WELL-BOUNDED OPERATORS AND THE GEOMETRY OF BANACH SPACES

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Presented for the degree of
Doctor of Philosophy

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May, 1988
**ABSTRACT**

Well-bounded operators possess a type of spectral decomposition which is somewhat analogous to that for self-adjoint and scalar-type spectral operators. This spectral decomposition is of a much simpler form when the functional calculus is weakly compact and this case has usually been treated separately to the general case. We give here new proofs which show how similar methods can be used to produce the decompositions in both cases, and which highlight the importance of weak compactness.

It is often of interest to know whether a well-bounded operator is scalar-type spectral. We prove that a sufficient condition for a well-bounded operator on an $L^p$ space ($1 < p < \infty$) to be scalar-type spectral is that its functional calculus be contractive. This generalises a result of Fong and Lam for Hilbert spaces. A major tool in the proof is a result of Dor and Odell and others which states that monotone Schauder decompositions in real $L^p$ spaces ($1 < p < \infty$) are unconditional. This in turn depends on some deep facts about the behaviour of martingale transforms on these spaces. We show Dor and Odell's theorem also holds for complex $L^p$ spaces and supply some of those details of the proof which do not appear in the literature. In order to illustrate how the structure of a Banach space effects the spectral behaviour of the operators on that space, an example is given of a linear transformation which defines a well-bounded operator on $L^p[0,1]$ for $1 \leq p \leq \infty$, the properties of which vary markedly depending on $p$. We also show that spaces which do not contain a copy of $c_0$ can be characterised by the condition that every well-bounded operator whose decomposition of the identity
is of bounded variation is scalar-type spectral. This extends a result of Berkson and Dowson for weakly complete spaces.

The final section takes a more abstract look at the relationship between properties of contractive projections and properties of well-bounded operators. In particular, we show that on a reflexive Banach space, every operator with a contractive absolutely continuous functional calculus is scalar-type spectral if and only if the space has what we call the bilateral unconditionality property for contractive projections. This is a property similar to, but in general stronger than, that of having unconditionality for all monotone Schauder decompositions.
DECLARATION

The material contained within this work is original, except where explicitly mentioned to the contrary.

This thesis has been composed by myself.
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ACKNOWLEDGEMENTS

I would like to express my gratitude to:

My supervisor, Dr. T.A. Gillespie, for his constant help and encouragement: our many hours of discussion helped improve both my Mathematics and my English;

My fellow students, David Blecher, Mathew Penrose and Steve White for listening patiently, and for solving many of my troublesome little problems;

My family and friends for their support and encouragement (and for not trying to convince me to earn an honest living!);

The University of Edinburgh for supplying a stimulating environment in which to do this study, and for its financial support through a Postgraduate Studentship;

The Overseas Research Students Award Scheme for its financial support.
"It should be noted that it is by no means the case that all the familiar eigenvalue expansions of classical analysis are unconditionally convergent. Indeed, there are many examples, such as the Fourier series expansions in the space $L_p(0,2\pi)$ with $1 < p < \infty$ where the expansion converges, but only conditionally. This seems to indicate that further developments in spectral theory will include a theory of conditionally convergent expansions associated with discrete and continuous spectra. ... Nevertheless, the cases where one does have unconditionally convergent eigenvalue expansions are of sufficient importance to justify studying them for their own sake. It is this fact which lends importance to the problem of discovering which operators are spectral operators."

N. Dunford, 1958 [Dun3, p.239].

This thesis examines the relationships between spectral decompositions of operators, the functional calculi that operators admit and Banach space structure. The deep connections between the first two of these concepts has long been known. For example one can use the fact that a normal operator on a Hilbert space $\mathcal{H}$ possesses a functional calculus for the continuous functions on its spectrum to show that such an operator can be represented as an integral with

\[ \text{\footnotesize It is clear that we should exclude } p = 2 \text{ here.} \]
respect to a spectral measure. This in turn allows one to define a functional calculus for all the bounded Borel functions on the spectrum.

Since the 1950's, much work has been done showing how spectral decompositions for operators on Banach spaces can be obtained by assuming the existence of a suitable functional calculus. In Chapter 1 we give a brief overview of some of the theory of spectral and prespectral operators. In particular we list some of the properties of scalar-type spectral operators, a class of operators which can be characterised by the existence of a weakly compact continuous functional calculus.

In 1960, Smart [Sm] introduced well-bounded operators. It was known that many operators which are self-adjoint on \( L^2 \) gave rise to bounded linear operators on the other \( L^p \) spaces having conditionally, rather than unconditionally, convergent eigenfunction expansions. Well-bounded operators, which are those which possess a functional calculus for the absolutely continuous functions on some compact interval of the real line, were shown to give rise to a type of spectral decomposition which covers this conditional convergence.

In Chapter 3 we prove the spectral theorem for well-bounded operators, both in the general case, and in the case when the functional calculus is weakly compact. Well-bounded operators with a weakly compact absolutely continuous functional calculus (and these include all well-bounded operators on reflexive spaces) are said to be of type (B). Well-bounded operators of type (B) allow a much simpler type of decomposition than in the general case and this will be important in the later chapters. Our proofs are somewhat different to the original proofs of Smart, Ringrose and Spain [Sm,
Rin1, Rin2, Sp2] in that we show that essentially the same methods can be used to obtain both decomposition theorems. Some of the necessary preliminary theory for these theorems is given in Chapter 2.

As was noted by Dunford, it is often of interest to know whether a well-bounded operator is scalar-type spectral. The spectral theorems for well-bounded operators characterise these operators in terms of uniformly bounded, increasing families of projections known as decompositions of the identity. In [BD2], Berkson and Dowson found a condition on the decompositions of the identity associated with a well-bounded operator which is sufficient on a weakly complete Banach space to ensure that the operator is scalar-type spectral. We show in Chapter 3 that the condition is in fact sufficient precisely when the Banach space on which the operator acts does not contain a copy of the sequence space $c_0$.

The remainder of the thesis is largely devoted to an examination of how the geometry of a Banach spaces affects the relationship between well-bounded and scalar-type spectral operators on that space. The main result in Chapter 5 is that if $X$ is a reflexive $L^p$ space and $T \in B(X)$ possesses a contractive absolutely continuous functional calculus, then $T$ is scalar-type spectral. This generalises a result for Hilbert spaces due to Fong and Lam [FL], which we prove as a simple corollary of the theorems in Chapter 3. The result for $L^p$ spaces requires a theorem due to Dor and Odell which states that monotone decompositions of reflexive $L^p$ spaces are unconditional. This theorem in turn rests on some deep facts from probability theory about martingale transforms on real $L^p$ spaces. In the generality we require it, Dor and Odell's theorem is not readily
available in the literature, so this is presented in Chapter 4. In particular we show that their theorem can be extended to cover complex, as well as real, spaces.

As an application of the results of Chapter 5, we give an example of a linear transformation which defines a self-adjoint operator on \( L^2 \), a scalar-type spectral operator on \( L^p \) \((1 < p < \infty)\), a well-bounded operator of type \((B)\) on \( L^1 \) and a well-bounded operator which is not of type \((B)\) on \( L^\infty \).

In the final chapter, we discuss several properties relating to the behaviour of contractive projections on a Banach space. On reflexive spaces, it is shown that one of these properties, the bilateral unconditionality property for contractive projections, is equivalent to the property that every operator with a contractive absolutely continuous functional calculus is scalar-type spectral. There are many unanswered questions in this chapter however. For example, whilst it is easy to construct reflexive spaces without this property, we are unable to find any infinite dimensional examples of spaces with the bilateral unconditionality property for contractive projections apart from the reflexive \( L^p \) spaces.

A major inconvenience in much of this work is that whilst spectral theory is traditionally done on complex Banach spaces, the results we need from probability theory and the geometry of Banach spaces are usually only given for real spaces. Suitably interpreted however, most of the results hold for either scalar field, so unless comment is made to the contrary, our spaces may be taken to be either real or complex. Where reference is made to one of the classical Banach spaces (for example \( c_0 \)), this should be interpreted as being real or complex as appropriate. Note however, that \( \mathbb{N} \) will
always denote a complex, separable, infinite dimensional Hilbert space.

Much of our notation and terminology is standard and is therefore not formally introduced in the text. We have however included a list of some of the notation at the end of the thesis.

We note here that some of the results in Chapters 3, 4 and 5 have been submitted for publication [Doul, Dou2].
CHAPTER 1 BACKGROUND

In this chapter we shall present some of the basic background material about self-adjoint, normal and spectral operators. Our aim is to give a framework into which our later results will fit rather than to present a thorough exposition of the theory of these operators. Such an exposition is done more than adequately in the works of Dunford and Schwartz [DS1, DS2, DS3] and Dowson [Dow], and it is to these references that we shall direct the reader for most of the proofs. The literature covering self-adjoint and spectral operators is now quite extensive so we have made no attempt to give a full account of the history of this area of mathematics. The interested reader can find excellent accounts of the history of the spectral theorem in the book of Dieudonné [Dieu] and the paper of Steen [St], whilst the survey paper of Dunford [Dun3] includes some interesting background reading on spectral operators. The Notes and Remarks sections of Dunford and Schwartz [DS1, DS2, DS3] and Dowson [Dow] also include much interesting historical material.

Our theme throughout this chapter is to emphasise the close relationships between the functional calculus for an operator and its spectral resolution.

§ 1.1. The Hilbert space theory

We begin by introducing some notation. Throughout this thesis $\mathcal{H}$ will denote a complex, separable, infinite dimensional Hilbert
space with inner product $(\cdot | \cdot)$. We shall denote the algebra of all bounded linear transformations, or operators, on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. If $T \in \mathcal{B}(\mathcal{H})$, the adjoint operator to $T$ is the unique operator $T^* \in \mathcal{B}(\mathcal{H})$ for which $(Tx | y) = (x | T^*y)$ for all $x, y \in \mathcal{H}$. An operator $T$ is said to be normal if $T^*T = TT^*$ and self-adjoint if $T = T^*$. As is well-known, $T$ is self-adjoint if and only if it is normal and the spectrum of $T$, $\sigma(T)$, is contained in the real line.

By the end of the 19th century it was known that every self-adjoint (or even normal) matrix acting on $\mathbb{C}^n$ could be diagonalised (see, for example, [St, p. 367]). One way of interpreting this is to say that if $T$ is a self-adjoint matrix on $\mathbb{C}^n$ then there exist real numbers $\lambda_1, \ldots, \lambda_n$ and orthogonal projections $E_1, \ldots, E_n$ such that

i) $E_i E_j = \begin{cases} E_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

ii) $\sum_{j=1}^{n} E_j = I$

iii) $Tx = \sum_{j=1}^{n} \lambda_j E_j x$ for all $x \in \mathbb{H}$.

Note that this decomposition allows us to give a simple formula for $g(T)$ for any polynomial $g$. In fact if $f$ is any function defined on the points $\{\lambda_1, \ldots, \lambda_n\}$, then setting

$f(T) = \sum_{j=1}^{n} f(\lambda_j) E_j$

gives a homomorphism from the algebra of all such functions into the algebra of matrices acting on $\mathbb{C}^n$.

In 1906 Hilbert [Hil] proved an infinite dimensional analogue of the above decomposition on the spaces $L^2[0,1]$ and $l^2$, which we now call the spectral theorem for self-adjoint operators. A more
abstract description of this result had to wait until von Neumann's
axiomatic definition of Hilbert space in \[vN\], published in 1930.
In that paper he defines decompositions of unity on \(\mathcal{H}\).

1.1.1. Definition. A decomposition of unity on \(\mathcal{H}\) is a family
\[\{E(\lambda)\}_{\lambda \in \mathbb{R}}\]
of orthogonal projections on \(\mathcal{H}\) which satisfy

i) \(E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min\{\lambda, \mu\})\) for all \(\lambda, \mu \in \mathbb{R}\);

ii) \(E\) is right continuous in the strong operator topology;

iii) \(E(\lambda) \to 0\) (respectively \(E(\lambda) \to 1\)) in the strong operator topology as \(\lambda \to -\infty\) (\(\lambda \to \infty\)).

If there is a compact interval \([a, b] \subset \mathbb{R}\) such that \(E(\lambda) = 0\) for all \(\lambda < a\) and \(E(\lambda) = 1\) for all \(\lambda > b\), then we say that \(\{E(\lambda)\}\) is concentrated on \([a, b]\).

In von Neumann's terminology the spectral theorem for (bounded)
self-adjoint operators is as follows.

1.1.2. THEOREM \([vN, \text{Satz 36}]\). There is a one-to-one
correspondence between bounded self-adjoint operators on \(\mathcal{H}\) and
decompositions of unity on \(\mathcal{H}\) which are concentrated on a compact
interval of \(\mathbb{R}\). For such a decomposition of unity \(\{E(\lambda)\}\), the
function \(\lambda \mapsto (E(\lambda)x|y)\) is of bounded variation for all \(x, y \in \mathcal{H}\)
and, if \(T\) is the self-adjoint operator corresponding to \(\{E(\lambda)\}\),
then
\[(Tx|y) = \int_{\mathbb{R}} \lambda \ d(E(\lambda)x|y)\]
for all \(x, y \in \mathcal{H}\).
The integral appearing in Theorem 1.1.2 is a Riemann-Stieltjes integral. A full description of these integrals and of Theorem 1.1.2 is given by Stone [Sto, Chapter V], who also discusses the unbounded case (due independently to Stone and von Neumann). It is customary now however to state the spectral theorem in a slightly different form involving spectral measures. As we shall need this concept frequently later, we shall state the definition for a general real or complex Banach space $X$. The algebra of bounded linear operators on $X$ will be denoted by $B(X)$, and the set of idempotents, or projections, in $B(X)$ by $\text{Proj}(X)$.

1.1.3. Definition. Suppose that $\mathcal{A}$ is a $\sigma$-algebra of subsets of a set $\Omega$. A spectral measure for $(\Omega, \mathcal{A}, X)$ is a projection-valued function $\mathcal{S}: \mathcal{A} \to \text{Proj}(X)$ such that

i) $\mathcal{S}(\Delta_1 \cup \Delta_2) = \mathcal{S}(\Delta_1) + \mathcal{S}(\Delta_2) - \mathcal{S}(\Delta_1) \mathcal{S}(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathcal{A}$;

ii) $\mathcal{S}(\Delta_1 \cap \Delta_2) = \mathcal{S}(\Delta_1) \mathcal{S}(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathcal{A}$;

iii) $\mathcal{S}(\Omega \setminus \Delta) = 1 - \mathcal{S}(\Delta)$ for all $\Delta \in \mathcal{A}$;

iv) $\mathcal{S}(\Omega) = 1$;

v) there is a constant $M > 0$ such that $\| \mathcal{S}(\Delta) \| \leq M$ for all $\Delta \in \mathcal{A}$.

A spectral measure on $\mathcal{A}$ will be described as self-adjoint if $\mathcal{S}(\Delta)^* = \mathcal{S}(\Delta)$ for all $\Delta \in \mathcal{A}$. A spectral measure is said to be countably additive in the strong operator topology if, for every sequence $\{\Delta_j\}_{j=1}^{\infty}$ of disjoint elements of $\mathcal{A}$,

$$\mathcal{S}\left(\bigcup_{j=1}^{\infty} \Delta_j\right)x = \sum_{j=1}^{\infty} \mathcal{S}(\Delta_j)x$$

for all $x \in X$ (where the right hand side converges in the norm topology on $X$). We
shall say that a spectral measure $\mathcal{S}$ is supported on a set $\Omega_0 \subseteq \Omega$ if $\mathcal{S}(\Omega_0) = 1$.

There is a simple procedure for integrating bounded $\mathcal{A}$-measurable scalar-valued functions on $\Omega$ against a spectral measure $\mathcal{S}$ on $(\Omega, \mathcal{A}, \mathcal{X})$. Let $\chi_A$ denote the characteristic function of the set $A \in \mathcal{A}$. The $\mathcal{A}$-simple functions are those of the form $f = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$ for disjoint $A_1, \ldots, A_n \in \mathcal{A}$ and scalars $\alpha_1, \ldots, \alpha_n$. We define the integral of such an $f$ with respect to $\mathcal{S}$ by

$$\int_{\Omega} f(\lambda) \mathcal{S}(d\lambda) = \sum_{j=1}^{n} \alpha_j \mathcal{S}(A_j).$$

Note that this does not depend on the way we have represented $f$ as a combination of characteristic functions.

Now, if $x \in X$ and $x^* \in X^*$, the scalar-valued set function $\mu$ defined by $\mu(A) = \langle \mathcal{S}(A)x, x^* \rangle$ is clearly finitely additive on $\mathcal{A}$. The variation of such a function is defined to be

$$\text{var} \mu = \sup \sum_{j=1}^{n} |\mu(A_j)|,$$

where the supremum is taken over all finite sequences of disjoint sets in $\mathcal{A}$. A fundamental property of finitely additive set functions is that $\text{var} \mu \leq 4 \sup_{A \in \mathcal{A}} |\mu(A)|$ (see [DS1, III.1.5]). Thus, if $f$ is a simple measurable function represented as above

$$|\langle \int_{\Omega} f(\lambda) \mathcal{S}(d\lambda) x, x^* \rangle| = |\sum_{j=1}^{n} \alpha_j \langle \mathcal{S}(A_j)x, x^* \rangle| \leq \sup_{\omega \in \Omega} |f(\omega)| \text{var} \mu \leq 4 \sup_{\omega \in \Omega} |f(\omega)| \sup_{A \in \mathcal{A}} \|\mathcal{S}(A)\| \|x\| \|x^*\|$$

and so
\[ \left\| \int_{\Omega} f(\lambda) \, \mathcal{G}(d\lambda) \right\| \leq 4M \sup_{\omega \in \Omega} |f(\omega)| \]

where \( M = \sup_{A \in \mathcal{A}} \| \mathcal{A}(A) \| \).

Let \( \mathcal{B}(\Omega, \mathcal{A}) \) denote the Banach algebra of all bounded \( \mathcal{A} \)-measurable functions on \( \Omega \) (under the supremum norm \( \| \cdot \|_\infty \)), and \( \mathcal{S}(\Omega, \mathcal{A}) \) the dense subalgebra of \( \mathcal{B}(\Omega, \mathcal{A}) \) consisting of the simple \( \mathcal{A} \)-measurable functions. The map \( \psi: \mathcal{S}(\Omega, \mathcal{A}) \rightarrow \mathcal{B}(X) \) defined by

\[ \psi f = \int_{\Omega} f(\lambda) \, \mathcal{G}(d\lambda) \]

is an algebra homomorphism which satisfies

\[ \| \psi f \| \leq 4M \| f \|_\infty, \]

so we can extend \( \psi \) to all of \( \mathcal{B}(\Omega, \mathcal{A}) \). This theory is presented in much more detail in [DS1, X.1].

We have now a second formulation of the spectral theorem. For a Borel subset \( \Omega \) of \( \mathbb{C} \) we shall let \( \mathcal{B}(\Omega) \) (or simply \( \mathcal{B} \) if the set \( \Omega \) is understood) denote the \( \sigma \)-algebra of Borel subsets of \( \Omega \).

1.1.4. THEOREM. Suppose that \( T \in \mathcal{B}(\mathcal{H}) \) is self-adjoint. Then there exists a unique countably additive self-adjoint spectral measure \( \mathcal{G} \) for \((\mathcal{C}, \mathcal{B}, \mathcal{H})\) (supported on \( \sigma(T) \)) such that

\[ T = \int_{\sigma(T)} \lambda \, \mathcal{G}(d\lambda). \]

This spectral measure will be called the resolution of the identity for \( T \).

In this formulation the theorem also holds for the larger class of normal operators on \( \mathcal{H} \).
§ 1.2. Functional calculi

One of the main applications of the spectral theorem is that it allows us to construct a large functional calculus for self-adjoint operators.

1.2.1. Definition. Suppose that \( \mathcal{U} \) is a Banach algebra of scalar-valued functions defined on some set \( \Omega \subseteq \mathbb{C} \) and that \( \mathcal{U} \) contains the polynomials. For \( n = 0,1,\ldots \), let \( e_n(z) = z^n \). A \( \mathcal{U} \)-functional calculus for an operator \( T \in B(\mathcal{X}) \) is an \( \sigma \)-norm continuous algebra homomorphism \( \psi: \mathcal{U} \to B(\mathcal{X}) \) for which \( \psi(e_n) = T^n \) for \( n = 0,1,\ldots \). Some authors prefer the term operational calculus to functional calculus. We shall say that a \( \mathcal{U} \)-functional calculus is compact (respectively weakly compact) if, for all \( x \in \mathcal{X} \), the operator \( \psi_x: \mathcal{U} \to \mathcal{X} \) defined by \( \psi_x(f) = \psi(f)x \) is compact when \( \mathcal{X} \) is given its norm (weak) topology.

For a topological space \( \Omega \), we shall denote the space of continuous, scalar valued functions on \( \Omega \) by \( C(\Omega) \). The first step in the spectral theorem for self-adjoint operators is usually to show that any such operator \( T \in B(\mathcal{X}) \) possesses an isometric \( C(\sigma(T)) \)-functional calculus. The following theorem (essentially a Riesz Representation theorem) says that this is what is required to construct a spectral measure for \( T \).

1.2.2. THEOREM [Con, IX.1.14]. Suppose that \( \Omega \) is a compact Hausdorff space. If \( \pi: C(\Omega) \to B(\mathcal{X}) \) is a unital \( * \)-homomorphism, then there is a unique countably additive self-adjoint spectral measure \( \mathcal{E} \).
defined on the Borel subsets of $\Omega$ such that
\[ \pi(f) = \int_{\Omega} f(\lambda) \, \mathcal{E}(d\lambda) \]
for all $f \in C(\Omega)$.

As we saw in § 1.1, given a spectral measure $\mathcal{E}$, we can integrate bounded measurable functions against $\mathcal{E}$. Suppose then that $T \in \mathcal{B}(X)$ is self-adjoint with resolution of the identity $\mathcal{E}$. For $f \in BM(\sigma(T),\mathcal{B})$, setting
\[ f(T) = \int_{\sigma(T)} f(\lambda) \, \mathcal{E}(d\lambda) \]
defines a $BM(\sigma(T),\mathcal{B})$ functional calculus for $T$. We refer the reader to [DS2, X.2.8] or [Con, IX.2.3] for the details.

§ 1.3. Spectral and prespectral operators

We now turn to the spectral operators introduced by Dunford and his co-workers [Dun2, Bad1, Bad2, Kak, Schw, Wer], and their generalisation, the prespectral operators. The rather algebraic definition for normal operators ensures that the $C^*$-algebra generated by such an operator, its adjoint and the identity is commutative. These algebraic properties can be used to construct a continuous functional calculus and a spectral measure for the operator. Dunford began by first assuming the existence of the spectral measure. Throughout this section, $X$ will denote a complex Banach space.
1.3.1. Definition. i) A subspace $\Gamma$ of $X^*$ is total if, whenever $x \in X$ and $<x, x^*> = 0$ for all $x^* \in \Gamma$, then $x = 0$.

ii) Suppose that $\Gamma$ is a total subspace of $X^*$. A spectral measure defined on $B(\mathbb{C})$ is said to be of class $\Gamma$ if the map $\Delta \mapsto <\mathcal{G}(\Delta)x, x^*>$ is countably additive for each $x \in X$ and $x^* \in \Gamma$. In other words, $\mathcal{G}$ is of class $\Gamma$ if, whenever $\{\Delta_j\}_{j=1}^{\infty}$ is a sequence of disjoint elements of $B(\mathbb{C})$, then

$$<\mathcal{G}\left(\bigcup_{j=1}^{\infty} \Delta_j\right)x, x^*> = \sum_{j=1}^{\infty} <\mathcal{G}(\Delta_j)x, x^*>$$

for all $x \in X$ and $x^* \in \Gamma$.

iii) We shall say that $T \in B(X)$ is a prespectral operator of class $\Gamma$ if there exists a spectral measure $\mathcal{G}$ of class $\Gamma$ with values in $B(X)$ such that for all $\Delta \in B$, $T\mathcal{G}(\Delta) = \mathcal{G}(\Delta)T$ and $\sigma(T|\mathcal{G}(\Delta)X) \subseteq \Delta$. $\mathcal{G}$ is called a resolution of the identity (of class $\Gamma$) for $T$.

iv) Suppose that $T \in B(X)$ is a prespectral operator of class $\Gamma$ with resolution of the identity $\mathcal{G}$. We shall say that $T$ is spectral if the vector-valued measure $\nu(\Delta) = \mathcal{G}(\Delta)x$, $\Delta \in B$, is countably additive for each $x \in X$.

Theorem 1.3.2 lists some of the important properties of resolutions of the identity.

1.3.2. Theorem. i) [Dow, Theorem 5.13] A prespectral operator has a unique resolution of the identity of any given class.
ii) [Dow, Theorem 6.5] An operator $T \in B(X)$ is spectral if and only if it is prespectral of class $X^*$.

iii) [Dow, Theorem 6.6] (The Commutativity Theorem) Let $T \in B(X)$ be a spectral operator with resolution of the identity $\mathcal{E}$ of class $X^*$. Then every operator $A \in B(X)$ which commutes with $T$ commutes with $\mathcal{E}(\Delta)$ for all $\Delta \in \mathcal{B}$.

In light of part (i) of this theorem we shall henceforth refer to the resolution of the identity of class $\Gamma$ for a prespectral operator $T$. We shall also refer to the resolution of the identity of class $X^*$ for a spectral operator as the spectral resolution or spectral measure for that operator.

The basic decomposition theorem for prespectral operators is the following result due to Dunford [Dun2].

1.3.3. THEOREM [Dow, Theorem 5.10]. Suppose that $T \in B(X)$ is a prespectral operator with resolution of the identity $\mathcal{E}$ of class $\Gamma$, and that $S = \int_{\sigma(T)} \lambda \mathcal{E}(d\lambda)$. Then $S$ is a prespectral operator with resolution of the identity $\mathcal{E}$ of class $\Gamma$, and $N = T - S$ is quasinilpotent.

If $T = S + N$ is the decomposition given in the theorem, we shall say that $T$ is a scalar-type prespectral operator (of class $\Gamma$) if $N = 0$.

As we have seen above, scalar-type spectral operators possess a rich functional calculus, which can be extended to include the bounded Borel functions on $\sigma(T)$. One might however, try to develop
a spectral theory by first assuming the existence of a suitable functional calculus. Berkson and Dowson [BD1] proved that an analogue of Theorem 1.2.2 holds on general Banach spaces. The following theorem shows that a $C(\Omega)$-functional calculus allows one to construct a spectral measure, but only on $X^*$.

1.3.4. **THEOREM** [Dow, Theorem 5.21]. Suppose that $\Omega$ is a compact Hausdorff space and that $\pi: C(\Omega) \to B(X)$ is a continuous unital algebra homomorphism. Then there exists a spectral measure $\mathcal{G}$ of class $X$ with values in $\text{Proj}(X^*)$ such that

$$\pi(f)^* = \int_{\Omega} f(\lambda) \mathcal{G}(d\lambda)$$

for all $f \in C(\Omega)$.

The natural class of operator to which a $C(\sigma(T))$-functional calculus leads is thus the scalar-type operators. Spain [Sp1] has shown that we can characterise the scalar-type spectral operators by their functional calculi.

1.3.5. **THEOREM** [Dow, Theorem 6.24]. An operator $T \in B(X)$ is scalar-type spectral if and only if it has a weakly compact $C(\sigma(T))$-functional calculus.

The application of the theorems in this section is much simplified by the following theorems concerning Banach spaces which do not contain isomorphic copies of certain sequence spaces. Note that in particular these theorems apply to reflexive Banach spaces.
1.3.6. THEOREM [Gill]. Let $X$ be a complex Banach space. Then every prespectral operator on $X$ is spectral if and only if $X$ does not contain a subspace isomorphic to $\ell^\infty$.

1.3.7. THEOREM [Pell]. Suppose that $\Omega$ is a compact Hausdorff space and that $Y$ is a real or complex Banach space which does not contain a subspace isomorphic to $c_0$. Then every bounded operator $T: C(\Omega) \to Y$ is weakly compact.

Most of the later chapters will be chiefly concerned with operators which, like self-adjoint operators, have spectra which are contained in the real line.

1.3.8. Definition. A scalar-type spectral operator $T$ will be described as real scalar-type spectral if $\sigma(T) \subseteq \mathbb{R}$.

Again we refer the reader who would like further details of the theory of spectral and prespectral operators to the works of Dunford and Schwartz [DS3] and Dowson [Dow].

§ 1.4. Scalar-type spectral operators on real Banach spaces

The theory of spectral operators is, for technical reasons, usually only developed on complex Banach spaces. As much of our later work involves real Banach spaces however, we shall discuss here briefly some of the appropriate concepts on these spaces.
Suppose that $X$ is a real Banach space. The complexification of $X$, written $X_c$, is the complex space $X \times X$ equipped with the scalar multiplication $(\alpha+i\beta)(x,y) = (\alpha x - \beta y, \alpha y + \beta x)$. Under the norm $\| (x,y) \| = \| x \| + \| y \|$, $X_c$ is a Banach space. We shall, however, often use an equivalent norm where this seems more natural (for example when regarding a complex $L^p$ space as the complexification of the corresponding real $L^p$ space). We may regard $X$ as lying inside $X_c$ as the subspace $\{ (x,0) : x \in X \}$ and we shall denote the natural projection of $X_c$ onto $X$ by $P$.

Given $T \in B(X)$, we can define the complexification of $T$, written $T_c$, by $T_c(x,y) = (Tx,Ty)$. It is clear that $T_c \in B(X_c)$. A fuller discussion of complexification is given in [DS3, p. 2130] and [Ric, Chapter 1].

1.4.1. Definition. Suppose that $T \in B(X)$ where $X$ is a real Banach space. We shall say that $T$ is **real scalar-type spectral** if $T_c$ is a real scalar-type spectral operator (in the sense of definition 1.3.8) on $X_c$.

Thus $T$ is real scalar-type spectral if there exists a spectral measure $\mathcal{S}$ on $X_c$, supported on a compact subset of $\mathbb{R}$, such that $T_c = \int \lambda \mathcal{S}(d\lambda)$. Now clearly $PT_c = T_cP$, so by the commutativity theorem (Theorem 1.3.2 (iii)) for scalar operators, $P\mathcal{S}(\Delta) = \mathcal{S}(\Delta)P$ for all $\Delta \in B$. We may thus define the set function $\mathcal{S}_R$ taking values in $B(X)$ by $\mathcal{S}_R(\Delta) = \mathcal{S}(\Delta)|_X$. It is easy to check that $\mathcal{S}_R$ is a spectral measure on $X$ and that $T = \int \lambda \mathcal{S}_R(d\lambda)$. Conversely, every spectral measure supported on a compact subset of $\mathbb{R}$ and taking values
in $B(X)$ will define a real scalar-type spectral operator on $X$ in the above sense.

Thus, for example, the operator $(Tf)(t) = tf(t)$ will be regarded as a real scalar-type spectral operator on both real and complex $L^1[0,1]$, with spectral measure $(\mathcal{M}(\Delta)f)(t) = \chi_{\Delta}(t)f(t)$, $\Delta \in \mathcal{B}$, $t \in [0,1]$. In most of our examples the scalar field may be taken to be either the real or complex numbers.
CHAPTER 2 PRELIMINARIES

In this chapter we have collected together a miscellany of results which will be needed later. Most of these results are minor variations of known theorems, but are included here as they are of some independent interest, and because of the vital role they play in the later chapters.

§ 2.1. Operator topologies

Many of our results will require us to make a judicious choice of topology on $B(X)$. Apart from the three standard topologies (the norm, strong operator and weak operator topologies), we shall also need to introduce a further weak topology on the bounded operators on the dual of $X$.

2.1.1. Definition. The weak-* operator topology on $B(X^*)$ is the weak topology on $B(X^*)$ induced by the family $\mathcal{F}$ of linear functionals $\rho(x,x^*): B(X^*) \to \mathbb{C}$ defined by $\rho(x,x^*)(A) = \langle x, Ax^* \rangle$ ($x \in X$, $x^* \in X^*$).

If $\rho(x,x^*)(A) = 0$ for all $x \in X$ and $x^* \in X^*$, then $A = 0$, so $\mathcal{F}$ is a separating family of linear functionals on $B(X^*)$. It follows that the weak-* operator topology is a locally convex topology on $B(X^*)$ determined by the seminorms $|\rho(x,x^*)|$.

A net of operators $\{T_\alpha\}$ converges to an operator $T$ in the weak-*
operator topology if \( \lim_{\alpha} < x, T_\alpha x^* > = < x, T x^* > \) for all \( x \in X \) and \( x^* \in X^* \). The weak-* operator topology is thus clearly weaker than the weak operator topology on \( B(X^*) \). Obviously if \( X \) is reflexive then the weak operator and weak-* operator topologies agree. The following theorem gives the most important property of the weak-* operator topology.

2.1.2. THEOREM. The closed unit ball of \( B(X^*) \), \( B_1(X^*) \), is weak-* operator topology compact.

Proof. The proof is almost identical to the proof that the closed unit ball of \( B(\mathbb{X}) \) is weak operator topology compact. Our proof is based on [HR, Theorem 5.1.3].

For \( x \in X \) and \( x^* \in X^* \), let \( D(x,x^*) \) be the closed disc of radius \( \| x \| \| x^* \| \) in the complex plane. Define

\[
\Gamma : B_1(X^*) \rightarrow \Pi \{ D(x,x^*) : x \in X, x^* \in X^* \}
\]

by

\[
\Gamma(T) = \{ < x, T x^* > : x \in X, x^* \in X^* \}.
\]

Then \( \Gamma \) is a homeomorphism of \( B_1(X^*) \) with the weak-* operator topology onto \( W = \Gamma(B_1(X^*)) \) with the topology induced by the product topology on \( \Pi \{ D(x,x^*) \} \). In this topology, \( \Pi \{ D(x,x^*) \} \) is a compact Hausdorff space, so \( W \) is compact if it is closed.

Suppose that \( b \in \text{cl}(W) \), that \( \alpha \in \mathbb{C} \), that \( x_1, x_2 \in X \) and \( x_1^*, x_2^* \in X^* \) and that \( \epsilon > 0 \). We shall denote the coordinate of \( b \) lying in \( D(x,x^*) \) by \( b(x,x^*) \). We have then that there exists \( T \in B_1(X^*) \) such that for \( j,k = 1,2 \),

\[
| \alpha b(x_j,x_k^*) - \alpha < x_j, T x_k^* > | < \epsilon,
\]

\[
| b(x_j,x_k^*) - < x_j, T x_k^* > | < \epsilon.
\]
\[ b(\alpha x_1 + x_2, x_k^*) - < \alpha x_1 + x_2, Tx_k^* > | < \varepsilon \quad \text{and} \]
\[ b(x_j, \alpha x_1^* + x_2^*) - < x_j, T(\alpha x_1^* + x_2^*) > | < \varepsilon. \]

Thus
\[ b(\alpha x_1 + x_2, x_1^*) - \alpha b(x_1, x_1^*) - b(x_2, x_1^*) | < 3\varepsilon \quad \text{and} \]
\[ b(x_1, \alpha x_1^* + x_2^*) - \alpha b(x_1, x_1^*) - b(x_1, x_2^*) | < 3\varepsilon. \]

Therefore
\[ b(\alpha x_1 + x_2, x_1^*) = \alpha b(x_1, x_1^*) + b(x_2, x_1^*) \quad \text{and} \]
\[ b(x_1, \alpha x_1^* + x_2^*) = \alpha b(x_1, x_1^*) + b(x_1, x_2^*). \]

Also \(|b(x, x^*)| \leq \|x\| \|x^*\|\) since \(b(x, x^*) \in D(x, x^*)\), so \(b\) is a bounded bilinear functional on \(X \times X^*\), with bound 1. Define
\[ T_0 \in B(X^*) \text{ by} \]
\[ < x, T_0 x^* > = b(x, x^*). \]

Then \(\|T_0\| \leq 1\) and so \(b \in W\). This implies that \(W\) is closed and hence compact, and so \(B_1(X^*)\) must be compact in the weak-* operator topology.

2.1.3. COROLLARY. The weak-* operator topology and the weak operator topology on \(B(X^*)\) coincide if and only if \(X\) is reflexive.

Proof. This is an immediate consequence of the fact that if \(Y\) is a Banach space then \(B_1(Y)\) is weak operator compact if and only if \(Y\) is reflexive (see [DS1, p. 512]).

In later sections we shall often abbreviate the strong operator, the weak operator and the weak-* operator topologies by SOT, WOT and W*OT respectively.
2.2. \( BV[a,b] \) as a Banach algebra

Much of this thesis is concerned with functions of bounded variation. Suppose that \( J = [a,b] \) is a compact interval contained in the real line. We shall denote by \( \mathcal{P} = \mathcal{P}(J) \) the set of finite partitions \( \Lambda = \{ a = \lambda_0 < \lambda_1 < \ldots < \lambda_n = b \} \) of \( J \). If \( f \) is a scalar-valued function on \( J \), then the variation of \( f \) over \( J \) is defined to be

\[
\text{var } f = \sup_{J} \sum_{\Lambda \in \mathcal{P}} |f(\lambda_j) - f(\lambda_{j-1})|.
\]

If \( \text{var } f < \infty \), then \( f \) is said to be of bounded variation over \( J \).

Let \( BV[a,b] \) denote the set of functions of bounded variation over \( [a,b] \) and suppose that \( c \in [a,b] \). It is well-known (and easy to show) that

\[
\|f\|_{[a,b],c} = |f(c)| + \text{var } f
\]

is a norm on \( BV[a,b] \) which makes that space into a Banach space.

Somewhat harder is to show that \( BV[a,b] \) forms a Banach algebra under this norm (and pointwise multiplication).

2.2.1 THEOREM. \( (BV[a,b],\|\cdot\|_{[a,b],c}) \) is a Banach algebra.

Proof. We shall only show that \( \|\cdot\|_{[a,b],c} \) is submultiplicative. Checking the other properties is routine.

Suppose that \( [\alpha,\beta] \) is a compact interval of \( \mathbb{R} \) and let

\[
B_0 = \{ f \in BV[\alpha,\beta] : f(\alpha) = 0 \}.
\]

The first step in the proof is to show that the norm \( \|\cdot\|_{[\alpha,\beta],\alpha} \) is submultiplicative on \( B_0 \).

Suppose that \( f, g \in B_0 \) and that \( \Lambda = \{ \alpha = \lambda_0 < \ldots < \lambda_n = \beta \} \in \mathcal{P}[\alpha,\beta] \). Then
\[ \sum_{j=1}^{n} |f g(\lambda_j) - f g(\lambda_{j-1})| \]

\[ = \sum_{j=1}^{n} |f(\lambda_j)(g(\lambda_j) - g(\lambda_{j-1})) + g(\lambda_{j-1})(f(\lambda_j) - f(\lambda_{j-1}))| \]

\[ \leq \sum_{j=1}^{n} \left[ \sum_{k=1}^{j} |f(\lambda_k) - f(\lambda_{k-1})| \right] |g(\lambda_j) - g(\lambda_{j-1})| \]

\[ + \sum_{j=2}^{n} \left[ \sum_{k=1}^{j-1} |f(\lambda_k) - f(\lambda_{k-1})| \right] |g(\lambda_j) - g(\lambda_{j-1})| \]

\[ \leq \sum_{j=1}^{n} \sum_{k=1}^{j} |f(\lambda_k) - f(\lambda_{k-1})| |g(\lambda_j) - g(\lambda_{j-1})| \]

\[ + \sum_{j=1}^{n} \sum_{k=j+1}^{n} |f(\lambda_k) - f(\lambda_{k-1})| |g(\lambda_j) - g(\lambda_{j-1})| \]

\[ = \left( \sum_{k=1}^{n} |f(\lambda_k) - f(\lambda_{k-1})| \right) \left( \sum_{j=1}^{n} |g(\lambda_j) - g(\lambda_{j-1})| \right) \]

\[ \leq \text{var} f \text{ var} g. \]

Thus

\[ ||fg||_{[\alpha,\beta],\alpha} = \text{var} f \text{ var} g_{[\alpha,\beta]} \]

\[ \leq \text{var} f \text{ var} g_{[\alpha,\beta]} \]

\[ = ||f||_{[\alpha,\beta],\alpha} ||g||_{[\alpha,\beta],\alpha}. \]

By symmetry, \( ||\cdot||_{[\alpha,\beta],\beta} \) is submultiplicative on the subspace of \( \text{BV}[\alpha,\beta] \) consisting of functions vanishing at \( \beta \).

Suppose then that \( f, g \in \text{BV}[a,b] \). We can write these functions as sums

\[ f = f_1 + f_2 + f_3 \text{ and} \]

\[ g = g_1 + g_2 + g_3 \]

where \( f_1 \) and \( g_1 \) are constant functions, \( f_2 \) and \( g_2 \) vanish on \([c,b]\) and \( f_3 \) and \( g_3 \) vanish on \([a,c]\). For notational convenience we shall write \( ||\cdot|| \) instead of \( ||\cdot||_{[a,b],c} \) in the remainder of the proof. It is easy to see that

\[ ||f|| = ||f_1|| + ||f_2|| + ||f_3|| \text{ and} \]

\[ ||g|| = ||g_1|| + ||g_2|| + ||g_3||. \]
\[ ||g|| = ||g_1|| + ||g_2|| + ||g_3||. \]

We have then that
\[ ||fg|| = ||(f_1 + f_2 + f_3)(g_1 + g_2 + g_3)|| \]
\[ \leq \sum_{i,j=1}^{3} ||f_i g_j||. \]

The first part of the proof implies that \[ ||f_2 g_2|| \leq ||f_2|| \ ||g_2|| \]
and \[ ||f_3 g_3|| \leq ||f_3|| \ ||g_3||. \] That \[ ||f_i g_j|| \leq ||f_i|| \ ||g_j|| \]
for the other values of \( i \) and \( j \) follows for straightforward reasons.

Thus
\[ ||fg|| \leq \sum_{i,j=1}^{3} ||f_i|| \ ||g_j|| \]
\[ = \left( \sum_{i=1}^{3} ||f_i|| \right) \left( \sum_{j=1}^{3} ||g_j|| \right) \]
\[ = ||f|| \ ||g||. \]

\[ \square \]

**Remark.** Proofs of this in the case that \( c \) is one of the endpoints are given in [Sill] and [Ral].

It is easily seen that all of the norms \( ||\cdot||_{[a,b],c} \), \( c \in [a,b] \), are equivalent. Where there seems no risk of confusion, we shall often use the notation \( ||\cdot||_{[a,b]} \) or \( ||\cdot||_b \) to stand for \( ||\cdot||_{[a,b],b} \). If \( \Psi : BV[a,b] \to B(X) \) is a bounded algebra homomorphism under one of these norms, then it is clearly bounded under all of them. We shall denote the norm of \( \Psi \) acting on \( BV[a,b] \), equipped with the norm \( ||\cdot||_c \), by \( ||\Psi||_c \).
§ 2.3. Integration with respect to spectral families

In this section we shall present the integration theory for spectral families developed by Spain in [Sp2]. Our treatment of the theory will more closely follow that given in [BG2] however.

2.3.1 Definition. A spectral family of projections in a Banach space \( X \) is a projection-valued function \( E: \mathbb{R} \to B(X) \) such that

i) \( E \) is right continuous in the strong operator topology and has a strong left hand limit at each point in \( \mathbb{R} \);

ii) \( E \) is uniformly bounded, i.e. there exists \( K \) such that

\[
\| E(\lambda) \| \leq K \quad \text{for all} \quad \lambda \in \mathbb{R};
\]

iii) \( E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min\{\lambda, \mu\}) \) for all \( \lambda, \mu \in \mathbb{R} \);

iv) \( E(\lambda) \to 0 \) (respectively \( E(\lambda) \to I \)) in the strong operator topology as \( \lambda \to -\infty \) (respectively \( \lambda \to \infty \)).

If \( E(\lambda) = 0 \) for all \( \lambda < a \in \mathbb{R} \) and \( E(\lambda) = I \) for all \( \lambda > b \in \mathbb{R} \), then we say that \( E \) is concentrated on \( [a, b] \).

Spectral families give rise to a Riemann-Stieltjes integration theory which we shall now describe. Suppose that \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) is a spectral family of projections in \( X \) which is concentrated on \( J = [a, b] \). For \( g \in BV(J) \) and \( \Lambda = \{ a = \lambda_0 < \ldots < \lambda_n = b \} \in \mathcal{P} \), let

\[
\mathcal{S}(g, \Lambda) = g(a)E(a) + \sum_{j=1}^{n} g(\lambda_j)(E(\lambda_j) - E(\lambda_{j-1}))
\]

\[
= g(b)E(b) - \sum_{j=1}^{n} (g(\lambda_j) - g(\lambda_{j-1}))E(\lambda_{j-1}).
\]

Clearly \( \| \mathcal{S}(g, \Lambda) \| \leq \| g \|_b \sup \{ \| E(\lambda) \| : \lambda \in \mathbb{R} \} \). For \( x \in X \), define

\[
\mathcal{S}(g, \Lambda) x = g(a)E(a)x + \sum_{j=1}^{n} g(\lambda_j)(E(\lambda_j) - E(\lambda_{j-1}))x
\]

\[
= g(b)E(b)x - \sum_{j=1}^{n} (g(\lambda_j) - g(\lambda_{j-1}))E(\lambda_{j-1})x.
\]
\[ \omega(x, \Lambda) = \max \sup_{1 \leq j \leq n} \{ \| E(\lambda)x - E(\lambda_{j-1})x \| : \lambda \in [\lambda_{j-1}, \lambda_j) \} . \]

Note that the set \( \mathcal{P} \) of partitions of \( J \) is partially ordered and directed by refinement. We shall write \( \Lambda_1 \geq \Lambda_2 \) to show that \( \Lambda_1 \) is a refinement of \( \Lambda_2 \).

2.3.2. LEMMA [Sp2, Lemma 4]. Fix \( x \in X \). Then \( \lim_{\Lambda \in \mathcal{P}} \omega(x, \Lambda) = 0 \).

Proof [Sp2]. Fix \( \epsilon > 0 \). For each \( s \in [a, b) \) there exists (since \( E \) is strongly right continuous) a point \( r_s \in (s, b) \) such that
\[ \| E(t)x - E(t')x \| \leq \epsilon \quad \text{for all} \quad t, t' \in [s, r_s). \]

Also, since \( E \) possesses a strong left limit everywhere, there exists for each \( s \in (a, b] \) a point \( \ell_s \in (a, s) \) such that
\[ \| E(t)x - E(t'x) \| \leq \epsilon \quad \text{for all} \quad t, t' \in [\ell_s, s). \]

The sets \([a, r_a), (\ell_b, b), (\ell_{s_j}, r_{s_j}) \quad (a < s < b)\), form an open covering for \( J \), so by compactness there is a finite subcovering \([a, r_a), (\ell_b, b), (\ell_{s_j}, r_{s_j}) \quad (j = 1, \ldots, n)\). Let \( \Lambda_0 \) be the partition with points \( a, b, r_a, \ell_a, s_j, r_{s_j}, \ell_{s_j}, r_{s_j} \quad (j = 1, \ldots, n) \), and let \( \Lambda = \{ \lambda_k \}^{m}_{k=0} \) be any partition which refines \( \Lambda_0 \). Then each interval \([\lambda_{k-1}, \lambda_k)\) is a subset of one of \([a, r_a), (\ell_b, b), (\ell_{s_j}, s_j) \) or \([s_j, r_{s_j}) \) for some \( j \). Thus
\[ \| E(t)x - E(\lambda_{k-1})x \| \leq \epsilon \quad \text{for all} \quad t \in [\lambda_{k-1}, \lambda_k) \]
and so \( \omega(x, \Lambda) \leq \epsilon \). The result follows easily.

2.3.3 LEMMA [BG2, Lemma 2.5]. Fix \( x \in X \). Suppose that \( \Lambda_1, \Lambda_2 \in \mathcal{P} \) with \( \Lambda_2 \geq \Lambda_1 \) and that \( g \in BV(J) \). Then
\[ \| S(g, \Lambda_2)x - S(g, \Lambda_1)x \| \leq \omega(x, \Lambda_1) \var \ g. \]
Proof. Suppose that \( \Lambda_1 = \{ \lambda_0 < \ldots < \lambda_{n_1} \} \) and that \( \Lambda_2 = \{ \mu_0 < \ldots < \mu_{n_2} \} \). Then
\[
\mathcal{S}(g, \Lambda_2) - \mathcal{S}(g, \Lambda_1)
\]
\[
= g(b)E(b) - \sum_{j=1}^{n_2} (g(\mu_j) - g(\mu_{j-1}))E(\mu_{j-1})
- g(b)E(b) + \sum_{k=1}^{n_1} (g(\lambda_k) - g(\lambda_{k-1}))E(\lambda_{k-1})
\]
\[
= - \sum_{k=1}^{n_1} \left( \sum_{j \in I_k} (g(\mu_j) - g(\mu_{j-1}))E(\mu_{j-1}) \right)
- (g(\lambda_k) - g(\lambda_{k-1}))E(\lambda_{k-1})
\]
where \( I_k = \{ j : \lambda_{k-1} < \mu_j \leq \lambda_k \} \). Thus
\[
\mathcal{S}(g, \Lambda_2) - \mathcal{S}(g, \Lambda_1)
\]
\[
= - \sum_{k=1}^{n_1} \left( \sum_{j \in I_k} (g(\mu_j) - g(\mu_{j-1}))E(\mu_{j-1}) - E(\lambda_{k-1}) \right)
- (g(\lambda_k) - g(\lambda_{k-1}))E(\lambda_{k-1})
\]
Each term in the second sum over \( k \) is zero, so
\[
\| \mathcal{S}(g, \Lambda_2)x - \mathcal{S}(g, \Lambda_1)x \| 
\]
\[
= \left\| \sum_{k=1}^{n_1} \sum_{j \in I_k} (g(\mu_j) - g(\mu_{j-1}))(E(\mu_{j-1}) - E(\lambda_{k-1}))x \right\|
\]
\[
\leq \sum_{k=1}^{n_1} \sum_{j \in I_k} |g(\mu_j) - g(\mu_{j-1})| \left\| E(\mu_{j-1})x - E(\lambda_{k-1})x \right\|
\]
\[
\leq (\text{var } g) \omega(x, \Lambda_1).
\]

The preceding lemmas now allow us to make the following definition for the integral of a function of bounded variation with respect to a spectral family.
2.3.4 Definition. For \( g \in BV(J) \) and \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) a spectral family concentrated on \( J \), the integral of \( g \) with respect to \( \{E(\lambda)\} \), written \( \int_J \Theta g(\lambda) \, dE(\lambda) \) or \( \int_J g \, dE \), is defined by
\[
\int_J \Theta g(\lambda) \, dE(\lambda) = \text{SOT-lim}_{\Lambda \in \mathcal{P}} S(g, \Lambda).
\]

2.3.5. Proposition [BG2, Proposition 2.6]. The mapping \( g \mapsto \int_J \Theta g \, dE \) is an identity preserving algebra homomorphism of \( BV(J) \) into \( B(X) \).

Furthermore, if \( g \in BV(J) \), \( \Lambda \in \mathcal{P}(J) \), and \( x \in X \), then
\[
\begin{align*}
&i) \quad \left\| \int_J \Theta g \, dE \right\| \leq \|g\|_J \sup \{ \| E(\lambda) \| : \lambda \in \mathbb{R} \}; \\
&ii) \quad \left\| \left[ \int_J \Theta g \, dE \right] x - S(g, \Lambda) x \right\| \leq \omega(x, \Lambda) \text{ var } g.
\end{align*}
\]

Proof. Consider the maps \( \psi_{\Lambda}: BV(J) \rightarrow B(X) \), \( g \mapsto S(g, \Lambda) \) (for \( \Lambda \in \mathcal{P} \)). Each \( \psi_{\Lambda} \) is an identity preserving algebra homomorphism. The only property that is not trivial to check is that \( \psi_{\Lambda} \) is multiplicative. Suppose that \( f, g \in BV(J) \) and that \( \Lambda = \{\lambda_j\}_{j=0}^{n} \).

Then
\[
\begin{align*}
f(\lambda_j)(E(\lambda_j) - E(\lambda_{j-1}))g(\lambda_k)(E(\lambda_k) - E(\lambda_{k-1})) \\
= f(\lambda_j)g(\lambda_k) \left[ E(\lambda_j)E(\lambda_k) - E(\lambda_j)E(\lambda_{k-1}) \\
- E(\lambda_{j-1})E(\lambda_k) + E(\lambda_{j-1})E(\lambda_{k-1}) \right] \\
= \begin{cases} 
0 & \text{if } k \neq j \\
f(\lambda_j)g(\lambda_j)(E(\lambda_j) - E(\lambda_{j-1})) & \text{if } k = j
\end{cases}
\end{align*}
\]

by the properties of spectral families. It follows then that
\[
S(fg, \Lambda) = S(f, \Lambda)S(g, \Lambda) \quad \text{for all } f, g \in BV(J).
\]

For \( g \in BV(J) \), let \( \psi(g) = \text{SOT-lim}_{\Lambda \in \mathcal{P}} \psi_{\Lambda}(g) = \int_J \Theta g \, dE \). Inequality (i) follows from the bound on \( \| S(g, \Lambda) \| \) given on page 21 and the fact that closed balls in \( B(X) \) are complete in the strong operator norm.
topology (see [HR, Proposition 2.5.11]). Inequality (ii) follows from the bound given in Lemma 2.3.3.

That \( \psi \) is an identity preserving algebra homomorphism is easy to check. Again the only non-trivial point is to show that
\[
\psi(fg) = \psi(f)\psi(g).
\]
However this follows since multiplication in \( B(X) \) is jointly strong operator continuous when restricted to bounded sets (see [HR, § 2.5]).

2.3.6. COROLLARY [BG2, Equation 2.9]. For all \( f, g \in BV(J) \), \( x \in X \) and \( \Lambda \in \mathcal{P} \),
\[
\left\| \left( \int_J \phi f \, dE - \int_J \phi g \, dE \right) x \right\| \leq (\var(x, \Lambda) + \var(x, \Lambda)) \left\| \mathcal{S}(f, \Lambda)x - \mathcal{S}(g, \Lambda)x \right\|.
\]

Proof. \[
\int_J \phi f \, dE - \int_J \phi g \, dE = \left( \int_J \phi f \, dE - \mathcal{S}(f, \Lambda) \right) - \left( \int_J \phi g \, dE - \mathcal{S}(g, \Lambda) \right)
\]
\[
+ (\mathcal{S}(f, \Lambda) - \mathcal{S}(g, \Lambda)).
\]

The following interesting result will be needed in Chapter 3 to prove the uniqueness of the spectral family associated with a well-bounded operator of type (B).

2.3.7. PROPOSITION [Sp2, Theorem 2; BG2, Proposition 2.10]. Let \( \{g_\alpha\} \) be a net in \( BV(J) \) and let \( g \) be a scalar-valued function on \( J \) such that

i) \( \sup_{\alpha} \var(x, \Lambda) < \infty \), and

ii) \( g_\alpha \to g \) pointwise on \( J \).
Then \( g \in BV(J) \) and \( \int_J g_\alpha \, dE \) converges to \( \int_J g \, dE \) in the strong operator topology.

**Proof.** That \( g \in BV(J) \) follows from the fact that if

\[
\Lambda = \{ \lambda_j \}_{j=0}^n \in \mathcal{P}, \quad \text{then}
\]

\[
\sum_{j=1}^n |g(\lambda_j) - g(\lambda_{j-1})| = \lim_{\alpha \to \Lambda} \sum_{j=1}^n |g_\alpha(\lambda_j) - g_\alpha(\lambda_{j-1})| = \sup_{\alpha \in \Lambda} \text{var } g_\alpha.
\]

To show that \( \int_J g_\alpha \, dE \) converges to \( \int_J g \, dE \) in the strong operator topology, note that for \( x \in X \),

\[
\left\| \left( \int_J g \, dE - \int_J g_\alpha \, dE \right) x \right\| \leq \text{var } g + \text{var } g_\alpha \omega(x, \Lambda)
\]

\[+ \left\| S(g, \Lambda)x - S(g_\alpha, \Lambda)x \right\| \]

for any partition \( \Lambda \in \mathcal{P} \). Since \( \text{var } g + \text{var } g_\alpha \) is bounded by a constant (independent of \( \alpha \)), we can (by Lemma 2.3.2) make the first term in the upper bound arbitrarily small by choosing a suitable partition. Having fixed this partition, the limit over \( \alpha \) of the second term is zero. \( \square \)

In the final part of this section we present a variant of a result due to Mazur [Mazur]. This result will enable us to show in Chapter 3 that an AC-functional calculus constructed via Proposition 2.3.5 is weakly compact.
2.3.8. Definition. Suppose that \( S \subseteq X \). The **absolutely convex hull** of \( S \), written \( \text{aco}(S) \), is defined by
\[
\text{aco}(S) = \left\{ \sum_{j=1}^{n} \alpha_j x_j : x_1, \ldots, x_n \in S \text{ and } \alpha_1, \ldots, \alpha_n \text{ are scalars such that } \sum_{j=1}^{n} |\alpha_j| \leq 1 \right\}.
\]
The **closed absolutely convex hull** of \( S \), \( \overline{\text{aco}(S)} \), is the norm closure of \( \text{aco}(S) \).

For \( x_0 \in X \) and \( \varepsilon > 0 \), let
\[
\text{Ball}(x_0, \varepsilon) = \{ x \in X : \| x - x_0 \| \leq \varepsilon \}.
\]
Recall that a subset \( S \subseteq X \) is said to be **totally bounded** if, for every \( \varepsilon > 0 \), there exists \( x_1, \ldots, x_n \in S \) such that
\[
S \subseteq \bigcup_{j=1}^{n} \text{Ball}(x_j, \varepsilon).
\]

2.3.9. **Theorem** [Dow, Lemma 17.13]. Suppose that \( S \subseteq X \) is totally bounded. Then \( \overline{\text{aco}(S)} \) is compact.

§ 2.4. Ordered nets of operators

The results in the section will be useful in allowing us to deduce the existence of strong operator topology limits for increasing and decreasing nets of operators.

2.4.1. **Definition.** A net \( \{T_\alpha\}_{\alpha \in A} \) of operators on \( X \) is **naturally ordered** if \( T_\alpha T_\beta = T_\beta T_\alpha = T_\alpha \) for all \( \beta > \alpha \).
The following theorem is due to Barry [Bar].

2.4.2. THEOREM. Suppose that \( \{T_\alpha\} \) is a naturally ordered, uniformly bounded net of operators on \( X \). Then \( \{T_\alpha\} \) converges in the strong operator topology if and only if, for every \( x \in X \),

\[
\bigcap_{\alpha} \text{wk-} \cl \{ T_\beta x : \beta \geq \alpha \} \neq \emptyset.
\]

We refer the reader to [Bar] or [Dow, Theorem 6.4] for a proof.

In practice we shall use the theorem via the following corollary.

2.4.3. COROLLARY. Let \( \{T_\alpha\}_{\alpha \in A} \) be a naturally ordered and uniformly bounded net of operators on \( X \) and suppose that there exists a weak operator topology compact subset \( S \subseteq B(X) \), such that \( T_\alpha \in S \) for all \( \alpha \in A \). Then \( \{T_\alpha\}_{\alpha \in A} \) converges in the strong operator topology.

Proof. Suppose that \( x \in X \) and \( \alpha \in A \). Then as \( S \) is weak operator topology compact, there exists a subnet \( \{T_{\gamma}\}_{\gamma \in \Gamma} \) of \( \{T_\alpha\}_{\alpha \in A} \) with weak operator limit \( T \) say. Thus, for all \( x^* \in X^* \),

\[
<T_{\gamma} x, x^* > \rightarrow < T x, x^* >,
\]

and so, by the properties of subnets,

\( T x \in \text{wk-} \cl \{ T_\beta x : \beta \geq \alpha \} \). In other words,

\( T x \in \bigcap_{\alpha} \text{wk-} \cl \{ T_\beta x : \beta \geq \alpha \} \), and so by the theorem, \( T_\alpha \) converges in the strong operator topology.

Note that if \( X \) is reflexive then every uniformly bounded naturally ordered net of operators on \( X \) lies within a weak operator compact subset of \( B(X) \), and so all such nets converge in the strong
operator topology. The existence of limits on reflexive spaces was first proved by Lorch [Lor].
A crucial step in proving the spectral theorem for self-adjoint or normal operators on a Hilbert space is showing that such operators possess a continuous functional calculus. As we saw in § 1.3, scalar-type spectral operators, whose spectral properties are patterned on those of normal operators, can also be shown to possess such a functional calculus. In [Sm] Smart introduced operators with a smaller functional calculus — one for the absolutely continuous functions on some interval of the real line. He and Ringrose [Sm, Rin1, Rin2] showed that this functional calculus suffices to enable one to construct a "spectral decomposition of the identity" for such operators, of a type reminiscent of that for self-adjoint and real scalar-type spectral operators. The properties of this decomposition are somewhat more complicated for operators on non-reflexive spaces (or more precisely operators which are not of type (B)), so the two cases have usually been treated separately. In this chapter we shall adopt a line of attack towards proving the existence of these decompositions which shows clearly the similarities and differences between the two cases.
§ 3.1. Basics

3.1.1. Definition. An operator $T \in B(X)$ is well-bounded if there exists a compact interval $[a, b] \subset \mathbb{R}$ and a real constant $K$ such that for all polynomials $g$

$$\| g(T) \| \leq K \left\{ |g(b)| + \int_a^b |g'(t)| \, dt \right\} = K \| g \|_{[a, b], b}$$

As $AC[a, b]$ is the closure in $BV[a, b]$ of the polynomials, $T$ is well-bounded if and only if it has an $AC[a, b]$ functional calculus. In other words, there must exist a norm continuous algebra homomorphism from $AC[a, b]$ into $B(X)$ which sends the identity map to $T$ and the constant function $1$ to $I$. As we have noted previously, choosing a different evaluation point $c \in [a, b]$ gives rise to an equivalent norm on $AC[a, b]$, so the definition is independent of the norm $\| . \|_{[a, b], c}$ we choose.

The following proposition is an easy consequence of the definition and shows that we should consider well-bounded operators as being akin to self-adjoint and real scalar-type spectral operators rather than normal operators or general scalar-type spectral operators.

3.1.2. Proposition. Let $X$ be a complex Banach space and suppose that $T \in B(X)$ is well-bounded with $\| g(T) \| \leq K \| g \|_{[a, b]}$ for all $g \in AC[a, b]$. Then $\sigma(T) \subset [a, b]$.

Proof. Suppose that $\lambda \not\in [a, b]$. Then $g(t) = t-\lambda$ and $f(t) = (t-\lambda)^{-1}$ define absolutely continuous functions on $[a, b]$. Let
$S = f(T)$. Then

$$(T - \lambda)S = g(T)f(T) = gf(T) = 1(T) = 1.$$ 

Similarly $S(T - \lambda) = I$ so $\lambda \notin \sigma(T)$.

\[ \square \]

The idea of the proofs of the decomposition theorems is to construct projections $E(\lambda)$ as the limit (in a suitable topology) of the images under the functional calculus of functions of the form

$$g_{\lambda, \delta}$$

as $\delta \to 0^+$, and then to show that $T$ possesses some sort of integral representation with respect to the family of projections $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$. Unfortunately this does not quite work in general. Given a suitable extra condition on the functional calculus, the weak operator topology limit of $g_{\lambda, \delta}(T)$ does exist for each $\lambda \in [a, b)$ and this gives rise to a spectral family $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ for which

$$T = \int_{[a,b]} \lambda \, dE(\lambda).$$

This is done in § 3.2 below. In general, however, we must consider the nets $\{g_{\lambda, \delta}(T)^*\}_{\delta > 0}$ and construct projections $E(\lambda)$ on $X^*$ by taking limits in the weak-$*$ operator topology. The resulting family is not a spectral family, but what is known as a weak decomposition of the identity for $T$. This has the property that

$$<Tx, x^* > = b <x, x^* > - \int_a^b <x, E(\lambda)x^* > \, d\lambda$$

for all $x \in X$ and $x^* \in X^*$. 
§ 3.2. Well-bounded operators of type (B)

We observed in § 1.3 that $T \in B(X)$ is scalar-type spectral if and only if $T$ has a weakly compact $C(\sigma(T))$-functional calculus. In light of this we make the following definition.

3.2.1. Definition. A well-bounded operator is said to be of type (B) if it possesses a weakly compact $AC(J)$-functional calculus for some compact interval $J \subset \mathbb{R}$.

In particular then, every well-bounded operator on a reflexive space is of type (B). This extra condition on the functional calculus, although difficult to check in practice on non-reflexive spaces, is the right one to ensure that we can represent the operator as an integral with respect to a spectral family of projections on $X$.

3.2.2. Theorem [Sp2]. Suppose that $T \in B(X)$. Then $T$ is well-bounded of type (B) if and only if there exists a spectral family $E$ of projections on $X$ such that for some compact interval $J \subset \mathbb{R}$, $E$ is concentrated on $J$ and $T = \int_{J}^{\oplus} \lambda \, dE(\lambda)$. If this is the case then the spectral family is uniquely determined.

Remark. The necessity part of the following proof is somewhat different from the corresponding parts in the existing proofs of this theorem. The main feature of this new proof is that it is relatively "elementary", requiring neither ultrafilters [Sp2, Theorem 10; Dow,
Theorem 17.14] nor the use of convexity arguments and the
Krein-Milman theorem [BG, p. 42]. Furthermore, since much of the
proof can be adapted to show the existence of a type of decomposition
when the functional calculus is not weakly compact, the importance
of this compactness condition is emphasised.

**Proof. (Necessity).** Suppose that $T$ has a weakly compact $\text{AC}(J)$
functional calculus $\psi: \text{AC}(J) \to B(X)$, where $J = [a, b] \subset \mathbb{R}$. For
$\lambda \in [a, b)$ and $0 < \delta < (b-\lambda)$ let $\mathcal{F}_{\lambda, \delta}$ be the set of all real
valued functions $f \in \text{AC}[a, b]$ such that $f = 1$ on $[a, \lambda]$, $f = 0$ on $\text{[}\lambda + \delta, b\text{]}$ and $f$ is decreasing on $[\lambda, \lambda + \delta]$. Define

$$K_{\lambda, \delta} = \text{WOT-} \text{cl} \{ \psi(f) : f \in \mathcal{F}_{\lambda, \delta} \} \subset B(X).$$

Each $\mathcal{F}_{\lambda, \delta}$ is non-empty and norm bounded ($||f||_b \leq 1$ for $f \in \mathcal{F}_{\lambda, \delta}$), so it follows that each $K_{\lambda, \delta}$ is a non-empty, weakly compact subset of $B(X)$. Also, since $\delta_1 < \delta_2$ implies that

$$K_{\lambda, \delta_1} \subset K_{\lambda, \delta_2},$$

it follows by compactness that $K_{\lambda} = \bigcap_{\delta > 0} K_{\lambda, \delta}$ is non-empty and weakly compact.

Let $M_\lambda = \{ x \in X : \psi(f)x = 0, \text{ for all } f \in \bigcup_{\delta > 0} (1-\mathcal{F}_{\lambda, \delta}) \}$. We shall show that $M_\lambda$ is the range of every element of $K_{\lambda}$. Suppose that $x \in M_\lambda$ and $E \in K_{\lambda}$. If we fix $\delta > 0$, then there exists a net

$$\{g_\alpha\}_{\alpha \in A} \text{ in } \mathcal{F}_{\lambda, \delta} \text{ such that, for all } x^* \in X^*,$$

$$< Ex, x^* > = \lim_{\alpha \in A} < \psi(g_\alpha)x, x^* >$$

$$= \lim_{\alpha \in A} < (1-\psi(f_\alpha))x, x^* > \text{ where } f_\alpha = 1 - g_\alpha \in 1 - \mathcal{F}_{\lambda, \delta}.$$ 

Thus $< Ex, x^* > = < x, x^* >$. It follows that $Ex = x$ and so

$x \in \text{Ran } E$. Suppose now that $Ey = x$ and that for some $\delta > 0$,

$f \in 1 - \mathcal{F}_{\lambda, \delta}$. To show that $x \in M_\lambda$, we must show that $\psi(f)x = 0$. 


Fix $\varepsilon > 0$. Then, as $f$ is a continuous, increasing function, there exists $\delta_0 > 0$ such that $0 \leq f(t) \leq \varepsilon/2$ for $t \in [\lambda, \lambda + \delta_0]$. Thus, as $E \in \mathcal{K}_{\lambda, \delta_0}$, there exists a net $\{g_\alpha\}_{\alpha \in A}$ in $\mathcal{F}_{\lambda, \delta_0}$ such that $E = \text{WOT-lim } \psi(g_\alpha)$. For all $\alpha \in A$,

$$|| fg_\alpha ||_b = \int_a^b |(fg_\alpha)'|$$

$$= \int_a^b |f'g_\alpha + fg_\alpha'|$$

$$\leq \int_\lambda^{\lambda + \delta_0} |f'g_\alpha| + \int_\lambda^{\lambda + \delta_0} |fg_\alpha'|$$

$$\leq \int_\lambda^{\lambda + \delta_0} |f'| + \varepsilon/2 \int_\lambda^{\lambda + \delta_0} |g_\alpha'|$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

as $\int_\lambda^{\lambda + \delta_0} |f'| = \int_\lambda^{\lambda + \delta_0} f' = f(\lambda + \delta_0)$. Thus, for all $x^* \in X^*$,

$$|< \psi(f)x, x^*>| = |< \psi(f)Ey, x^*>|$$

$$= |< Ey, \psi(f)x^*>|$$

$$= |\lim_{\alpha \in A} < \psi(g_\alpha)y, \psi(f)x^*>|$$

$$= |\lim_{\alpha \in A} < \psi(fg_\alpha)y, x^*>|$$

$$\leq \sup_{\alpha \in A} || \psi(fg_\alpha) || || y || || x^* ||.$$

We have then that

$$|< \psi(f)x, x^*>| \leq || y || || x^* || || \psi ||_b \varepsilon,$$

and so $\psi(f)x = 0$. This completes the proof that for $E \in \mathcal{K}_\lambda$, $M_\lambda = \text{Ran } E$. Note also that as $Ex = x$ for all $x \in \text{Ran } E$, $E^2 = E$.

The next step is to show that each set $\mathcal{K}_{\lambda, \delta}$ and hence $\mathcal{K}_\lambda$, is a commutative multiplicative semigroup. Suppose that $K_1$ and $K_2$ are elements of $\mathcal{K}_{\lambda, \delta}$. Then $K_1 = \text{WOT-lim } \psi(f_\alpha)$ and $K_2 = \text{WOT-lim } \psi(g_\beta)$ for some nets $\{f_\alpha\}_{\alpha \in A}$, $\{g_\beta\}_{\beta \in B}$ in $\mathcal{F}_{\lambda, \delta}$. Thus, if $x \in X$ and $x^* \in X^*$, then
Thus $K_1 K_2 = K_2 K_1$. As two commuting projections with the same range must be identical, this implies that each set $\mathcal{K}_\lambda$ contains just one element, which we shall denote as $E(\lambda)$. If we now set $E(\lambda) = 0$ for $\lambda < a$ and $E(\lambda) = 1$ for $\lambda \geq b$, we have defined a set of projections on $X$, $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$.

We proceed by showing that $\{E(\lambda)\}$ is a spectral family. Clearly for $\lambda \in [a, b)$ and $\delta \in (0, b-a)$

$$\mathcal{K}_{\lambda, \delta} \subset \text{WOT-cl}\{ \psi(f) : ||f||_b \leq 1 \} \subset \{ U \in \mathcal{B}(X) : ||U|| \leq ||\psi||_b \}$$

which shows that $||E(\lambda)||$ is uniformly bounded for $\lambda \in \mathbb{R}$.

Suppose now that $a \leq \lambda < \mu < b$. That $E(\lambda)E(\mu) = E(\mu)E(\lambda)$ follows from the fact that for $\delta$ large enough, both $E(\lambda)$ and $E(\mu)$ are elements of the commutative set $\mathcal{K}_{\lambda, \delta}$. We shall show now that $E(\lambda)E(\mu) = E(\lambda)$. Let $\delta = \mu - \lambda$ and $\rho = b - \mu$. As $E(\lambda) \in \mathcal{K}_{\lambda, \delta}$, there exists a net $\{g_\alpha\}_{\alpha \in A}$ of functions in $\mathcal{F}_{\lambda, \delta}$ such that $E(\lambda) = \text{WOT-lim} \psi(g_\alpha)$. Similarly, there exists a net $\{h_\beta\}_{\beta \in B}$ of
functions in \( \mathcal{F} \), such that \( E(\mu) = \text{WOT-lim} \, \psi(h_\beta) \). Thus, for \( x \in X \)

and \( x^* \in X^* \)

\[
\langle E(\lambda) E(\mu) x, x^* \rangle = \lim_{\alpha \in A} \langle \psi(g_\alpha) E(\mu) x, x^* \rangle
\]

\[
= \lim_{\alpha \in A} \langle E(\mu) x, \psi(g_\alpha)^* x^* \rangle
\]

\[
= \lim_{\alpha \in A} \left\{ \lim_{\beta \in B} \langle \psi(h_\beta) x, \psi(g_\alpha)^* x^* \rangle \right\}
\]

\[
= \lim_{\alpha \in A} \left\{ \lim_{\beta \in B} \langle \psi(g_\alpha h_\beta) x, x^* \rangle \right\}
\]

\[
= \lim_{\alpha \in A} \left\{ \lim_{\beta \in B} \langle \psi(g_\alpha) x, x^* \rangle \right\}
\]

as \( g_{\alpha \beta} = g_\alpha \) for all \( \alpha \in A \) and \( \beta \in B \). Thus

\[
\langle E(\lambda) E(\mu) x, x^* \rangle = \langle E(\lambda) x, x^* \rangle
\]

and so \( E(\lambda) E(\mu) = E(\mu) E(\lambda) = E(\lambda) \).

We need now only show that \( \{E(\lambda)\} \) has a strong left limit and is strongly right continuous everywhere. The existence of strong left and right limits follows from Corollary 2.4.3. Fix \( \lambda \in [a,b) \) and let \( 0 < \delta < b - \lambda \). We shall denote \( \text{SOT-lim} \, E(\mu) \) by \( E(\lambda^+) \). It is clear that for every \( \mu \in (\lambda, \lambda + \delta) \) we can choose a \( \rho \) such that

\[ [\mu, \mu + \rho] \subset [\lambda, \lambda + \delta], \quad \text{so } E(\mu) \in K_{\lambda, \delta} \]

for all such \( \mu \). Now \( K_{\lambda, \delta} \) is weakly compact so the weak limit (if it exists) of the \( E(\mu) \)'s as \( \mu \to \lambda^+ \) must lie in this set - and hence in \( K_{\lambda} \). But \( E(\mu) \to E(\lambda^+) \) in the strong operator topology, and so in the weak operator topology. Thus \( E(\lambda^+) \in K_{\lambda} = \{E(\lambda)\} \).

We complete the necessity proof by showing that \( T = \int_J \theta \lambda \, dE(\lambda) \).

Fix \( x \in X \) and \( x^* \in X^* \). The mapping \( f \mapsto \langle \psi(f)x, x^* \rangle \) is a continuous linear functional on \( AC(J) \). Since \( AC(J) \) is linearly isometric to \( L^1(J) \otimes C \), there must be some \( \varphi(x, x^*) \in L^\infty(J) \) and \( c(x, x^*) \in C \) such that

\[
\langle \psi(f)x, x^* \rangle = \int_J f'(t) \varphi(x, x^*) (t) \, dt + c(x, x^*) f(b)
\]
for all \( f \in AC(J) \). Taking \( f \equiv 1 \) shows that \( c(x, x^*) = < x, x^* > \).

Fix \( \lambda \in [a, b) \). For \( 0 < \delta < b - \lambda \), define \( g_{\lambda, \delta} \in F_{\lambda, \delta} \) by setting

\[
g_{\lambda, \delta}(t) = \begin{cases} 1 & \text{for } t \in [a, \lambda] \\ 0 & \text{for } t \in [\lambda + \delta, b] \\
\end{cases}
\]

and making \( g_{\lambda, \delta} \) linear on \([\lambda, \lambda + \delta]\). It is easy to see that

\[
< \psi(g_{\lambda, \delta}) x, x^* > = -\frac{1}{\delta} \int_{\lambda}^{\lambda + \delta} \varphi(x, x^*)(t) \, dt.
\]

It follows then, that \( \{\psi(g_{\lambda, \delta})\}_{\delta \in (0, b - \lambda)} \) is a net which is eventually in each of the weakly compact sets \( K_{\lambda, \rho} \) \((\rho > 0)\). It must therefore have weak cluster points which lie in all of these sets, and hence in their intersection, \( K_\lambda \). Thus every weak cluster point must be \( E(\lambda) \), and so the net must have weak limit \( E(\lambda) \) as \( \delta \to 0^+ \).

The Lebesgue differentiation theorem (see for example [Roy, pp. 102-103]) implies that the right hand side of the above equation converges to \(-\varphi(x, x^*)(\lambda)\) almost everywhere as \( \delta \to 0^+ \). Thus \( \varphi(x, x^*)(\lambda) = -< E(\lambda) x, x^* > \) for almost all \( \lambda \in [a, b) \). We have then that

\[
< \psi(f) x, x^* > = f(b) < x, x^* > - \int_{a}^{b} f'(\lambda) < E(\lambda) x, x^* > \, d\lambda
\]

for all \( f \in AC(J) \). Now

\[
< \left[ \int_{J} f \, dE \right] x, x^* > \\
= \lim_{\Lambda \in \mathcal{F}} \left\{ < f(b) E(b) x, x^* > - \sum_{\Lambda} (f(\lambda_j) - f(\lambda_{j-1})) E(\lambda_j) x, x^* > \right\} \\
= f(b) < x, x^* > - \lim_{\Lambda \in \mathcal{F}} \left\{ \sum_{\Lambda} (f(\lambda_j) - f(\lambda_{j-1})) < E(\lambda_j) x, x^* > \right\} \\
= f(b) < x, x^* > - \int_{a}^{b} f'(\lambda) < E(\lambda) x, x^* > \, d\lambda.
\]

Thus

\[
< \psi(f) x, x^* > = < \left[ \int_{J} f \, dE \right] x, x^* >.
\]
Since \(x\) and \(x^*\) are arbitrary, this equation, with \(f(\lambda) = \lambda\), shows that
\[
T = \int_j \lambda \, dE(\lambda).
\]

(Sufficiency). Suppose that there is a spectral family \(\{E(\lambda)\}_{\lambda \in \mathbb{R}}\) of projections on \(X\) such that \(\{E(\lambda)\}\) is concentrated on some compact interval \(J = [a, b] \subseteq \mathbb{R}\) and that \(T = \int_j \lambda \, dE(\lambda)\).

Define \(\gamma: AC(J) \to B(X)\) by \(\gamma(f) = \int_j f \, dE\). By Proposition 2.3.5, \(\gamma\) is an \(AC(J)\) functional calculus for \(T\), and so \(T\) is well-bounded.

To show that \(T\) is of type \((B)\), we need to show that for every \(x \in X\), the map \(f \mapsto \gamma(f)x\) is weakly compact.

Fix \(x \in X\). Define \(\Omega_x = \{ E(\lambda)x : \lambda \in [a, b] \}\). As \(\{E(\lambda)\}\) is concentrated on \([a, b]\), we have that for any \(\delta > 0\),
\[
\Omega_x = \{ E(\lambda)x : \lambda \in [a - \delta, b] \}\.
\]

Fix \(\varepsilon > 0\). Then by Lemma 2.3.2, we can pick \(\delta > 0\) and a partition \(\Lambda = \{ s_0 = a - \delta < s_1 = a < s_2 < \ldots < s_k = b \}\) of \([a - \delta, b]\) such that
\[
\omega(x, \Lambda) = \max_{1 \leq j \leq k} \sup_{\lambda \in [s_{j-1}, s_j]} \| E(\lambda)x - E(s_{j-1})x \| = \varepsilon.
\]

Thus
\[
\Omega_x = \bigcup_{j=0}^n \text{Ball}(E(s_j)x, \varepsilon)
\]

and so \(\Omega_x\) is totally bounded. By Theorem 2.3.9 we have that \(\overline{\text{aco}(\Omega_x)}\) is compact, and hence so is \(\Gamma_x = \{ \alpha y : |\alpha| \leq 1, y \in \overline{\text{aco}(\Omega_x)} \}\).

Suppose that \(f \in AC(J)\) and that \(\| f \|_b \leq 1\). Then
\[
\gamma(f)x = \lim_{u \in P} S(f, u). \quad \text{Now } f(b)x = f(b)E(b)x \in \overline{\text{aco}(\Omega_x)} \text{ as}
\]
\[
|f(b)| = \| f \|_b \leq 1. \quad \text{Similarly, for any}
\[
\Lambda = \{ \lambda_1, \ldots, \lambda_n \} \in \mathcal{P}, \quad \sum_{j=1}^n (f(\lambda_j) - f(\lambda_{j-1})) E(\lambda_j)x \in \overline{\text{aco}(\Omega_x)}.
\]
Thus
\[ S(f, u)x = f(b)x - \sum_{j=1}^{n} (f(\lambda_j) - f(\lambda_{j-1})) E(\lambda_j)x \in \Gamma_x. \]

Now \( \Gamma_x \) is compact, so \( \gamma(f)x \in \Gamma_x \). This shows that the map
\[ f \mapsto \gamma(f)x \]
is compact and hence weakly compact, for every \( x \in X \).
Thus \( T \) is well-bounded of type (B).

(Uniformity of \( \{E(\lambda)\} \)). Suppose that
\[ T = \int_{J_1} \lambda \ dE_1(\lambda) = \int_{J_2} \lambda \ dE_2(\lambda). \]

Fix \( \lambda_0 \in \mathbb{R} \). We shall show that \( E_1(\lambda_0) = E_2(\lambda_0) \). Choose \( M > 0 \) such that \( J_1 \cup J_2 \cup \{\lambda_0\} \subset (-M, M) \). Let \( J = [-M, M] \). Then
\[ T = \int_{J} \lambda \ dE_1(\lambda) = \int_{J} \lambda \ dE_2(\lambda). \]

Let \( f = \chi_{[-M, \lambda_0]} \), the characteristic function of \( [-M, \lambda_0] \). It is easy to check that \( \int_{J} f \ dE_k(\lambda) = E_k(\lambda_0) \) for \( k = 1, 2 \). Choose a sequence \( \{f_n\} \) of polynomials uniformly bounded in \( AC(J) \) (so \( \sup_n \varphi f_n < \infty \)) such that \( f_n \to f \) pointwise on \( J \). By Proposition 2.3.7,
\[ \int_{J} f_n \ dE_k \to \int_{J} f \ dE_k = E_k(\lambda_0), \]
in the strong operator topology for \( k = 1, 2 \). But by Proposition 2.3.5,
\[ \int_{J} f_n \ dE_k = f_n(T) \] (where \( f_n(T) \) has its elementary meaning), which is independent of \( k \).

Remark. Definition 3.2.1 is not the original definition of well-bounded operators of type (B). In [BD2], Berkson and Dowson defined these to be those well-bounded operators whose decompositions of the identity (see Definition 3.3.1 below) are the adjoints of projections on \( X \) and which satisfy certain continuity conditions.
Our aim here however, is to emphasise how the functional calculus for an operator determines its structure. The equivalence of the two definitions is shown in [BD2].

For well-bounded operators on a Hilbert space $\mathcal{H}$ it is more natural to use inner products rather than evaluation of linear functionals. Note that by Corollary 2.4.3, every decomposition of unity on $\mathcal{H}$ (in the sense of Definition 1.1.1) has a strong left hand limit at each point in $\mathbb{R}$, and so is a spectral family. The following spectral theorem for well-bounded operators on $\mathcal{H}$ should be compared with the version of the spectral theorem for self-adjoint operators given in Theorem 1.1.2.

3.2.3. COROLLARY. Suppose that $T \in \mathcal{B}(\mathcal{H})$ and that there exists a constant $K$ and a compact interval $J = [a, b] \subset \mathbb{R}$ such that

$$\| g(T) \| \leq K \| g \|_J$$

for all polynomials $g$. Then there exists a spectral family $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$, concentrated on $J$, such that $T = \int_J \lambda \, dE(\lambda)$. Thus, for $x, y \in \mathcal{H}$,

$$(Tx|y) = b(x|y) - \int_a^b (E(\lambda)x|y) \, d\lambda.$$  

Proof. The first statement is immediate from Theorem 3.2.2. The second follows from a change of notation in the last part of the necessity proof of that theorem.

In Proposition 2.3.5 we showed that the norm of the AC-functional calculus for a well-bounded operator of type (B) is
bounded by the supremum of the norms of its spectral family. The proof of Theorem 3.2.2 allows us to go the other way and give bounds on the norms of the projections which depend on the norm of the functional calculus.

3.2.4. COROLLARY. Suppose that $T \in B(X)$ is a well-bounded operator of type (B) and that $\|g(T)\| \leq K \|g\|_{[a,b],c}$ for all polynomials $g$. If $T$ has spectral family $\{E(\lambda)\}$, then

i) $\|E(\lambda)\| \leq K$ for all $\lambda < c$;

ii) $\|1-E(\lambda)\| < K$ for all $\lambda \geq c$.

Proof. Following the notation in the proof of Theorem 3.2.2, we have that for $\lambda \in [a,b)$, $E(\lambda) = \text{WOT-lim}_{\delta \to 0^+} \psi(g_{\lambda,\delta})$. If $c > a$ and $\lambda \in [a,c)$ then $\|g_{\lambda,\delta}\|_c = 1$ for $\delta < c-\lambda$. By hypothesis $\|\psi\|_c \leq K$, so $\|\psi(g_{\lambda,\delta})\| \leq K$ for such $\delta$. The bound on $\|E(\lambda)\|$ for $\lambda \in [a,c)$ follows since the unit ball is closed in the weak operator topology.

If $c < b$, the bound for $\lambda \in [c,b)$ is a consequence of the fact that $\|1-g_{\lambda,\delta}\|_c = 1$ for $\delta < \lambda-c$. As $K$ must be at least 1, the bounds for $\lambda \notin [a,b)$ follow trivially since $E(\lambda) = 0$ for $\lambda < a$, and $E(\lambda) = 1$ for $\lambda \geq b$. \qed
§ 3.3 The general case

In § 3.2 we showed that the construction of a spectral family for a well-bounded operator requires that its functional calculus is weakly compact. In general this will not be the case. We can, however, proceed with a similar construction by using the fact that, if $T$ is well-bounded, then $T^*$ possesses a weak-* operator compact $\Lambda C$ functional calculus. This is just a trivial consequence of Theorem 2.1.2. The passage from the weak operator topology to the weak-* operator topology does, however, introduce some additional complications.

3.3.1. Definition A weak decomposition of the identity (for $X$) is a family of projections $\{ E(\lambda) \in B(X^*) : \lambda \in \mathbb{R} \}$ such that

i) $E$ is concentrated on some compact interval $J = [a, b] \subset \mathbb{R}$, i.e. $E(\lambda) = 0$ for all $\lambda < a$ and $E(\lambda) = I$ for all $\lambda \geq b$;

ii) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min\{\lambda, \mu\})$ for all $\lambda, \mu \in \mathbb{R}$;

iii) $E$ is uniformly bounded, i.e. there exists $K$ such that $\| E(\lambda) \| \leq K$ for all $\lambda \in \mathbb{R}$;

iv) For all $x \in X$ and all $x^* \in X^*$, the function $\lambda \mapsto <x, E(\lambda)x^*>$ is Lebesgue measurable;

v) For all $x \in X$, $\gamma_x : X^* \to L^\infty[a,b]$, $x^* \mapsto <x, E(.)x^*>$ is continuous when $X^*$ and $L^\infty[a,b]$ are given their weak-* topologies as duals of $X$ and $L^1[a,b]$.

A decomposition of the identity (for $X$) is a weak decomposition of the identity such that
vi) If \( x \in X, \ x^* \in X^*, \ s \in [a,b) \) and the map 
\[
    t \mapsto \int_a^t < x, E(\lambda)x^* > \, d\lambda
\]
is right differentiable at \( s \), then the value of its right 
derivative at that point is \( < x, E(s)x^* >. \)

Decompositions of the identity were introduced by Ringrose in
[Ring2], where he characterised well-bounded operators in terms of
such objects. Ringrose's definition was partly motivated by a desire
to be able to recover the results for reflexive spaces (obtained in
[Sm, Ring1]) from his general theory. On such spaces, conditions (iv)
and (v) are consequences of conditions (i), (ii) and (iii), whilst
(v) implies that \( \{E(\lambda)\} \) is strongly right continuous everywhere. In
applications however, it appears to be important to consider
well-bounded operators of type (B) (on general Banach spaces) rather
than just the smaller class of well-bounded operators on reflexive
spaces (see, for example [RG2]). Our aim here is to show that if we
are willing to give up our requirement to recover the reflexive
theory, then it suffices to consider weak decompositions of the
identity. The main disadvantage of doing this is the loss of the
theorems which allow one to deduce the uniqueness of the
decomposition of the identity associated with a well-bounded operator
under certain circumstances. The payoff, of course, is that we
need to check one less "technical" condition before we can say that a
family of projections defines a well-bounded operator. As the
following proposition shows however, the other two technical
conditions, conditions (iv) and (v), are required to ensure that a
weak decomposition of the identity does give rise to a well-bounded operator on $X$.

3.3.2. PROPOSITION [Rin2, Theorem 1]. Each weak decomposition of the identity for $X$ defines a unique operator $T \in B(X)$ such that

$$< Tx, x^* > = b < x, x^* > - \int_a^b < x, E(\lambda)x^* > d\lambda$$

for all $x \in X$ and $x^* \in X^*$. We shall say that \{E(\lambda)\} is a weak decomposition of the identity associated with $T$.

Proof [Rin2]. Suppose that \{E(\lambda)\}$_{\lambda \in \mathbb{R}}$ satisfies conditions (i) to (v) for a weak decomposition of the identity. Fix $x \in X$. Then

$$L_x(x^*) = b < x, x^* > - \int_a^b < x, E(\lambda)x^* > d\lambda$$

clearly defines a linear functional on $X^*$ which has norm at most $\{bI + K(b-a)\} \|x\|$. Condition (v) implies that $x^* \mapsto \int_a^b < x, E(\lambda)x^* > d\lambda$ is a weak-* continuous map from $X^*$ to $\mathbb{C}$, and, as $x^* \mapsto < x, x^* >$ is also weak-* continuous, $L_x$ shares this property. Thus, by [Bo, Proposition 1, p. 50], there exists $y \in X$ such that

$$L_x(x^*) = < y, x^* >$$

for all $x^* \in X^*$. Let $Tx = y$. It is easy to check that $T$ is linear and bounded by $\{bI + K(b-a)\}$. Thus $T \in B(X)$ and

$$< Tx, x^* > = b < x, x^* > - \int_a^b < x, E(\lambda)x^* > d\lambda$$

for all $x \in X$ and $x^* \in X^*$. Uniqueness is immediate. \qed
3.3.3. THEOREM. Suppose that $T \in B(X)$. Then $T$ is well-bounded if and only if there exists a weak decomposition of the identity for $X$, 

$\{E(\lambda)\}_{\lambda \in \mathbb{R}}$, concentrated on $J = [a, b]$, such that

$$<Tx, x^* > = b <x, x^* > - \int_a^b <x, E(\lambda)x^* > d\lambda$$

for all $x \in X$ and $x^* \in X^*$.

Proof. (Necessity). Suppose that $T$ has an $AC(J)$ functional calculus $\xi: AC(J) \to B(X)$. Define $\psi: AC(J) \to B(X^*)$ by $\psi(f) = \xi(f)^*$. It is easy to check that $\psi$ is a W*OT compact $AC(J)$ functional calculus for $T^* \in B(X^*)$. For $\lambda \in [a, b)$ and $\delta \in (0, b-\lambda)$ define the sets

$\mathcal{F}_{\lambda, \delta} \in AC(J)$ as in the proof of Theorem 3.2.2. Now let

$\mathcal{K}_{\lambda, \delta} = W^{*OT-cl} \{ \psi(f) : f \in \mathcal{F}_{\lambda, \delta} \} \subset B(X^*)$

and, as before,

$\mathcal{K}_\lambda = \bigcap_{\delta > 0} \mathcal{K}_{\lambda, \delta}$.

The W*OT compactness of the unit ball in $B(X^*)$ ensures that the sets $\mathcal{K}_{\lambda, \delta}$ are all W*OT compact and uniformly bounded, and consequently that the $\mathcal{K}_\lambda$ sets are all non-empty, W*OT compact and uniformly bounded. We now proceed as in 3.2.2.

Let $M_\lambda = \left\{ x^* \in X^* : \psi(f)x^* = 0, \text{ for all } f \in \bigcup_{\delta > 0} (1-\mathcal{F}_{\lambda, \delta}) \right\}$.

We will show that $M_\lambda$ is the range of every element of $\mathcal{K}_\lambda$. Suppose that $x^* \in M_\lambda$ and $E \in \mathcal{K}_\lambda$. If we fix $\delta > 0$, then there exists a net $\{g_\alpha\}_{\alpha \in A}$ in $\mathcal{F}_{\lambda, \delta}$ such that for all $x \in X$,

$$<x, Ex^* > = \lim_{\alpha \in A} <x, \psi(g_\alpha)x^* >$$

$$= \lim_{\alpha \in A} <x, (1-\psi(f_\alpha))x^* > \text{ where } f_\alpha = 1 - g_\alpha \in 1 - \mathcal{F}_{\lambda, \delta}.$$ 

Thus $<x, Ex^* > = <x, x^* >$. It follows that $Ex^* = x^*$ and so $x^* \in \text{Ran } E$. Suppose now that $Ey^* = x^*$ and that for some $\delta > 0$, 

...
$f \in 1 - F_{\lambda, \delta}$. To show that $x^* \in M_\lambda$, we must show that $\psi(f)x^* = 0$.

Fix $\varepsilon > 0$. Then, as $f$ is a continuous, increasing function, there exists $\delta_0 > 0$ such that $0 \leq f(t) \leq \varepsilon/2$ for $t \in [\lambda, \lambda + \delta_0]$.

Thus, as $E \in \mathcal{K}_{\lambda, \delta_0}$, there exists a net $\{g_\alpha\}_{\alpha \in \Lambda}$ in $F_{\lambda, \delta_0}$ such that $E = W^{*}\text{OT-lim} \psi(g_\alpha)$. As in 3.2.2, $\|fg_\alpha\|_b \leq \varepsilon$ for all $\alpha \in \Lambda$. Thus, for all $x \in X$

$$< x, \psi(f)x^* > = < x, \psi(f)Ey^* >$$

$$= |< \xi(f)x, Ey^* >|$$

$$= |\lim_{\alpha \in \Lambda} < \xi(f)x, \psi(g_\alpha)y^* >|$$

$$= |\lim_{\alpha \in \Lambda} < x, \psi(fg_\alpha)y^* >|$$

$$\leq \sup_{\alpha \in \Lambda} \|\psi(fg_\alpha)\| \|x\| \|y^*\|$$

$$\leq \|x\| \|y^*\| \|\psi\|_b \varepsilon,$$

and so $\psi(f)x^* = 0$. This completes the proof that for $E \in \mathcal{K}_\lambda$, $M_\lambda = \text{Ran} E$. Again, as $Ex^* = x^*$ for all $x^* \in \text{Ran} E$, $E^2 = E$.

Note however, that because the elements of $\mathcal{K}_\lambda$ need not be the adjoints of operators on $X$, we cannot proceed as in 3.2.2 to show that this set is commutative and so contains just one projection.

Suppose now that $a \leq \lambda < \mu < b$, and that $E_\lambda \in \mathcal{K}_\lambda$ and $E_\mu \in \mathcal{K}_\mu$. We shall show that $E_\lambda E_\mu = E_\lambda$. Let $\delta = \mu - \lambda$ and $\rho = b - \mu$. Then, as $E_\lambda \in \mathcal{K}_\lambda \subset \mathcal{K}_{\lambda, \delta}$, there exists a net $\{g_\alpha\}_{\alpha \in \Lambda}$ of functions in $F_{\lambda, \delta}$ such that $E_\lambda = W^{*}\text{OT-lim} \psi(g_\alpha)$. Similarly, there exists a net $\{h_\beta\}_{\beta \in \mathcal{B}}$ of functions in $F_{\mu, \rho}$ such that $E_\mu = W^{*}\text{OT-lim} \psi(h_\beta)$. Thus, for $x \in X$, $x^* \in X^*$

$$< x, E_\lambda E_\mu x^* > = \lim_{\alpha \in \Lambda} < x, \psi(g_\alpha)E_\mu x^* >$$

$$= \lim_{\alpha \in \Lambda} < \xi(g_\alpha)x, E_\mu x^* >$$
\[ = \lim_{\alpha \in \mathcal{A}} \left\{ \lim_{\beta \in \mathcal{B}} \langle \xi(g_{\alpha})x, \psi(h_{\beta})x^* \rangle \right\} \]
\[ = \lim_{\alpha \in \mathcal{A}} \left\{ \lim_{\beta \in \mathcal{B}} \langle \xi(h_{\beta})\xi(g_{\alpha})x, x^* \rangle \right\} \]
\[ = \lim_{\alpha \in \mathcal{A}} \left\{ \lim_{\beta \in \mathcal{B}} \langle \xi(g_{\alpha})x, x^* \rangle \right\} \]

as \( g_{\alpha} h_{\beta} = g_{\alpha} \) for all \( \alpha \in \mathcal{A} \) and \( \beta \in \mathcal{B} \). Thus

\[ < x, E_{\lambda} E_{\mu} x^* > = < x, E_{\lambda} x^* > \]

and so \( E_{\lambda} E_{\mu} = E_{\lambda} \). An almost identical proof shows that \( F_{\mu} F_{\lambda} = F_{\lambda} \).

We are going to construct our weak decomposition of the identity \( \{E(\lambda)\} \) by choosing \( E(\lambda) \in \mathcal{K}_\lambda \) for \( \lambda \in [a, b] \) and setting \( E(\lambda) = 0 \) for \( \lambda < a \) and \( E(\lambda) = 1 \) for \( \lambda \geq b \). Any such choice will satisfy conditions (i), (ii) and (iii) for a weak decomposition of the identity (and of course \( E(\lambda) \in \text{Proj}(X^*) \) for all \( \lambda \)). We will show now that we can make our choice so as to satisfy conditions (iv) and (v).

For every \( x \in X \) and \( x^* \in X^* \), the linear functional on \( AC(J) \)
\( \psi(x, x^*) : f \mapsto < \psi(f)x, x^* > \) is continuous. As in the proof of 3.2.2, there must be some \( \varphi(x, x^*) \in L^\infty(J) \) and \( c(x, x^*) \in C \) such that

\[ < x, \psi(f)x^* > = \int f(t)\varphi(x, x^*)(t) \, dt + c(x, x^*) f(b) \]

for all \( f \in AC(J) \). Taking \( f \equiv 1 \) shows that \( c(x, x^*) = < x, x^* > \).

Fix \( \lambda \in [a, b] \). For \( 0 < \delta < b - \lambda \), define \( g_{\lambda, \delta} \in \mathcal{I}_{\lambda, \delta} \) as in 3.2.2. For each \( \lambda \) then, \( \{g_{\lambda, \delta}\}_{\delta \in (0, b-\lambda)} \) is a net which is eventually in each of the \( W*OT \) compact sets \( \mathcal{K}_{\lambda, \rho} \) (\( \rho > 0 \)). It must therefore have a \( W*OT \) cluster point, \( E_0^{\lambda} \) say, which lies in the intersection of these sets, \( \mathcal{K}_\lambda \). Thus there exists a subnet \( \{\psi(g_{\lambda, \delta_\nu}) : \nu \in A_\lambda\} \) with \( W*OT \) limit \( E_{\lambda}^0 \). We have then that

\[ \lim_{\nu \in A_\lambda} < x, \psi(g_{\lambda, \delta_\nu})x^* > = < x, E_{\lambda}^0 x^* > \]

for all \( x \in X \) and \( x^* \in X^* \). But
\[ < x, \psi(g_{\lambda}, \delta_v)x^* > = < x, x^* > g_{\lambda, \delta_v}(b) + \int_a^b g'_{\lambda, \delta_v}(t) \varphi(x, x^*)(t) \, dt \]
\[ = -1/\delta_v \int_\lambda^{\lambda+\delta} \varphi(x, x^*)(t) \, dt. \]

By the Lebesgue differentiation theorem, the nets
\[ \left\{ -1/\delta \int_\lambda^{\lambda+\delta} \varphi(x, x^*)(t) \, dt : \delta > 0 \right\} \]
converge to \(-\varphi(x, x^*)(\lambda)\) for almost all \(\lambda\). Thus the above subnets also converge to these limits, i.e.
\[ < x, E^0_\lambda x^* >= \lim_{n \to \infty} < x, \psi(g_{\lambda, \delta_v})x^* > \]
\[ = -\varphi(x, x^*)(\lambda) \quad (a.e.). \]

Thus, if we set \(E(\lambda) = E^0_\lambda\), then the function \(\lambda \mapsto < x, E(\lambda)x^* >\) is measurable for all \(x \in X\) and \(x^* \in X^*\). We also have that
\[ < x, \psi(f)x^* > = f(b) < x, x^* > - \int_a^b f'(\lambda) < x, E(\lambda)x^* > \, d\lambda \quad (1) \]
for all \(f \in AC(J), x \in X\) and \(x^* \in X^*\). Substituting \(f : t \mapsto t\) we find that
\[ < T x, x^* > = < x, T^* x^* > \]
\[ = b < x, x^* > - \int_a^b < x, E(\lambda)x^* > \, d\lambda \]
for all \(x \in X\) and \(x^* \in X^*\).

We need now only verify that condition (v) holds. Given \(u \in L^1(J)\), define \(f_u = \int_a^b u(t) \, dt\). Clearly \(f_u \in AC(J)\) and
\[ f'_u(\lambda) = -u(\lambda) \quad (a.e.). \]
Fix \(x \in X\) and define \(A_x : L^1(J) \to X\) by
\[ A_x(u) = \xi(f_u)x. \]
\(A_x\) is clearly a continuous linear map. Now by (1) above,
\[ < A_x u, x^* > = < \xi(f_u)x, x^* > = \int_a^b u(\lambda) < x, E(\lambda)x^* > \, d\lambda, \]
so \(A^*_x : X^* \to L^0(J)\) satisfies
\[ \langle u, A^*_x x^* \rangle = \int_a^b u(\lambda) \langle x, E(\lambda)x^* \rangle \, d\lambda. \]

Consequently, \( A^*_x \) is the map \( \gamma_x \) in the definition, and as it is the adjoint of a continuous linear mapping from \( L^1(J) \) into \( X \), it has the required continuity property.

**(Sufficiency).** Suppose that \( \{E(\lambda)\} \) is a weak decomposition of the identity for \( X \) and that \( T \) is the unique operator such that

\[ \langle Tx, x^* \rangle = b \langle x, x^* \rangle - \int_a^b \langle x, E(\lambda)x^* \rangle \, d\lambda \]

for all \( x \in X \) and \( x^* \in X^* \). A simple induction proof using Fubini's Theorem shows that for \( n = 0, 1, 2, \ldots \)

\[ \langle T^n x, x^* \rangle = b^n \langle x, x^* \rangle - \int_a^b \langle x, E(\lambda)x^* \rangle n^1 \, d\lambda \]

and so if \( g \) is a polynomial

\[ \langle g(T)x, x^* \rangle = g(b) \langle x, x^* \rangle - b \int_a^b \langle x, E(\lambda)x^* \rangle g'(\lambda) \, d\lambda. \]

Therefore

\[ \| g(T) \| \leq |g(b)| + b \int_a^b K |g'(\lambda)| \, d\lambda \]

where \( K > 1 \) is any upper bound on \( \{ \| E(\lambda) \| : \lambda \in \mathbb{R} \} \). Thus

\[ \| g(T) \| \leq K \| g \|_b \]

and so \( T \) is well-bounded.

A representation of this type for well-bounded operators was first obtained by Ringrose [Rin2, Theorem 1 and Theorem 6] in terms of decompositions of the identity. Combining Ringrose's theorem with Theorem 3.3.3 gives the following.
3.3.4. **THEOREM.** Suppose that \( T \in B(X) \). The following are equivalent:

(i) \( T \) is well-bounded.

(ii) There exists a weak decomposition of the identity for \( X \),

\[ \{E(\lambda)\}_{\lambda \in \mathbb{R}}, \text{ concentrated on } J = [a,b], \text{ such that} \]

\[ < Tx, x^* > = b < x, x^* > - \int_a^b < x, E(\lambda) x^* > d\lambda \]

for all \( x \in X \) and \( x^* \in X^* \).

(iii) There exists a decomposition of the identity for \( X \),

\[ \{E(\lambda)\}_{\lambda \in \mathbb{R}}, \text{ concentrated on } J = [a,b], \text{ such that} \]

\[ < Tx, x^* > = b < x, x^* > - \int_a^b < x, E(\lambda) x^* > d\lambda \]

for all \( x \in X \) and \( x^* \in X^* \).

The following fact, which will be used frequently later, is a simple consequence of the sufficiency part of Theorem 3.3.3.

3.3.5. **COROLLARY.** Suppose that \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) is a weak decomposition of the identity associated with the well-bounded operator \( T \in B(X) \). Then, for all \( g \in \text{AC}[a,b] \), \( x \in X \) and \( x^* \in X^* \),

\[ < g(T)x, x^* > = g(b) < x, x^* > - \int_a^b < x, E(\lambda) x^* > g'(\lambda) d\lambda. \]

**Proof.** We have already seen that the corollary holds for \( g \) a polynomial. The passage from polynomials to general absolutely continuous functions follows from the usual limiting arguments. \( \Box \)

It is easy to see that weak decompositions of the identity and decompositions of the identity are indeed different classes. The
difference is akin to that between arbitrary representatives of an $L^\infty$ equivalence class and ones which are "as right continuous as possible." For example, if $E \in \text{Proj}(X)$ then

$$E(\lambda) = \begin{cases} 0 & \text{for } \lambda < 0 \\ E^* & \text{for } \lambda \in [0,1) \\ I & \text{for } \lambda \geq 1 \end{cases}$$

is the unique decomposition of the identity associated with the well-bounded operator $E$. We can however redefine $E(0)$ to be any projection $E_0 \in \mathcal{B}(X^*)$ satisfying $E^*E_0 = E_0E^* = E_0$ and still have a weak decomposition of the identity for $E$. Showing that one can always find a decomposition of the identity associated with $T$ requires showing that there are "nice" elements in each $L^\infty$ equivalence class. We shall see in § 3.5 that even decompositions of the identity need not be uniquely determined.

Ringrose [Rin2, Theorem 8] has shown, however, that if each element of a decomposition of the identity is the adjoint of an operator on $X$, then the decomposition of the identity is unique. Thus, for example, there is a unique decomposition of the identity associated with each well-bounded operator of type (B), consisting of the adjoints of its spectral family. The representation obtained in Theorem 3.3.3 may formally be considered as coming from integrating the spectral family representation by parts.
3.4. Well-bounded and scalar-type spectral operators

If \( T \in B(X) \) is real scalar-type spectral, with spectral measure \( \mathcal{E} \), then it is simple to check that \( E(\lambda) = \mathcal{E}(-\infty, \lambda] \) forms a spectral family for \( T \) (see [BD2, Theorem 5.4]). Thus all real scalar-type spectral operators are well-bounded of type (B). Given a decomposition of the identity \( \{E(\lambda)\} \), one can attempt to define a spectral measure \( \mathcal{E} \) by first setting \( \mathcal{E}(a_1, b_1] = E(a_1) \setminus E(b_1) \). It may not, however, be possible to extend this to a countably additive set function. A necessary and sufficient condition for this procedure to succeed was found by Berkson and Dowson [BD2].

3.4.1. Definition. A weak decomposition of the identity is said to be of bounded variation if, for all \( x \in X \) and all \( x^* \in X^\ast \), the function \( \lambda \mapsto \langle x, E(\lambda)x^* \rangle \) is of bounded variation. If the adjoints of a spectral family form a decomposition of the identity of bounded variation then we shall say that the spectral family is of bounded variation.

3.4.2. Theorem ([BD2, Theorem 5.2]). Suppose that \( X \) is a complex Banach space and that \( T \in B(X) \). Then the following are equivalent.

(i) \( T \) is well-bounded and possesses a decomposition of the identity of bounded variation.

(ii) There is a compact interval \( [a, b] \) and a constant \( M > 0 \) such that for all polynomials \( g \)

\[
\| g(T) \| \leq M \sup_{[a,b]} |g(t)|.
\]
(iii) $T^*$ is a real scalar-type pre-spectral operator of class $X$. Under these conditions $T$ has a unique decomposition of the identity.

We refer the reader to [BD2] or [Dow, Theorem 16.15] for a proof. More useful for us will be the following modification of [BD2, Theorem 5.3].

3.4.3. THEOREM. Suppose that $X$ is a real or complex Banach space which does not contain a subspace isomorphic to $c_0$, and that $T \in B(X)$. Then the following are equivalent.

(i) $T$ is well-bounded and possesses a weak decomposition of the identity of bounded variation.

(ii) $T$ is well-bounded and possesses a decomposition of the identity of bounded variation.

(iii) $T$ is well-bounded of type (B) with a spectral family of bounded variation.

(iv) There exists a compact interval $[a,b] \subset \mathbb{R}$ and a constant $M > 0$ such that for all polynomials $g$

$$\|g(T)\| \leq M \sup_{[a,b]} |g(t)|.$$  

(v) $T$ is real scalar-type spectral.

Proof. Suppose first that $X$ is a complex Banach space. We shall show that (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iv). This follows from Theorem 3.4.2.

(iv) $\Rightarrow$ (v). By the Stone-Weierstrass theorem, $T$ admits a $C[a,b]$-functional calculus, which must, by Theorem 1.3.7 be weakly compact. The result follows by Theorem 1.3.5.
(v) $\Rightarrow$ (iii). This follows from [BD2, Theorem 5.4] and Theorem 3.4.2.

(iii) $\Rightarrow$ (i). Obvious.

(i) $\Rightarrow$ (ii). Suppose that \{E(\lambda)\} is a weak decomposition of the identity associated with T which is of bounded variation. By Theorem 3.3.4 there also exists a decomposition of the identity, \{F(\lambda)\}, say, associated with T. Our aim is to show that \{F(\lambda)\} is of bounded variation. Fix $x \in X$ and $x^* \in X^*$. Define $e: \lambda \mapsto < x, E(\lambda)x^* >$ and $f: \lambda \mapsto < x, F(\lambda)x^* >$. As $e$ is of bounded variation, for each $\lambda \in \mathbb{R}$, $e(\lambda^+) = \lim_{\mu \to \lambda^+} e(\mu)$ exists. Let $\eta(t) = \int_a^t e(\lambda) \, d\lambda$. An appropriate choice of function in Corollary 3.3.5 shows that

$$\eta(t) = \int_a^t f(\lambda) \, d\lambda \quad \text{for all } t \in \mathbb{R}.$$ 

Fix $t \in \mathbb{R}$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that $e(\lambda) \in (e(t^+) - \epsilon, e(t^+) + \epsilon)$ for all $\lambda \in (t, t+\delta)$. Thus, by the mean value theorem, $1/\rho \int_t^{t+\rho} e(\lambda) \, d\lambda \in (e(t^+) - \epsilon, e(t^+) + \epsilon)$ for all $\rho \in (0, \delta)$. Hence $\eta$ is right differentiable at $t$, with right derivative $e(t^+)$. By condition (vi) for a decomposition of the identity we must have that

$$f(t) = e(t^+) \quad \text{for all } t \in \mathbb{R}. \quad (1)$$

Next we shall show that $f$ is of bounded variation. Fix $\epsilon > 0$ and let $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ be some finite partition in $\mathbb{R}$. Then by (1) there exists $\delta > 0$ such that $|f(\lambda_k) - e(\lambda_k + \delta)| < \epsilon/n$ for $k = 0, 1, \ldots, n$. Thus

$$\sum_{k=1}^n \left| f(\lambda_k) - f(\lambda_{k-1}) \right| \leq \sum_{k=1}^n \left| f(\lambda_k) - e(\lambda_k + \delta) \right|$$

$$+ \sum_{k=1}^n \left| e(\lambda_k + \delta) - e(\lambda_{k-1} + \delta) \right|$$

$$+ \sum_{k=1}^n \left| e(\lambda_{k-1} + \delta) - f(\lambda_{k-1}) \right|$$
which proves that \( \text{var } f \leq \text{var } e \) and hence that \( T \) possesses a decomposition of the identity of bounded variation.

This completes the proof in the case that \( X \) is a complex Banach space. Suppose now that \( T \) is an operator on the real Banach space \( X \) with complexification \( T_C \in B(X_C) \). Again we shall show that

\[
(ii) \implies (iv) \implies (v) \implies (iii) \implies (i) \implies (ii).
\]

\( (ii) \implies (iv) \). It is easily seen that if \( T \) satisfies \( (ii) \) then so does \( T_C \). By the complex case, \( T_C \) must satisfy \( (iv) \) and it is readily verified that this ensures that \( T \) satisfies \( (iv) \).

\( (iv) \implies (v) \). If \( T \) satisfies \( (iv) \). As in the complex case, \( T \) must have a weakly compact \( C[a,b] \)-functional calculus. It is simple to check that \( T_C \) must also have a weakly compact \( C[a,b] \)-functional calculus, and so \( T_C \) is real scalar-type spectral. It follows (by definition) that \( T \) is real scalar-type spectral.

\( (v) \implies (iii) \). Suppose that \( T \) is real scalar-type spectral. Then by definition \( T_C \) is real scalar-type spectral with resolution of the identity \( \mathcal{E} \) say. From the complex case we have that \( T_C \) satisfies condition \( (iii) \) and that \( E(\lambda) = \mathcal{E}(\mathbb{C}(\lambda)) \) forms the spectral family for \( T_C \). Since \( \mathcal{E}(\Delta) \) leaves \( X \) invariant for all \( \Delta \in B \) (see § 1.4), we can define \( E'(\lambda) = E(\lambda)|_X (\lambda \in \mathbb{R}) \). It is easy to check that \( E' \) must form a spectral family for \( T \) and must be of bounded variation.

\( (iii) \implies (i) \). Obvious.

\( (i) \implies (ii) \). The proof that was used in the complex case also works here.

Some of the above equivalences hold more generally. For example, an examination of the proof of Theorem 3.4.3 shows that
conditions (i) and (ii) are equivalent on any Banach space. The implication (i) \( \Rightarrow \) (v) however, which allows us to decide when a well-bounded operator is scalar-type spectral, holds only when \( X \) does not contain a copy of \( c_0 \).

3.4.4. THEOREM. Suppose that \( X \) is a Banach space. Then the following are equivalent:

(i) \( X \) does not contain a subspace isomorphic to \( c_0 \).

(ii) Every well-bounded operator on \( X \) with a decomposition of the identity of bounded variation is real scalar-type spectral.

Proof. (i) \( \Rightarrow \) (ii). This has been proved in Theorem 3.4.3.

(ii) \( \Rightarrow \) (i). Suppose that \( X \) contains a subspace \( Y \) which is isomorphic to \( c_0 \). Then \( Y \) is also isomorphic to the space \( c \) of all convergent sequences. We shall follow the methods of [Gi12] to embed a suitable operator on \( c \) into \( B(X) \). If we regard \( Y \) as being identified with \( c \), we can equip it with either of the two equivalent norms; \( \| \cdot \|_\infty \) from \( c \) and \( \| \cdot \|_X \) from \( X \). We shall assume that \( K^{-1} \| \cdot \|_X \leq \| \cdot \|_\infty \leq K \| \cdot \|_X \).

Let \( e_n \in Y \) be the \( n \)-th standard basis element of \( c \), and let \( \varphi_n' \) be the \( n \)-th coordinate functional on \( c \) (i.e. \( \langle \sum_{m=1}^{\infty} \alpha_m e_m, \varphi_n' \rangle = \alpha_n \)). It is easy to check that \( | \langle y, \varphi_n' \rangle | \leq K \| y \|_X \) for all \( y \in Y \).

We shall denote by \( \varphi_n \) a fixed norm-preserving extension of \( \varphi_n' \) to all of \( X \).

Define \( S \) on \( X \) by \( Sx = \sum_{n=1}^{\infty} \frac{1}{n} \langle x, \varphi_n \rangle e_n \). Note that for \( x \in X \),

\[
| \langle x, \varphi_n \rangle | \leq K \| x \|_X ,
\]

so that \( \frac{1}{n} \langle x, \varphi_n \rangle \) tends to zero as \( n \) tends to \( \infty \). Thus \( Sx \in Y \) for all \( x \in X \). Now
\[ \| Sx \|_X \leq K \| Sx \|_\infty \]
\[ = K \sup_n \frac{1}{n} < x, \varphi_n > \]
\[ \leq K^2 \sup_n \frac{1}{n} \| x \|_X \]

so \( S \) is a bounded operator on \( X \). Further, if \( g \) is a polynomial then
\[
g(S)x = \sum_{n=1}^{\infty} \left( g\left( \frac{1}{n} \right) - g(0) \right) < x, \varphi_n > e_n + g(0)x.
\]

Thus
\[
\| g(S)x \|_X \leq K \sup_n \frac{1}{n} \left| \left( g\left( \frac{1}{n} \right) - g(0) \right) < x, \varphi_n > \right| + \| g(0)x \|_X
\]
\[ \leq (2K^2 + 1) \sup_{[0,1]} |g(t)| \| x \|_X.
\]

By Theorem 3.4.2 then, \( S \) is a well-bounded operator which possesses a decomposition of the identity of bounded variation. However, as is shown in [Dow, Example 14.6], \( SY \) is not a scalar-type spectral operator. Note that \( \sigma(S|Y) = \mathbb{C} \{ 1/n : n = 1, 2, \ldots \} \subset \mathbb{R} \). Thus, by [Dow, Theorem 12.16] \( S \) is not a scalar-type spectral operator on \( X \).

\[ \square \]

\S 3.5. Examples

In this section we shall present some examples of well-bounded operators which will illustrate some of the theory given in this chapter.

3.5.1 Example. For \( \lambda \in [0,1) \) define \( E(\lambda) \in BL^1[0,1] \) by
\[
(E(\lambda)f)(t) = \begin{cases} f(t) & \text{for } t \in [0,\lambda) \\
1/(1-\lambda) \int_{\lambda}^{1} f(u) \, du & \text{for } t \in [\lambda,1). \end{cases}
\]

If we set \( E(\lambda) = 0 \) for \( \lambda < 0 \) and \( E(\lambda) = 1 \) for \( \lambda \geq 1 \), then it
is easily checked that \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) is strongly right continuous and has a strong left hand limit at each point in \( \mathbb{R} \). It is thus clear that \( \{E(\lambda)\} \) forms a spectral family on \( L^1[0,1] \) and so determines a unique well-bounded operator \( T \in B(L^1[0,1]) \). For \( f \in L^1[0,1] \) and \( \varphi \in L^\infty[0,1] \)

\[
< Tf, \varphi > = < f, \varphi > - \int_0^1 < E(\lambda) f, \varphi > d\lambda
\]

\[
= \int_0^1 f(t) \varphi(t) \, dt - \int_0^1 \left\{ \int_0^\lambda f(t) \varphi(t) \, dt + \int_\lambda^1 \frac{1}{1-\lambda} \int_0^1 f(u) \, du \, \varphi(t) \, dt \right\} d\lambda
\]

\[
= \int_0^1 f(t) \varphi(t) \, dt - \int_0^1 \int_0^\lambda f(t) \varphi(t) \, dt \, d\lambda
\]

\[
+ \int_0^1 \int_0^1 \frac{1}{1-\lambda} \chi_{[\lambda,1]}(u) \chi_{[\lambda,1]}(t) f(u) \varphi(t) \, du \, dt \, d\lambda.
\]

Let \( h(u,t,\lambda) = (1-\lambda)^{-1} \chi_{[\lambda,1]}(u) \chi_{[\lambda,1]}(t) f(u) \varphi(t) \). Our next step will be to apply Fubini's theorem to the last two integrals. It is not immediately clear however that \( h \) is integrable, so we shall proceed by verifying this. It suffices to show that \( |h| \) is integrable. By Tonelli's theorem

\[
\int_{[0,1]^3} |h| \, d\lambda
\]

\[
= \int_0^1 \left\{ \int_0^1 \left\{ \int_0^1 |h| \, du \right\} \, dt \right\} \, d\lambda
\]

\[
= \int_0^1 \frac{1}{1-\lambda} \left\{ \int_0^1 |f(u)| \chi_{[\lambda,1]}(u) \, du \right\} \left\{ \int_0^1 |\varphi(t)| \chi_{[\lambda,1]}(t) \, dt \right\} \, d\lambda
\]

\[
\leq \int_0^1 \frac{1}{1-\lambda} \left\| f \right\|_1 \left\| \varphi \right\|_\infty (1-\lambda) \, d\lambda
\]

\[
= \left\| f \right\|_1 \left\| \varphi \right\|_\infty .
\]

It follows that \( h \) is integrable, so applying Fubini's theorem gives that
$< Tf, \varphi >$

$$= \int_0^1 f(t)\varphi(t) \, dt - \int_0^1 \int_0^1 f(t)\varphi(t) \, d\lambda \, dt$$

$$- \int_0^1 \int_0^1 \min\{u,t\} \frac{1}{1-\lambda} f(u)\varphi(t) \, d\lambda \, du \, dt$$

$$= \int_0^1 f(t)\varphi(t) \, dt - \int_0^1 (1-t)f(t)\varphi(t) \, dt$$

$$- \int_0^1 \int_0^1 (-\log(1-\min\{u,t\})) f(u)\varphi(t) \, du \, dt$$

$$= \int_0^1 \left\{ tf(t) + \int_0^1 \log(1-\min\{u,t\}) f(u) \, du \right\} \varphi(t) \, dt.$$

This implies that $(Tf)(t) = tf(t) + \int_0^1 \log(1-\min\{u,t\}) f(u) \, du$ (a.e.). This gives an example of a well-bounded operator of type (B) on a non-reflexive Banach space. We shall now use Theorem 3.4.3 to show that $T$ is not scalar-type spectral. It is interesting to note that Corollary 5.2.2 will show that $T$ is scalar-type spectral on $L^p[0,1]$ for $1 < p < \infty$.

We shall show that there exist $f \in L^1[0,1]$, $\varphi \in L^\infty[0,1]$ and an increasing sequence $0 \leq \lambda_1 < \lambda_2 < \ldots < 1$ such that

$$\sum_{j=1}^n | < (E(\lambda_j)-E(\lambda_{j-1}))f, \varphi > | \rightarrow \infty \text{ as } n \rightarrow \infty.$$

It is slightly simpler to do this indirectly. Let

$$S_n = \sum_{j=1}^n (-1)^j (E(\lambda_j)-E(\lambda_{j-1})).$$

If we can find $\varphi \in L^\infty[0,1]$ such that $\lim_{n \rightarrow \infty} \| S_n^* \varphi \| = \infty$, then by the principle of uniform boundedness there exists $f \in L^1[0,1]$ such that

$$\sup_n | < S_n f, \varphi > | = \infty.$$ But

$$| < S_n f, \varphi > | = | < \sum_{j=1}^n (-1)^j (E(\lambda_j)-E(\lambda_{j-1}))f, \varphi > |$$

$$\leq \sum_{j=1}^n | < (E(\lambda_j)-E(\lambda_{j-1}))f, \varphi > |.$$
Thus \[ \lim_{n \to \infty} \sum_{j=1}^{n} | \langle E(\lambda_j) - E(\lambda_{j-1})f, \varphi \rangle | = \infty. \]

It is easy to check that
\[
(E^*(\lambda) \varphi)(t) = \begin{cases} 
\varphi(t) & \text{for } t \in [0, \lambda) \\
1/(1-\lambda) \int_{\lambda}^{1} \varphi(u) \, du & \text{for } t \in [\lambda, 1).
\end{cases}
\]

For \( j = 0, 1, 2, \ldots \), let \( \lambda_j = (3^j-1)/3^j \). Define \( \varphi \in L^\infty[0,1] \) by \( \varphi(t) = (-1)^k \) for \( t \in [\lambda_{k-1}, \lambda_k) \). Then
\[
\frac{1}{1-\lambda_j} \int_{\lambda_j}^{1} \varphi(u) \, du = 3^j \sum_{k=0}^{\infty} (-1)^{j+1} (2/3)^{j+1} (-1/3)^k 
= (-1)^{j+1} (2/3) (3/4) 
= \frac{1}{2} (-1)^{j+1}.
\]

Thus
\[
(E^*(\lambda_j) \varphi)(t) = \begin{cases} 
(-1)^m & \text{for } t \in [\lambda_{m-1}, \lambda_m), \ m < j \\
(-1)^{j+1}/2 & \text{for } t \in [\lambda_j, 1].
\end{cases}
\]

Therefore, for \( j \geq 1 \)
\[
((E^*(\lambda_j) - E^*(\lambda_{j-1})) \varphi)(t) = \begin{cases} 
0 & \text{for } t \in [0, \lambda_{j-1}) \\
(-1)^{j}/2 & \text{for } t \in [\lambda_{j-1}, \lambda_j) \\
(-1)^{j+1} & \text{for } t \in [\lambda_j, 1].
\end{cases}
\]

From this it follows that \( \| S_n^* \varphi \| = n \).

We have shown then that there exists \( f \) and \( \varphi \) such that the function \( \lambda \mapsto \langle E(\lambda)f, \varphi \rangle \) is not of bounded variation, and so \( T \) is not scalar-type spectral.

3.5.2. Example. Our second example, due to Ringrose [Rin2, § 6], is of a well-bounded operator on \( L^1[0,1] \) which is not of type (B) and which admits many different decompositions of the identity. This example will be presented in some detail as it illustrates the extra complications involved in considering decompositions of the identity. Define \( T \in B(L^1[0,1]) \) by
(Tf)(t) = tf(t) + \int_0^t f(u) \, du.

It is easy to see that if \( g \) is a polynomial then

\[
(g(T)f)(t) = g(t)f(t) + g'(t) \int_0^t f(u) \, du
\]

so \( \| g(T) \| \leq \sup_{t \in [0,1]} |g(t)| + \text{var}_g \)[0,1] \leq 2 \| g \|_1. \)

For \( f \in L^1[0,1] \) and \( \varphi \in L^\infty[0,1] \) we have

\[
< g(T)f, \varphi > = \int_0^1 \left\{ g(t)f(t) + g'(t) \int_0^t f(u) \, du \right\} \varphi(t) \, dt
\]

\[
= \int_0^1 g(1)f(t)\varphi(t) \, dt - \int_0^1 \int_0^1 g'(u) f(t)\varphi(t) \, dt \, du + \int_0^1 \int_0^t g'(t)f(u)\varphi(t) \, du \, dt
\]

\[
= \int_0^1 g(1)f(t)\varphi(t) \, dt - \int_0^1 \int_0^t g'(t)f(u)\varphi(u) \, du \, dt + \int_0^1 \int_0^t g'(t)f(u)\varphi(t) \, du \, dt
\]

(by Fubini's theorem)

\[
= g(1) \int_0^1 f(t)\varphi(t) \, dt
\]

\[
- \int_0^1 g'(\lambda) \left\{ \int_0^\lambda f(u)\varphi(u) \, du - \varphi(\lambda) \int_0^\lambda f(u) \, du \right\} d\lambda.
\]

Thus any decomposition of the identity \{E(\lambda)\}_{\lambda \in \mathbb{R}} will satisfy

\[
< f, E(\lambda)\varphi > = \int_0^\lambda \varphi(u)f(u) \, du - \varphi(\lambda) \int_0^\lambda f(u) \, du \quad \text{(a.e.)}
\]

for \( f \in L^1[0,1] \) and \( \varphi \in L^\infty[0,1] \). Formally, one would like to set

\[
(E(\lambda)\varphi)(t) = (\varphi(t)-\varphi(\lambda))\chi_{[0,\lambda]}(t),
\]

but this definition is dependent on the representative of the \( L^\infty \) equivalence class of \( \varphi \). To overcome this difficulty, Ringrose \cite{Ringrose2} showed that there exists a set of representatives which depend linearly at each point in \([0,1]\) on their equivalence classes and which possess the appropriate properties to ensure that we can define a projection on \( L^\infty \).
3.5.3. **Lemma** [Rin2, Lemma 4]. Let $\mathcal{F}$ be an ultrafilter on $(0,\infty)$ which converges to 0 in the usual topology on $\mathbb{R}$, and suppose that $u$ is an essentially bounded Lebesgue measurable function on $[0,1]$. Then for all $t \in [0,1)$,

$$u_{\mathcal{F}}(t) = \lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} u(s) \, ds$$

exists. Set $u_{\mathcal{F}}(1) = 0$. Then if $u, v$ and $w$ are essentially bounded Lebesgue measurable functions then

(i) $u_{\mathcal{F}} = u$ (a.e.), and if $u = v$ (a.e.) then $u_{\mathcal{F}} = v_{\mathcal{F}}$;

(ii) $\sup_{[0,1]} |u_{\mathcal{F}}(t)| = \|u\|_{\infty}$;

(iii) If $u$ is continuous on the right at $t \in [0,1)$, then $u_{\mathcal{F}}(t) = u(t)$;

(iv) If $\alpha$ and $\beta$ are scalars and $\alpha u + \beta v = w$ (a.e.), then $\alpha u_{\mathcal{F}} + \beta v_{\mathcal{F}} = w_{\mathcal{F}}$;

(v) If $w = uv$ (a.e.) and $v_{\mathcal{F}}$ is right continuous at $t \in [0,1)$ then $w_{\mathcal{F}}(t) = u_{\mathcal{F}}(t)v(t)$.

**Remark.** In Lemma 3.5.3, $u = v$ (a.e.) means that $u$ and $v$ are in the same $L^{\infty}[0,1]$ equivalence class, whereas $u = v$ means that $u$ and $v$ are identical as functions. For a proof of the Lemma see [Rin2] or [Dow, Lemma 15.13].

We shall now see how this Lemma allows us to construct a decomposition of the identity for $T$. For $\lambda \in [0,1)$, define $E(\lambda) \in B(L^{\infty}[0,1])$ by

$$(E(\lambda)\varphi)(t) = (\varphi(t) - \varphi(\lambda)) \chi_{[0,\lambda]}(t) \quad t \in [0,1].$$

Let $E(\lambda) = 0$ for $\lambda < 0$ and $E(\lambda) = 1$ for $\lambda \geq 1$. That $E(\lambda)$ is a
projection for \( \lambda \in [0,1) \) follows from the fact that if \( u(t) = 0 \) for \( t \in [\lambda,1] \), then \( u_{\lambda}(\lambda) = 0 \). Similarly it is easy to check that \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) satisfies conditions (i), (ii) and (iii) of Definition 3.3.1.

Fix \( f \in L^1[0,1] \) and \( \varphi \in L^\infty[0,1] \), and let

\[
\psi(\lambda) = \langle f, E(\lambda)\varphi \rangle = \int_0^\lambda f(t)\varphi(t) \, dt - \varphi_{\lambda}(\lambda) \int_0^\lambda f(t) \, dt.
\]

As the two integrals are continuous functions of \( \lambda \) and \( \varphi_{\lambda} \) is Lebesgue measurable, \( \psi \) must also be Lebesgue measurable, and so condition (iv) is satisfied. The final two conditions require a little more work however.

Suppose that \( g \in L^1[0,1] \) and that \( \gamma_g : L^\infty[0,1] \to L^\infty[0,1] \) is defined by

\[
(\gamma_g \varphi)(t) = \int_0^t g(u)\varphi(u) \, du - \varphi_{\lambda}(t) \int_0^t g(u) \, du.
\]

To check that condition (v) is satisfied we need to show that if the net of functions \( \{\varphi_\alpha\} \) has weak-* limit \( \varphi \) in \( L^\infty[0,1] \), then \( \{\gamma_g \varphi_\alpha\} \) has weak-* limit \( \gamma_g \varphi \). Suppose then that \( f \in L^1[0,1] \). Then

\[
\begin{align*}
\langle f, \gamma_g \varphi_\alpha \rangle &= \int_0^1 f(t) \left\{ \int_0^t g(u)\varphi_\alpha(u) \, du - (\varphi_\alpha)_{\lambda}(t) \int_0^t g(u) \, du \right\} \, dt \\
&= \int_0^1 \int_0^t f(t)g(u)\varphi_\alpha(u) \, du \, dt \\
&\quad - \int_0^1 \int_0^t f(t)g(u)(\varphi_\alpha)_{\lambda}(t) \, du \, dt \\
&= \left( \int_0^1 f(t)g(u) \, dt \right) \varphi_\alpha(u) \, du \\
&\quad - \int_0^1 \left( \int_0^t f(t)g(u) \, du \right) (\varphi_\alpha)_{\lambda}(t) \, dt.
\end{align*}
\]

Both of the functions \( h_1(u) = \int_0^1 f(t)g(u) \, dt \) and
\[ h_2(t) = \int_0^t f(t)g(u) \, du \text{ are in } L^1[0,1], \text{ so} \]

\[
\lim_{\alpha \in A} \langle f, \gamma g \varphi \rangle = \lim_{\alpha \in A} \left\{ \int_0^1 h_1(u)\varphi(u) \, du - \int_0^1 h_2(t)(\varphi_\alpha f(t)) \, dt \right\}
= \int_0^1 f(t)(\gamma g \varphi)(t) \, dt
= \langle f, \gamma g \varphi \rangle.
\]

We have thus verified that condition (v) holds.

Fix \( f \in L^1[0,1] \) and \( \varphi \in L^\infty[0,1] \). We need finally to show that if \( \eta: t \mapsto \int_0^t \langle f, E(\lambda)\varphi \rangle \, d\lambda \) is right differentiable at \( s \) then its derivative is \( \langle f, E(s)\varphi \rangle \). Now

\[
\eta(t) = \int_0^t \int_0^\lambda f(u)\varphi(u) \, du \, d\lambda - \int_0^\lambda \varphi(\lambda) \int_0^t f(u) \, du \, d\lambda
= \int_0^t h_3(\lambda) \, d\lambda - \int_0^t \varphi(\lambda)h_4(\lambda) \, d\lambda
\]
say. It is clear that \( h_3 \) and \( h_4 \) are continuous functions of \( \lambda \). Thus

\[
\frac{\eta(s+h)-\eta(s)}{h} = \frac{1}{h} \int_s^{s+h} h_3(\lambda) \, d\lambda - \frac{1}{h} \int_s^{s+h} \varphi(\lambda)h_4(\lambda) \, d\lambda.
\]

Hence if \( \eta \) is right differentiable at \( s \), then the value of that derivative must be

\[
(h_3)_T(s) - (\varphi_\lambda h_4)_T(\lambda) = h_3(s) - \varphi(\lambda)h_4(\lambda) \quad \text{(by the Lemma)}
= \langle f, E(s)\varphi \rangle.
\]

We have thus shown that \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) is a decomposition of the identity for \( T \). Note however that the functions \( \varphi_\lambda \) depend on our choice of ultrafilter. As is shown in [Rin2], we may construct a different decomposition of the identity for \( T \) by making a different choice of \( \mathcal{F} \).
As the reader can see, decompositions of the identity are much more difficult to work with than spectral families. Fortunately in most of our later work we shall mainly consider reflexive spaces, so all of the well-bounded operators will be of type (B).
CHAPTER 4  CONTRACTIVE PROJECTIONS ON $L^p$ SPACES

On a Hilbert space $\mathcal{H}$, the contractive projections are just the usual orthogonal, or self-adjoint, projections. One might hope then, that contractive projections on other Banach spaces may share some of the properties of orthogonal projections. Much work has been done characterising the contractive projections on $L^p$ spaces.

Following Grothendieck [Gr] and Douglas [Doug], Ando [An] showed that for $1 < p < \infty$, $p \neq 2$, such operators could be related to conditional expectation operators. This lead to the theorem of Dor and Odell [DO] which states that all monotone Schauder decompositions of $L^p$ spaces ($1 < p < \infty$) are unconditional.

This result holds for arbitrary (positive) measure spaces and for real or complex functions. The details for the more general case are somewhat sketchy in the literature however, so our aim in this chapter is to present a full account of these extensions. We shall end the chapter with a discussion of some interesting unsolved problems.

§ 4.1. Definitions and notation

Before we move on to considering $L^p$ spaces we shall introduce some general definitions.
4.1.1. Definition. Suppose that $X$ is a Banach space.

(i) A **contractive projection** on $X$ is an idempotent of norm at most 1. We shall denote the set of all contractive projections on $X$ by $\text{Proj}_1(X)$.

(ii) An **increasing family of projections** (on $X$) is a map $P: \mathbb{R} \to \text{Proj}(X)$ satisfying

$$P(\lambda)P(\mu) = P(\min\{\lambda, \mu\})$$

for $\lambda, \mu \in \mathbb{R}$.

(iii) An **increasing sequence of projections** is a finite or infinite sequence of projections $\{P_i\}$ satisfying

$$P_iP_j = P_{\min\{i, j\}}$$

for all $i$ and $j$. We shall allow infinite sequences indexed by either $\mathbb{Z}$ or $\mathbb{Z}^+$. For a sequence $\{P_j\}_{j=1}^\infty$ we shall employ the convention that $P_0 = 0$.

In what follows we shall often need to distinguish between real and complex spaces, so it is necessary to be a little more explicit with our notation. Let $(\Omega, \mathcal{A}, \mu)$ be a (positive) measure space. For $1 \leq p < \infty$ we shall denote by $L^p(\Omega, \mathcal{A}, \mu; \mathbb{R})$ (respectively $L^p(\Omega, \mathcal{A}, \mu; \mathbb{C})$) the usual Banach space of (equivalence classes of) real (respectively complex) valued $p$-integrable, $\mathcal{A}$-measurable functions on $\Omega$. $L^\infty(\Omega, \mathcal{A}, \mu; \mathbb{R})$ and $L^\infty(\Omega, \mathcal{A}, \mu; \mathbb{C})$ will denote the spaces of essentially bounded $\mathcal{A}$-measurable functions on $\Omega$. Where there is no possibility of confusion we may omit one or more of the parameters. We will use $L^p(\Omega, \mathcal{A}, \mu)$ whenever statements are valid for both real and complex spaces.

We shall also be considering finite dimensional $L^p$ spaces. Let $\Omega$ denote the $n$-point set $\{1, 2, \ldots, n\}$, which we shall equip with the
power set $\sigma$-algebra $P(\Omega_k)$ and the measure $\mu$ for which $\mu(\{k\}) = 1$, $k = 1, \ldots, n$. For $1 \leq p \leq \infty$ we shall write $\ell^p(n)$ for $L^p(\Omega_n, P(\Omega_n), \mu)$. Again we shall use the notation $\ell^p(n; \mathbb{R})$ or $\ell^p(n; \mathbb{C})$ if we need to distinguish between real and complex spaces.

If $\Omega_0 \in A$, we shall denote by $A|_{\Omega_0}$ the restriction of the $\sigma$-algebra $A$ to $\Omega_0$, and we shall identify $L^p(\Omega_0, A|_{\Omega_0}, \mu)$ with those functions in $L^p(\Omega, A, \mu)$ vanishing off $\Omega_0$.

§ 4.2. A theorem of Dor and Odell

The following theorem, due to Dor and Odell, shows that the contractive projections on real $L^p$ spaces ($1 < p < \infty$) possess a similar orthogonality property to those on $L^2$. For $1 < p < \infty$, let $p^* = \max\{p, p/(p-1)\}$.  

4.2.1. THEOREM ([DO, Theorem 2.1], [PR]). Let $(\Omega, A, \mu)$ be a measure space, let $1 < p < \infty$, and let $\{P_j\}_{j=1}^\infty$ be an increasing sequence of contractive projections on $L^p(\Omega, A, \mu; \mathbb{R})$. Then, for any sequence $\{a_j\}_{j=1}^\infty$ of real numbers with $|a_j| \leq 1$ and any $f \in L^p(\Omega, A, \mu; \mathbb{R})$, the series $\sum a_j (P_j - P_{j-1})f$ converges in norm and

$$\left\| \sum_{j=1}^\infty a_j (P_j - P_{j-1})f \right\| \leq (p^* - 1) \left\| f \right\|_p.$$

This theorem has a long history, going back to the work of Paley and Marcinkiewicz, who proved that the Haar basis is unconditional on $L^p$ (see [Sil, § 14; Pal; Har]). If $\{h_j\}_{j=1}^\infty$ is the normalised Haar basis for $L^p$ then the operators
Theorem 4.2.1 is no greater for the Haar system than for any other sequence of "martingale differences" with values in $L^P$. The proof of Theorem 4.2.1 given in [DO] combines two main ingredients. The first is showing that increasing sequences of contractive projections on $L^P$ spaces can be related to martingales; the second is Burkholder's theorem that martingale transforms are bounded on these $L^P$ spaces.

In [DO] Dor and Odell remark that Theorem 4.2.1 also holds on general measure spaces. In this chapter we shall supply the details of the reduction from a general measure space to a finite one, and show that the result also holds for complex spaces.

It should be noted that the result for complex $L^P$ spaces is not an immediate consequence of the real result. If we regard $L^P(\mu;\mathbb{C})$ as the real space $L^P(\mu;\mathbb{R}) \oplus L^P(\mu;\mathbb{R})$ then, because of the complex linearity requirements, we can write $P \in \text{Proj}_1(L^P(\mu;\mathbb{C}))$ as

$$
\begin{pmatrix}
A_1 & -A_2 \\
A_2 & A_1
\end{pmatrix}
$$

with $A_1, A_2 \in B(L^P(\mu;\mathbb{R}))$. The problem confronted is that $A_1$ and $A_2$ need not be projections. For example, $P = \begin{pmatrix} k_2 & -k_2 i \\ k_2 i & k_2 \end{pmatrix}$ acting on $L^P(2;\mathbb{C})$ has $A_1 = \begin{pmatrix} k_2 & 0 \\ 0 & k_2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & -k_2 \\ k_2 & 0 \end{pmatrix}$. We shall see below, however, that contractive projections on complex $L^P$ spaces can be "straightened out" to a form where $A_1 \in \text{Proj}_1(L^P(\mu;\mathbb{R}))$ and $A_2 = 0$. 

$P_n(\sum_{j=1}^{\infty} a_j h_j) = \sum_{j=1}^{n} a_j h_j$ are contractive projections for each $n$, and the sequence $\{P_j\}_{j=1}^{\infty}$ is clearly increasing. The importance of this example was shown by Maurey [Maur] who showed that the constant required in Theorem 4.2.1 is no greater for the Haar system than for any other sequence of "martingale differences" with values in $L^P$. 

$\begin{pmatrix}
A_1 & -A_2 \\
A_2 & A_1
\end{pmatrix}$ with $A_1, A_2 \in B(L^P(\mathbb{R};\mathbb{R}))$. The problem confronted is that $A_1$ and $A_2$ need not be projections. For example, $P = \begin{pmatrix} k_2 & -k_2 i \\ k_2 i & k_2 \end{pmatrix}$ acting on $L^P(2;\mathbb{C})$ has $A_1 = \begin{pmatrix} k_2 & 0 \\ 0 & k_2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & -k_2 \\ k_2 & 0 \end{pmatrix}$. We shall see below, however, that contractive projections on complex $L^P$ spaces can be "straightened out" to a form where $A_1 \in \text{Proj}_1(L^P(\mu;\mathbb{R}))$ and $A_2 = 0$. 

$P_n(\sum_{j=1}^{\infty} a_j h_j) = \sum_{j=1}^{n} a_j h_j$ are contractive projections for each $n$, and the sequence $\{P_j\}_{j=1}^{\infty}$ is clearly increasing. The importance of this example was shown by Maurey [Maur] who showed that the constant required in Theorem 4.2.1 is no greater for the Haar system than for any other sequence of "martingale differences" with values in $L^P$. The proof of Theorem 4.2.1 given in [DO] combines two main ingredients. The first is showing that increasing sequences of contractive projections on $L^P$ spaces can be related to martingales; the second is Burkholder's theorem that martingale transforms are bounded on these $L^P$ spaces. In [DO] Dor and Odell remark that Theorem 4.2.1 also holds on general measure spaces. In this chapter we shall supply the details of the reduction from a general measure space to a finite one, and show that the result also holds for complex spaces.
§ 4.3. The reduction to finite measure spaces

The reduction from arbitrary measure spaces is essentially due to Tzafriri [Tz]. We shall show in this section that his methods allow us to work with a sequence of projections simultaneously. The first step is to reduce from an arbitrary measure space to a σ-finite measure space. The following lemma is a slight generalisation of [Tz, Lemma 1] where the result is proved for a single operator (see also [DS1, III.8.5]).

4.3.1. Lemma. Suppose that \((\Omega, \mathcal{A}, \mu)\) is an arbitrary measure space, that \(1 < p < \infty\), and that \(f \in L^p(\Omega, \mathcal{A}, \mu)\). Suppose also that 
\[
\{P_j\}^\infty_{j=1}
\]
is an increasing sequence of contractive projections on 
\(L^p(\Omega, \mathcal{A}, \mu)\). Then there exists \(\Omega' \in \mathcal{A}\) and a sub-σ-ring \(\mathcal{A}'\) of \(\mathcal{A}|\Omega'\) such that

(i) \((\Omega', \mathcal{A}', \mu)\) is σ-finite;

(ii) \(f \in L^p(\Omega', \mathcal{A}', \mu)\);

(iii) \(L^p(\Omega', \mathcal{A}', \mu)\) is invariant under \(P_j\) for \(j = 1, 2, \ldots\).

Proof. Let \(g_n = \Sigma_{i=1}^{m_n} a_{n,i} \chi_{\mathcal{A}_n,i}\) be a sequence of simple functions whose \(L^p\) closure contains \(f\). By simple functions we are requiring that \(\mu(\mathcal{A}_{n,i}) < \infty\) for all \(n\) and \(i\). Let 
\[
\mathcal{A}_1 = \{ A_{n,i} : i = 1, \ldots, m_n ; n = 1, 2, \ldots \}
\]
and \(\mathcal{B}_1\) be the smallest subring of \(\mathcal{A}\) containing \(\mathcal{A}_1\). By a similar construction to that used in the proof of [DS1, III.8.4], one can show that \(\mathcal{B}_1\) is countable and \(\mu(\mathcal{A}) < \infty\) for all \(\mathcal{A} \in \mathcal{B}_1\). It is also
clear that
\[ C_1 = \{ P_j \chi_B : B \in B_1 \} \subseteq L^P(\Omega, \mathcal{A}, \mu) \]
is countable.

We shall proceed to define \( \mathcal{A}_{k+1}, B_{k+1} \) and \( C_{k+1} \) by recursion. Suppose that we have constructed the countable collections of sets \( \mathcal{A}_k \) and \( B_k \), and the countable subset of \( L^P(\Omega, \mathcal{A}, \mu) \), \( C_k \). Let
\[ h_n = \sum_{i=1}^{r_n} b_{n,i} \chi_{B_{n,i}} \]
be a sequence of simple functions whose \( L^P \)
closure contains \( C_k \). We define
\[ \mathcal{A}_{k+1} = \{ B_{n,i} : i = 1, \ldots, r_n; n = 1, 2, \ldots \} \]
and \( B_{k+1} \) to be the subring generated by \( \mathcal{A}_{k+1} \) and \( B_k \). As above we can show that \( \mathcal{A}_{k+1} \) and \( B_{k+1} \) are both countable collections and that
\[ \mu(B) < \infty \text{ for all } B \in B_{k+1} \]
We can define then
\[ C_{k+1} = \{ P_j \chi_B : B \in B_{k+1}, j = 1, 2, \ldots \} \]
We thus obtain an increasing sequence of countable subrings
\[ B_1 \subseteq B_2 \subseteq \cdots \]
and an increasing sequence of collections of functions
\[ C_1 \subseteq C_2 \subseteq \cdots \subseteq L^P \]. Let \( B = \bigcup_{k=1}^{\infty} B_k \), \( \Omega' = \bigcup_{A \in B} A \) and let \( \mathcal{A}' \) be the sub-\( \sigma \)-ring generated by \( B \). Since \( \mu(A) < \infty \) for all \( A \in B \), it follows that \( (\Omega', \mathcal{A}', \mu) \) is \( \sigma \)-finite. Statement (ii) is just an immediate consequence of the fact that \( B_1 \subseteq \mathcal{A}' \).

It just now remains for us to prove (iii). Suppose that \( j \in \mathbb{N} \) and that \( A \in B_k \) for some \( k \). Now \( P_j \chi_A \in C_k \), so it can be approximated (in \( L^P \) norm) by simple functions over \( B_{k+1} \subset \mathcal{A}' \). Thus \( P_j \) maps simple functions over \( B \) into \( L^P(\Omega', \mathcal{A}', \mu) \). But the simple function over \( B \) are dense in \( L^P(\Omega', \mathcal{A}', \mu) \) (see [DS1, 11.8.3]), so it follows that \( P_j \) leaves \( L^P(\Omega', \mathcal{A}', \mu) \) invariant. \( \Box \)
Passing from $\sigma$-finite to finite measures is standard (see [Tz, Lemma 2]).

4.3.2. Lemma. Suppose that $(\Omega, \mathcal{A}, \mu)$ is a $\sigma$-finite measure space and that $1 < p < \infty$. Suppose also that $\{P_j\}_{j=1}^\infty$ is an increasing sequence of contractive projections on $L^p(\Omega, \mathcal{A}, \mu)$. Then there exists a finite measure $\nu$ on $(\Omega, \mathcal{A})$ and an invertible isometry

$$S : L^p(\Omega, \mathcal{A}, \mu) \to L^p(\Omega, \mathcal{A}, \nu)$$

such that $Q_j = SP_jS^{-1}$, $j = 1, 2, \ldots$, is an increasing sequence of contractive projections on $L^p(\Omega, \mathcal{A}, \nu)$.

Proof. We may split $\Omega$ up as a countable disjoint union $\Omega = \bigcup_{i \in \mathbb{N}} \Omega_i$, with $0 < m_i = \mu(\Omega_i) < \infty$ for each $i \in \mathbb{N}$. Choose positive real numbers $a_i$ ($i \in \mathbb{N}$) such that $\sum_{i \in \mathbb{N}} 1/a_i = 1$.

For $x \in \Omega_i$, define $k(x) = (a_i m_i)^{-1/p}$. Then $\|k\|_p = 1$ in $L^p(\Omega, \mathcal{A}, \mu)$ and $k$ is non-zero for all $x \in \Omega$. Define $\nu$ on $(\Omega, \mathcal{A})$ by $\nu(A) = \int_A k^p \, d\mu$, and $S : L^p(\Omega, \mathcal{A}, \mu) \to L^p(\Omega, \mathcal{A}, \nu)$ by $(Sf)(x) = f(x)/k(x)$. As is easily checked, $S$ is an invertible isometry. Easy calculations show that each of the operators $Q_j$ is a contractive projection.

Combining Lemmas 4.3.1 and 4.3.2 gives the following.

4.3.3. Lemma. Suppose that $(\Omega, \mathcal{A}, \mu)$ is an arbitrary measure space, that $1 < p < \infty$, and that $f \in L^p(\Omega, \mathcal{A}, \mu)$. Suppose also that $\{P_j\}_{j=1}^\infty$ is an increasing sequence of contractive projections on $L^p(\Omega, \mathcal{A}, \mu)$. Then there exists $\Omega' \in \mathcal{A}$, a sub-$\sigma$-ring $\mathcal{A}'$ of $\mathcal{A}|\Omega'$, a finite measure $\nu$ on $(\Omega', \mathcal{A}')$ and an invertible isometry
$S: L^P(\Omega', A', \mu) \to L^P(\Omega', A', \nu)$ such that

(i) $f \in L^P(\Omega', A', \mu)$;

(ii) $L^P(\Omega', A', \mu)$ is invariant under $P_j$ for $j = 1, 2, \ldots$;

(iii) For $j = 1, 2, \ldots$, $Q_j = S(P_j \mid L^P(\Omega', A', \mu))S^{-1}$ is a contractive projection on $L^P(\Omega', A', \nu)$.

§ 4.4. Conditional expectation operators

4.4.1. Definition. Suppose that $(\Omega, A, \mu)$ is a finite measure space, and that $A_0$ is a sub-$\sigma$-ring of $A$. The greatest element of $A_0$ is defined to be $\Omega_0 = \bigcup_{A \in A_0} A$. The conditional expectation $E f = E(f \mid A_0, \mu)$ of $f \in L^1(\Omega, A, \mu)$ with respect to $A_0$ and $\mu$ is defined to be the unique $g \in L^1(\Omega_0, A_0, \mu)$ satisfying $\int_A g \, d\mu = \int_A f \, d\mu$ for all $A \in A_0$.

The existence of such a $g$ is guaranteed by the Radon Nikodym theorem. As is well-known, conditional expectation operators are contractive projections of $L^P(\Omega, A, \mu)$ onto $L^P(\Omega_0, A_0, \mu)$ for every $p \geq 1$. Note that we may consider a conditional expectation operator as acting on either the real or complex $L^P$ spaces. The reader is directed to [Ste, Chapter 4] or [EG, Chapter 5] for background on these operators.

In [Am] Ando characterised contractive projections on $L^P(\Omega, A, \mu)$ ($1 < p < \infty$, $p \neq 2$, $\mu(\Omega) \leq \infty$) in terms on conditional expectation operators. He first showed that the range of a contractive
projection, \( P \), on such an \( L^p \) space contains a positive function, \( k \) say, of maximal support (i.e. a function whose support contains the supports of all other functions in the range of \( P \)). It is then a relatively easy task to show that the operator
\[
Qf = \frac{P(kf)}{k}
\]
is a contractive projection on \( L^p(\Omega, \mathcal{A}, \nu) \), where \( \nu \) is the measure given by \( \nu(A) = \int_A k^p \, d\mu \). To deal with increasing sequences of contractive projections we need the following generalisation due to Dor and Odell.

4.4.2. THEOREM ([DO, Theorem 2.1]). Suppose that \((\Omega, \mathcal{A}, \mu)\) is a finite measure space, that \( 1 < p < \infty \), \( p \neq 2 \), and that \( \{Q_j\}_{j=1}^{\infty} \) is an increasing sequence of contractive projections on \( L^p(\Omega, \mathcal{A}, \mu) \). Then there is a sequence of sub-\( \sigma \)-rings of \( \mathcal{A} \), \( \mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots \subset \mathcal{A} \), a finite measure \( \nu \) on \((\Omega, \mathcal{A})\) and an invertible isometry \( U \) of \( L^p(\Omega, \mathcal{A}, \mu) \) onto \( L^p(\Omega, \mathcal{A}, \nu) \) such that
\[
Q_j = U^{-1} E_j U \quad j = 1, 2, \ldots
\]
where \( E_j \) is the conditional expectation operator with respect to \( \mathcal{A}_j \) and \( \nu \).

It should be noted that this result is valid for both real and complex spaces.

The final part of the jigsaw is the following deep result due to Burkholder and Gundy, which states that martingale difference sequences are unconditional on \( L^p(\Omega, \mathcal{A}, \mu; \mathbb{R}) \) if \( \mu(\Omega) < \infty \) and \( 1 < p < \infty \). Suppose that \( \mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots \) are sub-\( \sigma \)-algebras of \( \mathcal{A} \) and
that $\mathbb{E}_j$ is the conditional expectation operator with respect to $\mathcal{A}_j$.

For notational convenience we shall always set $\mathbb{E}_0 = 0$. For $f \in L^p(\Omega,\mathcal{A},\mu)$ and $j = 1,2,\ldots$, let $f_j = \mathbb{E}_j f$. The sequence $\{f_j\}_{j=1}^{\infty}$ forms what is known as a martingale. The martingale difference sequence associated with $\{f_j\}$ is the sequence $d_j = f_j - f_{j-1}$, $j = 1,2,\ldots$. Suppose that $\{a_j\}_{j=1}^{\infty}$ is a sequence of scalars with $|a_j| \leq 1$ for each $j$. The sequence $g_n = \sum_{j=1}^{n} a_j d_j$ is known as the martingale transform of $f$ with respect to $\{\mathcal{A}_j\}$ and $\{a_j\}$. Burkholder's theorem states that for $f \in L^p(\Omega,\mathcal{A},\mu;\mathbb{R})$, $\|g_n\|_p \leq (p^*-1) \|f\|_p$ for all $n$.

The original proof (which is probabilistic in nature) that $\|g_n\|_p$ is bounded is given in [Burk1] (see also [BGum]). Burkholder later showed that this bound is $(p^*-1) \|f\|_p$ and gave a more elementary proof of the unconditionality of the difference sequence (see [Burk2, Burk3]). In view of Theorem 4.4.2, it is necessary to recast Burkholder's theorem in terms of $\sigma$-rings.

4.4.3. THEOREM. Suppose that $(\Omega,\mathcal{A},\mu)$ is a finite measure space, that $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots$ is an increasing sequence of sub-$\sigma$-rings of $\mathcal{A}$, and that $1 < p < \infty$. Let $\mathbb{E}_j$ denote the conditional expectation with respect to $\mathcal{A}_j$ and $\mu$. Then for any sequence of real numbers $\{a_j\}$ with $|a_j| \leq 1$,

$$\|\sum_{j=1}^{n} a_j (\mathbb{E}_j - \mathbb{E}_{j-1}) f\|_p \leq (p^*-1) \|f\|_p$$

for all $f \in L^p(\Omega,\mathcal{A},\mu;\mathbb{R})$.
Proof. [DG, Corollary 2.2].

The following extension to complex $L^p$ spaces is probably well-known, but little about complex martingale transforms appears in the literature.

4.4.4. THEOREM. Suppose that $(\Omega, \mathcal{A}, \mu)$ is a finite measure space, that $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots$ is an increasing sequence of sub-$\sigma$-rings of $\mathcal{A}$, and that $1 < p < \infty$. Let $E_j$ denote the conditional expectation with respect to $\mathcal{A}_j$ and $\mu$. Then for any sequence of complex numbers $\{a_j\}$ with $|a_j| \leq 1$,

$$\left\| \sum_{j=1}^{n} a_j (E_j - E_{j-1}) f \right\|_p \leq 2(p^*-1) \left\| f \right\|_p$$

for all $n \in \mathbb{N}$.

Proof. We have

$$\sum_{j=1}^{n} a_j (E_j - E_{j-1}) = \sum_{j=1}^{n} (\text{Re } a_j)(E_j - E_{j-1}) + i \sum_{j=1}^{n} (\text{Im } a_j)(E_j - E_{j-1})$$

$$= T_1 + iT_2 \text{ say.}$$

An important property of conditional expectation operators is that they map real functions to real functions and imaginary functions to imaginary functions. Indeed the conditional expectation with respect to a sub-$\sigma$-ring $\mathcal{A}_0 \subset \mathcal{A}$, on $L^p(\Omega, \mathcal{A}, \mu; \mathbb{C})$ is just the natural complexification of the conditional expectation with respect to $\mathcal{A}_0$ on the corresponding real space. Thus $T_1$ is the natural complexification of the operator $U_1 = \sum_{j=1}^{n} (\text{Re } a_j)(E_j - E_{j-1})$ acting on $L^p(\Omega, \mathcal{A}, \mu; \mathbb{R})$. By Theorem 4.4.3, $\left\| U_1 \right\| \leq p^*$ so, as complexification does not increase norms (see [FIP, Corollary 1.3]),
\[ \| T_1 \| \leq p^*-1 \] as well. The same clearly holds for \( T_2 \). Thus
\[ \left\| \sum_{j=1}^{n} a_j (E_j - E_{j-1}) \right\| \leq 2(p^*-1). \]

\[ \square \]

Remark. Burkholder [Burk2, Theorem 15.1] has shown that the constant \( p^*-1 \) in Theorem 4.4.3 is sharp (i.e. it is the minimum value for which the theorem is true for all measure spaces). We can improve on the bound \( 2(p^*-1) \) in the complex case by noting that if \( \{a_j\} \) and \( \{E_j\} \) are as above, then
\[ \left\| \sum_{j=1}^{n} a_j (E_j - E_{j-1}) \right\| \leq 1 \]
on \( L^2(\mu;\mathbb{C}) \). The Riesz-Thorin interpolation theorem [BL, Theorem 1.1.1] then gives an improved bound near \( p = 2 \). To be more precise, one can show that for \( r \in (1,2) \) and \( p \in (r,2) \),
\[ \left\| \sum_{j=1}^{n} a_j (E_j - E_{j-1}) \right\|_p \leq \left( \frac{2}{r-1} \right)^{(r(p-2))/(r-2)p} \]
If \( p \) is fixed, \( r \) may be varied to minimise the right hand side of this equation. One can show that there exists a value \( r_0 \) (approximately 1.213) for which the above expression gives an improvement on the bound given in Theorem 4.4.4 for all \( p > r_0 \). The minimum value of this expression is achieved by choosing \( r = r_0 \); we shall spare the reader the tedious calculations. It has been conjectured that the sharp complex bound is also given by \( p^*-1 \), but this question is still open (see [Pe12, § 2] for a related question).
§ 4.5. Unconditionality for monotone decompositions on complex $L^p$ spaces

We are now in a position to prove the extended version of Theorem 4.2.1.

4.5.1. THEOREM. For each $p$ with $1 < p < \infty$, there exists a constant $K_p$ such that if $(\Omega, \mathcal{A}, \mu)$ is any measure space, $\{P_j\}_{j=1}^{\infty}$ is a increasing sequence of contractive projections on $L^p(\Omega, \mathcal{A}, \mu)$, and $\{a_j\}_{j=1}^{\infty}$ is any sequence of scalars with $|a_j| \leq 1$, then the series $\sum_j a_j (P_j - P_{j-1}) f$ converges in norm with

$$\left\| \sum_{j=1}^{\infty} a_j (P_j - P_{j-1}) f \right\|_p \leq K_p \| f \|_p$$

Furthermore, $K_p \leq 2(p^* - 1)$.

Proof. For $p = 2$, the result follows easily from general Hilbert space theory (with $K_p = 1$).

Suppose now that $p \neq 2$. Fix $f \in L^p(\Omega, \mathcal{A}, \mu)$ and let $(\Omega', \mathcal{A}', \nu)$ and $S$ be as constructed in Lemma 4.3.3. By that Lemma, $Q_j = S(P_j |L^p(\Omega', \mathcal{A}', \mu)) S^{-1}$ is a contractive projection on $L^p$ of a finite measure space. Thus, by Theorem 4.4.2,

$$P_j f = S^{-1} U^{-1} \sum_j U S f$$

for some invertible isometry $U$ and an increasing sequence of conditional expectation operators $\{E_j\}$. Clearly then,

$$\sum_{j=1}^{n} a_j (P_j - P_{j-1}) f = S^{-1} U^{-1} \left( \sum_{j=1}^{n} a_j (E_j - E_{j-1}) \right) U S f$$

and so

$$\left\| \sum_{j=1}^{n} a_j (P_j - P_{j-1}) f \right\|_p = \left\| \sum_{j=1}^{n} a_j (E_j - E_{j-1}) U S f \right\|_p.$$
Theorems 4.4.3 and 4.4.4 show that $2(p^*-1)$ is an absolute bound on the norm of operators of the form $\sum_{j=1}^{n} b_j(E_j - E_{j-1})$, where $\{b_j\}$ is a sequence with $|b_j| \leq 1$. This shows that the series $\sum (P_j - P_{j-1})f$ is weakly unconditionally Cauchy. As our spaces are reflexive, a theorem of Bessaga and Pelczynski [BP, Theorem 5] implies that $\sum a_j(P_j - P_{j-1})f$ converges (in norm) for any sequence $\{a_j\}$ with $|a_j| \leq 1$. Clearly then,

$$\left\| \sum_{j=1}^{\infty} a_j(P_j - P_{j-1})f \right\|_p \leq \sup_n \left\| \sum_{j=1}^{n} a_j(P_j - P_{j-1})f \right\|_p \leq 2(p^*-1) \left\| f \right\|_p$$

and the proof is complete. \(\square\)

§ 4.6 Some open questions

As discussed in § 4.4, the sharp constant appearing in Theorem 4.5.1 for complex $L^P$ spaces is unknown. Rather than try to calculate the smallest constant which holds for all measure spaces, one might try to calculate the sharp constant for a particular measure space $(\Omega, \mathcal{A}, \mu)$. The simplest case is to consider measure spaces which contain only a finite number of points.

4.6.1. Definition. Suppose that $X$ is a Banach space. Define

$$K(X) = \sup_{\{P_j\}} \sup_{\{a_j\}} \sup_n \left\| \sum_{j=1}^{n} a_j(P_j - P_{j-1}) \right\|$$

where the supremum is taken over all increasing sequences of contractive projections and all scalars $a_j$ with $|a_j| \leq 1$. 
Proposition 4.6.2 is an easy fact which we shall use repeatedly.

4.6.2. PROPOSITION. (i) $K(X) \leq K(X^*)$.

(ii) If $X$ is reflexive then $K(X) = K(X^*)$.

Proof. (i) If $\{P_j\}$ is an increasing sequence of contractive projections on $X$ then $\{P_j^*\}$ is an increasing sequence of contractive projections on $X^*$.

(ii) $K(X) \leq K(X^*) \leq K(X^{**}) = K(X)$.

For certain $p$, the calculation of $K(\ell^p(n))$ is easy; $K(\ell^2(n))$ is clearly 1 for all $n$ as $\ell^2(n)$ is a Hilbert space. The projections

$$P_j(x_1, \ldots, x_n) = (x_1, \ldots, x_{j-1}, \sum_{k=j}^{n} x_k, 0, \ldots, 0)$$

are all contractive on $\ell^1(n)$; and as

$$\left\| \sum_{j=1}^{n} (-1)^j (P_j - P_{j-1})(0, \ldots, 0, 1) \right\|_1 = 2n-1$$

the triangle inequality shows that $K(\ell^1(n)) = 2n-1$. Note that this does not depend on the scalar field. It is immediate that $K(\ell^\infty(n)) = 2n-1$ as well.

For $p \neq 1, 2, \infty$, the situation is rather less clear. Let $p'$ denote the conjugate index to $p$ (i.e. $1/p + 1/p' = 1$). Define

$$r(p) = \begin{cases} 
(2-p)/p & \text{for } 1 \leq p \leq 2 \\
(2-p')/p' & \text{for } 2 < p \leq \infty.
\end{cases}$$

The following theorem gives an upper bound on the constants $K(\ell^p(n))$. 
4.6.3. PROPOSITION. Suppose that $1 \leq p \leq \infty$. Then

$$K(\ell^p(\mathbb{R})) \leq K(\ell^p(\mathbb{C})) \leq (2n-1)^{r(p)}.$$ 

Proof. The first inequality is an easy consequence of the fact that complexification of operators on $L^p$ spaces does not increase their norm (and so each contractive projection on $\ell^p(\mathbb{R})$ induces one on $\ell^p(\mathbb{C})$). For $p = 1, 2, \infty$, the second has been shown above. Suppose then that $1 < p < 2$, and that $(P_j)$ is an increasing sequence of contractive projections on $\ell^p(\mathbb{C})$. By Theorem 4.4.2 we know that there is a second measure $\nu$ on $\Omega_n$ such that each $P_j$ is simultaneously and isometrically similar to a conditional expectation operator $E_j$ on $L^p(\Omega_n, \nu)$. Thus

$$K(\ell^p(\mathbb{C})) \leq \sup_{\nu} \sup_{\{E_j\}} \sup_{\{a_j\}} \max \left\| \Sigma a_j (E_j - E_{j-1}) \right\|_p,$$

where the supremum is taken over all finite measures $\nu$ on $\Omega_n$, all sequences of conditional expectation operators $(E_j)$ on $L^p(\Omega_n, \nu)$ and all scalars $a_j$ with $|a_j| \leq 1$. As conditional expectation operators are defined on $L^p(\Omega_n)$ for all $p$, if we fix $\nu$, $(E_j)$ and $\{a_j\}$ then the Riesz-Thorin interpolation theorem will imply that

$$\max \left\| \Sigma a_j (E_j - E_{j-1}) \right\|_p \leq \max \left\| \Sigma a_j (E_j - E_{j-1}) \right\|_1 \max \left\| \Sigma a_j (E_j - E_{j-1}) \right\|_2^{1-t},$$

where $1/p = t/1 + (1-t)/2$, or $t = (2-p)/p$. As

$$\max \left\| \Sigma a_j (E_j - E_{j-1}) \right\|_2 = 1,$$

it suffices to show that

$$\sup_{\nu} \sup_{\{E_j\}} \sup_{\{a_j\}} \max \left\| \Sigma a_j (E_j - E_{j-1}) \right\|_1 = 2n-1.$$

Fix $\epsilon \in (0, 1)$. Set $\nu(\{1\}) = \epsilon^{n-1}$, $\nu(\{2\}) = \epsilon^{n-2} - \epsilon^{n-1}$, ..., $\nu(\{n\}) = 1 - \epsilon$. If we let $x_0 = (\epsilon^{1-n}, 0, \ldots, 0)$, then $\|x_0\|_1 = 1$ (in $L^1(\nu)$). Define $\sigma$-algebras

$$\mathcal{A}_j = \left\{ A \in \mathcal{P}(\Omega_n) : A \cap \{1, 2, \ldots, j\} = \emptyset \text{ or } A \cap \{1, 2, \ldots, j\} = \{1, 2, \ldots, j\} \right\}$$
and conditional expectation operators $E_j = E(\cdot | A_j, \nu)$. It is easy to see that $E_j x_0 = (\epsilon^{j-n}, \ldots, \epsilon^{j-n}, 0, \ldots, 0)$ (where there are $j$ non-zero elements) and so

$$\sum_{j=1}^{n} (-1)^j (E_j - E_{j-1}) x_0 = (\epsilon^{1-n} + (-1)^2 (\epsilon^{2-n} - \epsilon^{1-n}) + \ldots + (-1)^n (1-\epsilon),$$

$$(-1)^2 (\epsilon^{2-n} + (-1)^3 (\epsilon^{3-n} - \epsilon^{2-n}) + \ldots + (-1)^n (1-\epsilon),$$

$$\ldots, (-1)^{n-1} (\epsilon^{-1}) + (-1)^n (1-\epsilon),$$

$$(-1)^n 1 )$$

$$(y_1, \ldots, y_n) \text{ say.}$$

We have then that $\nu({\{1\}}) |y_1| = |2 + g_1(\epsilon)|$, $\nu({\{2\}}) |y_2| = |2 + g_2(\epsilon)|$, $\ldots$, $\nu({\{n\}}) |y_n| = |1 + g_n(\epsilon)|$, where each $g_j$ is a polynomial whose coefficients depend only on $n$, and whose constant coefficients are zero. It is clear that

$$\sum_{j=1}^{n} \nu(A_j) (E_j - E_{j-1}) a_j \to 2n-1 \text{ as } \epsilon \to 0. \text{ Again, the triangle inequality ensures that sup \sup \sup \sum_{j=1}^{n} \nu(A_j) (E_j - E_{j-1}) a_j \to 2n-1.}$$

The result for $2 < p < \infty$ follows by duality.

It should be noted that the bound given in Proposition 4.6.3 is not sharp (except for $p = 1, 2, \infty$). Indeed for large $n$ and $p$ near 2, $(2n-1)^{r(p)} > (p^+ - 1)$, so the general bounds found in § 4.5 are better. Taking the largest log-convex function smaller than $\min\{(2n-1)^{r(p)}, (p^+ - 1)\}$ obviously gives a further improvement. In general however, it seems very difficult to give an exact value for $K(q^p(n))$ - even for $n = 2$. Diagram 4.6.4 shows the functions $(2n-1)^{r(p)}$ and $(p^+ - 1)$ for $n = 2$ and $1 \leq p \leq 2$.\end{document}
Diagram 4.6.4.

A second line of investigation leads one to consider vector-valued $L^p$ spaces. We shall denote the Lebesgue-Bochner space of (equivalence classes of) $p$-integrable, $\mathcal{A}$-measurable, $X$-valued functions on $(\Omega,\mathcal{A},\mu)$ by $L^p(\Omega,\mathcal{A},\mu;X)$. These spaces are equipped with the usual norms

$$
\| f \|_p = \left( \int_\Omega \| f(\omega) \|_p^p \, d\mu \right)^{1/p}
$$

for $1 < p < \infty$,

$$
\| f \|_\infty = \text{ess sup} \| f(\omega) \|_{\omega \in \Omega}
$$

We refer the reader to [DU, chapter II] for more background on these Banach spaces. Little seems to be known about the general form of contractive projections on Lebesgue-Bochner spaces, even when $X$ is finite dimensional. For example the answer to the following question is unknown (except for the trivial case $p = 2$).

4.6.5. Question. Is $K(L^p([0,1];\mathbb{R}^2)) < \infty$ for $1 < p < \infty$.

Sundaresan [Sund1, Sund2] has shown that despite the fact that the Radon Nikodym theorem does not extend to all Banach spaces, one
can define conditional expectation operators on Lebesgue-Bochner spaces.

4.6.6. **Theorem** ([Sundi, Proposition 4]). Suppose that $X$ is a Banach space, that $(\Omega, \mathcal{A}, \mu)$ is a finite measure space and that $\mathcal{A}_0$ is a sub-$\sigma$-ring of $\mathcal{A}$. Then there exists a unique $E \in \text{Proj}_1(L^p(\Omega, \mathcal{A}, \mu; X))$ such that for all $f \in L^p(\Omega, \mathcal{A}, \mu; X)$ and all $A \in \mathcal{A}_0$, 
\[ \int_A Ef \, d\mu = \int_A f \, d\mu. \] We shall call $E$ the conditional expectation operator with respect to $\mathcal{A}_0$ and $\mu$.

One can define vector-valued martingales and martingale transforms on $L^p([0,1]; X)$, just as one does in the scalar case.

4.6.7. **Definition.** A Banach space $X$ is said to have **UMD** (the unconditionality property for martingale differences), or be a **UMD** space, if, for $1 < p < \infty$, there is an absolute bound (depending only on $p$) on the norms of martingale transforms on $L^p([0,1]; X)$.

**Remark.** As is now well-known, it actually suffices to check that martingale transforms are bounded (with an absolute bound on the norm) on just one of the spaces $L^p([0,1]; X)$ with $1 < p < \infty$ (see, for example, [Boug]).

In recent years there has been much interest in these spaces. See, for example [BGN, Introduction]. The class of UMD spaces includes many of the classical reflexive Banach spaces, including $L^p$ spaces ($1 < p < \infty$) and the von Neumann-Schatten $p$-classes.
(1 < p < \infty). It is clear that for \( K(L^p([0,1];X)) \) to be finite, \( X \) must be UMD. However, if we are considering the behaviour of general contractive projections on \( L^p([0,1];X) \) we must at least take into account the behaviour of the \( X \)-valued projections as well. We shall have more to say about UMD spaces in Chapter 6.

4.6.8. Definition. Suppose that \( X \) is a Banach space and that \((\Omega,\mathcal{A},\mu)\) is a finite measure space. A projection multiplier function is an element \( P \) of \( \text{Proj}(L^p(\Omega,\mathcal{A},\mu;X)) \) such that for all \( \omega \in \Omega \), there exists \( Q(\omega) \in \text{Proj}(X) \) such that for all \( f \in L^p(\Omega,\mathcal{A},\mu;X) \),

\[
(Pf)(\omega) = Q(\omega)(f(\omega)) \quad \text{(a.e.)}.
\]

By considering the constant functions, it is easy to see that the operators \( Q(\omega) \) are uniquely determined up to sets of measure zero. We shall say that a projection multiplier function is contractive if \( Q(\omega) \in \text{Proj}_1(\omega) \) for almost all \( \omega \). If \( \mathcal{A}_0 \) is a sub-\( \sigma \)-ring of \( \mathcal{A} \), we shall say that \( P \) is \( \mathcal{A}_0 \)-measurable if \( Pf \) is \( \mathcal{A}_0 \)-measurable for all \( f \in L^p(\Omega,\mathcal{A},\mu;X) \). If the set of projections \( \{ Q(\omega) \in \text{Proj}(X) : \omega \in \Omega \} \) defines a projection multiplier function on \( L^p(\Omega,\mathcal{A},\mu;X) \), then we shall denote that operator by \( M_Q \).

Even for a Hilbert space \( \mathcal{H} \), the best we could hope for as a characterisation of contractive projections on \( L^p(\Omega,\mathcal{A},\mu;\mathcal{H}) \) is the following conjecture.

4.6.9. Conjecture. Suppose that \( \mathcal{H} \) is a Hilbert space, that \((\Omega,\mathcal{A},\mu)\) is a finite measure space, and that \( \{P_j\} \) is an increasing sequence of contractive projections on \( L^p(\Omega,\mathcal{A},\mu;\mathcal{H}) \) (1 < p < \infty, p \neq 2). Then there exists a finite measure \( \nu \) on \((\Omega,\mathcal{A})\), a sequence of sub-\( \sigma \)-rings
$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots \subset \mathcal{A}$ (with associated conditional expectation operators $E_1, E_2, \ldots$), an increasing sequence of $\mathcal{A}_1$-measurable projection multiplication functions $M Q_j$, and an isometric isomorphism $S: L^P(\Omega, \mathcal{A}, \mu; \mathbb{R}) \rightarrow L^P(\Omega, \mathcal{A}, \nu; \mathbb{R})$ such that

$$p_j = S^{-1} \sum_{j=1}^n M Q_j \quad j = 1, 2, \ldots .$$

If the conjecture does hold, then one can show that $K(L^P(\Omega, \mathcal{A}, \mu; \mathbb{R}^2)) < \infty$. Suppose that $\{P_j\}_{j=1}^\infty$ is an increasing sequence of contractive projections on $L^P(\Omega, \mathcal{A}, \mu; \mathbb{R}^2)$. It is not difficult to see that we could arrange the isometry $S$ of Conjecture 4.6.9 so that $Q_j(\omega)$ is either the identity, the projection onto the first coordinate of $\mathbb{R}^2$, or else the zero operator. If we write $L^P(\Omega, \mathcal{A}, \nu; \mathbb{R}^2)$ as $L^P(\Omega, \mathcal{A}, \nu; \mathbb{R}) \oplus L^P(\Omega, \mathcal{A}, \nu; \mathbb{R})$, then each $\sum_{j=1}^n M Q_j$ is of the form

$${\begin{bmatrix} A_j & 0 \\ 0 & B_j \end{bmatrix}}$$

where $A_j$ and $B_j$ are conditional expectation operators on $L^P(\Omega, \mathcal{A}, \nu; \mathbb{R})$. Thus, if $\{a_j\}$ is a sequence of real numbers with $|a_j| \leq 1$, then

$$\sum_{j=1}^n a_j (P_j - P_{j-1}) = S^{-1} \begin{bmatrix} \sum_{j=1}^n a_j (A_j - A_{j-1}) & 0 \\ 0 & \sum_{j=1}^n a_j (B_j - B_{j-1}) \end{bmatrix} S.$$

Applying Theorem 4.4.3 gives that

$$\left\| \sum_{j=1}^n a_j (P_j - P_{j-1}) \right\| = \left\| \sum_{j=1}^n a_j (E_j Q_j - E_{j-1} M Q_j) \right\| \leq 2(p^*-1).$$
In Chapters 1 and 3 we saw that a $C[a,b]$ functional calculus for an operator $T \in B(X)$ leads to a countably additive spectral measure, whilst an $AC[a,b]$ functional calculus allows one to construct a finitely additive decomposition of the identity for $X$. An important problem is to decide when well-bounded operators are scalar-type spectral. In this chapter we shall see that if a well-bounded operator on an $L^p$ space ($1 < p < \infty$) possesses a contractive AC functional calculus, then this is sufficient for the operator to be real scalar-type spectral. The advantage of this result (as opposed to Theorem 3.4.3) is that we do not need to explicitly know the spectral family for $T$.

§ 5.1. The Hilbert space case

In [FL], Fong and Lam, using the methods of spectral carriers, showed that if $T \in B(\mathbb{K})$, and $\| g(T) \| \leq |g(b)| + \int_a^b |g'(t)| \, dt$ for all polynomials $g$, then $T$ is self adjoint. This result is included in the following theorem, which is well-known to workers in the field.

5.1.1 THEOREM. Suppose that $T \in B(\mathbb{K})$. Then the following are equivalent:
(i) There exist $a \leq c \leq b$ such that
\[ \| g(T) \| \leq \| g \|_{[a,b],c} \]
for all polynomials $g$.

(ii) $T$ is well-bounded and every element of its associated spectral family $\{E(\lambda)\}$ is self-adjoint.

(iii) $T$ is self-adjoint.

Proof. (i) $\rightarrow$ (ii). From Theorem 3.2.2 and Proposition 3.2.8, we know that $T = \int_{[a,b]} \lambda \, dE(\lambda)$ and that $\| E(\lambda) \| \leq 1$ for $\lambda \leq c$ and $\| 1 - E(\lambda) \| \leq 1$ for $\lambda > c$. Thus either $E(\lambda)$ or $1 - E(\lambda)$ is a contractive, and hence self-adjoint projection. But if $1 - E(\lambda)$ is self-adjoint, then so is $E(\lambda)$.

(ii) $\rightarrow$ (iii). By Corollary 3.2.3 we have that for all $x,y \in \mathbb{H}$,
\[
(Tx|y) = b (x|y) - \int_{a}^{b} (E(\lambda)x|y) \, d\lambda \\
\quad = b (x|y) - \int_{a}^{b} (x|E(\lambda)y) \, d\lambda \\
\quad = (x|Ty)
\]
so $T = T^*$.

(iii) $\rightarrow$ (i). Suppose that $T$ is self-adjoint and that $\sigma(T) \subset [a,b]$. Then, if $g$ is a polynomial,
\[
\| g(T) \| = \sup \left\{ |\lambda| : \lambda \in \sigma(g(T)) \right\}
\]
because the norm of a self-adjoint operator is equal to its spectral radius. The spectral mapping theorem then implies that
\[
\| g(T) \| \leq \sup \left\{ |g(\lambda)| : \lambda \in \sigma(T) \right\} \\
\quad \leq \sup \left\{ |g(\lambda)| : \lambda \in [a,b] \right\} \\
\quad \leq \| g \|_{[a,b],b}.
\]
\end{document}
§ 5.2. Contractive AC functional calculi on $L^p$ spaces

Our aim in this section is to show that we can obtain a similar result to Theorem 5.1.1 on $L^p$ spaces with $1 < p < \infty$. The part of the theorem which interests us most here is the statement that if $T \in B(\mathcal{H})$ has a contractive AC functional calculus, then $T$ admits an integral representation with respect to a spectral measure. The following theorem uses the results of Chapter 4 to give a close analogue of this fact on the reflexive $L^p$ spaces.

5.2.1. THEOREM. Suppose that $T$ is a bounded linear operator on $L^p(\Omega, \mathcal{A}, \mu)$, where $(\Omega, \mathcal{A}, \mu)$ is an arbitrary measure space and $1 < p < \infty$. Suppose also that there exist real numbers $a < c < b$ such that for all polynomials $g$, $\quad \| g(T) \| \leq |g(c)| + \int_a^b |g'(t)| \, dt.$

Then $T$ is a scalar-type spectral operator.

Proof. $T$ is clearly well-bounded. Let $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ be its spectral family. As we have seen in the proof of Theorem 3.2.2, $\{E(\lambda)\}$ is concentrated on $[a, b]$.

Fix $f \in L^p(\Omega, \mathcal{A}, \mu)$ and $f^* \in L^p(\Omega, \mathcal{A}, \mu)^*$. By Theorem 3.4.3 it suffices to show that the function $\lambda \mapsto < E(\lambda)f, f^* >$ is of bounded variation.

Let $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ be a partition $\Lambda$ in $\mathbb{R}$. We will assume without loss of generality that $\lambda_0 < a$ and that $b < \lambda_n$. The variation of the above function over $\Lambda$ is then
\[ V_\Lambda = \sum_{j=1}^{n} \left| \langle E(\lambda_j) f, f^* \rangle - \langle E(\lambda_{j-1}) f, f^* \rangle \right| \]

\[ = \sum_{j=1}^{n} \left| \langle (E(\lambda_j) - E(\lambda_{j-1})) f, f^* \rangle \right| \]

\[ = \sum_{j=1}^{n} \alpha_j \langle (E(\lambda_j) - E(\lambda_{j-1})) f, f^* \rangle \]

for some sequence of unimodular constants \( \alpha_j \). Thus

\[ V_\Lambda = \langle \sum_{j=1}^{n} \alpha_j (E(\lambda_j) - E(\lambda_{j-1})) f, f^* \rangle \]

\[ \leq \left\| \sum_{j=1}^{n} \alpha_j (E(\lambda_j) - E(\lambda_{j-1})) \right\| \left\| f \right\| \left\| f^* \right\| . \]

The proof will be completed by finding a bound on

\[ \left\| \sum_{j=1}^{n} \alpha_j (E(\lambda_j) - E(\lambda_{j-1})) \right\| \text{ which is independent of the partition } \Lambda. \]

Suppose that \( k \) is the integer such that \( \lambda_{k-1} < c \leq \lambda_k \). Then

\[ \left\| \sum_{j=1}^{n} \alpha_j (E(\lambda_j) - E(\lambda_{j-1})) \right\| \leq \left\| \sum_{j=1}^{k-1} \alpha_j (E(\lambda_j) - E(\lambda_{j-1})) \right\| + \left\| \alpha_k (E(\lambda_k) - E(\lambda_{k-1})) \right\| + \left\| \sum_{j=k+1}^{n} \alpha_j (E(\lambda_j) - E(\lambda_{j-1})) \right\|. \]

By the hypothesis on the norm of \( g(T) \) and Lemma 3.2.4 we have that

\[ \left\| E(\lambda) \right\| \leq 1 \quad \text{for } \lambda < c \]

and

\[ \left\| I - E(\lambda) \right\| \leq 1 \quad \text{for } \lambda \geq c. \]

Thus \( \{E(\lambda_j)\}_{j=1}^{k-1} \) forms an increasing sequence of contractive projections. Then, by Theorem 4.5.1

\[ \left\| \sum_{j=1}^{k-1} \alpha_j (E(\lambda_j) - E(\lambda_{j-1})) \right\| \leq 2(p-1). \]

Similarly, \( 0 = (I - E(\lambda_n)), (I - E(\lambda_{n-1})), \ldots, (I - E(\lambda_{k+1})) \) forms an increasing sequence of contractive projections, so

\[ \left\| \sum_{j=k+1}^{n} \alpha_j (E(\lambda_j) - E(\lambda_{j-1})) \right\| = \left\| \sum_{j=k+1}^{n} \alpha_j ((I - E(\lambda_{j-1})) - (I - E(\lambda_j))) \right\| \]

\[ \leq 2(p-1). \]
Finally, \( \| E(\lambda_k) \| \leq 2 \) and \( \| E(\lambda_{k-1}) \| \leq 1 \), so
\[
\| \alpha_k (E(\lambda_k) - E(\lambda_{k-1})) \| \leq 3.
\]

Thus
\[
\left\| \sum_{j=1}^{n} \alpha_j (E(\lambda_j) - E(\lambda_{j-1})) \right\| \leq 4(p^*-1) + 3.
\]

The variation of \( \lambda \mapsto \langle E(\lambda)f, f^* \rangle \) is thus bounded by
\[
(4p^* - 1) \| f \| \| f^* \| < \infty,
\]
and the proof is complete.

5.2.2. **COROLLARY.** Suppose that \( 1 < p \leq \infty \) and that
\[
T = \int_{[a,b]} \lambda \, dE(\lambda)
\]
is a well-bounded operator on \( L^p(\Omega, \mathcal{A}, \mu) \). Suppose also that \( \| E(\lambda) \| \leq 1 \) for all \( \lambda \in \mathbb{R} \). Then \( T \) is real scalar-type spectral.

**Proof.** This is just a simple consequence of the fact (Proposition 2.3.5) that
\[
\| g(T) \| \leq \| g \|_{[a,b], b} \sup_{\lambda \in \mathbb{R}} \| E(\lambda) \|.
\]

§ 5.3. **Some examples**

In this section we give some examples which show that Theorem 5.2.1. cannot be extended to the non-reflexive \( L^p \) spaces.

5.3.1. **Example.** We give here an example of an operator on
\( L^1 = L^1(\mathbb{N}) \) which admits a contractive absolutely continuous functional calculus, but which is not real scalar-type spectral.

Suppose that \( 0 = d_1 < d_2 < \ldots < 1 = \lim_{k \to \infty} d_k \) and that \( T \) is represented
(with respect to the standard basis on $\ell^1$) by the matrix

$$
\begin{bmatrix}
d_1 & (d_1 - d_2) & (d_1 - d_2) & \ldots & (d_1 - d_2) & \ldots \\
d_2 & (d_2 - d_3) & \ldots & (d_2 - d_3) & \ldots \\
d_3 & \ldots & (d_3 - d_4) & \ldots \\
\vdots \\
0 & \ldots & & (d_{k-1} - d_k) & \\
d_k & & \ldots & & \\
\end{bmatrix}
$$

A simple induction proof shows that

$$
T^n =
\begin{bmatrix}
d_1^n & (d_1^n - d_2^n) & (d_1^n - d_2^n) & \ldots \\
d_2^n & (d_2^n - d_3^n) & \ldots \\
0 & \ldots \\
\end{bmatrix}
$$

and so if $g$ is a polynomial

$$
g(T) =
\begin{bmatrix}
g(d_1) & (g(d_1) - g(d_2)) & \ldots & (g(d_1) - g(d_2)) & \ldots \\
g(d_2) & \ldots & (g(d_2) - g(d_3)) & \ldots \\
0 & \ldots & (g(d_{k-1}) - g(d_k)) & \ldots \\
g(d_k) & & \ldots \\
\end{bmatrix}
$$

The norm of $g(T)$ is the supremum of the $\ell^1$ norms of the columns. Let $c_k$ denote the $k^{th}$ column. Then

$$
\| c_k \| = |g(d_1) - g(d_2)| + |g(d_2) - g(d_3)| + \ldots \\
+ |g(d_{k-1}) - g(d_k)| + |g(d_k)|.
$$

Thus, because of our choice of $\{d_k\}$, $\| g(T) \| \leq |g(1)| + \var g.$ [0,1]

We shall now show that $T$ is not scalar-type spectral. For $\lambda \in [d_k, d_{k+1})$, define $E(\lambda) \in B(\ell^1)$ by

$$
E(\lambda)(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_{k-1}, \sum_{j=k}^{\infty} x_j, 0, 0, \ldots).
$$
Extend $E$ to all of $\mathbb{R}$ by setting $E(\lambda) = 0$ for $\lambda < 0$ and $E(\lambda) = 1$ for $\lambda \geq 1$. It is easy to check that $\{E(\lambda)\}$ is a spectral family for $T$. Fix $x = (1, 1/4, 1/9, \ldots) \in \ell^1$ and $x^* = (1, -1, 1, -1, \ldots) \in \ell^\infty$. To show that $T$ is not scalar-type spectral, it suffices, by Theorem 3.4.3 to prove that the function $f(\lambda) = \langle E(\lambda)x, x^* \rangle$ is not of bounded variation over $[0,1]$. By considering, for each $n \in \mathbb{N}$, the partition $\{d_1, d_2, \ldots, d_n, 1\}$ we see that

$$\text{var } f \geq \sum_{j=2}^{n} \left| \sum_{j=2}^{n} \sum_{k=j}^{\infty} 1/k^2 \right|$$

$$= \sum_{j=2}^{n} (\sum_{k=j}^{\infty} 1/k^2) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

5.3.2. Example. It is even easier to show that Theorem 5.2.1 cannot be extended to $L^\infty[0,1]$. Define $T \in B(L^\infty[0,1])$ by $(Tf)(t) = tf(t)$. Then for any polynomial $g$

$$\| g(T) \| = \sup_{t \in [0,1]} |g(t)| \leq |g(0)| + \int_{0}^{1} |g'(t)| \, dt$$

so $T$ is well-bounded with a contractive $AC[0,1]$ functional calculus.

Note that $\sigma(T) = [0,1]$.

McCarthy and Tzafriri have shown [ST, Corollary 27] that every scalar-type spectral operator on $L^\infty[0,1]$ is similar to an operator of the form $Sf = fh$ for some simple function $h$. In particular, they all have only a finite number of points in their spectrum. Hence $T$ cannot be real scalar-type spectral.

Alternatively, it is easy to see that if $T$ were scalar-type spectral, then its spectral measure would have to be given by
\( \mathcal{S}(\Delta)f = \chi_\Delta f \) for \( \Delta \in \mathcal{B} \) \( f \in L^\infty[0,1] \). This is not a countably additive set function however.

### 5.3.3. Example

It would now be appropriate to re-examine the example we considered in 3.5.1:

\[
(Tf)(t) = tf(t) + \int_0^1 \log(1-\min\{u,t\}) f(u) \, du
\]

acting on \( L^1[0,1] \). Recall that the spectral family for \( T \) is given by

\[
(E(\lambda)f)(t) = \begin{cases} 
  f(t) & \text{for } t \in [0,\lambda) \\
  1/(1-\lambda) \int_\lambda^1 f(u) \, du & \text{for } t \in [\lambda,1).
\end{cases}
\]

for \( \lambda \in [0,1) \), \( E(\lambda) = 0 \) for \( \lambda < 0 \), and \( E(\lambda) = I \) for \( \lambda \geq 1 \).

Define \( \sigma \)-algebras \( \{\mathcal{A}_\lambda\}_{\lambda \in [0,1)} \) as follows.

\[
\mathcal{A}_\lambda = \left\{ A \in \mathcal{B}[0,1] : A \cap (\lambda,1] = \emptyset \text{ or } A \cap (\lambda,1] = (\lambda,1] \right\}.
\]

It is easy to check that for \( \lambda \in [0,1) \), \( E(\lambda) \) is the conditional expectation operator with respect to \( \mathcal{A}_\lambda \) and Lebesgue measure, and so \( \| E(\lambda) \|_p \leq 1 \) for \( 1 \leq p \leq \infty \). It is easily seen then that \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) forms a uniformly bounded, increasing family of projections on \( L^p[0,1] \) for all \( 1 \leq p < \infty \). Furthermore, for \( 1 \leq p < \infty \) and \( f \in L^p[0,1] \), the function \( \lambda \mapsto E(\lambda)f \) is right continuous and possesses a left limit everywhere, so \( \{E(\lambda)\} \) forms a spectral family. On these spaces \( \{E(\lambda)\} \) thus determines a unique well-bounded operator, \( S_p \in \mathcal{B}(L^p[0,1]) \). As we saw in Example 3.5.1, \( S_1 = T \). The proof that \( S_p = T \) for \( 1 < p < \infty \) is similar. We shall omit the details except to comment that in showing that the function \( h(u,t,\lambda) = (1-\lambda)^{-1}X_{[\lambda,1]}(u)X_{[\lambda,1]}(t)f(u)\varphi(t) \) is integrable whenever \( f \in L^p[0,1] \) and \( \varphi \in L^q[0,1] \), one needs to modify the previous argument as follows.
\[ \int_{[0,1]^3} |h| = \int_0^1 \frac{1}{1-\lambda} \left\{ \int_0^1 |f(u)| x_{[\lambda,1]}(u) \, du \right\} \left\{ \int_0^1 |\varphi(t)| x_{[\lambda,1]}(t) \, dt \right\} \, d\lambda \leq \int_0^1 \frac{1}{1-\lambda} \| f \|_p (1-\lambda)^{1/q} \| \varphi \|_q (1-\lambda)^{1/p} \, d\lambda \]

(by Hölder's inequality)

\[ = \| f \|_p \| \varphi \|_q < \infty. \]

We have then, that T acts as a well-bounded operator (of type (B)) on \( L^p[0,1] \) for \( 1 < p < \infty \). Indeed since \( \| E(\lambda) \|_p \leq 1 \) for all \( \lambda \), Corollary 5.2.2 implies that T is a scalar-type spectral operator on these spaces. It is also worth noting that T is self-adjoint on \( L^2[0,1] \); this is easily checked directly, or by applying Theorem 5.1.1.

To complete the picture, we examine the behaviour of T on \( L^\infty[0,1] \). It is easily verified that if \( \varphi \in L^\infty[0,1] \) and \( f \in L^1[0,1] \) then \( \langle Tf, \varphi \rangle = \langle f, T\varphi \rangle \), i.e. T acting on \( L^\infty[0,1] \) is the adjoint of T acting on \( L^1[0,1] \). It follows then that T must also be a well-bounded operator on \( L^\infty[0,1] \). However T cannot be of type (B) on this space since the well-bounded operators of type (B) on \( L^\infty[0,1] \) correspond to the real scalar-type spectral operators (see [Rick]). Again applying the result of McCarthy and Tzafriri [NT, Corollary 27] shows that this cannot be the case.
In this final chapter we shall take a more abstract look at the properties exhibited by the reflexive $L^p$ spaces in Chapters 4 and 5, and examine how these properties are related on more general Banach spaces. Although we shall see that there are strong links between the properties, many natural questions remain unanswered.

§ 6.1. Contractive projections

Much of this chapter will be concerned with the properties which the contractive projections on a Banach space exhibit. We shall see that these can tell us much about the structure of the space and the spectral theory of operators on that space. The following theorem for example, shows that we can tell whether a Banach space is reflexive just by examining its set of contractive projections. If $\mathcal{A} \subset B(X)$, let $\mathcal{A}^*$ denote the set $\{ A^* \in B(X^*) : A \in \mathcal{A} \}$.

6.1.1. THEOREM. A Banach space $X$ is reflexive if and only if $\text{Proj}_1(X)^* = \text{Proj}_1(X^*)$.

(Sufficiency). The sufficiency is an easy consequence of James' characterisation of reflexivity. Suppose that $X$ is not reflexive. Then, by [Dies, p. 12; Jam], there exists $x_0^* \in X^*$ of norm 1 which
does not achieve that norm. By the Hahn-Banach theorem, there exists \( x_0^{**} \in X^{**} \) such that \( \| x_0^{**} \| = 1 \) and \( < x_0^{**}, x_0^* > = 1 \). Clearly then, \( x_0^{**} \) is not in the image of \( X \) in \( X^{**} \). Define \( E \in B(X^*) \) by \( E(x^*) = < x^*, x_0^* > x_0^* \). Then

\[
E^2 x^* = < x^*, x_0^{**} > < x_0^{**}, x_0^* > x_0^*
= < x^*, x_0^{**} > < x_0^*, x_0^* > x_0^*
= < x^*, x_0^{**} > x_0^*
= E x^*
\]

and \( \| Ex^* \| \leq \| x^* \| \| x_0^{**} \| \| x_0^* \| = \| x^* \| \). Thus \( \| E \| = 1 \), and so \( E \in \text{Proj}_1(X^*) \). Now

\[
< Ex^*, x^{**} > = < (< x^*, x_0^{**} > x_0^*), x^{**} >
= < x^*, x_0^{**} > < x_0^*, x^{**} >
= < x^*, (< x_0^*, x^{**} > x_0^{**} ) >
\]

so \( E^* x^{**} = < x_0^*, x^{**} > x_0^* \). Since the image of this operator restricted to \( X \) does not lie in \( X \), \( E \) cannot be the adjoint of any operator on \( X \). Thus \( \text{Proj}_1(X)^* \neq \text{Proj}_1(X^*) \).

The set of contractive projections on \( X \) may be quite small. For example Bosnay and Garay [234] have shown that there are many norms on \( \mathbb{R}^n \) or \( \mathbb{C}^n \) for which the only contractive projections are what we might call the trivial ones; zero, the identity and the rank one projections guaranteed by the Hahn-Banach theorem.

6.1.2. Question (see [SiII, Problem 1.1; pp. 729-730]). Does there exist an infinite dimensional Banach space with just these trivial contractive projections? How small can \( \text{Proj}_1(X) \) be?
§ 6.2. Unconditionality properties for contractive projections

We begin this section with several definitions.

6.2.1. Definition. (i) An increasing system of projections on $X$ is a map $P$ from an totally ordered set $A$ to Proj($X$) such that $P(\lambda)P(\mu) = P(\min\{\lambda, \mu\})$ for all $\lambda, \mu \in A$. As in the earlier chapters, we shall call the system a family if $A = \mathbb{R}$, and a sequence if $A \subseteq \mathbb{Z}$. Such a system is strictly increasing if $\lambda \not= \mu$ implies that $P(\lambda) \not= P(\mu)$. Recall that if $\{P_i\}_{i=1}^{\infty}$ is an increasing sequence of projections we shall employ the convention that $P_0 = 0$.

(ii) An increasing system of projections $\{P(\lambda)\}_{\lambda \in A}$ is said to have the unconditionality property if there exists a constant $K \in [1, \infty)$ such that, for any $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ in $A$ and any scalars $|a_j| \leq 1$

$$\left\| \sum_{j=1}^{n} a_j (P(\lambda_j) - P(\lambda_{j-1})) \right\| \leq K.$$  

If such a $K$ exists, the smallest such one will be called the unconditionality constant for $\{P(\lambda)\}$.

(iii) A Banach space $X$ has the unconditionality property for contractive projections (UPCP) if every increasing sequence of contractive projections $\{P_j\}_{j=1}^{\infty}$ on $X$ has the unconditionality property.
iv) A Banach space $X$ has bilateral UPCP if every increasing sequence of contractive projections $\{P_j\}_{j=-\infty}^{\infty}$ on $X$ has the unconditionality property.

v) A Banach space $X$ has uniform UPCP if $X$ has UPCP and there is a uniform bound on the unconditionality constants for increasing sequences of contractive projections on $X$. In other word, $X$ has uniform UPCP if $K(X) < \infty$ (see Definition 4.6.1).

vi) A Schauder decomposition of an infinite dimensional Banach space $X$ is a strictly increasing sequence of projections $\{P_j\}_{j=1}^{\infty}$ such that $P_j \to 1$ in the strong operator topology. Such a decomposition is said to be monotone if $\|P_j\| \leq 1$ for all $j \in \mathbb{N}$. We shall regard a (Schauder) basis for $X$ as a Schauder decomposition for which $\text{rank}(P_j - P_{j-1}) = 1$ for all $j \in \mathbb{N}$.

This, of course, is not the usual way of introducing a basis. A sequence $\{x_j\}_{j=1}^{\infty}$ of elements of $X$ is a basis for $X$ if every $x \in X$ can be written in a unique way as $x = \sum_{j=1}^{\infty} \alpha_j x_j$ for some sequence of scalars $\{\alpha_j\}$. The corresponding "basis of projections" is given by $P_n \left[ \sum_{j=1}^{\infty} \alpha_j x_j \right] = \sum_{j=1}^{n} \alpha_j x_j$. Constructing a "basis of elements" from a basis of projections is also straightforward, so we shall often blur the distinction. Let $\Pi$ denote the set of all permutations of the set $\mathbb{N}$.

6.2.2. Definition. A Schauder decomposition $\{P_j\}_{j=1}^{\infty}$ of a Banach space $X$ is said to be unconditional if it satisfies any (and hence
all) of the following equivalent conditions:

(i) For every \( x \in X \), the series \( \sum_{j=1}^{\infty} (P(j) - P(j-1))x \) converges (in norm) for all \( x \in X \).

(ii) For every \( x \in X \) and every sequence of scalars \( \{\varepsilon_j\}_{j=1}^{\infty} \) with \( \varepsilon_j = \pm 1 \), the series \( \sum_{j=1}^{\infty} \varepsilon_j (P(j) - P(j-1))x \) converges.

(iii) For every \( x \in X \) and every sequence of scalars \( \{\alpha_j\}_{j=1}^{\infty} \) with \( |\alpha_j| \leq 1 \), the series \( \sum_{j=1}^{\infty} \alpha_j (P(j) - P(j-1))x \) converges.

The equivalence of these conditions is shown in [Si1, Lemma 16.1]. A conditional Schauder decomposition exists which is not unconditional.

We know from Chapter 4 that for \( 1 < p < \infty \), \( L^p(\Omega, \mathcal{A}, \mu) \) has uniform UPCCP. The following theorem shows the relationships between the above concepts.

6.2.3. THEOREM. The following implications hold for a general infinite dimensional Banach space \( X \):

\[
\begin{align*}
X & \text{ has uniform UPCCP} \\
\downarrow \\
X & \text{ has bilateral UPCCP} \\
\downarrow \\
X & \text{ has UPCCP} \\
\downarrow \\
\text{All monotone Schauder decompositions of } X & \text{ are unconditional} \\
\downarrow \\
\text{All monotone bases of } X & \text{ are unconditional.}
\end{align*}
\]
Proof. All the implications except the first are immediate. To see that uniform UPCP implies bilateral UPCP note that if \( \{ P_j \}_{j=-\infty}^{\infty} \) is an increasing sequence of contractive projections then \( \{ P_j \}_{j=m}^{\infty} \) forms a "unilateral" increasing sequence of contractive projections. Thus, for any sequence of scalars \( \{ a_j \} \) bounded in modulus by 1,

\[
\left\| \sum_{j=m}^{n} a_j (P_j - P_{j-1}) \right\| \leq K(X),
\]

for all \( n > m \). As \( K \) does not depend on \( m \) we are done.

It is well-known that the Haar basis forms a conditional monotone basis for \( L^1[0,1] \) and that the Schauder basis is a conditional monotone basis for \( C[0,1] \) [Si1, p. 215 and p. 396], so neither of these spaces have UPCP. Similarly \( c_0 \) does not have UPCP [Si1, pp. 634-635].

Some of the reverse implications in Theorem 6.2.2 fail trivially. For example, \( \ell^{\infty} \) has no Schauder decompositions (see [SiII; Theorem 15.2]), so "all" its monotone Schauder decompositions must be unconditional. However it is easy to construct increasing sequences of contractive projections on \( \ell^{\infty} \) which do not have the unconditionality property. An examination of Example 5.3.1 shows that the sequence of operators \( \{ F_j \}_{j=1}^{\infty} \in B(\ell^{\infty}) \) given by

\[
F_j(x_1,x_2,\ldots) = (x_1,\ldots,x_{j-1},x_j,x_{j},\ldots)
\]

does not have the unconditionality property since \( F_j = E(d_j)^{\delta} \). More difficult is the following result.
6.2.4. PROPOSITION. The trace class operators $C_1$ have no monotone basis, but do admit a conditional monotone Schauder decomposition. Thus $C_1$ does not have UPCP.

Proof. The first statement is due to Arazy and Friedman [AF, § 7]. For $n = 1, 2, \ldots$, let $P_n$ be the contractive projections on $\ell^2$ given by

$$P_n(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_n, 0, \ldots).$$

Then [Rin3, Corollary 2.3.11] shows that the operators $Q_{2n-1}: T \mapsto P_n T P_n$ and $Q_{2n}: T \mapsto P_{n+1} T P_n$ are contractive. They thus form an increasing sequence of contractive projections on $C_1$. As has been noted by Arazy and Friedman [AF, p. 159], this sequence defines a Schauder decomposition of $C_1$.

For $j = 1, 2, \ldots$, define the constants $a_j = (-1)^{j+1}$. Suppose that $T \in C_1$ has matrix $(t_{ij})_{i,j=1}^{\infty}$ with respect to the standard basis for $\ell^2$. Then the operator $S_n = \sum_{j=1}^{2n} a_j (Q_j - Q_{j-1})$ maps $T$ to

$$
\begin{bmatrix}
  t_{11} & t_{12} & \cdots & t_{1n} \\
  -t_{21} & t_{22} & \cdots & t_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  -t_{n1} & -t_{n2} & \cdots & t_{nn} \\
  0 & 0 & \cdots & 0
\end{bmatrix}
$$

It is well-known that $\| S_n \| \to \infty$ as $n \to \infty$. This follows, for example, from a calculation due to Davies [Dav, proof of Lemma 10].

Suppose that $\tilde{T}_n = (\tilde{t}_{ij})$ where

$$
\tilde{t}_{ij} = \begin{cases} 
1 & \text{for } i, j \leq n \\
0 & \text{otherwise.}
\end{cases}
$$

Then $\| \tilde{T}_n \|_1 = n$ as $\tilde{T}_n$ is $n$ times a rank one projection. However,
as Davies shows, \( \| S_n(T_n) \|_1 \) is asymptotically \( n \log n \) and so \( \| S_n \| \) grows at least as fast as \( \log n \).

6.2.5. **Remark.** It is interesting to note that the operators \( \{ Q_n \} \) defined in the above proof are also monotone Schauder decompositions on the von Neumann-Schatten \( p \) classes, \( C_p \), for \( 1 < p < \infty \).

However, for these values of \( p \), the sequence \( \{ \| S_n \|_p \}_{n=1}^{\infty} \) is bounded (see [GR]). It is not known whether \( C_p \) \( (1 < p < \infty) \) has UPCP (except, of course, for the Hilbert space \( C_2 \)).

The relationships between the various UPCP properties is more mysterious. The only spaces which we know to have UPCP (namely the reflexive \( L^p \) spaces and finite dimensional spaces) also have uniform UPCP.

6.2.6. **Question.** Are the three conditions UPCP, bilateral UPCP and uniform UPCP equivalent? If not, is there any general class of Banach spaces on which this is true?

The following proposition gives some simple facts about the unconditionality properties for contractive projections.

6.2.7. **PROPOSITION.** Suppose that \( X \) is a Banach space. Then

i) If \( X^\diamondsuit \) has UPCP (respectively bilateral UPCP; uniform UPCP), then \( X \) has UPCP (bilateral UPCP; uniform UPCP).

ii) If \( P \in \text{Proj}_1(X) \) and \( X \) has UPCP (respectively bilateral UPCP; uniform UPCP), then \( PX \) has UPCP (bilateral UPCP; uniform UPCP).
iii) If $L^p([0,1];X)$ has UPCP (respectively bilateral UPCP; uniform UPCP) for some $p (1 \leq p \leq \infty)$ then $X$ has UPCP (bilateral UPCP; uniform UPCP).

Proof. We shall just sketch the proof for UPCP. The proofs for bilateral UPCP and uniform UPCP are virtually identical.

(i) See Proposition 4.6.2.

(ii) If \( \{P_j\}_{j=1}^{\infty} \) is an increasing sequence of contractive projections on $PX$, then \( \{P_jP_j\}_{j=1}^{\infty} \) is an increasing sequence of contractive projections on $X$.

(iii) If \( \{P_j\}_{j=1}^{\infty} \) is an increasing sequence of contractive projections on $X$ then \( (Q_jf)(\omega) = P_j(f(\omega)) \) defines an increasing sequence of contractive projections on $L^p([0,1];X)$. The result follows by considering the constant functions.

Unfortunately none of the statements in the proposition allows us to construct any new examples of UPCP spaces, since the dual of a reflexive $L^p$ space is clearly another $L^p$ space and, as Ando and Tzafriri [Ando,Tz] have shown, the image of a contractive projection on an $L^p$ space \( (1 < p < \infty) \) is isometrically isomorphic to another such $L^p$ space. Proposition 6.2.7 does however show that the dual of a space without UPCP cannot have UPCP.

6.2.8. Questions. (i) If $X$ has UPCP, must $X^*$ have UPCP?

(ii) If $X$ has UPCP, must every (complemented) subspace of $X$ have UPCP?
Perhaps the most important unanswered question concerns reflexivity.

6.2.9. Question. If \( X \) has UPCP, must \( X \) be reflexive?

An affirmative answer to this question would make many of the other questions much easier.

§ 6.3. UPCP on reflexive spaces

Our aim in this section is to examine how the unconditionality properties for contractive projections relate to properties of the well-bounded operators on a Banach space.

6.3.1. Definition. i) A Banach space \( X \) has spectral UPCP if every spectral family \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) on \( X \) with \( \sup_{\lambda \in \mathbb{R}} \| E(\lambda) \| = 1 \) has the unconditionality property. Note that it is sufficient to check that this holds for spectral families concentrated on \([0,1]\) (i.e. \( E(\lambda) = 0 \) for \( \lambda < 0 \), and \( E(\lambda) = I \) for \( \lambda \geq 1 \)) as, given any spectral family, we can construct one concentrated on this interval which "contains" the same projections (i.e. has the same range as an operator valued function).

ii) We will say that \( X \) has the contractive functional calculus condition (CFCC) if every operator which possesses a contractive absolutely continuous functional calculus is scalar-type spectral.
More specifically, X has CFCC if a sufficient condition for an operator on X to be scalar-type spectral is that there exist real numbers \( a \leq c \leq b \) such that for all polynomials g,

\[
\| g(T) \| \leq |g(c)| + \int_{a}^{b} |g'(t)| \, dt.
\]

6.3.2. THEOREM. Suppose that X is an infinite dimensional Banach space. Then

\[
X \text{ has CFCC} \\
\downarrow
\]

X has spectral UPCP

\[
\downarrow
\]

All monotone Schauder decompositions of X are unconditional.

Proof. (CFCC + spectral UPCP). Suppose that \( \{E(\lambda)\} \) is a spectral family with \( \| E(\lambda) \| \leq 1 \) for all \( \lambda \in \mathbb{R} \). As noted in the definition, we may assume that \( \{E(\lambda)\} \) is concentrated on \([0,1]\). By Theorem 3.2.2, \( \{E(\lambda)\} \) determines a well-bounded operator

\[
T = \int_{[0,1]}^{\oplus} \lambda \, dE(\lambda),
\]

and so Proposition 2.3.5 ensures that

\[
\| g(T) \| \leq \| g \|_{[0,1],1} \sup_{\lambda \in \mathbb{R}} \| E(\lambda) \|
\]

for all polynomials g. By the hypothesis then, T is scalar-type spectral.

Suppose that \( A = (\lambda, \mu] \subset \mathbb{R} \). If we denote the resolution of the identity for T by \( \mathcal{E} \), then it is clear from the remarks at the start of § 3.4 and the definitions of \( \int_{\sigma(T)} \chi_{A}(\omega) \mathcal{E}(d\omega) \) (§ 1.1) and

\[
\int_{[0,1]}^{\oplus} \chi_{A}(\lambda) \, dE(\lambda) \) (§ 2.3) that

\[
\chi_{A}(T) = \mathcal{E}(\lambda, \mu]) = E(\mu) - E(\lambda).
\]
Suppose that $\lambda_0 < \ldots < \lambda_n$, and that $|a_j| \leq 1$ for $j = 1, \ldots, n$.

Let $S = \sum_{j=1}^{n} \alpha_j (E(\lambda_j) - E(\lambda_{j-1}))$. Then, by the discussion in § 1.1,

$$\| S \| = \left\| \sum_{j=1}^{n} \alpha_j (\lambda_j, \lambda_{j-1}] \right\|$$

$$\leq 4 \sup_{\lambda \in \mathbb{B}} \| \mathcal{A} (\lambda) \| \sup_{\omega \in [0,1]} \left| \sum_{j=1}^{n} \alpha_j X(\lambda_j, \lambda_{j-1}) (\omega) \right|$$

$$\leq K,$$

say.

As $K$ depends only on $T$ and not on $\{\lambda_j\}$ or $\{a_j\}$, $X$ must have spectral UPCP.

(Spectral UPCP + all monotone Schauder decompositions are unconditional) Suppose that $\{P_j\}_{j=1}^{\infty}$ forms a monotone Schauder decomposition for $X$. For $j = 1, 2, \ldots$, let $\lambda_j = 1 - 1/j$, and define $E(\lambda) \in \text{Proj}_1 (X)$ for $\lambda \in \mathbb{R}$ by

$$E(\lambda) = \begin{cases} 0 & \text{for } \lambda < 0 \\ P_j & \text{for } \lambda \in [\lambda_j, \lambda_{j+1}] \\ 1 & \text{for } \lambda \geq 1. \end{cases}$$

The fact that $P_j \to I$ in the strong operator topology ensures that

$\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ forms a spectral family. It is clear that

$$\sup_{\lambda \in \mathbb{R}} \| E(\lambda) \| = 1,$$

so by the hypothesis this spectral family has the unconditionality property, with unconditionality constant $K$ say. It suffices to show that if $n \in \mathbb{N}$ and $|a_j| \leq 1$ for $j = 1, \ldots, n$,

then $\left\| \sum_{j=1}^{n} a_j (P_j - P_{j-1}) \right\| \leq K$. This is an immediate consequence of the fact that

$$\left\| \sum_{j=1}^{n} a_j (P_j - P_{j-1}) \right\| = \left\| \sum_{j=1}^{n} a_j (E(\lambda_j) - E(\lambda_{j-1})) \right\|$$

for some $\lambda_0 = -1 < \lambda_1 < \ldots < \lambda_n$. \qed
If we restrict our attention to reflexive Banach spaces we can say rather more. On these spaces the conditions bilateral UPCP, spectral UPCP and CFCC are all equivalent. Before we prove this we need to give two lemmas. The first is a more sophisticated version of a construction we used in the proof of Theorem 6.3.2.

6.3.3. **Lemma.** Suppose that \( \{ P_j \}_{j=-\infty}^{\infty} \) is an increasing sequence of contractive projections on a reflexive Banach space \( X \). Define

\[
E(\lambda) = \begin{cases}
0 & \lambda < 0 \\
\text{SOT-lim}_{k \to \infty} P_k & \lambda = 0 \\
P_k & \lambda \in [-1/(k-1), -1/k), \ k = -1, -2, \ldots \\
P_0 & \lambda \in [1, 2) \\
P_k & \lambda \in [3-1/k, 3-1/(k+1)], \ k = 1, 2, \ldots \\
1 & \lambda \geq 3.
\end{cases}
\]

Then \( \{ E(\lambda) \} \) is a spectral family.

**Proof.** A quick check shows that we have in fact defined \( E(\lambda) \) for all \( \lambda \in \mathbb{R} \). That \( E(0) \) is well-defined follows from Corollary 2.4.3 and it is clear that \( \{ E(\lambda) \} \) is right-continuous at 0. Corollary 2.4.3 also implies that \( \{ E(\lambda) \} \) has a strong left-hand limit at 3. The right-continuity conditions and the existence of left-hand limits are trivial for all other \( \lambda \in \mathbb{R} \). The only other point which should be remarked on is that \( E(0)E(\mu) = E(\mu)E(0) = E(0) \) for all \( \mu > 0 \). This follows from the fact that left and right multiplication are continuous in the strong operator topology. \( \square \)
The second lemma may be viewed as saying that functions of unbounded variation must be of unbounded variation around some particular point.

6.3.4. **Lemma.** Suppose that $f : [0,1] \rightarrow \mathbb{R}$ is not of bounded variation. Then there exists $\{\lambda_j\}_{j=-\infty}^\infty$ such that

1. $0 \leq \lambda_j \leq \lambda_{j+1} \leq 1$ for all $j \in \mathbb{Z}$;
2. $\sum_{j=-\infty}^\infty |f(\lambda_j) - f(\lambda_{j-1})| = \infty$.

**Proof.** Choose $S_1 = (\alpha_1, \beta_1) \subseteq [0,1]$ and $R_1 = [\gamma_1, \delta_1] \subseteq [0,1] - S_1$ such that $\text{var } f > 1$ and $\text{var } f = \infty$. For $n > 1$, choose $S_n = [\alpha_n, \beta_n] \cup R_1$ inductively so that

1. $S_n \subseteq R_{n-1}$;
2. $\text{var } f > 1$;
3. $S_n \cap R_n = \emptyset$;
4. $\text{var } f = \infty$.

For $n = 1, 2, \ldots$ choose $t_n \in S_n$. Since $[0,1]$ is compact, $\{t_n\}_{n=1}^\infty$ has a limit point, $t$ say, and a monotone subsequence $\{t_{n_k}\}_{k=1}^\infty$ which approaches this limit.

Assume first that $\{t_{n_k}\}$ is increasing. For $k=1,2,\ldots$ we can, by the definition of the sets $S_n$, choose $\lambda_{k,1} < \lambda_{k,2} < \ldots < \lambda_k, \ell_k$ in $[\alpha_{n_k}, \beta_{n_k}]$ so that

$$\sum_{j=2}^{\ell_k} |f(\lambda_{k,j}) - f(\lambda_{k,j-1})| > 1$$

As $\{t_{n_k}\}$ is increasing and the sets $S_n$ are disjoint intervals, we have that
Thus, if we renumber these points as $\lambda_0, \lambda_1, \ldots$ then

$$\sum_{j=1}^{\infty} |f(\lambda_j) - f(\lambda_{j-1})| = \infty.$$ 

It suffices to set $\lambda_j = 0$ for $j < 0$ to get the required set $
\{\lambda_j\}_{j=-\infty}^{\infty}$.

The case when \( \{t_{nk}\} \) is decreasing is a "mirror image" of the above proof. In this case we get a sequence of points which we may label so that $0 \leq \ldots \leq \lambda_{-1} \leq \lambda_0 \leq 1$ and for which

$$\sum_{j=-\infty}^{0} |f(\lambda_j) - f(\lambda_{j-1})| = \infty.$$ 

Setting $\lambda_j = 1$ for $j > 0$ completes the proof.

6.3.5. THEOREM. Suppose that $X$ is a reflexive Banach space. Then the following are equivalent:

i) $X$ has bilateral UPCP.

ii) $X$ has spectral UPCP.

iii) $X$ has CFCC.

Proof. We have already seen in Theorem 6.3.2 that (iii) $\Rightarrow$ (ii), so we need only show that (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii).

(Spectral UPCP $\Rightarrow$ bilateral UPCP). Suppose that \( \{P_j\}_{j=-\infty}^{\infty} \) is an increasing sequence of contractive projections on $X$. Let \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) be the spectral family constructed in Lemma 6.3.3. The proof that \( \{P_j\} \) has the unconditionality property follows exactly as in Theorem 6.3.2.

(Bilateral UPCP $\Rightarrow$ CFCC). Suppose that there exists an operator $T \in B(X)$ which is not scalar-type spectral but for which there exists
a \leq c \leq b \text{ such that } \| g(T) \| \leq \| g \|_{[a,b],c} \text{ for all polynomials } g. \text{ Let } \{E(\lambda)\} \text{ denote the spectral family for } T. \text{ Then, by Theorem 3.4.3, there exists } x \in X \text{ and } x^* \in X^* \text{ such that } \\
\varphi : \lambda \mapsto <E(\lambda)x, x^*> \text{ is not of bounded variation. Clearly then, either } \varphi|_{[a,c]} \text{ or } \varphi|_{[c,b]} \text{ is not of bounded variation. }

Suppose that \varphi|_{[a,c]} \text{ is not of bounded variation. Then by Lemma 6.3.4, there exists an increasing sequence } \{\lambda_j\}_{j=-\infty}^{\infty} \in [a,c] \text{ such that } \\
\sum_{j=-\infty}^{\infty} |\varphi(\lambda_j) - \varphi(\lambda_{j-1})| = \infty.

Suppose that \lambda_j = c \text{ for some } j. \text{ In this case let } j_0 \text{ denote the smallest such } j \text{ (so that } \lambda_j = c \text{ for all } j \geq j_0 \). \text{ Otherwise let } j_0 = \infty. \text{ If we set } \\
\mu_j = \begin{cases} 
\lambda_j & \text{if } j < j_0 \\
\lambda_{j_0-1} & \text{if } j \geq j_0
\end{cases}

\text{then it is clear that } \mu_j \in [a,c] \text{ for all } j. \text{ A simple calculation shows that } \\
\sum_{j=-\infty}^{\infty} |\varphi(\mu_j) - \varphi(\mu_{j-1})| = \infty.

\text{For } j \in Z, \text{ let } P_j = E(\mu_j). \text{ Then } \{P_j\} \text{ is clearly an increasing sequence of projections. By Corollary 3.2.4 we know that } \\
\| E(\lambda) \| \leq 1 \text{ for all } \lambda \in [a,c], \text{ so all the elements of } \{P_j\} \text{ are contractive projections. Now for each } j \in Z \\
| <P_j x, x^*> - <P_{j-1} x, x^*> | = <\alpha_j(P_j - P_{j-1})x, x^*> \\
\text{for some unimodular scalar } \alpha_j. \text{ Thus } \\
\sum_{j=m}^{n} \alpha_j(P_j - P_{j-1}) \to \infty \text{ as } m \to -\infty, \text{ and } n \to \infty, \text{ so } X \text{ does not have bilateral UPCP.}

Suppose now that } \varphi|_{[c,b]} \text{ is not of bounded variation. Again
by Lemma 6.3.4 choose an increasing sequence \( \{\lambda_j\}_{j=-\infty}^{\infty} \in [c,b] \) such that
\[
\sum_{j=-\infty}^{\infty} |\varphi(\lambda_j) - \varphi(\lambda_{j-1})| = \infty.
\]
Now for \( j \in \mathbb{Z} \), define \( P_j = I - E(\lambda_{-j}) \). Corollary 3.2.4 again ensures that \( \{P_j\} \) is an increasing sequence of contractive projections. We have that
\[
|\varphi(\lambda_j) - \varphi(\lambda_{j-1})| = |< (I-P_{-j})x, x^\delta > - < (I-P_{1-j})x, x^\delta > |
\]
\[
= |< (P_{-j}-P_{1-j})x, x^\delta > |
\]
\[
= \alpha_j < (P_{-j}-P_{j-1})x, x^\delta >
\]
for some unimodular scalar \( \alpha_j \). The proof is then completed as above.

\[\Box\]

§ 6.4. UPCP and UMD

We return now to a discussion of the relationship between UPCP and UMD (see Definition 4.6.7). All the spaces we know to have UPCP also are UMD spaces. One can easily see however that the two properties are quite distinct, for whilst the UMD property is stable under equivalent renorming, this is not true for UPCP.

6.4.1. THEOREM. Let \( (X,\|\cdot\|) \) be a Banach space which contains an infinite dimensional, complemented subspace \( Y \) with a basis. Then there exists an equivalent norm \( \|\cdot\|_c \) on \( X \) such that \( (X,\|\cdot\|_c) \) does not have UPCP.
Proof. Pelczynski and Singer have shown [PS] that every Banach space with a basis admits a conditional basis. We may suppose then that \( Y \) has the conditional basis \( \{ e_n \}_{n=1}^{\infty} \). Let \( P_n \) be the projection on \( Y \) defined by
\[
P_n \left( \sum_{j=1}^{\infty} \alpha_j e_j \right) = \sum_{j=1}^{n} \alpha_j e_j
\]
and suppose that \( Y \) is the image of the projection \( P \) on \( X \). We can now define the norm \( \| \cdot \|_c \) on \( X \) by
\[
\| x \|_c = \sup_n \left\{ \| P_n P x \| + \| (1-P)x \| \right\}.
\]
Then
\[
\| x \|_c \leq \sup_n \left\{ \| P_n \| \| P_n P x \| + \| (1-P) x \| \right\} \leq K \| x \|
\]
say and
\[
\| x \| \leq \| P x \| + \| (1-P)x \|
\]
\[
= \lim_{n \to \infty} \| P_n P x \| + \| (1-P)x \|
\]
\[
\leq \sup_n \| P_n P x \| + \| (1-P)x \|
\]
\[
= \| x \|_c.
\]
The two norms are thus equivalent. We shall now define an increasing sequence on contractive projections on \( (X, \| \cdot \|_c) \) which does not have the unconditionality property.

Define the increasing sequence of projections \( \{ Q_j \}_{j=1}^{\infty} \) on \( X \) by
\[
Q_j x = P_j P x. \quad \text{Then}
\]
\[
\| Q_j x \|_c = \sup_n \left\{ \| P_n P_j P x \| + \| (1-P_j)P x \| \right\}
\]
\[
= \sup_n \left\{ \| P_{\min(n,j)} P x \| \right\}
\]
\[
\leq \| x \|_c
\]
so each of the projections \( Q_j \) is contractive under the new norm. As the basis \( \{ e_n \} \) is conditional there exists \( v = \sum_{j=1}^{\infty} \alpha_j e_j \in Y \) and a
sequence \( \{a_j\}_{j=1}^{\infty} \) of unimodular scalars such that

\[
\lim_{n \to \infty} \left\| \sum_{j=1}^{n} a_j \alpha_j e_j \right\| = \lim_{n \to \infty} \left\| \sum_{j=1}^{n} a_j \alpha_j e_j \right\| c = \infty.
\]

In other words,

\[
\lim_{n \to \infty} \left\| \sum_{j=1}^{n} a_j (Q_j - Q_{j-1}) \right\| c = \infty
\]

and so \( (X, \| \cdot \|_c) \) does not have UPCP.

6.4.2. COROLLARY. There exists a Banach space which has UMD but which does not have UPCP.

Proof. Take \( X \) to be \( (L^2[0,1], \| \cdot \|_2) \). Since \( X \) has a basis, we may give it an equivalent norm, \( \| \cdot \|_c \) say, so that \( (L^2[0,1], \| \cdot \|_c) \) does not have UPCP. This space does however have UMD, since \( (L^2[0,1], \| \cdot \|_2) \) does.

6.4.3. Remark. It was known to Banach (see [Ban, p. 238]) that every Banach space contains a subspace with a basis (for a proof see [SII, Theorem 1.1]). The question of whether we can always choose this subspace to be complemented appears to be very difficult.

The question of whether ever UPCP (or even uniform UPCP) space must be UMD is still open. It is interesting to note however the following result for Lebesgue-Bochner spaces.

6.4.4. PROPOSITION. Suppose that \( X \) is a Banach space and that for some \( 1 < p_0 < \infty \), \( L^{p_0}([0,1];X) \) has uniform UPCP. Then for \( 1 < p < \infty \), \( L^p([0,1];X) \) has UMD.
Proof. As was noted in Chapter 4, martingales in $L^0([0,1];X)$ correspond to increasing sequences of contractive projections (in fact, conditional expectation operators) on that space. Thus, by the unconditionality property for contractive projections on $L^0([0,1];X)$, martingale transforms are bounded. As we have assumed that $L^0([0,1];X)$ has uniform UPCP, the bound does not depend on the particular martingale. By the remark following Definition 4.6.7 then, that $X$ has UMD. As has been noted by Bourgain [Boug], this implies that $L^p([0,1];X)$ has UMD for $1 < p < \infty$. □

A more difficult proposition would be to prove the following.

6.4.5. Conjecture. If $X$ has UPCP, then for $1 < p < \infty$, $L^p(\Omega,\mathcal{A},\mu;X)$ has UPCP.
The following list includes notation which is either not defined in the body of the thesis or which is used in a different section to where it is defined.

\( X \)  
A real or complex Banach space

\( X^\circ \)  
The Banach space of continuous linear functionals on \( X \)

\( \langle x, x^\circ \rangle \)  
The linear functional \( x^\circ \in X^\circ \) evaluated at \( x \in X \)

\( \mathcal{H} \)  
A complex, separable, infinite-dimensional Hilbert space

\( (x|y) \)  
The inner product of \( x,y \in \mathcal{H} \)

\( \mathcal{B}(X) \)  
The space of bounded (i.e. continuous) linear transformations on \( X \)

\( \text{Proj}(X) \)  
The set of projections, or idempotent operators on \( X \)

\( \text{Proj}_1(X) \)  
The set of contractive projections on \( X \)

\( \mathcal{K}(X) \)  
See § 4.6

\( \sigma(T) \)  
The spectrum of an operator \( T \)

\( T_C \)  
The complexification of an operator on a real Banach space

\( \mathcal{C}(\Omega) \)  
The space of continuous functions on the Hausdorff space \( \Omega \)

\( \mathcal{B} \) or \( \mathcal{B}(\Omega) \)  
The Borel subsets of \( \Omega \)

\( \mathcal{P} \) or \( \mathcal{P}(I) \)  
The set of finite partitions of an interval \( I \subseteq \mathbb{R} \)
AC(J) the space of absolutely continuous functions on the compact interval \( J \subset \mathbb{R} \)

BV(J) the space of functions of bounded variation over the interval \( J \)

\( \chi_A \) the characteristic function of a set \( A \)

\((\Omega, \mathcal{A}, \mu)\) a positive measure space

\( L^p(\Omega, \mathcal{A}, \mu) \) the space of equivalence classes of \( p \)-integrable, \( \mathcal{A} \)-measurable functions on \( \Omega \) (\( 1 \leq p < \infty \))

\( L^\infty(\Omega, \mathcal{A}, \mu) \) the space of equivalence classes of essentially bounded \( \mathcal{A} \)-measurable functions on \( \Omega \)

\( C_p \) the von-Neumann-Schatten \( p \) class of compact operators on \( \mathcal{H} \)

\( \| f \|_p \) the norm of \( f \in L^p(\Omega, \mathcal{A}, \mu) \) (or \( f \in L^p(\Omega, \mathcal{A}, \mu; X) \))

\( \| T \|_p \) the norm of the operator \( T \in B(L^p(\Omega, \mathcal{A}, \mu)) \) (or occasionally of \( T \in C_p \))

\( \mathbb{R}, \mathbb{C} \) the real and complex scalar fields

\( \mathbb{Z}, \mathbb{N} \) the integers and the positive integers

SOT the strong operator topology on \( B(X) \)

WOT the weak operator topology on \( B(X) \)

\( W^{*}OT \) the weak--\( * \) operator topology on \( B(X^*) \)
REFERENCES


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