Kazhdan's Property (T) and Related Properties of Locally Compact and Discrete Groups.

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Declaration.

This thesis was composed by myself and has not been submitted for any other degree or professional qualification. All work not otherwise attributed is original.
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Abstract.

In this thesis we look at a number of properties related to Kazhdan’s property (T), for a locally compact, metrisable, σ-compact group. For such a group, G, the following properties are equivalent.

1. Kazhdan’s definition of property (T): the trivial representation is isolated in the unitary dual of G (with the Fell topology).

2. The group, G, is compactly generated and for every compact generating set, K, there is a positive constant, ε, such that if π is a unitary representation of G on a Hilbert space, H, and ζ is a unit vector in H such that

$$\sup_{g \in K} \| \pi(g)\zeta - \zeta \| < \varepsilon$$

then π fixes some non-zero vector in H. This is often taken as the definition of property (T).

3. Every conditionally negative type function on G is bounded.

4. For a discrete group, G is finitely generated and for every finite generating set, K, zero is an isolated point in the spectrum of the Laplacian,

$$\Delta = |K \cup K^{-1}|e - \sum_{g \in K \cup K^{-1}} g, \text{ in } C^*(G).$$

From 2 we can define the Kazhdan constant, the largest possible value of ε for a given G and K. In Chapter 2 we investigate how to calculate these constants. In Chapter 4 we look at the bound on conditionally negative type functions and use its existence to extend a result of A.Connes and V.Jones about the von Neumann algebras of property (T) groups. The first half of Chapter 3 examines the spectrum of the Laplacian for a discrete group and finite generating set and compares its least positive element to the Kazhdan constant.
Non-compact property (T) groups are all non-amenable. However, the standard example of a non-amenable group, $F_2$, does not have property (T). The second half of Chapter 3 looks at the spectrum of $\lambda(\Delta)$ for $F_2$ with various generating sets, where $\lambda$ is the left regular representation of $G$ on $l^2(G)$. For any non-amenable group, the smallest element of $\text{Sp} \lambda(\Delta)$ is positive. Chapter 5 is an attempt to extend various results about $F_2$ to other non-amenable groups.
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Chapter 1

Introduction.

Property (T) was introduced by D.A. Kazhdan in 1966 [Kaz]. His definition says that a locally compact group, $G$, has property (T) if the trivial representation is isolated in the unitary dual, $\hat{G}$, of $G$. The unitary dual is the space of (equivalence classes of) unitary representations of $G$ on Hilbert space, with the Fell topology (described in Section 2.1). Some equivalent definitions of property (T) appear in Chapter 2. The one with which we shall work says that $G$ has property (T) if and only if $G$ is compactly generated and for any compact generating set, $K$, there is a positive number, $\varepsilon$, such that if $\pi : G \to U(H)$ is a unitary representation of $G$ on a Hilbert space, $\mathcal{H}$, and $\zeta \in \mathcal{H}$ with $\|\zeta\| = 1$ is such that

$$\sup_{g \in K} \| \pi(g)\zeta - \zeta \| < \varepsilon$$

then $\pi$ has a non-zero invariant vector. The largest value of $\varepsilon$ corresponding to a given $G$ and $K$ is the Kazhdan constant, $\varepsilon(G, K)$. In Chapter 2 we are concerned with how to calculate these constants. We look at some simple examples, including cyclic groups and the circle group, and we establish rules which should ease more general calculations. The eventual aim is to find rules which will make it possible to calculate Kazhdan constants for more complicated groups and, ultimately, to link the Kazhdan constant to well documented information about groups, such as group characters.
One use of Kazhdan constants is in indicating how efficient Cayley graphs are as networks for transmitting information. This can assist in the design of telephone networks and computer circuitry, where the aim is to transmit information from one point to another as quickly as possible while minimising both the cost of materials and the space occupied. The construction of efficient graphs is discussed in the survey article by F.Bien [Bien] and the forthcoming book by A.Lubotzky [Lub].

Other constants associated with finite generating sets of discrete groups are useful in determining the efficiency of networks and are discussed in the same literature. These are the least positive values in the spectra of the Laplacian, \( \Delta \), and of \( \lambda(\Delta) \), where \( \lambda \) is the left regular representation of \( G \) on \( L^2(G) \). If \( K \) is a finite generating set for \( G \) such that \( K = K^{-1} \) then the Laplacian is the element of the group \( C^* \)-algebra given by \( \Delta = |K|e - \sum_{g \in K} g \). In the first half of Chapter 3 we look at the spectrum of \( \Delta \) and its relationship to the Kazhdan constant. In the case of certain finite groups, this should make it possible to calculate Kazhdan constants by computer, since programmes exist to calculate eigenvalues of Laplacians. In the second half of the chapter we examine the spectrum of \( \lambda(\Delta) \) for free groups, with a variety of generating sets. The spectrum of the Laplacian has been studied in [HRV], where it is shown that a finitely generated, discrete group has property (T) if and only if zero is an isolated point in the spectrum of the Laplacian.

Another property equivalent to property (T) is the absence of unbounded conditionally negative type functions ([HaV] Ch.5 Theorem 20). In Section 4.5 we establish a bound on real valued conditionally negative type functions on a property (T) group, \( G \), in terms of the supremum on a compact generating set, \( K \), and the Kazhdan constant \( \varepsilon(G, K) \). Conditionally negative type functions are more usually referred to as continuous negative definite functions, in
English. We use the European terminology, stressing that a conditionally negative type function is not simply the negative analogue of a positive type (continuous positive definite) function. In particular, real valued conditionally negative type functions are strictly non-negative.

Examples of conditionally negative type functions are provided by distance functions for the actions of groups on trees and $\mathbb{R}$-trees; that is how far each element of the group moves a chosen point of the tree or $\mathbb{R}$-tree. Sections 4.6 to 4.10 deal with some of these examples. If the group has property (T) then such an action must have a fixed point ([HaV] Ch.6 Propositions 4 and 11). We show in Section 4.5 that the bound on the distance function is then its supremum on any compact generating set. This is clearly the best possible bound, while there is no reason to suppose the same of the more general bound for real valued conditionally negative type functions. One possible direction for further work is to look at precisely which functions the tighter bound applies to and to establish the best possible bound in the more general case.

In the final sections of Chapter 4 we apply the boundedness of conditionally negative type functions on Kazhdan groups to von Neumann algebras arising from the groups. The results relate closely to those in [C&J], where property (T) for von Neumann algebras was first defined.

Non-compact Kazhdan groups are known not to be amenable ([HaV] Ch.1 Example 5(i)), but the classic example of a non-amenable group, $F_2$, does not have property (T). This follows from the fact that the word length on $F_2$ is an unbounded conditionally negative type function (see Section 4.6). Chapter 5 is an attempt to generalise to other discrete, non-amenable groups various results about $F_2$ which are known to fail for amenable groups. The first result we look at is a theorem of A.G.Robertson and R.R.Smith (Theorem 3.2 of [R&S]) which says that there is a Hilbert space, $\mathcal{H}$, such that for each positive integer, $n,$
there is an $n$-positive unital map from $C^*_\lambda(F_2)$ into $B(\mathcal{H})$ which has no positive extension to $B(L^2(F_2))$. The second is a theorem of S. Wasserman ([Was]) which says that the identity map on $C^*_\lambda(F_2)$ does not have a completely positive lifting to $C^*(F_2)$. These results and our generalisations of them depend on the fact that $F_2$ is residually finite. We do not know of any examples of residually finite, non-amenable groups which do not contain $F_2$.

1.1 Notation and Conventions.

Throughout this work we are concerned with locally compact groups with a countable base for the topology. These are what are meant whenever we refer to groups, unless otherwise stated. In particular, the discrete groups will all be countable. Note that a locally compact group has a countable base for its topology if and only if it is metrisable and is a countable union of compact sets, that is metrisable and $\sigma$-compact ([BTG] page IX.21). The identity element of a group, $G$, will be denoted by $e$ or $e_G$.

A representation, $\pi$, of a group, $G$, will be a strongly continuous, unitary representation of $G$ on a Hilbert space, $\mathcal{H}_\pi$. That is, if $U(\mathcal{H})$ denotes the space of unitary operators on a Hilbert space $\mathcal{H}$, then $\pi$ is a homomorphism from $G$ into $U(\mathcal{H}_\pi)$ and the map $G \times \mathcal{H}_\pi \to \mathcal{H}_\pi$ defined by $(g, \zeta) \mapsto \pi(g)\zeta$ ($g \in G$, $\zeta \in \mathcal{H}_\pi$) is continuous. Two representations, $\pi$ and $\rho$, of $G$ are equivalent if there is an isometric isomorphism $U : \mathcal{H}_\rho \to \mathcal{H}_\pi$ such that, $\rho(g) = U^{-1}\pi(g)U$ for each $g \in G$. Then $\|\rho(g)\zeta - \zeta\| = \|U^{-1}\pi(g)U\zeta - \zeta\| = \|\pi(g)U\zeta - U\zeta\|$ and $\|U\zeta\| = \|\zeta\|$. This shows that in the various calculations that follow it is sufficient to consider equivalence classes of representations.

If $\pi$ and $\rho$ are two representations of the group $G$, then their direct sum,
\( \pi \oplus \rho \), is the representation of \( G \) on the Hilbert space \( \mathcal{H}_\pi \oplus \mathcal{H}_\rho \) defined by

\[
(\pi \oplus \rho)(g)(\zeta \oplus \eta) = \pi(g)\zeta \oplus \rho(g)\eta \quad g \in G, \ \zeta \in \mathcal{H}_\pi, \ \eta \in \mathcal{H}_\rho.
\]

Let \( G \) be a group and let \( \pi \) be a representation of \( G \). A closed subspace, \( \mathcal{K} \), of \( \mathcal{H}_\pi \) is said to be invariant if \( \pi(G)\mathcal{K} = \mathcal{K} \). The representation \( \pi \) contains another representation, \( \rho \), if \( \rho \) is (equivalent to) the restriction of \( \pi \) to a closed invariant subspace, \( \mathcal{H}_\rho \), of \( \mathcal{H}_\pi \). Then \( \pi = \rho \oplus \sigma \) where \( \sigma \) is the restriction of \( \pi \) to the orthogonal complement of \( \mathcal{H}_\rho \) in \( \mathcal{H}_\pi \). The representation \( \rho \) is called a subrepresentation of \( \pi \).

The representation \( \pi \) is irreducible if \( \mathcal{H}_\pi \) has no proper invariant closed subspace.

A vector \( \zeta \in \mathcal{H}_\pi \) is invariant if \( \pi(g)\zeta = \zeta \), for all \( g \in G \).

**Notation.**

The set of (equivalence classes of) representations of \( G \) on separable Hilbert spaces will be denoted by \( \hat{G} \). We shall refer to these as separable representations.

The set of (equivalence classes of) separable representations of \( G \) with no non-zero invariant vector will be denoted by \( \hat{G}^* \).

The set of (equivalence classes of) irreducible representations of \( G \) will be denoted by \( \hat{G} \).

The trivial representation of \( G \) on \( \mathbb{C} \) is \( \pi_0 \), where \( \pi_0(g)\zeta = \zeta \) for all \( \zeta \in \mathbb{C} \).

We shall denote \( \hat{G} \setminus \{\pi_0\} \) by \( \hat{G}^* \).

Note that an irreducible representation has non-zero invariant vectors precisely if it is (equivalent to) \( \pi_0 \), since if \( \zeta \) is a non-zero invariant vector for the representation \( \pi \), then \( \zeta \) spans a 1-dimensional, invariant subspace of \( \mathcal{H}_\pi \).
Chapter 2

Kazhdan Constants for Compact and Discrete Groups.

We begin this chapter by explaining Kazhdan's definition of property (T). We shall then introduce equivalent definitions (found in Chapter 1 of [HaV]) which will enable us to define the Kazhdan constant for a given group and compact generating set. The remainder of the chapter is an investigation of Kazhdan constants, including their calculation in some simple cases and results which should aid more complicated calculations.

2.1 Definitions of Kazhdan's Property (T).

Let $G$ be a group (locally compact, metrisable and $\sigma$-compact, as in the introduction). Recall, from the introduction, that $\hat{G}$ is the space of irreducible representations of $G$ and that $\hat{G}$ is the space representations of $G$ on separable Hilbert spaces. Observe first that $\hat{G} \subseteq \hat{G}$, that is if $\pi \in \hat{G}$ then $\mathcal{H}_\pi$ is a separable Hilbert space. This follows from the fact that $G$ is metrisable and the countable union of compact sets, since this guarantees that $G$ has a countable dense subset, and so the strong continuity of $\pi$ means that the same is true of $\pi(G)\zeta$ for each $\zeta \in \mathcal{H}_\pi$; but the closed linear span of $\pi(G)\zeta$ is an invariant subspace of $\mathcal{H}_\pi$ and so must be the whole of $\mathcal{H}_\pi$ (if $\zeta \neq 0$); hence the countable dense subset of $\pi(G)\zeta$ can be used to construct such a subset in the linear span.
of $\pi(G)\zeta$, which is itself dense in $\mathcal{H}_\pi$.

The Fell Topology.

We topologize $\hat{G}$ as follows. If $\pi \in \hat{G}$, $K$ is a compact subset of $G$, $\varepsilon > 0$ and $\zeta_1, \ldots, \zeta_n$ are orthonormal vectors in $\mathcal{H}_\pi$, then we define the neighbourhood $N(\pi, K, \varepsilon, \zeta_1, \ldots, \zeta_n)$ of $\pi$ to consist of those representations $\rho \in \hat{G}$ for which there are orthonormal vectors $\eta_1, \ldots, \eta_n \in \mathcal{H}_\rho$ such that

$$\sup_{g \in K} |\langle \eta_i, \rho(g)\eta_j \rangle - \langle \zeta_i, \pi(g)\zeta_j \rangle| \leq \varepsilon \quad 1 \leq i, j \leq n.$$ 

These sets form a base of neighbourhoods for the Fell topology on $\hat{G}$.

The Fell topology on $\hat{G}$ is the subspace topology induced by the Fell topology on $G$. Kazhdan used this topology in his definition of property (T).

**Definition 2.1.1 (Kazhdan’s Property (T))**:

A group, $G$, has property (T), or is a Kazhdan group, if and only if $\pi_0$ is an isolated point in $\hat{G}$ (with the Fell topology).

**Definition 2.1.2**:

Let $\pi$ be a representation of $G$. Suppose $K$ is a compact subset of $G$ and $\varepsilon > 0$. A unit vector $\zeta \in \mathcal{H}_\pi$ is $(\varepsilon, K)$-invariant if

$$\sup_{g \in K} \|\pi(g)\zeta - \zeta\| < \varepsilon.$$ 

Since $K$ is compact, the strong continuity of $\pi$ implies that the above supremum is in fact a maximum.

**Definition 2.1.3**:

We say that $\pi$ almost has invariant vectors if $\pi$ has an $(\varepsilon, K)$-invariant unit vector for every number $\varepsilon > 0$ and every compact set $K \subseteq G$.

The basic neighbourhoods of $\pi_0$ in the Fell topology on $\hat{G}$ are

$$N(\pi_0, K, \varepsilon, 1) = \left\{ \pi \in \hat{G} : \exists \zeta \in \mathcal{H}_\pi, \|\zeta\| = 1 \text{ s.t. } \sup_{g \in K} |\langle \zeta, \pi(g)\zeta \rangle - 1| \leq \varepsilon \right\}$$
where $\varepsilon > 0$ and $K$ is a compact subset of $G$. Suppose that $\pi \in \hat{G}$ and $\pi_0 \in \{\pi\}$. Then for any positive number, $\varepsilon$, and any compact subset, $K$, of $G$, it follows that $\pi \in N(\pi_0, K, \varepsilon^2/3, 1)$ and so there is a unit vector $\zeta \in \mathcal{H}_\pi$ such that

$$\sup_{g \in K} \|\pi(g)\zeta - \zeta\|^2 = 2 \sup_{g \in K} (1 - \Re \langle \zeta | \pi(g)\zeta \rangle)$$

$$\leq 2 \sup_{g \in K} |\langle \zeta | \pi(g)\zeta \rangle - 1|$$

$$< \varepsilon^2;$$

that is $\zeta$ is an $(\varepsilon, K)$-invariant unit vector. Hence $\pi$ almost has invariant vectors.

Conversely, suppose $\pi \in \hat{G}$ and $\pi$ almost has invariant vectors. Then, for any compact $K \subset G$ and $\varepsilon > 0$, $\pi$ has an $(\varepsilon, K)$-invariant unit vector, $\zeta$, and so

$$\sup_{g \in K} |\langle \zeta | \pi(g)\zeta \rangle - 1| = \sup_{g \in K} |\langle \zeta | \pi(g)\zeta - \zeta \rangle|$$

$$\leq \sup_{g \in K} \|\pi(g)\zeta - \zeta\|$$

$$< \varepsilon$$

and $\pi \in N(\pi_0, K, \varepsilon, 1)$. We conclude that a representation $\pi \in \hat{G}$ almost has invariant vectors if and only if $\pi_0$ lies in the closure of $\{\pi\}$.

An alternative definition of property (T) is given in [HaV].

**Proposition 2.1.4 ([HaV] Chapter 1, Proposition 14):**

A group $G$ is a Kazhdan group if and only if every representation of $G$ which almost has invariant vectors in fact has a non-zero invariant vector.

Suppose $G$ satisfies this new definition, $\pi \in \hat{G}$ and $\pi_0 \in \{\pi\}$. Then $\pi$ almost has invariant vectors and so $\pi$ has a non-zero invariant vector. Since $\pi$ is irreducible it follows that $\pi = \pi_0$. Hence $\pi_0$ is isolated in $\hat{G}$ and $G$ satisfies Kazhdan's definition of property (T). The reverse implication of the proposition is more complicated and is proved in [HaV] (Ch.1, Proposition 14, (iii) $\Rightarrow$ (i)).
Separable Spaces.

We have already seen that if \( \pi \) is a representation of \( G \) and \( \zeta \in \mathcal{H}_\pi \) then \( \text{span} \, \pi(G)\zeta \) is a separable, invariant subspace of \( \mathcal{H}_\pi \). Then \( \pi \) is the direct sum of its restrictions to this separable Hilbert space and to its orthogonal complement in \( \mathcal{H}_\pi \). If \( \zeta \) is an \((\varepsilon, K)\)-invariant unit vector (resp. non-zero invariant vector) for \( \pi \) then it is also an \((\varepsilon, K)\)-invariant unit vector (resp. non-zero invariant vector) for the restriction of \( \pi \) to the separable space \( \text{span} \, \pi(G)\zeta \). Conversely, if any separable subrepresentation of \( \pi \) has an \((\varepsilon, K)\)-invariant unit vector (resp. non-zero invariant vector) then this provides an \((\varepsilon, K)\)-invariant unit vector (resp. non-zero invariant vector) for \( \pi \). Since \( G \) is a countable union of compact sets it follows that \( \pi \) almost has invariant vectors if and only if we can find a countable direct sum of the separable subrepresentations which almost has invariant vectors. The countable direct sum of separable Hilbert spaces is separable, so we see that \( \pi \) almost has invariant vectors if and only if it has a subrepresentation on a separable Hilbert space which also almost has invariant vectors. Hence \( G \) has property (T) if and only if every representation of \( G \) on a separable Hilbert space which almost has invariant vectors actually has a non-zero invariant vector. In the space of such representations, with the Fell topology, we see that a group has property (T) if and only if every representation, \( \pi \), such that \( \pi_0 \in \{\pi\} \) has a non-zero invariant vector.

2.2 Kazhdan Constants.

Proposition 2.2.1 ([HaV] Chapter 1, Lemma 15):

Let \( G \) be a locally compact group generated by a compact set \( K \).

(a) If \( G \) has property (T), then there exists a number \( \varepsilon > 0 \) such that each representation \( \pi : G \to \mathcal{U}(\mathcal{H}) \) which has an \((\varepsilon, K)\)-invariant unit vector also has a non-zero invariant vector.
(b) Suppose there exists a number \( \delta > 0 \) such that every irreducible representation of \( G \) which has a \((\delta, K)\)-invariant unit vector has a non-zero invariant vector. Then \( G \) has property (T).

Part (b) is immediate, since the existence of such a \( \delta \) implies that there is a basic neighbourhood, \( N(\pi_0, K, \frac{\delta^2}{3}, 1) \cap \hat{G} \), of \( \pi_0 \) in \( \hat{G} \) which contains only \( \pi_0 \). For a proof of part (a) see [HaV] Chapter 1, Lemma 15.

It is clear that the proposition will still hold if we restrict (a) to irreducible representations and/or extend (b) to all representations. Thus we see that each of the two conditions is equivalent to property (T).

If the group \( G \) has property (T) then [HaV] Chapter 1, Lemma 10 says that \( G \) is compactly generated. Hence we can rewrite Proposition 2.2.1 to state two more alternative definitions of property (T).

**Proposition 2.2.2** The following are equivalent.

(i) The group \( G \) has Kazhdan's property (T).

(ii) The group \( G \) has a compact generating set, \( K \), for which there is a number \( \varepsilon > 0 \) such that each representation of \( G \) which has an \((\varepsilon, K)\)-invariant unit vector also has a non-zero invariant vector.

(iii) The group \( G \) is compactly generated and for every compact generating set, \( K \), of \( G \) there is a number \( \varepsilon > 0 \) such that each representation of \( G \) which has an \((\varepsilon, K)\)-invariant unit vector also has a non-zero invariant vector.

Part (iii) is the version of property (T) with which we shall work.

Let \( G \) be a Kazhdan group and let \( K \) be a compact generating set for \( G \). It is clear that if, for this given \( G \) and \( K \), a particular value of \( \varepsilon \) will work in Proposition 2.2.1 (a) then so will any smaller positive value. Hence our interest is in finding the largest possible value of \( \varepsilon \) for given \( G \) and \( K \).
Definition 2.2.3:

We define the Kazhdan constant, $\varepsilon(G,K)$, of $G$ and $K$ to be the supremum of the set of possible values of $\varepsilon$ in Proposition 2.2.1 (a).

Lemma 2.2.4:

The above supremum is in fact a maximum.

Proof:

Let $\pi$ be a representation of $G$. If $\zeta$ is an $(\varepsilon(G,K),K)$-invariant unit vector, that is $\sup_{g \in K} \|\pi(g)\zeta - \zeta\| < \varepsilon(G,K)$, then there is a positive number, $\varepsilon$, such that

$$\sup_{g \in K} \|\pi(g)\zeta - \zeta\| < \varepsilon < \varepsilon(G,K),$$

so that $\zeta$ is also $(\varepsilon,K)$-invariant. But, by the definition of $\varepsilon(G,K)$, any representation of $G$ which has an $(\varepsilon,K)$-invariant unit vector also has a non-zero invariant vector. Hence $\pi$ must have a non-zero invariant vector and $\varepsilon(G,K)$ is a possible value of $\varepsilon$ in Proposition 2.2.1 (a).

$\square$

Recall that a representation of $G$ has an $(\varepsilon,K)$-invariant unit vector or a non-zero invariant vector if and only if the same is true of one of its separable subrepresentations. Hence we do not alter the possible values of $\varepsilon$ in Proposition 2.2.1 (a) by restricting attention to separable representations. Then

$$\varepsilon(G,K) = \inf_{\pi \in \hat{G}^*} \inf_{\kappa \in \mathcal{H}_\pi, \|\kappa\| = 1} \sup_{g \in K} \|\pi(g)\zeta - \zeta\|. $$

Recall from Chapter 1 that $\hat{G}^*$ is the set of representations of $G$ on separable Hilbert spaces without non-zero invariant vectors and that $\hat{G}^*$ is the set of non-trivial irreducible representations of $G$.

Note that if $K_1$ and $K_2$ are both compact generating sets for $G$ and $K_1 \subseteq K_2$, then $\varepsilon(G,K_1) \leq \varepsilon(G,K_2)$. 

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**Definition 2.2.5**:

For convenience of notation we define, for each representation \( \pi \) of \( G \),

\[
\varepsilon(\pi, G, K) = \inf_{\zeta} \left\{ \sup_{g \in K} \| \pi(g) \zeta - \zeta \| : \zeta \in \mathcal{H}_\pi, \| \zeta \| = 1 \right\}.
\]

This definition is meaningful even if \( G \) does not have property (T) (as long as \( K \) is a compact generating set) and we can extend the definition of \( \varepsilon(G, K) \) to all compactly generated groups by

\[
\varepsilon(G, K) = \inf \left\{ \varepsilon(\pi, G, K) : \pi \in \hat{G}^* \right\}.
\]

Notice that if \( G \) is not a Kazhdan group then \( \varepsilon(G, K) = 0 \).

**Lemma 2.2.6**:

If \( \pi_1 \) and \( \pi_2 \) are representations of \( G \) and \( \pi = \pi_1 \oplus \pi_2 \), then

\[
\varepsilon(\pi, G, K) \leq \varepsilon(\pi_1, G, K).
\]

**Proof**:

If \( \zeta \in \mathcal{H}_{\pi_1} \) and \( \eta \) is the vector in \( \mathcal{H}_\pi = \mathcal{H}_{\pi_1} \oplus \mathcal{H}_{\pi_2} \) with component \( \zeta \) in \( \mathcal{H}_{\pi_1} \) and 0 in \( \mathcal{H}_{\pi_2} \), then \( \| \eta \| = \| \zeta \| \) and, for each \( g \in G \), \( \| \pi(g) \eta - \eta \| = \| \pi_1(g) \zeta - \zeta \| \).

\( \square \)

**Definition 2.2.7**:

We also define

\[
\varepsilon(G, K) = \inf \left\{ \varepsilon(\pi, G, K) : \pi \in \hat{G}^* \right\}.
\]

Then \( \varepsilon(G, K) \leq \varepsilon(G, K) \). In general the inequality is strict (see the end of Section 2.6 below for an example).

For every unitary representation \( \pi \) of \( G \), every \( g \in G \) and every \( \zeta \in \mathcal{H}_\pi \),

\[
\| \pi(g) \zeta - \zeta \| \leq 2\| \zeta \|. \quad \text{So } \varepsilon(\pi, G, K) \leq 2 \text{ for each } \pi \in \hat{G}^*. \quad \text{Hence}
\]

\[
\varepsilon(G, K) \leq 2
\]

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unless every representation of $G$ has a non-zero invariant vector, in which case $G$ is the trivial group, $\{e\}$, and $\varepsilon(G,K)$ is infinite. The upper bound is attained; for example, if $C_2$ denotes the cyclic group with 2 elements, $\varepsilon(C_2,C_2) = 2$, as will be seen below (Section 2.6).

If $H$ is a compact group then $H$ has property (T) and $\varepsilon(H,H) \geq \sqrt{2}$ ([HaV] Chapter 1, Theorem 5(ii)). We show later that this value is attained for the circle group (see Section 2.7). An amenable group has property (T) if and only if it is compact ([HaV] 1.7(i)).

Kazhdan constants are known in some very simple cases such as the cyclic group with a single generator or the dihedral group with two generators of order 2. They have recently been calculated in some less simple cases. In [BaH] R.Bacher and P.de la Harpe calculate the Kazhdan constant for the symmetric group, $S_n$, with generating set $\{(1,2),(2,3),(3,4),\ldots,(n-1,n)\}$. M.Burger has found Kazhdan constants for $SL(3,\mathbb{Z})$ with various generating sets. The results of this chapter include the calculation of $\varepsilon(G,G)$ when $G$ is a cyclic group and when $G$ is the circle group.

2.3 A Simple Example.

For each integer $n \geq 2$, let $C_n = \{e, g, g^2, \ldots, g^{n-1}\}$ be the cyclic group of order $n$, generated by the single element $g$.

**Lemma 2.3.1** ([HaV] Chapter 1, Section 17):

*For each integer $n \geq 2$,*

$$\varepsilon(C_n,\{g\}) = 2 \sin \frac{\pi}{n}.$$

**Proof:**

The irreducible representations of $G$ are $\pi_0, \pi_1, \ldots, \pi_{n-1}$, where

$$\pi_j(g^k)\zeta = \omega^{jk}\zeta \quad \zeta \in \mathbb{C} ; \quad j, k \in \{0,1,\ldots,n-1\}$$
and

\[ \omega = e^{\frac{2\pi i}{n}}. \]

By [NaS] Chapter IV, 2.7 VII, any representation of a compact group is the direct sum of finite dimensional, irreducible representations. Hence, if \( \pi \in \hat{G}_n \), then \( \pi \) is a direct sum of copies of the \( \pi_j \),

\[ \pi = \bigoplus_{\alpha \in A} \pi_{j_\alpha} \]

where each \( j_\alpha \in \{1, 2, \ldots, n - 1\} \).

Then

\[
\varepsilon \left( G_n, \{g\} \right)^2 = \inf_{\pi} \inf_{\zeta} \left\{ \| \pi(g) \zeta - \zeta \|^2 : \pi \in \hat{G}_n, \zeta = \bigoplus_{\alpha \in A} \zeta_\alpha \in \mathcal{H}_\pi, \| \zeta \| = 1 \right\}
\]

\[
= \inf_{(j_\alpha)_{\alpha \in A}} \inf_{(\zeta_\alpha)_{\alpha \in A}} \left\{ \sum_{\alpha \in A} |\zeta_\alpha|^2 |\omega^{j_\alpha} - 1|^2 : j_\alpha \in \{1, \ldots, n - 1\}, \zeta_\alpha \in \mathbb{C}, \sum_{\alpha \in A} |\zeta_\alpha|^2 = 1 \right\}
\]

\[
= \inf \left\{ |\omega^j - 1|^2 : j \in \{1, 2, \ldots, n - 1\} \right\}
\]

\[ = |\omega - 1|^2. \]

Whence

\[ \varepsilon \left( G_n, \{g\} \right) = |\omega - 1| = 2 \sin \frac{\pi}{n}. \]

\( \Box \)

We notice here that \( \varepsilon \left( G_n, \{g\} \right) = \varepsilon(C_n, \{g\}) \) and is attained for the irreducible representation \( \pi_1 \).

### 2.4 Diversion; a Lower Bound.

In the above example, the calculation was simple because the generating set had only one element. This meant that we were able to restrict attention to the
irreducible representations of $C_n$. Here we obtain a lower bound for $\varepsilon(G, K)$ which also requires us to consider only irreducible representations.

Let $G$ be a compactly generated group and let $K$ be a compact generating set for $G$. Let $\nu$ be a positive Borel measure on $G$ for which $\nu(K) = 1$. Let

$$\delta(G, K, \nu) = \inf_{\pi} \inf_{\zeta} \left\{ \int_K \|\pi(g)\zeta - \zeta\|^2 d\nu(g) : \pi \in \hat{G}, \zeta \in \mathcal{H}_\pi, \|\zeta\| = 1 \right\}.$$ 

For any $\pi \in \hat{G}$ and any $\zeta \in \mathcal{H}_\pi$ with $\|\zeta\| = 1$,

$$\delta(G, K, \nu) \leq \int_K \|\pi(g)\zeta - \zeta\|^2 d\nu(g) \leq \sup_{g \in K} \|\pi(g)\zeta - \zeta\|^2.$$ 

So

$$\delta(G, K, \nu) \leq \varepsilon(G, K)^2.$$ 

The lower bound, $\delta(G, K, \nu)$, may be zero, in which case this does not give us any information about the Kazhdan constant, since we already know that $\varepsilon(G, K) \geq 0$.

Now define

$$\tilde{\delta}(G, K, \nu) = \inf_{\pi} \inf_{\zeta} \left\{ \int_K \|\pi(g)\zeta - \zeta\|^2 d\nu(g) : \pi \in \hat{G}, \zeta \in \mathcal{H}_\pi, \|\zeta\| = 1 \right\}.$$ 

Then $\tilde{\delta}(G, K, \nu) \geq \delta(G, K, \nu)$.

The following result will be used later (Section 2.8).

Lemma 2.4.1:

With the above definitions, $\delta(G, K, \nu) = \tilde{\delta}(G, K, \nu)$.

Proof:

Theorem 8.5.2 of [Dix] says that if $A$ is a $C^*$-algebra and $\pi$ is a representation of $A$ on a separable Hilbert space, $\mathcal{H}_\pi$, then there is a measure space $(Z, \mu)$, where $\mu$ is a bounded positive measure on $Z$, and for each $z \in Z$ there is an irreducible representation $\pi_z$ of $A$ on a Hilbert space $\mathcal{H}_z$ such that

$$\mathcal{H}_\pi \cong \int_Z \mathcal{H}_z d\mu(z).$$
\[ \pi \text{ is equivalent to } \int_Z \pi_z d\mu(z). \]

From [Dix] 18.7.6 the same is true for any representation of the group \( G \).

Suppose \( \delta(G, K, \nu) > \delta(G, K, \nu) \). Then there exist a representation \( \pi \in \hat{G}^* \) and a unit vector \( \zeta \in \mathcal{H}_\pi \) for which

\[
\delta(G, K, \nu) > \int_K \|\pi(g)\zeta - \zeta\|^2 d\nu(g)
\]
\[
= \int_K \int_Z \|\pi_z(g)\zeta_z - \zeta_z\|^2 d\mu(z) d\nu(g)
\]
\[
\geq \int_Z \delta(G, K, \nu) \|\zeta_z\|^2 d\mu(z)
\]
\[
= \delta(G, K, \nu)
\]

where the measure space \((Z, \mu)\) and the representations \( \pi_z \) of \( G \) on Hilbert spaces \( \mathcal{H}_z \) are as described above, \( \zeta_z \) is the component of \( \zeta \) in \( \mathcal{H}_z \) and

\[
1 = \|\zeta\|^2 = \int_Z \|\zeta_z\|^2 d\mu(z). \]

Thus we arrive at a contradiction.

\[ \square \]

We conclude

Lemma 2.4.2:

With the above definitions

\[ \epsilon(G, K') \geq \epsilon(G, K) \geq \sqrt{\delta(G, K, \nu)}. \]

\[ \square \]

2.5 Compact Abelian Groups.

Our next aim is to restrict the set of representations we need to consider in calculating Kazhdan constants for a large class of groups. In the case of a compact abelian group it turns out that we need work with only one representation. Notice that an abelian group has property (T) if and only if it is
compact, since we have already observed that non-compact amenable groups do not have property (T).

Throughout this section, $G$ will be a compact abelian group and $K$ a compact generating set for $G$. Recall that $\pi_0$ is the trivial representation of $G$ on $\mathbb{C}$ and let the other inequivalent irreducible representations of $G$ be denoted by $\pi_\alpha$, $\alpha \in A$, for some indexing set $A$. We define the representation $\pi$ of $G$ by

$$\pi = \bigoplus_{\alpha \in A} \pi_\alpha.$$ 

**Theorem 2.5.1**: Let $G$ be a compact abelian group and let $K$ be a compact generating set for $G$. Then

$$\varepsilon(G, K) = \varepsilon(\pi, G, K).$$ 

**Proof**: We know from the definitions that $\varepsilon(\pi, G, K) \geq \varepsilon(G, K)$. We need to show that $\varepsilon(\pi, G, K) \leq \varepsilon(G, K)$.

Since $G$ is abelian, the irreducible representations of $G$ are all one dimensional, and so we may regard each $\pi_\alpha(g)$ ($g \in G$) as a complex number of modulus 1, acting by multiplication on the one dimensional Hilbert space $\mathcal{H}_{\pi_\alpha} = \mathbb{C}$.

Any representation, $\rho$, of $G$ may be written as

$$\rho = \bigoplus_{\beta \in B} \pi_{\alpha\beta} : G \to \mathcal{U}(l^2(B)) \quad \alpha \beta \in A \cup \{0\}, \beta \in B$$

for some indexing set $B$ ([NaS] Chapter IV, 2.7 VII says that any representation of a compact group is equivalent to the direct sum of multiples of its finite dimensional, irreducible representations.)

If $\zeta = (\zeta_\beta)_{\beta \in B} \in l^2(B) = \mathcal{H}_\rho$ then

$$\|\zeta\|^2 = \sum_{\beta \in B} |\zeta_\beta|^2$$

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and, for any \( g \in G \),

\[
\|\rho(g)\zeta - \zeta\|^2 = \sum_{\beta \in B} \|\pi_{\alpha_\beta}(g)\zeta_\beta - \zeta_\beta\|^2 = \sum_{\beta \in B} |\pi_{\alpha_\beta}(g) - 1|^2 |\zeta_\beta|^2.
\]

For each \( \alpha \in A \cup \{0\} \) set

\[
B_\alpha = \{ \beta \in B : \alpha_\beta = \alpha \}.
\]

So that

\[
B = \bigcup_{\alpha \in A \cup \{0\}} B_\alpha.
\]

Note that \( \rho \) has no non-zero invariant vector precisely when \( B_0 = \emptyset \). From now on we consider only such \( \rho \).

Take \( \zeta \in H_\rho \) with \( \|\zeta\| = 1 \). Then

\[
1 = \|\zeta\|^2 = \sum_{\beta \in B} |\zeta_\beta|^2 = \sum_{\alpha \in A} \sum_{\beta \in B_\alpha} |\zeta_\beta|^2
\]

and for any \( g \in G \)

\[
\|\rho(g)\zeta - \zeta\|^2 = \sum_{\beta \in B} |\pi_{\alpha_\beta}(g) - 1|^2 |\zeta_\beta|^2 = \sum_{\alpha \in A} |\pi_{\alpha}(g) - 1|^2 \sum_{\beta \in B_\alpha} |\zeta_\beta|^2.
\]

Now define \( \eta = (\eta_\alpha)_{\alpha \in A} \in l^2(A) = H_\pi \) by

\[
\eta_\alpha = \sqrt{\sum_{\beta \in B_\alpha} |\zeta_\beta|^2}.
\]

(We could equally well have taken \( \eta_\alpha \) to be any complex number of modulus \( \sqrt{\sum_{\beta \in B_\alpha} |\zeta_\beta|^2} \).) Then

\[
\|\eta\|^2 = \sum_{\alpha \in A} |\eta_\alpha|^2 = \|\zeta\|^2 = 1
\]

and for each \( g \in G \)

\[
\|\pi(g)\eta - \eta\|^2 = \sum_{\alpha \in A} |\pi_{\alpha}(g) - 1|^2 |\eta_\alpha|^2 = \|\rho(g)\zeta - \zeta\|^2.
\]

We have shown that for any representation \( \rho \in \hat{G}^* \) and any unit vector \( \zeta \in H_\rho \) there is a unit vector \( \eta \in H_\pi \) such that

\[
\|\pi(g)\eta - \eta\| = \|\rho(g)\zeta - \zeta\| \quad \forall \ g \in G.
\]

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We conclude
\[ \varepsilon(G, K)^2 = \inf_{\eta} \left\{ \sup_{g \in K} \| \pi(g) \eta - \eta \| : \eta \in \mathcal{H}, \| \eta \| = 1 \right\} \]
\[ = \varepsilon(\pi, G, K)^2 \]

\( \Box \)

Corollary 2.5.2:

If K is a compact generating set for the compact, abelian group G then
\[ \varepsilon(G, K)^2 = \inf_{(\mu_\alpha)_{\alpha \in A}} \sup_{g \in K} \sum_{\alpha \in A} \mu_\alpha |\pi_\alpha(g) - 1|^2 \]
where the infimum is now over all sequences of non-negative real numbers \( \mu_\alpha, \alpha \in A \), such that \( \sum_{\alpha \in A} \mu_\alpha = 1 \).

To see this we set \( \mu_\alpha = |\eta_\alpha|^2 \) in the above proof.

\( \Box \)

Let \( \rho \) denote the right-regular representation of G on \( L^2(G) \) defined by
\[ (\rho(g) \zeta)(h) = \zeta(hg) \quad g, h \in G, \zeta \in L^2(G). \]

Chapter IV, 2.5 of [NaS] says that the right regular representation of a compact group is the direct sum of copies of its finite dimensional, irreducible representations, each with multiplicity equal to its dimension. Then, for the compact abelian group G, \( \rho \) is the direct sum of the irreducible representations of G;
\[ \rho = \pi \oplus \pi_0. \]

Let \( \mu \) denote normalised Haar measure on G.

Define a subspace \( \mathcal{H} \) of \( L^2(G) \) by
\[ \mathcal{H} = \{1 \}^\perp = \left\{ \zeta \in L^2(G) : \int_G \zeta \, d\mu = 0 \right\}. \]
Then \( \mathcal{H} \) is the orthogonal complement of the subspace of \( L^2(G) \) generated by the constant functions, which are precisely the invariant vectors of \( \rho \). Now define a representation \( \sigma \) of \( G \) on \( \mathcal{H} \) by \( \sigma(g) = \rho(g)|_{\mathcal{H}}, g \in G \). Since \( \{1\} = \mathcal{H}_{\pi_0} \), we see that \( \mathcal{H} = \mathcal{H}_\pi \) and
\[
\sigma = \pi.
\]
Since \( \varepsilon(G,K) = \varepsilon(\pi,G,K) \), we draw the following conclusion.

**Corollary 2.5.3**:

In the calculation of \( \varepsilon(G,K) \) we need consider only the representation \( \sigma \).

That is
\[
\varepsilon(G,K) = \varepsilon(\sigma,G,K).
\]

### 2.6 \( \varepsilon(C_n,C_n) \).

As in Section 2.3, \( C_n \) is the cyclic group of order \( n \), generated by a single element, \( g \).

**Proposition 2.6.1**:

For each integer \( n \geq 2 \),
\[
\varepsilon(C_n,C_n) = \sqrt{\frac{2n}{n-1}}.
\]

We have already remarked that \( \varepsilon(C_1,C_1) \) is infinite, which is consistent with allowing \( n = 1 \) in the proposition.

Before we can prove the proposition, we need the following simple result.

**Lemma 2.6.2**:

For any integers \( n,m \) such that \( 1 \leq m < n \),
\[
\sum_{j=1}^{n-1} |\omega_n^{mj} - 1|^2 = 2n
\]
where
\[
\omega_n = e^{\frac{2\pi i}{n}}.
\]

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Proof:

For $1 \leq m < n$,

$$\sum_{j=1}^{n-1} |\omega_n^{mj} - 1|^2 = 2 \sum_{j=1}^{n-1} (1 - \Re \omega_n^{mj})$$

$$= 2(n - 1) - 2 \Re \left( \sum_{j=1}^{n-1} e^{\left( \frac{2\pi i}{n}\right)j} \right)$$

$$= 2(n - 1) - 2 \Re \left( \frac{e^{2\pi i} - e^{\frac{2\pi i}{n}}}{e^{\frac{2\pi i}{n}} - 1} \right)$$

$$= 2(n - 1 - (-1))$$

$$= 2n.$$

Now we prove the proposition.

Take an integer $n \geq 2$ and let $\omega = e^{\frac{2\pi i}{n}}$. Using Theorem 2.5.1 and the representation $\pi$ defined in Section 2.5

$$\varepsilon(C_n, C_n) = \inf_{\zeta} \left\{ \sup_{h \in G} \|\pi(h)\zeta - \zeta\| : \zeta \in \mathbb{C}^{n-1}, \|\zeta\| = 1 \right\}.$$

Take $\zeta = (\zeta_1, \ldots, \zeta_{n-1}) \in \mathbb{C}^{n-1}$ with $\|\zeta\| = 1$. For each integer $m$ such that $1 \leq m \leq n - 1$

$$\|\pi(g^m)\zeta - \zeta\|^2 = \sum_{j=1}^{n-1} |\omega^{jm}\zeta_j - \zeta_j|^2$$

$$= \sum_{j=1}^{n-1} |\omega^{jm} - 1|^2 |\zeta_j|^2.$$

Then

$$\sum_{m=1}^{n-1} \|\pi(g^m)\zeta - \zeta\|^2 = \sum_{m=1}^{n-1} \sum_{j=1}^{n-1} |\omega^{jm} - 1|^2 |\zeta_j|^2$$

$$= \sum_{j=1}^{n-1} 2n |\zeta_j|^2$$

$$= 2n.$$
Hence

\[ \sup \{ \| \pi(h)\zeta - \zeta \| \, : \, h \in G \} \geq \frac{2n}{n-1} \]

and so

\[ \varepsilon(C_n, C_n) \geq \sqrt{\frac{2n}{n-1}}. \] \tag{2.1}

But for each \( m \in \{1, 2, \ldots, n-1\} \),

\[ \| \pi(g^m) \left( \frac{1}{\sqrt{n-1}}, \ldots, \frac{1}{\sqrt{n-1}} \right) - \left( \frac{1}{\sqrt{n-1}}, \ldots, \frac{1}{\sqrt{n-1}} \right) \|^2 \]

\[ = \frac{1}{n-1} \sum_{j=1}^{n-1} |\omega^{jm} - 1|^2 \]

\[ = \frac{2n}{n-1} \]

so that

\[ \varepsilon(C_n, C_n) \leq \sqrt{\frac{2n}{n-1}}. \] \tag{2.2}

Combining inequalities (2.1) and (2.2) we conclude

\[ \varepsilon(C_n, C_n) = \sqrt{\frac{2n}{n-1}}. \]

\[ \square \]

Note that

\[ \varepsilon(C_n, C_n) = \sup \{ |\omega^m - 1| : m \in \{1, 2, \ldots, n-1\} \} \]

\[ = |\omega^{\lfloor \frac{n}{2} \rfloor} - 1| \]

\[ \geq \sqrt{3} \]

\[ > \varepsilon(C_n, C_n) \quad \text{for } n > 3. \]
2.7 Kazhdan Constant for the Circle.

It is known that for any compact group, $G$, $\varepsilon(G, G) \geq \sqrt{2}$ ([HaV] Chapter 1, Theorem 5(ii)), while Section 2.6 tells us that $\varepsilon(C_n, C_n) \to \sqrt{2}$ as $n \to \infty$. These two facts suggest that if we replace $C_n$ by the circle group then the Kazhdan constant we obtain will be $\sqrt{2}$. We show here that this is indeed the case.

Let $T$ denote the circle group;

$$T = \{ z \in \mathbb{C} : |z| = 1 \}.$$ 

The group operation is multiplication of complex numbers.

**Theorem 2.7.1 :**

*The Kazhdan constant $\varepsilon(T, T)$ is $\sqrt{2}$.*

**Proof :**

Since $T$ is compact, we know that $T$ is a Kazhdan group and $\varepsilon(T, T) \geq \sqrt{2}$ (result quoted above) and we need only show that $\varepsilon(T, T) \leq \sqrt{2}$. We do this by finding a representation, $\sigma$, of $T$, with no non-zero invariant vector, for which $\varepsilon(\sigma, T, T) \leq \sqrt{2}$.

Let the representation $\sigma$ be defined as at the end of Section 2.5. So $\sigma$ represents $G$ on the Hilbert space $\mathcal{H}$ which is the orthogonal complement in $L^2(T)$ of the constant functions. The action is given by

$$(\sigma(\omega)g)(z) = g(\omega z) \quad z, \omega \in T; \ g \in \mathcal{H}.$$ 

We recall that Corollary 2.5.3 says that $\varepsilon(T, T) = \varepsilon(\sigma, T, T)$, but this emerges from our current proof anyway.

We now translate the problem to one about continuous functions on $[0, 2\pi]$. The Hilbert space $\mathcal{H}$ has an orthonormal basis consisting of the functions $z \mapsto z^j$, $j \in \mathbb{Z} \setminus \{0\}$. Consider $g \in \mathcal{H}$ such that $\|g\| = 1$. Then

$$g(z) = \sum_{j \in \mathbb{Z} \setminus \{0\}} \alpha_j z^j,$$
where \( \alpha_j \in \mathbb{C} \) for each \( j \in \mathbb{Z} \setminus \{0\} \). Let \( \mu \) denote normalised Haar measure on \( T \).

Then

\[
1 = \|g\|^2 = \int_T |g(z)|^2 \, d\mu(z) = \sum_{j \in \mathbb{Z} \setminus \{0\}} |\alpha_j|^2 .
\]

Now, for \( \theta \in [0, 2\pi] \),

\[
\|\sigma(e^{i\theta}) g - g \|^2 = \left\| \sum_{j \in \mathbb{Z} \setminus \{0\}} \alpha_j z^j e^{ij\theta} - \sum_{j \in \mathbb{Z} \setminus \{0\}} \alpha_j z^j \right\|^2 = \sum_{j \in \mathbb{Z} \setminus \{0\}} |\alpha_j|^2 |e^{ij\theta} - 1|^2 = 2 \sum_{j \in \mathbb{Z} \setminus \{0\}} |\alpha_j|^2 \left( 1 - \Re(e^{ij\theta}) \right) = 2 \left( 1 - \sum_{j \in \mathbb{Z} \setminus \{0\}} |\alpha_j|^2 \cos j\theta \right).
\]

For \( j = 1, 2, \ldots \), set \( \beta_j = |\alpha_j|^2 + |\alpha_{-j}|^2 \). Since \( g \in L^2(T) \), \( \sum_{j \in \mathbb{Z} \setminus \{0\}} |\alpha_j|^2 \) must converge and so, given \( \delta > 0 \), there is a positive integer \( n_\delta \) such that

\[
\sum_{j=N}^{M} \beta_j < \delta \text{ whenever } M \geq N \geq n_\delta .
\]

Then, for every \( \theta \in [0, 2\pi] \) and \( M \geq N \geq n_\delta \), \( \sum_{j=N}^{M} |\beta_j \cos j\theta| \leq \sum_{j=N}^{M} \beta_j < \delta \). So \( \sum_{j=1}^{\infty} \beta_j \cos j\theta \) converges uniformly and absolutely. Now define a function \( \hat{g} : [0, 2\pi] \rightarrow \mathbb{R} \), by

\[
\hat{g}(\theta) = \sum_{j=1}^{\infty} \beta_j \cos j\theta \quad \theta \in [0, 2\pi] .
\]

Note that \( \hat{g} \) is continuous because it is the uniform limit of a sequence of continuous functions.

Let

\[
\mathcal{G} = \left\{ f : [0, 2\pi] \rightarrow \mathbb{R} \mid f(\theta) = \sum_{j=1}^{\infty} \gamma_j \cos j\theta, \text{ where } \gamma_j \geq 0, \ j = 1, 2, \ldots, \text{ and } \sum_{j=1}^{\infty} \gamma_j = 1 \right\} .
\]

Given \( f \in \mathcal{G} \), \( f(\theta) = \sum_{j=1}^{\infty} \gamma_j \cos j\theta \), we can define \( \hat{f} \in \mathcal{H} \) by

\[
\hat{f}(z) = \sum_{j=1}^{\infty} \sqrt{\gamma_j} z^j .
\]
We have \( \| \hat{f} \|^2 = \sum_{j=1}^{\infty} \gamma_j = 1 \) and \( \hat{f} = f \). Take \( g \in \mathcal{H} \) with \( \| g \| = 1 \). Then, for every \( \theta \in [0, 2\pi] \),

\[
\| \sigma(e^{i\theta})g - g \|^2 = 2(1 - \hat{g}(\theta)) = \| \sigma(e^{i\theta}) \hat{g} - \hat{g} \|^2 .
\]

This tells us that the problem of showing that

\[
\inf_g \left\{ \sup_{\omega \in \Gamma} \| \sigma(\omega)g - g \| : g \in \mathcal{H}, \| g \| = 1 \right\} \leq \sqrt{2} \quad (2.3)
\]

is equivalent to showing that

\[
\sup_f \left\{ \inf_{\theta \in [0, 2\pi]} f(\theta) : f \in \mathcal{G} \right\} \geq 0 .
\]

We now examine this translated problem. The following lemma tells us that the infimum in (2.3) is not attained and we must look instead for functions \( g \in \mathcal{H} \) which give values of \( \sup_{\omega \in \Gamma} \| \sigma(\omega)g - g \| \) arbitrarily close to \( \sqrt{2} \).

**Lemma 2.7.2**

*The infimum in (2.3) cannot be attained and, equivalently, there is no \( f \in \mathcal{G} \) for which

\[
\inf \{ f(\theta) : \theta \in [0, 2\pi] \} \geq 0 .
\]

**Proof**:

Suppose we have such a function \( f \). Then

\[
f(\theta) \equiv \sum_{j=1}^{\infty} \gamma_j \cos j\theta \quad \gamma_j \geq 0, \quad \sum_{j=1}^{\infty} \gamma_j = 1,
\]

and so

\[
\int_{0}^{2\pi} f(\theta) \, d\theta = 0 .
\]

But \( f \) is nowhere negative, and so \( f \) cannot be positive outside a set of measure zero. Hence \( f \) is zero almost everywhere, and in fact \( f \) is zero everywhere since it is continuous.
This gives, for $j = 1, 2, \ldots,$

$$\gamma_j = \int_0^{2\pi} f(\theta) \cos j \theta \, d\theta = 0,$$

contradicting $\sum \gamma_j = 1.$

Thus

$$\inf \{ f(\theta) : \theta \in [0, 2\pi] \} < 0$$

for all $f \in G$ and, equivalently,

$$\sup \{ \| \sigma(\omega) g - g \| : \omega \in T \} > \sqrt{2}$$

for all $g \in H$ with $\| g \| = 1.$

$\square$

To solve the problem, we seek, for arbitrary $\varepsilon > 0,$ a function $f_\varepsilon \in G$ such that

$$\inf \{ f_\varepsilon(\theta) : \theta \in [0, 2\pi] \} \geq -\varepsilon.$$

We construct these functions as follows.

Let $f = f_\varepsilon : [0, 2\pi] \to \mathbb{R}$ be the continuous piecewise linear function shown
above, where \( \delta \) is chosen (depending on \( \varepsilon \)) so that \( \int_0^{2\pi} f(x) \, dx = 0 \). We see that
\[
\int_0^{2\pi} f(x) \, dx = 0 \iff \delta = \varepsilon (\pi - \delta)
\]
so define
\[
\delta = \frac{\varepsilon \pi}{1 + \varepsilon}.
\]

**Lemma 2.7.3:**

*The function \( f = f_\varepsilon \) defined above belongs to \( G \).*

**Proof:**

By symmetry
\[
\int_0^{2\pi} f(x) \sin jx \, dx = 0 \quad j = 1, 2, \ldots.
\]

For \( j \in \mathbb{Z}^+ \), set
\[
\beta_j = \int_0^{2\pi} f(x) \cos jx \, dx.
\]

Then
\[
\frac{1}{2} \beta_j = \int_0^\delta \frac{\delta - x}{\delta} \cos jx \, dx + \int_\delta^\pi \frac{\varepsilon (\delta - x)}{\pi - \delta} \cos jx \, dx
\]
\[
= \left[ \frac{\delta - x}{\delta} \frac{1}{j} \sin jx \right]_0^\delta - \int_0^\delta \left( -\frac{1}{\delta} \right) \frac{1}{j} \sin jx \, dx
\]
\[
+ \left[ \frac{\varepsilon (\delta - x)}{\pi - \delta} \frac{1}{j} \sin jx \right]_\delta^\pi - \int_\delta^\pi \left( \frac{-\varepsilon}{\pi - \delta} \right) \frac{1}{j} \sin jx \, dx
\]
\[
= \frac{1}{j\delta} \left[ -\frac{1}{j} \cos jx \right]_0^\delta + \frac{\varepsilon}{j(\pi - \delta)} \left[ -\frac{1}{j} \cos jx \right]_\delta^\pi
\]
\[
= \frac{1 - \cos j\delta}{j^2\delta} + \frac{\varepsilon (\cos j\delta - (-1)^j)}{j^2(\pi - \delta)}.
\]

For every \( N \in \mathbb{Z}^+ \) and \( \theta \in [0, 2\pi] \),
\[
\sum_{j=1}^N |\beta_j \cos j\theta| \leq \sum_{j=1}^N |\beta_j|
\]

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\[
\frac{1}{2} j^2 \delta (\pi - \delta) \beta_j = (\pi - \delta) + (\delta - \pi) \cos j \delta + \varepsilon \delta \cos j \delta - \varepsilon (\pi - 1) j
\]  
(2.4)

Substituting \( \varepsilon = \frac{\delta}{\pi - \delta} \) into equation (2.4) gives

\[
\frac{1}{2} j^2 \delta (\pi - \delta)^2 \beta_j = \pi^2 - 2\pi \delta + \delta^2 - \delta^2 (-1)^j + (-\pi^2 + 2\pi \delta - \delta^2 + \delta^2) \cos j \delta
\]
\[
= (\pi^2 - 2\pi \delta)(1 - \cos j \delta) + \delta^2 (1 - (-1)^j)
\]
\[
\geq 0
\]
provided \( \delta \leq \frac{\pi}{2} \).

But \( \delta \leq \frac{\pi}{2} \) if and only if \( \varepsilon \leq 1 \). Thus, if \( 0 < \varepsilon \leq 1 \) then \( \beta_j \geq 0 \) for each \( j \in \mathbb{Z}^+ \).

For each \( \varepsilon \in (0, 1] \), we have a function \( f_{\varepsilon} \in \mathcal{G} \) defined by

\[
f_{\varepsilon}(\theta) = \sum_{j=1}^{\infty} \beta_j \cos j \theta
\]

such that

\[
\inf \{ f_{\varepsilon}(\theta) : \theta \in [0, 2\pi] \} = -\varepsilon .
\]

This shows that

\[
\sup_{f \in \mathcal{G}} \inf_{\theta \in [0, 2\pi]} f(\theta) \geq 0
\]
as desired.

We have now solved the translated problem. The final step is to translate this solution back to the original problem.
Define $g_\epsilon \in \mathcal{H}$ by
\[ g_\epsilon(z) = \hat{f}_\epsilon(z) = \sum_{j=1}^{\infty} \sqrt{\beta_j} z^j. \]
Then
\[ \|g_\epsilon\| = \sum_{j=1}^{\infty} \beta_j = 1. \]
For every $\theta \in [0, 2\pi]$
\[ \left\| \sigma(e^{i\theta}) g_\epsilon - g_\epsilon \right\|^2 = 2(1 - f_\epsilon(\theta)) \leq 2(1 + \epsilon). \]
From this we conclude that for every $\epsilon \geq 0$ there is a $g_\epsilon \in \mathcal{H}$ such that
\[ \|g_\epsilon\| = 1 \]
and
\[ \sup \{ \|\sigma(\omega)g_\epsilon - g_\epsilon\| : \omega \in T \} \leq \sqrt{2(1 + \epsilon)}. \]
Thus
\[ \inf \left\{ \sup_{\omega \in T} \|\sigma(\omega)g - g\| : g \in \mathcal{H}, \|g\| = 1 \right\} \leq \sqrt{2}, \]
that is $\varepsilon(\sigma, T, T) \leq \sqrt{2}$ as desired.

This proves Theorem 2.7.1.

\[ \square \]

### 2.8 Symmetrically Presented Groups.

We now consider a set of groups for which, for particular generating sets, we need consider only the irreducible representations in the calculation of Kazhdan constants.

**Definition 2.8.1:**

A group, $G$, is symmetrically presented with generating set $K$ ($K$ finite), if
\[ G = \langle K \mid \text{some relations} \rangle \]
where the set of relations is unchanged by permutations of the elements of $K$. 

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Examples:

1. The dihedral group

\[ D_n = <a, b | a^2 = b^2 = (ab)^n = e > = <a, b | a^2 = b^2 = (ab)^n = (ba)^n = e > \]

Exchanging \( a \) and \( b \) changes the first set of conditions but does not change the second, which is deduced from the first set. Hence \( D_n \) is symmetrically presented with generating set \{a, b\}.

2. The symmetric group \( S_n \) has a symmetric presentation with generating set \{(1 2), (1 3), (1 4), ..., (1 n)\}.

3. The Burnside group

\[ B(m,n) = <a_1, ..., a_m : w^n = e > \]

where \( w \) ranges over all words in \( a_1, ..., a_m \).

We recall from Section 2.2 that a finite group always has property (T).

**Theorem 2.8.2**:

*Suppose the discrete group \( G \) is symmetrically presented with finite generating set \( K \). Then*

\[ e(G,K) = \inf_{\rho} \inf_{\zeta} \left\{ \left( \frac{1}{|K|} \sum_{h \in K} \| \rho(h)\zeta - \zeta \|^2 : \rho \in \hat{G}^*, \zeta \in \mathcal{H}_\rho, \| \zeta \| = 1 \right) \right\} \]

**Remarks**

1. We prove that

\[ \inf_{\pi} \inf_{\zeta} \left\{ \sup_{g \in K} \| \pi(g)\zeta - \zeta \| : \pi \in \hat{G}^*, \zeta \in \mathcal{H}_\pi, \| \zeta \| = 1 \right\} \]

\[ = \inf_{\rho} \inf_{\zeta} \left\{ \frac{1}{|K|} \sum_{h \in K} \| \rho(h)\zeta - \zeta \|^2 : \rho \in \hat{G}^*, \zeta \in \mathcal{H}_\rho, \| \zeta \| = 1 \right\} \] \hspace{1cm} (2.5)

This proves the result by the extended definition of \( e(G,K) \).
2. Notice that the right hand side of (2.5) is $\delta(G, K, \nu)$ of Section 2.4 where the measure $\nu$ is given by $\nu(\{g\}) = \frac{1}{|K|}$, $g \in G$. So, in this case,

$$\varepsilon(G, K)^2 = \delta(G, K, \nu).$$

**Proof of Theorem 2.8.2:**

Let $G$ be a group, symmetrically presented with finite generating set $K$, with $n$ elements. Let $S_K$ denote the group of permutations of $K$. Recall that $|S_K| = n!$.

If $t \in S_K$ then $t$ extends to an automorphism of $G$ (also denoted $t$) defined by

$$t(g_1 g_2 \ldots g_m) = t(g_1) t(g_2) \ldots t(g_m)$$

for any positive integer $m$ and $g_1, g_2, \ldots, g_m \in K$. This is well defined since, if $g_1 g_2 \ldots g_m = h_1 h_2 \ldots h_r$ ($g_1, \ldots, g_m, h_1, \ldots, h_r \in K$), then the fact that the cancellation relations are unchanged by applying $t$ to $K$ means that

$$t(g_1) t(g_2) \ldots t(g_m) = t(h_1) t(h_2) \ldots t(h_r).$$

Consider a representation $\pi$ of $G$.

For each $t \in S_K$ we define a representation $\tilde{\pi}(t)$ of $G$ by

$$\tilde{\pi}(t) = \pi \circ t^{-1}.$$ 

Note that $\tilde{\pi}(t)$ represents $G$ on the same Hilbert space as $\pi$, so every vector $\zeta \in H_\pi$ may be considered as an element of $H_{\tilde{\pi}(t)}$.

We need to remember at this point that the elements of $\tilde{G}$ are equivalence classes of representations, each represented by one element of the equivalence class. Suppose that $\tilde{\pi}(t)$ belongs to the equivalence class represented by the element $\rho$. Then there is an isometric isomorphism $U : H_{\tilde{\pi}(t)} \to H_\rho$ such that, for every $g \in G$ and every $\eta \in H_{\tilde{\pi}(t)}$,

$$\tilde{\pi}(t)(g)\eta = U^{-1} \rho(g) U \eta.$$ 

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Then
\[ \| \tilde{t}(\pi)(g) \eta - \eta \| = \| U^{-1} \rho(g) U \eta - \eta \| = \| \rho(g) U \eta - U \eta \| . \]

We then define
\[ t(\pi) = \rho \]
and for each vector \( \zeta \in \mathcal{H}_\pi \)
\[ t(\zeta) = U \zeta . \]

**Lemma 2.8.3**: It follows from the above that for every \( t \in S_K \) and \( \zeta \in \mathcal{H}_\pi \)
\[ ||t(\zeta)|| = ||U\zeta|| = ||\zeta|| \]
and
\[ ||\pi(t^{-1}g)\zeta - \zeta|| = ||\tilde{t}(\pi)(g)\zeta - \zeta|| = ||\rho(g)U\zeta - U\zeta|| = ||t(\pi)(g)t(\zeta) - t(\zeta)|| . \]

\( \Box \)

If \( \mathcal{H} \) is a closed subspace of \( \mathcal{H}_\pi \), we define a Hilbert space
\[ t(\mathcal{H}) = \{ t(\zeta) : \zeta \in \mathcal{H} \} . \]

Note that a closed subspace, \( \mathcal{H} \), of \( \mathcal{H}_\pi \) is invariant under \( \pi \) if and only if \( t(\mathcal{H}) \) is invariant under \( t(\pi) \). This follows from the fact that
\[ \{ \tilde{t}(\pi)(g) : g \in G \} = \{ \pi(g) : g \in G \} \]
since then \( \mathcal{H} \) is invariant under \( \pi \) if and only if it is invariant under \( \tilde{t}(\pi) \), and hence if and only if \( U\mathcal{H} \) is invariant under \( t(\pi) \). Thus irreducibility of \( t(\pi) \) is equivalent to irreducibility of \( \pi \), and \( t(\pi) \) has
non-zero invariant vectors precisely when $\pi$ does. Since $K = t(K)$, we have $\varepsilon(\pi, G, K) = \varepsilon(t(\pi), G, K)$ for every $t \in S_K$.

Define a representation $S(\pi)$ of $G$ by

$$S(\pi) = \bigoplus_{t \in S_K} t(\pi).$$

Then $S(\pi)$ represents $G$ on the Hilbert space $\mathcal{H}_{S(\pi)} = \bigoplus_{t \in S_K} \mathcal{H}_{t(\pi)} \cong \mathcal{H}_{|S_K|}$. The representation $S(\pi)$ has a non-zero invariant vector if and only if some $t(\pi)$ does, and hence if and only if $\pi$ does. Since $S(\pi)$ contains $\pi$,

$$\varepsilon(S(\pi), G, K) \leq \varepsilon(\pi, G, K)$$

(see Lemma 2.2.6). Taking infima over $\pi \in \mathcal{G}^*$, this gives immediately that

$$\varepsilon(G, K) = \inf \left\{ \varepsilon(S(\pi), G, K) : \pi \in \mathcal{G}^* \right\}. \tag{2.6}$$

For every vector $\zeta \in \mathcal{H}_0$ define a vector $S(\zeta) \in \mathcal{H}_{S(\pi)} = \bigoplus_{t \in S_K} \mathcal{H}_{t(\pi)}$ by

$$S(\zeta) = \bigoplus_{t \in S_K} \left( \frac{1}{\sqrt{n!}} \; t(\zeta) \right).$$

Then

$$\|S(\zeta)\|^2 = \frac{1}{n!} \sum_{t \in S_K} \|t(\zeta)\|^2 = \|\zeta\|^2.$$

Lemma 2.8.4:

Let $\pi \in \mathcal{G}^*$ and $\zeta \in \mathcal{H}_0$. Then, for each $g \in K$

$$\|S(\pi)(g)S(\zeta) - S(\zeta)\|^2 = \frac{1}{n} \sum_{h \in K} \|\pi(h)\zeta - \zeta\|^2.$$

In particular, for every $f, g \in K$,

$$\|S(\pi)(f)S(\zeta) - S(\zeta)\| = \|S(\pi)(g)S(\zeta) - S(\zeta)\|$$

and

$$\|S(\pi)(g)S(\zeta) - S(\zeta)\| = \max \left\{ \|S(\pi)(h)S(\zeta) - S(\zeta)\| : h \in K \right\}. $$
Proof:

Fix \pi \in \tilde{G}^*, \zeta \in \mathcal{H}_\pi and g \in K. Then

\[
\|S(\pi)(g)S(\zeta) - S(\zeta)\|^2 = \frac{1}{n!} \sum_{t \in S_K} \|t(\pi)(g)t(\zeta) - t(\zeta)\|^2
\]

by Lemma 2.8.3

\[
= \frac{1}{n!} \sum_{t \in S_K} \|\pi \left( t^{-1}(g) \right) \zeta - \zeta \|^2
\]

\[
= \frac{1}{n} \sum_{h \in K} \|\pi(h)\zeta - \zeta\|^2.
\]

\[\Box\]

We have seen above, in (2.6), that

\[\varepsilon(G, K) = \inf \left\{ \varepsilon(S(\pi), G, K) : \pi \in \tilde{G}^* \right\}.
\]

The next lemma takes this further.

Lemma 2.8.5:

"For each g \in K

\[\varepsilon(G, K) = \inf_{\pi} \inf_{\zeta} \left\{ \|S(\pi)(g)S(\zeta) - S(\zeta)\| : \pi \in \tilde{G}^*, \zeta \in \mathcal{H}_\pi, \|\zeta\| = 1 \right\}.
\]

Proof:

Fix g \in K, \pi \in \tilde{G}^* and \zeta \in \mathcal{H}_\pi with \|\zeta\| = 1.

We can find h \in K such that

\[\|\pi(h)\zeta - \zeta\| = \max \left\{ \|\pi(f)\zeta - \zeta\| : f \in K \right\}.
\]

Then, for each t \in S_K,

\[
\max \left\{ \|t(\pi(f))t(\zeta) - t(\zeta)\| : f \in K \right\}
\]

\[
= \max \left\{ \|t(\pi(t(f)))t(\zeta) - t(\zeta)\| : f \in K \right\}
\]

\[
= \max \left\{ \|\pi(f)\zeta - \zeta\| : f \in K \right\} \quad \text{Lemma 2.8.3}
\]

\[
= \|\pi(h)\zeta - \zeta\|
\]

\[
= \|t(\pi(t(h)))t(\zeta) - t(\zeta)\| \quad \text{Lemma 2.8.3}.
\]

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Also
\[ \|S(\pi)(h)S(\zeta) - S(\zeta)\|^2 = \frac{1}{n!} \sum_{t \in S_K} \|t(\pi)(h)t(\zeta) - t(\zeta)\|^2 \leq \frac{1}{n!} \sum_{t \in S_K} \|t(\pi)(t(h))t(\zeta) - t(\zeta)\|^2 \]
\[ = \|\pi(h)\zeta - \zeta\|^2. \]

This, together with Lemma 2.8.4, tells us that
\[ \max \{ \|S(\pi)(f)S(\zeta) - S(\zeta)\| : f \in K \} = \|S(\pi)(g)S(\zeta) - S(\zeta)\| \]
\[ = \|S(\pi)(h)S(\zeta) - S(\zeta)\| \leq \|\pi(h)\zeta - \zeta\| \]
and hence that
\[ \varepsilon(G, K) \leq \inf \left\{ \|S(\pi)(g)S(\eta) - S(\eta)\| : \eta \in \mathcal{H}_\pi, \|\eta\| = 1 \right\} \quad (2.7) \]
\[ \leq \inf \left\{ \max_{f \in K} \|\pi(f)\eta - \eta\| : \eta \in \mathcal{H}_\pi, \|\eta\| = 1 \right\} \quad (2.8) \]

Taking infima over \( \pi \in \tilde{G}^* \) in (2.7) and (2.8) gives
\[ \varepsilon(G, K) \leq \inf_{\pi} \inf_{\eta} \left\{ \|S(\pi)(g)S(\eta) - S(\eta)\| : \pi \in \tilde{G}^*, \eta \in \mathcal{H}_\pi, \|\eta\| = 1 \right\} \]
\[ \leq \inf_{\pi} \inf_{\eta} \left\{ \max_{f \in K} \|\pi(f)\eta - \eta\| : \pi \in \tilde{G}^*, \eta \in \mathcal{H}_\pi, \|\eta\| = 1 \right\} \]
\[ = \varepsilon(G, K). \]

□

We now complete the proof of the theorem.

Combining Lemmas 2.8.4 and 2.8.5 gives
\[ \varepsilon(G, K)^2 = \inf_{\pi} \inf_{\zeta} \left\{ \frac{1}{n} \sum_{h \in K} \|\pi(h)\zeta - \zeta\|^2 : \pi \in \tilde{G}^*, \zeta \in \mathcal{H}_\pi, \|\zeta\| = 1 \right\}. \]

But \( \mu = \frac{1}{n} \times \text{counting measure} \) is a positive Borel measure on \( G \) with \( \mu(K) = 1 \). So, by Lemma 2.4.1,
\[ \inf_{\pi} \inf_{\zeta} \left\{ \frac{1}{n} \sum_{h \in K} \|\pi(h)\zeta - \zeta\|^2 : \pi \in \tilde{G}^*, \zeta \in \mathcal{H}_\pi, \|\zeta\| = 1 \right\} \]

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\[= \inf_{\rho} \inf_{\zeta} \left\{ \frac{1}{n} \sum_{h \in K} \|\rho(h)\zeta - \zeta\|^2 : \rho \in \hat{G}^*, \zeta \in \mathcal{H}_\rho, \|\zeta\| = 1 \right\}.\]

\[\square\]

Corollary 2.8.6:

If a finite group \(G\) is symmetrically presented with generating set \(K\) then

\[\varepsilon(G, K) = \inf_{\zeta} \left\{ \sqrt{\frac{1}{|K|} \sum_{h \in K} \|\sigma(h)\zeta - \zeta\|^2 : \zeta \in \mathcal{H}_\sigma, \|\zeta\| = 1} \right\}\]

where, as in Section 2.5, \(\sigma\) is the restriction of the right regular representation of \(G\) on \(L^2(G)\) to the orthogonal complement in \(L^2(G)\) of the constant functions.

Proof:

As already remarked in Section 2.5, \(\sigma\) contains every \(\rho \in \hat{G}^*\). Using Theorem 2.8.2, its proof and the proof of Lemma 2.2.6, we have

\[\varepsilon(G, K)^2 = \inf_{\pi} \inf_{\zeta} \left\{ \frac{1}{n} \sum_{h \in K} \|\pi(h)\zeta - \zeta\|^2 : \pi \in \hat{G}^*, \zeta \in \mathcal{H}_\pi, \|\zeta\| = 1 \right\}\]

\[\leq \inf_{\zeta} \left\{ \frac{1}{n} \sum_{h \in K} \|\sigma(h)\zeta - \zeta\|^2 : \zeta \in \mathcal{H}_\sigma, \|\zeta\| = 1 \right\}\]

\[\leq \inf_{\rho} \inf_{\zeta} \left\{ \frac{1}{n} \sum_{h \in K} \|\rho(h)\zeta - \zeta\|^2 : \rho \in \hat{G}^*, \zeta \in \mathcal{H}_\rho, \|\zeta\| = 1 \right\}\]

\[= \varepsilon(G, K)^2.\]

\[\square\]

2.9 Dihedral Groups.

We now look at an example of a symmetrically presented group. For \(n \geq 2\), the dihedral group with \(2n\) elements,

\[D_n = < a, b \mid a^2 = b^2 = (ab)^n = e > \]

\[= < a, b \mid a^2 = b^2 = (ab)^n = (ba)^n = e > ,\]
is symmetrically presented with generating set \( \{a, b\} \). So, by Theorem 2.8.2

\[
\varepsilon ( D_n, \{a, b\} )^2
= \frac{1}{2} \inf_R \inf_{\zeta} \left\{ \|\rho(a)\zeta - \zeta\|^2 + \|\rho(b)\zeta - \zeta\|^2 : \rho \in \hat{G}^*, \zeta \in H_{\rho}, \|\zeta\| = 1 \right\}.
\]

Let \( \omega = e^{\frac{2\pi i}{n}} \).

If \( n \) is even, the irreducible representations of \( D_n \) are \( \psi_1, \psi_2, \psi_3, \psi_4, \rho_1, \ldots, \rho_{\frac{n}{2}-1} \) defined as follows. The representations \( \psi_1, \psi_2, \psi_3, \psi_4 \) are 1 dimensional and

\[
\begin{align*}
\psi_1(a) &= 1 & \psi_1(b) &= 1 \\
\psi_2(a) &= -1 & \psi_2(b) &= -1 \\
\psi_3(a) &= 1 & \psi_3(b) &= -1 \\
\psi_4(a) &= -1 & \psi_4(b) &= 1.
\end{align*}
\]

The representations \( \rho_1, \ldots, \rho_{\frac{n}{2}-1} \) are 2 dimensional. For each \( j \in \{1, \ldots, \frac{n}{2} - 1\} \)

\[
\rho_j(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho_j(b) = \begin{pmatrix} 0 & \omega^{-j} \\ \omega^j & 0 \end{pmatrix}.
\]

If \( n \) is odd, the irreducible representations of \( D_n \) are \( \psi_1, \psi_2, \rho_1, \ldots, \rho_{\frac{n}{2}-1} \) with the same definitions as above ([Ser] Part I, 5.3).

The trivial representation is \( \psi_1 \).

For any \( \zeta \in \mathbb{C} \) with \( |\zeta| = 1 \),

\[
\begin{align*}
|\psi_2(a)\zeta - \zeta| &= 2 & |\psi_2(b)\zeta - \zeta| &= 2 \\
|\psi_3(a)\zeta - \zeta| &= 0 & |\psi_3(b)\zeta - \zeta| &= 2 \\
|\psi_4(a)\zeta - \zeta| &= 2 & |\psi_4(b)\zeta - \zeta| &= 0.
\end{align*}
\]

So

\[
\begin{align*}
\frac{1}{2} \left( |\psi_2(a)\zeta - \zeta|^2 + |\psi_2(b)\zeta - \zeta|^2 \right) &= 4 \\
\frac{1}{2} \left( |\psi_3(a)\zeta - \zeta|^2 + |\psi_3(b)\zeta - \zeta|^2 \right) &= 2 \\
\frac{1}{2} \left( |\psi_4(a)\zeta - \zeta|^2 + |\psi_4(b)\zeta - \zeta|^2 \right) &= 2.
\end{align*}
\]

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Take any positive integer \( j < \frac{n}{2} \). For any \( x, y \in \mathbb{C} \) with \( |x|^2 + |y|^2 = 1 \)

\[
\rho_j(a) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}
\]

\[
\rho_j(b) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & \omega^{-j} \\ \omega^{j} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \omega^{-j} y \\ \omega^{j} x \end{pmatrix}
\]

\[
\left\| \rho_j(a) \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 = 2|x-y|^2
\]

\[
\left\| \rho_j(b) \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 = |\omega^{-j}y - x|^2 + |\omega^{j}x - y|^2
\]

\[
= |\omega^{-j}(y - \omega^{j}x)|^2 + |\omega^{j}x - y|^2
\]

\[
= 2|\omega^{j}x - y|^2.
\]

So

\[
\frac{1}{2} \left( \left\| \rho_j(a) \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 + \left\| \rho_j(b) \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 \right) = |x - y|^2 + |\omega^{j}x - y|^2
\]

We seek to minimise \( |x - y|^2 + |\omega^{j}x - y|^2 \) subject to \( x, y \in \mathbb{C} \), \( |x|^2 + |y|^2 = 1 \).

Take \( x, y \in \mathbb{C} \) with \( |x|^2 + |y|^2 = 1 \). Let \( \theta = \text{arg } x - \text{arg } y \). Then

\[
|x - y|^2 = |x|^2 + |y|^2 - 2|x||y| \cos \theta
\]

\[
= 1 - 2|x|\sqrt{1 - |x|^2} \cos \theta
\]

and

\[
|\omega^{j}x - y|^2 = 1 - 2|x|\sqrt{1 - |x|^2} \cos \left( \theta + \frac{2j\pi}{n} \right).
\]

So

\[
|x - y|^2 + |\omega^{j}x - y|^2 = 2 - 2|x|\sqrt{1 - |x|^2} \left( \cos \theta + \cos \left( \theta + \frac{2j\pi}{n} \right) \right)
\]

\[
= 2 - 4|x|\sqrt{1 - |x|^2} \cos \left( \theta + \frac{j\pi}{n} \right) \cos \frac{j\pi}{n}.
\]
This is minimised over \( \theta \in [-\pi, \pi] \) by

\[
\theta = -\frac{j \pi}{n}
\]

which gives

\[
|x - y|^2 + |\omega^j x - y|^2 = 2 - 4|x|\sqrt{1 - |x|^2} \cos \frac{j \pi}{n}.
\]

This is minimised over \( |x| \in [0, 1] \) when \( |x|\sqrt{1 - |x|^2} \) is maximised, that is when

\( |x| = \frac{1}{\sqrt{2}} \). This gives

\[
\frac{1}{2} \inf \left\{ \| \rho_j(a)\zeta - \zeta \|^2 + \| \rho_j(b)\zeta - \zeta \|^2 : \zeta \in \mathcal{H}_{\rho_j}, \| \zeta \| = 1 \right\}
\]

\[
= 2 \left( 1 - \cos \frac{j \pi}{n} \right)
\]

\[
= 2 \left( 1 - \left( 1 - 2 \sin^2 \frac{j \pi}{2n} \right) \right)
\]

\[
= 4 \sin^2 \frac{j \pi}{2n}.
\]

Thus, if \( n \) is even

\[
\varepsilon(D_n, \{a, b\}) = \min \left\{ 2, \sqrt{2}, 2 \sin \frac{\pi}{2n}, 2 \sin \frac{2\pi}{2n}, \ldots, 2 \sin \left( \frac{n-2}{2} \frac{\pi}{2n} \right) \right\}
\]

\[
= 2 \sin \frac{\pi}{2n}
\]

and if \( n \) is odd

\[
\varepsilon(D_n, \{a, b\}) = \min \left\{ 2, 2 \sin \frac{\pi}{2n}, 2 \sin \frac{2\pi}{2n}, \ldots, 2 \sin \left( \frac{n-1}{2} \frac{\pi}{2n} \right) \right\}
\]

\[
= 2 \sin \frac{\pi}{2n}.
\]

We conclude that, for all integers \( n \geq 2 \),

\[
\varepsilon(D_n, \{a, b\}) = 2 \sin \frac{\pi}{2n}.
\]

This result is quoted without proof in [HaV] Chapter 1, Section 17.
Notice that
\[
\left\| \rho_1(a) \left( \frac{1}{\sqrt{2}} e^{i \pi \frac{1}{n}} \right) - \left( \frac{1}{\sqrt{2}} e^{i \pi \frac{1}{n}} \right) \right\| = \left\| \rho_1(b) \left( \frac{1}{\sqrt{2}} e^{i \pi \frac{1}{n}} \right) - \left( \frac{1}{\sqrt{2}} e^{i \pi \frac{1}{n}} \right) \right\|
\]
\[
= \left| 1 - e^{i \pi \frac{1}{n}} \right|
\]
\[
= 2 \sin \frac{\pi}{2n}
\]
so
\[
\varepsilon(D_n, \{a,b\}) = \varepsilon(D_n, \{a,b\}).
\]

2.10 Direct Products.

Let \(G\) be a Kazhdan group with compact generating set \(K\). For every positive integer, \(m\), let \(G^m\) denote the direct product of \(m\) copies of \(G\) and let \(K^m\) be the compact generating set for \(G^m\) given by

\[
K^m = \{i^g : g \in K; 1 \leq i \leq m\}
\]

where \(i^g\) denotes the element \((e, \ldots, e, g, e, \ldots, e)\) of \(G^m\), with \(g\) in the \(i^{th}\) place.

We know from [HaV] Chapter 1, Proposition 9 (ii) that the direct product of two groups has property (T) if and only if the groups do, so \(G^m\) has property (T).

For every subset, \(D\), of \(G\) and each \(i \in \{1, \ldots, m\}\) define the set \(iD = \{i^g : g \in D\}\).

Proposition 2.10.1:

(a) Looking at irreducible representations, \(\varepsilon(G^m, K^m) \leq \varepsilon(G, K)\).

(b) Suppose \(e \notin K\), if necessary replacing \(K\) by the compact generating set \(K \setminus \{e\}\).

Suppose \(\nu\) is a positive Borel measure on \(G\) with \(\nu(K) = 1\), as in Section 2.4. For each \(i \in \{1, \ldots, m\}\), define a measure, \(\nu_i\), on \(i^G\) by \(\nu_i(iD) = \nu(D)\) for every \(\nu\)-measurable subset, \(D\), of \(G\). Now define a positive Borel measure, \(\mu_0\), on \(G^m\) by \(\mu_0(F) = \frac{1}{m} \sum_{i=1}^{m} \nu_i(F \cap i^K)\) for every subset, \(F\), of \(G^m\) for which the sets \(F \cap i^K\) are \(\nu_i\)-measurable. (Notice that \(K^m = \bigcup_{i=1}^{m} i^K\).) Let \(\mu\) be any positive Borel measure on \(G^m\) which agrees with \(\mu_0\) on \(K^m\).

Then \(\mu(K^m) = \frac{1}{m} \sum_{i=1}^{m} \nu_i(i^K) = 1\) and \(\delta(G^m, K^m, \mu) \leq \frac{1}{m} \delta(G, K, \nu)\).

(c) The Kazhdan constant \(\varepsilon(G^m, K^m) \leq \frac{1}{\sqrt{m}} \varepsilon(G, K)\).
Proof:

(a) Let \( \{ \pi_\alpha : \alpha \in A \} \) be the set irreducible representations of \( G \), with \( \pi_0 \) the trivial representation. Then the irreducible representations of \( G^m \) are

\[
\{ \pi_{\alpha_1} \otimes \pi_{\alpha_2} \otimes \ldots \otimes \pi_{\alpha_m} : \alpha_1, \ldots, \alpha_m \in A \}
\]

([Ser] 3.2 Theorem 10), where the trivial representation is \( \pi_0 \otimes \ldots \otimes \pi_0 \). Let \( B = A^m \setminus \{(0,0,\ldots,0)\} \).

Then

\[
\hat{\varepsilon}(G^m, K_m) = \inf \left\{ \sup_{g \in K, 1 \leq i \leq m} \| (\pi_{\alpha_1} \otimes \ldots \otimes \pi_{\alpha_m}) (\zeta_1 \otimes \ldots \otimes \zeta_m) - (\zeta_1 \otimes \ldots \otimes \zeta_m) \| : (\alpha_1, \ldots, \alpha_m) \in B; \ z \in \mathbb{R}^m, \| z \| = 1 \right\}
\]

\[
\leq \inf \left\{ \sup_{\zeta_j \in \mathcal{H}_{\alpha_j}, 1 \leq j \leq m} \| (\pi_{\alpha_1} \otimes \ldots \otimes \pi_{\alpha_m})(g)(\zeta_1 \otimes \ldots \otimes \zeta_m) - (\zeta_1 \otimes \ldots \otimes \zeta_m) \| : (\alpha_1, \ldots, \alpha_m) \in B; \ z \in \mathbb{R}^m, \| z \| = 1 \right\}
\]

\[
= \inf \left\{ \sup_{\zeta_j \in \mathcal{H}_{\alpha_j}, 1 \leq j \leq m} \| \zeta_1 \otimes \ldots \otimes \zeta_{j-1} \otimes (\pi_{\alpha_1}(g)\zeta_j - \zeta_j) \otimes \zeta_{j+1} \otimes \ldots \otimes \zeta_m \| : (\alpha_1, \ldots, \alpha_m) \in B; \ z \in \mathcal{H}_{\alpha_j}, \| \zeta_i \| = 1 \right\}
\]

\[
= \inf \left\{ \sup_{\zeta_j \in \mathcal{H}_{\alpha_j}, 1 \leq j \leq m} \| \pi_{\alpha_1}(g)\zeta_j - \zeta_j \| : (\alpha_1, \ldots, \alpha_m) \in B; \ z \in \mathcal{H}_{\alpha_j}, \| \zeta_j \| = 1 \right\}
\]

\[
= \inf \left\{ \sup_{\zeta_j \in \mathcal{H}_{\alpha_j}, 1 \leq j \leq m} \| \pi_{\alpha_1}(g)\zeta_j - \zeta_j \| : (\alpha_1, \ldots, \alpha_m) \in B; \ z \in \mathcal{H}_{\alpha_j}, \| \zeta_j \| = 1 \right\}
\]

\[
= \hat{\varepsilon}(G, K).
\]

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Similarly

\[ \delta \left( G^m, K_m, \mathcal{M} \right) = \inf \left\{ \sum_{i=1}^{m} \int_K \| \pi_{\alpha_i}(g) \zeta_i - \zeta_i \|^2 \frac{1}{m} \nu(g) : \right. \]

\( (\alpha_1, ..., \alpha_m) \in B, \zeta_j \in \mathcal{H}_{\alpha_j}, \| \zeta_j \| = 1, j = 1, ..., m \} \]

\[ = \frac{1}{m} \inf \left\{ \int_K \| \pi_\alpha(g) \zeta - \zeta \|^2 \nu(g) : \alpha \in A \setminus \{0\}, \zeta \in \mathcal{H}_\alpha, \| \zeta \| = 1 \right\} \]

\[ = \frac{1}{m} \delta(G, K, \nu) \]

since we minimise the right hand side of (2.9) by taking all but one of the \( \alpha_j \) to be zero.

Finally,

\[ \varepsilon(G^m, K_m) \]

\[ = \inf \left\{ \sup_{g \in K, 1 \leq i \leq m} \| \rho(\phi)\eta - \eta \| : \rho \in \hat{G}_\rho^*, \eta \in \mathcal{H}_\rho, \| \eta \| = 1 \right\} \]

\[ \leq \inf \left\{ \sup_{g \in K, 1 \leq i \leq m} \left\| \left( (\pi \otimes \pi_0 \otimes ... \otimes \pi_0) \oplus (\pi_0 \otimes \pi \otimes ... \otimes \pi_0) \oplus ... \right) \left( \frac{1}{\sqrt{m}} \left( (\zeta \otimes 1 \otimes ... \otimes 1) \oplus ... \oplus (1 \otimes ... \otimes 1 \otimes \zeta) \right) \right) - \left( \frac{1}{\sqrt{m}} \left( (\zeta \otimes 1 \otimes ... \otimes 1) \oplus ... \oplus (1 \otimes ... \otimes 1 \otimes \zeta) \right) \right) \right\} : \]

\[ \pi \in \hat{G}^*, \zeta \in \mathcal{H}_\pi, \| \zeta \| = 1 \}

\[ = \inf \left\{ \sup_{g \in K} \left\| \pi(g) \frac{1}{\sqrt{m}} \zeta - \frac{1}{\sqrt{m}} \zeta \right\| : \pi \in \hat{G}^*, \zeta \in \mathcal{H}_\pi, \| \zeta \| = 1 \right\} \]

\[ = \frac{1}{\sqrt{m}} \varepsilon(G, K) . \]

\[ \square \]
Proposition 2.10.2:

If $G$ is a symmetrically presented, discrete, Kazhdan group with finite generating set $K$, then

$$\varepsilon(G^m, K_m) = \frac{1}{\sqrt{m}} \varepsilon(G, K).$$

Proof:

Let $\nu$ be counting measure on $G$ normalised by a factor $\frac{1}{|K|}$. By Theorem 2.8.2 and Remark 2 following it,

$$\delta(G, K, \nu) = \varepsilon(G, K)^2$$

and by Proposition 2.10.1 (b),

$$\delta\left(G^m, K_m, \frac{1}{m} \nu\right) = \frac{1}{m} \delta(G, K, \nu)$$

so that

$$\frac{1}{m} \varepsilon(G, K)^2 = \delta\left(G^m, K_m, \frac{1}{m} \nu\right)$$

$$\leq \varepsilon(G^m, K_m)^2$$

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$$\leq \frac{1}{m} \varepsilon(G, K)^2$$

Proposition 2.10.1(c).

Corollary 2.10.3:

Let $m$ and $n$ be positive integers with $n \geq 2$.

(a) For every generator, $g$, of the cyclic group of order $n$,

$$\varepsilon\left(C_n^m, \{g\}_m\right) \leq \frac{2}{\sqrt{m}} \sin \frac{\pi}{n}.$$  

(b) If $a$ and $b$ generate the dihedral group $D_n$ subject to $a^2 = b^2 = (ab)^n = e$ then

$$\varepsilon\left(D_n^m, \{a, b\}_m\right) \leq \frac{2}{\sqrt{m}} \sin \frac{\pi}{2n}.$$  

\[\square\]
Chapter 3

The Spectrum of the Laplacian.

In the previous chapter we looked at Kazhdan’s definition of property (T) and at an equivalent “quantitative” definition. We investigated how to calculate the positive constants provided by this definition. Another property equivalent to property (T) for finitely generated, discrete groups is that zero is an isolated point in the spectrum of the Laplacian, \( \Delta \), a positive operator in \( C^\ast(G) \) depending on the choice of finite generating set. So the infimum of the non-zero part of this spectrum, like the Kazhdan constant, is positive precisely when the group has property (T). The first half of this chapter looks at the spectrum of \( \Delta \). The second half looks at the spectrum of \( \lambda(\Delta) \), whose infimum is positive precisely when \( G \) is non-amenable.

3.1 The Spectrum of the Laplacian.

Consider the group \( C^\ast\)-algebra, \( C^\ast(G) \), of a discrete group, \( G \). We identify the elements of \( G \) with unitary operators in \( C^\ast(G) \). Representations of \( C^\ast(G) \) are obtained from representations of \( G \), by the universal property of \( C^\ast(G) \).

Suppose that \( G \) has a finite generating set, \( S \), not containing the identity, such that \( S = S^{-1} \), where \( S^{-1} \) denotes \( \{ g^{-1} : g \in S \} \). We shall refer to such a set
as a symmetric generating set. The Laplacian, $\Delta$, for $S$ is defined by

$$\Delta = |S| e - \sum_{g \in S} g.$$  

(The identity element, $e$, of $G$ corresponds to the identity operator in $C^*(G)$.) The Laplacian is a positive operator in $C^*(G)$ since it is the sum of positive operators of the form $2e - (g + g^{-1})$ (if $g^2 \neq e$) or $e - g$ (if $g^2 = e$), so $\text{Sp}(\Delta) \subseteq [0, \|\Delta\|] \subseteq [0, 2|S|]$. We are interested in finding

$$\inf \{ t \in \text{Sp}(\Delta) : t > 0 \},$$

which we will denote by $\lambda_1(G, S)$. Since $\text{Sp}(\Delta)$ is closed, $\lambda_1(G, S) \in \text{Sp}(\Delta)$. This value is of interest since it is an indication of how efficient the Cayley graph of $G$ with respect to $S$ is as a network for transmitting information [Bien]. It is shown in [HRV] (Introduction, Proposition III and Section D, Remark (2) after proof of Proposition III) that $\lambda_1(G, S) > 0$ if and only if $G$ has property (T). So $\lambda_1(G, S) > 0$ if and only if $\varepsilon(G, S) > 0$.

Now suppose that $G$ has property (T) and that $K$ is a finite generating set for $G$. Let

$$S = K \cup K^{-1} \setminus \{e\}.$$  

Then $S$ is also a finite generating set for $G$, $S = S^{-1}$ and $e \notin S$. Further,

$$\varepsilon(G, S) = \varepsilon(G, K)$$

since, for every representation $\pi$ of $G$, every $g \in G$ and every $\zeta \in \mathcal{H}_\pi$,

$$\|\pi(g^{-1})\zeta - \zeta\| = \|\pi(g)\left(\pi(g^{-1})\zeta - \zeta\right)\| = \|\pi(g)\zeta - \zeta\|$$

and $\|\pi(e)\zeta - \zeta\| = 0$.

Similarly, $\varepsilon(G, S) = \varepsilon(G, K)$. We know from [HRV] (Section D Remark (2)) that $\text{Sp}(\Delta) \subseteq \{0\} \cup \left[\frac{\varepsilon(G, S)^2}{2}, \|\Delta\|\right]$ so

$$\lambda_1(G, S) \geq \frac{\varepsilon(G, S)^2}{2}$$

(3.1)

and (Section A Remark after the proof of Proposition I) that if $S$ does not contain any element of order 2 then

$$\lambda_1(G, S) \geq \varepsilon(G, S)^2.$$  

(3.2)
The operator $h$ of [HRV] is $\frac{1}{|S|} \sum_{g \in S} g = (e - \frac{1}{|S|}\Delta)$, so that

$$\lambda_1(G, S) = |S|(1 - \sup\{t \in \text{Sp}(h) : t < 1\}).$$

Remark (2) (after proof of Proposition III) in Section D of [HRV] says that if $\delta > 0$ and $\text{Sp}(\Delta) \subseteq \{0\} \cup [|S|\delta, 2|S|]$, then $\varepsilon(G, K) \geq \sqrt{2\delta}$. It follows that

$$\lambda_1(G, S) \leq \frac{\varepsilon(G, S)^2 |S|}{2}. \tag{3.3}$$

In the case of the cyclic group of order $n > 2$ with generating set $K = \{g\}$, a single generator, $S = \{g, g^{-1}\}$ and the right hand sides of (3.2) and (3.3) are equal, so

$$\lambda_1(C_n, \{g, g^{-1}\}) = \varepsilon(C_n, \{g, g^{-1}\})^2 = \left(2 \sin \frac{\pi}{n}\right)^2.$$

If $n = 2$ then $S = K$ and the right hand sides of (3.1) and (3.3) are equal, so that

$$\lambda_1(C_2, \{g\}) = \frac{\varepsilon(C_2, \{g\})^2}{2} = 2.$$

Recall the following definitions from Section 2.10. Consider a group, $G$, and a positive integer, $m$. The direct product of $m$ copies of $G$ is denoted by $G^m$. If $g \in G$ and $1 \leq i \leq m$ then $i g$ will denote the element $(e, \ldots, e, g, e, \ldots, e)$ of $G^m$ with $g$ in the $i$th place and $e$ everywhere else. If $S$ is a subset of $G$ then

$$S_m = \{i g : g \in S, 1 \leq i \leq m\}.$$

Recall also that $\hat{G}$ is the set of irreducible representations of $G$ and that $\hat{G}^* = \hat{G} \setminus \pi_0$.

**Lemma 3.1.1 :**

Let $G$ be a discrete group, generated by a finite set $K$ and let $m$ be a positive integer. Let $S, \Delta, G^m, S_m$ and $i g$ $(g \in G, i = 1, 2, \ldots, m)$ be defined as above.

Let

$$\Delta_m = m|S| e_m - \sum_{i=1}^{m} \sum_{g \in S} i g$$
where \( e_m = (e, \ldots, e) \in G^m \). Let \( \nu \) denote counting measure on \( G \) normalised by a factor \( \frac{1}{|S|} \) and let \( \delta(G, S, \nu) \) be as defined in Section 2.4. Then

(a) \( \lambda_1(G, S) \geq \frac{|S|}{2} \delta(G, S, \nu) \);

(b) if \( G \) is abelian then \( \lambda_1(G, S) = \frac{|S|}{2} \delta(G, S, \nu) \);

(c) \( \| \Delta \| = \sup S_p \Delta = \frac{1}{2} \sup \left\{ \sum_{g \in S} \| \pi(g) \zeta - \zeta \| : \pi \in \hat{G}; \zeta \in \mathcal{H}_\pi, \| \zeta \| = 1 \right\} \);

(d) \( \| \Delta_m \| = \sup S_p \Delta_m = m \sup S_p \Delta = m \| \Delta \| \);

(e) \( \lambda_1(G^m, S_m) \leq \lambda_1(G, S) \).

**Proof:**

(a) We follow [Rob2]. Since \( \Delta \) is a normal element of \( C^*(G) \), every element of the spectrum of \( \Delta \) is equal to \( p(\Delta) \) for some pure state, \( p \), of \( C^*(G) \) ([K&R] 4.4.4). For every state, \( q \), of \( C^*(G) \), [K&R] 4.5.2 says that there is a cyclic representation, \( \pi \), of \( C^*(G) \) and a unit cyclic vector \( \xi \in \mathcal{H}_\pi \) such that \( q(\Delta) = \langle \pi(\Delta) \xi \mid \xi \rangle \). Then [K&R] 10.2.3 tells us that the representation \( \pi \) of \( C^*(G) \) is irreducible if and only if \( q \) is a pure state. In that case \( \pi \) arises from an irreducible representation of \( G \), also denoted \( \pi \). In particular, there is an irreducible representation, \( \pi \), of \( C^*(G) \) and a unit vector \( \xi \in \mathcal{H}_\pi \) such that \( \lambda_1(G, S) = \langle \pi(\Delta) \xi \mid \xi \rangle \). So

\[
\lambda_1(G, S) \geq \inf_{\rho \in \mathcal{C}} \{ \langle \rho(\Delta) \xi \mid \xi \rangle : \rho \text{ is an irreducible representation of } C^*(G), \\
\rho \neq \pi_0, \xi \in \mathcal{H}_\rho, \| \xi \| = 1 \}
\]

\[
= \frac{1}{2} \inf_{\rho \in \mathcal{C}} \sum_{g \in S} \langle \rho(2e - g - g^{-1}) \xi \mid \xi \rangle
\]

\[
= \frac{1}{2} \inf_{\rho \in \mathcal{C}} \sum_{g \in S} (2 - 2 \Re \langle \rho(g) \xi \mid \xi \rangle)
\]

\[
= \frac{1}{2} \inf_{\rho \in \mathcal{C}} \left\{ \sum_{g \in S} \| \rho(g) \zeta - \zeta \|^2 : \rho \in \hat{G}^*, \zeta \in \mathcal{H}_\rho, \| \zeta \| = 1 \right\}
\]

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(b) Suppose that $G$ is abelian. By [K&R] 3.2.11

$$\text{Sp}(\Delta) = \{ p(\Delta) : p \text{ is a multiplicative linear functional on } C^*(G) \}$$

and by [K&R] 4.4.1 $p$ is a multiplicative linear functional if and only if $p$ is a pure state of $C^*(G)$. As in part (a), [K&R] 4.5.2 and 10.2.3 then tell us that

$$\text{Sp}(\Delta) = \{ \rho(\Delta) : \rho \in \hat{G} \}$$

noting that the irreducible representations of $G$ are all 1 dimensional. Since

$$\rho(\Delta) = \frac{1}{2} \sum_{g \in S} \rho \left(2e - g - g^{-1}\right)$$

and

$$\rho \left(2e - g - g^{-1}\right) \geq 0 \quad \forall g \in S, \rho \in \hat{G}$$

we know that

$$\rho(\Delta) = 0 \iff \rho \left(2e - g - g^{-1}\right) = 0 \quad \forall g \in S$$

$$\iff \rho(g) = 1 \quad \forall g \in S \quad \text{(since } |\rho(g)| = |\rho(g^{-1})| = 1\text{)}$$

$$\iff \rho(g) = 1 \quad \forall g \in G \quad \text{(since } S \text{ generates } G\text{)}$$

$$\iff \rho = \pi_0.$$ 

Hence

$$\lambda_1(G, S) = \inf_{\rho \in \hat{G}^*} \rho(\Delta)$$

$$= |S| - \sup_{\rho \in \hat{G}^*} \sum_{g \in S} \rho(g)$$

$$= \frac{1}{2} \inf_{\rho \in \hat{G}^*} \sum_{g \in S} |\rho(g) - 1|^2$$

$$= \frac{|S|}{2} \delta(G, S, \nu)$$

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as required.

(c) For every self-adjoint element, a, of a C*-algebra, either \(\|a\|\) or \(-\|a\|\)
lies in \(\text{Sp} \ a\) ([K&R] 3.1.15). Since \(\Delta\) is a positive operator, \(\text{Sp} \ \Delta \subseteq [0, \|\Delta\|]\), so

\[
\|\Delta\| = \sup \text{Sp} \ \Delta.
\]

We have observed in (a) ([K&R] 4.4.4) that

\[
\text{Sp} \ \Delta \subseteq \{p(\Delta) : p \text{ is a pure state of } C^*(G)\}
\]

\[
\subseteq \{(\rho(\Delta)\zeta | \zeta) : \rho \in \hat{G}, \zeta \in \mathcal{H}_\rho, \|\zeta\| = 1\}
\]

\[
= \left\{ \frac{1}{2} \sum_{g \in S} \|\rho(g)\zeta - \zeta\|^2 : \rho \in \hat{G}, \zeta \in \mathcal{H}_\rho, \|\zeta\| = 1 \right\}.
\]

For every representation, \(\pi\), of \(C^*(G)\) and every unit vector \(\zeta \in \mathcal{H}_\pi\), the linear
functional \(a \mapsto (\pi(a)\zeta | \zeta) : C^*(G) \to \mathbb{C}\) is positive and has norm 1 i.e. it is a
state of \(C^*(G)\). Another application of [K&R] 10.2.3 (quoted in (a)) then shows
that

\[
\{p(\Delta) : p \text{ is a pure state of } C^*(G)\} = \{(\rho(\Delta)\zeta | \zeta) : \rho \in \hat{G}, \zeta \in \mathcal{H}_\rho, \|\zeta\| = 1\}.
\]

Since states have norm 1, it follows that

\[
\|\Delta\| \geq \sup \{p(\Delta) : p \text{ is a pure state of } C^*(G)\}
\]

\[
= \sup \left\{ \frac{1}{2} \sum_{g \in S} \|\rho(g)\zeta - \zeta\|^2 : \rho \in \hat{G}, \zeta \in \mathcal{H}_\rho, \|\zeta\| = 1 \right\}
\]

\[
\geq \sup \text{Sp} \ \Delta
\]

\[
= \|\Delta\|.
\]

(d) Again, since both \(\Delta\) and \(\Delta_m\) are positive operators we have

\[
\|\Delta\| = \sup \text{Sp} \ \Delta \text{ and } \|\Delta_m\| = \sup \text{Sp} \ \Delta_m.
\]

By (c)

\[
\sup \ \text{Sp} \ \Delta_m = \sup \{(\pi(\Delta_m)\zeta | \zeta) : \pi \in \hat{G}^m; \zeta \in \mathcal{H}_\pi, \|\zeta\| = 1\}
\]

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\[ \begin{align*}
\geq \sup \{ \langle (\pi_1 \otimes \cdots \otimes \pi_m) (\Delta_m) (\zeta_1 \otimes \cdots \otimes \zeta_m) | (\zeta_1 \otimes \cdots \otimes \zeta_m) \rangle : \\
\pi_i \in \hat{G}, \zeta_i \in \mathcal{H}_{\pi_i}, \|\zeta_i\| = 1, i = 1, \ldots, m \} \\
= \sup_{\pi_i, \zeta_i} \left( m |S| - \sum_{i=1}^{m} \sum_{g \in S} \langle \pi_i(g) \zeta_i | \zeta_i \rangle \right) \\
= \sup_{\pi_i, \zeta_i} \left( \sum_{i=1}^{m} \langle \pi_i(\Delta) \zeta_i | \zeta_i \rangle \right) \\
= m \sup \text{Sp} \Delta.
\end{align*} \]

For each \( \pi \in \hat{G} \), let \( \{ \eta^i_\pi : i \in I_\pi \} \) be an orthonormal basis for \( \mathcal{H}_{\pi} \). Then, for all \( \pi_1, \ldots, \pi_m \in \hat{G} \), the set of vectors

\[ \{ \eta^i = \eta^{i_1}_\pi \otimes \cdots \otimes \eta^{i_m}_\pi : i = (i_1, \ldots, i_m) \in I_{\pi_1} \times \cdots \times I_{\pi_m} = I \} \]

is an orthonormal basis for \( \mathcal{H}_{\pi_1 \otimes \cdots \otimes \pi_m} = \mathcal{H}_{\pi_1} \otimes \cdots \otimes \mathcal{H}_{\pi_m} \) (see [K&R] Section 2.6, Theorem 2.6.2 and the paragraph after Theorem 2.6.4). So if \( \zeta \in \mathcal{H}_{\pi_1 \otimes \cdots \otimes \pi_m} \) then

\[ \zeta = \sum_{i \in I} \left( \zeta | \eta^i \right) \eta^i. \]

Hence

\[ \{ \zeta \in \mathcal{H}_{\pi_1 \otimes \cdots \otimes \pi_m} : \|\zeta\| = 1 \} = \left\{ \sum_{i \in I} \beta_i \eta_i : \beta_i \in \mathbb{C} \forall i \in I; \sum_{i \in I} |\beta_i|^2 = 1 \right\}. \]

Now

\[ \begin{align*}
\sup \text{Sp} \Delta_m \\
= \sup \left\{ \langle \pi(\Delta_m) \zeta | \zeta \rangle : \pi \in \hat{G}^m; \zeta \in \mathcal{H}_{\pi}, \|\zeta\| = 1 \right\} \\
= \sup \left\{ \left( \pi_1 \otimes \cdots \otimes \pi_m \right) (\Delta_m) \sum_{i \in I} \beta_i \eta^i \left| \sum_{j \in I} \beta_j \eta^j \right| : \\
\pi_1, \ldots, \pi_m \in \hat{G}; \beta_i \in \mathbb{C} \forall i \in I; \sum_{i \in I} |\beta_i|^2 = 1 \right\}
\end{align*} \]

55 (a)
Let $D = \left\{ |S| e_m - \sum_{g \in S}^i g : i = 1, \ldots, m \right\} \cup \{ e_m \} \subset C^*(G^m)$
and let $C^*(D)$ denote the $C^*$-algebra generated by the elements of $D$. Note that
the elements of $D$ commute with each other so that $C^*(D)$ is an abelian
$C^*$-algebra and that $\Delta_m \in C^*(D)$. By $C^*(\Delta)$ we shall mean the $C^*$-subalgebra
of $C^*(G)$ generated by $\Delta$ and $e$. Since $C^*(\Delta)$ and $C^*(D)$ are abelian
$C^*$-algebras, we see as in (b) that

$$\text{Sp} C^*(\Delta) \Delta = \{ p(\Delta) : p \text{ is a pure state of } C^*(\Delta) \}$$

and

$$\text{Sp} C^*(D) \Delta_m = \{ p(\Delta_m) : p \text{ is a pure state of } C^*(D) \} .$$

Proposition 4.5.1 of [K&R] says that that if $a$ is an element of a $C^*$-subalgebra,
$B$, of a $C^*$-algebra, $A$, then $\text{Sp}_A a = \text{Sp}_B a$, so

$$\text{Sp} \Delta = \text{Sp} C^*(\Delta) \Delta \text{ and } \text{Sp} \Delta_m = \text{Sp} C^*(D) \Delta_m .$$

Theorem 4.3.13 (iv) of [K&R] says that if $A$ is a $C^*$-algebra with unit $I$ and $B$
is a self-adjoint subspace of $A$ containing $I$ then every pure state of $B$ extends
to a pure state of $A$. So every pure state of $C^*(\Delta)$ extends to a pure state of
$C^*(G)$ and every pure state of $C^*(D)$ extends to a pure state of $C^*(G^m)$.

Recalling from (c) that the pure states of $C^*(G)$ are the linear functionals $a \mapsto \langle \pi(a) \zeta | \zeta \rangle$ where $\pi$ is an irreducible representation of $C^*(G)$, $\zeta \in \mathcal{H}_\pi$ and $\|\zeta\| = 1$, we see that

$$\lambda_1(G, S) = \inf \left\{ \langle \pi(\Delta) \zeta | \zeta \rangle : \pi \in \hat{G}; \text{ the extension of } \pi \text{ to } C^*(G) \right. \left. \text{restricts to an irreducible representation of } C^*(\Delta) \right\} \quad \zeta \in \mathcal{H}_\pi, \|\zeta\| = 1 \right\} \quad (3.4)$$

and that

$$\lambda_1(G^m, S_m) = \inf \left\{ \langle (\pi_1 \otimes \ldots \otimes \pi_m) (\Delta_m) \zeta | \zeta \rangle : \pi_1, \ldots, \pi_m \in \hat{G}, \text{not all } \pi_0; \text{the extension of } \pi_1 \otimes \ldots \otimes \pi_m \text{ to } C^*(G^m) \text{ restricts to an irreducible representation of } C^*(D) \right\} \quad \zeta \in \mathcal{H}, \|\zeta\| = 1 \right\} \quad (3.5)$$

The following result is standard. Suppose that $\pi \in \hat{G}$ gives an irreducible representation of $C^*(\Delta)$. We shall show that $\pi \otimes \pi_0 \otimes \ldots \otimes \pi_0$ gives an irreducible representation of $C^*(D)$. Let $\zeta_1 \in \mathcal{H}_\pi \setminus \{0\}$, and $\zeta_2, \ldots, \zeta_m \in \mathbb{C} \setminus \{0\}$. Then

$$(\pi \otimes \pi_0 \otimes \ldots \otimes \pi_0)(C^*(D)) (\zeta_1 \otimes \zeta_2 \otimes \ldots \otimes \zeta_m) = (\pi(C^*(\Delta))\zeta_1) \otimes C \otimes \ldots \otimes C.$$

So every invariant subspace for $(\pi \otimes \pi_0 \otimes \ldots \otimes \pi_0)(C^*(D))$ must be of the form $\mathcal{K} \otimes \mathbb{C} \otimes \ldots \otimes \mathbb{C}$ where $\mathcal{K}$ is a subspace of $\mathcal{H}_\pi$ invariant for $\pi(C^*(\Delta))$. Since $\pi$ is irreducible as a representation of $C^*(\Delta)$ it follows that $\pi \otimes \pi_0 \otimes \ldots \otimes \pi_0$ is irreducible as a representation of $C^*(D)$. Hence, using (3.5) and (3.4),
\[ \lambda_1(G^m, S_m) \leq \inf \left\{ \left( \pi \otimes \pi_0 \otimes \cdots \otimes \pi_0 \right) \left( \Delta_m \right) \left( \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_m \right) : \pi \text{ is as in (3.4); } \zeta_1 \in \mathcal{H}_\pi, \zeta_2, \ldots, \zeta_m \in \mathcal{C}, \|\zeta_1\| = \|\zeta_2\| = \cdots = |\zeta_m| = 1 \right\} \]
\[ = \inf \left\{ \left( \pi(\Delta)\zeta \right) : \pi, \zeta \text{ are as in (3.4)} \right\} \]
\[ = \lambda_1(G, S). \quad (3.6) \]

Now suppose that \( \pi_1, \ldots, \pi_m \in \hat{G} \) are not all \( \pi_0 \), and that \( \pi_1 \otimes \cdots \otimes \pi_m \) gives an irreducible representation of \( C^*(D) \). We shall show that each \( \pi_i \) \( (i = 1, \ldots, m) \) gives an irreducible representation of \( C^*(\Delta) \). Suppose that \( \mathcal{K} \) is a subspace of \( \mathcal{H}_{\pi_1} \) invariant under the action of \( \pi_1(C^*(\Delta)) \). Then
\[ (\pi_1 \otimes \cdots \otimes \pi_m)(C^*(D))(\mathcal{K} \otimes \mathcal{H}_{\pi_2} \otimes \cdots \otimes \mathcal{H}_{\pi_m}) = \mathcal{K} \otimes \mathcal{H}_{\pi_2} \otimes \cdots \otimes \mathcal{H}_{\pi_m} \]
and so \( \mathcal{K} \otimes \mathcal{H}_{\pi_2} \otimes \cdots \otimes \mathcal{H}_{\pi_m} \) is invariant for \( (\pi_1 \otimes \cdots \otimes \pi_m)(C^*(D)) \). Hence \( \mathcal{K} \) is either \( \{0\} \) or \( \mathcal{H}_{\pi_1} \) and \( \pi_1 \) is irreducible as a representation of \( C^*(\Delta) \). Similarly, \( \pi_2, \ldots, \pi_m \) give irreducible representations of \( C^*(\Delta) \). It follows that
\[ \lambda_1(G^m, S_m) = \inf \left\{ \sum_{i=1}^m \left( \pi_i(\Delta)\zeta_i \right) : \pi_1, \ldots, \pi_m, \zeta_1, \ldots, \zeta_m \text{ are as in (3.5)} \right\} \]
\[ \geq \inf \left\{ \left( \pi(\Delta)\zeta \right) : \pi \text{ and } \zeta \text{ are as in (3.4)} \right\} \]
\[ = \lambda_1(G, S). \quad (3.7) \]

minimising the sum by taking all but one of the \( \pi_i \)'s to be \( \pi_0 \).

We conclude from (3.6) and (3.7) that
\[ \lambda_1(G^m, S_m) = \lambda_1(G, S). \]
Corollary 3.1.2:

If $m$ and $n$ are positive integers with $n \geq 3$ then

$$\lambda_1 \left( C_n^m, \{g, g^{-1}\} m \right) \leq \left( 2 \sin \frac{\pi}{n} \right)^2$$

and

$$\lambda_1 \left( C_2^m, \{g\} m \right) \leq 2.$$

We now consider symmetrically presented groups (see Section 2.8).

Proposition 3.1.3:

Suppose that the discrete group $G$ ($G \neq \{e\}$) has property $(T)$ and is symmetrically presented with finite generating set $K$. Let $S = K \cup K^{-1}$ as above. (Notice that $e \notin K$.) Then

$$\lambda_1 (G, S) = \frac{\varepsilon(G, K)^2 |S|}{2}$$

$$= \begin{cases} \frac{\varepsilon(G, K)^2 |K|}{2} & \text{if $K$ consists of elements of order 2} \\ \varepsilon(G, K)^2 |K| & \text{for some } a \in G \\ \varepsilon(G, K)^2 |K| & \text{otherwise} \end{cases}$$

Proof:

Since $G$ is symmetrically presented with generating set $K$, every element of $K$ must have the same order, and so the same is true of $S$. Furthermore, if $a$ and $a^{-1}$ are distinct elements of $K$ and $b \in K$ with $b \neq a$ then exchanging $b$ and $a^{-1}$ does not alter the relations on $K$ so $ba = e$ and $b = a^{-1}$. Hence we have one of the following three cases

(a) every element of $K$ has order 2 and $S = K$
(b) every element of \( K \) has order greater than 2 and \( K \cap K^{-1} = \emptyset \) so that 
\[ S = K \cup K^{-1} \]

(c) \( K = \{a, a^{-1}\} \) for some \( a \) of order greater than 2; this gives the same \( S, \Delta \)
and \( \varepsilon(G, K) \) as \( K = \{a\} \).

Let \( \nu \) denote counting measure on \( G \). In cases (a) and (c) \( S = K \) while in

case (b)

\[
\delta \left( G, S, \frac{1}{|S|} \nu \right)
\]
\[
= \frac{1}{2|K|} \inf_{\rho, \zeta} \left\{ \sum_{g \in K} \| \rho(g) \zeta - \zeta \|^2 + \sum_{g \in K} \| \rho(g^{-1}) \zeta - \zeta \|^2 : \rho \in \hat{G}^*, \zeta \in \mathcal{H}_\rho, \| \zeta \| = 1 \right\}
\]
\[
= \frac{1}{2|K|} \inf_{\rho, \zeta} \left\{ 2 \sum_{g \in K} \| \rho(g) \zeta - \zeta \|^2 : \rho \in \hat{G}^*, \zeta \in \mathcal{H}_\rho, \| \zeta \| = 1 \right\}
\]
\[
= \delta \left( G, K, \frac{1}{|K|} \nu \right)
\]

Then in all three cases

\[
\frac{|S|}{2} \varepsilon(G, K)^2 \geq \lambda_1(G, S)
\]

by (3.3)

\[
\geq \frac{|S|}{2} \delta \left( G, S, \frac{1}{|S|} \nu \right)
\]

by Lemma 3.1.1 (a)

\[
= \frac{|S|}{2} \delta \left( G, K, \frac{1}{|K|} \nu \right)
\]

by the above

\[
= \frac{|S|}{2} \varepsilon(G, K)^2
\]

Theorem 2.8.2.

Notice that the proof of the first equality in the above proposition works for
any discrete Kazhdan group, \( G \), and finite, symmetric generating set, \( S \), for
which \( \delta(G, S, \frac{1}{|S|} \nu) = \varepsilon(G, S)^2 \), where \( \nu \) is counting measure.
Corollary 3.1.4:

Let \( m, n \) be positive integers with \( n \geq 2 \). For the dihedral group

\[
D_n = \langle a, b \mid a^2 = b^n = (ab)^n = e \rangle.
\]

we have

\[
\lambda_1(D_n^m, \{a, b\}_m) \leq \left(2 \sin \frac{\pi}{2n}\right)^2 = \lambda_1(D_n, \{a, b\}).
\]

\( \square \)

From [Lub] (8.2) if \( G \) is a finite group and \( S \) is a union of conjugacy classes then

\[
\lambda_1(G, S) = |S| - \max_{\rho \in \hat{G}^*} \frac{1}{d_{\rho}} \sum_{g \in S} \chi_{\rho}(g)
\]

where \( d_{\rho} \) is the dimension of the irreducible representation \( \rho \) and \( \chi_{\rho} \) is its character. If, further, \( S = G \setminus \{e\}, \) it follows from the orthogonality relations that \( \sum_{g \in S} \chi_{\rho}(g) = -\chi_{\rho}(e) = -d_{\rho} \) for each \( \rho \in \hat{G}^* \) and so \( \lambda_1(G, S) = |S| - (-1) = |G| \). Using (3.3), this gives

\[
\frac{|G| - 1}{2} \varepsilon(G, G)^2 \geq |G|
\]

and so

\[
\varepsilon(G, G) \geq \sqrt{\frac{2|G|}{|G| - 1}}.
\]

Recall that if \( G \) is cyclic then \( \varepsilon(G, G) = \sqrt{\frac{2|G|}{|G| - 1}} \) (Section 2.6).

3.2 Laplacians for Free Groups.

In the previous Section, we remarked that the least positive value, \( \lambda_1(G, S) \), in the spectrum of the Laplacian, \( \Delta \), gives an indication of the efficiency, as a network for transmitting information, of the Cayley graph of a discrete group,
$G$, with respect to a symmetric generating set, $S$. In fact, the Cayley graph contains only information about the left regular representation of the group so this efficiency is better indicated by the least positive value in the spectrum of $\lambda(\Delta)$ in the regular group $C^*$-algebra, $C^*_\lambda(G)$, where $\lambda$ is the left regular representation of $G$ on $l^2(G)$; $\lambda(g)\zeta(h) = \zeta(g^{-1}h)$, $g, h \in G$, $\zeta \in l^2(G)$. We will denote this value by $\lambda_\lambda(G, S)$. For a finite group $\lambda(G, S) = \lambda_1(G, S)$, since $C^*(G)$ and $C^*_\lambda(G)$ coincide. In general, $\lambda_\lambda(G, S) \geq \lambda_1(G, S)$ since $\text{Sp } \pi(\Delta) \subseteq \text{Sp } \Delta$ for any representation, $\pi$, of $G$.

While we have observed (Section 3.1) that $\lambda_1(G, S) > 0$ if and only if $G$ has property (T), there are non-Kazhdan groups for which $\lambda_\lambda(G, S) > 0$. Indeed, [HRV] (Consequence (a) of Proposition 1(1)) tells us that $0 \in \text{Sp } \lambda(\Delta)$ if and only if $G$ is amenable and so $\lambda_\lambda(G, S) > 0$ precisely when $G$ is not amenable. In this section we are concerned with the calculation of $\lambda_\lambda$ for the free group on a finite set of generators. The method follows [Coh].

Let $F_d$ denote the free group on $d$ generators, $s_1, \ldots, s_d$. For each positive integer $k$, let $S_k$ be the set of reduced words of length $k$ in $s_1, \ldots, s_d, s_1^{-1}, \ldots, s_d^{-1}$. So

$$S_0 = \{e\}$$
$$S_1 = \{s_1, \ldots, s_d, s_1^{-1}, \ldots, s_d^{-1}\}$$
$$S_2 = \{s_1^2, s_1s_2, \ldots\} = \{s_is_j, s_i^{-1}s_j, s_is_j^{-1}, s_i^{-1}s_j^{-1} : i, j \in \{1, 2, \ldots, d\}\}$$

and

$$|S_0| = 1$$
$$|S_1| = 2d$$
$$|S_2| = 2d(2d - 1)$$
$$|S_t| = 2d(2d - 1)^{t-1} \quad t \geq 1.$$
Also, let
\[ \chi_k = \sum_{s \in S_k} s \in C^*(F_d) \]
and let
\[ \chi = \chi_1 . \]

So
\[
\begin{align*}
\chi_0 &= e \\
\chi &= \chi_1 = s_1 + \ldots + s_d + s_1^{-1} + \ldots + s_d^{-1} \\
\chi^2 &= \chi_2 + 2de \\
\chi \chi_k &= \chi_{k+1} + (2d-1)\chi_{k-1} \quad k \geq 2.
\end{align*}
\]

This is Theorem 1 of [Coh]. These relations are solved in the same paper.

From [HRV] Section A, example 2.

\[ Sp \chi = [-2d, 2d] \]

while it is a result of Kesten ([Pat] 4.31) that

\[ Sp \lambda(\chi) = [-2\sqrt{2d-1}, 2\sqrt{2d-1}] . \]

Hence

\[ \lambda(\chi) = 2d - 2\sqrt{2d-1} , \]

since \( \Delta = 2de - \chi \).

Now let
\[ T_k = S_0 \cup S_1 \cup \ldots \cup S_k = \{ \text{reduced words of length } \leq k \} \]
and let
\[ \phi_k = \sum_{s \in T_k} s = \chi_0 + \chi_1 + \ldots + \chi_k . \]

Then \( T_k \) generates \( F_d \), \( T_k = T_k^{-1} \) and
\[
\begin{align*}
\lambda(\chi, T_k \setminus \{e\}) &= \inf \ Sp \lambda(|T_k|e - \phi_k) \\
&= |T_k| - \sup \ Sp \lambda(\phi_k) \\
&= |T_k| - \sup \ Sp \lambda(\chi_0 + \chi_1 + \ldots + \chi_k) .
\end{align*}
\]

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Also
\[ \text{Sp} \lambda(\phi_1) = \left[ 1 - 2\sqrt{2d - 1}, 1 + 2\sqrt{2d - 1} \right] , \]
\[ \text{Sp} \frac{\lambda(\phi_1)}{|T_1|} = \left[ \frac{1 - 2\sqrt{2d - 1}}{1 + 2d}, \frac{1 + 2\sqrt{2d - 1}}{1 + 2d} \right] . \]

We calculate
\[ |T_k| = 1 + 2d \sum_{j=0}^{k-1} (2d - 1)^j \]
\[ = 1 + 2d \left( \frac{1 - (2d - 1)^k}{1 - (2d - 1)} \right) \]
\[ = \frac{d(2d - 1)^k - 1}{d - 1} . \]

From 3.8, for each integer \( k \geq 0 \), \( \chi_k \) is a monic polynomial of degree \( k \) in \( \chi \), with integer coefficients, say \( \chi_k = q_k(\chi) \). It follows that \( \phi_k = p_k(\chi) \) where \( p_k \) is also a monic polynomial of degree \( k \) with integer coefficients and
\[ p_k = 1 + \sum_{j=1}^{k} q_k . \]

Then
\[ \text{Sp} \lambda(\phi_k) = \text{Sp} \lambda(p_k(\chi)) \]
\[ = p_k(\text{Sp} \lambda(\chi)) \]
\[ = p_k \left( \left[ -2\sqrt{2d - 1}, 2\sqrt{2d - 1} \right] \right) . \]

By [Coh] each \( q_k(x) \) (for \( k \geq 1 \)) is maximised over \( x \in \text{Sp} \lambda(\chi) \) by \( x = 2\sqrt{2d - 1} \) and the maximum value is
\[ (k + 1)\sqrt{2d - 1}^k - (k - 1)\sqrt{2d - 1}^{k-2} . \]

Since the same value of \( x \) maximises every \( q_k(x) \) it also maximises the \( p_k(x) \) so that

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\[
\sup \text{Sp} \lambda(\phi_k) = p_k(2\sqrt{2d-1}) \\
= 1 + \sum_{j=1}^{k} q_j(2\sqrt{2d-1}) \\
= 1 + \sum_{j=1}^{k} ((j + 1)\sqrt{2d-1}^j - (j - 1)\sqrt{2d-1}^{j-2}) \\
= \sqrt{2d-1}^k \left( \left(1 + \frac{1}{\sqrt{2d-1}}\right) k + 1 \right)
\]

and

\[
\text{sup } \text{Sp} \lambda\left(\frac{\phi_k}{|T_k| - 1}\right) = \frac{\sqrt{2d-1}^k \left(k \left(1 + \frac{1}{\sqrt{2d-1}}\right) + 1\right)}{2d \left(\frac{1-(2d-1)^k}{2-2d}\right)} \\
= \frac{(d - 1)\sqrt{2d-1}^k \left(k \left(1 + \frac{1}{\sqrt{2d-1}}\right) + 1\right)}{d ((2d - 1)^k - 1)} \\
= \frac{(d - 1) \left(1 + \frac{1}{\sqrt{2d-1}} + \frac{1}{k}\right)}{d \left(\frac{\sqrt{2d-1}^k}{k} - \frac{1}{k\sqrt{2d-1}^k}\right)} \\
\rightarrow 0 \quad \text{as } k \rightarrow \infty.
\]

So

\[
\frac{\lambda_{\lambda}(F_d, T_k \setminus \{e\})}{|T_k \setminus \{e\}|} = 1 - \sup \text{Sp} \lambda\left(\frac{\phi_k}{|T_k| - 1}\right) \\
\rightarrow 1 \quad \text{as } k \rightarrow \infty
\]

and

\[
\lambda_{\lambda}(F_d, T_k \setminus \{e\}) \rightarrow \infty \quad \text{as } k \rightarrow \infty.
\]

If we fix \(k\) and look at different values of \(d\)

\[
\sup \text{Sp} \lambda\left(\frac{\phi_k}{|T_k| - 1}\right) = \frac{(d - 1) \left(k \left(1 + \frac{1}{\sqrt{2d-1}}\right) + 1\right)}{d \left(\sqrt{2d-1}^k - \sqrt{2d-1}^{1-k}\right)}
\]
Finally, for $d = 2$ we have, to calculator accuracy

\[
\begin{align*}
\text{Sp} \lambda \left( \frac{\phi_1}{|T_1|} \right) &= [-0.492802323, 0.892820323] \\
\text{Sp} \lambda \left( \frac{\phi_2}{|T_2|} \right) &= [-0.224137931, 0.733182447] \\
\text{Sp} \lambda \left( \frac{\phi_3}{|T_3|} \right) &= [-0.222351126, 0.561973769].
\end{align*}
\]
Chapter 4

Positive and Negative Type Functions.

4.1 Background.

In this chapter, we are concerned with the effect of Kazhdan's property (T) on the existence and nature of certain functions on and actions of a group. The interest arises from three theorems to be found in [HaV].

Theorem 4.1.1 ([HaV] 5.11):

Let $G$ be a locally compact group. The following properties are equivalent:

(a) $G$ has property (T).

(b) Let $(\phi_n)_{n \geq 1}$ be a sequence of functions of positive type (see definitions below) on $G$, normalised by $\phi_n(e) = 1$ for all $n$. If $(\phi_n)_{n \geq 1}$ converges to the constant function 1 uniformly on every compact subset of $G$, then $(\phi_n)_{n \geq 1}$ converges to 1 uniformly on $G$.

This result has been strengthened by A.G.Robertson (unpublished work). He has shown that in condition (b) above pointwise convergence is sufficient, rather than uniform convergence on compacta.

Theorem 4.1.2 ([HaV] 5.20):

Let $G$ be a locally compact group. The following properties are equivalent:
(a) $G$ has property (T).

(b) Every function of conditionally negative type (definition below) on $G$ is bounded.

The version of property (T) provided by Theorem 4.1.2 has proved fruitful to work with and provides the link between property (T) and most of the work in this chapter. In Section 4.5 we evaluate a bound for real valued conditionally negative type functions on a Kazhdan group in terms of the supremum on a compact generating set and the Kazhdan constant. Sections 4.11 and 4.12 look at the consequences of the existence of such a bound for von Neumann algebras arising from the group.

**Theorem 4.1.3 ([HaV] 6.4):**

If a locally compact group, $G$, has property (T) then every action of $G$ on a tree fixes a point.

This theorem is a consequence of Theorem 4.1.2. In Sections 4.6 to 4.9 we look at examples of actions of groups on trees and $\mathbb{R}$-trees (defined below).

The definitions and results in the next three sections do not require the assumptions that the groups concerned are locally compact and $\sigma$-compact.

### 4.2 Kernels and Functions of Positive Type.

**Definition 4.2.1:**

A kernel on a topological space $X$ is a continuous function from $X \times X$ into $\mathbb{C}$. 

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Definition 4.2.2:

A kernel, \( \Phi \), on a topological space \( X \) is of positive type if for each positive integer \( n \), all \( x_1, \ldots, x_n \in X \) and all \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \)

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \Phi(x_i, x_j) \geq 0.
\]

We also define a real kernel of positive type, \( \Phi : X \times X \to \mathbb{R} \), by taking the \( \lambda_i \in \mathbb{R} \) in the above and requiring that \( \Phi \) is symmetric. In the complex case, a kernel of positive type is easily seen to be Hermitian, but in the real case symmetry does not follow from the rest of the definition.

Definition 4.2.3:

A real or complex valued function, \( \phi \), on a topological group, \( G \), is of positive type if it is continuous and the kernel \( \Phi : G \times G \to \mathbb{R} \) or \( \mathbb{C} \) defined by

\[
\Phi(g, h) = \phi(g^{-1}h)
\]

g, h \in G

is of positive type (with the appropriate definition as \( \phi \) is real or complex).

Functions of positive type are more usually referred to as positive definite functions in work in English. Our terminology is common in Europe and is consistent with [HaV].

Proposition 5.8 of [HaV] says that a function \( \phi : G \to \mathbb{R} \) or \( \mathbb{C} \) on a topological group, \( G \), is of positive type precisely if there is a Hilbert space \( \mathcal{H}_\phi \) (real or complex as the range of \( \phi \)), a unitary representation \( \pi_\phi \) of \( G \) on \( \mathcal{H}_\phi \) and a vector \( \zeta_\phi \in \mathcal{H}_\phi \) which is cyclic for \( \pi_\phi \) (that is \( \mathcal{H}_\phi \) is topologically generated by \( \pi_\phi(G)\zeta_\phi \)) and

\[
\phi(g) = \langle \pi_\phi(g)\zeta_\phi | \zeta_\phi \rangle \quad \text{g} \in G.
\]

Furthermore, the triple \((\mathcal{H}_\phi, \pi_\phi, \zeta_\phi)\) is determined up to isomorphism by these conditions. We will refer to this triple as the cyclic representation of \( G \) induced by \( \phi \).
Lemma 4.2.4 (Well known):

For every function, \( \phi \), of positive type on a group, \( G \),

(a) \( \phi(g^{-1}) = \overline{\phi(g)} \) for all \( g \in G \),

(b) \( \sup_{g \in G}|\phi(g)| = \phi(e) \).

Proof:

(a) By the definition of positive type (with \( n = 1, \lambda_1 = 1 \) and \( g_1 = e \))

\[
\phi(e) \geq 0.
\]

By the same definition with \( n = 2, \lambda_1 = \lambda_2 = 1, g_1 = e, g_2 = g \in G \)

\[
2\phi(e) + \phi(g) + \phi(g^{-1}) \geq 0
\]

so that \( \phi(g) + \phi(g^{-1}) \) is real.

Changing \( \lambda_2 \) to \( i \) gives

\[
2\phi(e) + i\phi(g) - i\phi(g^{-1}) \geq 0
\]

so that \( \phi(g) - \phi(g^{-1}) \) is purely imaginary.

Thus \( \phi(g^{-1}) = \overline{\phi(g)} \).

(b) Take any \( g \in G \) and let \( \lambda \in \mathbb{C} \) be such that \( \lambda \phi(g) = |\phi(g)| \). By the definition of positive type

\[
0 \leq 2\phi(e) - \lambda \phi(g) - \overline{\lambda} \phi(g^{-1})
\]

\[
= 2(\phi(e) - |\phi(g)|).
\]

So \( |\phi(g)| \leq \phi(e) \).
4.3 Kernels and Functions of Conditionally Negative Type.

Definition 4.3.1:
A real kernel \( \Psi : X \times X \rightarrow \mathbb{R} \) is of conditionally negative type if

(a) \( \Psi \) is zero on the diagonal

(b) \( \Psi \) is symmetric

(c) for every integer \( n \geq 2 \), all \( x_1, \ldots, x_n \in X \) and all \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) summing to zero
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \Psi(x_i, x_j) \leq 0.
\]

Definition 4.3.2:
In the complex case, the definition becomes

(a) \( \Psi \) is Hermitian

(b) for every integer \( n \geq 2 \), all \( x_1, \ldots, x_n \in X \) and all \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \) summing to zero
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \bar{\lambda}_i \lambda_j \Psi(x_i, x_j) \leq 0.
\]

Definition 4.3.3:
If \( G \) is a group, a function \( \psi : G \rightarrow \mathbb{R} \) (resp. \( \mathbb{C} \)) is of conditionally negative type if it is continuous and the kernel \( \Psi : X \times X \rightarrow \mathbb{R} \) (resp. \( \mathbb{C} \)) defined by
\[
\Psi(g, h) = \psi(g^{-1}h)
\]
g, h \in G

is of conditionally negative type.

If a function, \( \psi : G \rightarrow \mathbb{R} \), is of conditionally negative type then \( \psi(e) = 0 \) since the kernel \( \Psi(g, h) = \psi(g^{-1}h) \) is zero on the diagonal. Also \( \psi \) must be non-negative throughout \( G \), since for every \( \lambda \in \mathbb{R} \) and each \( g \in G \),
0 \geq \lambda(-\lambda)\psi(g^{-1}e) + \lambda\lambda\psi(g^{-1}g) + (-\lambda)\psi(e^{-1}g) + (-\lambda)(-\lambda)\psi(e^{-1}e) \\
= -2\lambda^2\psi(g).

Functions of conditionally negative type are often referred to as negative definite functions. This terminology can be misleading since the concept is not the negative version of positive definite and, in particular, as we have just seen that functions of conditionally negative type are not negative valued.

One connection between functions of positive type and those of conditionally negative type is Schoenberg\'s Theorem ([HaV] Theorem 5.16, 5.17). This says that a (real or complex) function, $\psi$, on a group, $G$, which is zero on the identity, is of conditionally negative type if and only if, for every $t > 0$, the function $e^{-t\psi}(\cdot: G \to \mathbb{R}(\text{resp.} \mathbb{C}))$ is of positive type.

### 4.4 Actions of Groups on Trees and R-Trees.

#### 4.4.1 Trees.

**Definition 4.4.1:**

A tree is a connected, simply connected, one dimensional simplicial-complex, in which each one dimensional face has a direction.

In other words, a tree, $\Gamma$, consists of a set of vertices, $\Delta_0^\Gamma$, joined by a set of directed edges, $\Delta_1^\Gamma$, in such a way that, if we ignore the direction of the edges, there is precisely one path joining any given pair of vertices.

**Definition 4.4.2:**

An action of a group, $G$, on a tree, $\Gamma$, is a homomorphism from $G$ into the group of simplicial automorphisms of $\Gamma$ which preserve the orientation of the edges of $\Gamma$, where the group of elements of $G$ fixing a given vertex of $\Gamma$ is an open (and hence closed) subgroup of $G$.  

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Definition 4.4.3:

We define the distance function $d_{\Gamma} : \Delta_\Gamma^0 \times \Delta_\Gamma^0 \to \mathbb{N}$ by setting $d_{\Gamma}(x, y)$ equal to the number of edges in the path joining $x$ and $y$, where $x, y \in \Delta_\Gamma^0$.

Suppose a group, $G$, acts on a tree, $\Gamma$. Take a vertex $x \in \Delta_\Gamma^0$ and define a function $D_x : G \to \mathbb{N}$ by $D_x(g) = d_{\Gamma}(x, gx)$ for $g \in G$. In [HaV] Proposition 6.2 it is shown that the distance function $d_{\Gamma}$ is a conditionally negative type kernel. Since $d_{\Gamma}$ is invariant under the action of $G$, $D_x(g^{-1}h) = d_{\Gamma}(gx, hx)$ and it follows that the function $D_x$ is of conditionally negative type. This is the source of the close links between actions of groups on trees and conditionally negative type functions on the groups.

4.4.2 R-trees

Consider a metric space $X$.

Definition 4.4.4:

An arc in $X$ is a subspace of $X$ homeomorphic to a compact interval in $\mathbb{R}$.

Definition 4.4.5:

A segment in $X$ is a subspace of $X$ isometric to an interval in $\mathbb{R}$.

Definition 4.4.6:

An R-tree, $Y$, is a metric space in which for any two points, $x, y \in Y$, there is a unique arc, $A \subset Y$, with extreme points $x$ and $y$, and this arc is a closed segment. We denote this segment by $[x, y]$.

Let $d : Y \times Y \to \mathbb{R}$ denote the metric on $Y$. Proposition 6.9 of [HaV] tells us that $d$ is a conditionally negative type kernel and hence that if a group, $G$, acts by isometries on $Y$, then for each $x \in Y$ the function $D_x$ defined on $G$ by $D_x(g) = d(x, gx)$ (for $g \in G$) is of conditionally negative type. Proposition 6.11

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of [HaV] says that if $G$ is a Kazhdan group then every action of $G$ by isometries on a complete $\mathbb{R}$-tree has a fixed point.

Any tree gives an $\mathbb{R}$-tree, the metric being given by the distance between points measured along the edges of the tree.

### 4.5 A Bound for Real Functions of Conditionally Negative Type on Kazhdan Groups.

Consider a real valued function, $\psi$, of conditionally negative type on a group with property (T). We know from [HaV] 5.20 that $\psi$ is bounded. Here we find a bound for $\psi$ as a function of the supremum of $\psi$ on a compact generating set and the Kazhdan constant for that generating set.

We begin with a lemma, the proof of which is taken from the proof of [HaV] 1.16.

**Lemma 4.5.1**

Let $G$ be a group with property (T), $K$ a compact generating set for $G$ and $\delta$ a positive number. Let $\varepsilon = \varepsilon(G, K)$. If $\pi$ is a representation of $G$ on a Hilbert space, $\mathcal{H}$, and $\zeta \in \mathcal{H}$ is an $(\varepsilon\delta, K)$-invariant unit vector for $\pi$ then

$$\|\pi(g)\zeta - \zeta\| < 2\delta$$

for every $g \in G$.

**Proof**

Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a representation with $(\varepsilon\delta, K)$-invariant unit vector $\zeta$. If $\delta > 1$ then $\|\pi(g)\zeta - \zeta\| \leq 2\|\zeta\| < 2\delta$, since $\zeta$ is a unit vector. Suppose $\delta \leq 1$. Then, since $\varepsilon$ is the Kazhdan constant for $G$ and $K$, $\pi(G)$ has some non-zero invariant vector. Let $\mathcal{H}_0$ be the subspace of $\mathcal{H}$ consisting of vectors invariant by $\pi(G)$ and let $\mathcal{H}_1$ be the orthogonal complement of $\mathcal{H}_0$ in $\mathcal{H}$. Let $\zeta = \zeta_0 + \zeta_1$ be the decomposition of $\zeta$ into components in $\mathcal{H}_0$ and $\mathcal{H}_1$. If $\zeta_1 = 0$ then $\zeta$ is
invariant and the lemma is true. Suppose $\zeta_1 \neq 0$. Then

$$\sup_{g \in K} \left\| \pi(g) \frac{\zeta_1}{\|\zeta_1\|} - \frac{\zeta_1}{\|\zeta_1\|} \right\| = \frac{1}{\|\zeta_1\|} \sup_{g \in K} \|\pi(g)\zeta - \zeta\| < \frac{\varepsilon \delta}{\|\zeta_1\|}.$$ 

Since $\mathcal{H}_1$ does not contain any non-zero invariant vectors it cannot contain any $(\varepsilon, K)$-invariant unit vectors and so $\frac{\varepsilon \delta}{\|\zeta_1\|} > \varepsilon$. Hence $\|\zeta_1\| < \delta$. Then, for all $g \in G$,

$$\|\pi(g)\zeta - \zeta\| = \|\pi(g)\zeta_1 - \zeta_1\| \leq 2\|\zeta_1\| < 2\delta.$$ 

\[ \square \]

**Theorem 4.5.2:**

Let $G$ be a group with property $(T)$, $K$ a compact generating set for $G$ and $\psi : G \to \mathbb{R}$ a conditionally negative type function on $G$. Then for every $g \in G$

$$0 \leq \psi(g) \leq \left( \frac{2}{\varepsilon(G, K)} \right)^2 \sup_{h \in K} \psi(h).$$

**Proof:**

We already know that real valued conditionally negative type functions take only non-negative values.

Let $N = \sup \{ \psi(g) : g \in K \}$ and let $\varepsilon = \varepsilon(G, K)$. We know that $N$ is well defined since $\psi$ is continuous and $K$ is compact. Take $\delta$ such that $0 < \delta < \frac{1}{\sqrt{2}}$.

If $N = 0$ take any $t > 0$. Otherwise take $t$ such that $0 < t < -\frac{1}{N} \ln \left( 1 - \frac{(\varepsilon \delta)^2}{2} \right)$.

Notice that $\frac{(\varepsilon \delta)^2}{2} < 1$ since $\varepsilon \leq 2$. By Schoenberg's Theorem ([HaV] 5.16) we may define a function, $\phi : G \to \mathbb{R}$, of positive type by

$$\phi(g) = e^{-t\psi(g)} \quad (g \in G).$$

Let $(\mathcal{H}, \pi, \zeta)$ be the cyclic representation of $G$ induced by $\phi$. So

$$\phi(g) = \langle \pi(g)\zeta | \zeta \rangle$$

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for all $g \in G$. The representation acts on a real Hilbert space since $\phi$ is real valued. Note that $\|\zeta\|^2 = (\pi(e)\zeta | \zeta) = \phi(e) = e^{-t\psi(e)} = 1$ and that, for every $g \in G$, $0 < \phi(g) \leq \phi(e) = 1$ (Lemma 4.2.4).

Now
\[ -Nt > \ln \left(1 - \frac{(\epsilon \delta)^2}{2}\right) \]
\[ \iff e^{-Nt} > 1 - \frac{(\epsilon \delta)^2}{2} \]
\[ \iff 1 - e^{-Nt} < \frac{(\epsilon \delta)^2}{2} \]
and, for each $g \in K$,
\[ 0 \leq \psi(g) \leq N \]
\[ \iff 1 \geq \phi(g) = e^{-t\psi(g)} \geq e^{-Nt} \]
\[ \iff 0 \leq 1 - \phi(g) \leq 1 - e^{-Nt}. \]

Hence
\[ \sup_{g \in K} (1 - \phi(g)) < \frac{(\epsilon \delta)^2}{2} \]
and
\[ \sup_{g \in K} \|\pi(g)\zeta - \zeta\|^2 = \sup_{g \in K} (2 - 2(\pi(g)\zeta | \zeta)) \]
\[ = \sup_{g \in K} 2(1 - \phi(g)) \]
\[ < (\epsilon \delta)^2. \]

It follows by Lemma 4.5.1 that
\[ \|\pi(g)\zeta - \zeta\| < 2\delta \quad \forall g \in G \]
and hence that
\[ 0 \leq 1 - \phi(g) = \frac{1}{2} \|\pi(g)\zeta - \zeta\|^2 < 2\delta^2 \quad \forall g \in G. \]
We then have, for each \( g \in G \),
\[
1 - e^{-tq(g)} = 1 - \phi(g) < 2\delta^2
\]
\[
\iff e^{-tq(g)} > 1 - 2\delta^2
\]
\[
\iff t \psi(g) < -\ln(1 - 2\delta^2)
\]
\[
\iff \psi(g) < -\frac{1}{t} \ln(1 - 2\delta^2)
\]

If \( N > 0 \), this holds for every \( t \in (0, -\frac{1}{N} \ln \left(1 - \frac{(\epsilon \delta)^2}{2}\right)) \) so, for every \( \delta \in (0, \frac{1}{\sqrt{2}}) \) and every \( g \in G \),
\[
0 \leq \psi(g) \leq \frac{N \ln(1 - 2\delta^2)}{\ln(1 - \frac{(\epsilon \delta)^2}{2})}
\]

But
\[
\lim_{\delta \to 0} \frac{\ln(1 - 2\delta^2)}{\ln(1 - \frac{(\epsilon \delta)^2}{2})} = \left(\frac{2}{\epsilon}\right)^2
\]

So, for every \( g \in G \),
\[
0 \leq \psi(g) \leq \left(\frac{2}{\epsilon}\right)^2 N
\]

If \( N = 0 \) then, for every \( g \in G \) and every \( \delta \in \left(0, \frac{1}{\sqrt{2}}\right) \),
\[
0 \leq \psi(g) < -\frac{1}{t} \ln(1 - 2\delta^2)
\]

for all positive \( t \). Hence, for every \( g \in G \)
\[
\psi(g) = 0
\]

and \( 0 \leq \psi(g) \leq \left(\frac{2}{\epsilon}\right)^2 N \).
\( \Box \)

If instead of general conditionally negative type functions we consider distance functions obtained from the actions of groups on trees we can produce a tighter bound.
Theorem 4.5.3:

Let \( G \) be a compactly generated group acting on a tree \( \Gamma \) and suppose that the action has at least one fixed point (as is the case if \( G \) has property \((T)\)). Then, for every compact generating set, \( K \), for \( G \) and every \( x \in \Delta_0^0 \),

\[
\sup_{g \in G} d_\Gamma(x,gx) = \sup_{g \in K} d_\Gamma(x,gx)
\]

and \( G \) fixes some \( y \in \Delta_0^0 \) such that

\[
d_\Gamma(x,y) = \frac{1}{2} \sup_{g \in K} d_\Gamma(x,gx) .
\]

This \( y \) is the unique closest fixed point to \( x \). The path from \( x \) to any other fixed point passes through \( y \).

Proof:

Let \( x \in \Delta_0^0 \). If the action of \( G \) fixes \( x \) then the result is clearly true. We shall assume that \( x \) is not a fixed point.

Let \( K \) be a compact generating set for \( G \). Note that \( \sup \{d_\Gamma(x,gx) : g \in K\} \) is finite since the function \( g \mapsto d_\Gamma(x,gx) \) is continuous.

Since \( K \) generates \( G \), any point in \( \Delta_0^0 \) is fixed by \( G \) if and only if it is fixed by \( K \). Let

\[
D = \inf \{d_\Gamma(x,y) : y \text{ is fixed by } G\} .
\]

This infimum is a minimum since \( d_\Gamma(x,y) \) is a non-negative integer for all \( y \in \Delta_0^0 \), so we can find some \( y \in \Delta_0^0 \) which is fixed by \( G \) such that \( d_\Gamma(x,y) = D \). Then, for every \( g \in G \),

\[
d_\Gamma(gx,y) = d_\Gamma(gx,gy) = d_\Gamma(x,y) = D
\]

and

\[
d_\Gamma(x,gx) \leq d_\Gamma(x,y) + d_\Gamma(gx,y) = 2D .
\]
There is no fixed point (for \( G \) or equivalently \( K \)) on the path from \( x \) to \( y \), since if \( z \) were such a point we would have \( d_\Gamma (x, z) < D \), contradicting the definition of \( D \). In particular, if \( w \) is the vertex nearest to \( y \) on the path from \( x \) to \( y \) then some element \( h \in K \) does not fix \( w \). Suppose \( h \) fixes some point \( u \) between \( x \) and \( y \). Since \( h \) must fix \( y \), \( h \) also fixes the path from \( u \) to \( y \), and this path contains \( w \), giving a contradiction. Hence \( h \) does not fix any point strictly between \( x \) and \( y \). The action of \( h \) maps the path between \( x \) and \( y \) to the path between \( hx \) and \( hy \) and these two paths meet only at the fixed point \( y = hy \). Then

\[
d_\Gamma (x, hx) = d_\Gamma (x, y) + d_\Gamma (y, hx) = d_\Gamma (x, y) + d_\Gamma (hy, hx) = 2D.
\]

Hence

\[
\sup_{g \in K} d_\Gamma (x, gx) \leq \sup_{g \in G} d_\Gamma (x, gx) \leq 2D \leq \sup_{g \in K} d_\Gamma (x, gx).
\]

Now let \( z \) be a fixed point different from \( y \). Suppose that the path from \( x \) to \( z \) does not pass through \( y \).

Then the path from \( y \) to \( z \) must pass through some point \( u \), where either \( u = x \) or \( u \) lies strictly between \( x \) and \( y \). In either case \( d_\Gamma (x, u) < D \). Since \( y \) and \( z \) are both fixed, the path between \( y \) and \( z \) must be fixed and so \( u \) is fixed, contradicting the definition of \( D \). So the path from \( x \) to \( z \) must pass through \( y \) and \( d_\Gamma (x, z) > d_\Gamma (x, y) \).

\( \square \)
Corollary 4.5.4:

If $G$ and $\Gamma$ are as above, the subtree generated by the orbit under $G$ of any point in $\Delta^0_\Gamma$ has finite diameter. In particular if $G$ acts transitively on $\Gamma$ then $\text{diam}(\Gamma)$ is finite.

\[ \square \]

In the case of a property (T) group, this corollary and the existence of a fixed point follow from the fact that conditionally negative type functions are bounded.

Theorem 4.5.5 (Theorem 4.5.3 for a Complete $\mathbb{R}$-tree):

Let $G$ be a compactly generated group acting by isometries on a complete $\mathbb{R}$-tree, $Y$, and suppose that the action has at least one fixed point (as is the case if $G$ has property (T)). Then, for every compact generating set, $K$, of $G$ and every $x \in Y$,

\[
\sup_{g \in G} d_Y(x, gx) = \sup_{g \in K} d_Y(x, gx)
\]

and $G$ fixes some $y \in Y$ such that

\[
d_Y(x, y) = \frac{1}{2} \sup_{g \in K} d_Y(x, gx).
\]

This $y$ is the unique closest fixed point to $x$. The path from $x$ to any other fixed point passes through $y$.

Proof:

Let $x \in Y$. Again the result is clear if $x$ is a fixed point so we shall assume that it is not.

For any two points $u, v \in Y$ let $[u, v]$ denote the unique arc joining them in $Y$. Let $H$ denote a homeomorphism of $[u, v]$ onto a compact interval in $\mathbb{R}$. If $g \in G$ then $g$ is an isometry of $[u, v]$ onto $g([u, v])$; in particular it is a homeomorphism. Then $Hg^{-1}$ is a homeomorphism of $g([u, v])$ onto $H([u, v])$,}

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which is a compact interval in \( \mathbb{R} \), and so \( g([u,v]) \) is an arc in \( Y \) joining \( g(u) \) to \( g(v) \). By uniqueness

\[
g([u,v]) = [g(u), g(v)] \quad \forall \ u, v \in Y.
\]

It follows that if \( u \) and \( v \) are fixed points for the action of \( G \) on \( Y \) then the whole of \([u,v]\) is fixed.

Let \( w \in Y \) be fixed by the action of \( G \) and let

\[
D = \inf \{ d_Y(x,u) : u \in [x,w]; u \text{ is fixed by } G \}.
\]

Now \([x,w]\) is isometric to an interval in \( \mathbb{R} \). There is precisely one point \( y \in [x,w] \) such that \( d_Y(x,y) = D \). Every point strictly between \( y \) and \( w \) is fixed and no point strictly between \( x \) and \( y \) is fixed. For all \( \epsilon > 0 \) there is a fixed \( u \) between \( w \) and \( y \) with \( D \leq d_Y(x,u) < D + \epsilon \) and hence \( d_Y(y,u) < \epsilon \). Then, for each \( g \in G \),

\[
d_Y(y,g(y)) \leq d_Y(y,u) + d_Y(u,g(y)) < 2\epsilon.
\]

Since this holds for all \( \epsilon > 0 \), \( y \) is fixed by the action of \( G \).

Let \( K \) be a compact generating set for \( G \). As in Theorem 4.5.3, a point of \( Y \) is fixed by \( G \) if and only if it is fixed by \( K \). For all \( m < D \), there is a \( v \in Y \) strictly between \( x \) and \( y \) such that \( D > d_Y(x,v) > m \). Then there is some \( g \in K \) which does not fix \( v \) and so does not fix any point in \([x,v]\) (since if \( g \) fixed \( u \in [x,v] \) then \( g \) would also fix \([u,y]\), which contains \( v \)). Let \( z \) be the point in \([x,y]\) nearest to \( v \) fixed by \( g \). The point \( z \) exists for the same reasons as \( y \).

Then \( d_Y(x,z) > m \) and \( z \) is the nearest point of \([x,y]\) to \( x \) fixed by \( g \), so

\[
d_Y(x,g(x)) = d_Y(x,z) + d_Y(z,g(x)) > 2m
\]

and

\[
\sup_{g \in K} d_Y(x,g(x)) \geq 2D.
\]

The result now follows as for Theorem 4.5.3.

\( \Box \)
4.6 The Free Group on 2 Generators; an Example of the Action of a Group on a Tree.

A simple example of a group action on a tree is the action of the free group on two generators, $F_2 = \langle a, b \rangle$, on the following infinite tree.

Suppose $g, h \in G$. Then the action of $g$ maps the point $hx$ of the tree to the point $ghx$. The distance function with base point $x$ gives $d(x, gx)$ equal to the reduced word length of $g$. In [Hag] Lemma 1.2, it is shown directly that the word length is a conditionally negative type function on a free group with any
finite number of generators. The proof in [HaV] that the distance function on a
tree is a conditionally negative type kernel is based on this proof.

4.7 \( SL(2, \mathbb{Z}) \) and \( PSL(2, \mathbb{Z}); \) an Action on the
Upper Half Plane.

The special linear group, \( SL(2, \mathbb{Z}) \), is generated by two elements, one of order 4
and the other of order 3. It may be considered as the multiplicative matrix
group generated by

\[
A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.
\]

These matrices suggest an action of the group on the upper half plane:

\[
A : z \mapsto \frac{-1}{z} \quad B : z \mapsto \frac{z - 1}{z} = 1 - \frac{1}{z}.
\]

The group \( PSL(2, \mathbb{Z}) \) also acts on the upper half plane by

\[
a : z \mapsto \frac{-1}{z} \quad b : z \mapsto 1 - \frac{1}{z}.
\]

There is a homomorphism from \( SL(2, \mathbb{Z}) \) onto \( PSL(2, \mathbb{Z}) \) given by

\[
A \mapsto a \quad B \mapsto b.
\]

Composing this with the action of \( PSL(2, \mathbb{Z}) \) on the upper half plane gives the
original action of \( SL(2, \mathbb{Z}) \). Hence the action of \( PSL(2, \mathbb{Z}) \) on the \( \mathbb{R} \)-tree
described below also gives an action of \( SL(2, \mathbb{Z}) \) on the same \( \mathbb{R} \)-tree.

Every non-identity element of \( SL(2, \mathbb{Z}) \) has reduced word form either \( A, A^2, \)
\( A^3 \) or \( A^{p_1} B^{p_2} A^{q_1} B^{q_2} A^{r_1} B^{r_2} \cdots B^{q_{n-1}} A^{r_{n-1}} B^{q_n} A^{p_2} \) for some positive integer \( n \), where
\( p_1, p_2 \in \{0, 1, 2, 3\}, q_1, \ldots, q_n \in \{1, -1\} \) and \( r_1, \ldots, r_{n-1} \in \{1, 2, 3\} \).

For \( g \in PSL(2, \mathbb{Z}) \) we define \( |g| \) to be the shortest word length of a word in
\( a, b, b^{-1} \) equivalent to \( g \), that is the length of the reduced word form of \( g \). This
reduced word form is either $e$, $a$ or $a^{p_1}b^{q_1}a^{p_2} \ldots b^{q_{n-1}}a^{p_n}a^{p_2}$ for some positive integer $n$, where $p_1, p_2 \in \{0, 1\}$ and $q_1, q_2, \ldots, q_n \in \{1, -1\}$. We have $|e| = 0$, $|a| = 1$ and if $g$ is neither $e$ nor $a$ then $|g| = 2n - 1 + p_1 + p_2$.

4.8 An Action of $PSL(2, \mathbb{Z})$ on an $R$-tree.

We use the above action of $PSL(2, \mathbb{Z})$ on the upper half plane to construct an $R$-tree, $\Gamma$, as follows. Consider the upper half plane as a hyperbolic metric space, and let $L$ be the directed circular arc from $i$ to $e^{i \frac{\pi}{3}}$. Then $\Gamma$ consists of the images of $L$ under the action on the upper half plane of $PSL(2, \mathbb{Z})$ described above. The action of the group on the $R$-tree is the restriction of the action on the upper half plane.

The distance between two points of $\Gamma$ is the hyperbolic distance measured along the $R$-tree, normalised by a factor $\frac{2}{\ln 3}$ to make the length of each image of $L$ equal to 1. The distance is invariant under the action of the group. That is, for every $z, w \in \Gamma$ and each $g \in PSL(2, \mathbb{Z})$, $d(z, w) = d(gz, gw)$.

Let $z \in \Gamma$ and choose $g \in PSL(2, \mathbb{Z})$ so that $z \in gL$. This $g$ will be unique unless $z$ is an image of either $i$ or $e^{i \frac{\pi}{3}}$, in which case there is a choice of 2 or 3
elements of $PSL(2, \mathbb{Z})$.

We have

\[
d(gi, z) = d(i, g^{-1}z) = d(b^{-1}i, b^{-1}g^{-1}z)
\]

\[
d(ge^{i\frac{\pi}{3}}, z) = d(e^{i\frac{\pi}{3}}, g^{-1}z) = d(b^{-1}e^{i\frac{\pi}{3}}, b^{-1}g^{-1}z) = d(e^{i\frac{\pi}{3}}, b^{-1}g^{-1}z).
\]

Let $\theta = \arg(g^{-1}z)$. Then

\[
g^{-1}z = e^{i\theta}
\]

\[
b^{-1}g^{-1}z = \frac{1}{1 - e^{i\theta}} = \frac{1 - e^{-i\theta}}{2(1 - \cos \theta)}
\]

\[
\Re(b^{-1}g^{-1}z) = \frac{1 - \cos \theta}{2(1 - \cos \theta)} = \frac{1}{2}
\]

\[
\Im(b^{-1}g^{-1}z) = \frac{\sin \theta}{2(1 - \cos \theta)}
\]

which gives

\[
d(gi, z) = d(b^{-1}i, b^{-1}g^{-1}z)
\]

\[
= \frac{2}{\ln 3} \int_{\frac{1}{2}}^{\frac{\sin \theta}{2(1 - \cos \theta)}} \frac{1}{y} \, dy
\]

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\[ \frac{2}{\ln 3} \ln \left( \frac{\sin \theta}{1 - \cos \theta} \right) \]

and

\[ d(ge^{i\pi}, z) = d(e^{i\pi}, b^{-1}g^{-1}z) \]

\[ = \frac{2}{\ln 3} \int_{\frac{\sin \theta}{2(1 - \cos \theta)}}^{\frac{\sqrt{3}}{2}} \frac{1}{y} \, dy \]

\[ = \frac{2}{\ln 3} \left( \ln \sqrt{3} - \ln 2 - \ln \left( \frac{\sin \theta}{1 - \cos \theta} \right) + \ln 2 \right) \]

\[ = \frac{2}{\ln 3} \ln \left( \frac{\sqrt{3}(1 - \cos \theta)}{\sin \theta} \right). \]

Hence

\[ d(i, e^{i\pi}) = \frac{2}{\ln 3} \ln \left( \frac{\sin \frac{\pi}{3}}{1 - \cos \frac{\pi}{3}} \right) = 1. \]

Now consider a point \( x \in L; x = e^{i\theta}, \theta \in \left[ \frac{\pi}{3}, \frac{\pi}{2} \right] \). We know that the function \( f : \text{PSL}(2, \mathbb{Z}) \to \mathbb{R}_{\geq 0} \) defined by \( f(h) = d(x, hx) \) for \( h \in \text{PSL}(2, \mathbb{Z}) \) is of conditionally negative type. We can now evaluate this function. Note first that if \( h = b^{\pm 1}ab^{\pm 1}a...b^{\pm 1}a^r \in \text{PSL}(2, \mathbb{Z}) \), where \( r \in \{0, 1\} \), then

\[ f(b^{\pm 1}ah) = f(h) + 2; \]

and if \( h = ab^{\pm 1}ab^{\pm 1}a...b^{\pm 1}a^r \in \text{PSL}(2, \mathbb{Z}) \), where \( r \in \{0, 1\} \), then \( f(ab^{\pm 1}h) = f(h) + 2 \). The value of the function for each element of the group depends on the structure of the reduced word. We divide the group into the following cases.

\[ f(e) = 0 \]

\[ f(a) = 2d(x, i) = \frac{4}{\ln 3} \ln \left( \frac{\sin \theta}{1 - \cos \theta} \right) \]

\[ f(b^{\pm 1}) = 2d(x, e^{i\pi}) = \frac{4}{\ln 3} \ln \left( \frac{\sqrt{3}(1 - \cos \theta)}{\sin \theta} \right) \]

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\[
\begin{align*}
    f(b^{\pm 1}a b^{\pm 1}a b^{\pm 1}) &= (\text{word length} - 1) + f(b^{\pm 1}) \\
    &= (\text{w.l.} - 1) + \frac{4}{\ln 3} \ln \left( \frac{\sqrt{3}(1 - \cos \theta)}{\sin \theta} \right) \\
    f(ab^{\pm 1}a b^{\pm 1}ab^{\pm 1}ab^{\pm 1}a) &= (\text{word length} - 1) + f(a) \\
    &= (\text{w.l.} - 1) + \frac{4}{\ln 3} \ln \left( \frac{\sin \theta}{1 - \cos \theta} \right) \\
    f(b^{\pm 1}ab^{\pm 1}ab^{\pm 1}ab^{\pm 1}a) &= (\text{word length}) \\
    f(ab^{\pm 1}ab^{\pm 1}ab^{\pm 1}ab^{\pm 1}) &= (\text{word length})
\end{align*}
\]

### 4.9 Action of $PSL(2, Z)$ on a Tree.

The graph obtained in the previous section may be considered simply as a tree, $\gamma$, with vertices the images of $i$ and $e^{i\frac{\pi}{3}}$ and edges the images of $L$. The action of the group on the tree is again the restriction of the action on the upper half plane. We define two functions, $\omega$ and $\phi$, on $PSL(2, Z)$ by

\[
\begin{align*}
    \omega(g) &= d_\gamma (i, gi) \\
    \phi(g) &= d_\gamma (e^{i\frac{\pi}{3}}, ge^{i\frac{\pi}{3}}) \quad g \in PSL(2, Z).
\end{align*}
\]

We know from Section 4.4 that these functions are of conditionally negative type. We evaluate them in terms of the reduced word froms of the elements of $PSL(2, Z)$. Let $g \in PSL(2, Z)$ and suppose that $g$ is neither $e$ nor $a$. Let the reduced word form for $g$ be $g_1 g_2 \ldots g_m = a^{p_1} b^{q_1} a b^{p_2} a b^{q_2} \ldots b^{q_{n-1}} a b^{q_n} a b^{p_2}$ where $m = |g|$, $n \in \mathbb{Z}^+$, $p_1, p_2 \in \{0, 1\}$, and $q_1, \ldots, q_n \in \{1, -1\}$. From the previous section, taking $x = i = e^{i\frac{\pi}{3}}$,

\[
\begin{align*}
    \omega(e) &= 0 \\
    \omega(a) &= 0
\end{align*}
\]
\[ \omega(g) = |g| + 1 - \delta_a(g_1) - \delta_a(g_m) \]

and taking \( x = e^{i\xi} \),

\[ \phi(e) = 0 \]
\[ \phi(a) = 2 \]
\[ \phi(g) = |g| - 1 + \delta_a(g_1) + \delta_a(g_m) . \]

4.10 A Result Using Actions of Groups on Trees.

Consider a non-compact group, \( G \) (locally compact and \( \sigma \)-compact as in Section 1.1). From [A&W], the following properties are equivalent and imply that \( G \) does not have property (T).

(A) \( C_0(G) \) has an approximate identity consisting of functions of positive type.

(B) There is a conditionally negative type function, \( \psi \), on \( G \) for which for any \( m > 0 \) there is a compact set \( K_m \subset G \) such that \( |\psi(x)| > m \) for all \( x \in G \setminus K_m \).

A compact group has property (T) and both properties (A) and (B). The approximate identity is the constant function 1, the conditionally negative type function is the constant function 0 and \( K_m \) is the whole group for each \( m \).

Lemma 4.10.1:

The word length on a free group on a finite number of generators and the functions \( \omega \) and \( \phi \) on \( PSL(2, \mathbb{Z}) \) defined in Section 4.9 satisfy condition (B) above.

Proof:

We already know that these functions are of conditionally negative type, since they are distance functions for the actions of the groups on trees. Take a
positive number $m$. For the free group, $|g| > m$ for all $g$ outside the finite set of elements with reduced words of length at most $m$. For $PSL(2, \mathbb{Z})$, $|\phi(g)| > m$ and $|\omega(g)| > m$ for all $g$ outside the finite set of elements with reduced words of length not exceeding $m + 1$.

□

Lemma 4.10.2:

Suppose a discrete group $G$ has the equivalent properties (A) and (B). Then so does every subgroup of $G$. In particular, no infinite subgroup of $G$ has property (T).

Proof:

We have already remarked that all finite groups satisfy (A) and (B). Let $H$ be an infinite subgroup of $G$. Take a function $\psi$ which demonstrates property (B), a constant $m > 0$ and a compact subset $K_m \subseteq G$ such that $|\psi(x)| > m$ for all $x \in G \setminus K_m$. Since $G$ is discrete, $K_m$ is finite and so $K_m \cap H$ is a finite, and hence compact, subset of $H$. For every $g \in H \setminus (H \cap K_m)$, $|\psi(g)| > m$.

□

Proposition 4.10.3:

Every $F_n$ ($n \in \mathbb{Z}^+$) and every infinite subgroup of $PSL(2, \mathbb{Z})$ has properties (A) and (B) above and so does not have property (T).

Proof:

This follows immediately from Lemmas 4.10.1 and 4.10.2.

□

(Every non-trivial subgroup of $F_n$ is itself a free group and so the subgroups of $F_n$ need not be considered separately from the whole group.)
4.11 Application to Group von Neumann Algebras.

In [C&J], A.Connes and V.Jones define the concept of property (T) for von Neumann algebras and prove (Theorem 2) that the group von Neumann algebra, \( VN(G) \), of a countable, discrete, ICC group, \( G \), has property (T) if and only if the group does. An ICC group is one in which every conjugacy class apart from the identity has infinitely many members, for example a non-abelian free group. The group von Neumann algebra of a countable, discrete, ICC group is a type II\(_1\)-factor ([K&R] 6.7.5). Connes and Jones also show (Theorem 3) that if \( M \) is a type II\(_1\)-factor with property (T) then the identity map on \( M \) cannot be approximated (pointwise in \( L^2 \)) by a sequence, \((\psi_n)_{n \in \mathbb{N}}\) of normal, completely positive, trace decreasing maps with compact extensions to \( L^2(M) \) (defined below) such that \( \psi_n(I_M) \leq I_M \), where \( I_M \) is the identity element of \( M \).

The requirement in [C&J] Theorem 2 that the maps be trace decreasing ensures that they have bounded extensions to \( L^2(M) \). In [Mgo] Section 1, J.A.Mingo demonstrates that a completely positive, normal operator on a von Neumann algebra, \( M \), need not have a bounded extension to \( L^2(M) \), even if the operator has finite rank. M.Choda has shown ([Cho]) that if \( M \) is \( VN(G) \) for some countable, discrete, ICC, Kazhdan group, \( G \), the requirements that \( \psi_n(I_M) \leq I_M \) and that \( \psi_n \) is trace decreasing can be dropped in favour of a requirement that \( \psi_n \) is bounded in the \( L^2 \) topology. A.G.Robertson has extended Choda's result to all type II\(_1\)-factors with property (T) in [Rob](Theorem). Theorem 4.11.1 below includes the extension of this result to \( VN(G) \) where \( G \) is any infinite, countable, discrete, group with property (T). J.A.Mingo has proved a similar, stronger result for von Neumann algebras not of type I with property (T) ([Mgo] Theorem 5).
Property (T) for von Neumann Algebras.

The definition of property (T) for von Neumann algebras is analogous to that for groups, but it concerns correspondences rather than representations. If $M$ is a von Neumann algebra, a correspondence from $M$ to $M$ is a Hilbert space, $\mathcal{H}$, which is a $M$-bimodule such that the left and right actions of $M$ on $\mathcal{H}$ are normal and commute with each other, so that $a\zeta b = a(\zeta b) = (a\zeta)b$ for all $a, b \in M$ and $\zeta \in \mathcal{H}$. Two correspondences, $\mathcal{H}_1$ and $\mathcal{H}_2$ are equivalent if there is an isometric isomorphism $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that $U^{-1}xU\zeta = x\zeta$ and $U^{-1}((U\zeta)x) = \zeta x$ for all $x \in M$ and $\zeta \in \mathcal{H}_1$. We define a topology on the set of (equivalences classes of) correspondences from $M$ to $M$ as follows. Given a correspondence $\mathcal{H}$, $\varepsilon > 0$, $\zeta_1, ..., \zeta_n \in \mathcal{H}$ and $x_1, ..., x_p, y_1, ..., y_q \in M$, let $U(\varepsilon, (\zeta_i), (x_j), (y_k))$ be the set of correspondences, $\mathcal{H}'$, from $M$ to $M$ which contain vectors $\eta_1, ..., \eta_n$ such that $|\langle x_j \eta_i y_k, \eta_l \rangle - \langle x_j \zeta_l y_k, \zeta_i \rangle| < \varepsilon$ for each $i, l \in \{1, ..., n\}$, $j \in \{1, ..., p\}$ and $k \in \{1, ..., q\}$. These sets form a base of neighbourhoods of $\mathcal{H}$. The identity correspondence, $id_M$, is the space of the standard representation of $M$ defined in [Hag2]. A standard representation of $M$ is a $*$-isomorphism, $\pi$, of $M$ into $B(\mathcal{H}_\pi)$ for some Hilbert space, $\mathcal{H}_\pi$, for which there exist an anti-linear, isometric involution $J : \mathcal{H}_\pi \rightarrow \mathcal{H}_\pi$ and a self-dual cone $P$ in $\mathcal{H}_\pi$ (that is $\{\zeta \in \mathcal{H}_\pi : \langle \zeta | \eta \rangle \geq 0 \forall \eta \in P\} = P$ ) such that

1. $J\pi(M)J = \pi(M)'$, the commutant of $\pi(M)$ in $B(\mathcal{H}_\pi)$,

2. $J\pi(c)J = \pi(c^*)$ for all $c$ in the centre of $M$,

3. $J\zeta = \zeta$ for all $\zeta \in P$,

4. $aJaJ(P) \subseteq P$ for all $a \in M$.

The left and right actions of $M$ on $\mathcal{H}_\pi$ are given by $x\zeta = \pi(x)\zeta$ and $\zeta x = J\pi(x^*)J\zeta$ for all $\zeta \in \mathcal{H}_\pi$ and $x \in M$. The extension of the left regular representation of $G$ on $l^2(G)$ is a standard representation of $VN(G)$.

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U. Haagerup shows in [Hag2] Theorem 1.6 that every von Neumann algebra has such a representation. In Theorem 2.3 of the same paper he shows the if \(\pi\) and \(\rho\) are two such representations then the correspondences \(\mathcal{H}_\pi\) and \(\mathcal{H}_\rho\) are equivalent, so we have defined \(id_M\) uniquely in the set of equivalence classes of correspondences from \(M\) to \(M\). We say that \(M\) has property (T) if there is a neighbourhood of \(id_M\) all of whose elements contain \(id_M\) as a direct summand.

**Definition of \(L^2(M)\).**

Suppose the von Neumann algebra, \(M\), has a faithful, unital, trace, \(\tau\). That is

(a) \(\tau\) is a linear functional on \(M\),

(b) \(\tau(X^*X) = \tau(XX^*) > 0\) for all non-zero \(X \in M\),

(c) \(\tau(I_M) = 1\) where \(I_M\) is the identity element of \(M\).

(If such a trace exists then \(M\) is a finite von Neumann algebra.) We can use this trace to define an inner product on \(M\):

\[
(X | Y) = \tau(XY^*) \quad X, Y \in M.
\]

The topology induced by this inner product is called the \(L^2\) topology on \(M\).

The completion of \(M\) with respect to the \(L^2\) topology is a Hilbert space denoted \(L^2(M)\).

For a general von Neumann algebra such a trace need not exist or need not be unique, so it may not be possible to define a \(L^2\) topology or there may be more than one possible such topology. In the case of a finite von Neumann factor such a trace always exists and is unique ([K&R] Theorem 8.2.8) so the \(L^2\) topology is well defined. If \(M\) is the group von Neumann algebra of a countable, discrete group, \(G\), we make the choice of trace and construct \(M\) and \(L^2(M)\) as follows. In this case we shall see that \(L^2(M)\) is isometrically isomorphic to \(l^2(G)\).

Let \(\lambda\) denote the left-regular representation of \(G\) on \(l^2(G)\);
\((\lambda(g)\zeta)(h) = \zeta(g^{-1}h)\) for all \(g, h \in G, \zeta \in \ell^2(G)\). Let \((. \mid .)\) denote the usual inner product in \(\ell^2(G)\). Let \(\{\delta_t : t \in G\}\) be the orthonormal basis of \(\ell^2(G)\) given be \(\delta_t(t) = 1, \delta_t(s) = 0\) if \(s \neq t, s, t \in G\). Then \(\lambda(s)\delta_t = \delta_{st}\) for all \(s, t \in G\). The group von Neumann algebra \(M = VN(G) = C_r^*(G)''\) is the closure of \(\text{span}(\lambda(G))\) in the strong, weak or ultraweak operator topology on \(B(\ell^2(G))\).

Recall that the strong, weak and ultraweak closures of a \(*\)-algebra in \(B(\mathcal{H})\) (where \(\mathcal{H}\) is a Hilbert space) are all the same ([Dix2] Ch.1, Sect.3, Theorem 2).

Suppose \(X \in M \subseteq B(\mathcal{H})\). Then \(X\) is in the strong operator closure of \(\text{span}(\lambda(G))\), so for any \(h \in G\) and any \(\varepsilon > 0\) there exist a finite subset \(K\) of \(G\) and complex numbers \(\mu_g\) for each \(g \in K\) such that

\[
\left\| \left( X - \sum_{g \in K} \mu_g \lambda(g) \right) \delta_h \right\| < \varepsilon
\]

and

\[
\left\| \left( X - \sum_{g \in K} \mu_g \lambda(g) \right) \delta_e \right\| < \varepsilon.
\]

Hence

\[
\varepsilon^2 > \sum_{g \in G \setminus K} |(X \delta_e \mid \delta_g)|^2 + \sum_{g \in K} |(X \delta_e \mid \delta_g) - \mu_g|^2
\]

and

\[
\varepsilon^2 > \sum_{g \in G} \left| \left( X \delta_h - \sum_{k \in K} \mu_k \delta_{kh} \right) \delta_{gh} \right|^2
\]

\[
= \sum_{g \in G \setminus K} |(X \delta_h \mid \delta_{gh})|^2 + \sum_{g \in K} |(X \delta_h \mid \delta_{gh}) - \mu_g|^2.
\]

So for each \(g \in G\),

\[
|(X \delta_e \mid \delta_g) - (X \delta_h \mid \delta_{gh})| < 2 \varepsilon.
\]

Since this holds for all \(\varepsilon > 0\), we see that

\[
(X \delta_e \mid \delta_g) = (X \delta_h \mid \delta_{gh}) \quad \forall g, h \in G.
\]
Hence, for every $h \in G$,

$$X \delta_h = \sum_{g \in G} (X \delta_h | \delta_{gh}) \delta_{gh}$$

$$= \sum_{g \in G} (X \delta_e | \delta_g) \lambda(g) \delta_h .$$

Since $\{\delta_g : g \in G\}$ spans $l^2(G)$, it follows that

$$\sum_{g \in G} (X \delta_e | \delta_g) \lambda(g) = X.$$ 

So $X$ can be written as

$$X = \sum_{g \in G} \mu_g \lambda(g)$$

for some complex numbers $\mu_g$ such that $\sum \mu_g \delta_g = X \delta_e \in l^2(G)$. As an operator on $l^2(G)$

$$X^* = \sum_{g \in G} (\delta_{g^{-1}} | X \delta_e) \lambda(g) .$$

We define a trace, $\text{tr}$, on $M$ by

$$\text{tr}(X) = (X \delta_e | \delta_e)$$

for $X \in M$. We check that $\text{tr}$ is a faithful, unital, trace on $M$.

(a) The function $\text{tr}$ is clearly a linear functional on $M$.

(b) For each $X \in M$

$$\text{tr} (XX^*) = (XX^* \delta_e | \delta_e)$$

$$= (X^* \delta_e | X^* \delta_e)$$

$$= \|X^* \delta_e\|^2$$

$$= \sum_{g \in G} |(X^* \delta_e | \delta_g)|^2$$

$$= \sum_{g \in G} |(\delta_{g^{-1}} | X^* \delta_e)|^2$$
\[
\begin{align*}
&= \sum_{g \in G} | (X \delta_e | \delta_g)|^2 \\
&= \|X \delta_e\|^2 \\
&= \text{tr} (X^* X)
\end{align*}
\]
and \( \text{tr} (X X^*) > 0 \) unless \((X \delta_e | \delta_g) = 0 \ \forall \ g \in G; \) in which case \( X = 0. \)

(c) The trace is unital since \( \text{tr} \lambda(e) = 1. \)

With the inner product and consequent norm defined from the above trace \( M \) may be isometrically embedded in the convolution algebra \( L^2(G) \) as follows.

We map \( X = \sum (X \delta_e | \delta_g) \lambda(g) \in M \subseteq B(L^2(G)) \) to
\[
\sum (X \delta_e | \delta_g) \delta_g = X \delta_e \in L^2(G).
\]
This is clearly a linear map from \( M \) into \( L^2(G) \).

We have already seen that \( X \delta_e = 0 \) if and only if \( X = 0 \), so the map \( X \mapsto X \delta_e \) is an embedding of \( M \) into \( L^2(G) \). We know that \( \lambda(g)\lambda(h) = \lambda(gh) \) and \( \delta_g \ast \delta_h = \delta_{gh} \), so that the embedding is a homomorphism. Also, for \( X, Y \in M, \)
\[
\langle X | Y \rangle = \text{tr} (XY^*)
\]
\[
= (XY^* \delta_e | \delta_e)
\]
\[
= (Y^* \delta_e | X^* \delta_e)
\]
\[
= \sum_{g \in G} (Y^* \delta_e | \delta_g) \ (\delta_g | X^* \delta_e)
\]
\[
= \sum_{g \in G} (\delta_{g^{-1}} | Y \delta_e) \ (X \delta_e | \delta_{g^{-1}})
\]
\[
= (X \delta_e | Y \delta_e),
\]
so that the embedding is an isometry. If \( \xi \in L^2(G), \ h \in G \) and \( X \in M \) then
\[
((X \delta_e) \ast \xi)(h) = \sum_{k \in G} \sum_{g \in G} (X \delta_e | \delta_g) \delta_g(hk^{-1}) \xi(k)
\]
\[
= \sum_{g \in G} (X \delta_e | \delta_g) \xi(g^{-1}h)
\]
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\[ = \left( \sum_{g \in G} (X \delta_e \mid \delta_g) \lambda(g) \right)(\xi) \right) \](h)
\[ = (X \xi)(h). \]

The subspace \( M \delta_e \) of \( L^2(M) \) acts on \( L^2(G) \) by convolution and

\[ M \delta_e = \{ \eta \in L^2(G) : \eta \ast \xi \in L^2(G) \text{ for all } \xi \in L^2(G) \}. \]

We have seen above that if \( X \in M \) and \( \xi \in L^2(G) \) then \( (X \delta_e) \ast \xi = X \xi \in L^2(G) \), since \( M \subseteq B(L^2(G)) \). Conversely, if \( \sum \mu_g \delta_g \in L^2(G) \) and \( (\sum \mu_g \delta_g) \ast \xi \in L^2(G) \) for all \( \xi \in L^2(G) \) then \( \sum \mu_g \lambda(g) \) is in the strong operator closure of \( \text{span} \{ \lambda(g) : g \in G \} \) in \( B(L^2(G)) \) i.e. \( M \), and \( (\sum \mu_g \lambda(g)) \delta_e = \sum \mu_g \delta_g \). The completion, \( L^2(M) \), of \( M \) in the \( L^2 \) topology is isometrically isomorphic to the completion of \( M \delta_e \) as a subspace of \( L^2(G) \) and this completion is the whole of \( L^2(G) \).

**Theorem 4.11.1:**

Let \( G \) be an infinite, countable, discrete group and let \( M \) denote \( VN(G) \).

The following properties are equivalent:

(A) The space \( C_0(G) \) has an approximate identity consisting of functions of positive type.

(B) There is a conditionally negative type function, \( \psi \), on \( G \) for which for any \( m > 0 \) there is a compact set \( K_m \subset G \) such that \( |\psi(x)| > m \) for all \( x \in G \setminus K_m \).

(C) There is a net of normal, completely positive operators \( (T_\alpha : M \to M)_{\alpha \in A} \), converging pointwise (in \( L^2 \)) to the identity on \( M \), where the operators have bounded extensions to \( L^2(M) \) which are compact operators.

(D) There is a net of completely positive operators \( (T_\alpha : M \to M)_{\alpha \in A} \), converging pointwise (in \( L^2 \)) to the identity on \( M \), where the operators have bounded extensions to \( L^2(M) \) which are compact operators.
If $G$ has these properties then neither $G$ nor any infinite subgroup of $G$ has property $(T)$ and every von Neumann subalgebra, $N$, of $M$ (containing the identity element $I_M$) also satisfies condition (C) (and hence condition (D)), the $L^2$ topology on $N$ being its topology as a subspace of $L^2(M)$. In particular, no type $II_1$-factor with property $(T)$ can be unitally embedded in $VN(G)$.

Remarks

1. We know from the previous section that $F_n \ (n \in \mathbb{Z}^+)$ and $PSL(2, \mathbb{Z})$ satisfy (B).

2. We could replace any or all of the nets in the theorem by sequences, for the following reason. The proof in [A&W] that (B) implies (A) gives the approximate identity in $C_0(G)$ as $(e^{-t\psi})_{t \in \mathbb{R}}$ (a net) where $\psi$ is the conditionally negative type function of (B), but we could equally well take it to be $(e^{-t\psi})_{t \in \mathbb{N}}$ (a sequence). In the proof of our theorem, this would then give sequences in (C) and (D).

Proof of Theorem 4.11.1:

Let $G$ be an infinite, countable, discrete group. Let $I_M$ be the identity map on $l^2(G)$.

As already remarked, the equivalence of (A) and (B) is proved in [A&W]. It is clear that (C) implies (D). We prove first that (A) implies (C), then that (D) implies (A) and finally that (D) is inherited by von Neumann subalgebras of $M$ (with the subspace $L^2$ topology).

Stage 1, (A) $\Rightarrow$ (C).

Suppose that $G$ satisfies (A). Then there is a net $\{\phi_\alpha\}_{\alpha \in A}$ of functions on $G$ of positive type such that for each $\alpha \in A$ and every $\varepsilon > 0$ there is a compact (and hence finite) subset $K_{\alpha, \varepsilon} \subseteq G$ such that

$$|\phi_\alpha(g)| < \varepsilon \quad \forall \ g \in G \setminus K_{\alpha, \varepsilon}$$
and for every compact subset $H \subset G$ and every $\varepsilon > 0$ there is an $\alpha_{\varepsilon,H} \in A$ for which

$$\sup_{g \in H} |\phi_{\alpha}(g) - 1| < \varepsilon \quad \forall \alpha \geq \alpha_{\varepsilon,H}.$$ 

**Lemma 4.11.2:**

*We may assume $\phi_{\alpha}(e) = 1$ for all $\alpha \in A.$*

**Proof:**

By Lemma 4.2.4, $\phi_{\alpha}(e) > 0$ for each $\alpha \in A,$ unless $\phi_{\alpha} \equiv 0.$ We may discard any $\alpha$ for which $\phi_{\alpha} \equiv 0$ without altering the required properties of the net, so without loss of generality $\phi_{\alpha}(e) > 0$ for all $\alpha \in A.$

For each $\alpha \in A,$ define a function $\psi_{\alpha} : G \to \mathbb{C}$ by

$$\psi_{\alpha}(g) = \frac{\phi_{\alpha}(g)}{\phi_{\alpha}(e)} \quad g \in G.$$ 

Since $\phi_{\alpha}(e)$ is positive, $\psi_{\alpha}$ is of positive type:

$$\sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j \psi_{\alpha}(g_i^{-1}g_j) = \frac{1}{\phi_{\alpha}(e)^2} \sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j \phi_{\alpha}(g_i^{-1}g_j) \geq 0.$$ 

For every $\varepsilon > 0$ and every $\alpha \in A$

$$|\psi_{\alpha}(g)| < \varepsilon \quad \forall g \in G \setminus K_{\alpha,\varepsilon\phi_{\alpha}(e)}.$$ 

Fix $\delta > 0$, a compact set $H \subset G$ and $\alpha \geq \max \{\alpha_{\varepsilon,H}, \alpha_{\varepsilon,1}, \alpha_{1/2}\}$. For each $g \in H$

$$|\psi_{\alpha}(g) - 1| = \frac{1}{\phi_{\alpha}(e)} |\phi_{\alpha}(g) - \phi_{\alpha}(e)|$$

$$\leq \frac{|\phi_{\alpha}(g) - 1| + |1 - \phi_{\alpha}(e)|}{\phi_{\alpha}(e)}$$

$$\leq \frac{\delta + \frac{\delta}{2}}{1 - \frac{1}{2}}$$

$$= \delta.$$ 

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So $(\psi_\alpha)_{\alpha \in A}$ satisfies all the properties required of $(\phi_\alpha)_{\alpha \in A}$ and $\psi_\alpha(e) = 1$ for all $\alpha \in A$.

Following [Hag] Lemma 1.1, we define $T_\alpha$ for each $\alpha \in A$ as follows. Let $(\pi_\alpha, \mathcal{H}_\alpha, \zeta_\alpha)$ be the cyclic representation of $G$ induced by $\phi_\alpha$. So, for each $g \in G$,

$$\phi_\alpha(g) = \langle \pi_\alpha(g) \zeta_\alpha \mid \zeta_\alpha \rangle.$$  

The Hilbert space $\mathcal{H}_\alpha$ is spanned by $\{\pi_\alpha(g)\zeta_\alpha : g \in G\}$ and so $\mathcal{H}_\alpha$ has a countable orthonormal basis, since $G$ is countable. Let $(e_i)_{i \in I}$ be such a basis, where $I$ is either $\mathbb{Z}^+$ or $\{1, 2, \ldots, n\}$ for some finite $n$.

For each $g \in G$ and each $i \in I$ set

$$a_i(g) = \langle e_i \mid \pi_\alpha(g)^* \zeta_\alpha \rangle.$$  

Then, as shown in [Hag], for every $g, h \in G$

$$\sum_{i \in I} a_i(g)\overline{a_i(h)} = \sum_{i \in I} \langle e_i \mid \pi_\alpha(g)^* \zeta_\alpha \rangle \langle \pi_\alpha(h)^* \zeta_\alpha \mid e_i \rangle = \langle \pi_\alpha(h)^* \zeta_\alpha \mid \pi_\alpha(g)^* \zeta_\alpha \rangle = \langle \pi_\alpha(g h^{-1}) \zeta_\alpha \mid \zeta_\alpha \rangle = \phi(g h^{-1}). \tag{4.1}$$

Also $a_i \in l^\infty(G)$ for each $i \in I$ (since $|a_i(g)| \leq \|\zeta_\alpha\|_{\mathcal{H}}$ for all $g \in G$) and

$$\sum_{i \in I} |a_i(g)|^2 \leq \|\zeta_\alpha\|_{\mathcal{H}}^2 = \phi_\alpha(e) = 1 \text{ for each } g \in G.$$

Since $a_i \in l^\infty(G)$, $a_i$ acts by multiplication as a bounded linear operator on $l^2(G)$ i.e. if $a_i : l^2(G) \to l^2(G)$ is defined by $(a_i \xi)(g) = a_i(g) \xi(g)$ for $\xi \in l^2(G)$ and $g \in G$, then $a_i \in B(l^2(G))$. The operator norm of $a_i$ is equal to its $l^\infty$ norm.

Considering $a_i$ in this way we define $T_\alpha$ acting on $B(l^2(G))$: for each $x \in B(l^2(G))$, set

$$T_\alpha x = \sum_{i \in I} a_i x a_i^*.$$  

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Lemma 4.11.3:

For each $\alpha \in A$, $T_\alpha$ is a well defined bounded linear operator on $B(P^2(G))$.

Proof:

If $I$ is finite then this is clear. Suppose $I$ is infinite.

If $w \in B(P^2(G))$ is positive then there is an operator $y \in B(P^2(G))$ such that $w = yy^*$. Then, for each $i \in \mathbb{Z}^+$,

$$a_iwa_i^* = a_iyy^*a_i^* = (a_iy)(a_iy)^* \geq 0 \quad (4.2)$$

and

$$\|w\|a_i^*a_i - a_iwa_i^* = a_i(\|w\|I_M - w)a_i^* \geq 0$$

by (4.2) with the positive operator $(\|w\|I_M - w)$ replacing $w$. Thus

$$0 \leq a_iwa_i^* \leq \|w\|a_i^*a_i.$$

Hence, for all positive integers $m < n$

$$0 \leq \sum_{i=m}^{n} a_iwa_i^* \leq \|w\| \sum_{i=m}^{n} a_i^*a_i$$

and

$$0 \leq \left\| \sum_{i=m}^{n} a_iwa_i^* \right\| \leq \|w\| \left\| \sum_{i=m}^{n} a_i^*a_i \right\| \quad (4.3)$$

If $x \in B(P^2(G))$ then $x$ can be written as $x = x_1 - x_2 + ix_3 - ix_4$ where $x_1$, $x_2$, $x_3$ and $x_4$ are positive operators in $B(P^2(G))$ and $x_1 - x_2 = \frac{1}{2}(x + x^*)$,

$x_3 - x_4 = \frac{1}{2i}(x - x^*)$. By [K&R] 4.2.3 we can choose $x_1$, $x_2$, $x_3$, $x_4$ so that

$$\|x_1 - x_2\| = \max\{\|x_1\|, \|x_2\|\}$$

and

$$\|x_3 - x_4\| = \max\{\|x_3\|, \|x_4\|\}.$$
Combining this with (4.3), for all positive integers \( m < n \)
\[
\left\| \sum_{i=m}^{n} a_i x a_i^* \right\| \leq \sum_{j=1}^{4} \left\| \sum_{i=m}^{n} a_i x_j a_i^* \right\| \leq 4 \| x \| \left\| \sum_{i=m}^{n} a_i a_i^* \right\| \tag{4.4}
\]
and this last quantity tends to 0 as \( m \to \infty \) since \( 1 = \left\| \sum_{i=1}^{\infty} a_i a_i^* \right\|_\infty = \left\| \sum_{i=1}^{\infty} a_i a_i^* \right\| \). So \( \sum a_i x a_i^* \) converges (in norm) in \( B(l^2(G)) \) and \( T_\alpha : B(l^2(G)) \to B(l^2(G)) \) is well defined.

Since \( T_\alpha \) is the sum of compositions of linear operators it is clearly linear.

Notice that, since the sum converges in operator norm,
\[
\left( \sum_{i \in I} a_i x a_i^* \right) \zeta = \sum_{i \in I} (a_i x a_i^*) \zeta \text{ for each } x \in B(l^2(G)) \text{ and each } \zeta \in l^2(G).
\]

From (4.4), for all \( n \in \mathbb{Z}^+ \) and all \( x \in B(l^2(G)) \)
\[
\left\| \sum_{i=1}^{n} a_i x a_i^* \right\| \leq 4 \| x \| \left\| \sum_{i=1}^{n} a_i a_i^* \right\| \to 4 \| x \| \quad \text{as } n \to \infty .
\]
Hence \( \| T_\alpha x \| \leq 4 \| x \| \) for all \( x \in B(l^2(G)) \) and \( T_\alpha \) is a bounded linear operator on \( B(l^2(G)) \).

\[\square\]

**Lemma 4.11.4 :**

For each \( \alpha \in A \), \( T_\alpha : B(l^2(G)) \to B(l^2(G)) \) is completely positive.

**Proof :**

From [Tak] IV 3.4, if \( A \) and \( B \) are \( C^* \)-algebras and \( \psi : A \to B \) is a linear map then \( \psi \) is \( n \)-positive if and only if
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} y_i^* \psi(x_i^* x_j) y_j \geq 0
\]
for all \( x_1, \ldots, x_n \in A \) and all \( y_1, \ldots, y_n \in B \).

Take \( n \in \mathbb{Z}^+ \) and \( x_1, \ldots, x_n, y_1, \ldots, y_n \in B(l^2(G)) \).

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} y_i^* T_\alpha(x_i^* x_j) y_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k \in I} y_i^* a_k x_i^* x_j a_k^* y_j
\]
\[
= \sum_{k \in I} \left( \left( \sum_{i=1}^{n} x_i a_k^* y_i \right)^* \left( \sum_{j=1}^{n} x_j a_k^* y_j \right) \right)
\]
\[
\geq 0
\]

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So $T_\alpha$ is $n$-positive for every $n \in \mathbb{Z}^+$, i.e. $T_\alpha$ is completely positive.

□

Lemma 4.11.5:

For each $\alpha \in A$, $T_\alpha : B(l^2(G)) \to B(l^2(G))$ is ultraweakly continuous, that is normal.

Proof:

A net, $(S_\beta)_{\beta \in B}$, in $B(l^2(G))$ converges ultraweakly to $S$ if and only if

$$\sum_{i=1}^{\infty} (S_\beta \xi_i \mid \eta_i) \to \sum_{i=1}^{\infty} (S \xi_i \mid \eta_i)$$

for all $\xi_i, \eta_i \in l^2(G)$ such that $\sum (\|\xi_i\|^2 + \|\eta_i\|^2) \leq 1$, where $(\cdot \mid \cdot)$ denotes the inner product on $l^2(G)$.

Let $(S_\beta)_{\beta \in B}$ be a net of operators in $B(l^2(G))$ converging to $S$ in the ultraweak topology on $B(l^2(G))$. Take two sequences $(\xi_j)_{j=1}^{\infty}$ and $(\eta_j)_{j=1}^{\infty}$ of elements of $l^2(G)$ such that $\sum (\|\xi_j\|^2 + \|\eta_j\|^2) \leq 1$. Then

$$\sum_{j=1}^{\infty} ( (T_\alpha S_\beta) \xi_j \mid \eta_j ) = \sum_{j=1}^{\infty} ( (T_\alpha S) \xi_j \mid \eta_j )$$

$$= \sum_{j=1}^{\infty} \left( \sum_{i \in I} a_i (S_\beta - S) a_i^* \xi_j \mid \eta_j \right)$$

$$= \sum_{j=1}^{\infty} \sum_{i \in I} ( a_i (S_\beta - S) a_i^* \xi_j \mid \eta_j )$$

$$= \sum_{j=1}^{\infty} \sum_{i \in I} ( (S_\beta - S) a_i^* \xi_j ) a_i^* \eta_j$$

$$= \sum_{(i,j) \in I \times \mathbb{Z}^+} ( (S_\beta - S) b_{(i,j)} \mid c_{(i,j)} )$$

where $b_{(i,j)} = a_i^* \xi_j$ and $c_{(i,j)} = a_i^* \eta_j$, $i \in I$, $j \in \mathbb{Z}^+$. 

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The indexing set $I \times \mathbb{Z}^+$ is countable and

$$
\sum_{(i,j) \in I \times \mathbb{Z}^+} \left( \|b_{(i,j)}\|^2 + \|c_{(i,j)}\|^2 \right)
= \sum_{i \in I} \sum_{j \in \mathbb{Z}^+} \left( \|a_i^x \xi_j\|^2 + \|a_i^y \eta_j\|^2 \right)
= \sum_{j \in \mathbb{Z}^+} \sum_{g \in G} \left( \|\xi_j(g)\|^2 + |\eta_j(g)|^2 \right) \sum_{i \in I} |a_i^x(g)|^2
= \sum_{j \in \mathbb{Z}^+} \sum_{g \in G} \left( \|\xi_j(g)\|^2 + |\eta_j(g)|^2 \right) \|\pi^\alpha(g)\zeta\|^2
= \sum_{j \in \mathbb{Z}^+} \left( \|\xi_j\|^2 + \|\eta_j\|^2 \right)
\leq 1.
$$

Hence $\sum_{(i,j) \in I \times \mathbb{Z}^+} \left( (S_\beta - S)b_{(i,j)} \mid c_{(i,j)} \right) \rightarrow 0$ (since $S_\beta \rightarrow S$ ultraweakly) and $T_\alpha S_\beta \rightarrow T_\alpha S$ ultraweakly. Thus $T_\alpha$ is ultraweakly continuous on $B(l^2(G))$.

\[ \square \]

**Lemma 4.11.6**

For each $s \in G$ and each $\alpha \in A$

$$
T_\alpha(\lambda(s)) = \phi_\alpha(s)\lambda(s).
$$

**Proof**:

For each $i \in I$ and each $r, t \in G$

$$(a_i \delta_i)(r) = a_i(r)\delta_i(r) = a_i(t)\delta_i(r).$$

So

$$a_i \delta_i = a_i(t) \delta_t.$$
For each \( s,t \in G \)

\[
(T_\alpha \lambda(s))(\delta_t) = \sum_{i \in I} a_i \lambda(s)(a_i^* \delta_t)
= \sum_{i \in I} \overline{a_i(t)} a_i \lambda(s) \delta_t
= \sum_{i \in I} \overline{a_i(t)} a_i \delta_{st}
= \sum_{i \in I} a_i(st) \overline{a_i(t)} \delta_{st}
= \phi_\alpha(s) \delta_{st} \quad \text{by (4.1)}
= \phi_\alpha(s) \lambda(s) \delta_t
\]

So

\[
T_\alpha \lambda(s) = \phi_\alpha(s) \lambda(s)
\]

\[\square\]

**Lemma 4.11.7 :**

For each \( \alpha \in A \), \( T_\alpha(M) \subseteq M \).

**Proof :**

For each \( \alpha \in A \), \( T_\alpha(\text{span}(\lambda(G))) \subseteq \text{span}(\lambda(G)) \) by Lemma 4.11.6. But \( M \) is the ultraweak closure of \( \text{span}(\lambda(G)) \) and \( T_\alpha \) is ultraweakly continuous, so \( T_\alpha(M) \subseteq M \).

\[\square\]

**Lemma 4.11.8 :**

For each \( \alpha \in A \), \( T_\alpha : M \to M \) extends to a bounded linear operator on \( L^2(M) \) which is compact.
Proof:

For each \( \alpha \in A \), \( T_\alpha \) is a bounded linear operator on \( \mathcal{B}(l^2(G)) \) and so \( T_\alpha : L^2(M) \to \mathcal{B}(l^2(G)) \) is a bounded linear extension of \( T_\alpha : M \to M \). By Lemma 4.11.6, if \( \sum \mu_g \lambda(g) \in L^2(M) \) then

\[
T_\alpha \left( \sum_{g \in G} \mu_g \lambda(g) \right) = \sum_{g \in G} \mu_g \phi_\alpha(g) \lambda(g).
\]

Since \( |\phi_\alpha(g)| \leq 1 \) for all \( g \in G \), \( T_\alpha(L^2(M)) \subseteq L^2(M) \) and

\[
\|T_\alpha : L^2(M) \to L^2(M)\| \leq 1.
\]

For each \( n \in \mathbb{Z}^+ \) define an operator \( N_n : L^2(M) \to L^2(M) \) by

\[
N_n \left( \sum_{g \in G} \mu_g \lambda(g) \right) = \sum_{g \in K_{\alpha, \frac{1}{n}}} \mu_g \phi_\alpha(g) \lambda(g).
\]

Then

\[
\left\| (T_\alpha - N_n) \left( \sum_{g \in G} \mu_g \lambda(g) \right) \right\|^2_{L^2} = \left\| \sum_{g \in G \setminus K_{\alpha, \frac{1}{n}}} \phi_\alpha(g) \mu_g \lambda(g) \right\|^2_{L^2} = \sum_{g \in G \setminus K_{\alpha, \frac{1}{n}}} |\phi_\alpha(g)|^2 |\mu_g|^2 \leq \frac{1}{n^2} \sum_{g \in G} |\mu_g|^2 \lambda(g)^2_{L^2}.
\]

So \( \|T_\alpha - N_n\| \leq \frac{1}{n} \). Hence \( N_n \to T_\alpha \) as \( n \to \infty \). Since each \( N_n \) is of finite rank \( |K_{\alpha, \frac{1}{n}}| \), it follows that \( T_\alpha \) is compact.

\[ \square \]

Lemma 4.11.9:

For each \( x \in L^2(M) \), \( T_\alpha x \to x \), that is the net \( (T_\alpha)_{\alpha \in A} \) converges pointwise to the identity on \( L^2(M) \).
Proof:

Take non-zero $\sum_{g} \mu_{g} \lambda(g) \in L^{2}(M)$ and $\varepsilon > 0$. Then there is a finite set $J \subseteq G$ such that

$$\sum_{g \in G \setminus J} |\mu_{g}|^{2} < \varepsilon$$

and there is an $\alpha_{0} \in A$ such that for any $\alpha \geq \alpha_{0}$

$$\sup_{g \in J} |\phi_{\alpha}(g) - 1| < \frac{\varepsilon}{\sum_{h \in G} |\mu_{h}|^{2}}.$$ 

Then, for any $\alpha \geq \alpha_{0}$

$$\left\| T_{\alpha} \sum_{g \in G} \mu_{g} \lambda(g) - \sum_{g \in G} \mu_{g} \lambda(g) \right\|_{L^{2}}^{2}$$

$$= \left\| \sum_{g \in G} \mu_{g} \left(\phi_{\alpha}(g) - 1\right) \lambda(g) \right\|_{L^{2}}^{2}$$

$$= \sum_{g \in G} |\mu_{g}|^{2} |\phi_{\alpha}(g) - 1|^{2}$$

$$= \sum_{g \in J} |\mu_{g}|^{2} |\phi_{\alpha}(g) - 1|^{2} + \sum_{g \in G \setminus J} |\mu_{g}|^{2} |\phi_{\alpha}(g) - 1|^{2}$$

$$< \varepsilon + \varepsilon (1 + 1)^{2}$$

$$= 5\varepsilon$$

since $|\phi_{\alpha}(g)| \leq 1$ for all $g \in G$.

So $(T_{\alpha} : L^{2}(M) \to L^{2}(M))_{\alpha \in A}$ converges pointwise to the identity on $L^{2}(M)$.

$\square$

This concludes the proof that (A) implies (C).

Stage 2, (D) $\Rightarrow$ (A).

Now suppose $G$ satisfies (D). So we have a net, $(T_{\alpha})_{\alpha \in A}$, of completely positive operators on $M$ which extend to bounded, compact operators on $L^{2}(M)$ and satisfy $\|T_{\alpha}x - x\|_{L^{2}} \to 0$ for each $x \in M$.
For each $\alpha \in A$ we define a function $\phi_\alpha : G \to \mathbb{C}$ by

$$\phi_\alpha(g) = \text{tr}(T_\alpha(\lambda(g)) \lambda(g)^*) = \langle T_\alpha(\lambda(g)) | \lambda(g) \rangle$$

recalling that the trace on $M$ is defined by $\text{tr}(x) = \langle x\delta_e | \delta_e \rangle$ and that the inner product on $M$ is defined by $\langle a | b \rangle = \text{tr}(ab^*)$ for $a, b \in M$.

**Lemma 4.11.10:**

For each $\alpha \in A$, $\phi_\alpha \in C_0(G)$.

**Proof:**

Fix $\alpha \in A$ and write $T$ for $T_\alpha$ and $\phi$ for $\phi_\alpha$.

Take $\varepsilon > 0$. Since $T$ extends to a compact map on $L^2(M)$ there is a finite rank operator $S : L^2(M) \to L^2(M)$ such that

$$\|T(x) - S(x)\|_{L^2} < \varepsilon$$

for all $x \in L^2(M)$ with $\|x\|_{L^2} \leq 1$. Then

$$\phi(g) = \langle T(\lambda(g)) - S(\lambda(g)) | \lambda(g) \rangle + \langle S(\lambda(g)) | \lambda(g) \rangle$$

and

$$|\phi(g)| \leq \varepsilon + |\langle S(\lambda(g)) | \lambda(g) \rangle| .$$

The set $\lambda(G)$ is an orthonormal basis for $L^2(M)$ so for each $x \in L^2(M)$

$$\sum_{g \in G} |\langle x | \lambda(g) \rangle|^2 = \|x\|_{L^2}^2 .$$

Let $z_1, z_2, \ldots, z_n$ be an orthonormal basis for the range of $S$. For each $z_i$ there is a finite set $K_i \subset G$ such that

$$\sum_{g \in G \setminus K_i} |\langle z_i | \lambda(g) \rangle|^2 < \left( \frac{\varepsilon}{n\|S\|} \right)^2 .$$
Set $K = \bigcup_{i=1}^{n} K_i$. Notice that $K$ is finite, and hence compact in $G$. Then, for all $g \in G \setminus K$ and each $i \in \{1, \ldots, n\}$, $|\langle z_i | \lambda(g) \rangle| < \frac{\varepsilon}{n\|S\|}$. For each $g \in G \setminus K$

$$\left|\langle S(\lambda(g)) | \lambda(g) \rangle\right| = \left| \sum_{i=1}^{n} \langle S(\lambda(g)) | z_i \rangle \langle z_i | \lambda(g) \rangle \right|$$

$$\leq \left( \sum_{i=1}^{n} \left| \langle S(\lambda(g)) | z_i \rangle \right|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} \left| \langle z_i | \lambda(g) \rangle \right|^2 \right)^{\frac{1}{2}}$$

$$= \|S(\lambda(g))\| \left( \sum_{i=1}^{n} \left| \langle z_i | \lambda(g) \rangle \right|^2 \right)^{\frac{1}{2}}$$

$$< \|S\| \frac{\varepsilon}{n\|S\|}$$

$$= \varepsilon.$$ 

Hence, for all $g \in G \setminus K$

$$|\phi(g)| < 2\varepsilon$$

and $\phi \in C_0(G)$.

$\Box$

**Lemma 4.11.11 :**

For each $g \in G$, $\phi_\alpha(g) \to 1$ as $\alpha$ runs over $A$.

**Proof :**

Fix $g \in G$.

$$|\phi_\alpha(g) - 1| = |\langle T_\alpha(\lambda(g)) | \lambda(g) \rangle - 1|$$

$$= |\langle T_\alpha(\lambda(g)) - \lambda(g) | \lambda(g) \rangle|$$

$$\leq \|T_\alpha(\lambda(g)) - \lambda(g)\|_{L^2}$$

$$\to 0.$$ 

$\Box$ 

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Lemma 4.11.12:

Each $\phi_\alpha$ is of positive type.

Proof:

Fix $\alpha \in A$ and write $T$ for $T_\alpha$ and $\phi$ for $\phi_\alpha$.

Take $n \in \mathbb{N}$, $\beta_1, \ldots, \beta_n \in \mathbb{C}$ and $g_1, \ldots, g_n \in G$.

$$\sum_{i,j=1}^n \overline{\beta}_i \beta_j \phi(g_i^{-1} g_j) = \sum_{i,j=1}^n \overline{\beta}_i \beta_j \left( T(\lambda(g_i^{-1})\lambda(g_j)) \mid \lambda(g_i^{-1})\lambda(g_j) \right)$$

$$= \sum_{i,j=1}^n \overline{\beta}_i \beta_j \left( \lambda(g_i)T(\lambda(g_i)^*\lambda(g_j)) \mid \lambda(g_j) \right)$$

$$= \sum_{i,j=1}^n \overline{\beta}_i \beta_j \text{tr} \left( \lambda(g_i)T(\lambda(g_i)^*\lambda(g_j))\lambda(g_j)^* \right)$$

$$= \text{tr} \left( \sum_{i,j=1}^n \left( \beta_i \lambda(g_i^{-1}) \right)^* T(\lambda(g_i)^*\lambda(g_j)) \left( \beta_j \lambda(g_j^{-1}) \right) \right)$$

$$\geq 0$$

since $T$ is completely positive.

This concludes the proof that (D) implies (A) and hence of the equivalence of (A), (B), (C) and (D).

Suppose $G$ satisfies the equivalent conditions (A), (B), (C) and (D).

Since $G$ satisfies condition (B) it has an unbounded conditionally negative type function and so does not have property (T) ([HaV] Theorem 5.20 quoted at the beginning of this chapter). We know from Lemma 4.10.2 that no infinite subgroup of $G$ can have property (T).

The trace, $\text{tr}$, on $M$ is normal, that is ultraweakly continuous. This follows from the definition of ultraweak convergence (see Lemma 4.11.5) which implies that $(x_{\beta} \delta_e | \delta_e) \rightarrow (x \delta_e | \delta_e)$, that is $\text{tr} x_{\beta} \rightarrow \text{tr} x$, if the net $(x_{\beta})_{\beta \in B}$ converges ultraweakly to $x$ in $M$. Since $G$ is countable, $M$ is countably decomposable, i.e. each orthogonal family of subprojections of $I_M$ in $M$ is countable. So the fact

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that every von Neumann subalgebra of $M$ satisfies (C) and cannot be a type II$_1$-factor with property (T) follows from Proposition 4.11.13 below. (We already know from Lemma 4.10.2 that properties (A) and (B) are inherited by subgroups of $G$.)

□

**Proposition 4.11.13:**

Let $M$ be a countably decomposable von Neumann algebra with a faithful, unital, normal trace, $\tau$, and suppose $M$ has property (C) above. Let $N$ be a von Neumann subalgebra of $M$, containing the identity element of $M$, and use $\tau$ to define inner product topologies on $M$ and $N$ as described above. Then $N$ also has property (C). In particular, $N$ cannot be a type II$_1$-factor with property (T).

**Remarks.**

1. Again, we may replace the nets in (C) by sequences.

2. The proof below also shows that if $M$ satisfies the weaker condition (D) then so does $N$.

**Proof:**

The trace, $\tau$, is a normal, faithful, tracial state on $M$. Hence, by [Sak] 4.4.23 and its proof, there is a normal linear mapping, $E$, from $M$ onto $N$, called the trace preserving conditional expectation, which has the following properties.

1. If $x \in M$ and $x \geq 0$ then $E(x) \geq 0$ with equality if and only if $x = 0$.
2. $\|E(x)\| \leq \|x\|$ for all $x \in M$. (This is the operator norm.)
3. $E(axb) = aE(x)b$ for all $a, b \in N$, $x \in M$.
4. $E(x^*E(x)) \leq E(x^*x)$ for all $x \in M$.
5. $\tau(xa) = \tau(E(x)a)$ for all $x \in M$, $a \in N$. 

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(6) \( \tau(x) = \tau(E(x)) \) for all \( x \in M \) (taking \( a = I_M \) in (5)).

(7) \( E \) extends to a bounded linear map \( E : L^2(M) \to L^2(N) \), with norm 1. This is because, for all \( x \in M \),

\[
\| Ex \|_{L^2}^2 = \tau((Ex)(Ex)^*) \
\leq \tau(E(xx^*)) \quad \text{by (4)}
\]

\[
= \tau(xx^*) \quad \text{by (6)}
\]

\[
= \| x \|_{L^2}^2 .
\]

(8) \( E(I_M) = I_M \) since \( E(I_M) \) is the unique positive element of \( N \) such that \( \tau(y) = \tau(E(I_M)y) \) for all \( y \in N \), so \( E(I_M) = I_N = I_M \) since \( I_M \in N \).

(9) \( E(a) = a \) for all \( a \in N \) (taking \( b = x = I_M \) in (3) and using (8)).

Let \( (T_\alpha)_{\alpha \in A} \) be a net of normal, completely positive linear operators on \( M \) demonstrating condition (C). So the operators, \( T_\alpha \), have bounded extensions, \( \overline{T_\alpha} \), to \( L^2(M) \) which are compact and \( \| T_\alpha x - x \|_{L^2} \to 0 \) for each \( x \in M \). For each \( \alpha \in A \), \( E \circ \overline{T_\alpha} : L^2(N) \to L^2(N) \) is a bounded extension of \( E \circ T_\alpha : N \to N \).

**Lemma 4.11.14**:

Each \( E \circ T_\alpha : N \to N \) is completely positive.

**Proof**:

Fix \( \alpha \in A \) and write \( T \) in place of \( T_\alpha \). Take \( n \in \mathbb{Z} \) and \( x_1, \ldots, x_n, y_1, \ldots, y_n \in N \).

\[
\sum_{i,j=1}^{n} y_i^* E \circ T(x_i^* x_j)y_j = \sum_{i,j=1}^{n} E(y_i^* T(x_i^* x_j)y_j) \quad \text{by (3)}
\]

\[
= E \left( \sum_{i,j=1}^{n} y_i^* T(x_i^* x_j)y_j \right) \quad \text{linearity}
\]

\[
\geq 0
\]

since \( T \) is completely positive and \( E \) is positive.

\( \square \)
Lemma 4.11.15:

Each $E \circ T_{\alpha} : L^2(N) \to L^2(N)$ is compact.

Proof:

Each $T_{\alpha} : L^2(M) \to L^2(M)$ is compact (by condition (C)) and $E : L^2(M) \to L^2(N)$ is continuous in the $L^2$ topology.

□

Lemma 4.11.16:

The net $(E \circ T_{\alpha})_{\alpha \in A}$ converges pointwise to the identity in the $L^2$ topology on $N$.

Proof:

Take $x \in N$.

$$
\| (E \circ T_{\alpha}) x - x \|_{L^2} = \| E(T_{\alpha} x - x) \|_{L^2} \leq \| T_{\alpha} x - x \|_{L^2} \to 0.
$$

□

Lemma 4.11.17:

Each $E \circ T_{\alpha}$ is normal.

Proof:

$E \circ T_{\alpha}$ is the composition of two ultraweakly continuous functions.

□

We have defined a net, $(E \circ T_{\alpha})_{\alpha \in A}$, of completely positive, normal operators on $N$, with compact, bounded extension to $L^2(N)$, which converges pointwise (in $L^2$) to the identity on $N$. Thus $N$ satisfies condition (C).

If $N$ is a type $II_1$-factor then it has a unique trace which must therefore be the restriction of the trace on $M$. If, further, $N$ has property (T), the theorem
in [Rob] tells us that there is no sequence of normal, completely positive maps on \( N \) which are compact on \( L^2(N) \), converging pointwise in \( L^2 \) to the identity on \( N \). The proof of that theorem works for a net as well as a sequence. Hence we reach a contradiction and conclude that no von Neumann subalgebra of \( M \) can be a type II\(_1\)-factor with property (T).

\( \Box \)

The fact that an infinite discrete group with properties (A) to (D) does not have property (T) follows from the fact that conditionally negative type functions on a Kazhdan group are bounded ([HaV] 5.20). We give an alternative proof, not using conditionally negative type functions, that a non-compact Kazhdan group cannot have property (A) above. We use [HaV] 5.11 which says that a locally compact group, \( G \), has property (T) if and only if every net, \((\phi_\alpha)_{\alpha \in A}\), of positive type functions on \( G \) with \( \phi_\alpha(e) = 1 \) for each \( \alpha \in A \) which converges to the constant function 1 uniformly on compacta also converges to 1 uniformly on \( G \). The theorem is stated in [HaV] only for sequences, but the same proof goes through for nets.

**Proposition 4.11.18** :

*Let \( G \) be a non-compact group with property (T). Then \( C_0(G) \) cannot have an approximate identity consisting of functions of positive type.*

**Proof** :

Suppose \( \phi_\alpha \) is an approximate identity in \( C_0(G) \) consisting of functions of positive type. Then \( \phi_\alpha \to 1 \) uniformly on compacta. As before we may assume that \( \phi_\alpha(e) = 1 \) for each \( \alpha \in A \). Hence, by the above theorem from [HaV], \( \phi_\alpha \to 1 \) uniformly on \( G \), contradicting the fact that each \( \phi_\alpha \) tends to zero at infinity.

\( \Box \)
4.12 Other von Neumann Algebras.

In Theorem 4.11.1, we saw that a type II$_1$-factor with property (T) cannot be contained in the group von Neumann algebra, $VN(G)$, of a countable, infinite, discrete group, $G$, with the equivalent properties (A) to (D). This strengthens the fact that $VN(G)$ cannot be a type II$_1$-factor with property (T), which is a special case of [C&J] Theorem 2. That theorem says, further, that certain other von Neumann algebras arising from the group cannot be factors with property (T). In this section we generalise Theorem 4.11.1 by replacing $VN(G)$ with these other algebras. Before we can define these algebras we need to look at the cohomology of $G$.

4.12.1 Cohomology.

(The general definition is taken from [Z-M].)

Let $G$ be a discrete group. Suppose that $H$ is an abelian group and that $\alpha : G \to \text{Aut } H$ is a homomorphism from $G$ into the group of automorphisms of $H$. (We make no assumptions about the topology of $H$.) For each integer $n \geq 0$ an $n$-cochain on $G$ with values in $H$ is a function $\sigma : G^n \to H$. With the pointwise product, the $n$-cochains form a group, $C^n(G; H)$. This group is abelian because $H$ is abelian. The coboundary map $d_n : C^n(G; H) \to C^{n+1}(G; H)$ is defined by

$$(d_n \sigma)(g_1, \ldots, g_{n+1}) = \alpha(g_1) \left(\sigma(g_2, \ldots, g_{n+1})\sigma(g_1g_2, \ldots, g_{n+1})^{-1}\sigma(g_1, g_2g_3, \ldots, g_{n+1})\right)$$

$$\ldots \left(\sigma(g_1, g_2, \ldots, g_n, g_{n+1})\right)^{(-1)^n}\left(\sigma(g_1, g_2, \ldots, g_n)\right)^{(-1)^{n+1}}$$

for $\sigma \in C^n(G; H)$ and $g_1, \ldots, g_{n+1} \in G$. The $n$-cocycles on $G$ with values in $H$ are the elements of the group

$$Z^n(G; H) = \ker d_n .$$

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For positive integers, \( n \), the \( n \)-coboundaries on \( G \) with values in \( H \) are the elements of the group

\[
B^n(G; H) = d_{n-1}(C^{n-1}(G; H))
\]

The composition \( d_n d_{n-1} \) maps the whole of \( C^{n-1}(G; H) \) to the constant function at \( e_H \). Hence \( B^n(G; H) \) is a subgroup of \( Z^n(G; H) \). It must be a normal subgroup since \( Z^n(G; H) \) is contained in the abelian group \( C^n(G; H) \). The \( n \)th cohomology group of \( G \) with values in \( H \) is the quotient group

\[
H^n(G; H) = Z^n(G; H)/B^n(G; H).
\]

The elements of \( H^n(G; H) \) are called cohomology classes. Two \( n \)-cocycles are said to be cohomologous if they belong to the same cohomology class.

An \( n \)-cochain \( \sigma \in C^n(G; H) \) is normalised if

\[
\sigma(g_1, \ldots, g_n) = e_H
\]

whenever one (or more) of \( g_1, \ldots, g_n \) is \( e_G \). If \( \sigma \) and \( \tau \) are two normalised \( n \)-cochains in \( C^n(G; H) \) and \( g_1, \ldots, g_n \in G \) with at least one of \( g_1, \ldots, g_n \) equal to \( e_G \) then

\[
\sigma^{-1}\tau(g_1, \ldots, g_n) = e_H,
\]

so \( \sigma^{-1}\tau \) is a normalised \( n \)-cochain. Hence the normalised \( n \)-cochains form a subgroup, \( C^n_{\text{norm}}(G; K) \), of \( C^n(G; H) \); and similarly the normalised \( n \)-cocycles from a subgroup, \( Z^n_{\text{norm}}(G; H) \), of \( Z^n(G; H) \) and the normalised \( n \)-coboundaries from a subgroup, \( B^n_{\text{norm}}(G; H) \), of \( B^n(G; H) \). Also, if \( \sigma \in C^n_{\text{norm}}(G; K) \) and \( g_1, \ldots, g_{n+1} \in G \)

\[
d_n\sigma(e_G, g_2, \ldots, g_{n+1}) = \alpha(e_G)\left(\sigma(g_2, \ldots, g_{n+1})\sigma(e_G g_2, \ldots, g_{n+1})^{-1}\right) = e_H,
\]

\[
d_n\sigma(g_1, \ldots, g_n, e_G) = \alpha(g_1)\left(\sigma(g_1, \ldots, g_n e_G)^{(-1)^n}\sigma(g_1, \ldots, g_n)^{(-1)^{n+1}}\right) = e_H
\]
and if \( g_i = e_G \) for some \( i \in \{2, \ldots, n\} \) then

\[
d_n \sigma(g_1, \ldots, g_{n+1}) = \alpha(g_1) \left( \sigma(g_1, \ldots, g_{i-1}e_G, g_i+1, \ldots, g_{n+1})^{(-1)^{i-1}} \right) \\
\,
\sigma(g_1, \ldots, g_{i-1}, e_G g_{i+1}, \ldots, g_{n+1})^{(-1)^{i}}
\]

\[
= e_H ;
\]

so \( d_n \) maps \( C^{n}_{\text{norm}}(G; H) \) into \( C^{n+1}_{\text{norm}}(G; H) \). The group of normalised \( n \)-cocycles \( Z^{n}_{\text{norm}}(G; H) \) is \( \ker d_n \cap C^{n}_{\text{norm}}(G; H) = \ker (d_n : C^{n}_{\text{norm}}(G; H) \to C^{n+1}_{\text{norm}}(G; H)) \). The group of normalised \( n \)-coboundaries \( B^{n}_{\text{norm}}(G; H) \) contains \( d_n C^{n-1}_{\text{norm}}(G; H) \). Clearly every coset of \( B^{n}_{\text{norm}}(G; H) \) in \( Z^{n}_{\text{norm}}(G; H) \) is contained in a coset of \( B^{n}(G; H) \) in \( Z^{n}(G; H) \). Conversely, if two normalised \( n \)-cocycles, \( \sigma, \tau \in Z^{n}_{\text{norm}}(G; H) \), are cohomologous then the \( n \)-coboundary \( \sigma^{-1} \tau \) is normalised, so each coset of \( B^{n}(G; H) \) in \( Z^{n}(G; H) \) contains a unique coset of \( B^{n}_{\text{norm}}(G; H) \) in \( Z^{n}_{\text{norm}}(G; H) \).

We are interested in normalised 2-cochains, 2-cocycles and 2-coboundaries on \( G \) with values in \( T \), the unit circle in \( \mathbb{C} \), where the action of \( G \) on \( T \) is trivial, that is \( \alpha(g) \mu = \mu \) for all \( g \in G \) and all \( \mu \in T \). From now on, these are what we shall mean when we refer to cochains, cocycles and coboundaries. We shall return to writing \( e \) in place of \( e_G \).

Suppose \( \sigma \in C^2(G, T) \). The cochain \( \sigma \) is normalised if and only if

\[
\sigma(e, g) = \sigma(g, e) = 1 \quad \forall \ g \in G . \tag{4.5}
\]

Further, \( \sigma \in Z^2_{\text{norm}}(G, T) \) if and only if \( \sigma \) satisfies (4.5) and, for every \( g, h, k \in G \),

\[
1 = (d_2 \sigma)(g, h, k) \\
= \sigma(h, k)(\sigma(gh, k))^{-1} \sigma(g, hk)(\sigma(g, h))^{-1} ,
\]

that is

\[
\sigma(g, h)\sigma(gh, k) = \sigma(g, hk)\sigma(h, k) . \tag{4.6}
\]

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Finally, $\sigma \in B^2_{\text{norm}}(G; T)$ if and only if $\sigma$ satisfies (4.5) and there is some $\rho \in C^1(G; T)$ such that for all $g, h \in G$

$$\sigma(g, h) = (d_1 \rho)(g, h) = \rho(h)(\rho(gh))^{-1}\rho(g). \quad (4.7)$$

Since $\sigma$ is normalised it follows that, for all $g \in G$,

$$1 = \sigma(g, e) = \rho(g)(\rho(g))^{-1}\rho(e) = \rho(e)$$

so $\rho$ is also normalised. We have already seen that if $\rho \in C^1_{\text{norm}}(G; T)$ then $d_1 \rho \in B^2_{\text{norm}}(G; T)$. So $B^2_{\text{norm}}(G; T) = d_1(C^1_{\text{norm}}(G; T))$.

**Lemma 4.12.1 :**

*Each cohomology class in $Z^2_{\text{norm}}(G; T)$ contains an element, $\tau$, such that*

$$\tau(g, g^{-1}) = 1 \quad \forall g \in G.$$  

**Proof :**

Let $\sigma \in Z^2_{\text{norm}}(G; T)$. For each $g \in G$ set

$$\rho(g) = \sigma(g, g^{-1})^{-\frac{1}{2}}$$

where the square root is chosen to have non-negative real part and a unique choice of $(-1)^{\frac{1}{2}}$ is made. By (4.6), for every $g \in G$

$$\sigma(g^{-1}, gg^{-1})\sigma(g, g^{-1}) = \sigma(g^{-1}, g)\sigma(g^{-1}g, g^{-1})$$

and so by (4.5)

$$\sigma(g, g^{-1}) = \sigma(g^{-1}, g)$$

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\[
\rho(g^{-1}) = \rho(g).
\]

By (4.5), \(\rho(e) = 1\) so \(\rho \in C^1_{\text{norm}}(G; T)\). The cocycle \((d_1\rho)\sigma\) is cohomologous to \(\sigma\) and

\[
((d_1\rho)\sigma)(g, g^{-1}) = \sigma(g, g^{-1})^{-\frac{1}{2}}\sigma(g^{-1}, g)^{-\frac{1}{2}}\sigma(e, e)^{\frac{1}{2}}\sigma(g, g^{-1}) = 1.
\]

\[\square\]

4.12.2 Cocycle Representations.

Let \(\sigma : G \times G \to T\) be a normalised 2-cocycle. A cocycle representation with cocycle \(\sigma\), or a \(\sigma\)-representation, of \(G\) is a function \(S : G \to \mathcal{U}(\mathcal{H})\) for some Hilbert space \(\mathcal{H} = \mathcal{H}_S\), such that

\[
S(e) = I_{\mathcal{H}},
\]

where \(I_{\mathcal{H}}\) is the identity operator on \(\mathcal{H}\), and

\[
S(g)S(h) = \sigma(g, h)S(gh) \quad \forall \ g, h \in G.
\]

If \(g \in G\) then \(I_{\mathcal{H}} = S(e) = \overline{\sigma(g, g^{-1})}S(g)S(g^{-1})\), so \(S(g)^* = \overline{\sigma(g, g^{-1})}S(g^{-1})\).

Then, for any \(g, h \in G\), \(S(g)S(h) \in \mathcal{C}(G)\) and \(S(g)^* \in \mathcal{C}(G)\) so span \(S(G)\) is a self-adjoint algebra and the weak operator closure of span \(S(G)\) in \(\mathcal{B}(l^2(G))\) is a von Neumann algebra, \(VN(S(G)) = (S(G))^"\).

Two cocycle representations, \(S\) and \(T\), are equivalent if there exist a linear, isometric bijection \(F : \mathcal{H}_S \to \mathcal{H}_T\) and a function \(\rho : G \to T\) such that for each \(g \in G\)

\[
T(g) = \rho(g)F S(g) F^{-1}.
\]

If this is the case, then \(\text{span } T(G) = \text{span } FS(G)F^{-1}\) and so the mapping \(x \mapsto FXF^{-1}\) is an isomorphism from \(VN(S(G))\) onto \(VN(T(G))\). Hence when
looking at the isomorphism class of \( VN(S(G)) \) we need consider only one representative of each equivalence class of cocycle representations.

**Lemma 4.12.2**:

Let \( \sigma, \tau \in Z^2_{\text{norm}}(G; T) \) and let \( S \) be a \( \sigma \)-representation of \( G \). The cocycles \( \sigma \) and \( \tau \) are cohomologous if and only if there is a \( \tau \)-representation of \( G \) equivalent to \( S \).

**Proof**:

Suppose that \( \sigma, \tau \in Z^2_{\text{norm}}(G; T) \), \( S \) is a \( \sigma \)-representation, \( T \) is a \( \tau \)-representation and \( S \) and \( T \) are equivalent. Let \( F \) be a linear isometry from \( \mathcal{H}_S \) onto \( \mathcal{H}_T \) and \( \rho \) a map from \( G \) into \( T \) such that \( T(g) = \rho(g) FS(g) F^{-1} \) for each \( g \in G \). Then \( \rho(e) = 1 \) and, for all \( g, h \in G \),

\[
\tau(g, h)T(gh) = T(g)T(h)
\]

\[
= \rho(g)FS(g)F^{-1}\rho(h)FS(h)F^{-1}
\]

\[
= \rho(g)\rho(h)\overline{\rho(gh)}\rho(gh)FS(g,h)S(gh)F^{-1}
\]

\[
= \rho(g)\overline{\rho(gh)}\rho(h)\sigma(g,h)T(gh)
\]

so

\[
\tau(g, h) = \rho(g)\overline{\rho(gh)}\rho(h)\sigma(g,h)
\]

\[
= ((d_1 \rho) \sigma)(g,h)
\]

and \( \sigma \) and \( \tau \) are cohomologous.

Conversely, let \( \sigma \) and \( \tau \) be cohomologous elements of \( Z^2_{\text{norm}}(G; T) \) and \( S \) a \( \sigma \)-representation of \( G \). If \( \rho \in C^1_{\text{norm}}(G; T) \) is such that \( \tau = (d_1 \rho) \sigma \) and \( F : \mathcal{H}_S \to \mathcal{H}_S \) is the identity map on \( \mathcal{H}_S \) then we define a map \( T : G \to \mathcal{U}(\mathcal{H}_S) \) by

\[
T(g) = \rho(g)FS(g)F^{-1} \quad \forall g \in G.
\]

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Then, for all $g, h \in G$,

$$
T(g)T(h) = \rho(g)\rho(h)FS(g)S(h)F^{-1}
$$

$$
= \rho(g)\rho(h)\overline{\rho(gh)\sigma(g, h)}\rho(gh)FS(gh)F^{-1}
$$

$$
= \tau(g, h)T(gh)
$$

and so $T$ is a $\tau$-representation equivalent to $S$.

\[\square\]

Since we have already seen that the von Neumann algebras arising from equivalent cocycle representations are isomorphic we see that in examining isomorphism classes of von Neumann algebras generated by cocycle representations we need consider only one representative of each cohomology class. Recalling (Lemma 4.12.1) that each cohomology class contains a cocycle, $\sigma$, such that $\sigma(g, g^{-1}) = 1$ for all $g \in G$, we see that we need consider only such $\sigma$.

### 4.12.3 The Left $\sigma$-regular Representation.

Let $\sigma$ be a cocycle in $Z^2_{\text{norm}}(G; T)$. We define a mapping $\lambda_{\sigma} : G \to \mathcal{U}(l^2(G))$ by

$$(\lambda_{\sigma}(g)\zeta)(h) = \sigma(h^{-1}, g)\zeta(g^{-1}h)$$

for all $g, h \in G$ and $\zeta \in l^2(G)$. (It is clear that $\lambda_{\sigma}(g)$ is a linear operator on $l^2(G)$. It is unitary since $|\sigma(h^{-1}, g)| = 1$ for all $g, h \in G$.)

For every $\zeta \in l^2(G)$ and $g, h, k \in G$

$$(\lambda_{\sigma}(g)\lambda_{\sigma}(h)\zeta)(k) = \sigma(k^{-1}, g)(\lambda_{\sigma}(h)\zeta)(g^{-1}k)$$

$$
= \sigma(k^{-1}, g)\sigma(k^{-1}g, h)\zeta(h^{-1}g^{-1}k)
$$

$$
= \sigma(k^{-1}, gh)\sigma(g, h)\zeta((gh)^{-1}k)
$$

$$
= \sigma(g, h)(\lambda_{\sigma}(gh))\zeta(k)
$$
so $\lambda_\sigma$ is a $\sigma$-representation of $G$. It is known as the left $\sigma$-regular representation of $G$.

**Lemma 4.12.3**

Let $\sigma, \tau \in Z^2_{\text{norm}}(G; T)$. Then $\lambda_\sigma$ and $\lambda_\tau$ are equivalent cocycle representations if and only if $\sigma$ and $\tau$ are cohomologous.

**Proof:**

By Lemma 4.12.2, if $\lambda_\sigma$ and $\lambda_\tau$ are equivalent then $\sigma$ and $\tau$ are cohomologous.

Now suppose $\sigma$ and $\tau$ are cohomologous and let $\rho \in C^1_{\text{norm}}(G; T)$ be such that $\tau = (d_1\rho)\sigma$. Define $F : l^2(G) \to l^2(G)$ by

$$ (F \zeta)(g) = \rho(g^{-1})\zeta(g) \quad \forall \zeta \in l^2(G), \, g \in G. $$

Then $F$ is a linear isometry onto $l^2(G)$. Take $\zeta \in l^2(G)$ and $g, h \in G$. Then

$$ \rho(g) \left( F \lambda_\sigma(g)F^{-1}\zeta \right)(h) = \rho(g)\rho(h^{-1})\sigma(h^{-1}, g)\rho(h^{-1}g)\zeta(g^{-1}h) $$

$$ = \tau(h^{-1}, g)\zeta(g^{-1}h) $$

$$ = \left( \lambda_\tau(g)\zeta \right)(h) $$

so $\lambda_\sigma$ and $\lambda_\tau$ are equivalent.

\[ \square \]

It follows that the isomorphism class of $VN(\lambda_\sigma(G))$ is determined by the cohomology class of $\sigma$ and so (by Lemma 4.12.1) we need only consider those $\sigma$ for which $\sigma(g, g^{-1}) = 1$ for all $g \in G$. From now on we shall assume that this condition holds for all cocycles.

Now let $G$ be a countable, infinite, discrete group. Let $\sigma$ be a cocycle. We shall denote $VN(\lambda_\sigma(G))$ by $M_\sigma$. 

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Lemma 4.12.4:

If \( x \in M_\sigma \) then there are complex numbers \( \mu_g, g \in G \), such that

\[
\sum_{g \in G} |\mu_g|^2 < \infty \quad \text{and} \quad x = \sum_{g \in G} \mu_g \lambda_\sigma(g).
\]

Then

\[
\mu_g = (x_\delta_e | \delta_g)
\]

for each \( g \in G \).

Proof:

Take \( x \in M_\sigma \), \( h \in G \) and \( \varepsilon > 0 \). Since \( x \) is in the strong operator closure of

span \( \lambda_\sigma(G) \) in \( B(l^2(G)) \), there are a finite subset \( K \) of \( G \) and complex numbers \( \kappa_g, g \in K \), such that

\[
\left\| \left( x - \sum_{g \in K} \kappa_g \lambda_\sigma(g) \right) \delta_e \right\| < \varepsilon \quad (4.8)
\]

and

\[
\left\| \left( x - \sum_{g \in K} \kappa_g \lambda_\sigma(g) \right) \delta_h \right\| < \varepsilon. \quad (4.9)
\]

For \( g \in G \setminus K \), set \( \kappa_g = 0 \).

We know that

\[
\| x_\delta_e \|^2 = \sum_{f \in G} |(x_\delta_e | \delta_f)|^2. \quad (4.10)
\]

From (4.8)

\[
\varepsilon^2 > \sum_{f \in G} |(x_\delta_e | \delta_f) - \left( \sum_{g \in G} \kappa_g \delta_g \right) \delta_f|^2
\]

\[
= \sum_{f \in G} |(x_\delta_e | \delta_f) - \kappa_f|^2 \quad (4.11)
\]

and from (4.9)

\[
\varepsilon^2 > \sum_{f \in G} |(x_\delta_h | \delta_f) - \left( \sum_{g \in G} \kappa_g \sigma(h^{-1}g^{-1}, g) \delta_{gh} \right) \delta_f|^2
\]

\[
= \sum_{f \in G} |(x_\delta_h | \delta_f) - \kappa_{fh^{-1}} \sigma(f^{-1}h)\delta_f|^2. \quad (4.12)
\]

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For each \( f \in G \), (4.10) tells us that

\[
| (x \delta_e | \delta_f) | \leq \| x \delta_e \| .
\]

Combining this with (4.11)

\[
| \kappa_f | < \| x \delta_e \| + \varepsilon
\]

and combining this with (4.12)

\[
| (x \delta_h | \delta_f) | < \| x \delta_e \| + 2\varepsilon .
\]

The same argument works for every \( h \in G \) and all \( \varepsilon > 0 \), so for all \( h, f \in G \)

\[
| (x \delta_h | \delta_f) | \leq \| x \delta_e \| .
\] (4.13)

For every finite \( J \subset G \), \( (x - \sum_{g \in J} (x \delta_e | \delta_g) \lambda_\sigma(g)) \in M_\sigma \). Then, by (4.13), for all \( f, h \in G \)

\[
\left| \left( \left( x - \sum_{g \in J} (x \delta_e | \delta_g) \lambda_\sigma(g) \right) \delta_h \right| \delta_f \right|^2 \leq \left\| x \delta_e - \sum_{g \in J} (x \delta_e | \delta_g) \delta_g \right\|^2 \\
= \sum_{g \in G \setminus J} \| (x \delta_e | \delta_g) \|^2 .
\]

This last quantity may be made as small as we choose by making \( J \) suitably large. Hence, for all \( f, h \in G \)

\[
(x \delta_h | \delta_f) = \left( \sum_{g \in G} (x \delta_e | \delta_g) \lambda_\sigma(g) \right) \delta_h | \delta_f
\]

and so

\[
x = \sum_{g \in G} (x \delta_e | \delta_g) \lambda_\sigma(g) .
\]

Since \( x \in B(l^2(G)) \),

\[
\sum_{g \in G} (x \delta_e | \delta_g) \delta_g = x \delta_e \in l^2(G)
\]

so

\[
\sum_{g \in G} |(x \delta_e | \delta_g)|^2 < \infty .
\]

\( \square \)
Lemma 4.12.5:

The function \( \text{tr} : M_\sigma \to \mathbb{C} \) defined by

\[
\text{tr} (x) = (x\delta_e | \delta_e) \quad \forall x \in M_\sigma
\]

is a faithful, unital, normal trace on \( M_\sigma \).

Proof:

As for the trace on \( VN(G) \), the function \( \text{tr} \) is clearly normal (i.e. ultraweakly continuous).

We check that \( \text{tr} \) is a faithful, unital trace.

(a) It is clear that \( \text{tr} \) is a linear functional on \( M_\sigma \).

(b) Recall that for each \( g \in G \),

\[
\lambda_\sigma(g)^* = \sigma(g,g^{-1}) \lambda_\sigma(g^{-1})
\]

\[
= \lambda_\sigma(g^{-1}).
\]

For \( x \in M_\sigma \),

\[
\text{tr}(xx^*) = (xx^*\delta_e | \delta_e)
\]

\[
= (x^*\delta_e | x^*\delta_e)
\]

\[
= \| x^*\delta_e \|^2
\]

\[
= \sum_{g \in G} |(x^*\delta_e | \delta_g)|^2
\]

\[
= \left\| \sum_{g \in G} (x^*\delta_e | \delta_g) \lambda_\sigma(g^{-1})\delta_e \right\|^2
\]

\[
= \| x\delta_e \|^2
\]

\[
= \text{tr}(x^*x).
\]

Then \( \text{tr}(x^*x) \geq 0 \) for all \( x \in M_\sigma \) and, by (4.13), \( \text{tr}(x^*x) = 0 \) if and only if \( x = 0 \).
So \( \text{tr} \) is a faithful trace.

(c) The trace is unital since \( \text{tr} \lambda_\sigma(e) = 1 \).

As in the case of \( VN(G) \), we form an inner product on \( M_\sigma \) by

\[
(x | y) = \text{tr} (xy^*) \quad x, y \in M_\sigma
\]

and let \( L^2(M_\sigma) \) denote the completion of \( M_\sigma \) in the inner product topology. Then \( L^2(M_\sigma) \) is a Hilbert space with orthonormal basis \( \lambda_\sigma(G) \).

4.12.4 The Theorem.

We are now in a position to state and prove an extension of Theorem 4.11.1.

Theorem 4.12.6:

Let \( G \) be an infinite, countable, discrete group and let \( \sigma : G \times G \to T \) be a cocycle. Then properties (A), (B), (C) and (D) of Theorem 4.11.1 are equivalent to each of the following properties:

(E) There is a net of normal, completely positive operators \( (T_\alpha : M_\sigma \to M_\sigma)_{\alpha \in \Lambda} \), converging pointwise (in \( L^2 \)) to the identity on \( M_\sigma \), where the operators have bounded extensions to \( L^2(M_\sigma) \) which are compact operators. That is \( M_\sigma \) has property (C) above.

(F) There is a net of completely positive operators \( (T_\alpha : M_\sigma \to M_\sigma)_{\alpha \in \Lambda} \), converging pointwise (in \( L^2 \)) to the identity on \( M_\sigma \), where the operators have bounded extensions to \( L^2(M_\sigma) \) which are compact operators. That is \( M_\sigma \) has property (D) above.

If \( G \) has these properties then any von Neumann subalgebra of \( M_\sigma \) (containing the identity) has properties (C) and (D). In particular, no von Neumann subalgebra of \( M_\sigma \) can be a type \( II_1 \)-factor with property (T).
Again, the nets can be replaced by sequences.

**Proof:**

It is clear that (E) implies (F). We use suitable alterations to the proof of Theorem 4.11.1.

We start with the proof that (A) implies (E). Given an approximate identity, \((\phi_\alpha)_{\alpha \in A}\), for \(C_0(G)\) consisting of functions of positive type, we define completely positive, normal operators \((T_\alpha : \mathcal{B}(l^2(G)) \to \mathcal{B}(l^2(G)))_{\alpha \in A}\) exactly as in the proof of Theorem 4.11.1. Everything up to and including Lemma 4.11.5 goes through without alteration. We need to translate Lemma 4.11.6. Take \(g, h \in G\).

Recall that
\[
\sum_{i \in I} a_i(gh)\overline{a_i(h)} = \phi_\alpha(g)
\]
and
\[
T_\alpha x = \sum_{i \in I} a_i x a_i^*.
\]

Then
\[
T_\alpha (\lambda_\sigma(g)) (\delta_h) = \sum_{i \in I} \overline{a_i(h)} a_i \lambda_\sigma(g) \delta_h
\]
\[
= \sum_{i \in I} \overline{a_i(h)} \sigma(h^{-1} g^{-1}, g) a_i \delta_{gh}
\]
\[
= \sum_{i \in I} \overline{a_i(h)} a_i(gh) \sigma(h^{-1} g^{-1}, g) \delta_{gh}
\]
\[
= \phi_\alpha(g) \sigma(h^{-1} g^{-1}, g) \delta_{gh}
\]
\[
= \phi_\alpha(g) \lambda_\sigma(g) \delta_h
\]
so
\[
T_\alpha (\lambda_\sigma(g)) = \phi_\alpha(g) \lambda_\sigma(g) \quad \forall g \in G.
\]

Using this result, the rest of the proof that (A) implies (C) goes through replacing \(M\) and \(\lambda\) with \(M_\sigma\) and \(\lambda_\sigma\). This concludes the proof that (A) implies (E).
The proof that (D) implies (A) translates directly into the proof that (F) implies (A), replacing $M$ and $\lambda$ with $M_\sigma$ and $\lambda_\sigma$ and using the trace on $M_\sigma$ defined above.

As for $M$, the trace on $M_\sigma$ is a faithful, normal, tracial state and $M_\sigma$ is countably decomposable. Hence the rest of the proof follows exactly as for Theorem 4.11.1.

\□
Chapter 5

Non-Amenable Groups; Lifting and Extension Problems

5.1 Introduction; the Banach-Tarski Paradox.

In this chapter we study discrete, countable, non-amenable groups, remembering that non-compact Kazhdan groups are not amenable. The free group on two generators, $F_2$, is a well studied example of a non-amenable group. Indeed, the existence of non-amenable groups not containing $F_2$ was not demonstrated until the 1980s. However, we have already seen that $F_2$ is not a Kazhdan group. We seek to generalise various results about $F_2$ to larger classes of non-amenable groups. The basis of these results is the Banach-Tarski paradox.

Definition 5.1.1:

A paradoxical decomposition of a group, $G$, is a set of pairwise disjoint subsets $A_1, \ldots, A_m, B_1, \ldots, B_n$ of $G$ and a collection of elements $g_1, \ldots, g_m, h_1, \ldots, h_n$ of $G$ such that

$$G = \bigcup_{i=1}^m g_i A_i = \bigcup_{j=1}^n h_j B_j.$$ 

Note that each $A_i$ is disjoint from each $B_j$ as well as from the other $A_j$s. If $G$ has a paradoxical decomposition then we can choose $A_1, \ldots, A_m, B_1, \ldots, B_n \subset G$. 

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and \( g_1, \ldots, g_m, h_1, \ldots, h_n \in G \) so that
\[
G = \bigsqcup_{i=1}^{m} A_i \cup \bigsqcup_{j=1}^{n} B_j = \bigsqcup_{i=1}^{m} g_i A_i = \bigsqcup_{j=1}^{n} h_j B_j .
\]

(See [Wag] definition 1.1 and the paragraph preceding it.) Tarski's Theorem, [Wag] Theorem 9.2, tells us that a group is non-amenable if and only if it has a paradoxical decomposition.

A paradoxical decomposition in \( F_2 = \langle a, b \rangle \) is given by

\[
S = \{ \text{reduced words in } a, a^{-1}, b, b^{-1} \text{ starting with } a \text{ or } a^{-1} \} \\
A_1 = b^{-1}S \\
A_2 = b^{-2}S \\
B_1 = bS \\
B_2 = b^2S
\]

Indeed the sets \( b^nS \), where \( n \) ranges over all integers, are mutually disjoint subsets of \( F_2 \) so that, for any positive integer \( m \),
\[
\bigsqcup_{i=-m}^{m} b^i S \subset F_2 = S \cup aS = S \cup a^{-1}S .
\]

This paradox is easy to work with because all the sets involved are translations of a single set and because nothing is changed by exchanging \( a \) with \( a^{-1} \) or \( b \) with \( b^{-1} \). Using this specialised form of the paradox, \( F_2 \) has been shown to exhibit certain behaviour known not to occur in amenable groups. We would like to extend some of these results to all non-amenable groups, but the general paradox has proved too hard to adapt to our methods of proof. Consequently we have looked at other specialisations of the paradox. One such is the Powers' property.

**Definition 5.1.2 ([HaS] Section 1):**

A group, \( G \), is a **Powers' group** if, for every non-empty, finite subset \( F \) of \( G \) and every positive integer \( n \), there are a partition of \( G \), \( G = A \cup B \), and \( n \) elements, \( g_1, \ldots, g_n \), of \( G \) such that
(a) \( fA \cap hA = \emptyset \) for any distinct \( f, h \in F \) and

(b) \( g_i B \cap g_j B = \emptyset \) for any distinct \( i, j \in \{1, \ldots, n\} \).

### 5.2 Residually Finite Groups.

Another property of \( F_2 \) on which our proofs depend is that it is residually finite ([L&S] Chapter IV, before Theorem 4.6). This means that there is a chain of normal subgroups of \( F_2 \), \( F_2 = H_0 \supset H_1 \supset H_2 \supset \ldots \), such that each \( |F_2 / H_i| < \infty \) and \( \cap_{i=1}^{\infty} H_i = \{e\} \). Following [Was2] (1.6), if \( G \) is a residually finite group then there is unital *-isomorphism from \( C_1^\ast(G) \) onto a subset of the type II_1-factor \( M/J \) described below.

Let

\[ \mathcal{H} = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \ldots \]

and

\[ M = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \ldots \]

where \( M_n(\mathbb{C}) \) denotes the \( n \times n \) complex matrices. \( M \) is a injective von-Neumann subalgebra of \( B(\mathcal{H}) \), so there is a completely positive projection \( p \) from \( B(\mathcal{H}) \) onto \( M \) and whenever \( A \) is a \( C^\ast \)-algebra acting on a Hilbert space \( \mathcal{K} \) and \( \phi \) is a completely positive linear map from \( A \) into \( M \), \( \phi \) extends to a completely positive linear map from \( B(\mathcal{K}) \) into \( M \).

Let \( \mathcal{U} \) be a free ultra filter on \( \mathbb{N} \) and let \( J \) be the maximal two-sided ideal of \( M \) given by

\[ J = \left\{ (x_n)_{n \in \mathbb{N}} \in M : x_n \in M_n; \lim_{\mathcal{U}} \tau_n(x_n^*x_n) = 0 \right\} \]

where \( \tau_n \) is the trace on \( M_n \) normalised so that the identity has trace 1. Let \( q \) denote the quotient map from \( M \) onto \( M/J \). Then \( M/J \) is a II_1-factor with trace \( \tau \) such that

\[ \tau q((x_n)_{n \in \mathbb{N}}) = \lim_{\mathcal{U}} \tau_n x_n \]
In particular,\
\[ \tau(I) = 1 \]
where \( I \) is the identity element in \( M/J \).

5.3 Liftings and Extensions.

Suppose that \( A \) and \( B \) are \( \text{C}^* \)-algebras faithfully represented on Hilbert spaces \( \mathcal{K} \) and \( \mathcal{F} \) respectively and let \( L \) be a closed, two sided ideal in \( B \). Let \( q \) denote the quotient map from \( B \) into \( B/L \) and let \( \phi \) be a linear map from \( A \) into \( B/L \). A linear map, \( \psi \), from \( A \) into \( B \) is said to be a lifting of \( \phi \) if \( \phi = q\psi \), that is if the following diagram commutes.

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & B/L \\
\downarrow{\psi} & & \downarrow{q} \\
& B & \\
\end{array}
\]

Theorem 2.6 of [R&S] says that if \( A \) is separable and has the \textit{n-positive approximation property} then every \textit{n-positive} linear map \( \phi : A \to B/L \) has an \textit{n-positive} lifting \( \psi : A \to B \) and \( \psi \) may be chosen unital if \( \phi \) is unital. In [C&E], M.D. Choi and E.Effros show that if \( A \) is separable and either \( A, B \) or \( B/L \) is nuclear (see Section 5.5) then completely positive maps from \( A \) into \( B/L \) always have completely positive lifts. In particular, this holds if \( A \) is \( C^*_\Lambda(G) \) for some countable, discrete, amenable group \( G \), since \( C^*_\Lambda(G) \) is nuclear precisely when \( G \) is amenable ([Lce] Theorem 4.2). S.Wasserman shows in [Was] that this result fails if we replace the amenable group \( G \) with \( F_2 \). We show that the result also fails if we use any countable, residually finite, non-amenable, discrete group.
Definition 5.3.1:

A separable $C^\star$-algebra, $A$, has the $n$-positive approximation property if there is a sequence $(T_i : A \to A)_{i \in \mathbb{N}}$ of finite rank, $n$-positive operators converging pointwise to the identity map on $A$.

A.G. Robertson and R.R. Smith use their lifting theorem (2.6) to demonstrate (for each positive integer $n$) the existence of $n$-positive, unital maps from $C^\star_\lambda(F_2)$ into $C^\star(F_2)$ which do not extend to positive maps from $B(l^2(F_2))$ into $B(K)$, where $C^\star(F_2)$ is faithfully represented on $K$ ([R&S] Theorem 3.2). At the same time, their Proposition 3.1 says that if $G$ is a discrete, amenable group and $K$ is a Hilbert space then every $n$-positive, unital map from $C^\star_\lambda(G)$ into $B(K)$ extends to an $n$-positive map from $B(l^2(G))$ into $B(K)$. Again we seek to negate this result for a larger class of non-amenable groups.

5.4 Non-extendibility.

Here we use the techniques of [Rob3] to obtain $n$-positive maps on $C^\star(G)$ which do not have positive extensions.

Suppose $G$ is a countable, discrete group and recall that $\lambda : C^\star(G) \to C^\star(G)$ is the left regular representation of $C^\star(G)$ on $l^2(G)$ extending the map given by $\lambda(g)\zeta(h) = \zeta(g^{-1}h), \zeta \in l^2(G), g, h \in G$.

For each subset $S \subseteq G$ we define a projection $E(S) \in B(l^2(G))$ by

$$E(S)\zeta(g) = \begin{cases} \zeta(g) & \text{if } g \in S \\ 0 & \text{if } g \in G \setminus S \end{cases}$$

where $g \in G, \zeta \in l^2(G)$. Notice that

(a) $E(S)^2\zeta = E(S)\zeta$ for all $\zeta \in l^2(G)$ so that $E(S)$ is indeed a projection, and in particular $E(S)$ is positive;

(b) if $S$ and $T$ are subsets of $G$ then

$$E(S) + E(T) = E(S \cup T) + E(S \cap T) \geq E(S \cup T) \quad (5.1)$$
(c) if $\zeta \in l^2(G)$ and $g, h \in G$ then

\[
(\lambda(g)E(S)\lambda(g)^*)\zeta(h) = \left( E(S)\lambda(g^{-1}) \right) \zeta(g^{-1}h) \\
= \begin{cases} 
\lambda(g^{-1})\zeta(g^{-1}h) & \text{if } g^{-1}h \in S \\
0 & \text{if } g^{-1}h \in G \setminus S 
\end{cases} \\
= \begin{cases} 
\zeta(h) & \text{if } h \in gS \\
0 & \text{if } h \in G \setminus gS 
\end{cases} \\
= E(gS)\zeta(h)
\]

so that

\[
\lambda(g)E(S)\lambda(g)^* = E(gS) \, . \tag{5.2}
\]

Recall from Section 5.2 the definitions of $M, \mathcal{H}, M/J$, the quotient map $q : M \to M/J$, the completely positive projection $p : \mathcal{B}(\mathcal{H}) \to M$, and the trace $\tau$ on $M/J$.

**Theorem 5.4.1:**

Let $G$ be a discrete, countable, residually finite group containing a Powers group, $H$, and suppose that $C^*_s(G)$ has the $n$-positive approximation property for some positive integer, $n$. Then there is an $n$-positive map $\phi_n : C^*_s(G) \to \mathcal{B}(\mathcal{H})$ which does not extend to a positive map from $\mathcal{B}(l^2(G))$ into $\mathcal{B}(\mathcal{H})$.

**Proof:**

Let $F$ be a finite subset of $H$ with at least three elements, such that $F = F^{-1}$. Suppose the Powers’ property is demonstrated for $H$ and $F$ by the partition $H = C \sqcup D$ and the elements $g_1, g_2 \in H$. Following the construction of [Wag] 1.10 and 1.11 we choose, by the axiom of choice, a set $M \subset G$ containing exactly one element of each right coset of $H$ in $G$. Then

\[
\quad gM \cap hM = \emptyset \quad \text{whenever } g \text{ and } h \text{ are distinct elements of } H \tag{5.3}
\]

and

\[
G = \bigsqcup_{h \in H} hM.
\]

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Now set
\[ S = \bigcup_{h \in C} hM \quad \text{and} \quad T = \bigcup_{h \in D} hM. \]

It follows that
\[ G = S \cup T; \quad (5.4) \]

if \( f \) and \( g \) are distinct elements of \( F \) then
\[
fs \cap gs = f \left( \bigcup_{h \in C} hM \right) \cap g \left( \bigcup_{h \in C} hM \right) \\
= (fC \cap gC)M \quad \text{by (5.3), since } F, C \subset H \\
= \emptyset \quad \text{by the definition of } C \tag{5.5}
\]

and, similarly,
\[ g_1T \cap g_2T = \emptyset \]
so that
\[ g_2^{-1}g_1T \cap T = g_1^{-1}g_2T \cap T = \emptyset. \tag{5.6} \]

From (5.4) and (5.1)
\[ E(G) = E(S) + E(T). \tag{5.7} \]

From (5.5) and (5.2)
\[
E(G) \geq \sum_{f \in F} E(fs) \\
= \sum_{f \in F} E(f^{-1}s) \\
= \sum_{f \in F} \frac{1}{2} \left( E(f^{-1}s) + E(fs) \right) \\
= \sum_{f \in F} \{ \lambda(f)E(S)f^{*} \} \tag{5.8}
\]

where \( \{xyz\} = \frac{1}{2}(xyz + zyx) \), the Jordan triple product on \( B(P(G)) \). Similarly from (5.6)
\[ E(G) \geq E(T) + E(g_2^{-1}g_1T) \]
and

\[ E(G) \geq E(T) + E(g_1^{-1}g_2T) \]

and so from (5.2)

\[ E(G) \geq E(T) + \left\{ \lambda(g_1^{-1}g_2)E(T)\lambda(g_1^{-1}g_2)^* \right\} . \tag{5.9} \]

Now let \( \pi \) denote the unital *-isomorphism from \( C^*_\lambda(G) \) into \( M/J \) described in Section 5.2. Suppose that \( \pi \) extends to a positive map, \( \bar{\pi} \), into \( M/J \) from the \( C^* \)-subalgebra, \( A \), of \( B(L^2(G)) \) generated by \( C^*_\lambda(G), E(S) \) and \( E(T) \). Let \( \circ \) denote the the Jordan product, \( x \circ y = \frac{1}{2}(xy + yx) \). For every \( g \in G \)

\[ \bar{\pi}(\lambda(g)^* \circ \lambda(g)) = \bar{\pi}(\lambda(e)) = I = \bar{\pi}(\lambda(g))^* \circ \bar{\pi}(\lambda(g)) \]

where \( I \) denotes the identity element in \( M/J \). Then, by [Rob3] Lemma 1, our equations (5.8) and (5.9) give

\[ I = \bar{\pi}(E(G)) \geq \sum_{f \in F} \{ \bar{\pi}(\lambda(f))\bar{\pi}(E(S))\bar{\pi}(\lambda(f))^* \} \tag{5.10} \]

and

\[ I \geq \bar{\pi}(E(T)) + \left\{ \bar{\pi}(\lambda(g_1^{-1}g_2))\bar{\pi}(E(T))\bar{\pi}(\lambda(g_1^{-1}g_2))^* \right\} . \tag{5.11} \]

Recall that \( M/J \) has a trace, \( \tau \), and that \( \tau(I) = 1 \). Applying the trace to (5.7), (5.10) and (5.11) we find

\[ 1 = \tau \bar{\pi}(E(S)) + \tau \bar{\pi}(E(T)) , \tag{5.12} \]

\[ 1 \geq \sum_{f \in F} \tau(\bar{\pi}(\lambda(f))\bar{\pi}(\lambda(f))^* \bar{\pi}(E(S))) = |F| \tau \bar{\pi}(E(S)) \tag{5.13} \]

and

\[ 1 \geq \tau \bar{\pi}(E(T)) + \tau \left( \bar{\pi}(\lambda(g_1^{-1}g_2))\bar{\pi}(\lambda(g_1^{-1}g_2))^* \bar{\pi}(E(T)) \right) = 2\tau \bar{\pi}(E(T)) . \tag{5.14} \]

Form (5.14), \( \tau \bar{\pi}(E(T)) \leq \frac{1}{2} \) and so (5.12) tell us that \( \tau \bar{\pi}(E(S)) \geq \frac{1}{2} \), but this contradicts (5.13) since \( |F| > 2 \). We conclude that \( \pi \) does not extend to a positive map from \( A \) into \( M/J \).
From [R&S] Theorem 2.6, $\pi$ has a unital $n$-positive lifting $\phi_n : C_\pi^*(G) \to M \subseteq B(H)$. Suppose that $\phi_n$ extends to a positive map $\psi : B(l^2(G)) \to B(H)$. Recall that $p$ is a completely positive projection from $B(H)$ onto $M$. Then the following diagram commutes

\[
\begin{array}{ccc}
C_\pi^*(G) & \xrightarrow{\pi} & M/J \\
\downarrow i & & \downarrow q \\
B(l^2(G)) & \xrightarrow{\psi} & B(H) \\
\downarrow \phi_n & & \downarrow p \\
M & & \end{array}
\]

where $i$ is the natural embedding of $C_\pi^*(G)$ into $B(l^2(G))$. Consequently, $qp\psi : B(l^2(G)) \to M/J$ is a positive extension of $\pi : C_\pi^*(G) \to M/J$. But we have already shown that such an extension cannot exist, so we conclude that $\phi_n$ does not extend to a positive map from $B(l^2(G))$ into $B(H)$.

\[\square\]

Since completing this work, I have received [JoV] whose Proposition 2 is the same result for groups containing a free group rather than a Powers group. The proof follows the same line of argument. Theorem 5.4.1 appears to be more general, but I do not know of any example of a group which satisfies the conditions of Theorem 5.4.1 without containing a free group. It is shown in the same paper that there are groups other than free groups which satisfy the conditions of its Proposition 2.

**Theorem 5.4.2**

*Let $G$ be a countable, discrete, residually finite, non-amenable group and*
suppose that $C^*_r(G)$ has the $n$-positive approximation property for some positive integer, $n$. Then there is an $n$-positive map $\phi_n : C^*_r(G) \to B(\mathcal{H})$ which does not extend to a 2-positive map from $B(L^2(G))$ into $B(\mathcal{H})$.

Proof:

As before, $\pi : C^*_r(G) \to M/J$ is a unital $*$-isomorphism with $n$-positive lifting $\phi_n : C^*_r(G) \to M \subset B(\mathcal{H})$ and we suppose that $\phi_n$ has a 2-positive extension $\psi : B(L^2(G)) \to B(\mathcal{H})$, so that we have the same commuting diagram as before, but with $\psi$ 2-positive. Thus $q\psi : B(L^2(G)) \to M/J$ is a 2-positive extension of $\pi : C^*_r(G) \to M/J$.

Since $G$ is not amenable it has a paradoxical decomposition

$$G = \biguplus_{i=1}^m A_i \cup \bigcup_{j=1}^r B_j = \bigsqcup_{i=1}^m g_i A_i = \bigsqcup_{j=1}^r h_j B_j$$

where $A_1, \ldots, A_m, B_1, \ldots, B_r \subset G, g_1, \ldots, g_m, h_1, \ldots, h_r \in G$. Using (5.1) and (5.2), it follows that

$$E(G) = \sum_{i=1}^m E(A_i) + \sum_{j=1}^r E(B_j)$$

$$E(G) = \sum_{i=1}^m E(g_i A_i)$$

$$= \sum_{i=1}^m \lambda(g_i) E(A_i) \lambda(g_i)^*$$

$$E(G) = \sum_{j=1}^r E(h_j B_j)$$

$$= \sum_{j=1}^r \lambda(h_j) E(B_j) \lambda(h_j)^*.$$  (5.15)

(5.16)

(5.17)

Let $A$ be the $C^*$-subalgebra of $B(L^2(G))$ generated by $C^*_r(G)$ and $E(A_1), \ldots, E(A_m), E(B_1), \ldots, E(B_r)$ and suppose that $\tilde{\pi} : A \to M/J$ is a 2-positive extension of $\pi$. As before, if $g \in G$ then $I = \tilde{\pi}(\lambda(g)^* \lambda(g)) = \tilde{\pi}(\lambda(g) \lambda(g)^*) = \tilde{\pi}(\lambda(g))^* \tilde{\pi}(\lambda(g)) = \tilde{\pi}(\lambda(g)) \tilde{\pi}(\lambda(g))^*$, so by [Choi] Theorem 3.1 (taking

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involutions where necessary)

\[ \tilde{\pi}(\lambda(g)x) = \tilde{\pi}(\lambda(g))\tilde{\pi}(x) \quad \text{and} \quad \tilde{\pi}(x\lambda(g)) = \tilde{\pi}(x)\tilde{\pi}(\lambda(g)) \]

for all \( g \in G \) and \( x \in A \). Hence, applying \( \tau\tilde{\pi} \) to (5.15), (5.16) and (5.17),

\[ 1 = \sum_{i=1}^{m} \tau\tilde{\pi}E(A_i) + \sum_{j=1}^{r} \tau\tilde{\pi}E(B_j) \]

(5.18)

\[ 1 = \sum_{i=1}^{m} \tau(\tilde{\pi}(\lambda(g_i))\tilde{\pi}(E(A_i))\tilde{\pi}(\lambda(g_i))^*) = \sum_{i=1}^{m} \tau\tilde{\pi}(E(A_i)) \]

(5.19)

\[ 1 = \sum_{j=1}^{r} \tau(\tilde{\pi}(\lambda(h_j))\tilde{\pi}(E(B_j))\tilde{\pi}(\lambda(h_j))^*) = \sum_{j=1}^{r} \tau\tilde{\pi}(E(B_j)) \] .

(5.20)

Adding (5.19) and (5.20) gives

\[ 2 = \sum_{i=1}^{m} \tau\tilde{\pi}E(A_i) + \sum_{j=1}^{r} \tau\tilde{\pi}E(B_j) \]

which contradicts (5.18). We conclude that \( \pi \) does not extend to a 2-positive map from \( A \) into \( M/J \). But the restriction of \( q\pi\psi \) to \( A \) is just such an extension, so the 2-positive extension, \( \psi \), of \( \phi_n \) does not exist.

\[ \square \]

5.5 Liftings.

Our aim here is to generalise the result of [Was] about \( F_2 \) to all discrete, countable, residually finite, non-amenable groups, \( G \). We start by looking at tensor products. Let \( A \) and \( B \) be \( C^* \)-algebras. The algebraic tensor product of \( A \) and \( B \) will be denoted by \( A \odot B \). If \( \| \cdot \|_\beta \) is a \( C^* \)-norm on \( A \odot B \) then \( A \otimes_\beta B \) will denote the completion of \( A \odot B \) in the norm \( \| \cdot \|_\beta \). If \( \pi_1 \) and \( \pi_2 \) are faithful representations of \( A \) and \( B \) on Hilbert spaces \( \mathcal{F} \) and \( \mathcal{K} \) respectively then the minimal or spatial norm, \( \| \cdot \|_\alpha \), on \( A \odot B \) is given by

\[ \left\| \sum_i a_i \otimes b_i \right\|_\alpha = \left\| \sum_i \pi_1(a_i) \otimes \pi_2(b_i) \right\| \]

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\((a_i \in A, b_i \in B)\), where the norm on the right hand side is the operator norm on \(\mathcal{F} \otimes \mathcal{K}\) ([Lce] Section 1). So \(C^*_\lambda(G) \otimes \alpha C^*_\lambda(G)\) can be identified with the C*-subalgebra of \(B(l^2(G) \otimes l^2(G))\) generated by \(\{\lambda(g) \otimes \lambda(h) : g, h \in G\}\).

Further, \(\| \cdot \|_\alpha\) is the smallest C*-norm on \(A \circ B\). By definition, \(A\) is nuclear if, for each choice of \(B\), \(\| \cdot \|_\alpha\) is the only C*-norm on \(A \circ B\). C.Lance shows in [Lce] Theorem 4.2 that if \(G\) is a discrete group then \(C^*_\lambda(G)\) is nuclear if and only if \(G\) is amenable. Much of the work in this section is in constructing a different C*-norm, \(\| \cdot \|_\gamma\), on \(C^*_\lambda(G) \circ C^*_\lambda(G)\) when \(G\) is not amenable. The construction is based on that for \(\mathcal{F}_2\) in [Tak2] and [Was2].

**Lemma 5.5.1**

Let \(G\) be a discrete, countable, non-amenable group. There is a C*-norm, \(\| \cdot \|_\gamma\), on \(C^*_\lambda(G) \circ C^*_\lambda(G)\) distinct from the minimal C*-norm, \(\| \cdot \|_\alpha\), such that there is an isometric isomorphism from \(C^*_\lambda(G) \circ C^*_\lambda(G)\) into \(B(l^2(G))\).

**Proof:**

The first part of the proof follows [Tak2] exactly.

Let \(\rho\) denote the right regular representation of \(G\) on \(l^2(G)\),

\[
(\rho(g)\zeta)(h) = \zeta(hg) \quad g, h \in G, \zeta \in l^2(G)
\]

and let \(C^*_\rho(G) = \rho(C^*(G))\). We define a map \(w : l^2(G) \to l^2(G)\) by

\[
(w\zeta)(g) = \zeta(g^{-1}) \quad g \in G, \zeta \in l^2(G).
\]

Then \(w^2\) is the identity and \(\rho(g) = w\lambda(g)w\) for all \(g \in G\). The map \(\pi : C^*_\lambda(G) \circ C^*_\lambda(G) \to B(l^2(G))\) defined by

\[
\pi \left( \sum x_i \otimes y_i \right) = \sum x_iwy_iw
\]

is a homomorphism into the C*-subalgebra of \(B(l^2(G))\) generated by \(C^*_\lambda(G) \cup C^*_\rho(G)\). We define the C*-norm \(\| \cdot \|_\gamma\) on \(C^*_\lambda(G) \circ C^*_\lambda(G)\) by

\[
\| \sum x_i \otimes y_i \|_\gamma = \| \pi \left( \sum x_i \otimes y_i \right) \|
\]

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where the norm on the right is the operator norm in \( \mathcal{B}(L^2(G)) \), so that \( \pi \) is an isometry if we endow \( C_\alpha(G) \odot C_\alpha(G) \) with the norm \( \| . \|_\gamma \). We show that \( \| . \|_\gamma \) is different from \( \| . \|_\alpha \) by showing that \( \pi \) is not continuous if the norm on \( C_\alpha(G) \odot C_\alpha(G) \) is \( \| . \|_\alpha \).

Suppose that \( \pi \) is continuous with respect to the norm \( \| . \|_\alpha \) on \( C_\alpha(G) \odot C_\alpha(G) \). Then \( \pi \) extends to a continuous map from \( C_\alpha(G) \otimes_C C_\alpha(G) \) into \( \mathcal{B}(L^2(G)) \). Let \( A \) denote the weak closure in \( \mathcal{B}(L^2(G) \otimes L^2(G)) \) of \( C_\alpha(G) \otimes_C C_\alpha(G) \). Then \( \delta_e \otimes \delta_e \in L^2(G) \otimes L^2(G) \) is a separating vector for \( A \), that is if \( x \in A \) and \( x(\delta_e \otimes \delta_e) = 0 \) then \( x = 0 \), so by [K&R] Theorem 7.3.8 if \( \psi \) is a normal state of \( A \), then there is a unit vector \( \zeta \in L^2(G) \otimes L^2(G) \) such that \( \psi(x) = \langle x\zeta, \zeta \rangle \) for all \( x \in A \); that is, \( \psi \) is a vector state of \( A \). Exactly as in [Tak2], this means that the vector states of \( A \) are weak*-dense in the state space of \( A \). For each unit vector \( \zeta \in L^2(G) \) let \( \phi_\zeta : A \to \mathbb{C} \) denote the state \( x \mapsto \langle x\zeta, \zeta \rangle \) of \( A \). Then, for any finite collection of elements, \( x_1, \ldots, x_r \), of \( A \) and any \( \epsilon \geq 0 \), there is a unit vector \( \eta \in L^2(G) \otimes L^2(G) \) such that

\[
|\phi_\zeta(x_i) - \langle x_i\eta, \eta \rangle| < \epsilon \quad i = 1, 2, \ldots, r.
\]

Suppose we have a paradoxical decomposition of \( G \),

\[
G = \bigsqcup_{i=1}^m A_i \cup \bigsqcup_{j=1}^n B_j = \bigsqcup_{i=1}^m g_i A_i = \bigsqcup_{j=1}^n h_j B_j
\]

where \( g_1, \ldots, g_m, h_1, \ldots, h_n \in G \). Let \( 0 < \epsilon \leq \frac{1}{2(m+n)} \). Notice that for each \( g \in G \),

\[
\phi_{\delta_e}(\lambda(g) \otimes \lambda(g)) = \langle \pi(\lambda(g) \otimes \lambda(g))\delta_e, \delta_e \rangle
\]

\[
= \langle \lambda(g)\rho(g)\delta_e, \delta_e \rangle
\]

\[
= \langle \delta_e, \delta_e \rangle
\]

\[
= 1.
\]

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Then, by the previous paragraph, there is a unit vector \( \eta \in \ell^2(G) \otimes \ell^2(G) \) such that
\[
\left| 1 - \langle (\lambda(g_i^{-1}) \otimes \lambda(g_i^{-1})) \eta, \eta \rangle \right| < \frac{\varepsilon^2}{2} \quad i = 1, 2, \ldots, m,
\]
\[
\left| 1 - \langle (\lambda(h_j^{-1}) \otimes \lambda(h_j^{-1})) \eta, \eta \rangle \right| < \frac{\varepsilon^2}{2} \quad j = 1, 2, \ldots, n
\]
and
\[
\| \eta - (\lambda(g_i^{-1}) \otimes \lambda(g_i^{-1})) \eta \| = \sqrt{2 \Re \left( 1 - \langle (\lambda(g_i^{-1}) \otimes \lambda(g_i^{-1})) \eta, \eta \rangle \right)}
\]
\[
< \varepsilon \quad i = 1, 2, \ldots, m ,
\]
\[
\| \eta - (\lambda(h_j^{-1}) \otimes \lambda(h_j^{-1})) \eta \| < \varepsilon \quad j = 1, 2, \ldots, n.
\]

As in Section 5.4, for each subset \( S \subseteq G \times G \) we define a projection \( E(S) \in B(\ell^2(G) \otimes \ell^2(G)) \) by
\[
(E(S)\zeta)(g,h) = \begin{cases} 
\zeta(g,h) & (g,h) \in S \\
0 & (g,h) \in (G \times G) \setminus S
\end{cases}
\]
for all \( \zeta \in \ell^2(G) \otimes \ell^2(G) \) and \( g, h \in G \). Then, for \( \zeta \in \ell^2(G) \otimes \ell^2(G) \) and \( g, h, f, k \in G \)
\[
[E((f, k)S)\zeta](g,h) = \begin{cases} 
\zeta(g,h) & (f^{-1}g, k^{-1}h) \in S \\
0 & (f^{-1}g, k^{-1}h) \notin S
\end{cases}
\]
\[
= \begin{cases} 
[(\lambda(f^{-1}) \otimes \lambda(k^{-1})) \zeta](f^{-1}g, k^{-1}h) & (f^{-1}g, k^{-1}h) \in S \\
0 & (f^{-1}g, k^{-1}h) \notin S
\end{cases}
\]
\[
= [E(S)(\lambda(f) \otimes \lambda(k)))^{-1}\zeta](f^{-1}g, k^{-1}h)
\]
\[
= [(\lambda(f) \otimes \lambda(k)) E(S) (\lambda(f) \otimes \lambda(k)))^{-1}\zeta](g,h)
\]
so
\[
E((f, k)S) = (\lambda(f) \otimes \lambda(k)) E(S) (\lambda(f) \otimes \lambda(k))^{-1}.
\]
Then, for all subsets $S \subseteq G \times G$ and $i \in \{1, \ldots, m\}$,

$$\left| \langle E(S) \eta, \eta \rangle - \langle E((g_i, g_i)S) \eta, \eta \rangle \right|$$

$$= \left| \langle E(S) \eta, \eta \rangle - \langle (\lambda(g_i) \otimes \lambda(g_i)) E(S) \left( \lambda(g_i^{-1}) \otimes \lambda(g_i^{-1}) \right) \eta, \eta \rangle \right|$$

$$= \left| \langle E(S) \eta, \eta \rangle - \langle E(S) \left( \lambda(g_i^{-1}) \otimes \lambda(g_i^{-1}) \right) \eta, \left( \lambda(g_i^{-1}) \otimes \lambda(g_i^{-1}) \right) \eta \rangle \right|$$

$$\leq \left| \langle E(S) \eta, \left( \lambda(g_i^{-1}) \otimes \lambda(g_i^{-1}) \right) \eta \rangle \right|$$

$$= \left| \langle E(S) \eta, \left( \lambda(g_i^{-1}) \otimes \lambda(g_i^{-1}) \right) \eta \rangle \right| + \left| \langle E(S) \eta, \left( \lambda(g_i^{-1}) \otimes \lambda(g_i^{-1}) \right) \eta \rangle \right|$$

$$\leq \left\| E(S) \eta \right\| \left\| \eta - \left( \lambda(g_i^{-1}) \otimes \lambda(g_i^{-1}) \right) \eta \right\|$$

$$+ \left\| E(S) \left( \lambda(g_i^{-1}) \otimes \lambda(g_i^{-1}) \right) \eta \right\| \left\| \left( \lambda(g_i^{-1}) \otimes \lambda(g_i^{-1}) \right) \eta \right\|$$

$$< 2\varepsilon \quad (5.22)$$

from (5.21). Similarly, for $j \in \{1, \ldots, n\}$,

$$\left| \langle E(S) \eta, \eta \rangle - \langle E((h_j, h_j)S) \eta, \eta \rangle \right| < 2\varepsilon. \quad (5.23)$$

We know that

$$G \times G = \bigcup_{i=1}^{m} (A_i \times G) \cup \bigcup_{j=1}^{n} (B_j \times G) = \bigcup_{i=1}^{m} (g_i, g_i)(A_i \times G) = \bigcup_{j=1}^{n} (h_j, h_j)(B_j \times G)$$

so

$$1 = \left\| \eta \right\|^2 = \langle E(G \times G) \eta, \eta \rangle$$

$$= \sum_{i=1}^{m} \langle E(A_i \times G) \eta, \eta \rangle + \sum_{j=1}^{n} \langle E(B_j \times G) \eta, \eta \rangle \quad (5.24)$$

$$= \sum_{i=1}^{m} \langle E((g_i, g_i)(A_i \times G)) \eta, \eta \rangle \quad (5.25)$$

$$= \sum_{j=1}^{n} \langle E((h_j, h_j)(B_j \times G)) \eta, \eta \rangle \quad (5.26)$$
But, by (5.22),
\[ \sum_{i=1}^{m} \langle E \left((g_i, g_i)(A_i \times G)\right) \eta, \eta \rangle < \sum_{i=1}^{m} \langle E(A_i \times G) \eta, \eta \rangle + 2m \epsilon \]
and, by (5.23),
\[ \sum_{j=1}^{n} \langle E \left((h_j, h_j)(B_j \times G)\right) \eta, \eta \rangle < \sum_{j=1}^{n} \langle E(B_j \times G) \eta, \eta \rangle + 2n \epsilon \]
so, from (5.25) and (5.26),
\[ \sum_{i=1}^{m} \langle E(A_i \times G) \eta, \eta \rangle > 1 - 2m \epsilon \]
and
\[ \sum_{j=1}^{n} \langle E(B_j \times G) \eta, \eta \rangle > 1 - 2n \epsilon . \]
Combining these with (5.24) and recalling that \( 0 < \epsilon \leq \frac{1}{2(m+n)} \) we find that
\[ 1 = \sum_{i=1}^{m} \langle E(A_i \times G) \eta, \eta \rangle + \sum_{j=1}^{n} \langle E(B_j \times G) \eta, \eta \rangle > 2 - 2(n + m) \epsilon \geq 1 . \]
From this contradiction we conclude that \( \pi \) is not continuous with respect to the norm \( \| \cdot \|_\alpha \) on \( C^*_\lambda(G) \odot C^*_\lambda(G) \) and hence that \( \| \cdot \|_\gamma \neq \| \cdot \|_\alpha \).
\( \Box \)

For any discrete group \( G \), we identify \( C^*_\lambda(G) \) with the quotient space \( C^*(G)/\ker \lambda \) and think of the map \( \lambda : C^*(G) \to C^*_\lambda(G) \) as a quotient map.

**Theorem 5.5.2** :

Let \( G \) be a discrete, countable, residually finite, non-amenable group. The identity map, \( \text{id} \), on \( C^*_\lambda(G) \) does not have a completely positive lifting; that is there is no completely positive map \( \psi : C^*_\lambda(G) \to C^*(G) \) which makes the following diagram commute.

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Proof:

The Lemma in [Was] says that if $A$ and $B$ are unital $C^*$-algebras, $J$ is a closed two sided ideal in $B$, $\tau : B \to B/J$ is the quotient map and there is a $C^*$-norm on $(B/J) \odot (B/J)$, distinct from the minimal $C^*$-norm, with respect to which the map $\tau \odot \tau : B \odot B \to (B/J) \odot (B/J)$ is bounded, when $B \odot B$ has the minimal $C^*$-norm, then no completely positive $*$-isomorphism from $A$ onto $B/J$ has a completely positive lifting, mapping $A$ into $B$. Hence the result will follow if we can show that the map $\lambda \odot \lambda : C^*(G) \odot C^*(G) \to C^*_\lambda(G) \odot C^*_\lambda(G)$ is bounded with respect to the norms $\| \cdot \|_\alpha$ on $C^*(G) \odot C^*(G)$ and $\| \cdot \|_\gamma$ on $C^*_\lambda(G) \odot C^*_\lambda(G)$. But this is shown for $F_2$ in [Was2] (2.7) and exactly the same proof goes through for the residually finite group $G$. 

$\square$
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