phi_{ij}) \right\|_l \left\| (m_{st}) \right\|_r \left\| \bar{\xi} \right\| \left\| \bar{\eta} \right\|.$$

The last observation together with (5.2) and (5.3) show that

$$\left| \langle (\alpha(\phi_{ij})(m_{st}))(\bar{\xi}) \mid \bar{\eta} \right| \leq (1 + \varepsilon) \left\| (\phi_{ij}) \right\|_l \left\| (m_{st}) \right\|_r \left\| \bar{\xi} \right\| \left\| \bar{\eta} \right\|.$$

Since the previous inequality holds for all $\varepsilon > 0$ and all $\bar{\xi}, \bar{\eta} \in H^{tr}$,

$$\left\| (\alpha(\phi_{ij})(m_{st})) \right\|_r \leq \left\| (\phi_{ij}) \right\|_l \left\| (m_{st}) \right\|_r$$

which, by the definition of the standard norm on $\mathcal{M}_l (\mathcal{L}_{cb}^1(\mathcal{M}, \mathcal{M}))$, implies that

$$\left\| (\alpha(\phi_{ij})) \right\|_l \leq \left\| (\phi_{ij}) \right\|_l.$$

Hence $\alpha$ is completely bounded with $\|\alpha\|_{cb} \leq 1$. Moreover $\|\alpha\|_{cb} = 1$, since $\alpha(id_{\mathcal{M}}) = id_{\mathcal{M}}$.  

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To finish the proof, take \( l, r \in \mathbb{N} \), \((\phi_{ij}) \in M_l(L^1_{cb}(M, M)) \) and \((m_{st}) \in M_r(M)\). By Proposition 5.4.1(1), there exists a net \( \{\alpha_\lambda\}_{\lambda \in \Lambda} \) of maps like the ones considered in the second step of the proof, such that \( \rho(\phi)(m) \) is the ultra-weak limit of \( \{\alpha_\lambda(\phi)(m)\}_{\lambda \in \Lambda} \), for all \( \phi \in L^1_{cb}(M, M) \) and all \( m \in M \). Hence for all \( \xi = \sum_{(i,t) \in \{1, \ldots, l\} \times \{1, \ldots, r\}} \xi_t^i \), \( \eta = \sum_{(i,t) \in \{1, \ldots, l\} \times \{1, \ldots, r\}} \eta_t^i \) \( \in H^{lr} \),

\[
|< (\rho(\phi_{ij})(m_{st}))(\xi) | \eta > | = | \sum_{1 \leq i,j \leq 1} \sum_{1 \leq s,t \leq r} < \rho(\phi_{ij})(m_{st})(\xi_t^i) | \eta_t^i > |
\]

\[
= | \sum_{1 \leq i,j \leq 1} \sum_{1 \leq s,t \leq r} \lim_{\lambda \in \Lambda} < \alpha_\lambda(\phi_{ij})(m_{st})(\xi_t^i) | \eta_t^i > |
\]

\[
= \lim_{\lambda \in \Lambda} | < (\alpha_\lambda(\phi_{ij})(m_{st}))(\xi) | \eta > |
\]

By the second part of the proof,

\[
|< (\alpha_\lambda(\phi_{ij})(m_{st}))(\xi) | \eta > | \leq \| (\phi_{ij}) \|_1 \| (m_{st}) \|_r \| \xi \| \| \eta \|
\]

for all \( \lambda \in \Lambda \). Therefore

\[
|< (\rho(\phi_{ij})(m_{st}))(\xi) | \eta > | \leq \| (\phi_{ij}) \|_1 \| (m_{st}) \|_r \| \xi \| \| \eta \|
\]

which proves that \( \rho \) is completely bounded with \( \| \rho \|_{cb} \leq 1 \).

Now we can prove that the completely bounded Hochschild cohomology complex of \( M \), with coefficients in \( M \), splits (III).

**Proposition 5.4.3.** Let \( M \) be a von Neumann algebra acting on a Hilbert space \( H \). Then the completely bounded Hochschild cohomology complex of \( M \), with coefficients in \( M \), splits (III).

**Proof.** Let

\[
s_1 : \mathcal{L}_{cb}^1(M, M) \to M
\]

be defined by

\[
s_1(\phi) = -\rho(\phi)(1_M)
\]

for all \( \phi \in \mathcal{L}_{cb}^1(M, M) \). It follows immediately from the previous proposition that \( s_1 \) is completely bounded (with \( \| s_1 \|_{cb} = 1 \), since \( s_1(id_M) = -\rho(id_M)(1_M) = -1_M \)).

For \( n > 1 \) define

\[
s_n : \mathcal{L}_{cb}^n(M, M) \to \mathcal{L}_{cb}^{n-1}(M, M)
\]

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by

\[ s_n(\phi)(m_1, \ldots, m_{n-1}) = (-1)^n \rho(\phi_{m_1, \ldots, m_{n-1}})(1_M) \]

for all \( \phi \in \mathbb{L}_{cb}^n(\mathcal{M}, \mathcal{M}) \) and all \( m_1, \ldots, m_{n-1} \in \mathcal{M} \), where

\[ \phi_{m_1, \ldots, m_{n-1}}(m) = \phi(m_1, \ldots, m_{n-1}, m) \]

for all \( m \in \mathcal{M} \). The definition of \( \phi_{m_1, \ldots, m_{n-1}} \) implies that

\[ \left\| \left( \sum_{1 \leq k_1, \ldots, k_{n-2} \leq r} \phi_{ij} m_{k_{1}, \ldots, m_{k_{n-2}}} \right) \right\|_r \leq \left\| \phi_{ij} \right\|_r \left\| (m^1_{k_{1}}) \right\|_r \cdots \left\| (m^{n-1}_{k_{n-2}}) \right\|_r \]

for all \( l, r \in \mathbb{N} \), all \( (\phi_{ij}) \in \mathbb{M}_l(\mathbb{L}_{cb}^n(\mathcal{M}, \mathcal{M})) \) and all \( (m^1_{k_{1}}), \ldots, (m^{n-1}_{k_{n-2}}) \in \mathbb{M}_r(\mathcal{M}) \).

Hence, using the previous inequality and \( \|\rho\|_{cb} = 1 \), we get that

\[ \left\| \left( \sum_{1 \leq k_1, \ldots, k_{n-2} \leq r} s_n(\phi_{ij})(m^1_{k_{1}}, \ldots, m^{n-1}_{k_{n-2}}) \right) \right\|_r = \left\| \left( \sum_{1 \leq k_1, \ldots, k_{n-2} \leq r} \phi_{ij} m_{k_{1}, \ldots, m_{k_{n-2}}} \right) \right\|_r \leq \left\| \phi_{ij} \right\|_r \left\| (m^1_{k_{1}}) \right\|_r \cdots \left\| (m^{n-1}_{k_{n-2}}) \right\|_r \]

which shows that \( s_n \) is completely bounded (with \( \|s_n\|_{cb} = 1 \), since if \( \phi : \mathcal{M}^n \to \mathcal{M} \) is defined by \( \phi(m_1, \ldots, m_n) = m_1 \ldots m_n \) for all \( m_1, \ldots, m_n \in \mathcal{M} \), then \( \phi_{1, \mathcal{M}, \ldots, 1, \mathcal{M}} = id_{\mathcal{M}} \) and so \( s_n(\phi)(1_M, \ldots, 1_M)(1_M) = (-1)^n \rho(id_{\mathcal{M}})(1_M) = (-1)^n 1_M \).

Now in a manner similar to the proof of Theorem 7.1 in [CS5] we can prove that \( s_n \) is a splitting map of the third kind. Let \( \phi \in \mathbb{L}_{cb}^n(\mathcal{M}, \mathcal{M}) \) and \( m_1, \ldots, m_n \in \mathcal{M} \). If \( \alpha_{\lambda} \) are the maps of Proposition 5.4.1(1), then, since \( \phi \) is an \( n \)-cocycle, we have

\[ (-1)^n \phi(m_1, \ldots, m_n) v^*_{k,\lambda} = m_1 \phi(m_2, \ldots, m_n, v^*_{k,\lambda}) \]

\[ + \sum_{1 \leq i \leq n-1} (-1)^i \phi(m_1, \ldots, m_i m_{i+1}, \ldots, m_n, v^*_{k,\lambda}) \]

\[ + (-1)^n \phi(m_1, \ldots, m_n v^*_{k,\lambda}) \]

for all \( \lambda \in \Lambda \) and all \( k \in \mathbb{N} \). Multiplying both sides with \( v_{k,\lambda} \) and taking the sums over \( k \) (in the strong operator topology) we get

\[ (-1)^n \phi(m_1, \ldots, m_n) = m_1 \alpha_{\lambda}(\phi_{m_2, \ldots, m_n})(1_M) \]

\[ + \sum_{1 \leq i \leq n-1} (-1)^i \alpha_{\lambda}(\phi_{m_1, \ldots, m_i m_{i+1}, \ldots, m_n})(1_M) \]

\[ + (-1)^n \alpha_{\lambda}(\phi_{m_1, \ldots, m_{n-1}})(m_n) \]

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for all $\lambda \in \Lambda$. Taking ultraweak limits over $\lambda \in \Lambda$ in the right hand side and using 
$\rho(\phi_{m_{1},...,m_{n-1}}) \in L_{cb}^{1}(M,M)_{M}$ we get
\[ (-1)^{n}\phi(m_{1},...,m_{n}) = m_{1}\rho(\phi_{m_{2},...,m_{n}})(1_{M}) \]
\[ + \sum_{1 \leq i \leq n-1} (-1)^{i}\rho(\phi_{m_{1},...,m_{i+1},...,m_{n}})(1_{M}) \]
\[ + (-1)^{n}\rho(\phi_{m_{1},...,m_{n-1}})(m_{n}) \]
\[ = (-1)^{n}m_{1}s_{n}(\phi)(m_{2},...,m_{n}) \]
\[ + \sum_{1 \leq i \leq n-1} (-1)^{n+i}s_{n}(\phi)(m_{1},...,m_{i},m_{i+1},...,m_{n}) \]
\[ + (-1)^{2n}s_{n}(\phi)(m_{1},...,m_{n-1})m_{n} \]
\[ = (-1)^{n}(\partial^{n-1}s_{n})(\phi)(m_{1},...,m_{n}) \]
which shows that $\partial^{n-1}s_{n} = id_{Z_{cb}^{n}(M,M)}$ and so $s_{n}$ is a splitting map of the third kind.

The geometric characterisation of the third type of splitting obtained in Proposition 2.2.4 and Propositions 4.1.2 and 4.2.2 imply, respectively, parts (i), (ii) and (iii) of the following corollary.

**Corollary 5.4.1.** If $M$ is a von Neumann algebra, then the following hold:

(i) $Z_{cb}^{n}(M,M)$ is completely complemented in $L_{cb}^{n}(M,M)$, for all $n \in \mathbb{N}$.

(ii) The completely bounded Hochschild cohomology complex of $M$, with coefficients in $L_{cb}^{1}(Y,M)$, splits (III), for all matricially normed spaces $Y$.

(iii) If $M_{*}$ is the predual of $M$ equipped with the reversed tracial predual matricial norm structure, then the completely bounded Hochschild cohomology complex of $M_{*}$, with coefficients in $L_{cb}^{1}(M_{*},Y_{rt}^{*})_{rt}$, splits (III), for all matricially normed spaces $Y$.

**5.4.2 The bounded case**

**5.4.2.1 The Kadison-Sakai theorem**

As we mentioned in the introduction, the Kadison-Sakai theorem says, in cohomological terms, that the first bounded cohomology group of $M$, with coefficients in $M$, vanishes, for all von Neumann algebras $M$. In the following proposition we prove that it splits (III).

**Proposition 5.4.4.** Let $M$ be a von Neumann algebra. Then the first bounded cohomology group of $M$, with coefficients in $M$, splits (III).

**Proof.** As we proved in Proposition 5.4.3, the completely bounded cohomology complex of $M$, with coefficients in $M$, splits (III) and thus in particular
Thus, by Remark 2.2.4(ii), \( \mathcal{H}^1_{cb}(\mathcal{M}, \mathcal{M}) \) splits (I). From Proposition 2.2.13 that implies split (I) for \( \mathcal{H}^1_c(\mathcal{M}, \mathcal{M}) \). Combining this with the Kadison-Sakai theorem we get, by Proposition 2.1.11, that \( \mathcal{H}^1_c(\mathcal{M}, \mathcal{M}) \) splits (III).

\[ \square \]

### 5.4.2.2 Algebras with type \( II_1 \) central direct summand stable under tensoring with the type \( II_1 \) hyperfinite factor

We said in the opening discussion, that Christensen and Sinclair proved in [CS7], Theorem 5.1 that if \( \mathcal{M} \) is a von Neumann algebra, the type \( II_1 \) central direct summand of which is stable under tensoring with the type \( II_1 \) hyperfinite factor \( \mathcal{R} \), then \( \mathcal{H}^n_c(\mathcal{M}, \mathcal{M}) = \{0\} \), for all \( n \in \mathbb{N} \). To do that they used a result they established earlier jointly with Effros in [CES], Theorem 5.1, which says that if \( \mathcal{M} \) is a von Neumann algebra which is isomorphic either to \( \mathcal{M} \otimes \mathcal{R} \) or to \( \mathcal{M} \otimes \mathcal{B}(H) \), then \( \mathcal{H}^n_{wcb}(\mathcal{M}, X) \simeq \mathcal{H}^n_w(\mathcal{M}, X) \), for all normal operator completely bounded \( \mathcal{M} \)-bimodules \( X \) and all \( n \in \mathbb{N} \). The proof of this theorem relies on the following automatic complete boundedness result (for the original version see [CES], Lemma 5.2).

**Lemma 5.4.1.** ([SSm1], Lemma 6.2.1) Let \( \mathcal{M} \) be a von Neumann algebra acting on a Hilbert space \( H \). If there exists a hyperfinite subalgebra \( \mathcal{N} \) of \( \mathcal{M} \) which is isomorphic either to the hyperfinite type \( II_1 \) factor \( \mathcal{R} \) or to \( \mathcal{B}(H) \), such that \( \mathcal{M} \) is isomorphic to \( \mathcal{M} \otimes \mathcal{N} \), then \( \mathcal{L}^n_{wcb}(\mathcal{M}, X : /\mathcal{N}) = \mathcal{L}^n_w(\mathcal{M}, X : /\mathcal{N}) \), for all normal operator completely bounded \( \mathcal{M} \)-bimodules \( X \) and all \( n \in \mathbb{N} \).

Using Lemma 5.4.1 we can prove that the bounded Hochschild cohomology complex of an algebra \( \mathcal{M} \) with type \( II_1 \) central direct summand stable under tensoring with \( \mathcal{R} \) splits (III).

**Proposition 5.4.5.** Let \( \mathcal{M} \) be a von Neumann algebra acting on a Hilbert space \( H \). If the tensor product of the type \( II_1 \) central direct summand of \( \mathcal{M} \), \( \mathcal{M}_{II_1} \), with the hyperfinite type \( II_1 \) factor \( \mathcal{R} \) is isomorphic to \( \mathcal{M}_{II_1} \), then the bounded Hochschild cohomology complex of \( \mathcal{M} \), with coefficients in \( \mathcal{M} \), splits (III).

**Proof.** Let \( n > 1 \) and let \( \mathcal{M}_{II_{\infty}} \) and \( \mathcal{M}_{III} \) be respectively the type \( II_{\infty} \) and type \( III \) central direct summands of \( \mathcal{M} \). By Proposition 5.4.3, \( \mathcal{H}^n_{cb}(\mathcal{M}, \mathcal{M}) \) splits (III). Therefore, by Corollary 5.2.2, \( \mathcal{H}^n_{cb}(\mathcal{M}_{II_1}, \mathcal{M}_{II_1}), \mathcal{H}^n_{cb}(\mathcal{M}_{II_{\infty}}, \mathcal{M}_{II_{\infty}}) \) and \( \mathcal{H}^n_{cb}(\mathcal{M}_{III}, \mathcal{M}_{III}) \) split (III). Since \( \mathcal{M}_{II_1} \otimes \mathcal{R} \) is isomorphic to \( \mathcal{M}_{II_1} \), the isomorphic copy \( \mathcal{N}_1 \) of \( \mathcal{N} \) in \( \mathcal{M}_{II_1} \), is a hyperfinite subalgebra of \( \mathcal{M}_{II_1} \), isomorphic to \( \mathcal{R} \). The hyperfiniteness of \( \mathcal{N}_1 \) implies, by Proposition 5.1.1, the existence of a completely bounded linear map

\[
s_n : Z^n_{wcb}(\mathcal{M}_{II_1}, \mathcal{M}_{II_1} : /\mathcal{N}_1) \to \mathcal{L}^{n-1}_{wcb}(\mathcal{M}_{II_1}, \mathcal{M}_{II_1} : /\mathcal{N}_1)
\]

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with $\partial_{n-1}s_n = id_{\mathcal{L}_{\text{wcb}}(\mathcal{M}_{II_1}, \mathcal{M}_{II_1} : /N_1)}$. By Lemma 5.4.1, $\mathcal{L}_{\text{wcb}}(\mathcal{M}_{II_1}, \mathcal{M}_{II_1} : /N_1) = \mathcal{L}_{\text{w}}(\mathcal{M}_{II_1}, \mathcal{M}_{II_1} : /N_1)$. Combining those two results with Proposition 5.1.1 we get that $\mathcal{H}_c^n(\mathcal{M}_{II_1}, \mathcal{M}_{II_1})$ splits (III). Moving to $\mathcal{M}_{II_\infty}$, we have that $\mathcal{M}_{II_\infty} \hat{\otimes} \mathcal{B}(H)$ is isomorphic to $\mathcal{M}_{II_\infty}$, since $\mathcal{M}_{II_\infty}$ is properly infinite. Arguing as in the previous step, with the isomorphic copy $N_2$ of $\mathcal{C}_{1_{\mathcal{M}_{II_\infty}} \hat{\otimes} \mathcal{B}(H)}$ in $\mathcal{M}_{II_\infty}$ in the place of $N_1$ we get that $\mathcal{H}_c^n(\mathcal{M}_{II_\infty}, \mathcal{M}_{II_\infty})$ splits (III). Similarly we can prove that $\mathcal{H}_c^n(\mathcal{M}_{III}, \mathcal{M}_{III})$ splits (III). The result then follows from Corollary 5.2.2. □

5.4.2.3 Algebras with norming subalgebras

In a recent preprint Pop, Sinclair and Smith introduced the notion of a forming algebra and showed that if we have an inclusion of von Neumann algebras $\mathcal{A} \subseteq \mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{B}(H)$, where $\mathcal{M}$ is a type $II_1$ factor with separable predual, $\mathcal{N}$ is hyperfinite with $\mathcal{N} \cap \mathcal{M} = \mathbb{C}$ and $\mathcal{A}$ norms $\mathcal{M}$, then $\mathcal{H}_c^n(\mathcal{M}, \mathcal{M}) = \{0\}$ ([PopSSm], Theorem 6.1.(1)) and if moreover the unitary normaliser of $\mathcal{A}$ in $\mathcal{M}$ generates $\mathcal{M}$ as a von Neumann algebra, then $\mathcal{H}_c^n(\mathcal{M}, \mathcal{M})$ vanishes, for all $n \in \mathbb{N}$ ([PopSSm], Theorem 6.1.(3)). On the other hand if $\mathcal{M}$ is a type $II_1$ factor with separable predual, $\mathcal{N}$ is a hyperfinite subalgebra of $\mathcal{M}$ with $\mathcal{N} \cap \mathcal{M} = \mathbb{C}$ and there exists an abelian von Neumann subalgebra $\mathcal{B}$ of $\mathcal{M}$ such that $\mathcal{C}^*(\mathcal{N}, \mathcal{B})$ norms $\mathcal{B}(H)$, then $\mathcal{H}_c^n(\mathcal{M}, \mathcal{M})$ vanishes, for $n = 2, 3$ ([PopSSm], Theorem 6.1.(2)). We will show that in all three cases a splitting occurs.

We start by giving the definition of a forming algebra.

**Definition 5.4.1.** ([PopSSm], Definition 2.1) Let $H$ be a Hilbert space and $\mathcal{A}$ and $\mathcal{B}$ be unital $C^*$-subalgebras of $\mathcal{B}(H)$. We say that $\mathcal{A}$ norms $\mathcal{B}$ if, for each $n \in \mathbb{N}$ and each $(b_{ij}) \in \mathcal{M}_n(\mathcal{B})$,

$$\|(b_{ij})\|_n = \sup\{\|R(b_{ij})C\| : R \in \text{Row}_n(\mathcal{A}), C \in \text{Col}_n(\mathcal{A}), \|R\|_n, \|C\|_n \leq 1\}$$

In [PopSSm], Theorem 2.10 it was proved that if $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ are $C^*$-algebras such that $\mathcal{A}$ norms $\mathcal{B}$, then every bounded $\mathcal{A}$-module map $\phi : \mathcal{C} \to \mathcal{B}$ is completely bounded, with $\|\phi\|_{cb} = \|\phi\|$. We prove a result about automatic row boundedness of right module maps in a similar situation, which we will need in the proof of Proposition 5.4.7.

**Proposition 5.4.6.** Let $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2$ be $C^*$-algebras such that $\mathcal{A}$ is contained in both $\mathcal{B}_1$ and $\mathcal{B}_2$ and norms $\mathcal{B}_2$. If $\phi : \mathcal{B}_1 \to \mathcal{B}_2$ is a bounded right $\mathcal{A}$-module map, then it is row bounded, with $\|\phi\|_r = \|\phi\|_r$.

**Proof.** Suppose that $\|\phi\|_r > \|\phi\|$. Then there exist $n \in \mathbb{N}$ and a row $R \in \text{Row}_n(\mathcal{B}_1)$ with $\|R\|_n \leq 1$, such that $\|\phi_n(R)\|_n > \|\phi\|$. By [PopSSm], Lemma 2.4,
since \( \mathcal{A} \) norms \( B_2 \), for each \( X \in \text{Row}_n(B_2) \),
\[
\|X\|_n = \sup\{\|XC\| \mid C \in \text{Col}_n(\mathcal{A}), \|C\|_n \leq 1\}.
\]

Hence there exists \( C \in \text{Col}_n(\mathcal{A}) \) with \( \|C\|_n \leq 1 \), such that
\[
\|\phi_n(R)C\| > \|\phi\|
\]  
(5.4)

But \( \phi_n(R)C = \phi(RC) \), since \( \phi \) is a right \( \mathcal{A} \)-module map. Moreover \( \|RC\| \leq 1 \).
Thus (5.4) leads to a contradiction.

We start by proving that under the assumptions in [PopSSm], Theorem 6.1.(3) the bounded Hochschild cohomology complex of \( \mathcal{M} \) splits (III). Our proof is essentially the proof of Theorem 5.1 of [SSm2]. For convenience we remind the reader of the following definition.

**Definition 5.4.2.** Let \( \mathcal{M} \) be a von Neumann algebra with unitary group \( \mathcal{U}(\mathcal{M}) \) and \( \mathcal{A} \) be a \(*\)-subalgebra of \( \mathcal{M} \). The unitary normaliser of \( \mathcal{A} \) in \( \mathcal{M} \) is the set
\[
\mathcal{N}(\mathcal{A}) = \{ u \in \mathcal{U}(\mathcal{M}) \mid u^*Au = \mathcal{A} \}
\]

**Proposition 5.4.7.** Let \( \mathcal{M} \) be a type II\(_1\) von Neumann algebra and \( \mathcal{A} \subseteq \mathcal{N} \subseteq \mathcal{M} \) be von Neumann subalgebras of \( \mathcal{M} \) such that \( \mathcal{N} \) is hyperfinite. If \( \mathcal{A} \) norms \( \mathcal{M} \) and the unitary normaliser of \( \mathcal{A} \) in \( \mathcal{M} \) generates \( \mathcal{M} \) as a von Neumann algebra, then the bounded Hochschild cohomology complex of \( \mathcal{M} \), with coefficients in \( \mathcal{M} \), splits (III).

**Proof.** The case \( n = 1 \) follows from Proposition 5.4.4. So we may take \( n > 1 \). Let \( \mathcal{U} = \mathcal{N}(\mathcal{A}) \) be the unitary normaliser of \( \mathcal{A} \) in \( \mathcal{M} \). It is easy to see from its definition that \( \mathcal{U} \) is closed under multiplication and hence \( \text{Alg}(\mathcal{U}) = \text{Span}(\mathcal{U}) \). Moreover it is selfadjoint and thus the \( C^* \)-subalgebra of \( \mathcal{M} \) generated by \( \mathcal{U} \), \( C^*(\mathcal{U}) \), is the norm closure of \( \text{Alg}(\mathcal{U}) \). Since \( \mathcal{U} \) generates \( \mathcal{M} \) as a von Neumann algebra, \( C^*(\mathcal{U}) \) is weakly dense in \( \mathcal{M} \). Therefore, from Propositions 5.1.1 and 5.1.4, \( \mathcal{H}_c^n(\mathcal{M}, \mathcal{M}) \) splits (III) if and only if \( \mathcal{H}_c^n(C^*(\mathcal{U}), \mathcal{M}) \) splits (III).

Take \( \phi \in \mathcal{Z}_c^n(C^*(\mathcal{U}), \mathcal{M}) \). If \( J_n : \mathcal{L}_c^n(C^*(\mathcal{U}), \mathcal{M}) \to \mathcal{L}_c^{n-1}(C^*(\mathcal{U}), \mathcal{M}) \) is defined as in Lemma 5.1.5, then \( \phi - \partial^{n-1}J_n(\phi) \in \mathcal{Z}_w^n(C^*(\mathcal{U}), \mathcal{M}) \). Since \( C^*(\mathcal{U}) \) is weakly dense in \( \mathcal{M} \), \( \phi - \partial^{n-1}J_n(\phi) \) can be uniquely extended without change of norm to \( (\phi - \partial^{n-1}J_n(\phi))^- \in \mathcal{Z}_w^n(\mathcal{M}, \mathcal{M}) \). Moreover, since \( \mathcal{N} \) is a hyperfinite subalgebra of \( \mathcal{M} \), \( (\phi - \partial^{n-1}J_n(\phi))^- - \partial^{n-1}L_n((\phi - \partial^{n-1}J_n(\phi))^-) \in \mathcal{Z}_w^n(\mathcal{M}, \mathcal{M}: /\mathcal{N}) \), where \( L_n : \mathcal{L}_w^n(\mathcal{M}, \mathcal{M}) \to \mathcal{L}_w^{n-1}(\mathcal{M}, \mathcal{M}) \) is the map defined in Lemma 5.1.6. We will denote \( (\phi - \partial^{n-1}J_n(\phi))^- - \partial^{n-1}L_n((\phi - \partial^{n-1}J_n(\phi))^-) \) by \( \theta \). The norm estimates for
$J_n$ and $L_n$ together with $\|\phi - \partial^{n-1}J_n(\phi)\| = \|(\phi - \partial^{n-1}J_n(\phi))^\perp\|$ and $\|\partial^{n-1}\| \leq n+1$ show that

$$\|\theta\| \leq K\|\phi\|$$

(5.5)

where $K = (n+2)^n(1 + (n+1)((n+2)^n-1) + 2((n+2)^n-1)^2/(n+1))$.

For each $(n-1)$-tuple $(u_1, ..., u_{n-1}) \in \mathcal{U}^{n-1}$, we define

$\mu_{(u_1, ..., u_{n-1})} : \mathcal{M} \to \mathcal{M}$

by

$$\mu_{(u_1, ..., u_{n-1})}(m) = \theta(u_1, ..., u_{n-1}, m)$$

for all $m \in \mathcal{M}$. Take $k \in \mathbb{N}$ and $(m_{ij}) \in \mathbb{M}_k(\mathcal{M})$ with $\|(m_{ij})\|_k \leq 1$. It is easy to see that

$$(\mu_{(u_1, ..., u_{n-1})})(m_{ij}) = \theta_k(u_1 \otimes I_k, ..., u_{n-1} \otimes I_k, (m_{ij})).$$

Now let $R \in \text{Row}_k(\mathcal{A})$, $C \in \text{Col}_k(\mathcal{A})$ with $\|R\|_k, \|C\|_k \leq 1$. As in the part of the proof of [SSm2], Theorem 5.1 contained between (5.2) and (5.6), we can use the ($\mathcal{N}$- and thus) $\mathcal{A}$-modularity of $\theta$ and $u_1, ..., u_{n-1} \in \mathcal{U}$ to show that

$$R(\mu_{(u_1, ..., u_{n-1})})(m_{ij})C = \theta(u_1, ..., u_{n-1}, R_{n-1}(m_{ij})C)$$

where $R_{n-1} \in \text{Row}_k(\mathcal{A})$ with $\|R_{n-1}\|_k \leq 1$. Thus

$$\sup\{\|R(\mu_{(u_1, ..., u_{n-1})})(m_{ij})C\| \mid R \in \text{Row}_k(\mathcal{A}), C \in \text{Col}_k(\mathcal{A}), \|R\|_k, \|C\|_k \leq 1\}$$

is less than or equal to $\|\theta\|$. But $\mathcal{A}$ norms $\mathcal{M}$ and hence the previous supremum is equal to $\|(\mu_{(u_1, ..., u_{n-1})})(m_{ij})\|_k$. Therefore $\mu_{(u_1, ..., u_{n-1})}$ is completely bounded. Obviousy if $y_1, ..., y_{n-1} \in \text{Span}(\mathcal{U}) = \text{Alg}(\mathcal{U})$, then $\mu_{(y_1, ..., y_{n-1})} : \mathcal{M} \to \mathcal{M}$ defined in a similar manner is also completely bounded, being a linear combination of maps of the form $\mu_{(u_1, ..., u_{n-1})}$. Hence we can define

$$\alpha_{\phi} : \text{Alg}(\mathcal{U})^{n-1} \to \mathcal{M}$$

by

$$\alpha_{\phi}(y_1, ..., y_{n-1}) = (-1)^n \rho(\mu_{(y_1, ..., y_{n-1})})(1, \mathcal{M})$$

for all $y_1, ..., y_{n-1} \in \text{Alg}(\mathcal{U})$, where $\rho : \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \to \mathcal{L}^1_{cb}(\mathcal{M}, \mathcal{M}) \mathcal{M}$ is the projection we discussed in the first part of this section. To show that $\alpha_{\phi}$ is bounded
consider \( y_1, \ldots, y_{n-1} \in \text{Alg}(U) \). From [SSm2], Theorem 3.1.(3), \( \|\rho(\psi)\| \leq \|\psi\|_r \), for all \( \psi \in L_{cb}^1(M, M) \). Thus
\[
\|\alpha_\phi(y_1, \ldots, y_{n-1})\| \leq \|\mu(y_1, \ldots, y_{n-1})\|_r.
\]

An immediate consequence of the way that we defined \( \mu(y_1, \ldots, y_{n-1}) \) and the \( A \)-modularity of \( \theta \) is that \( \mu(y_1, \ldots, y_{n-1}) \) is a right \( A \)-module map. Hence, by Proposition 5.4.6, \( \|\mu(y_1, \ldots, y_{n-1})\|_r = \|\mu(y_1, \ldots, y_{n-1})\| \). Obviously \( \|\mu(y_1, \ldots, y_{n-1})\| \leq \|\theta\| \|y_1\| \cdots \|y_{n-1}\| \) and therefore \( \|\alpha_\phi\| \leq \|\theta\| \). Moreover \( \alpha_\phi \) can be extended to a map \( \tilde{\alpha}_\phi \in L_{cb}^{-1}(C^*(U), M) \), with
\[
\|\tilde{\alpha}_\phi\| \leq \|\theta\|. \tag{5.6}
\]

The next step is to prove that \( \theta(y_1, \ldots, y_{n-1}) = \partial^{n-1}(\tilde{\alpha}_\phi)(y_1, \ldots, y_{n-1}) \), for all \( y_1, \ldots, y_{n-1} \in C^*(U) \). To do that we will need two observations about the projection \( \rho \). The first one is Theorem 3.1.(3) of [SSm2] which says that if \( \phi \in L_{cb}^1(M, M) \), \( a \in A \) and \( \phi_a \in L_{cb}^1(M, M) \) is defined by \( \phi_a(m) = \phi(am) \), for all \( m \in M \), then \( \rho(\phi_a) = (\rho(\phi))_a \). The second one is that if \( \phi \in L_{cb}^1(M, M) \), \( a \in A \) and \( a\phi \in L_{cb}^1(M, M) \) is defined by \( (a\phi)(m) = a\phi(m) \), for all \( m \in M \), then \( \rho(a\phi) = a\rho(\phi) \); it is easy to see that by considering a net \( \{a_\lambda\}_{\lambda \in \Lambda} \) of maps approximating \( \rho \) as in Proposition 5.4.1.(1). (What the previous observations say in other words is that \( \rho \) is a left \( A \)-module map with respect to the module actions defined in Section 3.1 and a right \( A \)-module map with respect to the module actions defined in Section 3.2).

Consider \( y_1, \ldots, y_{n-1} \in \text{Alg}(U) \). Since \( \theta \) is a cocycle,
\[
y_1\theta(y_2, \ldots, y_n, m) + \sum_{1 \leq i \leq n-1} (-1)^i \theta(y_1, \ldots, y_iy_{i+1}, \ldots, y_n, m) + (-1)^n\theta(y_1, \ldots, y_n m) + (-1)^{n+1}\theta(y_1, \ldots, y_n) m = 0
\]
for all \( m \in M \). Thus
\[
y_1\mu(y_2, \ldots, y_n) + \sum_{1 \leq i \leq n-1} (-1)^i \mu(y_1, \ldots, y_iy_{i+1}, \ldots, y_n) + (-1)^n(\mu(y_1, \ldots, y_{n-1}))y_n + (-1)^{n+1}(m \mapsto \theta(y_1, \ldots, y_n) m) = 0.
\]

Using the two observations of the previous paragraph and the fact that the map \( m \mapsto \theta(y_1, \ldots, y_n) m \) belongs to \( L_{cb}^1(M, M)_M \) we get
\[
y_1\rho(\mu(y_2, \ldots, y_n)) + \sum_{1 \leq i \leq n-1} (-1)^i \rho(\mu(y_1, \ldots, y_iy_{i+1}, \ldots, y_n)) + (-1)^n(\rho(\mu(y_1, \ldots, y_{n-1}))y_n + (-1)^{n+1}(m \mapsto \theta(y_1, \ldots, y_n) m) = 0.
\]

If we apply both sides of the previous equality to \( 1_M \), we get \( \theta(y_1, \ldots, y_{n-1}) = \partial^{n-1}(\alpha_\phi)(y_1, \ldots, y_{n-1}) \).
Therefore
\[
\theta(y_1, \ldots, y_{n-1}) = \vartheta^{n-1}(\bar{\alpha}_\phi)(y_1, \ldots, y_{n-1})
\] (5.7)
for all \(y_1, \ldots, y_{n-1} \in C^*(U)\). An immediate consequence of (5.7) and the definition of \(\theta\) is that
\[
\phi = \vartheta^{n-1}(\bar{\alpha}_\phi + L_n((\phi - \vartheta^{n-1}J_n(\phi))^{-1}) + J_n(\phi)).
\] (5.8)

To finish the proof we define
\[
s_n : \mathcal{Z}^n_c(C^*(U), \mathcal{M}) \to \mathcal{L}^{n-1}_c(C^*(U), \mathcal{M})
\]
by
\[
s_n(\phi) = \bar{\alpha}_\phi + L_n((\phi - \vartheta^{n-1}J_n(\phi))^{-1}) + J_n(\phi).
\]
It is easy to see that if \(\phi_1, \phi_2 \in \mathcal{Z}^n_c(C^*(U), \mathcal{M})\) and \(\lambda_1, \lambda_2 \in \mathbb{C}\), then \(\lambda_1 \bar{\alpha}_\phi_1 + \lambda_2 \bar{\alpha}_\phi_2 = \lambda_1 \bar{\alpha}_\phi_1 + \lambda_2 \bar{\alpha}_\phi_2\) and hence \(s_n\) is linear. (5.5), (5.6), the boundedness of \(L_n, J_n, \vartheta^{n-1}\) and \(\|((\phi - \vartheta^{n-1}J_n(\phi))^{-1}) = \|\phi - \vartheta^{n-1}J_n(\phi)\|\) imply that \(s_n\) is bounded. By (5.8), \(\phi = \vartheta^{n-1}s_n(\phi)\), for all \(\phi \in \mathcal{Z}^n_c(C^*(U), \mathcal{M})\). Therefore \(\mathcal{H}^n_c(C^*(U), \mathcal{M})\) splits (III).

As a corollary of the previous proposition we will prove that the bounded Hochschild cohomology complex of \(\mathcal{M}\), with coefficients in \(\mathcal{M}\), splits (III) if \(\mathcal{M}\) is a type \(II_1\) von Neumann algebra with a Cartan subalgebra. We begin by recalling the definition and a property of Cartan subalgebras.

**Definition 5.4.3.** Let \(\mathcal{M}\) be a type \(II_1\) von Neumann algebra. A masa (maximal abelian selfadjoint subalgebra) \(A\) in \(\mathcal{M}\) is called a Cartan subalgebra of \(\mathcal{M}\) if the unitary normaliser of \(A\) in \(\mathcal{M}\) generates \(\mathcal{M}\) as a von Neumann algebra.

The following proposition is due to Feldmann and Moore ([FM], Proposition 2.9). It also follows as a corollary of a more general result by Popa ([Po2], Corollary 3.2).

**Proposition 5.4.8.** Let \(\mathcal{M}\) be a type \(II_1\) von Neumann algebra with faithful normal trace \(\text{tr}\) which is represented in standard form on the Hilbert space \(L^2(\mathcal{M}, \text{tr})\) with conjugate linear isometry \(J\). If \(A\) is a Cartan subalgebra of \(\mathcal{M}\), then \((A \vee JAJ)^{\prime\prime}\) is a masa in \(B(L^2(\mathcal{M}, \text{tr}))\).

To establish the splitting of the bounded cohomology complex of type \(II_1\) von Neumann algebras with Cartan subalgebras we will need two results about locally cyclic algebras.
Definition 5.4.4. Let $\mathcal{A}$ be a $C^*$-algebra acting on a Hilbert space $H$. We say that $\mathcal{A}$ is locally cyclic if for all $n \in \mathbb{N}$ and all $\xi_1, \ldots, \xi_n \in H$, there exists $\xi \in H$ with $\{\xi_1, \ldots, \xi_n\} \subseteq \mathcal{A}\xi$.

Proposition 5.4.9. ([PopSm], Lemma 2.2) Let $\mathcal{A}$ be an abelian $C^*$-algebra acting on a Hilbert space $H$. If $\mathcal{A}''$ is a masa in $\mathcal{B}(H)$, then $\mathcal{A}$ is locally cyclic.

Proposition 5.4.10. ([SSm2], Proposition 4.1) Let $\mathcal{A} \subseteq \mathcal{M} \subseteq \mathcal{B}(H)$ be von Neumann algebras and $\mathcal{B} \subseteq \mathcal{M}'$ be an abelian von Neumann algebra. If $C^*(\mathcal{A}, \mathcal{B})$ is locally cyclic, then $\mathcal{A}$ norms $\mathcal{M}$.

Remark 5.4.1. In [SSm2] the previous proposition is phrased with "has a cyclic vector" in the place of "is locally cyclic". It is trivial to see that it holds under that weaker assumption (see [Sm], Remark 2.2).

Now we can prove the result about algebras with a Cartan subalgebra.

Corollary 5.4.2. Let $\mathcal{M}$ be a type $II_1$ von Neumann algebra with a Cartan subalgebra $\mathcal{A}$. Then the bounded Hochschild cohomology complex of $\mathcal{M}$, with coefficients in $\mathcal{M}$, splits (III).

Proof. Since $\mathcal{A}$ is Cartan, it is abelian and thus $\mathcal{J}\mathcal{A}\mathcal{J}$ is an abelian von Neumann subalgebra of $\mathcal{M}'$. Moreover, by Proposition 5.4.8, $(\mathcal{A} \vee \mathcal{J}\mathcal{A}\mathcal{J})''$ is a masa. Thus $C^*(\mathcal{A}, \mathcal{J}\mathcal{A}\mathcal{J})$ is locally cyclic by Proposition 5.4.9. So $\mathcal{A}$ norms $\mathcal{M}$ by Proposition 5.4.10. To finish the proof just observe that the unitary normaliser of $\mathcal{A}$ in $\mathcal{M}$ generates $\mathcal{M}$ as a von Neumann algebra, since $\mathcal{A}$ is Cartan, and that $\mathcal{A}$ is hyperfinite being abelian and apply Proposition 5.4.7.

We already mentioned in the introduction that if for a type $II_1$ factor $\mathcal{M}$ there exist a hyperfinite subalgebra $\mathcal{R}$ of $\mathcal{M}$, an abelian-* subalgebra $\mathcal{A}$ of $\mathcal{M}$ and $\xi_1, \ldots, \xi_n \in L^2(\mathcal{M}, tr)$, with $\overline{\text{Span}}(\mathcal{A}\{\xi_1, \ldots, \xi_n\}\mathcal{R}) = L^2(\mathcal{M}, tr)$, then $\mathcal{H}_c^2(\mathcal{M}, \mathcal{M})$ vanishes. A special case of those factors is the crossed product $\mathcal{R} \times_{\alpha} G$ of the type $II_1$ hyperfinite factor $\mathcal{R}$ by a discrete countable group $G$ acting on $\mathcal{R}$ by outer automorphisms ([GPo], Theorem 1.4). It was proved in [PopSSm], Corollary 6.4 that if $\mathcal{M} = \mathcal{R} \times_{\alpha} G$ is as above and $\mathcal{N}$ is a type $II_1$ factor with separable predual, then the conditions of Proposition 5.4.7 hold for $\mathcal{N} \hat{\otimes} \mathcal{M}$. Hence we get the following corollary.

Corollary 5.4.3. Let $\mathcal{R}$ be the type $II_1$ hyperfinite factor, $G$ be a discrete countable group acting on $\mathcal{R}$ by outer automorphisms $\alpha : G \to \text{Aut}(\mathcal{R})$, $\mathcal{R} \times_{\alpha} G$ be their crossed product and $\mathcal{N}$ be a type $II_1$ factor with separable predual. Then the bounded Hochschild cohomology complex of $\mathcal{N} \hat{\otimes} \mathcal{M}$, with coefficients in $\mathcal{N} \hat{\otimes} \mathcal{M}$, splits (III).
We move now to the case studied in [PopSSm], Theorem 6.1.(1). We start with an automatic complete boundedness result and then prove that if a type $II_1$ von Neumann algebra $\mathcal{M}$ contains an injective norming subalgebra with trivial relative commutant, then $\mathcal{H}_c^2(\mathcal{M}, \mathcal{M})$ splits (III).

**Proposition 5.4.11.** Let $\mathcal{M}$ be a finite von Neumann algebra with centre $\mathcal{Z}$ and $\mathcal{N}$ be a hyperfinite subalgebra of $\mathcal{M}$ such that $\mathcal{N}$ norms $\mathcal{M}$ and $\mathcal{N} \cap \mathcal{M} = \mathcal{Z}$. If $\phi \in \mathcal{L}_w^2(\mathcal{M}, \mathcal{M} : /\mathcal{N})$, then $\phi$ is completely bounded with $\|\phi\|_{cb} \leq 2\|\phi\|$.

**Proof.** Let $\phi \in \mathcal{L}_w^2(\mathcal{M}, \mathcal{M} : /\mathcal{N})$, $n \in \mathbb{N}$, $(m_{ij}^1), (m_{ij}^2) \in \mathcal{M}_n(\mathcal{M})$ with $\|(m_{ij}^1)\|_n \leq 1$ and $\|(m_{ij}^2)\|_n \leq 1$ and $\varepsilon > 0$. Since $\mathcal{N}$ norms $\mathcal{M}$, there exist $(a_1, \ldots, a_n) \in \text{Row}_n(\mathcal{N})$ and $(b_1, \ldots, b_n)^T \in \text{Col}_n(\mathcal{N})$ with $\|(a_1, \ldots, a_n)\|_n, \|(b_1, \ldots, b_n)^T\|_n \leq 1$ such that

$$\|\phi_n((m_{ij}^1), (m_{ij}^2))\|_n - \varepsilon < \|(a_1, \ldots, a_n)\phi_n((m_{ij}^1), (m_{ij}^2))(b_1, \ldots, b_n)^T\|_n.$$ 

By the $\mathcal{N}$-modularity of $\phi$ the right hand side of the previous inequality is equal to $\|\sum_{1 \leq k \leq n} \phi((\sum_{1 \leq i \leq n} a_i m_{ik}^1, \sum_{1 \leq j \leq n} m_{kj}^2 b_j))\|$, which is less than or equal to $2\|\phi\|$, from [CPopSSm], Theorem 2.3. Thus $\phi$ is completely bounded, with $\|\phi\|_{cb} \leq 2\|\phi\|$. $\square$

**Corollary 5.4.4.** Let $\mathcal{M}$ be a type $II_1$ von Neumann algebra with centre $\mathcal{Z}$. If $\mathcal{M}$ contains a hyperfinite norming subalgebra $\mathcal{N}$ with $\mathcal{N} \cap \mathcal{M} = \mathcal{Z}$, then $\mathcal{H}_c^2(\mathcal{M}, \mathcal{M})$ splits (III).

**Proof.** By the previous proposition $Z_{wcb}^2(\mathcal{M}, \mathcal{M} : /\mathcal{N}) = Z_w^2(\mathcal{M}, \mathcal{M} : /\mathcal{N})$ and the result follows immediately from Proposition 5.4.3 and Proposition 5.1.1.$\square$

To finish let’s see what happens with the case studied in [PopSSm], Theorem 6.1.(2). It follows immediately from the proof of [CPopSSm], Proposition 5.3 and [PopSSm], Theorem 2.7 that if $\mathcal{M}$ is a type $II_1$ von Neumann algebra with centre $\mathcal{Z}$, $\mathcal{N}$ is a hyperfinite subalgebra of $\mathcal{M}$ with $\mathcal{N} \cap \mathcal{M} = \mathcal{Z}$ and $\mathcal{B}$ is an abelian von Neumann subalgebra of $\mathcal{M}$ such that $\mathcal{C}^*(\mathcal{N}, \mathcal{B})$ norms $\mathcal{B}(\mathcal{H})$, then $Z_{wcb}^3(\mathcal{M}, \mathcal{M} : /\mathcal{N}) = Z_w^3(\mathcal{M}, \mathcal{M} : /\mathcal{N})$. Combining this observation with Propositions 5.1.1 and 5.4.3 we get the following corollary for $n = 3$. For $n = 2$ it follows from Corollary 5.4.4, Proposition 5.4.10 and [PopSSm], Theorem 2.7.

**Corollary 5.4.5.** Let $\mathcal{M}$ be a type $II_1$ von Neumann algebra with centre $\mathcal{Z}$. If there exist a hyperfinite subalgebra $\mathcal{N}$ of $\mathcal{M}$ with $\mathcal{N} \cap \mathcal{M} = \mathcal{Z}$ and an abelian von Neumann subalgebra $\mathcal{B}$ of $\mathcal{M}$ such that $\mathcal{C}^*(\mathcal{N}, \mathcal{B})$ norms $\mathcal{B}(\mathcal{H})$, then $\mathcal{H}_c^2(\mathcal{M}, \mathcal{M})$ splits (III), for $n = 2, 3$. 

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Combining Propositions 5.4.4, 5.4.5 and 5.4.7 and Corollaries 5.4.2, 5.4.3, 5.4.4 and 5.4.5 with the geometric characterisations of the third type of splitting of a bounded cohomology group (Proposition 2.1.7) and of the bounded Hochschild cohomology complex (Proposition 2.1.24) and with the results of Chapter 4 (Proposition 4.1.1, Corollary 4.1.1 and Proposition 4.2.1) we get results similar to those of Corollary 5.4.1 for the bounded case.
References


[H5] U. Haagerup, Decomposition of completely bounded maps on operator algebras, unpublished manuscript.


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