STRUCTURAL RESPONSE OF LATTICE TOWER STRUCTURES

TO

THE NATURAL WIND

by

MICHEL J. DAREAU, B.Sc.(Hons)

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Lattice tower set up to study wind response in the field
ABSTRACT

The broad motivation behind this thesis lies in the existing problem of determining accurately the effect of the natural wind on buildings and structures. It is thought that as a first step towards this end the experimental observation of the actual phenomena involved in the simplest of structures is important and necessary. For this reason this thesis deals almost exclusively with the study of the effect of the natural turbulent wind on a relatively small and aerodynamically straightforward structure consisting of a thirty foot lattice tower.

A major part of the work presented deals with the problems of obtaining meaningful records of simultaneous total response and total causative wind; briefly, this objective is achieved, on the structural side, by resolution of the two-dimensional behaviour of the lattice tower into uni-directional components along the principal axes of moment of inertia and on the wind side, by similar resolution of the horizontal wind speed fluctuation about the same axes. The transducing instrumentation devised makes use of strain gauges in measuring the structural response and a hot-wire anemometer combined with a resolving vane in recording the wind fluctuation.

The records of structural response and wind speed actually obtained show that the tower is subjected to both quasi-steady displacement and resonant vibration at both its fundamental frequencies. Remarkable agreement is obtained between the quasi-steady structural response and the 'quasi-steady component' of wind speed. Attention is drawn to the possibility of fluctuations
in wind direction inducing substantial resonant vibrations in the structure which are transverse to the mean wind flow direction.

In an attempt to reduce the resonant part of the structural response certain auxiliary 'dampers' are devised, one such damper being built and fitted to the tower and proving extremely effective.

The thesis concludes with a discussion of the records actually obtained, stressing the importance of determining accurate wind energy spectra and the necessity of using certain statistical concepts.
CHAPTER 1.

Introduction
INTRODUCTION

The problems involving the effect of wind on engineering structures have in recent years given rise to much discussion and thought on the part of a large number of researchers. Several international conferences have been held in an attempt to compare, discuss and reconcile contemporary research on the subject. However, the practising engineer is still far from being provided with a reasonably comprehensive and simple design procedure. The reason for this almost certainly lies in the complexity underlying the different aspects of the problem.

In the past the design wind loads for buildings were deduced exclusively from steady flow wind-tunnel experimentation on models. However, today, with ever-increasing accuracy in the design of structures (together with economic pressures) it follows that a correspondingly refined method of determining the ultimate wind loading is essential.

A consideration of the problem from very first principles is instructive. For instance, it is immediately clear from a glance at a wind record (i.e., a record of the wind velocity with respect to time at a particular point in the air flow) that the wind velocity is not steady but random in its nature. It would thus not seem unreasonable to presuppose that the corresponding response of a bluff object at that point would also be of a random nature.

1 See REFS 1, 2 and 3.
2 REF 4: forming the basis of the British Code of Functional Requirements for Buildings - CP3, chapter V, for instance.
3 See records of wind on page 163.
This tends to suggest at the very outset that any comprehensive theoretical approach must be a quasi-statistical one. A further inspection of the wind record will reveal that the wind velocity variation does obey certain physical laws. For instance, it will be noted that the velocity fluctuations over periods of approximately one minute are in general larger than those occurring over shorter intervals of time. In other words, there is more energy in the wind at certain recurrent frequencies than at others. This phenomenon has led to the drawing up of energy-spectra for the wind.

The above considerations have led to what is by far the most significant theoretical advance made so far. Given originally by Davenport, it combines a statistical approach with the use of energy-spectra. Briefly, the method involves the building up of the structural response energy-spectrum from the wind velocity energy-spectrum through successive multiplication by the aerodynamic admittance (a measure of how much of the wind the structure actually 'sees') and the mechanical admittance (a measure of the dynamic amplification of the structure). Using a statistical formulation which includes the area of the response spectrum (which equals the variance of the response) the largest likely response of the structure within a given period of time is determined. This is then superimposed on the highest averaged response for the particular time interval considered which is obtained on the familiar steady wind basis from the corresponding

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1 see REF 5. (Application of Statistical Concepts to the Wind Loading of Structures. by A. G. Davenport) and REF 31. (The Response of Structures to Gusts, by R. I. Harris.)

REF 6: The Spectrum of Horizontal Gustiness near the Ground in High Winds by A. G. Davenport
highest averaged wind\textsuperscript{1} likely in, say, fifty years.

Unfortunately, the method as first put forward by Davenport appears somewhat involved to the practising engineer and although in later\textsuperscript{2,3} papers Davenport produces design curves from which it is possible to read off actual numerical values, the quantities that these represent take on a somewhat abstract character. Even before these values can be read off, however, it is necessary to make rather difficult assumptions about the topography and surface roughness of the locality in order to obtain the necessary energy-spectrum, boundary layer profile and design velocity particular to the wind at the proposed site. Further, it is necessary to make a blind assumption as to the amount of structural damping present - quite apart from the problem of estimating the natural frequency or stiffness of the proposed building. These uncertainties in the theory as presented by Davenport, alone, are sufficient to necessitate an extensive programme of experimental investigation with, possibly, ensuing theoretical amplification and clarification.

From the experimental point of view, it will be realized that because of the large variety of engineering structure types - ranging from suspension bridges to cooling towers - and because of the extent of the problem - ranging from the need for a full understanding of the wind itself to a similarly full knowledge of the physical properties of the particular structure - individual researchers have had to restrict themselves to the study of one particular aspect of the whole. This has resulted

\textsuperscript{1} see REF 7. (Rationale for Determining Design Wind Velocities, by A. G. Davenport.)
\textsuperscript{2} see REF 8. (Gust Loading Factors, by A. G. Davenport.)
\textsuperscript{3} see REF 9. (The Treatment of Tall Buildings, by A. G. Davenport.)
in a rather fragmented approach¹ with some fields having been explored more fully than others.

A large proportion of the experimental work performed up till now has been devoted to the intensive study of models in wind tunnels. Much of this work has been carried out under steady or uniform flow conditions — dealing, in the main, with problems concerning aerodynamic pressure coefficients and vortex shedding². Some attempts have been made at simulating the boundary layer profiles and the turbulent conditions found in the natural wind³ but more research is necessary before it can be judged just how effectively such conditions can be applied to individual models and how faithfully the response of the prototype in natural wind is reproduced⁴. In view of these present uncertainties, the only valid alternative means of advance towards a fuller understanding of the behaviour of life-sized structures is the conducting of field tests on these structures themselves — an alternative which is not only important from the point of view of verifying or prompting theoretical work but is in any case necessary in the verification of wind-tunnel simulations. It will be realized that the popularity and desirability of wind tunnel experimentation on models is due to the fact that the conditions under which these experiments are performed can be scientifically controlled — whereas field tests on real structures are very much subject to the whims of nature; quite apart of course from the problems associated with working on a much larger scale, with accessibility and so on. In fact, understandably, very

¹Though the conferences held (REFs 1,2,3) and some surveys (such as REF 10) have helped to reconcile this.
²See, in particular SESSIONS 3 & 4 of REF 2.
³See REF 10, notably, Elder, Whitbread, Vickery and Davenport and Harris.
⁴See, for instance, REF 2, PAPER 3: (Wind Loading of a Tall Building in an Urban Environment a Comparison of Full Scale and Wind Tunnel Tests. by Newberry, Eaton, and Mayne.)
few real structures have been comprehensively examined from the aspect of their full dynamic behaviour under natural wind conditions. Among the few structures that have been tentatively so examined may be mentioned a 120 foot floodlight tower situated at Santa Fé in New Mexico. The measurements taken of its motion in the natural wind were analyzed on the basis of the theory developed by Davenport. However, the final conclusions, as those of other structures analyzed in this way, appear somewhat too premature to be incorporated into a universally acceptable design procedure. In fact, some of the conclusions are more in the nature of tentative suggestion or even opinion.

Other experimental work has been concerned with the establishment of 'gust factors' (i.e. ratios of various short duration gust means to wind means taken over longer periods of time). Obtained from the study of long-term meteorological wind records, these gust factors have been incorporated into a straight-forward design procedure using wind tunnel model results. The method purports to make an allowance for the spatial characteristics of the wind (viz. the highest gust sufficient to envelop the whole building is considered) and in this respect it must be considered as a

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1 Note, for instance, the divergence between the following conclusions drawn from two separate papers in which the same (Santa Fé) structure is dealt with: REF 3, PAPER 26, (A Comparison of Theoretical and Experimental Determination of the Response of Elastic Structures to Turbulent Flow, by B. J. Vickery and A. G. Davenport.): "The aerodynamic damping of lattice masts can be predicted from steady aerodynamic coefficients and may be several times greater than damping of mechanical origin," and REF 11, (Comparison of the Predicted and Measured Dynamic Response of Structures to Wind (Santa Fé Tower), by A. J. Hartmann and A. G. Davenport.): "The damping observed appears to be entirely due to the mechanical damping present in the structure."

2 See for instance REF 3, PAPER 19, (Results of Some Recent Special Measurements in the United Kingdom Relevant to Wind Loading Problems, by H. C. Shellard.)

3 See REF 12, (Wind Loading on Buildings, B.R.S. digests 1 & 2.)
significant improvement on previous methods\textsuperscript{1}. It must be emphasized, however, that the method, which appears to have no rigorous theoretical backing, is not strictly applicable to structures which are caused to vibrate substantially about their resonant frequencies as a result of turbulence in the wind\textsuperscript{2}.

More specifically, then, everything seems to point to the need for more experimental work on real structures in the natural wind. It was, therefore, decided to attempt an analysis of one such structure and to let it form the basis of the thesis presented here.

The layman is familiar with everyday effects of the wind in, for instance, the vibration of lamp standards and the swaying of trees. Very simple and small structures are affected just as are more complex and larger ones, though, possibly, in different ways. This suggests the setting up of simple structures in the natural wind as a means of preliminary investigation into the response of larger and more complex structures. One such experiment was suggested by Hendry and was later performed by Dutt\textsuperscript{3} who considered the behaviour of a simple cantilever structure (consisting of a 4" diameter tube 9' 0" long surmounted by a circular sheet steel drum 2' 0" x 2' 0" diameter). Harris\textsuperscript{4}, in a later paper, considers a similar structure, with the exception that the drum is perforated, from a theoretical point of view.

\textsuperscript{1}See in particular REF 4. (British Code of Functional Requirements for Buildings - CP3 chapter V.
\textsuperscript{2}No extensive experimental or theoretical study of the dynamic behaviour of real structures in wind seems to have been made.
\textsuperscript{4}See REF 2, PAPER 15 (The Effect of Wind on Some Simple Structures by R.I. Harris).
By choosing a structure of this nature, some of the phenomena and problems associated with larger buildings are eliminated. This will be apparent from the following observations:

1. The drum is sufficiently small for the instantaneous wind pressure on it to be considered uniform — i.e. there is no deterioration in the spatial correlation of the wind as there would be if a large area were to be considered. This also means that the wind velocity measured close to the drum is essentially the same as that contributing to the instantaneous pressure on the drum.

2. The drum is uniformly cylindrical along a vertical axis. This means that though the drag will vary according to the wind direction and speed, (considering the drag as a vector quantity) the overall drag coefficient will be independent of wind direction (in any horizontal plane) notwithstanding possible drag coefficient variations in a fluctuating but uniform flow.

3. The tube on which the drum is mounted acts as a linear spring of (ideally) constant stiffness in all directions. The motion of the drum can then be considered as analogous to that of a one-degree-of-freedom system with a single resonant frequency and mode shape, with the important exception that the drum is free to move horizontally in any direction from one moment to the next.

4. Being of a manageable size, the structure may be dismantled and tested under controlled laboratory conditions — enabling, for instance, the steady flow pressure coefficient for the drum to be found from wind tunnel tests and, on the structural side, the obtaining of the natural frequency, stiffness and damping characteristics.
The results of the tests carried out by Dutt show that even with such a simplified system many difficulties of analysis remain. It was found that due to the random shedding of vortex pairs from the sides of the drum that a corresponding periodic force was exerted on the system in a direction always transverse to the direction of wind flow. Apart from this the straightforward turbulent buffeting action of the wind subjected the system to a generally smaller but nevertheless appreciable drag force in the same direction as that of the wind. One noteworthy feature concerning the response was the predominant oscillation of the system about its natural frequency in both the in-wind and cross-wind directions. Unfortunately, because of an added variable — the continuous variation in the wind direction — it could not be ascertained just what proportion of the motion, in line with the mean wind direction or across it, was due to which effect (i.e., fluctuating lift or drag). It is worth pointing out here that later in this thesis it will be demonstrated how even relatively small fluctuations in the direction of the wind can induce substantial oscillations in a structure, in a direction transverse to the mean direction of wind flow.

Clearly, if a successful experimental study of the fluctuating drag aspect of the wind alone is to be made, with the use of a simplified structure such as the one mentioned above, some means of controlling or allowing for the formation of vortices must be found. While the latter effect on models of tall cylindrical structures and, in particular, tall chimney stacks, is fairly well understood for a wide range of Reynolds' Number and surface roughnesses from the results of

---

1 The average direction taken over a fairly long period of time (1 hour, say) is considered here.
controlled wind-tunnel tests, it is not known with the same confidence what happens to the corresponding real structures exposed to the more turbulent conditions found in the natural wind\textsuperscript{1} - although it is well established qualitatively (from everyday experience of vibrating lamp standards and similar structures) that a certain amount of sideways motion resulting from vortex shedding does take place. Indeed, it would seem reasonable to suppose that all life-size bluff structures are, to an extent dependent on their effective surface roughness, shape and other factors, subject to quasi-steady or randomly oscillatory lift forces generated in one way or another by the flow of wind around them. Looking for exceptions it might be conjectured that a hypothetical structure in which several lift forces are brought into play and made to act in opposition to one another at the same time might not be subject to an overall lift force. Such a structure implies the existence of distinctly separate members around which an air flow is allowed to develop. This leads to the consideration of open or lattice-type structures composed of slender interconnected struts and ties. It seems reasonable to make the following predictions concerning such structures:

1. Though possibly common to all or some of the individual members of the lattice-network the frequency of any vortex shedding occurring will, in general, be randomly phased from member to member resulting in a null overall effect.

2. Applying, in particular, to lattice-tower-like structures

\textsuperscript{1}See, for instance, REF 2, PAPER 18: (The Oscillations of Model Circular Stacks Due to Vortex Shedding by L. R. Wootton.)
and irrespectively of whether the above prediction is valid or not, the total sum of the strengths of vortices arising from the individual members will, in general, be much smaller than the strength of the single but much larger vortex street which might arise from the 'equivalent' bluff structure (i.e. the solid formed by the outermost extremities of the lattice-work). This prediction is made in the knowledge that different strengths of vortices will arise from members having different sectional shapes and roughnesses - this qualification also applies to the 'equivalent' structure.

3. Because of a scale difference between that of a 'large diameter' bluff structure and the small effective diameter of the individual lattice members the shedding frequency of vortices will be much higher in the lattice structure than in the equivalent bluff structure. It is, therefore, less likely that the frequency of vortex shedding will tie itself to the natural frequency of the structure.

4. Implicit in all the above predictions is the assumption that the members are sufficiently well spaced apart for the motion of the whole structure to be due principally to the sum effect of the wind flowing through and around the structure and acting on individual members and not in some way acting on the structure as a whole.

Partly because of the predictable lack of problematic fluctuating lift effects and, hence, the improved possibility of studying experimentally the aspect of fluctuating drag and partly because the study of open structures in the natural wind is important in its own right it was considered profitable to proceed with a field

1 It may be possible for the oscillation of a structure about its natural frequency and the shedding of vortices from individual members to be mutually reinforcing - in which case, the sheddings of vortices from all members are in phase with each other - but see prediction 3.

2 For instance, a circular bluff tower of 2' 0" diameter and natural frequency, 3-c/s, might reasonably be expected to shed vortices at that same frequency in a 25-ft/sec. wind but an equivalent open tower (of the same frequency) consisting of sectionally circular members of 1" diameter should, in the same wind speed, shed vortices of a frequency of approximately 70-c/s.
investigation of such a structure. While deciding to enlarge on the scale of the structure used by Dutt - in order to make the proposed structure more compatible with those found in practice - it was nevertheless attempted to fulfill as far as was possible the four conditions laid down earlier (on page 7) for the cantilevered drum system. The design of a suitable structure was therefore undertaken.
CHAPTER 2

Design of Lattice Tower
2. Design of Lattice Tower

2.1. Criteria Affecting Design: The design of a lattice tower suitable for the purposes of the proposed study is controlled by criteria concerning size, symmetry, dynamic behaviour and continuity in physical properties.

2.1.1. Size of the Structure in Relation to the Size of Gusts.

Of importance is the shortest-lasting eddy likely just to envelop - in what can be considered to be an approximately uniform way - the whole structure. Were the effective duration of this gust to be equal to or greater than the period of natural oscillation of the (one-degree-of-freedom) structure, the whole frequency range of the response could be analysed on the basis of a simplified time-fluctuating, but uniform, flow - assuming that the pressure coefficient variation is known for the frequency range in question. Clearly, this particular state of affairs depends on the establishment of the correct relation between the size and dynamic behaviour of the structure and that of the turbulence impinging on the structure. In general, however, the size of practical full-scale structures is not controlled by the size of gusts - so that some means of allowing for the loss in spatial correlation must be found. Certain empirical expressions for the spatial correlation of the natural wind have been put forward but more recent measurements would seem to suggest that these expressions underestimate the spatial compass of gusts.

Davenport defines the "scale" (or "semi-scale") of

1See REF 1. (PAPER 2) (The Relationship of Wind Structure to Wind Loading by A. G. Davenport).
2See REF 2.
3See REF 2. (PAPER 3).
turbulence as proportional to the wavelength, viz. \( l/C \cdot \bar{V}/n \). (where 
'C' is a proportionality constant, '\( \bar{V} \)' the mean wind and 'n' the 
frequency in c/s.). The coefficient, C, is derived from experimentally obtained exponential decay distributions of the cross-
correlation (or co-variance) of the horizontal wind speed measured 
between two points separated by a distance \( \Delta z \). Plotted as a 
function of separation to wavelength ratio, \( n \cdot \Delta z/\bar{V} \), the curves 
fitting to the experimental distribution are of the nature 
\( \exp[-C(n \cdot \Delta z/\bar{V})] \). (see figure 1). In the vertical 
direction the 

Cross-correlation, \( R(n \Delta z/\bar{V}) \)

\[ \exp[-C(n \Delta z/\bar{V})] \]

'c.of g.' of correlation diagram at \( \Delta z = l/C \cdot \bar{V}/n \) 

FIG 1

corresponding to 'effective gust width'

value given to C varies between, C=8 for "strong turbulence", in a 
wind-tunnel, C = 7.7, for open grassland, and C = 6 for wooded 
country\(^2\). Although this suggests a possible extrapolation of

\(^1\)It must be noted that Davenport's curve fitting is somewhat 
puzzling. The actual distribution of measured values does not 
correspond particularly well to the fitted curve - it is suggested, 
from inspection, that an even function of the nature 
\( \exp[-k(n \cdot \Delta z/\bar{V})^2] \) would be more appropriate.

\(^2\)The separation \( \Delta z \) was \( \geq 150 \) ft. in both the field tests.
C = 3-5, for city centres which would at least tend toward greater compatibility with the observations of Newberry, Eaton and Mayne who were concerned with the full-scale experimental study of a tall building in central London (Royex House in the Barbican) where it was concluded that the spatial compass of gusts was much larger than anticipated - Davenport, nevertheless, appears to adopt a universal value of C = 8 in later papers. The discrepancy between the conclusions reached on the one hand by Davenport and those reached on the other by Newberry, et al. is possibly due, in part, to the contrasting observational approach employed in the two cases. Davenport bases his predictions on wind speed measurements taken from instrumentation mounted on a structure in such a way that the free wind flow is measured, whereas, Newberry, et al. draw their conclusions from the readings of pressure gauges mounted flush with the curtain walling of a large bluff structure and as such, the measured distribution is that of the wind as affected by the structure. In fact, in the latter case, a certain amount of 'spreading out' of gusts impinging on the large surface may take place - resulting in an observation of what appears to be gusts whose span may, indeed, be as wide as the building itself.\(^1\)

\(^1\)A tentative explanation of this effect is suggested in the following diagram. (Although shown in the horizontal, the effect may occur simultaneously in the vertical plant.)

![Diagram showing the effect of gust spreading out](image)

At 1: gust of effective 'width', a, and velocity, \(V_a\), (as measured by Davenport.)

At 2: after a small time interval \(\Delta t\) the same gust width is expanded to \(b(>a)\) and the velocity is reduced to \(V_b(=V_a)\) (as measured by Newberry, et al.)
This effect would not be expected to develop on lattice-type structures - in fact, the effective spatial correlation of gusts on such a structure should be approximately the same as that in the free wind, assuming that the structure is of a sufficiently open nature to allow a reasonable amount of through flow.

Assuming Davenport's approach to be correct, the 'effective width', taken vertically, of a gust having a componental frequency of 3-c/s., in a mean wind of 24-ft/sec., say, is 1-ft.\(^1\) For a frequency decrease to \(\frac{1}{2}\)-c/s. the 'effective width' increases linearly to 6-ft. If the mean (hourly) wind speed is increased to 88-ft/sec. (60-mph.) the 'effective width' at \(\frac{1}{2}\)-c/s. becomes 22-ft. This compares with a span of 20-m. (60 ft.) for a 2-sec. average gust as advocated for design purposes by a recent Building Research Station Digest on Wind Loading\(^2\) - a 'rule of thumb' based, it would appear, largely on the conclusions drawn by Newberry, et al. from the Royex House experiment. The BRS. recommendation seems to be independent of the mean wind velocity - which is clearly an important factor in Davenport's approach.

All of this points to the need for more field measurements, particularly in the determination of the cross-correlation of horizontal speeds in the free wind over terrains of different effective roughnesses (taking much lower separations, from 5-150 ft, say) and also, for similar measurements, of the cross-correlation, in the close proximity of large bluff structures (taking parallel planes 0-10 ft from the surface of the building, say, though this would depend on

\(^1\) with \(C = 8, \bar{V} = 24\)-ft/sec, \(n = 3\)-c/s, \(1/C.\bar{V}/n = 1\)-ft
\(^2\) See REF 12 (BRS. Digests 99 and 101).
the 'roughness' of the surface).

For the purposes of the design in question it was decided that a tower height of 30' 0" was a suitable compromise between the constraints of attempting a correlation of the wind over the whole structure for a frequency range of up to 2-4 c/s\(^1\), the desirability of dealing with a structure whose size has some practical significance, and the conditions imposed by other requirements such as those of manageability and dynamic behaviour.

2.1.2. Problems of Structural Symmetry

In general, the predominant velocity component of the free wind fluctuates randomly (both in magnitude and in direction) in the horizontal plane. A means of simplifying the analysis of the response of the proposed structure is to design it with continuous symmetry about a vertical axis such that the drag coefficient within any horizontal plane becomes directionally independent. Ideally, this means a structure of cylindrical shape which, in the case of a lattice structure, would necessitate the incorporation of curved members. This is clearly not a practical proposition, either from the point of view of achieving structural rigidity or of facilitating construction. It was considered a reasonable compromise to construct

\[ \text{FIG 2} \]

\(^{1}\text{See para. 2.1.3.}\)
a tower with a hexagonal section - formed of six uprights held apart by intermediate bracing (see plan in fig. 2). The geometrical width of the tower section varies from a maximum, D, to a minimum, 0.86D, with every 30° of rotation, and, assuming the bracing to be the same for each of the six faces, there are, for each direction taken, eleven other directions (marked 'a' in fig. 2) in which the aspect presented to the wind is the same.

Although it is assumed that the structure is of a sufficiently open nature for the total response to derive itself principally from the summation of the wind effect on individual members, it nevertheless remains true that there will be a certain amount of mutual interference between and 'shielding' of some members by others. For instance, it may be possible for members 1, 5 and 6 and the bracing 1-2 and 4-5\(^1\) to be 'shielded' (i.e. protected, to a greater or lesser extent from normal exposure to the free wind by being in the wake of windward members) from the effect of a wind impinging from direction 'A'; similarly, for direction 'B', members 5 and 6 may be shielded; for an intermediate direction, 'C', none of the uprights can be considered as sheltered. In an attempt to rectify the apparent imbalance in the shielding between directions 'A', 'B' and 'C', the geometry of the bracing has been devised such that, for direction 'B', the bracing on the windward face shields that on the leeward face while for direction 'A', the leeward face bracing experiences maximum exposure to the through wind (see fig. 9). Be this as it may, some variation in the solidity ratio\(^2\) with direction, with a

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\(^1\)Assuming the bracing to be in line with the centres of the uprights. The solidity ratio is defined as the ratio of the projected area of individual members of the lattice structure to the total area enclosed by the structure.
probable variation in the drag coefficient, is inherent in the design. Information on open structures, particularly that which relates to their aerodynamic behaviour, is very limited. However, certain facts do emerge to suggest patterns of behaviour. It seems that lattice structures consisting of sectionally circular members are much less prone to steady lift or side-drag (analogous to the continuous lift force experienced by an aerofoil) than are the equivalent structures constructed from sectionally non-symmetrical and flat-sided members.

For example, of five models of differently designed triangular section lattice antenna masts tested under steady wind-tunnel conditions by Pocock, the (steady) lift experienced by the three masts consisting of sectionally circular members was considerably less than the lift experienced by the remaining two masts consisting of members with flat-sided (angle and semi-hexagonal) sections. As a possible explanation of this it is suggested that for certain directions the aerofoil action of the wind on individual members may be additive toward the formation of an overall side-force on the structure. This is shown diagrammatically in fig. 3a, with the aid of stream-lines, for what would be expected to be the direction of maximum overall lift for the orientation and sectional shape of the members indicated (for the sake of simplicity, only the upright members are considered).

The equivalent case for circular members is shown in fig. 3b: as argued earlier, there should be random shedding of vortices, resulting in a null side-effect, (as there will also be in case - 1) but, unlike case - 1, this will not be associated with a steady lift force. The experimental data presented by Pocock, however, indicates that a small steady side-drag is experienced by the three round-membered models.

The angle of the wind direction at which maximum lift occurs (the same

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See REF 32: The Analysis of the Structural Behaviour of Guyed Antennae Masts under Wind and Ice Loading, Part II, Wind Loads by P. J. Pocock.
as that shown in fig. 3b) and the somewhat high solidity ratio (estimated as >0.75) suggest that, in fact, the models are acting partly as bluff structures, and that, therefore, the lift measured originates in the effect of the wind acting on the triangular section of the masts as a whole. This poses a problem of estimating the solidity ratio at which this effect becomes negligible. There seems to be very little

\[ \text{CASE-1} \]

\[ \text{CASE-2} \]

**FIG 3**

information on this particular aspect of lattice and open structures. For a single frame, Ower suggests that the summation of the forces on individual members is acceptable for a solidity ratio <0.5. Both this, what has been said previously, and the fact that if the proposed hexagonal tower were to act partly as a bluff structure the steady lift would be relatively much smaller than that for an equivalent triangular section tower (of the same solidity) - tending to act less as an aerofoil than as a vortex shedding cylinder - suggest that for the purposes of the present design no significant steady side-drag would occur if the solidity ratio were <0.5, assuming that circular members were used.

1. The lift may be transmitted to the whole section in the form of a torsion inducing couple dependent on the relative magnitude of the lift induced on individual members. The lift on 3 may be less than that on 1 and 2 due to shielding of 3 by 2.

2. See REF 17: *The Wind Resistance of Lattice Girder Bridges* by E. Ower.
In the same way it was felt that the solidity ratio condition was sufficient to prevent the proposed structure - acting as a bluff body - from shedding vortices. Hence in the structure as designed neither steady nor fluctuating side-drag phenomena were anticipated.

2.1.3. Dynamic Behaviour of the Structure.

A system with a flat response for the whole possible frequency range of applied dynamic loading is ideally suited to measuring the aerodynamic loading effect of the wind and is, in fact, the ideal aimed at in anemometric design. In practice, however, a real structure will show greater response at some frequencies of excitation than at others. The frequencies at which this happens - the natural frequencies - are those at which maximum energy is transferred to the system from the externally applied dynamic loading. A brief physical outline of this process is important at this stage.

Consider, firstly, a simple idealized system consisting of a concentrated or 'lumped' mass attached to a spring in which movement is permitted in one direction only - see fig. 4a. Subjected to an in-line impulse an initial velocity is transmitted to the previously stationary mass. But as the mass begins to move with this velocity it simultaneously begins to meet the (linearly increasing with deflection) resistance offered by the stiffness of the spring and so the mass undergoes a deceleration, eventually coming to rest, at which point the spring is in a position of being strained without being externally held in any way (possibly being deflected more, depending on the duration of the initial impulse, than if the impulse load had been applied statically). The system then attempts to regain its equilibrium, the

\[1\] I.e., a system whose response under a rapidly time-fluctuating load is the same as that were the same loading to be applied quasi-statically.
spring forcing the mass back, accelerating it back to its former position of equilibrium and restoring the initially imparted, though now directionally reversed, velocity – assuming that no frictional or other losses have been incurred – whereupon the half cycle described repeats itself in the opposite direction. The complete cycle will repeat itself indefinitely (with, it can be shown, a sinusoidally time-varying motion), the energy originally imparted – in a simple impulse-momentum transfer – remaining 'captive' in the system, but continuously changing in nature from strain (or elastic potential) energy (working against the stiffness of the spring) to kinetic energy (working against the inertia of the mass). The rate at which this internal energy transfer takes place – from strain to kinetic – clearly depends on the relative magnitudes of the stiffness and the mass and defines the natural frequency of the system. In the case of a real system the amplitude of the resulting 'free' vibration occurring at the natural frequency will gradually decrease as a proportion of the captive energy is lost in doing work against frictional, viscous and other forms of inherent resistance to motion which may be present. These non-recoverable forms of energy dissipation – which generally increase with amplitude – are generally known as 'damping losses'.

![Diagram showing dynamic load factor](a)

\[ n_0 = \omega_0 / 2\pi = 1 / 2\pi \sqrt{K/m} \text{ c/s} \]

![Frequency response curve](b)

max. response \( \omega_0 / 2\beta \text{ (viscous damping)} \)
If, instead of allowing the system to vibrate freely after initial subjection to an impulse a similar impulse is applied at the beginning of each subsequent or 'free' cycle of vibration, the energy imparted to the system builds up step by step, with a corresponding increase in the amplitude, theoretically resulting in, after an infinite time, an infinitely large oscillation (if no damping losses are present). The applied loading in this case can be considered as periodic, 'forcing' the system to vibrate, albeit at its natural frequency.

There is no reason why the transference of energy to the system should be restricted to the application of a discrete impulse at the beginning of each cycle. In fact, energy may be continuously fed to the system in the form of a continuous succession of impulse-momentum transfers. This leads to the consideration of sinusoidally applied loading which can be thought of as the envelope of a continuous succession of discrete impulses, each contributing to the overall response in the same way as described above. It can be shown for an undamped system that for a sinusoidal loading (whose frequency is that of resonance) the amplitude increases by a constant factor \((= \pi/2 \times \text{static deflection})\) with each successive oscillation which, again, theoretically, would lead to infinitely large oscillations after an infinite time. In practice, however, a finite amplitude is reached where the damping losses are so large as to dissipate all the 'extra' energy fed to the system\(^2\). If the frequency of the sinusoidal force is not quite the same as the natural frequency of the

\[1\] If the exciting force is \(P \cdot \sin 2\pi t\), the 'static deflection' is that due to force \(P\) applied statically.

\[2\] Here and throughout this discussion all motion will be assumed to take place within the elastic range.
system, the exciting force will periodically move out of phase with and oppose the previously built up oscillation resulting in a phenomenon known as 'beats'. This means that the motion about the natural frequency never fully develops, resulting in a smaller maximum response. For very low frequency excitation little momentum is imported to the system and the response is practically the same as if the force were applied statically. The variation of the maximum response attained by the system for different frequencies of sinusoidal excitation, $P \sin(2\pi f t)$, is shown in Fig. 4b. The static deflection due to $P$ (zero frequency) is taken as unity.

In physically possible or real structures most of the inertia is very often provided by the distributed mass of the strained members themselves. Approximating to one such structure by imagining it to consist of an equivalent assembly of spring-inter-connected finite elements of equal mass — each element having six degrees of freedom — two important factors emerge. Firstly, that since each surface element may be independently acted upon by an impulsive force, the total loading may be spatially distributed over the whole structure. Secondly, it can be reasoned that the structure taken as a whole will have a large number of ways or 'modes' of free vibration. For a cantilever the first few modes are shown in Fig. 5. The more complex the mode the higher is its associated natural frequency — this is easily understood if the behaviour between neighbouring node points in the higher modes is imagined to be similar in nature to that of the first mode development; although the mass per unit length and the flexural rigidity are the same as before, the length is shorter and so a higher frequency is anticipated. The extent to which motion in any particular mode develops depends entirely on the periodicity and spatial distribution of the applied loading, though for the same amount of total energy acceptance in any mode of
oscillation the mean square, $D_n^2$, (or RMS, $D_n$) deflection decreases with increase in mode ($n$). In other words, though the first mode deflection may appear to predominate, it does not necessarily follow that the highest energy acceptance, with the consequent maxima in shear and bending moment (the usual design criteria), is to be found within that mode.

*FIG 5*

It was considered that the incorporation of certain design features in the proposed tower structure would enable its behaviour to be essentially that of a 'fixed-free' or cantilever beam of uniform cross-section and uniformly distributed self-weight with a concentrated end load - at least for the first few modes of oscillation. That the width and general symmetry of the tower should be constant with height, apart from fulfilling certain aerodynamic conditions, was considered necessary in simulating constant stiffness. In order to further fulfil the cantilever-beam analogy it was thought desirable to ensure that the tower be reasonably slender (a slenderness

\[ \text{I.e., the mean square of the deflections summed along the length of the cantilever for the maximum modal deflection.} \]
ratio $\geq 15$ was considered suitable), that the height, $h$, of each lattice-work 'unit' be small compared to the overall height, $L$, of the tower, that welded joints be used throughout, and that the extremities of the tower be rigidly fixed to relatively stiff steel plates. The significance of the steel plates in helping the structure to behave as a cantilever is discussed in Appendix B. Of greater importance is the fact that the natural frequencies of the system can, to a certain extent, be controlled by judicious choice of both the base plate (the effective stiffness of which determines the extent to which the base acts as a spring) and the top plate (whose mass inertia acts as a retarding influence on the natural frequencies of the system). For this reason it was considered that the end plates should be bolted, not welded to the tower uprights so that if the need arose they could easily be replaced with others of different stiffness and mass.

\[ \text{FIG 6} \]

The equivalent cantilever system is shown in fig. 6. The system will have approximately the modes of free oscillation indicated in fig. 5. Whether a particular mode of vibration develops or not depends on the shape or the distribution of the applied loading in relation to the shape of the mode and whether there is a component of
Clearly a decrease in spatial correlation of the higher frequency gusts is favourable to the excitation of the higher modes and is implicit in the discussion concerning the various profiles shown in fig. 6a.

Clearly, if one of the initial conditions of the experiment is to be fulfilled - that the motion be that of a single-degree-of-freedom system - then only the first mode of vibration must be excited in the natural wind. Study of the reduced spectra in fig. 11A, Appendix A, will show that there may still be - at least for some site locations - considerable energy in the turbulence represented by the extreme right hand side of the spectrum, for frequencies of 2-4 c/s, say, at a wind speed of 100 ft/sec. (at a reference height of 10 m.). Extrapolation of the spectra would suggest that the energy level becomes negligible in all instances for frequencies greater than approximately 15-c/s. This suggests at once that, in view of the rise in the natural freq-

1In the vertical direction the joint acceptance is of the form proportional to \( \int_{0}^{Z_i} \int_{0}^{Z_i} \frac{V_z}{V_i} \frac{V_z'}{V_i'} \exp[-C|z-z'|] \phi(z) \phi(z') \, dz \, dz' \).

2This being relative to the size of the structure.
Frequency of a cantilever of approximately $n_0$ to $6.4n_0$ in going from the first to the second mode, the first mode natural frequency be fixed in the range 2-4 c/s. According to section 2.1.1., and using Davenport's correlation curve, this would mean the 'arriving' of gusts of 'effective width' 3-6 ft. in tune with the natural (first mode) frequency of the tower. Hence, for what is probably a conservative estimate of gust size, some resonant motion might be expected for a tower height of 30 ft. Because of the insignificant energy level and the very poor spatial correlation, at the second and higher modal frequencies, motion in the higher modes is not anticipated.

The motion of the system can then be considered as analogous to that of a single-degree-of-freedom system in two dimensions – see fig. 7c – (in which rotation of the base amounting to a free-end deflection, $\delta_{PB}$, is allowed for).

\[ K_{\text{total}} = \frac{K_T K_B}{(K_T + K_B)} \]

\[ \frac{1}{2} K_{\text{tot}} \]

\[ \frac{1}{2} K_{\text{tot}} \]

\[ (\text{elev}) \]

\[ (\text{plan}) \]

**FIG 7**

Theoretically, the stiffness of a hexagonal tower is independent of direction in the horizontal plane.

2.1.4 Continuity of Structural Properties
In order to fully understand the buffeting of the proposed structure in the natural wind, an accurate experimental investigation into the various parameters controlling its behaviour—stiffness, natural frequency, damping, and so on—is essential. It was considered most practicable in this instance to set up, preliminarily, the tower in a location where the above mentioned parameters could be most conveniently measured; and where transducers, eventually to be used in the measurement of the response of the tower in the wind, could be fitted, tested and calibrated. The tower would then be moved to an appropriate site (of sufficient exposure to the wind) where its response would be measured and recorded. In order that the parametric constants and the calibrations remain the same in moving from one location to the other, it is clear that certain provisions of continuity must be made—at the same time ensuring that the transfer is not too difficult an operation.

Apart from the more evident measures taken beforehand at the two sites, such as ensuring that the tower is truly vertical, the main criterion is that the conditions of support be identical. It was thought that the most satisfactory method of achieving this was to use a common base of sufficient rigidity and bulk to support the motion of the tower. To this base would also be fixed the transducers measuring the motion of the tower. By using bolts to fix the tower to the base, the system could be readily dismantled, transported in individual units, and reassembled—ensuring that bolt tensions, etc. are the same as before—whence the static and dynamic properties of the tower should be virtually the same as before.

It was considered that a 30 ft. tower and a 2-3 ton base would be within the capacities of normal transport facilities.
2.2. **Design of the Tower**

The principle design specifications for the proposed tower-structure are listed below; these are a direct outcome of the criteria discussed in the previous sections.

- **Tower height**: $30\text{\textprime\textquoteright\textquoteright'}\text{ O''}.$

- **Slenderness ratio** $= 15$, hence, mean width $= 2\text{\textprime\textquoteright\textquoteright'}\text{ O''}.$

- **Hexagonal cross-section**: six uprights with intermediate bracing

- **All members of circular (solid or hollow) section**.

- **Solidity ratio** $< 0.5$.

- **Welded joints**.

- **Structure bolted to end plates**.

- **Natural frequency** in range $2-4 \text{c/s.}$ (see APPENDIX B).

- **Provision for rigid base with holding-down bolts**.

The completed design was presented in the form shown in fig. 8. Figs. 9, 10, 11, 12, 13 show enlargements from this drawing. The various cross-sectional areas of members are given in APPENDIX B. For the purposes of comparison with design specifications certain photographic details of the tower as built are presented in APPENDIX C.
FIG 9  Detail from FIG 8: part of tower and base in elevation

2.0°%e

3.0°  (18° rad., 1/2" thick, steel plate).

9.5°x9.5° grid (from Fig. 19)
FIG 10 Detail from FIG 8 showing top half of tower

17" rad., 1/2" thick, steel plate.

6" dia., 3/8" thick, circular upright-end plates. The same at both ends of the tower. (see detail)

13.53° = distance from each start of curvilinear at centre of bar.

Note (for constructional purposes)

The 'basic' cross-sectional hexagonal shape of the structure is formed from two opposing equilateral Δs of sides 2.0" (all upright tubes).

5/8" Dia. (SOLID) STEEL

1 9/32" o.d. Dia., 8 SWG STEEL TUBING.
FIG 10A  3-D sketch showing alternate positioning of internal stiffeners
FIG 11 Detail of typical welded joint
FIG 12 Typical connection to end plates
The plan of the base plate showing the position of the 4 holes which are to accommodate the 1½" dia. (type "X") anchoring bolts (from test rig) and the ¾" dia. holes drilled in the intermediate plate (blue) allowing for bolt-heads.

8 rod tower base plate

Note: To avoid drilling holes in the test rig grid an intermediate (⅛" thick) plate has been inserted between the tower base plate and the grid. The bolts at Band E are reversed.

FIG 13 Plan of base plate showing holding-down bolt arrangement
CHAPTER 3.

Investigation of the Physical Properties of the Tower

Preliminary Tests
3. Investigation of the physical properties of the tower—Preliminary tests

3.1. Accessibility and other preliminary considerations

It was predicted in section 2.1.3. that if the tower possessed a first mode natural frequency in the range 2-4 c/s, it could eventually be analogized — as far as its motion resulting from exposure to the wind was concerned — to a single-degree-of-freedom, mass-spring system where the 'spring' stiffness, $K_{\text{total}}$, was equivalent to the overall stiffness of the tower (fitted to its base) defined as the inverse of the deflection per unit load applied at the top of the tower. The first consideration was to verify that the natural frequency of the whole system fitted with the base plate provided (18" radius x \( \frac{1}{8}'' \) thickness) fell within the specified range. (Were the natural frequency to fall outside this range the substitution of an accordingly thicker or thinner base plate would be necessary.)

The determination of the deflection, for a given load — and eventually for certain other experimental data, such as the calibration of transducers — necessitated access to the top of the tower, and for this reason a raised platform was built around the structure (in such a way that the supporting scaffold did not touch either the steel base of the tower or the tower itself) — see fig. 14. It was anticipated that the application of load to the top of the tower from this platform might result in some movement of the platform and scaffold. For this reason the structure was situated next to a permanent building in such a way that the deflection of the tower could be measured relative to a frame cantilevered from the side of the building — this is illustrated in fig. 15.

3.2. General physical properties

3.2.1. Measurement of deflection per unit load relationship
FIG 14  Tower set up for preliminary tests
FIG 15  Top platform showing independent 'datum' frame
A preliminary check\(^1\) established that the first mode (no other modal vibration was observed) natural frequency of the whole system was in the region of 3 c/s, thus satisfying the frequency range 2-4 c/s, condition discussed earlier. This permitted the determination of the stiffness, etc. of the system as it stood to be proceeded with.

(i) Method of load application

It was possible to apply a range of horizontal loads to the central point of the top plate by a straightforward procedure consisting of hanging weights from a nylon rope passing over a pulley system as illustrated in figs. 16, 17, 18. The pulley system was devised in such a way that it could be rotated round the tower allowing the application of load along any horizontal radius of the top plate.

(ii) Method of deflection measurement

A few preliminary measurements of the deflection of the top plate for a random selection of load directions established that, in general, the deflection was not coincident with the load direction - in other words, the overall stiffness of the system was not constant in the...

\(^1\)By placing a Cambridge vibrograph fitted with a low frequency attachment on the top plate of the tower it was possible to obtain some idea of the natural frequency of the system. The trace shown in the adjoining figure represents the motion of the tower system in the natural wind.
FIGS 16, 17 and 18 Details of load application and deflection measurement
horizontal plane, resulting in the observed unsymmetrical bending action. For a brief explanation of the statical properties of this phenomenon see APPENDIX D(i); the dynamical consequence is given in APPENDIX D(ii). Because of this certain care in the method employed in the measurement of the deflections was found to be necessary. Since substantially large deflections (within the elastic range of the system) could be produced, it was judged appropriate to make use of conventional dial gauges.

For a typical loading direction (P, say, at \( \phi \) to numbered direction '5') the deflection of the rim of the top plate was measured at four points, the complementary deflections \( \delta_A', \delta_B' \) in-line with the load direction and the corresponding lateral complementary deflections \( \delta_A, \delta_B \). — see fig. 19.

![Diagram](image)

**FIG 19**

In order to compensate for the curvature of the rim of the top plate small aluminium wedges with smooth flat external faces were attached to the rim at the points of measurement — see fig. 20.
The cantilever frame supporting the dial gauges was designed in such a way that with the use of magnetic bases the measurement of deflection of any point of the rim of the top plate was possible - see fig. 18.

(iii) Measurement of the deflection ellipse A typical load deflection relationship - the deflection being the mean of the dial gauge readings in-line with the applied load - is shown in fig. 21. The inverse of the slope of this (linear) relationship is equivalent to the stiffness of the system in the direction of the applied load. In order to construct the polar diagram of the deflection per unit load, it was found necessary to measure the deflections at every 5° interval along the rim of the top plate. Certain practical aspects concerning these measurements are presented briefly in APPENDIX E.

The deflection ellipse measured is shown in fig. 22. As an aid to understanding this diagram, the direction of the load causing a particular deflection (OA) has been plotted.

(iv) Preliminary conclusions The main conclusion to be drawn from the above measurements is that the tower structure as a whole behaves in a manner essentially similar to that of a cantilever having two mutually perpendicular axes of symmetry of different stiffnesses. Theoretically, this necessitates unsymmetrical bending about every other axis with the locus of the
IN-LINE STIFFNESS = 359.7 lbwt/in

FIG 21 Typical load-deflection relationship (for $\phi = 150^\circ$)
FIG 22 Polar variation of deflection per unit load
deflection per unit load following an elliptical path. By con-
structing the theoretical ellipse based on the measured deflec-
tions per unit load along the principal axes surprisingly good
agreement is reached with the ellipse measured entirely experiment-
ally. This means that the theory evolved in APPENDIX D(i) and
consequently that in APPENDIX D(ii) concerning the dynamic behav-
ior of the structure is valid in this case and, as will be shown
later, suggests a possible method for measuring the response to
wind.

It must be emphasized that although the ultimate behaviour
of the tower is the same as that of a cantilever beam of unsym-
metrical section, the fundamental causes of that behaviour are
somewhat different. In the former case the unsymmetrical bending
action arises not from unsymmetry of the tower section (as in the
latter case) but from unsymmetrical base conditions. Prelim-
inary indications of this derive themselves from the following
observations:

(i) As noted above, a preliminary investigation showed that the
natural frequency of the system was in the region of 3 c/s.
Since the computed minimum value was conservatively estimated as
2.9 c/s. (see APPENDIX B), this anomaly suggests that flexure of
the base plate must occur.

(ii) The hexagonal sectional nature of the tower is symmetrical
about more than one axis and as such the deflection per unit load
diagram is theoretically circular (not elliptical).

(iii) The major axes of the experimentally obtained ellipse

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1 In fact the two curves cannot be separated on the scale of fig. 22.
2 In the prototype tower certain imperfections in the symmetry
were noted but it was considered that these were not of suf-
ficient magnitude to warrant the ellipse measured.
virtually coincide with the mutually perpendicular axes XX and YY (see fig. 23) about which different base conditions prevail (brought about by superimposing a hexagonal on a square symmetry).

![Diagram of tower base with holding-down bolts](https://via.placeholder.com/150)

**FIG 23**

### 3.2.2. Unsymmetrical behaviour of tower base

In order to verify incontrovertably that flexure of the base plate was indeed responsible for the unsymmetrical bending measured it was considered that some direct measurements of the general strain pattern at the base of the tower were necessary. It was considered that the simultaneous measurement of strain in the six uprights near their point of attachment to the base plate would provide adequate information. Clearly the use of strain gauges was called for.

#### (i) Choice of strain gauges

In general, discrimination over a relatively small strain range (which was anticipated in this case) is more a function of the recording instrumentation used than the strain gauge

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1 Assuming symmetrical bending conditions a horizontal force of 100 lb. wt. applied at the top of the tower along axis XX (fig. 23) corresponds roughly to a force of ± 750 lb. wt. in each upright or roughly 40 microstrain units.
itself—assuming that the gauge is correctly bonded to the test surface, etc. In view of the possibility of making subsequent use of the strain gauges for dynamic recording, it was considered, however, desirable to achieve a relatively high output signal from the gauges. Reasonably large gauges, with an active length of $\frac{1}{2}''$ and half width grid, were therefore chosen.

Metalfilm (foil) constantan epoxy-fibrocellulose backed gauges (with leads) of the Budd-103 series type "for use with all cements" were used largely because of their relatively straightforward field application. The gauges were self-temperature compensated for steel (thereby dispensing with the need for 'dummy' gauges) and retained their performance characteristics within the temperature limits -70°C to 50°C. The gauges possessed the usual resistance, $120 \pm 0.2$ ohms, and gauge factor, $2.10 \pm \frac{1}{2}\%$.

(ii) Fitting of gauges, circuitry and calibration

The gauges were fitted vertically about 6" from the base plate, on the six uprights, with leads pointing downwards (see fig. 24 (i)). The fitting procedure consisted, briefly of the following operations:

1. Preparation of test surface (degreasing, abrading, neutralising).
2. Positioning and bonding of gauges and terminal strips.
3. Wiring up and leaving to cure.
5. Each gauge was incorporated into a classical full bridge circuit, one half of which constituted half of the internal strain gauge bridge (acting as an internal dummy bridge) of a

For more detail see REF 18, Foil Strain Gauges of the 101 Series, Bulletins 6001E, 619CE Budd Instruments (UK)
self-contained strain logger unit\(^1\); this unit also supplying the energising voltage to the external half bridge, and recording the imbalance of the active arm by means of a visual and printed record.

![Diagram](image)

**FIG 24**

In order to protect the data logger unit from the weather, it was situated in the laboratory building adjacent to the tower (as was all other electrical equipment used in these tests). This necessitated the running of long leads (25 yds.) to the strain gauges. Preliminary tests indicated substantial thermal effects on the leads exposed to the external weather conditions. For this reason, and in order to make use of the relatively fine potentiometric balancing facilities of the logger unit, the external half bridge was made up from a 3-wire, quarter bridge circuit incorporating the strain gauge (with the third wire distributing symmetrically the thermal effect in the two adjacent arms) and a dummy resistor, \(R_D\), (see fig. 24 (ii)). The size of the gauges warranted the use of the maximum energising voltage

\(^1\)Ten channel data logger type 10SL (Westland)
(10 Volts), giving a good compromise between stability and convenience of measurement. Prior adjustment and calibration of the gain of the logger's differential amplifier enabled the output to be printed directly in microstrain units.

Each of the six gauge circuits was fitted to a channel of the logger unit. Facilities for operation of the automatic mode, single scan control by remote control from the platform at the top of the tower were installed.

(iii) Strain measurements

Once having applied a given load, F, (at a given angle $\theta$ to direction '5') at the top of the tower it was possible for a single operator to trigger the data logger by remote control thereby obtaining a printed record of the resulting strain in the six uprights. By progressively increasing the load, F, it was possible to obtain the sort of relationship shown in fig. 25.

(iv) Conclusions

Fig. 25 verifies once more the linearity of the stress-strain relationship, at least for the load range in question. Theoretically, application of load in the direction shown would result in zero stress in uprights '3' and '6', and equal but opposite (in sign) stresses in '4' and '5', and in '1' and '2' respectively. This, however, assumes either a perfectly rigid base plate or a base plate whose flexural strength is the same about any diameter. In fact, neither conditions are met; the rigid base on to which the base plate is bolted introduces

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1 See fig. 6 in REF 18, Bulletin 6001E.
2 The calibration (given in the instruction manual for the data logger) was based on a corrected gauge factor, $K'$, allowing for the loss of sensitivity arising from the resistance of the long leads, viz,

$$K' = \frac{R_G}{(R_a + R_g) \times K} = \frac{120}{(120 + 4.5) \times 2.1} = 2.02$$

($R_G$, $K$, are the respective quoted gauge resistance and gauge factor)
FIG 25 Typical strain distribution in uprights resulting from free end load application
different conditions of support for the various uprights. In order to understand the nature of the resulting unsymmetrical bending, assume, firstly, that due to the load, P, symmetrically applied vis à vis the hexagon, an equal thrust develops on the base plate from uprights '4' and '5'. The resistance offered to '5' is the relatively large rigidity of the base, whereas the resistance offered to '4' is merely the flexural rigidity of the base plate cantilevered from the base and for this reason the abutment at '4' will subside slightly in relation to '5'. This 'subsidence' creates an overall deflection of the tower in a direction which is laterally perpendicular to that of the load application (3 will also subside slightly) such that with the main deflection due to the strain of the tower in the direction of the load a resultant deflection in some intermediate direction '6', say, will occur. This state of affairs is verified in fig. 25, with '3' experiencing compression, '6', tension, and '4', though deflecting more than '5', showing less strain than '5'. A similar behaviour is noted in the uprights experiencing tension: while the abutment '2' is held down by two bolts, '1' is held down by only one, resulting in '1' showing less tensile strain, but a greater movement or deflection than '2'.

3.3. General conclusions

One of the original criteria concerning the behaviour of the tower structure was that it possess constant stiffness about any horizontal diameter. In fact, because of unsymmetrical base conditions, this has been shown not to be the case. The system, however, follows a definite pattern of behaviour which

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1 This is in fact a simplification of what happens; some stress redistribution due to the effect of the bracing at the base would also occur, as well as some subsidence in the direction of the load.
does not at all complicate the problem and which, in fact, makes it of greater general validity. It is considered that either because of structural flaws, general structural unsymmetry, or non-uniform base conditions most tower structures behave, to a greater or lesser degree, in a similar manner to that of the tower considered here.

Sufficient information has been obtained to enable a reasonable prediction of the dynamic behaviour of the structure to be made. (See APPENDIX D(ii)). Because of this it was decided to design and proceed with the installation of the transducers measuring the dynamic response of the structure. The general pattern of the predicted behaviour could then be verified and measured.

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CHAPTER 4.

Instrumentation Used in Measurement of Response
4. Instrumentation used in measurement of response

4.1. Introduction

From what has been said in APPENDICES D(i) and D(ii) it is clear that any vibrational or quasi-static motion of the tower is a combination of two mutually perpendicular 'one-degree-of-freedom' motions. The foregoing chapter was principally devoted to the experimental determination of the deflection per unit load ellipse, the principle axes of which being those about which the motions described above take place. Measurement of the response along these axes thus offers a relatively simple means of measuring the somewhat complicated two dimensional lateral behaviour of the tower.

4.1.1. Selection of transducing element

It is not proposed to discuss here in detail merits of the vast range of response-measuring devices which are in current use. It was decided to avoid seismic or mass-spring transducers on the grounds that the response of these instruments drops significantly as the frequency of excitation approaches zero\(^1\). Apart from the more unorthodox instrumentation such as optical tracking methods this restricted the selection of a transducing element to fixed reference instruments (where one terminal of the instrument is fixed at some point in space and the other is fixed to the vibrating body). Further consideration of the low frequencies involved (approximately 0-3 c/s.) led to the conclusion that the measurement of displacement (as opposed to velocity or acceleration) was, in the case in point, the only type of response measurement practicable. In addition, the method of data analysis to be employed necessitated the obtaining of permanent,

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\(^1\) Low frequency variations in the deflection of the tower exposed to wind lasting several minutes or more were anticipated.
legible records of the response, a condition clearly favouring the use of a transducer with an electrical output allowing transmission of the signal over substantial distance to a sheltered environment where amplification and recording could be carried out.

The problem of providing a fixed reference point of attachment for the transducer is made difficult here because of the nature of the experiment to be performed. Two basic conditions have to be satisfied: that,

(i) although the tower is to be left free to respond in the wind, the reference point must remain stationary and unaffected by the wind.

(ii) Satisfying the above condition by the use of massive supports must not be detrimental to the exposure of the tower to the natural free-flow of the wind (by shielding effects, generation of vortices, increasing turbulence, etc.).

The above conditions, at first sight, seemed to preclude the possibility of taking any measurements at all. Further, consideration, however, on the nature of both the quasi-steady and dynamic response of the tower provided a compromise solution.

It must be remembered that observations to be taken are under strong to gale force wind conditions.
Fig. 26 shows, in elevation, a typical deflection profile (exaggerated) which, it was anticipated, the tower system would take in free vibration along one of the axes of proposed measurement.\(^1\)

The neutral position is 00. The modal shape, CD', will comprise a linear mode, OC', resulting from rotation of the base, together with the mode of the tower in flexure (considered as a fixed-free cantilever with an end load). It follows that knowledge of the deflection, \(\delta_H\), at some intermediate height, \(H\), will provide, after performing suitable operations, a full knowledge of the profile shape including the deflection of the top plate, \(\delta_L\). This assumes that the proportional contributions (from flexure of the base plate and from flexure of the tower) to the deflection, \(\delta_H\), are known.

The magnitude of the deflection, \(\delta_H\), relative to the displacement at the top of the tower, \(\delta_L\), will depend on the relative values of \(H\), and \(L\), the total height of the tower. Establishing the position of the fixed reference point was therefore a compromise between the necessity for obtaining tower displacements in the natural wind of measurable proportion (this depending on the sensitivity of the transducer used) and the need for maintaining the tower exposed to the natural wind as far as possible without obstruction from the necessarily bulky supports.

In view of the above considerations, and other considerations such as the necessity for reasonably good resolution over the relatively narrow displacement limits anticipated (eliminating, for example, the possibility of using variable inductance transducers)\(^2\), the use of either of two basic types of transducing elements was

\(^1\)This profile will be slightly different for quasi-steady response to various loading conditions - see APPENDIX F.

\(^2\)This is only true in general; certain specialized transducers of this type can, through a large multiplicity of windings, achieve good resolution.
seen to be feasible. These were:

(i) capacitance-type transducers

(ii) displacement cantilever transducers using bonded strain gauges.

Although either of these transducers could have performed the required function, the second was chosen because of the relatively straightforward auxiliary equipment required. The capacitance-type pick-up though simple and robust in itself necessitates the use of rather complex auxiliary equipment.

4.1.2. Description and Design of Transducer

The general idea behind the type of transducer to be used is that movement of the test member deforms an auxiliary mechanical member, the strain of which, linearly proportional to the displacement (within the elastic range of the material), is picked up by strain gauges bonded to that member. The salient features of the transducing system designed for measurement of the tower displacement are shown in fig. 27.

![Diagram of transducer system](image.png)
The vibrational or quasi-steady motion of the tower is transmitted laterally through a coupling rod to the free end of a fixed-free cantilever on to which two strain gauges (doubling the sensitivity, further enhancing compensation for temperature changes and cancelling axial strains) are bonded. The purpose of the universal joints is to permit rotation not only in the vertical plane of motion but also in the horizontal plane. It must be remembered that the tower moves laterally about two mutually perpendicular axes. Rotation of the coupling AB about B prevents (to a degree) the transference of one lateral motion to the other — see 5.1. for further discussion.

The position and various dimensions of the transducers were determined on the basis of certain preliminary observations (using dial gauges). It was noted that at a height of approximately 4' 6" from the base of the tower the application of a lateral load of 20 lbs. at the top of the tower produced deflections along the principal axes of approximately 0.0015" and 0.0030". The calculated surface strains at \( \frac{1}{2} \)" from the fixed end of a 4" 'spring steel' cantilever (of \( \frac{1}{16} " \times \frac{1}{4} " \) rectangular cross-section) for a free-end deflection of 0.0015" is approximately 15 microstrain units. In view of the auxiliary instrumentation to be used and the loading range anticipated in the natural wind (of the order of 0-50 lb, equivalent top load) the sensitivity produced by such a cantilever was considered suitable. One reason for not choosing a cantilever producing a much larger strain sensitivity for a given free-end displacement was the necessity for minimizing the loading effect of the transducer on the tower; for the system as designed the transducer exerted a load in the proportion of \( \frac{1}{4} \) lb. for a deflection of 0.0015", \( \frac{1}{4} \) lb. for a deflection of 0.0015",.
a load which, at a height of 4' 6"", was considered as producing negligible effects. The strain gauges used were of the same type as those used in the earlier base-strain experiment (see 3.2.2.(i)); similar bonding, weather-proofing, etc., techniques were used. Details of the transducer and its support are shown in figs. 28, 29.

4.2. Response and Performance of Transducer to Quasi-steady Tower Deflections

4.2.1. Instrumentation and Circuitry

In order to test the functioning and suitability of each of the two transducers under static displacement conditions, the same equipment and circuit as that shown in fig. 24 (3.2.2.) was used with the exception that in this case the external half bridge was made up of two active arms - each arm corresponding to one of the transducer strain gauges. The logger unit was adjusted to read directly in microstrain units (using a corrected gauge factor of 1,866\(^1\)).

4.2.2. Calibration

The transducer was calibrated by noting the output in microstrain units resulting from a range of input displacements. Care was taken to ensure that measurement of the displacements was in fact correctly located on the transducer, corresponding to the point of reception from the tower; to ensure this, a direct

\[^1\] The resistance of each of the 150' 0"" transducer leads was measured at 15 ohms whence the effective gauge factor, 
\[ K' = \frac{120}{120 + 15} \cdot K = 1.866. \]  
\( (K = 2.1). \)
FIGS 28 and 29  Installation of transducers along axes of principal M of I
connection by threading the shaft of the dial gauges used with the transducer universal joint was made. The strain-deflection relationship was identical for the two transducers built and is given in fig. 30.

After calibration the transducers with their supports were installed in their allotted positions at the base of the tower - along the axes of principal moment of inertia as found in CHAPTER 3. See fig. 22. (It will be noted that a protective coating of packing grease was applied to the transducers.)

In order to verify the predicted behaviour of the transducers as fitted to the tower, a further test was effected, consisting of noting the two transducer outputs (in microstrain units) for a range of lateral top-loads and load directions. The 'load-transducer output' relationship measured is given, firstly, for application of the top-load along the principal axes in fig. 31. The slopes of these and the similar linear relationships obtained for intermediate load directions are plotted in fig. 32.

4.2.3. Conclusions from Static Transducer Tests

The ostensible conclusion which can be drawn from the results of the above tests, in particular, that relating to fig. 32 is that the transducers respond only to the components of tower displacement which lie along their respective axes of sensitivity\(^1\), and do so linearly (at least for the displacement range in question). Both this and the excellent consistency of the experimental observations obtained were considered as clearly demonstrating the suitability of the transducers.

\(^1\)Possible error inducing exceptions are discussed in 5.1.
FIG 30 Calibration of transducers in terms of applied displacements.
Horizontal free-end load, \( P \), in lb wt.

**FIG 31** Calibration of transducers in terms of loading applied to tower.

- **AXIS-1 (min. stiffness)**
  - Sensitivity = 1.28 μ-strain/lb wt.

- **AXIS-2 (max. stiffness)**
  - Sensitivity = 0.625 μ-strain/lb wt.
FIG 32 Effect of rotation of free-end load on tower on transducer sensitivities
The calibration charts obtained in the tests above (fig. 30 and 31) enable an accurate determination of the relative contributions to the deflection/unit load ellipse (from flexure of the base plate and flexure of the tower) to be made by providing a knowledge of the deflections of the tower at its free-end and at the point of transducer measurement for a given lateral top-load. A typical calculation in the 'break-down' of the ellipse is given below.

From CHAPTER 3, the deflection per unit load at the free-end of the tower about the axis of minimum stiffness (AXIS 1) is,

\[(A_p)_1 = 0.00323 \text{ in/lb.}\]

From fig. 31 for AXIS 1, the transducer output is, for the application of a unit top-load to the tower,

\[1.28 \text{ microstrain/lb.}\]

From fig. 30 the displacement experienced by the transducer for a unit strain is,

\[0.000128 \text{ in/microstrain.}\]

From which, the displacement of the tower at the transducer for a unit top-load is,

\[(a_p)_1 = 1.28 \times 0.000128 = 0.000165 \text{ in/lb.}\]

Using the expressions developed in APPENDIX F (CASE 1), the displacement per unit load at the top of the tower, due to flexure of the tower alone is,

\[(A_{PT})_1 = K \left[(A_p)_1 - (a_p)_1 \times Z_1/Z_2 \right]/\left[K - Z_1/Z_2 \right]\]

where \(K = \frac{Z_1^3}{3Z_2^2} \left(Z_1 - Z_2/3 \right)\).

With \(Z_1 = 368 \text{ in.}\) and \(Z_2 = 51 \text{ in.}\), and the values of \((a_p)_1\), and \((A_p)_1\) found above,

\[(A_{PT})_1 = 0.00254 \text{ in/lb.} \quad \text{(or a stiffness of 393 lb/in).}\]

The deflection per unit load ellipse is shown in fig. 33.
FIG 33  Deflection ellipse showing contribution (shaded) from flexure of base-plate
The shaded portion represents the deflection due to flexure of the base plate. It is to be noted that the stiffness of the tower alone is constant with direction.

The following problem remains: given that the tower deflects under quasi-steady wind loading and produces a transducer reading in microstrain units; what is the effective wind load?

First of all, it is known from fig. 31 what lateral load \( P \) applied to the top of the tower would produce the given transducer strain. Hence the wind load can be imagined as an 'equivalent' top-load 'P'. Knowing 'P', the actual total wind load component, \( W \), can then be calculated from the conversion (see APPENDIX F(CASE 2)),

\[
W = \frac{Z_2/Z_1(2n+2)[(A_{PB})^1 + (A_{PT})^1]3Z_2/2Z_1^2(z_1-z_2/3)}{(A_{PB})(2n+1)Z_2/Z_1 + 3(A_{PT})[(2n+1)(Z_2/Z_1)^2 - (n+1)(Z_2/Z_1)^3 + 1/(2n+4)(2n+3)(Z_2/Z_1)^{2n+4}]
\]

where \((A_{PB})^1, (A_{PT})^1\), are the constituent deflections per units load from the ellipse in fig. 33 and \( n \) is the wind profile power index.

The above expression has been plotted for the two transducers for varying \( n \) values in fig. 34. Given a particular wind speed at the top of the tower (corresponding to a wind pressure, \( p_{z_1}/D \) per unit area, where \( D \) is the width of the structure) the equivalent free-end load 'P' is plotted for various profile shapes, in fig. 35.
FIG 3.4 Relationship between calibration load and load distributed in accordance with the wind speed profile.
FIG 35 Effect of profile shape in terms of calibration load, $P$, for a given pressure at top of structure.

P and the load distribution characterized by a pressure $p_{Z_1}$ at height $Z_1$ produce the same displacement at height $Z_2$. 

Calibration load, $P/p_{Z_1}Z_1$
4.3 Dynamical Behaviour

4.3.1. Selection of Equipment for Dynamic Strain Measurement

The tests described in the previous sections have been restricted to the obtaining of experimental data under static conditions, with, for instance, arbitrary time delays between successive observations. In order to effect a detailed study of the dynamical motion of the tower, however, it was considered necessary to obtain some form of legible oscillographic (uninterrupted) record of events. A brief review of oscillographic recording equipment indicated that this type of instrument requires substantially larger inputs (of the order of millivolts) than the sort of output anticipated from a strain-gauge transducer (which is of the order of microvolts). Clearly, some sort of intermediate amplification was necessary. Further consideration of the frequency range (0-3 c/s.) of the signal to be amplified and other factors such as the possibility of errors arising from the generation of thermo-electric emf's somewhat disfavoured the use of direct current to power the strain gauges. In any case, D.C. amplification is difficult. For this reason it was considered justifiable to use a carrier-wave system with the input transducer operated as part of a bridge current. The equipment used consisted of a standard commercial carrier-frequency amplifier, incorporating all the basic elements of a strain gauge instrumentation system, (such as Wheatstone bridge balancing facilities, stabilised a.c. power supply, etc.) and a visicorder oscillograph.

\[\text{For a displacement of 0.0015" the output of the transducer described in 4.1.2. was estimated to be about 75 microvolts (using a bridge voltage of 5V.).}\]
4.3.2. Description of Equipment and Circuitry

The basic operation of the system is best understood from a block diagram - see fig. 36.

The oscillator unit (A) supplies the bridge with a stable a.c. voltage. Consider, firstly, the instance when there is no strain in either of the strain gauges. Since strain gauges normally have a certain capacitance a phase shift between the a.c. voltages in the arms 'ab' and 'ac' (say) will occur. This means that though the amplitude of the voltages may have been equalized by the variable resistance, R, there will still be a resultant voltage across the bridge output 'bc' (corresponding to the subtraction of one wave from the other). In order to eliminate the phase difference from the whole bridge adjustment of the variable capacitor, X, is made. The bridge is then balanced with zero output voltage. If the bridge is now put out of balance by inducing strain in the gauges, a small a.c. voltage with amplitude proportional to the strain will appear across 'bc' in the form of

FIG 36
a modulated wave and be fed to the amplifier (D). The amplified voltage is then fed to a phase-sensitive rectifier (E), which converts the a.c. voltage signal into the appropriate d.c. voltage, the sign of which depends on the phase difference between the signal about to be demodulated and the original a.c. oscillator signal.

The carrier amplifiers used (HONEYWELL, single channel, TYPE 2506) possessed a recorder socket output impedance < 200 ohms, which made them suitable for direct connection to HONEYWELL BB-TYPE miniature mirror (TYPE BB 250A) galvanometers from which (using a HONEYWELL Visicorder (U.V.) oscillograph (TYPE 1705) a trace of the carrier-amplifier output signal on photosensitive paper could be obtained.

Duplication of the system as described above enabled the simultaneous, side-by-side recording of the lateral displacements along the mutually perpendicular 'principal' axes of the tower at the height of the transducers. As an extra visual aid monitoring of the two signals was carried out with the aid of an oscilloscope.

As anticipated considerable electrical interference was observed in the system when first set up. Decoupling of the two carrier-amplifiers (so as to reduce any possible cross-talk or beat effects) by operation the two units from the same oscillator did not appear to improve the situation. In fact most of the noise was attributable to pick-up from the exceptionally long leads making up the external strain gauge half bridge (three leads at 150° O°). Under normal circumstances this would have

\[^{1}\text{correctly damping the galvanometers, for optimum flat frequency response.}\]
warranted the use of cables with double shielding, but, in this case, in view of the low frequency range to be measured (approximately 0-3 c/s.) and the fact that the noise was of a high to audio frequency range, it was decided to retain the use of the single shielded cables in conjunction with a low pass filter consisting of a shunt capacitor across the oscilloscope output terminals of the carrier-amplifier. In order to further suppress any interference care was taken to avoid ground loops by avoiding multiple grounds on the signal circuit. The set up which gave the most satisfactory performance is shown in fig. 37.

\[\text{FIG 37}\]

4.3.3. Calibration

Because of the accessibility existing to the top of the tower it was possible to calibrate the entire system in one operation (circumventing the difficulties of combining the characteristics of all the system components) - although, it must be admitted, the calibration depended on there being perfectly windless conditions. The calibration was performed by applying along each of the principal axes in turn a lateral load to the top of the tower and noting, at the output end of the system, the corresponding shift of the U.V. trace on the photo-sensitive
oscillograph paper. As it was intended to accommodate at least four legible traces on recording paper only 120 mm. wide it was considered essential to be able to adjust the size of the response fluctuation in the wind to the proportions of the paper. For this reason calibration of the system was performed for a whole range of amplifier gains. A typical load-deflection calibration for a low gain (94% of complete attenuation) is given in fig. 38. All observations in the 85%-100% attenuation range were remarkably consistent and showed good stability with time (the oscillograph trace showing either a positive or negative drift of less than about 2% of the total deflection in a 30 min. time interval). For higher gains, however, the stability of the system deteriorated somewhat and considerable drifting was observed — although, the 'instantaneous' load-deflection relationship remained linear. The sensitivity of the system (defined here as the trace deflection — in mms. — per unit lateral load applied to the top of the tower — in lbs.) corresponding to the inverse of the slope in fig. 38 was plotted for the whole range of amplifier gains (0-100) and is given in fig. 39. An enlarged picture of the working range (85-100% attenuation) is given in fig. 40.
Load, $P$, applied about each principal axis in turn

**FIG 38** Typical calibration at a given carrier amplifier attenuation
Drift observed over 1 minute.

Sensitivity: Displacement of Visicorder UV trace in mm. per unit load, P, in the system.

Fig. 20: Sensitivity of transducer systems for whole range of carrier amplifier attenuations.
FIG 40 Shaded portion of FIG 39
CHAPTER 5.

Dynamical Properties of Tower Structure
5. **Dynamical Properties of Tower Structure**

In order to verify that the performance of the system described (in CHAPTER 4) was satisfactory under the dynamical conditions imposed by the tower structure, certain simple tests were carried out. From the results of these tests were deduced certain important dynamical properties and characteristics of the structure.

5.1. **Performance of Instrumentation**

By subjecting the tower to a sudden force or impulse and allowing the subsequent motion to decay naturally, the sort of trace shown in fig. 41 was obtained. When the instrumentation was first set up, however, the decay traces obtained were of the type shown in fig. 42. The traces shown in this figure represent the simultaneous decay of the free vibration of the tower (after initial excitation by a lateral impulse applied at approximately 45° to the principal stiffness axes of the tower) as picked up by the two transducers. A brief study of these traces will indicate that both of the motions are amplitude modulated and that the frequency of the modulation (approximately 0.3 c/s.) corresponds to twice the 'beat' frequency which would be expected from superimposition of the two main decay vibrations recorded (approximately 3 and 3.3 c/s.)\(^2\). Clearly this indicates some parasitic

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1 Each horizontal marking corresponds to a time interval of 1 sec.
2 Superimposition of two signals, \(a\sin wt\) and \(a\sin(w + \delta w)t\), where \(\delta w\) is small results in the motion, \(2a\sin(w + \delta w/2)t\cos\delta w/2.t\). This can be imagined as the reproduction of the original signal, \(a\sin wt \rightarrow a\sin(w + \delta w/2)t\), modulated by a low frequency wave, \(a\cos \delta w/2.t\).
FIG 42  Free vibrational decay in tower as picked up by transducers in slight misalignment with axes of principal M of I

FIG 41  Decay of free vibration in tower as picked up by correctly aligned transducer
interference between the two theoretically independent predominant motions of the structure. It was found that by altering the axes of sensitivity of the two transducers, the effect of the modulation could be either increased or decreased.

![Image of a diagram showing axes of principal M of I, vibration, a Sinωt, error, E = r - (r^2 - a^2)\frac{1}{2}, axes of principal moment of inertia of the tower system, and motion along the transverse axis alone.]

\[
E = r + a \cdot \sin\theta \pm \sqrt{(a \cdot \sin\theta)^2 + (r^2 - a^2)}
\]

(a) Correct alignment; (b) Incorrect alignment of transducer

Fig. 43 demonstrates two ways whereby the signal from one transducer may be affected by motion perpendicular to its intended axis of sensitivity. Fig. 43(a) shows a transducer correctly aligned with one of the axes of principal moment of inertia of the tower system. Consider motion along the transverse axis alone. To a displacement 'a', the transducer will respond by an amount 'E', which is dependent on both 'a' and 'r'. Because the displacement 'a' varies sinusoidally with time, and, in fact, represents the free vibration of the system along the transverse axis and because the induced error 'E' is independent of the sign...
of 'a', 'E' will also vary sinusoidally but with a frequency twice that of the transverse motion. For motion along the two principal axes occurring simultaneously, both transducer signals will be modulated by the sinusoidal variation 'E'. In fact the amplitude of 'E', expressed as a percentage of the amplitude 'a' for a correctly aligned transducer is exceedingly small (being less than 0.2% for a transverse free-end tower deflection of 1 in.)

It was found that practically all of the parasitic modulation was attributable to the incorrect alignment of the transducers - see fig. 43(b). The error produced in this case was found to be considerable, even for a small misalignment (theoretically being of the order of 5% of the transverse amplitude for a misalignment of $\theta = 3^\circ$) and accounted for the modulation shown in fig. 42.

By a process of trial and error the alignment of the transducers was corrected until the decay traces indicated no modulation. The traces obtained then were of the type shown in fig. 41.

It was possible to reconstitute the entire lateral motion of the tower in visual form by connecting the output of one transducer to the vertical axis of an oscilloscope and connecting the other to its horizontal axis (with the sweep switched off). By exciting the tower structure along both of its axes and then allowing the motion to decay naturally, the sort of pattern shown in fig. 44 was produced. For the sake of clarity only about half of the complete 'beat' cycle (the period of the whole cycle being approximately six seconds) is shown in fig. 44. The difference which will be noted between the lengths of the diagonals

1Although, in this case, the orientation of the principal axes was known with some accuracy a priori, the principle involved here suggests a simple method for accurately determining the orientation of the principal axes and thereby eliminating the need for an accurate determination of the stiffness or deflection/unit load ellipse.
FIG 44  Horizontal motion of tower in free vibration

FIG 45  Wind induced motion of tower
of the (Lissajous) figure correspond to the decay in the free vibrations of the structure as a whole during a half beat (3 secs.). This explains the diamond shape of the Lissajous pattern.

Fig. 45 shows a typical visual record of the lateral motion of the tower subjected to natural wind pressures - it is to be noted that the basic Lissajous pattern made up from resonant vibrations of the tower, though superimposed on quasi-static deflections, is nonetheless maintained.

5.2. Decay of Free Vibrations: Natural Frequency and Damping

For a whole range of initial displacements records of the decay in the free vibration of the tower were obtained. Fig. 41 shows a typical trace. From these traces was gauged the natural (first mode) frequency and the damping characteristics of the tower about each of its principal axes. The natural frequencies were measured as:

\[(n_0)_{axis1} = 2.97 \pm 0.01 \, \text{c/s}\] (about axis of minimum stiffness)

\[(n_0)_{axis2} = 3.29 \pm 0.01 \, \text{c/s}\] (about axis of maximum stiffness)

From preliminary study of the damping traces obtained, it was thought that the apparently exponential decay was due almost entirely to viscous damping (i.e. aerodynamic damping of the tower in still air). However, an attempt to measure the logarithmic decrement of the traces obtained showed consistent variations in the values obtained (decreasing with amplitude). This, it was estimated, was indicative of the presence of an additional form of damping. By making certain measurements of the decay envelopes it was firmly established that in addition to viscous damping (varying proportionately with the amplitude of vibration)
there was also present damping of a frictional nature - that is, damping consisting of a constant frictional force, independent of amplitude, but always acting in a direction so as to oppose the motion. The case of viscous and frictional (or Coulomb) damping combined is considered in APPENDIX G(ii).

Using the expressions developed in this Appendix, the damping along each axis was determined as follows. The values obtained showed reasonably good consistency (± 5%) for twenty-three recorded decay traces obtained. (The definition of critical damping for viscous damping is given in APPENDIX D(ii).)

(i) Damping about Axis-2 (axis of maximum stiffness)
The viscous damping was obtained from the decay traces using the expression:

\[ \beta = \frac{(n_0)_{\text{axis 2}}}{(n - m)} \ln \frac{(x_n)_T}{(x_m)_T} \]  
(see APPENDIX G (ii))

Values for a typical calculation are:

\( (x_n)_T = 0.85 \text{ cm.}, \) \( (x_m)_T = 0.275 \text{ cm.} \) \( \text{for } T = 40 \text{ cycles} \)

\( (n - m) = 75 \text{ cycles} \)

Whence,

\[ \beta = 3.29/75 \ln(0.85/0.275) = 0.0494. \]

The mean of all values so computed was (for motion about axis 2),

\[ \beta_2 = 0.0498 \]

or, expressed as a percentage of critical damping,

\[ \frac{c}{c_r} = 0.24\% \]

\( (c/c_r = \beta/2\pi(n_0)_{\text{axis 2}}) \)

The frictional damping was computed from the expression,

\footnote{These traces covered a range of different initial displacements.}
\[ F = K \left[ x_{N(n+1)} \cdot x_p \cdot \exp\left(-\beta / \omega_x'\right) \right] / \left[ 1 + \exp\left(-\beta / \omega_x'\right) \right] \]

The frictional damping was computed for displacements (e.g., \( x_{N(n+1)} \)) at the free-end of the tower; a typical calculation with \( \beta = 0.049 \) is given below:

Using the calibration curves (CHAPTER 4), an amplitude of 1.49 cm. on the recorded trace corresponds to a free-end tower deflection of \( 1.49 \cdot 0.00262 / 0.045 = 0.0865 \) in. (amplifier attenuation = 98.2%).

The slope of the decay envelope (at an amplitude \( x_p = 1.49 \) cm.) was measured as 0.68 cm. over 40 cycles, whence, \( x_{N(n+1)} = (x_p + 0.68/80) = -1.4985 \) cm. (APPENDIX G(ii)), and

\[ F = \frac{0.00262 \left[ (-1.4985 + 0.9925,1.49) / 1.9925 \right]}{0.00262 / 0.045} = 0.219 \text{ lb. wt.} \]

The mean of a number of values calculated in this way was

\[ F = 0.22 \text{ lb. wt.} \]

In order to establish some valid comparison between the magnitudes of the two types of damping, the viscous damping may be considered as that of the equivalent mass-spring system, of mass \( (M + 0.23m) \). The viscous damping force exerted on the motion of the system is proportional to the velocity, the constant of proportionality being \( C_2 = 2(M + 0.23m)\beta_2 \) (see APPENDIX D(ii)).

For axis-2,

\[ C_2 = 2 \left[ 146.5 + 0.23,836.7 \right] 0.0498 / 6 = 1 \text{ lb/ft/sec.} \]

Both viscous and frictional damping may then be represented graphically on the same diagram – see fig. 46. As an aid to further understanding of this diagram the amplitude of vibration (at the free-end of the tower) is also indicated. (For a particular amplitude of vibration, \( A \), say, the mean or average velocity of
the mass will be the distance covered in one complete cycle, \(4 \times A\), divided by the time taken \((2\pi/\omega)\), that is: \(2A\omega/\pi\).

(ii) Damping about Axis - 1 (axis of minimum stiffness):

By proceeding in a manner similar to that adopted above, it was established that the apportioning of the damping about the axis of minimum stiffness was,

Viscous damping: \(c/\omega = 0.38\%\)

or,

\[\beta = 0.071\]

and, Frictional damping: \(F = 1.0\) lb.

A similar diagram to that of fig. 46 may be drawn - see fig. 47.
5.3. Theoretical Determination of Natural Frequency

It was considered that comparison of the natural frequencies of the system obtained, on the one hand, on a theoretical basis using the experimentally obtained statical parameters of stiffness and mass distribution, and, on the other hand, from direct observation, would enable the assumptions concerning the distribution of mass and stiffness made so far to be verified. It was shown in APPENDIX F(iii) that the (first mode) natural frequency of the tower considered as a fixed-free cantilever (with no flexure of the base) was

\[ \omega_1 = \left( \frac{a^4 z_1^4 K}{3m} \right)^{\frac{1}{2}} \] ..........................(i)

where \( a \) is the lowest root of the expression,

\[
\left( \frac{I_m}{m} \right) \frac{a^4}{z_1} \sin a z_1 \cos h a z_1 - \left( \frac{M}{m} \right) \frac{a^3}{z_1} \cos a z_1 \sin h a z_1 - (I_m \frac{M}{m} \frac{a^4}{z_1} + 1) \cos a z_1 \cosh a z_1 = 0 \] .................(ii)

This equation allows for rotation of the mass at the free-end of the cantilever. The mass moment of inertia of the circular plate used in the tower-structure is:

\[ I_m = \frac{M d^2}{16} = 146.5 \text{ lb.in.}\sec^2 \]

and since it is known that \( \omega \) is in the region of 20 radian/sec. (3.3 c/s), which from (i) means that \( a \) is in the region of 0.0045, it is easily verified that the contributions from rotation of the plate are negligible\(^1\) and may therefore be ignored here. Equation (ii) reduces to

\[ \frac{M}{m} \frac{z_1 a}{z_1} \left( \sin a z_1 \cosh a z_1 - \cos a z_1 \sinh a z_1 \right) = (1 + \cos a z_1 \cosh a z_1) \] .................(iii)

By a process of successive iteration the lowest root of this equation is/\(^1\)

\(^1\)Viz, for the values quoted, \( I_m \frac{z_1 a^3}{z_1} \approx 0.0001, I_m \frac{M}{m} \frac{z_1 a^4}{z_1} \approx 0.00000002 \) compared to terms like \( \frac{M}{m} \frac{z_1 a}{z_1} \approx 0.3 \)
found to be (to three decimal places):

\((z_1 \sigma) = 1.64\)

It follows from this that the theoretical first mode natural frequency of the system, assuming a fixed base, is

\[ (n_1) = \frac{1}{2\pi} (1.64)^2 \cdot 393 / 2 \cdot 1675 = 3.32 \text{ c/s} \]

where \(K = 393 \text{ lb/in.} \) is the measured stiffness of the tower alone (CHAPTER and \(m = 2.1675 \text{ lb, in.}^{-1} \text{sec.}^2 \) is the mass of the tower (without top plate).

It was shown in APPENDIX F(iii) that the cantilever system was equivalent to a one-degree-of-freedom mass-spring system of mass \((M + \gamma m)\) where

\[ \gamma = \frac{1}{(z_1 \sigma)^4} - M/m \]

Substituting for \(z_1 = 1.64\) and \(M/m = 0.379 / 2 \cdot 1675 = 0.175\)

\[ \gamma = 0.238 \]

whence, the natural frequency of the equivalent system is

\[ n_0 = \frac{1}{2\pi} \sqrt{K/(M + 0.238m)} \text{ c/s}. \]

It was further shown in APPENDIX F(iii) that flexure of the base caused the decrease of the frequency of the system to a value given by

\[ (n_0)_{BT} = \frac{1}{2\pi} \left[ \frac{(K_T + K_B)/(K_T K_B)}{(M + \gamma m)} \right]^{\frac{1}{2}} \]

where \(K_B, K_T\) are the stiffnesses of the base plate and tower respectively with respect to the free-end of the tower. The natural frequencies about the principal axes of the system are therefore,

\((n_0)_{BT}, \text{ axis } 1 = \frac{1}{2\pi} (309.6/(0.379 + 0.238 \cdot 167))^\frac{1}{2} = 2.96 \text{ c/s}\)

\((n_0)_{BT}, \text{ axis } 2 = \frac{1}{2\pi} (381.7/(0.379 + 0.238 \cdot 167))^\frac{1}{2} = 3.28 \text{ c/s}\).

The second mode frequency of the tower structure were the base to be fixed would be: \((n_1) = \frac{1}{2\pi} (4.307^2 \cdot 393) / (3.2 \cdot 1675) = 22.95 \text{ c/s}\),

(this is based on the second root, \(z_1 = 4.307 \) of equation (iii)).
Having obtained the lowest root, 'a', of equation (iii), it is now possible to write the expression for the shape of the first mode of the system in free vibration, viz,

\[
\psi_1(x) = B_1 \left[ 0.6315(\sinh ax - \sin ax) + (\cosh ax - \cos ax) + \frac{2/3 \cdot K_1}{K_B} a^2 \cdot x \right]
\]

For a given unit displacement at the point of transducer measurement the above mode shape and the profiles for the static loads, \( P \) and \( W \), as defined in APPENDIX D are plotted in fig. 43 for the two principal axes of the structure.

5.4. Conclusions

It will be noted that the agreement between the observed natural (first mode) frequencies (see 5.2.) and those calculated (5.3.) from the statical parameters of the system is very good (the error being less than 1% for both principal axes). This vindicates the assumptions made in treating the tower system, firstly, as a fixed-free cantilever beam of constant stiffness and uniformly distributed self-weight with a concentrated load at the free-end and, secondly, as an equivalent mass-spring system consisting of two springs, the second spring allowing for flexure of the base. It is to be noted that the exact formula developed for the frequency of the system was (\( M = \text{'end' mass}, \ m = \text{cantilever mass} \)):

\[
n_o = \frac{1}{2\pi} \sqrt{\frac{K}{(M + 0.238m)}}
\]

This compares closely with the approximation usually made for a cantilever with a concentrated end mass,

\[
n = \frac{1}{2\pi} \sqrt{\frac{K}{(M + 0.23m)}}
\]

\[1\text{ This formula is based on Rayleigh's method in which an approximation to the true mode shape is made -- see REF. 19 Shock and Vibration Handbook Vol. 1 by C. M. Harris and C. E. Crede (editors).} \]
Deflection based on unity at $Z = 51$ in.

FIG 48 Different bending modes adopted by tower for the same transducer output
This is not surprising since this approximation is based on the displacement profile of the cantilever resulting from a lateral end load, \( P \); it will be noted from fig. 48 that the mode shape in free vibration is practically the same as that due to an end load \( P \). It will also be noted from fig. 48 that the quasi-static deflection profiles for distributed wind loading (of velocity profiles ranging in power index from 0.16 to 0.4) are practically identical although these profiles depart somewhat from the resonant or free-vibrational mode. Based on a unit deflection at the height of the transducer the difference in the modal shapes for the two principal axes, resulting from different flexures of the base plate along these axes, is clearly seen.

The damping characteristics found in 5.2. are interesting. It is shown that the damping in the system derives itself from two sources,

(i) viscous action of the air on the structure

(ii) frictional damping.

Consider, first, the case of viscous damping. It will be observed that this is different for the two axes of motion—being 0.24\% of critical about axis 2 (axis of maximum stiffness) and 0.38\% of critical about axis 1. This may appear somewhat surprising at first sight. Consider, for instance, resonant vibration about the two principal axes such that the respective free-end amplitudes of the tower are equal. Motion about the stiffer axis will have a slightly greater mean velocity (because of its higher frequency) and for this reason a more pronounced viscous decay than for the other axis might be expected. Further inspection of the system’s behaviour, however, shows that this effect is probably nullified; study of the mode shapes for a given
free-end deflection \( \delta \), say, shows that the lateral profile of the motion about the less stiff axis is more fully developed than in motion about the axis of maximum stiffness. The mean amplitude is thus smaller in the higher frequency case. In all probability the difference in viscous damping coefficients is due to the differing aerodynamic characteristics in operation about the two principle axes of the tower. Insofar as motion about the less damped axis is concerned, it is perhaps significant that the orientation of the bracing is such that its aerodynamic section is always elliptical. In the case of the more damped axis two entire panels of bracing present a circular section for lateral motion in the air. The constant frictional damping clearly originates for the most part in friction of the base plate on the base. (It is considered that the internal frictional damping for the whole structure is a small proportion of the damping present.) It is to be noted that most friction (1 lb.) is observed about the axis of most flexure of the base plate (axis-1) and least (0.22 lb.) about the axis of least flexure (axis-2). For very large amplitudes of vibration of the structure (greater than 2" at the free-end) the frictional effect of the base plate was quite audible!
It is worth noting that were the tower rigidly fixed at its base to the steel base frame, the frictional damping discussed above would practically all disappear leaving the extremely small air-viscous damping to control the size of resonant oscillations.
CHAPTER 6.

Instrumentation Used in Measurement of Wind
6. Instrumentation Used in Measurement of Wind

While the system described in CHAPTER 4 for measurement of the response of the structure was being devised, consideration was also given to the establishment of instrumentation for measuring the wind impinging on the structure. Since the first mode natural frequencies of the tower were of the order of 3 c/s, it was considered that the anemometric instrumentation to be used should be 100% responsive to gust fluctuations having componential frequencies of at least that magnitude. Further considerations, such as the necessity for portability and ease of setting-up will be discussed in the context of this chapter.

6.1. Choice of Instrumentation

A brief review of anemometers which have been used with more or less success in the natural wind permits them to be categorized as follows:

(i) cup anemometers
(ii) pressure spheres
(iii) hot wire anemometers
(iv) other types (vibrating reed, sonic, tethered balloons, thermistor, pressure-tube, windmill, pressure-plate, etc.)

Traditionally used, and still by far the most popular of these instruments is the cup anemometer which boasts many practical advantages such as simplicity of operation, reliability, durability and robustness; it is standard equipment in meteorological stations. As a transducing element for practical measuring of fluctuations in the natural wind, however, the performance of this type of instrument is limited, firstly, by its
inability to follow the more rapid wind fluctuations (mainly be-
cause of the inertial effect of the rotor tending to favour
a more rapid response to increasing wind speeds than to decreas-
ing speeds) and, secondly, because of its asymmetry, its sus-
ceptibility to error from the effect of vertical wind components.
Considerable study has been devoted to the performance of cup
anemometers and further reference may be made to publications by
Sheppard, Deacon, Jones\(^1\) and Gill\(^2\), among others. It may be
worth mentioning here a refinement by Jones\(^3\) on the normal cup
anemometer, giving a somewhat improved response. This anemometer
employs a photo-electric switching system in conjunction with
a light-weight rotor consisting of twelve polystyrene cups. The
instrument as described, however, is only suitable for low
operating speeds (less than 20 m.p.h.); for higher wind speeds
a more robust and, as a result, a more sluggish rotor is sub-
stituted. Everything being taken into account, it was concluded
that cup anemometers were inherently mean wind measuring devices
and as such were not suitable for the present purposes.

Attention was turned to pressure sphere sensors. The operation
of these instruments in the wind depends on measurement of the drag
force on a perforated sphere fixed to the free end of a cantilever system.
The drag force is usually calibrated in terms of the strain or
displacement produced in the cantilever. Many difficulties are
involved in refining this type of anemometer (in, for instance,
the elimination of parasitic resonant vibrations of the sphere)

\(^1\)See REF 20: *The Effect of Vertical Wind Fluctuations on the*
    \(^\)Response of a Sensitive Cup Anemometer by J.I.P. Jones.
\(^2\)See REF 21: *On the Dynamic Response of Meteorological Sensors*
    \(^\)and Recorders by G.C. Gill.
    pp. 414-417.
and, though a relatively fast response may be obtained\(^1\), the resultant cost, complexity and weight of a satisfactory instrument is considerable\(^2\). Clearly, the use of such instrumentation for the purposes of the experiment to be performed depended on the possibility of its commercial or other availability (it being considered impracticable to construct a suitable instrument from scratch) and since this did not appear particularly promising, the use of such an instrument was considered as something of a last resort.

Information on the field use of hot-wire anemometers is somewhat limited; these being normally used as laboratory instruments. Records of outdoor use, however, by Simmons, Beavans (1934), Swinbank (1950), and McIlroy (1961), among others, may be noted. The principle of operation of hot-wire sensors depends on the convective heat loss caused by a flow of air surrounding a thin, electrically heated wire suspended between two prongs. Unfortunately, it is found that continuous exposure, particularly to the intemperateness of the external atmosphere, even over relatively short periods of time, tends to cause chemical and other changes in the hot wire resulting in calibration changes. This apart, the fine diameter of the wire makes the instrument exceedingly fragile. These two disadvantages have been responsible for the lack of widespread use of what is otherwise an extremely sensitive and fast responding anemometer. Consideration of the basic requirements of the experiment to be performed in this

\(^1\) Reed III and Lynch (NASA) have used a simplified system with a maximum response of 15 c/s. — though difficulties were encountered with excessive resonance of the sphere under certain wind conditions — see REF 23: A Simple Fast Response Anemometer (1962) by W.H. Reed and J.W. Lynch, Journal of Applied Meteorology Vol. 2, pp. 412-416.

\(^2\) For instance, the ERA (prototype) Gust Anemometer described in ERA report C/T106 — see REF 24: The Design and Development of Three New Types of Gust Anemometer by H.H. Rosenbrock.
instance, however, led to the conclusion that only relatively brief periods of simultaneous wind and tower response measurement were necessary, allowing, for instance, the field installation of the anemometer immediately prior to use and after recent calibration\(^1\) - and avoiding exposure of the anemometer to adverse weather conditions, such as rain, snow, etc. Accurate anemometric observations would, of course, depend on a negligible calibration drift over a few hours.

A review of other forms of anemometers showed them all to be unsuitable in one way or another, being, in the main disadvantaged by having too slow a response. It was concluded from this brief survey that although use of a hot-wire anemometer was possible in this particular case, the existence of a simple wind transducer with a fast response (up to 20 c/s., say), showing little or no calibration drift and of sufficient robustness and durability for prolonged field use, was lacking.

6.1.1. Preliminary Test to Determine Suitability of Field Use of Hot-wire Anemometer

While it was thought, a priori, that the calibration of Disa Type hot-wire probes (55A25) would be maintained with reasonable stability over a period of field use lasting up to three or four hours, it was nevertheless decided to carry out a simple preliminary test to verify this assumption. Several probes were calibrated in a wind tunnel using the system shown in fig. 50.

\(^1\)Post calibration would be avoided on the grounds that a fragile probe may be damaged in the dismantling procedure immediately after use.
Full details of the Disa Type 55D50 constant-current (battery operated) anemometer are given in the appropriate manual\(^1\); it is sufficient to explain here that the hot-wire probe resistance is incorporated into a Wheatstone bridge circuit, the bridge being powered by a constant-current generator. By ensuring that the resistance of the bridge arm which includes the probe is much lower than in the opposite arm, most of the current flows through the probe wire, thereby heating it and increasing its resistance. After balancing the bridge, any air flow on the probe cools the wire, thereby decreasing its resistance; the resulting imbalance across the bridge is then either registered on a built-in sensitive moving-coil indicator or recorded externally with, say, a digital voltmeter.

Measurement of the pressure head (of water) recorded by a micromanometer connected to a Pitot-static tube situated next to the hot-wire probe (both probe and Pitot-static tube being normal to the air flow) enabled calculation of the air velocity

for a given anemometer output (in millivolts). After the first calibration (1), the anemometer was removed to a field position and put into continuous operation (in a substantial wind) for about four hours. The anemometer was then recalibrated (2) in the wind-tunnel (seven hours after the first calibration). An example of the two calibrations (for two probes) is shown in fig. 51. It may be worth noting that an ambient air temperature rise of about $3^\circ C$ was noted in the wind-tunnel laboratory between the two calibrations shown.

It was concluded from the above test that the probes retained their calibrations at least for a period of time adequate for tests to be made in conjunction with the tower system. The small change of temperature, $3^\circ C$, did not seem to affect the calibration.

6.1.2. Further Practical Considerations: Linearization, Probecable Lengths, etc.

It will be noticed from the calibration curves in fig. 51 that the output voltage of the anemometer is a non-linear function of the flow velocity; this is because the amount of heat transmitted by a heated body to the surrounding medium is itself a non-linear function of the flow velocity of the medium. It is clear that from the point of view of either obtaining a qualitative idea of the anticipated high degree of turbulence incident on the tower structure, or making a detailed comparison with the structural response of the tower in the form of a simultaneous

---

1Since two anemometers were available, two probes could be calibrated simultaneously.
FIG 51 Calibration stability of hot-wire probes
visual or oscillographic record, this is not very satisfactory. It was therefore decided to make use of a somewhat more refined anemometer, consisting of a constant-temperature battery-operated Disa Type 55D05 Anemometer in conjunction with a battery-operated Disa Type 55D15 Linearizer.

The principle of operation of the constant-temperature anemometer is similar to that of the constant-current anemometer, except that, instead of using a constant-current generator to power the bridge, an amplifier is used whose input is the voltage imbalance across the bridge caused by a change of resistance in the hot-wire. The amplifier output powering the bridge is designed to act in such a way that, being controlled by the bridge imbalance, it automatically compensates for that imbalance by supplying, accordingly, more or less current to the hot-wire. By this null seeking device the temperature of the hot-wire is kept constant. The power required by the servo-amplifier to maintain the hot-wire temperature constant represents the output of the system, which can now be measured in terms of volts (as opposed to millivolts in the constant-current anemometer).

The shape of the calibration curves produced by the constant-temperature anemometer is similar to that shown in fig. 51. It may be shown from a theoretical analysis that the heat loss, $Q$, of a hot-wire exposed to an air flow of velocity, $U$, is

$$Q = (A + B_n U^n)(T - T_0) \quad \text{(i)}$$

1For full details see REF 26: Disa Instruction and Service Manual for Type 55D05 Battery-operated CTA

2Apart from the substantially greater sensitivity, the constant-temperature anemometer has other advantages over the constant-current anemometer; such as, a minimization of the time lag, so increasing the upper frequency limit.
where $A$, $B$ and $n$ are constants, $T$ is the wire temperature, $T_0$ is the ambient temperature. The heat loss is proportional to the power needed to supply the bridge (and consequently, the probe), which is proportional to the square of the output voltage, $V$. Assuming $(T - T_0)$ to be constant the static-calibration curve will be of the form:

$$V^2 = (A_1 + B_1 U^n) \quad \text{(ii)}$$

where $A_1$, and $B_1$ are constants. Theoretically, for a two-dimensional heat transfer from a cylinder in an incompressible potential flow, $n = \frac{1}{2}$. In practice, however, there will be other forms of heat dissipation present tending to lower this value; for a hot-wire probe, $n$ lies between $1/2$ and $1/3$.

Linearization of the non-linear function is effected by operating electrically on the output voltage, $V_U$ (at flow velocity, $U$), according to the transfer function,

$$V_{\text{out}} = K[V_{\text{in}}^2 - C^2]^m$$

where $K$, $C$ and $m$ are constants. By adjusting the constant parameters, $C$ and $m$ in the linearizing circuit network, such that $C = V_{U=0}$, $m = 1/n$, ($K$ controls the sensitivity or gain of the linearized output), the linearizer performs the operation,

$$V_{\text{out}} = K[V_U^2 - V_{U=0}^2]^{1/n}$$

In practice the question arises: how do changes in ambient temperature affect the calibration? Consider an extreme ambient temperature rise of $30^\circ$C (say) after initial calibration. Let the power dissipation before and after the rise be $Q_1$, $Q_2$ corresponding to output voltages $V_1$ and $V_2$. From (i), $Q_2/Q_1 = (T - T_0 - 30)/(T - T_0)$. Assuming the operating temperature of the hot-wire to be in the region of $300^\circ$C, $Q_2/Q_1 = (V_2/V_1)^2 = 0.9$, whence, $V_2 = 95\% V_1$. This corresponds to a $5\%$ error on the original calibration.
which for input, (ii), gives,

\[ V_{\text{out}} = K_A + B_1 U^n - A_1 \frac{1}{n} = K_1 U \]

\( K_1 = \text{constant} \). This is a linear output of slope \( K_1 \), which is a function of the output gain of the linearizer.

Another practical requirement was that the hot-wire probe be situated in the field in a position as near as might be found feasible to the top of the tower structure, thus necessitating the use of a lengthy probe cable. The resistance of a typical probe of the Disa 55A25 series, at room temperature, is in the region of 3.5 ohms, which increases, for a given overheating ratio, to an operating resistance of 6.2 ohms. It is clear that, quite apart from causing parasitic oscillations and other effects, the line impedance of a lengthy cable may significantly increase the effective probe resistance as seen by the anemometer bridge. In order to counteract these effects it was found necessary to fit a compensating cable, of exactly the same type and length as the probe cable, to the opposite arm of the bridge. Simulation of the probe operating resistance was also effected with a non-inductive resistance connected to the end of the compensating cable. The bridge system could then be balanced in the normal way.

The length of the probe cable actually used was approximately 70 feet. Equalisation of resistance between probe and compensating cables was achieved by successive shortening of the latter. The overheating ratio, \( 'a' \), producing the recommended operating

\footnote{The resistance of cables used was measured as 0.016 ohm/ft., corresponding to a total resistance of 1.12 ohms for a 70 ft. length. This is about 20% of the probe operating resistance.}
temperature of the Disa Type 55A25 probes is 0.8. Measurement of the cold resistance, $R_0$, of each probe, showed that in all cases $R_0$ was close to 3.5 ohms. From this, the operating resistance, $R$, was calculated as,

$$ R = R_0(1 + a) = 3.5(1 + 0.8) = 6.3 \text{ ohms} $$

The compensating external resistance was made up according to this specification from preferred-value non-inductive resistors connected in parallel.

Once set-up the linearized anemometer was calibrated in the wind-tunnel in the manner described earlier (against a micro-manometric standard). The actual step-by-step procedure involved in operating the anemometer and linearizer combination is given in the appropriate instruction manual. A typical static-calibration curve is shown in fig. 52, for a linearizer exponent, $m = 3$. It will be noted that the response is linear for the velocity range extending above 6 mph. This was considered suitable for the required purposes.

6.2. Directional and Other Considerations

The preceding pages have been devoted to the description and adaptation of a high frequency response, linearized anemometer with a sensitive output and capable of being used in the field for a limited period of time. No mention has been made of wind direction. In fact, however, the hot-wire probes are sensitive to the angle, $\Theta$, between the direction of flow and the wire axis (see fig. 53). Calibration of the linearized anemometer output against the direction of flow, $\Theta$, shows that the directional characteristics of the hot-wire probes used is a cosinusoidal
FIG 52  Typical calibration of linearized anemometer system

Linearizer exponent, $m = 3$
function (for further details of this see Disa Instruction and Service Manual REF 26). Since the probe is not sensitive to directional changes in the plane perpendicular to the axis of the wire, this means that the instantaneous output of the system represents the instantaneous component of flow in the horizontal plane, viz, \( V(t) \cdot \cos(\theta) = V_H(t) \), say (assuming the wire axis to be vertical).

At this juncture attention was turned to the anticipated effect of the natural wind flow on the tower structure. Observation of the movements of smoke at the proposed field siting of the tower structure, on a windy day, suggested to the author that, although the wind direction was subject to sometimes rapid and substantial fluctuations\(^1\) (appearing to correspond with rapid changes in wind speed), these changes were reasonably well correlated over a height of at least that of the tower structure. (This condition was not observed in the immediate vicinity of large bluff structures, where different predominant wind directions appeared to co-exist at different heights.)

Further it was estimated, again from observations of smoke

\(^1\) the frequency of these fluctuations being representative of the micrometeorological turbulence.
and, more particularly, tree movements, that even under low mean hourly wind speed conditions, gusts with a duration of as low as one second were reasonably well correlated over a spatial compass whose vertical span was at least equivalent to the height of the tower structure. The term 'correlated' here refers only to the extent to which the basic frequency characteristic of a gust is adhered to over a given area normal to the mean flow; no indication of the shape of the vertical wind-speed distribution is implied. In fact, it is shown in 4.2.3. that for a given wind speed measured near the top of the tower structure, the variation in the effective load transmitted to the tower for various wind-speed profiles is relatively small. It was therefore tentatively put forward that the instantaneous loading on the tower structure (at least for those frequencies corresponding to quasi-static loading, if not to the dynamic or resonant loading) would be dependent, to a first degree of approximation, on a single parameter, namely, the instantaneous wind-speed at the top of the structure\(^1\).

In order to test this hypothesis and, secondly, for the more practical reason that only one linearized anemometer was available at the time, it was decided to measure the wind-speed at a single point and that near the top of the tower structure.

It was considered that the tower would respond laterally

\(^1\)It is important not to misinterpret this; clearly, for a given gradient wind-speed, the magnitude of the instantaneous wind-speed at a particular height will be dependent ultimately on the shape of the mean hourly profile which, in turn, is dependent on the roughness of the terrain and so on. What is being suggested here is that, given a particular site, the response of a tower structure of the type under consideration to gusting is dominated by the velocity of the gust at the top of the structure and that the variation of the wind velocity below this height about the shape of the mean profile may be ignored.
only to the horizontal component of the wind. As noted earlier, measurement of this component can be effected by orientating the axis of the hot-wire vertically (as shown in fig. 53). The problem still remaining, however, was how to correlate the measurement of the horizontal wind-speed at the top of the tower structure with its structural response, as measured along its two principal axes of stiffness. The observation stated above concerning the uniformity of the wind direction with height suggested that measurement of the instantaneous wind direction at a single point (close to the point of wind speed measurement at the top of the tower) would adequately correspond to the angle of incidence of the wind, at the same instant, over the tower as a whole.

\[ \begin{align*}
V_{H}(t) \sin \phi(t) &\quad V_{H}(t) \\
V_{H}(t) \cos \phi(t) &\quad \text{incident wind}
\end{align*} \]

FIG 54

Consider fig. 54 which shows the incidence at time, \( t \), of a horizontal wind component, \( V_H(t) \), at an angle \( \phi \) to AXIS 2 (maximum stiffness) of the tower structure. \( V_H(t) \) is the wind speed at the top of the tower as measured by the linearized anemometer. Assuming the aerodynamic drag coefficient of the tower to be independent of the angle \( \phi \) and taking into account all the other assumptions which have been made above, it is clear that the
response measured along AXIS-1, which, it was shown before, can only result from the application of a force on the tower in that direction, must be due entirely to the component of the effective wind pressure in that direction. Similarly, the response picked up by the transducer situated along AXIS-2 must result from the component of wind pressure acting along that axis. Since wind pressure is a function of wind velocity, it follows that the response along the principal axes is a function of the respective wind components, \( \dot{v}_H(t)_1 = v_H(t) \sin \theta(t) \), and \( \dot{v}_H(t)_2 = v_H(t) \cos \theta(t) \). (It will be noticed that the component velocities take into account the fluctuating angle of incidence, \( \theta(t) \), which measurement of \( \dot{v}_H(t) \) clearly does not.)

Should the assumptions made above turn out to be true, it was estimated that simple visual comparison of simultaneously obtained side-by-side oscillographic records of both the response and wind speed components would show a definite and marked correlation. As a brief summary of this section it might be said that the situation whereby three-dimensional turbulence acts to produce a two-dimensional response in the tower structure has been tentatively reduced into a much simpler situation composed of two mutually perpendicular single-degree-of-freedom systems with in-line wind loading.

6.2.1. Instrumentation for Resolution of Wind Components

In order to test the hypotheses made in the last section, it was considered necessary to obtain wind speed components, \( \dot{v}_H(t)_1 \), \( \dot{v}_H(t) \cos \theta(t) \), and \( \dot{v}_H(t)_2 \), \( \dot{v}_H(t) \sin \theta(t) \), in the form of separate continuous signals. A means of obtaining the fluctuating horizontal wind speed, \( \dot{v}_H(t) \), was described in section 6.1.1. A description of
the incorporation of this quantity with the measurement of $\phi(t)$ into the production of the appropriate signal outputs is undertaken in this section.

The classical method of measuring $\phi(t)$ is to use some form of simple electro-mechanical transducer consisting of a wind vane coupled to the shaft of a toroidally wound linear-law potentiometer. Rotation of the potentiometer shaft causes a moving contact or wiper to move over the resistance element, across which a constant voltage is applied, bringing more or less resistance into play and so delivering a voltage output linearly proportional to the angle of rotation (see fig. 55).

It will be noticed that two output voltages may be simultaneously obtained, $(V_{out})_1$ and $(V_{out})_2$. Because of the discontinuity at A, C the potentiometer is sometimes manually orientated towards the wind such that fluctuations about the mean wind direction occur in the region of B.

Although the sort of potentiometer described above was clearly inappropriate from the point of view of obtaining the cosinusoidal components, Sin$\phi(t)$ and Cos$\phi(t)$, it was nevertheless considered that the principle involved might be adapted to this
A suitable system producing simultaneous sine and cosine outputs is shown in fig. 56.

![Fig. 56](image)

Instead of an evenly spaced winding as before, the spacing between the windings is varied sinusoidally, one cycle of the variation corresponding to the complete toroidal winding. By introducing a second wiper at right angles to the first a phase difference of \( \pi/2 \) is produced between the two sinusoidal output voltages obtained, such that if one is designated to be the 'sine' of the supply voltage, the other will be the 'cosine'. If desired, simulation of the trigonometrical sign of the sinusoidal functions can be effected by biasing the supply voltage equally about zero voltage as shown\(^1\).

Although generally designed for use in a different manner to that intended here\(^2\), it was thought that commercially available precision wound potentiometers with sine/cosine laws as described above could be

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\(^1\) More generally, cosinusoidal outputs may be obtained for any voltage difference applied across AC, e.g., for voltage, \(+V\), at A, 0 at C, the output will be cosinusoidal about \(+V\).

\(^2\) In airborne navigational instruments, analogue computers, radar simulators, etc., where a constant rotation is applied to the shaft of the instrument, and the input and output voltages represent the two ends of the transducing operation.
used for the purpose in hand. Fortunately so, as the construction of an ad hoc device would have been impracticable. The position at this point, then, was as follows:

(i) By using the anemometer described earlier, the fluctuating wind speed, \( V_H(t) \), could be obtained in the form of a linearly proportional output voltage, \( V(t) \), say, and

(ii) By using a potentiometer of the type described above in conjunction with a wind-vane (near the point of measurement of \( V_H(t) \)) the directional components \( \sin \phi(t) \) and \( \cos \phi(t) \) also in the form of output voltages, \( V \cdot \sin \phi(t) \) and \( V \cdot \cos \phi(t) \), could be obtained.

It will be seen that a neat way of obtaining the components, \( V_H(t) \cdot \sin \phi(t) \) and \( V_H(t) \cdot \cos \phi(t) \), by substitution of \( V(t) \) for \( V \), or, more specifically, by supplying the potentiometer with the voltage output of the anemometer, \( V(t) \), instead of the constant supply voltage, \( V \), suggested itself. It was found, however, that this operation could not be put into direct effect. Since a potentiometer of the type described operates on the voltage difference which is supplied to it, producing a cosinusoidal output about the mean of that difference, it is clear that any variation in that mean will be transferred to the output. Were the anemometer output, \( V(t) \), to be applied directly to such a potentiometer, as shown in fig. 57, the individual outputs, referred to

\[ \text{FIG 57} \]
a fixed (zero) voltage level, would be, $\frac{1}{2}V(t) \cdot [1 + \cos \phi(t)]$ and $\frac{1}{2}V(t) \cdot [1 - \sin \phi]$. In order to compensate for this effect, it is necessary to convert the time-varying voltage, $V(t)$, into an equivalent voltage difference, $V(t)$, which operates about a fixed voltage level—see fig. 58.

Fed to the potentiometer in this form the signal $V(t)$ is readily converted into the required components, $V(t) \cdot \sin \phi(t)$ and $V(t) \cdot \cos \phi(t)$. Conversion of the anemometer output voltage into the appropriate voltage difference acceptable to the potentiometer can be directly effected with an operational difference amplifier; although not normally available commercially as a separate unit, it was found that the element was used in certain instruments produced commercially. A suitable amplifier, with suitable gain\(^1\), input and output impedances, and an input voltage range matched to the output voltage range of the linearised anemometer was accordingly selected.

\(^1\)Strictly speaking amplification is not essential, though it does increase the sensitivity of the whole system and as such helps to increase the signal to noise ratio (noise from leads, etc.).
6.2.2. Further Details of Circuitry, Wind Vane, etc.

As set up the instrumentation of the complete system (producing the desired horizontal wind speed components) consisted of the following elements ((i), (ii) and (iii) having been described previously).

(i) Disa Type 55A25 hot-wire probe.

(ii) Disa Type 55D05 battery-operated Constant-Temperature Anemometer (fitted with compensating cable and resistance).

(iii) Disa Type 55D15 battery-operated Linearizer.

(iv) Ferranti, Photo-cell Amplifier (C. 1809). The basic function of this all-transistorized printed-circuit element is to perform the operation shown in fig. 59.

![Diagram of circuit](#)

**FIG 59**

The 'difference' between two independent signals is amplified by a factor 'K' and is symmetrically distributed about a nominal d.c. voltage, $V_0$. For the amplifier specified, $V_0 = -120$ mV and $K = 5$. The output voltage of the linearizer being always positive (working above zero voltage or ground), one of the amplifier inputs was accordingly earthed.
(v) Power supply, consisting of Exide 'Drymax' 6V. batteries, but later replaced by more powerful 'Ecaram', 6LD5 Miniature 6V. accumulators. Used to power the difference amplifier (iv) and the 'emitter follower' circuit (see(viii)) the supply consisted of the two units connected 'back-to-back' (as shown in fig. 59) producing a 12V d.c. supply equally biased about ground. In order to protect the difference amplifier from possible damage caused by an imbalanced supply (when switching on, for instance), a single throw, double pole switch controlled the supply (as shown).

(vi) Ferranti, precision wire-wound sine/cosine single-gang potentiometer (Type 11HL2SC) with a total winding resistance of 40K ohm: (10K ohm/quadrant) and a nominal mechanical starting torque of 3 gm.cm. The physical operation of this element was described in 6.2.1.; for other details and specifications see the appropriate literature 1.

(vii) Wind Vane: The comprehensive design of a wind vane from the point of view of estimating theoretically its response characteristics, etc., is somewhat complicated. As with cup-anemometers, inertia/damping effects will determine and, in fact, restrict response to the more rapid wind fluctuations. For these reasons it was decided to build a prototype vane, using the lightest materials possible, and then to test its suitability in the wind-tunnel. The basic dimensions, materials, etc., of the prototype vane are shown in fig. 60.

Once built the vane was fitted to the shaft of the sine/cos. potentiometer and then exposed to a steady wind-tunnel air flow.

1REF 27: Ferranti data sheet L153 and pamphlet BR 063.
By deflecting the vane slightly (through about 15-30°) and then releasing it, the sort of decay shown in fig. 61 was obtained.

By varying the wind-tunnel velocity, $U$, it was found that the period, $T$, of the damped natural oscillation of the vane varied in accord, such that (as expected) if $U$ was doubled, say, $T$ was halved. The vane could therefore be characterized by its damped natural wavelength, $\mu' = U \cdot T$.

It can be shown (APPENDIX D(ii)) that the damped natural frequency of a single-degree-of-freedom system is, in terms of
its undamped natural frequency,

\[ \omega' = \omega \sqrt{1 - \left(\frac{c}{c_r}\right)^2} \]

\(\frac{c}{c_r}\%\) = \% critical damping\)

whence, dividing by \(2\pi U\),

\[ \frac{\omega}{2\pi U} = \frac{1}{\mu'} = \frac{\omega/2\pi U \sqrt{1 - \left(\frac{c}{c_r}\right)^2}}{1/\mu' \sqrt{1 - \left(\frac{c}{c_r}\right)^2}} = \frac{1}{\mu' \sqrt{1 - \left(\frac{c}{c_r}\right)^2}} \]

i.e., the undamped natural wavelength of the vane, assuming viscous damping is,

\[ \mu = \frac{\mu' \sqrt{1 - \left(\frac{c}{c_r}\right)^2}}{\mu' \sqrt{1 - \left(\frac{c}{c_r}\right)^2}} \]

(i) Consider the vane to be subjected to sinusoidal wind direction fluctuations of constant amplitude but varying frequency. It is clear that if the frequency approaches the undamped natural frequency of the vane, some resonant amplification may occur, the magnitude of which will be determined by the amount of damping present. The verification of two criteria was therefore necessary:

(i) that the aerodynamic damping of the prototype vane was sufficient to prevent parasitic resonant vibrations, and

(ii) that the undamped natural frequency was effectively higher than the upper limit of the required range (0-approx. 3-c/s. for the tower-structure system).

The damping was found by noting the 'overshoot', \(x_1\), of the vane in the wind tunnel (see fig. 61). For the vane described above, \(x_1\), was in the region of 15-20% of the initial deflection, \(x_0\). Calculation of the logarithmic decrement shows that this corresponds to a damping ratio, expressed as a percentage of critical damping, of 52-60%. Study of dynamical admittance or amplification curves for various damping ratios will show that only a small amount of 'overshooting' of the vane will occur for the damping ratios found (the response being practically flat and
then decreasing to zero, at resonance).

From the decay of the vane motion in the wind tunnel was as-
certained the damped natural frequency, for a range of wind tunnel
air speeds. From this it was estimated that the damped natural
wavelength of the vane was in the region of 4.4 ft. (i.e., wind-
tunnel air speed x frequency of decay). The undamped wavelength
was then computed from (i) as being in the region of 3.7 ft, for
the damping ratios found. This means that the wind vane is 100%
responsive to (sinusoidal) fluctuations in the direction of gusts
having a wavelength of 4 ft. (say) or greater. This in turn
means that any (sinusoidal\(^1\)) change in direction with a frequency
of 3 c/s. or lower of gusts carried along in a mean wind of approx-
imately 8 mph. or higher will be 'seen' by the vane. Since it
was intended to use the wind measuring system at substantially
higher wind speeds than this, the vane as constructed was con-
sidered to be suitable.

(viii) 'Emitter-follower' circuit: The output impedance of the
sine/cos. potentiometer will vary as a function of the vane pos-
ition (the resistance in each quadrant varying from 0-10K ohms).
While high input impedance recorders such as oscilloscopes, etc.,
may be directly connected to the output of the potentiometer, it
is clear that connection of a 'visicorder' galvanometer (requiring
an input impedance of approximately 250 ohms for correct damping)
via the appropriate resistive matching network, will, if the re-
sistances of the matching network be of the same order as or
lower than the potentiometer output impedance, effectively draw

\(^1\)Perfect sinusoidal variations in the natural wind will be rare;
the response will therefore probably be somewhat better than
specified.
current away from the potentiometric circuit and so falsify the measured voltage. For this reason the circuit shown in fig. 62 was devised, making use of the available ±6V power supply; the input impedance of the circuit being approximately 170K ohms.

All the elements of the system were connected together with screened leads of the appropriate lengths; the circuit which gave the minimum earthing difficulties is shown in fig. 63.
FIG 63 Lay-out of wind transducing system (eliminating ground-loop interference, etc.)
6.3. Calibration

In order to avoid the possibility of calibration drift (discussed previously) it was necessary to calibrate the system immediately prior to its use in the field. (quite apart from the fact that each individual probe, possessing heat dissipation characteristics of its own, requires individual calibration). The whole system was calibrated in the wind tunnel against the micromanometric dynamic head, as before, for a range of linearizer gains such that when in the field selection of a gain giving a legible and clear trace on the 'visicorder' oscilloscript paper was possible. When calibrating the system, the orientation of the vane was fixed such that one of the potentiometer outputs was always zero for any input wind speed as picked up by the hot-wires; under these conditions, \( \phi(t) = 0 \), say, the component outputs become, \( V(t) \cdot \sin \theta(t) = 0 \), and \( V(t) \cdot \cos \theta(t) = V(t) \), the speed of the wind tunnel air flow. For a given range of input air-speeds, \( V(t) \) (calculated in mph.), the deflections of the UV trace on the photo-sensitive paper (in cm.) were noted. A typical calibration for various linearizer gains is shown in fig. 64.
FIG 64 Typical calibration of anemometric system — immediately prior to test

Date: 25/6/69
Linearizer gain: coarse = G
fine = g

G = 51/59, g = 0
G = 42/51, g = 2
G = 42/51, g = 0
G = 33/42, g = 3

Wind speed in mph
Visicorder output : shift in UV trace in cm.
CHAPTER 7.

Damping Device
7. **Damping Device**

7.1. **Introduction**

It will be remembered (CHAPTER 5) that the rate of dissipation of vibrational energy per cycle stored in the tower structure when vibrating at resonance in still air was very low, requiring a time interval of at least 30 sec. or 90 cycles of vibration before a 90% decay in the amplitude of 'free' vibration was observed. In order to give some idea of the susceptibility of the structure to vibration at its natural frequencies, it may be worth mentioning the fact that in both its sitings, even in still air, a condition of resonant motion was frequently observed; a condition due, it was noticed, to spurious vibrations transmitted via the base of the structure from movement of vehicles, people, etc. in the immediate neighbourhood of the structure.

It was thought that, though large resonant motion of such a structure in general might not necessarily become so large in conditions of high wind as to lead to actual structural failure (by buckling of uprights, etc.), this state of affairs might not be tolerable on certain other accounts. For instance, structural failure could yet occur as a result of metal fatigue in structurally important members or weld joints. There may, equally, be other instances where excessive vibrations in lattice tower structures, such as those supporting antennae and other instrumentation, would not be permissible. Because of these and other factors it was decided to devise some form of 'damper' which when fitted to the tower structure under consideration would reduce its resonant motion by substantially increasing the rate of dissipation of energy accepted by the structure at that frequency.
7.2.1. Physical Considerations

In its field siting the tower structure under consideration will be exposed to the random loading of the wind and as such will be subjected to a full range of excitation frequencies. An auxiliary device which would merely alter the natural frequency of the tower system — such as the dynamic absorber discussed briefly in APPENDIX I (Case 1) — while very useful in other applications where a discrete vibration of constant frequency is to be eliminated, cannot be used here. What is in fact required is an auxiliary mass absorber which is tuned in such a way that it effectively minimizes any dynamic amplification of motion occurring in the primary system (i.e., the system whose motion is to be damped: the tower structure) at its natural frequency without introducing its own extra degree-of-freedom into the primary system. An optimum system of this type, consisting of a simple auxiliary spring-mass system with damping, was considered theoretically in APPENDIX I (Case 3). In order to damp out the resonant motion of the primary system to a required extent, it is necessary from the design point of view to know only the mass distribution and natural frequency of the primary system. The optimum mass, natural frequency and damping parameters of the auxiliary system may then be determined.

In order to test whether it was, in practical terms, possible to use this sort of auxiliary system in conjunction with the tower structure, it was decided to install, at least initially, a simpler system than the one mentioned above, this simpler system consisting of an auxiliary mass absorber coupled to the primary system by damping only. (This particular case is dealt with in the latter part of Case 3 - APPENDIX I.) Although less
effective than the more sophisticated absorber which includes the tuned spring, this type of 'damping only' absorber is considerably easier to tune and install.

The basic details of the absorber as fitted to the tower structure are shown in fig. 65. From a rigid support (A), firmly clamped to the tower structure, is suspended (by means of a simple universal joint (B)) a cylindrical perforated 'baffle' or damper\(^1\) (C) mounted firmly on a supporting 'pendulum' rod (D). Weights (E) can be placed at will above the damper as shown.

The damper is immersed in oil which is contained in a rigid-wall reservoir (F) which is firmly affixed to the tower structure.

The system described was fitted as close as was feasible to the top-end of the tower structure - see fig. 66.

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\(^1\)Use was made of a length of thermo-setting plastic tubing.
FIG 66 Auxiliary mass absorber fitted to the tower structure

FIG 67 Typical decay of free vibration in 'damped' system
treatment given in APPENDIX I. First of all use is made of the
fact that a suspended mass remains stationary if the natural
frequency of the 'pendulum' system which it forms is lower than
the frequency of lateral vibrations applied to the point of
suspension. Thus, in the device shown in fig. 65, though the
point of support (B) may vibrate with the same frequency as the
resonant motion of the tower (3 c/s.), the mass (E) will remain
stationary. In this way, a differential motion is set up bet-
ween the damper (C) and the oil which (like B) is constrained to
vibrate in accord with the motion of the structure. The oil
attempts to impart its motion to the damper which, because of
its mass inertia, attempts to remain stationary. In this way,
viscous shear stresses are set up in the oil with the ensuing
dissipation of energy.

A physical idea of the optimum system may be obtained from
the following considerations. Assume, firstly, a given mass
(E). It is clear that if a liquid of very low viscosity is
used (instead of oil), the damper will have no difficulty in
remaining stationary, and, although because of the low viscosity
of the liquid the relative motion between damper and liquid will
tend towards a maximum, little energy dissipation will take
place. At the other extreme, if the viscosity of the liquid
surrounding the damper is disproportionately high, both damper
and oil will move together, and, again, although as a result the
natural frequency of the primary system will be lowered slightly,
little or no energy dissipation will take place. It follows
that between these two extremes exists an optimum value for the
viscosity of the liquid, where the relative motion of the damper
and liquid is such that a maximum amount of energy is dissipated.
From a tuning point of view, the corollary to this is more useful.
Assume the damping capacity of the auxiliary system to be constant (ie., a fixed oil viscosity, size of damper, etc.). If the mass \(E\) is very low, the damper will move with the oil and little damping will ensue. If the mass \(E\) is progressively increased the damper will tend more and more to remain stationary and the relative motion between the oil and damper will be increased, calling into play more and more viscous dissipation of energy. A point will be reached, however, past which the addition of extra mass will produce very little extra damping. The auxiliary system is then tuned for optimum performance - obtaining the maximum damping capacity for the most economical use of mass \(E\).

From the above it is seen that it is not necessary to estimate the damping capacity of the auxiliary system for optimum tuning purposes. In this way it is possible to install an auxiliary absorber whose damping properties are totally unknown. An optimum performance is achieved by increasing progressively the inertia of the damper (by adding weights \(E\)) and noting the increase in damping in the primary system (which may be measured from, for instance, records of free-vibrational decay).

7.2.2. Performance of Absorber - Tuning

The auxiliary mass-absorber coupled by damping alone was first fitted to the top of the tower (underneath the top plate) with a small auxiliary mass \(E\) of 5 lb. A record of the decay of the motion of the tower-structure in free vibration was obtained by submitting the structure to an initial impulse (as described in CHAPTER 5). The mass \(E\) was then increased in 5 lb. increments to 40 lb., in each instance a record of the free-vibrational decay of the structure being obtained. With
each increase in mass up to 25 lb. a marked increase in damping in the tower structure was noted. Thereafter, the increases became smaller and were barely perceptible in the increases of mass between 30, 35 and 40 lb. It was concluded from this that the auxiliary mass absorber was correctly tuned (to the nearest ± 5 lb.) when the auxiliary mass was of magnitude 25 lb.

The trace shown in fig. 67 is a typical record of the decay in the tower structure in free vibration (for motion about the axis of lesser stiffness) with the auxiliary mass at 25 lb. From several such decay traces the mean logarithmic decrement was computed (assuming viscous damping and neglecting the small amount of frictional damping present in the tower structure). Expressed as a percentage of critical damping, the values obtained were as follows:

\[
\begin{align*}
\text{AXIS 1:} & \quad (c/c_{CR})\% = 1.93\% \\
\text{AXIS 2:} & \quad (c/c_{CR})\% = 1.90\%
\end{align*}
\]

7.3. Conclusions

Comparison of the damping values given above and, more particularly, the decay trace shown in fig. 67 with the corresponding values and traces presented in CHAPTER 5 will show that the installation of the auxiliary mass damper (coupled by damping only) described above has considerably increased the damping capacity of the tower structure. It will be noticed that the damping about the axis of maximum stiffness (AXIS-2) is slightly lower than that about AXIS-1; this probably reflects the difference in the relative amounts of 'natural' damping (ie., without the absorber) about these axes in the tower structure.
Comparison of the performance of the damping device fitted to the tower as measured experimentally with its theoretical behaviour as predicted in APPENDIX I is interesting. It was shown in APPENDIX I that the maximum response at any frequency of a primary system fitted with an auxiliary mass absorber connected by damping alone and tuned for optimum performance, was, theoretically,

\[(x_o/x_{st})_{\text{max}} = (2 + m_a/m_p)/(m_a/m_p)\]

where \(m_a\) = mass of auxiliary absorber, and \(m_p\) = mass of primary system. \(m_p\) is the mass of the primary system as seen by the absorber; in other words, \(m_p\) is the effective mass of the tower structure considered as a single-degree-of-freedom system at its free end, whence, as before,

\[(m_p)g = (M + 0.23m)g = 146.5 + 0.23 \times 340 = 340 \text{ lb.wt.}\]

As noted above,

\[(m_a)g = 25 \text{ lb.wt.}\]

and so

\[(x_o/x_{st})_{\text{max}} = (2 + 25/340)/(25/340) = 28\]

As seen in APPENDIX G(i) for viscous damping this corresponds to the quantity, \(\omega/2\beta\). Assuming the same value of damping for both axes, the damping of the structure fitted with the absorber was found to be, (by experiment)

\[(c/c_{CR})\% = 1.9\% = \beta/\omega \quad \text{(by definition)}\]

whence,

\[(x_o/x_{st})_{\text{max}} = \omega/2\beta = 100/2.19 = 26\]

In view of numerous secondary factors which have not been taken
into account in the above analysis, such as the damping present in the primary system before connection to the auxiliary system, rotation of the damper about a horizontal axis through the auxiliary mass, and so on, the agreement between the theoretical estimate of maximum response and that measured in the field is remarkably good, and certainly appears encouraging from a designer's point of view.

The total weight of the auxiliary system used in the experiment above represented about $7.4\%$ of the effective weight of the tower structure (referred to as a single-degree-of-freedom system), or, what is more important, approximately $2\%$ of the total weight of the structure. Clearly, if the damping capacity of the absorber was increased (by increasing the viscosity of the oil, or using a larger damper, say), which would entail for optimum damping the use of a larger auxiliary mass, the maximum response of the tower structure at any given frequency of excitation would be further reduced.

Using an auxiliary mass absorber connected to the structure with a spring (as well as damping), however, would enable further minimization of dynamic response of the structure at its natural frequency without any increase in the auxiliary mass. For instance, assume the auxiliary mass to be 25 lb. as before. From APPENDIX I, it follows that, for optimum performance, the natural frequency of the spring must be

$$(\omega_a)_{\text{opt}} = \frac{\omega_p}{(1 + m_a/m_p)} = \frac{\omega_p}{(1 + 25/340)} = 0.93 \omega_p \text{ rad/s},$$

(where $\omega_p$ - the natural frequency of the primary system - is in

\footnote{The damper was tuned for a lower value of damping (with the oil container approximately half-full of oil). Again, good correlation between theory and experiment was obtained.}

\footnote{The damper was tuned for a lower value of damping (with the oil container approximately half-full of oil). Again, good correlation between theory and experiment was obtained.
Fig. 68 Effect of 'dampers' on mechanical admittance of tower.
the region of 3-3.3 c/s.). The optimum value for damping required in the auxiliary system is,

\[ \beta_{opt} = \sqrt{\frac{3m_a/m_p}{8(1 + m_a/m_p)^3}} \times 11\% \text{ (% critical damping)} \]

An auxiliary system with the above tuned parameters would result in a maximum response at any frequency in the primary system of,

\[ (x_0/x_{st})_{opt} = \sqrt{1 + \frac{2m_p/m_a}{} = 5.3} \]

This type of damper needs to be accurately tuned to the particular structure to which it is fitted, a factor which depends, as mentioned above, on being able to obtain an accurate estimate of the mass distribution and natural frequencies of the structure. Fig. 69 suggests the basic form which such an auxiliary mass absorber, suitable for tower structures whose dynamic characteristics are known or can be easily measured, might take.

Though robust and durable, such a system could incorporate certain simple fine adjustment devices, enabling tuning of the controlling parameters (of mass, natural frequency and damping capacity).
The maximum response in the tower structure for a full range of excitation frequencies has been plotted in fig. 68 for the following cases of damping:

(i) Natural damping inherent in structure (experimentally obtained values).

(ii) Damping in structure arising from fitting of auxiliary (25 lb.) mass absorber coupled by damping only (experimental and theoretical values).

(iii) Damping in structure resulting from fitting of complete auxiliary (25 lb.) mass absorber (theoretical curve).

From the curves in fig. 68 can be gauged the maximum response (relative to the equivalent statical response) which would obtain in the tower structure at various discrete frequencies of excitation. The problem remains, however, of obtaining some idea of the response of such a structure to random excitation (such as, for instance, wind loading) — and, more particularly, how the various damping systems discussed affect this response.

Consider the subjection of a one-degree-of-freedom system to a purely random, 'white noise' excitation. As shown in APPENDIX A, such a signal can be thought of as composed of infinitely closely spaced sinusoidal waves of infinite length and which are, in this particular case, all of the same amplitude. This can be represented in spectral (energy density) form by a straight line as shown in fig. 70(a). From the Principle of Superposition it follows that each constituent wave of the random signal may be operated on individually, and that the response resulting from each such wave may be summed to give the overall pattern of response. Consider the effect of a typical sinusoidal wave of frequency \( \omega \) and amplitude \( c_n \) (such that its
energy contribution, \( \frac{1}{2} c_n^2 \) equals 'E' the spectral energy density ordinate at \( \omega' \). Subjecting the one-degree-of-freedom system to this sinusoidally varying force will produce a sinusoidally varying response of the same frequency, \( \omega' \), but with an amplitude dictated by the mechanical acceptance or admittance of the system at that frequency, viz, \( \left( \frac{x_o}{x_{st}} \right) \omega' \) or \( \left( \chi \right) \omega' \) see fig. 70(b)\(^1\).

Fig. 70

1The mechanical admittance represents the maximum response reached by the system for a given frequency of excitation. In practice, none of the large resonant amplitudes indicated by the peak in the mechanical admittance curve will be reached by a system subjected to random loading. At first sight this seems to contradict the assumption made above in which each constituent wave of the random excitation produces an amplitude of response in the system which is the maximum dictated by the mechanical admittance. It must be remembered, in explanation, that the phasing of the constituent waves is purely random and that as such the chance of obtaining the maximum response corresponding to the simple addition of all the constituent amplitudes is, in any given period of time, virtually zero. It might be further argued that the transient response of a system to a given loading is not accounted for by the above approach. In fact, the frequency component in (say) a 'sharp edge' excitation which causes a transient response is fully accounted for by the portion of the response spectrum at the natural frequency of the structure. (A simple physical explanation of the effect of the transient response is given in CHAPTER 8.) Because the constituent waves of the force spectra are (by definition of a continuous spectrum) infinite in length/
The amplitude of the response is \( (X_{\omega} \cdot c_n) \) which corresponds to an energy contribution of \( \frac{1}{2} (X_{\omega} \cdot c_n)^2 \), this is the ordinate of the of the response spectrum at the frequency, \( \omega' \). In this way, for the full range of excitation frequencies, the complete response spectrum may be built up - see fig. 70(c). It will be noticed that the response spectrum may be obtained directly from multiplication (ordinate by ordinate) of the force spectrum with the square of the mechanical admittance. It was shown in APPENDIX A that the mean square of a random signal was equal to the area under its energy density spectrum. The mean square of the response in the system under consideration is, therefore, equal to the total area, \( A_T \), under the response spectrum. In particular, the mean square of the response occurring at or near the natural frequency of the system will be the area, \( A_2 \), contained within the peak, at resonance, the width of which is defined by some pass-band \( (\omega_A < \omega_o < \omega_B) \) centred on \( \omega_o \). It is clear that if the ordinate \( E(\omega_o) \) of the force spectrum is constant within the limits of the pass-band then,

\[
A_2 = A_1 \cdot E(\omega_o)
\]

where \( A_1 \) is the area beneath the square of the mechanical admittance curve in the appropriate pass-band. The area \( A_1 \) was obtained in APPENDIX F(ii) for viscous damping, \( \beta \), viz,

\[
A_1 = (\pi \cdot \omega_o^2 / 4 \beta)
\]

length, it follows that the response of the system to each wave must be the maximum indicated by the mechanical admittance (i.e., the amplitude of response reached after an infinite time).
whence, the RMS of the resonant response may be written,

\[ Y = \sqrt{A_2} = \frac{1}{2} \omega_0 \cdot \sqrt{\pi E(\omega_0)/\beta} \]

Not unexpectedly, the RMS response\(^1\) of the system to random loading at its natural frequency is dependent on the amount of damping, \(\beta\), present in the system and the mean energy level of the excitation, \(E(\omega_0)\), causing the response. Assuming that the damping in the system is increased from \(\beta_1\) to \(\beta_2\), the RMS response will be decreased in the proportion, (assuming the same mean level of random excitation)

\[ \frac{Y_1}{Y_2} = \sqrt{\frac{\beta_2}{\beta_1}}. \]

Analogizing the response of the tower structure to that of a single-degree-of-freedom system and assuming that the effective loading of the natural wind on the structure is random in nature, the effect of the auxiliary mass 'dampers' may now be estimated.

It is convenient to adopt as a reference point the RMS statical response, \(Y_0\), which would result from quasi-static application of the random loading at the natural frequency, such that,

\[ Y_1 = \sqrt{\frac{\omega_0}{2\beta_1}} Y_0 \hspace{1cm} \text{.................. (i)} \]

where \((\omega_0/2.\beta_1)\) is the ordinate of the peak in the mechanical admittance curve. From the various values indicated in fig. 68 the following RMS responses of the structure are anticipated:

\(^1\)The RMS is considered here since it gives a clear idea of the mean 'level' of the resonant random response.
Natural damping\(^1\):

AXIS 1 \( (Y)_{1N} = \sqrt{132} Y_o = 11.5 Y_o \)

AXIS 2 \( (Y)_{2N} = \sqrt{209} Y_o = 12.0 Y_o \)

Auxiliary mass absorber coupled by damping only\(^2\):

Both AXES \( (Y)_{D1} = \sqrt{26} Y_o = 5.3 Y_o \)

Auxiliary mass absorber\(^3\):

Both AXES \( (Y)_{D2} = \sqrt{5.3} Y_o = 2.3 Y_o \)

Very simply, what these figures mean is as follows:

(i) Fitting of the (25 lb.) auxiliary mass absorber coupled by damping only to the tower structure should decrease the resonant motion of the structure in the wind by 55% along AXIS 1 and 67% along AXIS 2.

(ii) Fitting of the 25 lb. 'complete' auxiliary mass absorber should decrease the resonant motion by 80% along AXIS 1 and 82% along AXIS 2.

It must be remembered that increasing the mass of the dampers would improve theoretical performance figures given above — nevertheless the figures quoted for the auxiliary mass absorbers representing, in terms of weight, only 2% of the structure are encouraging.

As a final general conclusion it may be stated that the possibility seems to exist of using auxiliary mass absorbers in tower

---

\(^1\) The frictional damping inherent in the structure is not accounted for here, so that the figures quoted are probably higher than is in fact the case. The larger the vibrations of the structure, however, the less significant the frictional damping becomes.

\(^2\) The experimental value found for the damping is used.

\(^3\) Strictly speaking, it is not possible to apply equation (i) to this case, since the shape of the mechanical admittance departs somewhat from that which is normal for a single-degree-of-freedom system. The value indicated, however, serves as a rough estimate.
structures with low natural damping, with the intention, not so much of preventing a structure from ultimate collapse in a high wind where in any case aerodynamic damping may become so large as to effectively prevent this, but of reducing parasitic resonant vibrations at all levels of excitation. For such purposes the use of an auxiliary mass absorber coupled by damping only, needing very little tuning, and of very simple, robust construction, appears to be the most promising from the practical and economic points of view.

The whole concept of using dampers on tower structures has only been briefly considered in this chapter and it is felt that although first indications appear promising much more research is needed in this field before solid proposals are put forward.
CHAPTER 8.

Response of Tower Structure to Wind
8. Response of Tower Structure to Wind

8.1. Choice of Field Siting

It was noticed that the motion of the tower when subject to windy conditions in its initial siting (i.e., situated in a narrow 'air corridor' between two buildings and surrounded even more closely by a considerable amount of scaffolding, etc. - see fig. 14) was dominated by large amplitude (first mode) resonant vibrations. Study of simultaneous records of the horizontal wind speed near the top and at half the height of the tower structure (obtained from two non-linearized Disa Type 55D50 Constant Current anemometers - 6.1.1.) showed little or no correlation, except, as might have been expected, in the coarser turbulence (i.e., in the build up and subsequent decline of the larger gusts, lasting ten seconds or more). It was also noted that the wind speed at the lower station (15 ft.) was very often larger than at the top of the structure (30 ft.). From these and similar observations it was reckoned that the wind conditions at the site in question were so influenced by the presence and orientation of neighbouring buildings that they would bear little resemblance to the free wind flow conditions to be found in more open locations. While it was, of course, intended to test the response of the structure under wind conditions more representative of those found in general, it was nevertheless realized that practical structures might be positioned in restricted environments such as the above-mentioned. For this reason, several records of the tower response to the wind in its initial siting were obtained; it being considered that any general conclusions drawn from comparison of these records with those


obtained in the field might prove valuable.

The field position for the tower structure which was eventually considered to be the most suitable was in the relatively large expanse of flat open grassy terrain (250 x 100 yds.) situated centrally in The King's Buildings of Edinburgh University on the outskirts of Edinburgh. (A map of the area is given in fig. 71). In order to determine the best exact siting for the tower within this enclosure, rough notes of the wind direction were made for several months prior to its transfer. These showed the wind direction to be almost invariably in the region of WSW - SW\textsuperscript{1}. It was considered that the positioning of the tower fifty yards from the nearest NE building bordering the area concerned in which the instrumentation could conveniently be sheltered would allow for the most adequate exposure to the free wind - i.e., the turbulent wind characteristic of the general roughness of the terrain in the windward direction - without overwhelming influence from any nearby building. In fact, since the nearest buildings were relatively low (this being accentuated by the W-E slope of the land) it was thought that in its proposed site the tower would be reasonably well exposed to the 'free wind' from all directions.

The immediate lie of the land, relative size and position of surrounding buildings, etc. at the proposed site can be gauged from the photograph given in fig. 72. To the N and E the outlying terrain (up to one mile radius) may be described briefly as consisting of residential suburbs of a dense and even character,

\textsuperscript{1}The predominant direction, however, during the six subsequent months during which tests were made turned out to be almost exactly westerly.
FIG 72 Field siting of tower structure
built on slightly undulating land, but generally sloping away from The King's Buildings. To the south the land is hillier with only patches of built-up area while to the west the land, also hilly, gradually slopes upwards into the foothills of the Pentlands with, beyond the immediate precincts of The King's Buildings, relatively few buildings or other small scale obstructions.

8.2. Setting up of the Structure and Associated Instrumentation

As indicated in 2.1.4. matching of the support conditions at the base of the tower in its two sitings was effected by using the same steel base in both cases. (The square base was, in both cases, supported at its extremities on brick piers built up on concrete foundations.) It is not proposed to give a detailed account of the removal operation here: transfer and eventual reinstallation in its new position of the tower structure, together with its base and response transducers; it is sufficient to state that all possible precautions were taken to ensure continuity of the basic physical characteristics, calibrations, etc., of the structure as detailed in previous chapters. In fact, a brief check on the natural frequencies of the structure in its field siting showed them to be the same as before. From the photographs given in the frontispiece and figs. 73 and 74\(^1\) can be gauged the openness of the field siting as well as some idea of the disposition in the field of the tower structure.

Certain of the more important aspects of the latter are given below. For instance, sufficient head clearance for traffic

\(^1\)These photographs show the immediate lie of the terrain in the direction of the prevailing wind.
FIG 73  SW view of field set-up
Fig. 74  NW view of field set-up
in the intervening roadway (running between the field position of the tower and the building accommodating most of the instrumentation - the Cruden Laboratory) was ensured by suspending all the interconnecting cables from a 20 ft. raised platform (situated at about 30 yd. from the tower) - see fig. 75. A means of easy access to this raised platform was provided in order to facilitate the frequent removal and reinstallation of the anemometer-resolver-system cables; an operation necessary each time calibration of a hot wire probe was required. The structural response transducers being relatively robust and insensitive to adverse weather conditions were permanently installed at the base of the tower structure. In this way it was possible at any time (simply by energizing the transducer bridge) to record and observe visually the response of the structure without leaving the sheltered environment of the Cruden Laboratory. The wind transducers, however, being of a more fragile nature (quite apart from their requiring calibration in the wind tunnel prior to use) could not be left outside for any prolonged length of time. The field installation of the wind transducers had by necessity to be effected immediately prior to use; an operation which proved to be exceedingly trying under conditions of high wind. Apart from the running out of the cables from the laboratory to the tower, this operation involved the installation of the vane (fitted to the sin/cos potentiometer) and the hot wire probe to the top of a guyed mast\(^1\) situated close

\[^1\]In order to avoid any undesirable effects resulting from transfer of the tower motion to the anemometric instrumentation, an independent support for the anemometric instrumentation was used. The guyed aluminium mast used, while deflecting visibly in the wind, was characterized by a very low frequency of vibration, a factor which, it was felt, would have a negligible influence on the transducers at its free end.
FIG 75 Easterly view showing laboratory from which measurements were taken
to the tower. This was facilitated to a certain extent by fixing both the sin/cos potentiometer and the hot wire probe holder to the same support which could itself be fairly simply fixed to the top of the guyed mast — see figs. 76 and 77. Once this was done, however, it was necessary to align the principal axes of the potentiometer with the principal axes of the tower structure. This was performed by eye, aligning marks on the potentiometer with marks on the base of the tower.

Because of the limitation on the cable length to the hot wire probe, the anemometer and linearizer units were positioned in the raised platform 30 yd. from the tower, whence they could be put into operation. Once the anemometric instrumentation was set up as described above it was possible to obtain simultaneous records of wind and tower response from inside the shelter of the laboratory building — see fig. 78. More details of the procedure involved are given in the next section.

8.3. Simultaneous Response and Wind Measurements

In a period of six months (May-October 1969) only four occasions of high wind conditions (with an hourly mean speed of 20 m.p.h. or higher) in which both wind and structural response could be successfully obtained, presented themselves. Although there were more occasions of high winds than this, they were either accompanied by wet or rainy conditions or, if this was not the case, occurred during hours of darkness. (As it was, out of the instances in which successful observations were made, three occurred in the evening and one in the early morning — and all during weekends.) Although the local Meteorological Office did frequently accurately forecast windy conditions, it was somewhat
FIG 76  Anemometer-resolver set-up

FIG 77  Field installation
FIG 78  Control of experiment from within laboratory
rare that an accurate prediction of the strength of the wind was made; for this reason, during the six month period mentioned above, the instrumentation was kept in such a state that its use could be called upon at a moment's notice. At the first sign of increasing-to-strong wind conditions, the following procedure was embarked upon:

(i) Calibration of hot wire probes (normally two, in case of damage to one of them) at several linearizer gains. (This procedure involved the transportation of the wind-measuring instrumentation, including visicorder, etc. to the wind tunnel; the duration of this operation being about 2-3 hours.)

(ii) Installation of anemometric instrumentation in the field; \(\frac{1}{2}\)-1 hour.

(iii) Establishment of no-flow or wind zero position: this was performed by keeping the probe wire covered while the anemometric bridge was balanced and a suitable position for the galvanometer ultra-violet traces on the oscilloscript paper obtained (bearing in mind the prevailing direction and strength of the wind to be recorded; \(\frac{1}{2}\) hour.)

(iv) Exposure of hot wire to the free wind by removal of the probe cover; 5 min.

(v) Repetition of (iii) and (iv) if necessary. For instance, if it was found that the UV trace was operating off the oscilloscript paper either a different zero or linearizer gain would have to be selected.

After some experience had been acquired, it was possible to judge the strength of the wind such that calibration at only one linearizer gain was necessary.
(vi) Putting the structural response instrumentation into operation. This was normally carried out some considerable time before actual recording began (½-1 hour) in order to allow stabilization of the carrier amplifiers, etc.

Once the various steps in the above schedule were completed, it was possible to begin obtaining, in the form of side-by-side, continuous traces, both wind speed and structural response components\(^1\). Because it was intended to obtain permanent visual records of the response, it was necessary to take certain precautions against exposure of the oscillograph paper to light. As a result of this restriction, it was not possible to inspect the records obtained until some considerable time later (after latensification of the oscillograph paper). Partly because of this and partly because a direct visual impression of the structural response and wind velocity was thought instructive, both signals were monitored on oscilloscopes. In the same way that the two-dimensional character of the tower response could be reconstituted (by connection of one component signal to the horizontal axis of the oscilloscope and connection of the other to its vertical axis) so was it possible to represent the wind components in single vectorial form, consisting of a 'radius' vector continuously changing in magnitude \(v(t)\) and direction \(\phi(t)\) — see fig. 79.

\[
\begin{align*}
(l/p)_1 & \equiv v(t) \sin \phi(t) \\
(l/p)_2 & \equiv v(t) \cos \phi(t)
\end{align*}
\]

Fig. 79

\(^{1}\) On several occasions when this stage was reached it began to rain and the experiment had to be abandoned; a frustrating, if unavoidable, experience.
The time during which simultaneous recording of the wind velocity and structural response traces was possible was split into two parts.

In the first period the structural response measured was that of the tower in its natural state (ie., without any auxiliary damping), while in the second period was measured the structural response of the tower fitted with the auxiliary mass absorber coupled by damping alone, which was described in CHAPTER 7. As may be imagined, the setting up at the top of the tower of the damping device was, in high winds, a difficult procedure.

In figs 80 and 81 are given typical portions of the oscillographic paper on which are recorded traces of the structural response about the two mutually perpendicular principal axes (AXES 1 and 2) of the tower structure (traces $A_1$ and $A_2$ respectively), as measured by the transducers situated at the base of the structure, together with the traces of the components of the horizontal wind speed (traces $B_1$ and $B_2$) resolved about axes 1 and 2, as measured by the anemometer-resolver near the top of the structure.

Fig. 80 shows the structural response obtained with the auxiliary mass absorber, while fig. 81 shows the structural response without the external damper.

8.4. Discussion of Field Measurements

It must be stated at the outset that the conclusions arrived at in this section concerning the measured and predicted response of the tower structure under consideration are the result of a straightforward preliminary quantitative and qualitative study of the permanent visual records obtained: such as those presented
Fig. 80a  Typical record of structural response ($A_1, A_2$) and wind speed ($B_1, B_2$) — auxiliary damper fitted
Fig. 80b: As for Fig. 80a
Fig. 81a: As for Fig. 80 — no auxiliary damping
Fig. 81b: As for Fig. 81a
in 8.3. The possibility of recording the signals obtained on, say, magnetic tape, enabling a more thorough and mathematically complete investigation to be made through the use of frequency analysers, analogue-computers, etc., and so verifying incontrovertibly some of the ideas put forward, was, at the time of the experiment, not feasible. Nevertheless, it was felt that straightforward visual records of simultaneous wind speed and structural response on a continuous time base such as those obtained would, within the context of the present preliminary study, be extremely useful in helping to promote further insight into the fundamental nature of the problems involved, as well as enabling certain preliminary conclusions to be drawn.

Study of the records in 8.3. will reveal that the structural response along the two mutually perpendicular principal axes of the tower structure is characterized by a constant frequency 'ripple' of randomly varying amplitude. This ripple corresponds to the vibration of the structure about its resonant, first mode or fundamental frequency. Since no fluctuations of a higher frequency than this were obtained it follows that, as was anticipated, no excitation of the structure occurred in the second or higher modes of vibration.

For the purposes of the present discussion it is found convenient to treat as separate quantities the resonant and quasi-steady (or quasi-static components of the structural response. By 'quasi-steady' here is meant that part of the structural response which equals the total response minus the resonant 'ripple' - see fig. 82. The quasi-steady structural response may be reasonably accurately determined from the traces in 8.3. by sketching in the line which divides equally the area enclosed by the envelope of the crests and troughs of the 'ripple'.
FIG 82 Typical examples of records through which the 'quasi-steady' mean has been drawn.
The equivalent 'quasi-steady' wind is defined as that part of the wind turbulence from which results only a quasi-steady or flat-frequency structural response, which in the case of the tower structure would mean all components in the wind with frequencies of up to approximately 3-c/s. for AXIS 1 and 3.3-c/s. for AXIS 2. Study of the traces obtained shows that the size of the turbulent wind fluctuations above 3-c/s. are relatively small and become increasingly so with increased frequency. In fact, though there is clearly 'fine' turbulence present at frequencies well above 3 or 3.3-c/s., this, for the most part, forms, on the time scale of the traces obtained, a rather indistinct and blurry line. Quite apart from the evident fact that resonant excitation does take place in the tower structure, a brief glance at the wind traces will suffice to show that there are numerous random fluctuations which clearly include components having frequencies at or near the resonant frequency of the structure (ie., 3 or 3.3-c/s.). To plot the 'quasi-steady' wind from the wind traces in 8.3., however, by avoiding all components of frequency equal to or higher than 3 or 3.3-c/s., is extremely difficult especially when such components are part of a larger 'sharp-edged' gust fluctuation. As an approximation to the quasi-steady wind it was decided to draw in by eye a mean line through the higher frequency 'blur' and other well-defined fluctuations whose sharpness clearly identifies them as having componental frequencies

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1 The cut-off point would be immediately prior to those frequencies at which resonance would occur, allowing for the width of the resonance peak.
2 This 'fine' turbulence will be very poorly correlated over the height of the tower structure.
much higher than 3 or 3.3-c/s., but following faithfully all other fluctuations.

Inspection of the traces in 8.3. on the basis of the quasi-steady wind and the quasi-steady structural response as defined above for each of the two mutually perpendicular axes of resolution in turn reveals a striking correspondence\(^1\) between the response of the structure and the causal wind with the structure apparently sensitive to all gust fluctuations having a mean duration of at least as small as \(\frac{1}{2}\)-sec. It is worth mentioning here that a preliminary experiment in which a comparison was made between the horizontal wind speed, \(V(t)\), (measured directly, as a single fluctuation, using the anemometric system described earlier but without resolution through the sin/cos potentiometer) and the response of the tower structure along its two principal axes proved very unsatisfactory, with only a very coarse and, indeed, barely perceptible correspondence being noted. In view of the good correspondence achieved, as noted above, when using the resolving system, this, it would appear, was due to the fact that the fluctuating direction of the wind speed, \(\phi(t)\), was not being taken into account. Random fluctuations in the direction of the wind (about the mean hourly direction) observed visually as the oscilloscope's vectorial reconstruction, for the site in question, were surprisingly large and appeared to be composed from a wide range of componental frequencies, with rapid changes (\(< 1\) sec.) of up to \(\pm 45^\circ\) being frequently noted. Clearly, in the case of the tower structure such changes in wind direction

\(^1\)When comparing the traces, it must be remembered that the response will tend to vary with the square of the wind speed.
play a significant part in 'modulating' the wind speed seen by the structure; the wind speed components \( V(t) \cdot \cos(\phi(t)) \) and \( V(t) \cdot \sin(\phi(t)) \) to which the tower responds along its principal axes showing little resemblance to the horizontal wind speed, \( V(t) \). Comparison between energy spectra drawn up for the components \( V(t) \cdot \sin(\phi(t)) \) and \( V(t) \cdot \cos(\phi(t)) \) and for \( V(t) \) alone would, at least for the site in question, no doubt reveal considerable shape differences, with the former possessing relatively more energy at the higher frequency end of the spectrum than the latter. What this means in practical terms, as far as the quasi-steady response of the tower structure in its field siting is concerned, is that there will be more fluctuations in deflection along the principal axes in a given time than if the direction of the wind were relatively invariant.

Adopting a more quantitative approach in comparing the response of the tower structure to the causative wind, it was decided to plot the ordinates of the quasi-steady response obtained along the two principal axes of the structure against the corresponding ordinates of quasi-steady wind speed. A typical plot drawn from several 40 sec. lengths of records such as those given in 8.3., plotting all consecutive ordinates at one second intervals, is given in fig. 83. The response is given in terms of the calibration load '\( P_c \)' (ie., the load which, if applied horizontally to the top of the structure, would result in the same deflection at the point of response measurement (52" above the base plate) as that due to the overall force of the wind). The components of wind speed along the two principal axes (ie., \( V_c = V(t) \cdot \cos(\phi(t)) \) and \( V(t) \cdot \sin(\phi(t)) \)) are given in units of mph.

From the distribution of points in fig. 83 emerges quite
Fig. 83 Structural 'quasi-steady' response v 'quasi-steady' wind speed
clearly the nature of the physical law which governs the relationship between steady air flow and steady response; indeed, the correspondence between the mean curve drawn through the distribution and the square power law, \( P_c = \frac{1}{45} V_c^2 \), fitted to it (where \( \frac{1}{45} \) is an empirical constant) is excellent. The scatter of the distribution, in view of factors to be enumerated below, is surprisingly small, and appears to decrease slightly with increase in velocity such that although the slope of the mean curve increases with velocity, the largest likely error in the effective force exerted on the whole structure for a given quasi-steady wind speed as measured at the top of the structure, expressed as a percentage variation about the mean of the scatter, also decreases with increase in velocity. At a wind speed of 30 mph (this being the component of the horizontal wind speed resolved about one of the principal axes) the maximum range of the scatter is of the order of ± 2 lb. or ± 10\% of the mean 'cantilever-equivalent' force, \( P_c \), of 20 lb.wt. Some of the factors contributing to a greater or lesser degree to the scatter of fig. 83 may be enumerated briefly as follows:

(i) Directional variations in the drag coefficient of the lattice network: the likelihood of such variations in the tower structure under consideration was discussed in 2.1.2. Since the symmetry of the structure repeats itself with every rotation of 30°, it is clear that if there are any intermediate variations in the drag coefficient, these will be reflected in the general response pattern resulting from a turbulent wind showing directional fluctuations of up to ± 45°.

(ii) Variations in wind velocity distribution with height: while the long term (hourly mean, say) velocity profile will be
reasonably invariant and characteristic of the terrain roughness, the instantaneous wind profile as 'seen' by the structure, even in fairly slow gust fluctuations, lasting several seconds, will show departures from this mean.

(iii) Loss in spatial correlation: there may be, in the more rapid 'quasi-steady' turbulence with componental frequencies approaching 3 or 3.3-c/s., some reduction in spatial compass over the height of the structure, resulting in, say, the possibility of a gust of limited compass affecting only the lower regions of the tower structure, its presence being undetected by the anemometer at the top of the structure.

(iv) Variations in terrain roughness with direction: in this case the 'longterm' wind velocity profile as seen by the structure will vary with direction, a factor which will be reflected in the instantaneous profile shape of the incident turbulence with, as a result, variations in the response for a given wind velocity measured near the top of the structure.

(v) Differential rotation of gusts with height: the instantaneous pressure on the tower may not always act in the same direction over the whole height of the structure. The calculations made above are based on the assumption that the instantaneous wind direction is invariant and is that which is measured by the vane system situated near the top of the tower structure.

(vi) Frictional effects within the tower: the structure will never return to precisely its original position after deflection because of the residual amount of friction present within the system.

(vii) Transducing errors, such as calibration changes in the hot wire, overshoot of the wind vane at its natural frequency, etc.
(viii) Other experimental errors, such as misalignment of sine/cosine potentiometer with principal axes of the tower structure (an operation which, it will be recalled, is performed by eye), zero drift in carrier amplifiers, etc.

(ix) Errors involved in determining the 'quasi-steady' response and, more particularly, the quasi-steady wind by eye from traces such as those given in 8.3.

While a process of deduction may make it possible, after prolonged study of the various aspects presented by the traces (such as those to be found in 8.3.) to estimate how much of the scatter of the distribution in fig. 83 is due to each source of 'error' given above, this will not be attempted here\(^1\). Such an investigation would best be effected by an extension to the experiment described in this thesis which, set up, as it might be, to include two or more fast response anemometer-resolvers positioned at several heights in the immediate neighbourhood of the structure, to allow for the possibility of using auxiliary correlation instrumentation and to enable the isolation and independent measurement of the error inducing factors described above, would, apart from greatly facilitating the analysis, prove far more reliable and scientifically acceptable.

From the engineer's point of view, however, the sort of distribution given in fig. 83, with the incorporation of a suitable factor of safety, is acceptable as a basis for design. For practical purposes, the scatter of the distribution is such that factors (i) to (vi) above may be thought of as playing a secondary

\(^1\)It is, in any case, doubtful whether a sufficient quantity of records to permit this was obtained.
role in the quasi-steady response of the structure. As a result the quasi-steady force per unit area exerted on the structure at a height $Z$ may be assumed to follow the simple law:

$$ (F)_z = \frac{1}{2}\rho V_z^2 C $$  \hspace{1cm} (i)

where $V_z$ is the quasi-steady wind velocity at height $Z$ and varies with $Z$ according to the long term (hourly mean) vertical profile law characterized by the power index, $n$. The maximum bending moment (say) experienced by the structure due to the quasi-steady wind can then be easily calculated, viz. (from equations (i) and (ii), Case 2, APPENDIX F),

$$ M_{z=0} = \frac{1}{2}\rho V_z^2 CAZ [1/2(n+1)] $$

More generally, if the quasi-steady wind speed $V_{Z3}$ is measured at a reference height $Z_3$,

$$ M_{z=0} = \frac{1}{2}\rho V_{Z3}^2 (Z_1/Z_3)^{2n} CAZ [1/2(n+1)] \hspace{1cm} (ii) $$

If $(V_{Z3})_{max}$ is the highest quasi-steady wind speed likely in fifty years, say, the highest quasi-steady maximum moment likely in the structure (in fifty years) is,

$$ (M_{z=0})_{max} = \frac{1}{2}\rho (V_{Z3})_{max}^2 (Z_1/Z_3)^{2n} CAZ [1/2(n+1)] \hspace{1cm} (iii) $$

Faced with this expression the designer is put into the position of having to estimate the following parameters: $(V_{Z3})_{max}$, at

---

1. the projected area of the structure in the plane vertical to the direction of the wind
2. The question: To what types of structures in general can this simplified approach be applied? will be discussed later. At the moment the analysis is based on the results of the tower structure experimented upon.
a reference height $Z_3$, the power index, $n$, and the aerodynamic drag coefficient of the structure, $C$. For a given structure the other quantities, $A$, the plane area of the structure in projection and, $Z_1$, the height of the structure, will be known constants. Each will be considered in turn.

(i) Estimation of $(V_{Z_3})_{\text{max}}$

The quantity $(V_{Z_3})$ represents the 'quasi-steady' horizontal wind speed at height $Z_3$. It will be remembered that by 'quasi-steady wind' was meant that part of turbulence leading to a quasi-steady or flat frequency response in the given structure, in other words, the combination of all componental frequencies in the wind up to a frequency just below the resonant frequency of the structure.

The validity of assuming that a structure similar in size to the tower structure described in this thesis will respond quasi-statically (according to the formula given above) to a gust with a duration just slightly larger than the natural period of vibration of the structure (in its fundamental mode) may be questioned on the grounds that when this period is very small the spatial compass of a gust of corresponding period will also be small in relation to the size of the structure. The point to remember here is that although this may in fact be true the velocity fluctuation which a short duration, poorly correlated gust represents (at least on the spatial scale of the thirty foot tower structure) will, as a proportion of the velocity fluctuation of lower frequency gusts to which the structure is fully sensitive, also be small. The error invoked in assuming that the structure responds
to these 'high frequency' gust fluctuations is therefore secondary in magnitude. Further, it must be remembered that in conditions of extremely high wind (a condition for which, after all, the structure is being designed) the spatial correlation of the higher frequency gusts will increase (ie., spatially well correlated, slower turbulent fluctuations to which the structure responds quasi-statically under conditions of moderate wind, will, in a stronger wind, be 'carried along' at a higher mean wind speed and, as such, will be 'seen' by the structure at a higher range of frequencies). This last point may explain the decrease in percentage 'error' with increase in wind speed noted in the distribution in fig. 83.

On the basis of the experimental results obtained for the tower structure and in the context of what was said in the last paragraph it would appear reasonable to suggest that the largest quasi-steady response of any tower-like lattice structure with a height of up to 50-60 ft.\(^1\) and with a fundamental natural frequency greater than 3 c/s. may very roughly be calculated from equation (iii) on the basis that the structure will respond fully to the highest absolute wind speed likely to be obtained in any given period of time. Be this as it may, it is felt that a generally more valid and accurate estimate of the maximum quasi-steady response in any lattice tower structure with a height of up to 50-60 ft. would necessitate obtaining the highest likely quasi-steady wind speed as previously defined; this would, of course, require some estimate of the fundamental resonant frequency.

\(^1\)This is a conjectural extrapolation made from the results of the experiment actually performed. Clearly, the higher the structure, the more the need for taking into account the effect of loss in spatial correlation of gusts.
frequency of the structure to be made.

No meteorological or other wind records enabling the direct determination of the highest likely quasi-steady wind speed as required above, however, appear to be available. Indeed, it would seem that because of frequency limitations imposed by present anemometric instrumentation the shortest discriminatory period of wind speed either recorded or likely to be recorded in the immediate future on a widespread scale is of the order of as long as three seconds. Design of a tower structure such as the one experimented upon, say, on the assumption that the highest likely quasi-steady response would be that resulting from the highest likely wind speed averaged over three seconds might lead to serious error. In order to emphasise this point it is worthwhile describing one instance out of several when the tower structure situated in its field siting was subjected to what may be called 'freak' gusts of exceptionally large magnitude and short duration. A record of such a gust is shown in fig. 84 (marked 'A'). The peak of the gust shown corresponds to a horizontal wind speed (computed from the two components given in the record) in the region of 70 mph., and what is perhaps more significant, is that the duration of the gust (i.e., the time during which the same significant departure from the mean wind speed is observed) is of the order of one second. Though the response of the tower structure to this gust was not obtained on the oscillographic paper record¹, it was possible to make an estimate

¹In view of the mean wind speed at the time (about 30 mph. over the five minute period including the gust under discussion) this extreme was not anticipated.
Fig. 84 Record of response and wind speed with exceptionally large gust (auxiliary 'damper' fitted)
of the maximum quasi-steady response from the two-dimensional travel of the cathode-ray 'spot' which was seen to move (a somewhat startling distance!) to the edge of the screen of the monitoring oscilloscope. From this it was possible to sketch in roughly the quasi-steady response of the structure to the gust - see fig. 84. A simple calculation shows that the structure responded to a good degree of approximation to the gust according to the quasi-steady power law defined empirically from fig. 83 (ie., \( P_c = \frac{1}{45}V_c^2 \)): for the axis along which the direction of the gust is most closely aligned the maximum response obtained by direct measurement corresponds to the application of a free-end lateral load along that axis of \( P_c = 90 \text{ lb. wt} \); calculated by substituting the measured maximum quasi-steady wind speed along that axis into the empirical formula, \( P_c = \frac{1}{45}(67)^2 = 100 \text{ lb. wt.} \).

Were the highest wind speed seen quasi-statically by the structure to be taken as the highest average over three seconds, the response of the structure to the 'freak' one second gust as described above would not be accounted for. Indeed, the average wind speed over the three second interval which includes the freak gust (along the more exposed principal axis of the structure) being approximately 45 mph, would result in a computed response equivalent to that caused by a lateral end load, \( P_c = \frac{1}{45}(45)^2 = 45 \text{ lb. wt.} \); the discrepancy between this value and the true maximum response need hardly be emphasised!

The following method is suggested as a means of obtaining the highest likely quasi-steady wind speed \((V_{z_3})_{\text{max}}\) (at a reference height \(Z_3\)) in a given period of time. The method is based first of all on being able to predict, for the site in question, the highest mean hourly wind speed \((\overline{V_{z_3}})_{\text{max}}, \text{ say}\) which is likely to be

\[ \text{It may be worth noting that this corresponds to a lateral deflection of 0.24" at the top of the tower structure.} \]
obtained during the proposed life of the structure (which may be of the order of fifty years). Without going into detail, it would appear that a reasonably good estimate of \( (V_{Z_3})_{\text{max}} \) can at present be made for most localities on a straightforward interpolative and statistical basis from the extensive meteorological records of mean hourly wind speeds which have been accumulated over the years on a countrywide basis. (Alternatively, the 'gradient wind' approach may be used to estimate \( (V_{Z_3})_{\text{max}} \); this necessitates the estimation of the extreme mean hourly gradient wind speed in the locality of the proposed site, the gradient height and the surface roughness of the terrain – see APPENDIX A.)

Consider the turbulent fluctuations which will be associated with the extreme hourly mean wind speed, excluding all fluctuations corresponding to frequencies higher than or equal to the fundamental frequency of resonance of the structure – see fig. 85.

If the maximum deviation from the mean is \( (u_{Z_3})_{\text{max}} \), then it follows that the maximum quasi-steady wind speed (at height \( Z_3 \)) to
which the structure is subjected (in fifty years, say) is,

\[ (V_{Z3})_{\text{max}} = (V_{Z3})_{\text{max}} + (u_{Z3})_{\text{max}} \]

From purely physical considerations it can be argued that the quantity, \((u_{Z3})_{\text{max}}\), must, in a given time interval, ultimately depend on the energy level or 'amount' of turbulence present and on the frequency distribution of this turbulence - factors which will in turn be dependent on the mean hourly wind speed at a given height and the roughness of the terrain. The exact nature of the dependence of \((u_{Z3})_{\text{max}}\) on these factors is derived in APPENDIX H on a statistical basis assuming that the variation of the quasi-steady wind speed about the mean hourly wind speed is random and varies according to a Gaussian probability distribution. It is shown in APPENDIX H that the highest likely quasi-steady peak in the turbulent wind (expressed as a departure from the mean wind speed) in a period of duration \(T\), is given by,

\[ (u_{Z3})_{\text{max}} = \sqrt{2m_0} \left[ \log_e(N/\sqrt{\pi}) \right]^{1/2} \quad (N \geq 1000) \]

where \(N\) is the number of maxima in the time interval \(T\), and may be calculated from,

\[ N = \frac{1}{4\pi^2} \sqrt{m_4/m_2} \cdot T \]

where \(m_0\) is the area of the 'cut-off' energy spectrum characterizing the quasi-steady turbulent fluctuation and \(m_2, m_4\) are its second and fourth moments respectively, viz,

1If \(N < 1000\), the quantity \((u_{Z3})_{\text{max}}/\sqrt{m_0}\) can be read off directly from the graph in fig. H9.
\[ m_0 = \int_{0}^{n_0} E(f) \cdot df \]
\[ m_n = \int_{0}^{n_0} E(f) \cdot f^n \cdot df \quad (n = 2, 4) \]

By the 'cut-off' spectrum is meant the customary energy spectrum drawn up for the turbulent wind speed but with the exception that all energy contributions corresponding to the frequency range above the fundamental natural frequency of the structure, \( n_0 \), are excluded.

As it will be realized the above determination of \( (u_{z_3})_{\text{max}} \) depends essentially on being able to obtain the energy spectrum pertaining to the turbulent wind as governed by the surface roughness and the extreme mean hourly wind speed \( (V_{z_3})_{\text{max}} \) at the site in question. The only present means of estimating the wind spectrum relevant to a particular location (apart from the direct field determination of wind spectra at the location itself for a range of mean hourly wind speeds and subsequent extrapolation to find the spectrum apposite to the highest likely mean hourly wind speed) is to operate on Davenport's reduced or universal wind spectrum by means of the parameters controlling the turbulence at the site during the period of highest (likely) wind (i.e., the roughness of the terrain and \( (V_{z_3})_{\text{max}} \) at the reference height \( z_3 \)). The possibility of error incurred in using the reduced wind spectrum - especially in the higher frequency ranges - was discussed briefly in APPENDIX A. It is to be hoped that as more and more general information concerning wind spectra becomes available so it will become increasingly possible to make more accurate estimates of the influence of the nature of the terrain on the turbulent wind so enabling greater and more accurate use
of the reduced spectrum, the idea of which appears in theory to be basically sound.

(ii) **Estimation of the Power Law Exponent, n**

A rough estimate of the power index, $n$, at the site of the tower structure, for the prevailing westerly windward direction was obtained using two DISA TYPE 55D50 non-linearized hot wire anemometers positioned at 15 ft. and 30 ft. above ground level. The average value so obtained\(^1\) was in the region of $n = 1/3$. In view of the open character of the immediate foreground of the tower structure (in the direction of the prevailing wind) this value may, at first sight, appear somewhat high. In fact this exponent probably reflects the relatively high level of turbulence at the site, turbulence which, it is felt, is generated by the fairly rugged topography characterizing the lower slopes of the Pentland Hills and, on a smaller scale, the large buildings situated at a distance of 200-400 yds. from the tower structure - the advective influence of the flat foreground is probably quite small. In view of the nature of the general topography of the terrain and its disposition in relation to the direction of the prevailing wind, it would prove interesting to study the possibility of the occurrence of 'lee-wave' storms at the site in question and the connection, if any, with the 'freak' gusts of extremely short duration noted by the author. Such

\(^1\)Because of the rapid loss in sensitivity with rise in wind speed in the anemometers used (see 6.1.1. for a typical calibration curve), it was difficult to assess accurately the mean wind speed over a long period of time; for this reason the power index $'n'$ was estimated from fairly short period means (2-5 seconds).
a study would also involve determining the extent to which the 'freak' gusts measured are covered by the 'once-in-fifty-years' approach described above. (It might be found to be the case that the freak gusts observed are not 'freak' gusts at all and are, quite simply, the extremes expected from time to time in a random fluctuation.)

(iii) **Estimation of the Drag Coefficient, C**

It can be shown for the tower structure under consideration (see APPENDIX F, Case 2) that the quasi-steady equivalent (or calibration) free-end lateral load $P_c$ which results in the same response from the displacement transducers situated near the base of the tower as that resulting from the quasi-steady wind load distributed vertically according to a wind speed profile of exponent 'n' and defined by a quasi-steady velocity $V_{Z_3}$ at a height $Z_3$, is,

$$
P_c = \frac{1}{2} \rho V_{Z_3}^2 (Z_1/Z_3)^2 n \cdot CA \cdot \lambda (2n+1)^1
$$

where $Z_1$ is the total height of the structure, $C$ is the drag coefficient assumed constant with height, $A$ is the projected plane area enclosed by the frame of the structure, and $\lambda$ is a coefficient which relates the total wind load to the equivalent free-end load $P_c$ and is given for a range of power indices, $n$,

---

1 The reason for using this expression for the tower experiment lies in the fact that the response of the structure was measured as a deflection at a given height above the base (thereby requiring an allowance to be made for flexure of the base plate, etc.) and not in terms of (what might constitute a more usual criterion of response) the bending moment at the base, $M_{z=0}$. 
in fig. 34. The above expression enables the mean drag coefficient, $\overline{C}$, of the tower structure to be determined. Using the proportionality constant of the empirical 'mean' curve indicated in fig. 83 together with the known constants of the tower experiment ($Z_1 = 368$ in., $Z_3 = 342$ in., $\bar{A} = \bar{D}Z_1 \approx 68$ sq.ft.) the relationship between $\overline{C}$ and $n$ is that shown in fig. 86. It can be seen from fig. 86 that the magnitude of $\overline{C}$ is not particularly sensitive to variation in the exponent, $n$. This means that although only a rough estimate of the exponent was obtained ($n \approx 1/3$) at the site in question, the mean drag coefficient of the tower structure can be estimated with some accuracy, viz., from fig. 86, $\overline{C} \approx 0.32$.

This is an interesting result since numerically it corresponds closely to the mean solidity ratio of the structure - i.e., the ratio of the mean projected area of all individual members of the structure (inclusive of the projected area of the leeward members seen from the windward side) to the total mean area enclosed by the frame - $\psi \approx 1/3$. This means that the average drag coefficient of the individual members is approximately unity. The usual coefficient used for members of circular section is 1.2; the straightforward summation of the forces on the individual members of the plane projection of the structure (which automatically precludes any contribution from the hidden leeward members) calculated on this basis is tantamount to using

---

1 The mean drag coefficient, $\overline{C}$, is the mean of the variation in $C$ with rotation about the vertical axis of the structure.

2 See REF: 10. (The value given is for $DU_z < 50$, where $D$ is the transverse dimension of the structure normal to the flow (ft.) and $U_z$ is the flow speed (mph).
Fig. 86. Experimental determination of $\bar{C}$ for tower structure
an overall drag coefficient of $1.2 \times \frac{1}{3} = 0.4$. While this is not too large an overestimation, it must be emphasised that the basis on which the value is obtained is an oversimplification of what actually occurs. It must be remembered that a large proportion of the members constituting the bracing does not present itself normally to the wind and that, as such, the forces on these members will be substantially smaller than allowed for by the drag coefficient of $1.2$ as applied to the projection of the members in the plane normal to the wind direction. The force exerted on the leeward members in the wake of the windward members will compensate somewhat for this loss. An overall value of $0.32$ for the mean drag coefficient of the structure does not appear unreasonable in view of the above factors.

While for the tower structure in question the mean drag coefficient, $C$, could probably have been estimated fairly accurately on a semi-intuitive basis, this is not very satisfactory from a designer's point of view. In fact, it has been attempted (Flachsbart) to formulate the problem in terms of the spacing ratio between frames, solidity ratio, etc. on a semi-empirical basis (using wind tunnel measurements), this approach, however, appears to be restricted to multiple plane-frame structures and certainly could not be applied to the tower structure in question. Rather more simple empirical formulae involving only the total 'see-through' solidity ratio would perhaps be of greater use in the case of slender lattice-type tower structures. Alternatively, it may be that a direct estimation of the drag coefficient by measurement of the drag force exerted on a model of the proposed structure in the steady flow of a wind tunnel could be effected. In this context it would prove interesting to compare
the drag coefficient obtained in the field for the tower structure with that of a scaled-down model obtained in the wind tunnel. It can safely be concluded that considerable research is still needed in this domain.

8.4.2. The Resonant Component of Structural Response

This section deals with the vibration observed in the tower structure at its fundamental frequency when subjected to the buffeting of the wind in its field siting. It will be seen from the response records in 8.3. that this vibration takes the form of a 'ripple' of constant frequency but of randomly varying amplitude, superposed on the 'quasi-steady' response defined in the preceding section. Before attempting to discuss the magnitude of this ripple it is important to discuss certain aspects of the two-dimensional behaviour of the system involved. This can perhaps best be effected through consideration of certain hypothetical instances of external structural excitation. Consider subjection of the tower in question, or, for that matter, any other fixed-free cantilever structure showing a polar variation in stiffness, to the following cases of loading:

CASE 1 A time-varying load, \( p(t) \), applied in a fixed direction coincident with an axis of principal moment of inertia

For the sake of the present argument it is assumed that \( p(t) \) consists of a randomly varying concentrated load applied laterally to the free-end of the structure. It is further assumed that \( p(t) \) is formed from a wide range of constituent frequencies which includes the fundamental frequency of the
Since a condition of symmetrical bending exists about an axis of principal moment of inertia (the product of inertia being zero - see Appendix D(i)) it follows that any structural motion which takes place is constrained in direction to that of the applied loading (along XX in fig. 87). Displacement transducers coincident with the principal axes of the structure would be expected to pick up the sort of signal outputs shown; viz, zero response or displacement along YY and, assuming here that second and higher modes are not excited, a quasi-steady response with a superposed constant \(^1\) frequency resonant ripple along XX. If the long-term mean of the excitation \(p(t)\) is \(\overline{p(t)}\) then the mean of the structural response along XX expressed as a displacement will be equal to \(\frac{\overline{p(t)}}{K}\) where \(K\) is the stiffness of the structure with respect to its free end.

\(^1\)It is assumed that since all motion is well within the elastic range of the structure variations in the resonant frequency will be insignificant.
CASE 2  A constant load, $P$, fluctuating randomly in direction about a mean coincident with an axis of principal moment of inertia

Assume that (at time $t$) the direction in which $P$ is applied is at an angle $\phi(t)$ with the given axis of principal moment of inertia, such that the long-term mean of the fluctuation is $\overline{\phi(t)} = 0$.

As for the fluctuation $\rho(t)$ in CASE 1 it is assumed that the fluctuation $\phi(t)$ is composed from a wide range of constituent frequencies which includes the fundamental frequency of the structure about the relevant axis of principal moment of inertia. At time $t$ the loading applied to the free end of the structure, $P$, may be resolved as follows:

- $P \cos \phi(t)$, in line with the long-term mean direction, $\overline{\phi(t)} = 0$ and $P \sin \phi(t)$, transverse to the above direction.

If the directional fluctuation is assumed to be small, $\cos \phi(t) \rightarrow 1$ and $\sin \phi(t) \rightarrow \phi(t)$ and the load components become:

- $P$, along $XX$.
- $P \phi(t)$, along $YY$. 

![Diagram of two axes and fluctuation about an axis of principal moment of inertia](attachment:image.png)
In other words the sideways excitation (along YY) to which the structure is subjected approximates to a linear function of $\phi(t)$ and as such will display the same frequency characteristics. It follows that, although the amplitude of the directional fluctuations as expressed by $\phi(t)$ may be small, the product, $P\phi(t)$ resulting from a large value of $P$ may be sufficient to induce substantial resonant vibrations along the transverse axis of the structure; the exact amplitude of this sideways vibration will, of course, be dependent (for a given load $P$) on the energy content of the fluctuation $\phi(t)$ at the appropriate frequency and on the structural side, the amount of damping present. In line with the long-term mean direction of the applied loading, $P$, (along XX) it can be seen that for small fluctuations in $\phi(t)$ the structure is, to a first order of approximation, subjected to the steady load $P$ and will deflect accordingly. Displacement transducers picking up the motion along the two principal axes would be expected to produce an output similar in nature to that sketched in fig. 88 - that is, a relatively large resonant ripple superposed on a quasi-steady fluctuation possessing a mean zero response along YY and a large steady response showing small resonant and quasi-steady departures along XX.

It can be concluded from the two cases of applied loading considered so far that, in general, as far as the resonant part of the structural response is concerned, a random fluctuation in the magnitude of the applied load, $p(t)$, will tend to induce motion in line with the load while a directional fluctuation, $\phi(t)$, in the load will tend to promote a predominantly sideways vibration.
CASE 3 A time-varying load, $P(t)$, fluctuating randomly in direction, $\phi(t)$, such that the mean direction is coincident with an axis of principal moment of inertia.

Let the long-term mean of the loading, $P(t)$, be $\bar{P}(t)$ such that,

$$P(t) = \bar{P}(t) + p(t)$$

where $p(t)$ represents the variation about the mean. The instantaneous force on the system may be resolved along the axes of principal moment of inertia, assuming $\phi(t)$ to be small, as follows

along $XX$: $P(t) = \bar{P}(t) + p(t)$

along $YY$: $P(t)\phi(t) = \bar{P}(t)\phi(t) + p(t)\phi(t)$

It is seen that the nature of the excitation is a combination of CASES 1 and 2 considered above with the addition of the component $p(t)\phi(t)$. As far as the resonant response is concerned this latter component is interesting in so far as it brings into play frequency components of both the load and directional fluctuations which are not necessarily coincident with the (transverse) resonant frequency. It will be assumed here, however,
that both \( p(t) \) and \( \phi(t) \) represent small fluctuations in comparison with the magnitude of \( P(t) \) and that as such the product \( p(t) \cdot \phi(t) \), while contributing to the transverse resonant vibration, will only do so to a secondary degree. The sort of transducer response anticipated is that which is sketched in fig. 89. Quite apart from damping and other structural considerations, it is important to realize that the extent to which resonant motion develops, either in line or transverse to the mean wind direction, is dependent on the relative amounts of energy in the fluctuations \( p(t) \) and \( \phi(t) \) (and on the amplification of \( \phi(t) \) by the steady component \( P(t) \)); it is, for instance, quite possible for the transverse resonant vibration to be larger than the in line vibration. It must be remembered that, since both vibrations must occur simultaneously, the system will perform a Lissajous-type gyration.

### CASE 4

The same as for CASE 3 except that the mean of the directional fluctuation is not coincident with an axis of principal moment of inertia.

Apart from unsymmetrical bending action, resulting from the steady component \( P(t) \), the same sort of response as in CASE 3 would be anticipated at resonance, that is, with the directional fluctuation favouring sideways motion and the load fluctuation promoting in line vibration. The difference in this case lies in the fact that, since the transducers picking up the motion are aligned with the principal axis of the system, they will pick up components of vibration which arise from both the directional and load fluctuations. This makes it virtually impossible, at least from mere inspection of the transducer records,
to determine which vibration arises from which source.

Reverting to the discussion of the resonant motion of the lattice-tower structure, the question immediately arises: how much of the resonant vibration observed originates from transverse excitation (due to fluctuation in wind direction) and how much originates from in-line excitation (due to straightforward fluctuations in wind speed). As noted above (Case 4) it is not feasible to attempt an answer to this question for a wind fluctuation whose mean direction is not coincident with an axis of principal moment of inertia, which eliminates the use of such wind-response records as are presented in 8.3. In fact, during the period of experimentation it was possible to obtain only one brief record in which the mean wind direction was roughly in line with a principal axis of the tower - part of this record is shown in fig. 90. It will be noted that as in the previous records discussed the quasi-steady deflection of the structure, occurring predominantly about the stiffer axis (see trace A2), faithfully follows the quasi-steady wind resolved in the same direction (trace B2).

The wind speed recorded in the direction of the other principal axis (ie. trace B1) may be considered as fluctuating randomly about the wind speed zero and as such can be thought of as analogous to the quantity $P(t)\phi(t)$ defined above. While a certain proportion of the quasi-steady variations in the wind speed (B1) are, in magnitude, almost of the same order as the fairly low long-term mean wind-speed of the record (which is in the region of 20 mph.) the larger variations are slow in comparison with the natural frequencies of the structure. It follows, therefore, that a large proportion of the fluctuation B1 transverse to the mean flow of the turbulent wind
FIG 90  Record of response and wind speed with mean wind direction coincident with axis of principal moment of inertia
results purely from fluctuations in wind direction. That a large proportion of these fluctuations are rapid and occur within a frequency range which includes the transverse fundamental frequency is clear from inspection of the record presented. In fact, since the frequency response of the vane measuring the wind direction is dependent on the wind speed it is considered quite probable that under the low wind speed conditions of the test a proportion of higher frequency directional fluctuations would not be recorded. Inspection of the structural response in the transverse direction indicates considerable vibration at the resonant frequency. In fact, by measuring the amplitude of successive peaks and allowing for the difference in scale of the two structural response traces (Scale A1 = 1.7 Scale A2) the transverse vibration measured over a limited period of two minutes is found to be approximately 40% larger than the in line vibration. This is a very interesting result and suggests, if the above analysis is correct, that large side-ways vibrations in a structure may not necessarily be the result of vortex shedding but simply due to rapid directional fluctuations in the wind. Of course other factors may be involved, for instance, the aerodynamic damping of the structure may be less in the transverse direction than in line with the mean wind. (It must be remembered that the transverse excitation is operating about zero while the in line excitation at resonance is superimposed on the quasi-steady wind speed).

The above conclusion is drawn from a somewhat limited amount of information and must, therefore, be treated as tentative. It is nevertheless felt that the particular aspect of sideways excitation resulting from directional changes in the wind is
worthy of further investigation. An interesting series of experiments would be to repeat the above experiment with a smaller structure (ensuring full spatial correlation of directional changes) first of all comparing the spectral energy content of the directional and wind speed fluctuations over a wide range of frequencies and then relating this to the structural response (in spectral form) in both the in-line and transverse directions. Measurement of the total motion of the structure and the total (natural) wind in this way would enable a meaningful quantitative and qualitative study of the effect of aerodynamic damping, say, to be made. An extension of the experiment would be to progressively increase the size of the structure with the aim of determining the effect of loss of spatial correlation in both directional and wind speed fluctuations - this would not be the easiest of tasks!

As far as the overall largest response of a structure is concerned, vibrations occurring transversely to the mean wind flow must be considered in true perspective. For instance, assuming, for the sake of argument, that the largest likely simultaneous resonant displacements in the transverse and in line directions are of the same order of magnitude and that the largest likely quasi-steady response is at least as large as, if not larger than, the maximum resonant displacement; it will be
seen from fig. 91 that the extra contribution (a, say) is small in relation to the magnitude of the in-line displacement, A.

So far this section has been devoted to a discussion of the two-dimensional behaviour of the tower structure at resonance as developed by the natural wind. Attention is now turned on the problem of estimating the magnitude of the resonant vibration about any given axis of principal moment of inertia. To understand on a qualitative or physical basis how the sort of resonant ripple shown in the given records of structural response is produced as a result of buffeting from a turbulent wind whose wind speed variation shows no immediately apparent agreement with the structural resonant ripple is of great importance.

Consider the subjection of the tower structure to a sinusoidally varying lateral load (at its free end and along a principal axis) of the same frequency as the resonant frequency. The resonant motion of the structure will gradually build up from rest and continue increasing till such time as the damping losses (which usually increase with amplitude) become so large as to dissipate all extra energy supplied to the system, whence a maximum amplitude is reached; the rate at which the damping losses increase with amplitude controls the size of the maximum amplitude reached as well as the time interval before this state of dynamic equilibrium is approached. The first few vibrations of resonant build-up are shown in fig. 92 for a system with a natural frequency of 3.3 c/s. and a viscous damping ratio,

\[ \dot{c} = 0.3 \] (corresponding to approximately 1.5% critical damping)\(^1\).

\(^1\)Note that the response lags the exciting force by a quarter cycle and that in the given case the maximum response is about 35 times greater than the equivalent static response.
FIG 92
Effect of phase change at resonance

\[ f = \frac{\omega}{2\pi} = 3.282 \text{ c/s} \]

\[ t = \frac{1}{2\pi} \times \frac{3}{2} \text{ sec} = 0.175 \text{ sec} \]
Assume now that after a time interval during which the motion has built up over 5 cycles, say, the phase of the exciting force is suddenly changed (by \( \pi \), say). It can be shown that the resonant motion immediately embarks on a relatively rapid attenuation the envelope of which is marked 'H'. (It is worth noting that if the exciting force were suppressed altogether the subsequent motion would decay freely as indicated by the envelope marked 'F').

The phase change in the exciting force need not, of course, occur instantaneously; it may take place over several cycles resulting in a smooth transition from amplification to attenuation in the structural response.

While the response of a single-degree-of-freedom system to a phase change in an applied sinusoidal excitation represents only a simple model, the principle involved does nevertheless lead to an understanding of the response resulting from more complicated loading. Consider successive gusts impinging on a structure. These gusts may have constituent components with frequencies at or near the fundamental frequency of the structure. These components can be thought of as arriving as a succession of sinusoidal impulses (each discrete impulse probably being rarely greater than a single cycle in duration) all with randomly different duration, amplitude and phase. The structure will respond to each discrete impulse in turn in such a way that the

\[ (x_\omega) = x_{ST} \left[ e^{-\beta t} \left( \cos \omega t - \beta / \omega \sin \omega t \right) - \cos \omega t \right] \omega / 2 \beta \]

\[ + 2x_{ST} \left[ e^{-\beta (t+T)} \left( \cos \omega (t+T) - \beta / \omega \sin \omega (t+T) \right) - \cos \omega (t+T) \right] \omega / 2 \beta \]

with the phase change of \( \pi \) occurring at \( T = 11\pi / \omega \).
previously built up resonant motion is either reinforced or attenuated. In fact the sort of traces of structural response obtained represent, as far as the resonant 'ripple' is concerned, what may be called a continuous process of successive random amplification and attenuation. Because the structure only absorbs or loses 'resonant energy' at a fixed rate there will rarely be any immediately obvious correspondence between simultaneous records of resonant response and wind speed. Equally, it does not necessarily follow that during a period in which the wind turbulence reaches a peak energy level (as far as the frequency component at structural resonance is concerned) a peak amplitude in the resonant response will result. In fact it can be argued that whether a maximum response amplitude is obtained or not is a matter of chance.

Although the above offers a simple physical explanation of what happens at resonance it is clear that the quantitative problem of estimating the highest likely amplitude will be based first of all on a statistical knowledge over a fairly lengthy period of time of the fluctuation in wind speed at the appropriate frequency. It is then possible by using the parameters characterising the statistical record of the wind to operate (by the drawing up of spectra (as indicated in fig. 70) for instance) on the structural and aerodynamic parameters of the structure to obtain similar statistical parameters characterising the wind-induced resonant response over the same period of time. Then, once obtained, these parameters can in turn be operated upon according to the probability theory described in APPENDIX H — that is, knowing that the response is in the form of a random fluctuation following a recognised distribution and having
quantified this distribution a prediction as to the highest resonant peak or maximum likely to be obtained in a given period of time is made.

It is tempting at this stage to suggest that the highest likely total structural response will be the straightforward summation of the highest likely quasi-steady response as obtained in the preceding section and the highest likely maximum dynamic response, that is, the amplitude of the largest likely ripple obtained in the same interval of time. This, however, would be incorrect as the probability of both occurring simultaneously would in the sort of case being dealt with (and on the necessarily limited time scale involved) be very small indeed. At all events, it may be the case that a particular quasi-steady displacement occurring simultaneously with a particular dynamic oscillation, neither of which is, according to its respective kind, a maximum departure, combine to provide the maximum total displacement in the given period of time. It follows from the above argument that, as far as the determination of the highest likely response in a structure which simultaneously undergoes both quasi-steady displacement and resonant vibration is concerned, neither quasi-steady nor resonant components can be considered in isolation.

A rough check carried out on a number of records obtained from the lattice-tower experiment showed that while the quasi-

\[ \text{The highest likely mean wind speed in a time interval of one hour in a given number of years is usually considered.} \]
steady response fluctuation tended toward a normal distribution about the mean of the record concerned the randomness of the peaks of the resonant ripple (relative to the quasi-steady fluctuation) tended toward a Rayleigh distribution. (Although this was not checked, it was estimated that were discrete values of the whole ripple taken this would form a normal distribution.) The particular case of a fluctuation whose peaks obey a Rayleigh distribution superimposed on another fluctuation obeying a normal or Gaussian-type distribution is discussed in some detail in Appendix H. Particular reference is made to fig. H5 which shows how this type of fluctuation can be considered as transitional between a random fluctuation obeying a pure Rayleigh distribution and one obeying a pure Gaussian law.

Assuming that the type of fluctuation obtained experimentally from the lattice-tower structure is of the type given theoretical consideration in Appendix H — and though this needs further verification there appears to be no fundamental reason from the results actually obtained why this should not be the case — it

In the case of the quasi-steady response a histogram was built up from counting the number of occasions on which the trace fell between two adjacent oscillograph scale lines at regular intervals of 1-sec. Because of the very low frequencies involved (gusts building up over 30-secs. or more) it was considered that the number and duration of the records obtained was insufficient to obtain the full development of the distribution; however, on the basis of the histograms drawn it was felt that a normal distribution would eventually emerge. The drawing up of similar histograms for the peaks of the resonant ripple was rather more tedious as it involved the actual measuring of the amplitude of each peak. Again, it was felt that the records were not of sufficient duration to establish incontrovertibly a Rayleigh-type distribution, although the trend appeared to be in that direction; because of the low damping of the structure the decay and build-up of the resonant motion would have underlying trends lasting up to several minutes.
follows that the highest likely maximum in the response of such a structure in a period of time, \( T \), is given by the expression,

\[
R_{\text{max}} = \eta_{\text{max}} \sqrt{m_0} = \sqrt{2m_0} \left[ \log_e \left( \frac{1}{4\pi^2} \sqrt{\frac{m_2}{m_0} T} \right) \right]^\frac{1}{2} + \frac{1}{2} \gamma \left[ \log_e \left( \frac{1}{4\pi^2} \sqrt{\frac{m_2}{m_0} T} \right) \right]^\frac{1}{2}
\]

where \( m_0 \) is the area of the response spectrum and \( m_2 \) its second moment viz,

\[
m_0 = \int_{-\infty}^{\infty} E(f) \, df
\]

\[
m_2 = \int_{-\infty}^{\infty} E(f) f^2 \, df
\]

and \( \gamma = \) Euler's constant = 0.5772.

The problem remains, of course, of obtaining the response spectrum \( E(f) \). This ultimately depends on the establishment of the wind speed spectrum - this is discussed in APPENDIX A and in Chapter 8, page 184. In estimating the combined effect of quasi-steady and resonant response it is clear that the frequency base of the wind spectrum would have to extend at least as far as the highest natural frequency of the structural system at which any substantial resonant amplification is liable to occur. The response spectrum would then be built up, frequency by frequency, through successive multiplication of the mechanical admittance and the aerodynamic admittance - the latter taking into account any loss in spatial correlation of gusts with increase in frequency.

The above process would have to be carried out for both the in-line and transverse axes, so that the problem of estimating
the highest likely total displacement\(^1\) of the structure would have to be carried out on a two-dimensional basis; ending with a two-dimensional probability distribution such as that indicated in fig. H4 (ii).

\[\text{It may be of interest to determine what the largest likely resonant vibration about either axis of the structure will be in a given time interval, } T. \text{ This may be obtained from the expression,}\]

\[D_{\text{max}} = \sqrt{2m'_{\infty}} \left[ \log e \left( n_0 T \right) \right]^{\frac{1}{2}} + \frac{1}{2} \gamma \left[ \log e \left( n_0 T \right) \right]^{-\frac{1}{2}}\]

where \(m'_{\infty}\) is the area of the resonant part of the response spectrum, \(n_0\) is the natural frequency involved and \(\gamma\) is Euler's constant.
CONCLUSION

It is considered that the main contribution of this thesis toward the understanding of the interaction of wind and structures is the establishment of a relatively straightforward experimental approach which, when refined, will not only facilitate the verification of current theories, but because of the extra possibility offered of presenting data in an immediately recognizable form, will also help to promote wider understanding of the various physical phenomena involved.

The tower structure experimented upon in the course of this thesis is not, it must be understood, typical of the large majority of tower structures - it will be recalled that the structure in question was designed to eliminate certain wind effects with the intention of isolating others; this approach would not, of course, normally be used as a criterion for design. (It is worth recalling briefly that the size, shape and type of construction of the structure was chosen in order to eliminate the respective effects of loss of spatial compass of gusts with increased frequency, variation in aerodynamic drag in the horizontal plane and the generation of large vortices affecting the structure as a whole.)

It is felt that the popular notion held in some quarters of obtaining a large accumulation of experimental data from the extensive fitting of transducers on large, bluff and structurally complicated buildings, with the intention or hope that extensive sifting by computer will reveal some hitherto unknown relationship or favour a particular 'hunch' is basically illogical.
By attacking the problem from the wrong end this approach becomes self-defeating, with the sheer weight of information gathered getting out of hand and at best resulting in rather vague, but plentiful, qualitative generalizations. If an analytical approach is to be used at all in experimentation, it would seem sensible to begin with the study of the simplest (or even idealized) structures and then to go on to structures of increasing structural and aerodynamic complexity, building up knowledge on the basis of that previously acquired. Very roughly, the following programme is the one which might be envisaged.

Firstly, the repetition of the experiment described in this thesis with a smaller lattice tower would be undertaken. In fact by manipulation of the parameters of size and natural frequency full spatial correlation of the response inducing gusts (up to and including the relevant resonant frequencies) could be achieved. By simultaneous recording of both wind speed and structural response (resolved in both cases along the principal axes of moment of inertia - or any two mutually perpendicular axes in the horizontal plane, if the latter happen to be the same) the wind 'input' and structural response 'output' may be obtained on a frequency basis in spectral form (for instance by recording the sort of traces obtained in fig. 80 on magnetic tape loops and running these through a frequency analyzer). Having, a priori, obtained the mechanical admittance of the structure, it is then possible to calculate the aerodynamic admittance, so obtaining, on a frequency basis, a definite measure of the aerodynamic damping present, if any. The experiment would follow the normal course of determining which
variables affect the result obtained and, if so, according to which law. It must be appreciated that, because of the nature of the exciting force (the natural wind)\(^1\), this would not be the easiest of tasks and might involve the transportation of the structure or series of structures and associated instrumentation from site to site.

The next major step would be to determine the effect of the loss of spatial correlation of gusts with increased structural size. This would be constrained, initially, to the testing of a series of differently sized lattice structures of the type considered in this thesis. While the structural response could be obtained as before the measurement of the actual spatial compass of gusts would necessitate the use of several anemometer-resolvers with the associated analyzers. In fact, the experimental study of vertical and horizontal spatial correlation of turbulence would constitute a major programme in itself.

Thirdly, bluff structures would be considered. The problem could be tackled in the light of a natural progression of structural 'solidity' - beginning with a cylindrical lattice or open type of structure (as considered above) and then gradually building up the solidity of the structure till a stage was reached when the aspect presented to the wind was that of a bluff body with high surface roughness; a further sequence would be the reduction of the surface roughness until a given degree of smoothness was reached. From this study a meaningful understanding of the various stages at which vortex shedding becomes

\(^1\)In many ways unpredictable (e.g. when will the wind blow?) and in other ways tied implicitly to the particular nature of the surrounding terrain, etc.
significant in relation to the response of the structure would be obtained.

The last step would involve the generalization of the conclusions reached above to cover all shapes, sizes, and types of structure in all wind conditions.

While not wishing to appear pessimistic it is felt that the 'total' approach as described above is, at the present time, probably outside the bounds of feasibility. Until a major wind-induced disaster occurs it is doubtful whether the enormous cost of such an extensive programme would be acceptable. Further, some great improvement would be necessary in the equipment presently available for the type of field experimentation required - particularly as far as wind measuring transducers are concerned.

More optimistically, it may turn out to be the case that the above comprehensive programme is not entirely necessary. From tests such as those concerning auxiliary damping carried out on the tower structure dealt with in this thesis it may prove to be the case that lowly-damped structures may, by being fitted with auxiliary dampers, be effectively prevented from undergoing undue resonant vibration. By controlling the dynamic response of such structures in this way, it may even be possible to ignore the effect of resonant excitation altogether, it being small in comparison to the quasi-steady component. Quite apart from making the structure safer, this would considerably simplify the whole problem of estimating the highest likely response.

As a further point it may be mentioned that in view of the more conjectural factors involved in the buffering by the wind of a particular structure in a particular location - such as the
possible increase in turbulence generated by surrounding buildings which are not yet built (and which may or may not be built!) - it is doubtful whether an exact design procedure applicable to a particular case could ever be evolved. What is more likely is the establishment of a generally applicable set of rules incorporating certain factors of safety based on an exact knowledge of what actually happens in a given number of cases - cases which have been comprehensively tested in the natural wind.

It is considered that the measurements of simultaneous structural response and wind obtained in this thesis are among the most accurate obtained so far\textsuperscript{1}. The results actually obtained are, in many ways, entirely what one would expect, with the lower frequency wind fluctuations evoking a corresponding quasi-steady response in the structure and higher frequency gusts exciting vibration in the first mode natural frequencies of the structure. What becomes slightly more difficult is quantifying the response obtained. It is considered that the use of certain more abstract concepts (such as spectra, etc.) in the solution of estimating the largest combined quasi-steady and resonant response likely to be obtained (if not damped by the use of an auxiliary damper) offers the only means of solving the problem posed.

\textsuperscript{1}It is worth emphasizing again that the use of the anemometer-resolver devised not only enabled fairly high frequency directional fluctuations of the wind to be taken into account but effectively constrained the measurement of this fluctuation to the horizontal plane; it is doubtful whether this is the case with the only commercially available equipment presently in use which employs hot-wire X-probes. This last point is the result of a conversation with Dr. J. Morgan of the Department of Civil Engineering and Building Science at Edinburgh University.
This thesis has aimed, in its presentation, at providing a reasonably straightforward and readable introduction to anyone fresh to the subject and it is hoped that it will be a helpful contribution to the present state of knowledge in the field.
APPENDIX A.

An Explanation of the Use and Significance of Wind Energy Spectra
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An explanation of the use and significance of wind energy spectra

A brief study of a typical wind-speed record (taken at a standard height of 10m., say, in the free wind flow) will indicate that the higher frequency fluctuations in wind-speed have a definite statistical tendency to vary about a mean value - this value being part of a much slower fluctuation. It can be reasoned that these fluctuations belong to two quite separate types and scales of behaviour - one being termed 'micrometeorological,' originating from the mechanical generation of turbulence from surface obstructions and other small scale topographical irregularities of the terrain and the other, 'macrometeorological,' deriving itself from large scale climatic 'weather-map' fluctuations. The mean variation referred to above (at 10m.) is, in fact, a moderated macrometeorological wind variation; the macrometeorological (or 'gradient') wind, considered as the movement of air resulting from the pressure imbalance between neighbouring iso-bars on a large scale weather-map, is, strictly speaking, only operative at great heights (>1000ft.) away from the retarding influence of the frictional drag near the ground level. For a particular location the 'mean wind' near the ground level will be proportional to the gradient wind, and will, evidently, follow the same frequency fluctuation.

Both scales of turbulence contribute to the loading of a structure, though this appendix will deal principally with the representation of the small-scale micrometeorological wind.
It has been found from experimental observation that for a substantial range in periods, $T = 5 \text{ min-1 hour}$, say, the mean, $\bar{c}$, is part of the macrometeorological variation mentioned above - i.e. $\bar{c}$ will vary slowly over periods ranging from several hours to several days.

Consider a typical portion, $T$, of such a record - see fig. A1:

The mean is easily found by drawing $N$ sufficiently closely (equally) spaced ordinates ($a_n$) on some common base-line. Whence:

$$\bar{c} = \frac{1}{N} \sum_{n=1}^{N} a_n$$

If the mean, $\bar{c}$, is taken as the zero of the interval, $T$, it is possible to represent the signal, $f(t)$, as the sum of a trigonometrical series: the real part of the Fourier series, viz,
f(t) = \sum_{n=1,2,\ldots}^{\infty} |c_n| \cos(2\pi nt/T - \epsilon_n) \quad \text{1a,2}

where
\[ c_n = \frac{1}{T} \int_{0}^{T} f(t) \exp(-i.2\pi nt/T) \, dt \]

In other words, it is possible to reconstitute a finite portion, T, of a random signal \( f(t) \), by the simple addition of an infinite number of discrete sine waves whose frequencies are successive integral multiples of the frequency defined by the length, T, of the portion taken; this (lowest) frequency being \( 1/T_{cps} \). Because of the periodic nature of each sine wave and the fact that in each case a whole number of cycles fits exactly into the period, T, the whole signal, \( f(t) \), \( (0<t<T) \) will be reproduced in each subsequent period of duration, T. All of these features are shown in fig. A2 where the first few terms of the Fourier series are shown.

A more usual form is \( f(t) = a_0 + \sum_{n=1}^{\infty} c_n \cos(2\pi nt/T + \epsilon_n) \). This is the same as (1a) with \( c_n = (a_n^2 + b_n^2)^{1/2}, \epsilon_n = \tan^{-1}(b_n/a_n) \). Because the mean of the random signal has been taken as zero, \( a_0 = 0 \).

Only a finite number of max, and min, are permitted - see REF Fourier Series by I. N. Sneddon. In practice the finite number of max, or min, obtained in any actual record obtained is a function of the high frequency cut-off of the anemometric instrumentation used.

The randomness of the signal may be verified by plotting the frequency of occurrence of a particular velocity about the mean value; a normal or Gaussian distribution would be found. See fig. A1.
Having broken down the finite portion, T, of the random signal, \( f(t) \), into an equivalent series of simple sine waves, it is worthwhile to consider the properties of one such wave. By the Principle of Superposition, it is then possible, by simple summation of the effect of each wave, to deduce certain properties of the parent signal, \( f(t) \).

Let a typical wave be represented as

\[
S_n = c_n \cos(2\pi nt/T - \epsilon_n)
\]

which, with its square, is represented graphically in fig. A3.

FIG A3

It is clear from inspection, that the mean of the function is zero and that the mean of its square\(^1\) is \( \frac{1}{2}c_n^2 \). (RMS = \( c_n/\sqrt{2} \))

More rigorously for a typical cycle,

\[
\overline{S_n} = n/T \int_{t_p}^{t_{p+1}} c_n \cos(2\pi nt/T - \epsilon_n) \, dt = 0
\]

and,

\[
[S_n^2] = n/T \int_{t_p}^{t_{p+1}} \frac{1}{2}c_n^2 \, dt + n/T \int_{t_p}^{t_{p+1}} \frac{1}{2}c_n^2 \cos(2\pi nt/T - \epsilon_n) \, dt
\]

\[= \frac{1}{2}c_n^2
\]

\(^1\)Referred to usually as the mean square deviation, dispersion or variance.
where,
\[ t_p = [(4p - 3)\pi/2 + \epsilon_n].T/2\pi n \]
\[ t_{p+1} = [(4p + 1)\pi/2 + \epsilon_n].T/2\pi n, \quad p = 1,2,3,\ldots. \]

It is useful, here, to introduce the general concept of the power or energy associated with any sinusoidal time-varying motion. This is best done, from a physical point of view, by drawing an analogy with a simple mechanical system. It was remarked earlier (page 21) that the total energy within an undamped spring system subject to free oscillation remains constant with time, though fluctuating in its nature from elastic potential to kinetic energy. It follows that, at certain instances in time, when the system is not subject to any strain, all the energy within the system is kinetic, being contained in the velocity of the mass. In fact, it is easily shown that the mass performs a simple harmonic motion, the displacement, \( f(x) \), varying with time, \( t \), according to the function,

\[ f(x) = a.\sin2\pi ft. \]

where \( 'a' \) is the maximum displacement of amplitude, \( f \) is the natural frequency. The velocity fluctuation is also sinusoidal,

\[ \dot{f}(x) = 2\pi fa.\cos2\pi ft = v.\cos2\pi ft \quad (say) \]

Clearly when the system is not strained, \( f(x) = 0 \), and \( \dot{f}(x) = v \) is the (maximum) velocity of the mass, whence the energy within the system is \( \frac{1}{2}v^2 \) (for a unit mass). But since the energy within the system remains constant with time, \( \frac{1}{2}v^2 \) represents the mean energy level.
By analogy with this it can be seen that any physical quantity varying sinusoidally with time can be thought of as possessing 'energy' by virtue of that fluctuation equal to \( \frac{1}{2} \) (amplitude)\(^2\). In the case of the typical constituent wave, \( S_n \), discussed above, the associated energy level is \( \frac{1}{2} c_n^2 \).

As could be reasonably expected, this is the same as the mean square deviation of the wave. It then follows that the energy level of the parent signal, \( f(t) \), averaged over the period, \( T \), is equal to the summation of the energy contributed from each constituent wave, viz.,

\[
\frac{\langle f(t)^2 \rangle}{T} = E_T = \frac{c_1^2}{2} + \frac{c_2^2}{2} + \cdots + \frac{c_n^2}{2} + \cdots = \sum_{n=1}^{\infty} \frac{c_n^2}{2}
\]

This, of course, by the same argument, represents the same quantity as the mean square deviation of the fluctuation.

Having broken the fluctuation, \( f(t) \), into a number of frequency ordered sine waves suggests the building up of an energy 'spectrum' using a frequency base. Since each sine wave has its own characteristic energy level, \( \frac{1}{2} c_n^2 \), the energy spectrum may be built up - see fig. A4-(i)(b). So far the analysis has been centred on the characterization of a finite random signal of duration \( T \). The problem remains as to how the properties of a signal of infinite or non-specified duration and yet of the same basic nature and randomness as the one discussed above may be represented. Suppose a longer period, \( NT \), \( (N = 2, 3, \ldots) \) is considered. The finite signal can again be represented by a Fourier series, noting that now, for a given finite frequency range or bandwidth, there will be \( N \) equally-spaced
\[ \text{RMS} = \sqrt{E_T} = \sqrt{f(t)_{\infty}^2} = \sqrt{\sum_{n=1}^{\infty} c_n^2/2} \]

\[ \sqrt{E_{NT}} = \sqrt{f(t)_{NT}^2} = \sqrt{\sum_{n=1}^{\infty} d_{Nn}^2/2} \]

\[ \text{Energy density spectrum} \]

\[ \text{area} = E_T = \sum_{n=1}^{\infty} c_n^2/2 = f(t)_{\infty} \]

\[ \text{area} = E_{NT} = f(t)_{NT} = \sum_{n=1}^{\infty} d_{Nn}^2/2 \]

\[ \text{area} = E_{\infty} = f(t)_{\infty} = \int_0^\infty E(f) \, df \]
sinusoidal terms to every single term found previously. Clearly, the mean square deviation or the mean power of the signal will not have altered significantly from the shorter sample, so that,

\[ E_T \approx E_{NT} = \sum_{n=1}^{\infty} c_n^2/2 = \sum_{n=1}^{\infty} d_n^2/2 \]

This means that the sum of the energy contributions for the same given band-width in the two cases is approximately equal so that the individual energy coefficients, \( \frac{1}{2}d_n^2 \), making up the larger signal, \( f(t)_{NT} \), will be proportionately smaller, the average within the band-width being \( 1/N \) that of the average size of the coefficients, \( \frac{1}{2}c_n^2 \), in the band-width of the shorter signal. If the frequency interval between each successive contribution is \( \Delta f \) and the energy contribution within the band-width \( f \pm \frac{1}{2}\Delta f \) is denoted by \( \Delta[f(t)^2] \), then it is possible to write, at a typical frequency,

\[ \Delta[f(t)^2] / \Delta f_T = \frac{1}{2}c_n^2/(1/T) \]  

(for signal, T)

\[ \Delta[f(t)^2]_{NT} / \Delta f_{NT} = \frac{1}{2}d_n^2/(1/NT) = (\frac{1}{2}c_n^2/N)/(1/NT) = \frac{1}{2}c_n^2/(1/T) \]  

(for signal, NT)

The approximation in the second expression is due to the fact that the average size of the \( N \) energy contributions, \( \frac{1}{2}d_n^2 \), over the interval \( \Delta f_T = 1/T \) is taken as \( 1/N.c_n^2/2 \). This is apparent from fig. A5 where a typical increment, \( \Delta f_T = 1/T \), is considered.
In other words it is seen that the increment in energy divided by the increment in frequency is approximately the same in each case. For a finite signal length, the quantity

\[ E(f) = \frac{\Delta[f(t)^2]}{\Delta f} \]

is defined as the power spectral 'density'. This may be plotted as a step-function - see for instance fig. A4-(i)(c) and (ii)(c). If a signal of infinite duration, \( N_T \to \infty \), is considered then, clearly, \( \Delta f \to 0 \) and the energy density spectrum becomes continuous (see fig. A4-(iii) and

\[ E(f) = \lim_{\Delta f \to 0} \left[ \frac{\Delta[f(t)^2]}{\Delta f} \right] \Rightarrow d[f(t)^2]/df \]

In other words the continuous spectrum may be thought of as built up from the energy contributions of individual sine waves separated from each other by infinitely small intervals in frequency. It follows from this that the sum of these contributions which corresponds to the mean energy level or the mean square deviation of the signal is equal to the area under the power density spectrum, or, from the last expression,

\[ E_{NT \to \infty} = \int_{f_0}^{\infty} E(f) \cdot df = \int_0^{\infty} E(f) \cdot df = \left( = m_0 \right) \]

It is worth noting that if \( f(t) \) belongs to the same parent population in all three cases then, the areas of the power density spectra are approximately equal, viz,

\[ E_T \approx E_{NT} \approx E_{\infty} \]
In a later appendix it will be found convenient to make use of the moments of $E(f)$ about the origin, $f = 0$, such that the $n$th moment is given by:

$$m_n = \int_0^\infty E(f) f^n df$$

A method has thus been found for expressing the mean capacity for work or energy level (which is also the mean square deviation) of a random signal fluctuating with time. In practice, one method of obtaining the power spectral density (loosely called the energy spectrum) is to record on magnetic tape a sample, $f(t)$ of duration $T$ and by looping the tape it can be passed through, indefinitely, a Fourier analyser which picks out the contributions of the integral-multiple harmonics of the fundamental frequency defined by the length of the loop. A diagram such as that shown in fig. A4-(i)(b) can then be drawn and the power spectral density then obtained by dividing the energy coefficients by the frequency increment, $\Delta f = 1/T$ — fig. A4-(i)(c). The 'smoothness' of the spectrum so obtained is clearly dependent on the length of the record taken. A more satisfactory arrangement is to feed the magnetic tape into a spectrum

---

**FIG A6**

\[1\] See next page
analyser which filters out all frequencies outside the variable pass-band \( f \pm \frac{1}{2} \Delta f \). The mean energy level within the pass-band is then recorded on a mean-square meter and the power spectral density at \( f \) is obtained by dividing the meter reading by \( \Delta f \). The advantage of this method is that the width of the pass-band may be accommodated to suit the slope of the power spectrum\(^1\).

A typical power density spectrum for an actual micrometeorological wind fluctuation \( f(t) \) — normally called a gust spectrum — is shown in fig. A6\(^2\).

Many experimental measurements of gust spectra for different mean wind speeds, heights and terrain roughnesses have shown that the shape of the spectrum remains approximately the same but, that, under these different conditions, its size and position undergo considerable variations. This has led to the attempted establishment of a standardized or reduced spectrum of gustiness, which, when suitably operated upon by parameters involving the mean velocity, height and terrain roughness, would be converted into the spectrum appropriate to the particular location. Much work has been devoted to establishing a physical or semi-empirical understanding of the structure of the natural wind with the purpose of arriving at a valid universal reduced spectrum. Some of the physical aspects of wind turbulence are

\(^{1}\)There is another mathematically equivalent method where the power spectral density is determined from measurement of the auto-correlation function.

\(^{2}\)The frequency base is normally drawn to a logarithmic scale, in order to facilitate the finding of the area under the curve near the origin. This means that the ordinates plotted are \( f \cdot E(f) \) instead of \( E(f) \) whence,

\[
A = \int_0^\infty f \cdot E(f) d(\log f) = \int_0^\infty E(f) df \quad \text{as before.}
\]
briefly presented in the following pages.

For an explanation of the structure of turbulence in the natural wind recourse has been made to the more intensive studies made of turbulent flow in fluids. Analogies have been drawn with the rapid fluctuations in velocity, direction and magnitude found within the mean motion of the fluid parallel to the surface in turbulent boundary layer flow. These rapid fluctuations - or eddies - in fluid flow, which may originate from disruption of the laminar boundary flow due to irregularities in the surface, result in the mixing of some of the faster moving streams in the more remote main flow with the laminar boundary layer and so form a 'turbulent boundary layer'. Just as the molecular movements in laminar flow give rise to viscous shear stresses so it can be said that the eddies in turbulent flow give rise to eddy (or Reynold) shear stresses. Clearly, a similar mechanism can be considered to operate in the natural wind where the establishment of the mean velocity variation with height (the boundary layer profile) can be considered as resulting almost entirely from the retarding influence of eddy shear stresses - the original cause of the eddies being traceable to the mechanical disruption of what would have been laminar flow over the 'rough' surface of the terrain.

The disturbances within the flow may be imagined from an
elementary point of view by treating them as the arrival of 'wind elements' from slower moving substrata - this arrival being associated with a transfer of momentum. It is found, in fact, that for most predominantly longitudinal mean flows the momentum transport is angular (being described in the vertical plane of two-dimensional flow) - this results in the exertion of a negative tangential or shear stress on the stratum in question, tending to slow down its mean velocity. The mean shear stress (or Reynolds stress) will be directly proportional to the mean product of the arrival velocities - which can be considered as roughly equal to the mean flow velocities of the substratum whence the elements came. The velocities of the substrata will, for a given surface roughness, be proportional to the gradient or main flow velocity. This means that the eddy stresses increase in proportion to the square of the mean or gradient wind velocity.

It must be noted that, apart from disturbances due to eddy stresses, there will be disturbances in pressure - both will do work and as such both provide mechanisms whereby energy is transferred from the mean motion and vice versa. In fact, the energy contributions due to changes in pressure are comparatively small and may be ignored, which means that the mean energy level of the turbulence is contained essentially in the rate of work done by the eddy stresses. Some of this energy will eventually be dissipated by the viscosity of the air and converted into heat, some
will be subsequently absorbed by the buildings, trees, etc., forming the roughness of the terrain. As any real turbulence must develop in a three-dimensional field, a reasonable assumption is made that the energy spectrum for a longitudinal fluctuation, $f(t)$, is always approximately the same proportion of the total energy within the wind. It follows that the area of the gust spectrum of $f(t)$, proportional to the total energy of the turbulence in the wind, varies linearly with the magnitudes of the mean eddy stresses, which, in turn, vary according to the square of the mean velocity. Since experimentally obtained spectra appear to retain their characteristic form for all mean wind speeds, it follows that the energy ordinates—i.e., the contributions from the individual 'constituent sine waves'—vary, for a given surface roughness, in proportion to the square of the mean velocity.  

If the roughness of the terrain is altered in some way, some corresponding change in the amount of the ensuing turbulence is to be expected. The precise physical way in which this happens is not yet fully known, though the phenomenon has been the subject of considerable empirical study. A measure of the roughness of the terrain is clearly reflected in the shape variation of the turbulent boundary layer profile—i.e., the mean velocity variation with height. Many observations of this profile for the natural wind suggest the suitability of a simple exponential law of the type,

$$\bar{V}_z = k \cdot z^{1/\alpha}$$

1The mean velocity at any height of the boundary layer may be taken; a mean velocity which is often used is that measured at a reference height of 10m.
where \( \overline{\nu}_z \) = mean velocity at height \( Z \), and \( k \) and \( \alpha \) = constants (see fig. A7).

At a certain height, \( Z_G \), the gradient wind, \( V_G \), is reached. The analogy drawn with the fluid boundary layer flow implies that \( Z_G \) and \( \alpha \) vary with the roughness of the terrain. This has been shown to be true for \( \alpha \) where extensive measurement of the mean wind profile, particularly at the lower values of \( Z \), have shown a dependence on the general nature of the terrain. Partly based on these and other experimental observations and the fluid flow analogy, values for \( Z \) and \( \alpha \) for terrains of different characteristic roughnesses, and values for the extreme mean hourly gradient wind speed, \( V_{G,max} \) covering the British Isles have been put forward by Davenport. Knowledge of the parameters, \( V_{G,max} \), \( Z_G \) and \( \alpha \) enables the extreme mean hourly wind speed, \( \overline{\nu}_{Z,max} \), at any height \( Z \) to be obtained. Whilst the soundness of Davenport's approach cannot be questioned, it is felt that much more experimental verification, both in the postulated values of \( Z_G \) and \( \alpha \) for particular terrain roughnesses (a qualitative notion of the roughness is, in itself, difficult to assess) and in the values of \( \overline{\nu}_G \), is needed before a universal application of this method for estimating

\[ V_{Z,\text{max}} = k Z^{1/\alpha} = V_G Z_G^{1/\alpha} Z^{1/\alpha} \]

FIG A7

1 See REF 5.
\( V_{z,\text{max}} \) at a particular location is accepted.

The above empirical approach to the boundary layer profile does not explicitly give any indication as to how energy is transferred from the mean flow to the turbulence and vice versa in, for a fully developed stable profile, a process of dynamic equilibrium. The energy required to maintain the turbulence is provided by the work done by the Reynolds stresses on the mean flow, while at the same time, the mean flow is moderated by the diffusive action of the turbulence. The effect of the surface roughness will be to shed into the boundary layer eddies which will be related to the roughness size near the point of shedding. Consequently, for a given gradient wind and height above ground level, the Reynolds stresses will be related to the roughness size of the surface. The precise formulation of this relationship - for fluid flow - has been attempted by von Karman, Prandtl and others by making use of such concepts as the 'mixing length' theory. The classical development of these theories will not be given here. The important point to remember is that there is a (non-linear) dependence of the mean energy level of the turbulence (i.e. the rate of work done by the Reynolds stresses) on the surface roughness size. In fact it is usual to denote this non-linear relation by a simple drag coefficient, \( \kappa \), such that the mean energy level is directly proportional to \( \kappa \). The surface drag coefficient (usually referred to some mean velocity - at 10m., for instance) may be measured directly. Certain values for \( \kappa \) for terrains of different basic character have been put

1 By 'stable' here is meant that the fetch downwind of a given surface roughness is of sufficient length for the profile to have attained its full development. It is worthwhile noting here that, in the above discussion, conditions of neutral stability - i.e., the convective movements of air are small compared to the gusts of mechanical origin - prevail.

2 See REF 14: The Mechanics of Fluids by Duncan, Thom and Young.

The mixing length can be thought of as the mean distance travelled by an element of fluid before it surrenders its initial velocity and vorticity to its surroundings and is analogous to the mean free path of molecular movements.
forward by Davenport\(^1\).

Apart from an increase in the energy of the turbulence in the wind due to an increase in the mean velocity of the flow (with respect to some fixed height of the boundary layer) it might be expected that the rate at which the eddies are formed and shed from the surface irregularities into the boundary layer, with the consequent effect on the general eddy frequency set-up of the whole boundary layer, would also increase. To fix ideas, consider the disruption of laminar flow past a bluff object as 'seen' by a stationary point situated somewhere in the down-stream turbulent wake. If the means flow impinging on the obstruction is increased (from \( \overline{V} \) to \( \overline{V}' \), say) the eddies shed will be characterized by a greater vorticity or energy - this will be seen at \( P \) as a generally larger velocity fluctuation (in fact, \( \sigma_{(t)}^2 = (\overline{V}' - \overline{V})^2 \)). But these 'larger' eddies will be carried along with the mean flow of the wake whose velocity will have increased in direct proportion to the increase in the mean laminar flow to which the obstruction is subjected. This means that \( P \) will experience a more rapid succession of the now more powerful eddies than previously - i.e. the whole turbulence pattern as seen by \( P \) will have increased in frequency by an amount directly proportional to the increase in the mean flow velocity. In fact, the same result may be obtained by considering the sine waves building up the gustiness to be travelling with the velocity of the mean motion past the stationary point \( P \).

\(^1\)See REF 1, PAPER 2.
A typical wave is shown in fig. A8.

If the wavelength of the typical wave is \( X \), the frequency of the fluctuation as seen by \( P \) is \( f = \frac{\bar{V}}{X} \), whence, \( X \), the wavelength, is \( \frac{\bar{V}}{f} \).

The inverse wavelength or the reduced frequency is then \( \frac{1}{X} \) or \( \frac{f}{\bar{V}} \).

So far, no direct reference to the variation of the gust spectrum with height has been made. The fluid analogy here is interesting. Roughly speaking, the turbulent boundary layer in fluids can be subdivided as follows:

(i) The outer region, which is the outer part of the boundary layer characterized by large scale, low frequency, eddies and intermittent turbulence. The mean shear stress exerted by the turbulence is generally small. The thickness of this region is approximately defined by the limits 0.4\( \delta \)--1.0\( \delta \), where \( \delta \) is the boundary layer thickness (cf. gradient height).

(ii) The inner region is the fully turbulent region immediately below the outer region in which the shear stress is practically all attributable to turbulence (as opposed to viscosity in the laminar sub-layer). A wide range of turbulence frequencies is found with a preponderance of high frequency, small eddies near the surface ranging to larger, low frequency eddies away from the surface. There is evidence to suggest that the larger eddies extract energy from the mean flow and
pass it downwards to the smaller eddies, then dissipating it as heat. The eddy shear stress reaches a peak in a region where it is reasonably invariant - see region marked 'c', fig. A9(a) - and then decreases quite rapidly with distance from the surface.

(iii) The laminar sub-layer is a very narrow region of flow adjacent to the surface in which the shear stress is predominantly viscous, the turbulent fluctuations having become small in absolute magnitude.

Typical variations of shear stress and (total) turbulent energy with height for the boundary layer flow on a flat plate are shown in fig. A9. A similar variation in the turbulence energy - expressed as the area under the gust spectrum - might be anticipated in the natural wind. Also, from what was said concerning the nature of the turbulence in the inner region, some variation in the shape of the gust spectrum might be expected. An early semi-empirical expression for the gust spectrum by Davenport (1961) took some account of the variation of energy with height, but, it would appear, after consideration of more recently measured wind spectra and, in particular, work

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1 REF 15: (Schubauer and Klebanoff)
2 See REF 5.
\( \kappa = \text{surface drag} \)

\( \tau = \text{time, } t \)

\( f = \text{frequency} \)

\( f. E(f) = \text{spectral density function} \)

\( f. E(f)/\kappa. V_{10}^2 = \text{reduced spectrum} \)

\( f. E(f)'/\kappa'. V_{10}'^2 = \text{actual spectrum} \)

FIG. A10
by Berman, that the dependence on height has been dropped (1966). This is not altogether surprising. Analogizing directly with the fluid flow energy variation, it follows that if, for the wind, $\delta = 1000$ ft. (say) then $'c'$, the height for which the energy level (as a function of the shear stress) is reasonably constant, is of the order of 200 ft. This region of constant eddy stress could, of course, be much more extensive than the direct analogy suggests.

Assuming that the gust spectrum remains approximately constant with height (at least for the first few hundred feet) implies that the mean surface drag coefficient, $\kappa$, decreases in proportion to the inverse square of the mean profile velocity, which, for a given gradient wind velocity, is a function of height. Clearly some standardization in the values taken for $\kappa$, in moving from one location to another, is necessary. For this reason, $\kappa$, is usually taken as the drag coefficient at a standard height, $Z = 10$ m. This means that the mean velocity used must also be that of the profile at $Z = 10$ m, $\bar{V}_{10}$. The reduced spectrum may then be drawn for a variety of conditions — see fig. A10. Davenport bases his proposed (empirical) spectrum on the mean of reduced spectra so obtained — see fig. A11.

At first sight the agreement of the reduced gust spectra for different sites is surprisingly good. On further inspection, however, and remembering the way in which use is to be made of the reduced spectra, certain reservations must be made. Of primary importance is the total area under the reduced spectrum, which, when operated on by the parameters $\bar{V}_{10}$ and $\kappa_{10}$, is equal to the mean square deviation or mean
energy level of the turbulence at the site in question. Assuming the frequency of the first mode of response to fall within the extreme right hand side, low energy portion of the spectrum, the mean quasi-steady or non-resonant response of any structure due to the fluctuating component of the wind (about some hourly mean wind speed) will be directly proportional to the total area of the gust spectrum, though the higher frequency energy contributions of the spectrum may be less effective in building up the mean response due to a decay in spatial correlation.

\[ \text{SITE} \quad \text{ELEV. FT} \quad \text{SURFACE} \quad \kappa \]

1. SEVERN BRIDGE 100 RIVER ESTUARY .003
2. SALE 503
3. SALE 401 OPEN GRASSLAND .005
4. CARDINGTON 50 FEW TREES .008
5. ANN ARBOR 25-200 HEDGED FIELDS 30 FT. HIGH .015
6. CRANFIELD 50 SCRUB & TREES .030
7. BROOKHAVEN 300 URBAN AREA .015
8. LONDON, ONTARIO 150 URBAN AREA .030

\[ \text{FIG AII} \]

If, instead of the actual spectrum measured for the site concerned, the empirical curve were chosen, this would result in an overestimation of roughly 20% in the non-resonant response of a structure to turbulence at Cardington, and an underestimation of roughly 10% at Ann Arbor. While this sort of error is not unusual in engineering design, it is felt that many more reduced spectra need to be experi-

\[ \text{expressed in terms of the variance or (RMS)}^2 \]
mentally determined and compared with the empirical curve before it can be established that these errors are of a consistent nature.

While the outlook for the eventual establishment of a universal reduced spectrum for the purpose mentioned above is fairly optimistic, the same cannot be said of the use made of the spectrum in the determination of the resonant response of a structure. If the fundamental or first mode frequency of a structure is $n_0$, then, clearly, the magnitude of the resonant part of the response will be a function of the energy contained by those gusts arriving at that frequency, or, in other words, proportional to the ordinate of the gust spectrum at $n_0$. A natural frequency of the order of $n_0 = 1 \text{ c/s}$ is not unrepresentative of many structures. For a mean wind speed of 100 ft/sec, (~70 mph.) at a height of 10 m, the reduced wavelength of turbulence exciting structural resonance is $\sqrt[10]{n_0} = 100 \text{ ft}$. Looking at the reduced spectral energy density function at this wavelength results in the observation that at Ann Arbor the energy level is practically double that of the empirical curve, or, expressed in another way, the dynamic or resonant part of the response would, were the empirical curve to be used, be underestimated by a factor of two. At London, Ontario, the response would be overestimated by a factor of two. Further, a reasonable extrapolation of the actual spectrum for the site at Cardington would suggest a negligibly small or zero response, whereas a considerable response would be designed for using the empirical curve. Particularly for lightly damped structures - such as

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1 Strictly speaking, the ordinate represents the energy 'density'. A certain pass-band is usually defined from the width of the peak in the mechanical admittance curve. This pass-band defines an area of the energy 'density' spectrum. For a given structure, this is approximately proportional to the ordinate at the mid-point, $n_0$, of the pass-band.

2 expressed in terms of the variance
monolithically acting structures, for instance, all-welded lattice towers - where the resonant response may be as large as or larger than the non-resonant response and where the gusting at the natural frequency is reasonably well correlated over the whole structure, gross over-design or under-design may be the result if the empirical curve is used. Evidently, this is not a satisfactory state of affairs.

Possible reasons for variations in reduced spectra from site to site will not be discussed here. Briefly, it is suggested that, though the mean surface roughness size may be the same for the different sites, there may be variations within that mean, resulting in two reduced spectra of roughly the same area but showing substantial variations about the empirical mean curve. There may also be more or less variation of the spectrum with height despite what was said previously.

In practice, a designer would also be faced with other problems such as those posed by the possible variation of the actual gust spectrum with direction and future possible changes in the spectrum due to the alteration of the surrounding surface roughness by the removal of existing buildings or the construction of new ones, the leaf-shedding cycle of deciduous trees, etc.

In conclusion, the following points and suggestions may be made. If a structure is to be designed with a fair degree of accuracy from the point of view of its dynamic motion, then a measure of the fluctuating loading with time causing that motion is essential. This (wind) loading can only be characterized on a useful basis by the drawing up of the power spectral density for the turbulence at the site in question (together with the spatial correlation). From the gust-spectrum can be gauged not only the total mean square deviation of the turbulence (which could also be gauged from the drawing up of a simple Gaussian probability density distribution for the turbulence)
but also the proportion of turbulence which is effective in particular frequency band-widths. Because of the variation of the gust-spectrum with location the drawing of a reduced or universally acceptable gust-spectrum in terms of parameters involving surface roughness and mean wind speed has been attempted. It is here that important discrepancies have arisen. Three suggested courses of action are put forward as a possible remedy:

(i) An extensive programme of experimental observations at a series of 'different' sites, the primary aim of which would be the further understanding of the structure of turbulence particularly that dealing with the higher frequency part of the spectrum. If successful this would probably lead to a reduced gust-spectrum expressed as a rather complicated implicit function involving a selection of ground roughness parameters. The practising engineer would still be faced with the estimation of these parameters for a particular location.

(ii) The measurement of a vast number of spectra (not reduced) forming a closely-knit network covering a whole country. Several representative directional variations at each point of measurement would also be presented. Simple interpolation between points in the network for a particular site would give approximately the appropriate spectrum.

(iii) In view of the fact that the total area of the reduced empirical spectrum as presented by Davenport may, as suggested earlier, prove to be a reasonable consistency (± 25%, say) for a particular site, a brief measurement of the more elusive high frequency portion of the spectrum carried out at the site of the proposed structure might be undertaken. Just as a specialized engineer dealing with the soil conditions may be called in so would an expert be required to measure the turbulence. The disadvantage here is that while the soil (say) is always there the wind is not(!), though it probably would not be
necessary to measure a wind of greater than moderate strength.

It must be remembered that the dynamic loading on a structure is
defined by its correlation in space as well as time. The three courses
of action outlined above apply equally well to the accurate estimation
of the spatial correlation.
APPENDIX B.

Preliminary Investigation into the Natural Frequency of the Tower
Preliminary investigation into the natural frequency of the tower

That the proposed tower structure should have a first mode natural frequency greater than 2 c/s (or, more specifically, within the range 2-4 c/s.) was shown to be necessary in section 2.1.3. It was therefore essential that design-stage calculations be made in order to verify that this condition would be satisfied. These calculations are briefly presented here.

\[ K_s = K \]
\[ m_s = M + 0.23m \]
\[ n_o = 1/2\pi \sqrt{K_s/m_s} \]

It was predicted in section 2.1.3. that the behaviour of the tower would be essentially that of a simple cantilever and, that, if the first mode alone were to be excited in the wind, its motion could be further compared to that of a simple single-degree-of-freedom spring-mass system (see fig. B1.). The stiffness, \( K \), of the spring is equivalent to the overall stiffness of the tower, which can be defined as,

\[ K = \delta W/W \quad (= L^3/3EI, \text{for the cantilever}) \]
where $\delta_w$ is the deflection resulting from the application of a point load, $W$, at the top of the tower. The equivalent mass can be shown to be approximately,

$$m_s = M + 0.23m$$

Whence the natural frequency,

$$n_0 = \frac{1}{2\pi} \sqrt{\frac{k}{(M + 0.23m)}} \text{ c/s.}$$

While a reasonably accurate computation of the mass distribution in the tower presents no difficulty only a rough estimate of the deflection, $\delta_w$, for a given load, $W$, is possible. A rigorous analysis of an all-welded space-frame such as the one proposed is somewhat difficult, and is not, in fact, essential here.)

Before proceeding to a rough estimation of the stiffness, however, the static effect of the top plate is briefly considered.

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1The derivation of this approx. formula is obtained using Rayleigh's method.
By treating the structure as a pin-jointed assembly of struts and ties and by assuming half of the structure (e.g., the panels forming the faces ABC (see fig. B2) produces the same deflection for a given lateral end load as would an 'equivalent' plane frame or truss of unspecified width, D, the system is reduced to a more readily analysable form. Consideration of a particular plane frame having a low height to width ratio, with only two bracing stages, and with a bracing angle, $\beta = 45^\circ$, permits certain general conclusions to be drawn regarding similar frames of greater slenderness. The 'equilibrium' force diagram for a lateral load, $T$, applied at $D'$ is shown in fig. B3.

Fig B3

It is clear from the force diagram that, in general, the forces in the compression and tension members will not balance symmetrically about the central upright. However, if the member $A'B'C'$ is made relatively rigid (with respect to the other members), the load, $T$, is equally distributed about joints $A'$ and $C'$, whence, because of the frame symmetry, the forces in corresponding tension and
compression members balance and the redundant forces, \( Q, R = 0 \), \( P = \frac{T}{2} \). In other words, surmounting the structure with a relatively rigid member (a circular plate in the actual design) means that balanced compressive and tensile forces in the side uprights increase linearly with each bracing stage (from the top), and the central upright experiences zero stress. The deflection mode of the frame (at the junction of successive bracing stages) is therefore the same as that of a simple fixed-free cantilever beam.

Taken in perspective the distributive effect of the top plate (and the base plate) is probably quite small affecting only the ends of the tower — but it does, nevertheless, ensure that the stiffness of the tower is constant throughout its height.

Estimation of the width, 'D', of the equivalent frame discussed above is clearly not a simple matter and for that reason an alternative approach to determine the stiffness of the tower was used. It was assumed that the overall stiffness of the tower was basically the sum of the stiffness contributions from each of the six constituent panels acting individually (a panel consisting of two adjacent uprights with intermediate bracing).

The above assumption (and the fact that a pin-jointed approximation is used) will result in an underestimation of the stiffness of the tower; a fact which, were the tower base to be perfectly rigid, would enable the determination of the various member sizes such that the natural frequency was not below 2-c/s. However, since it was anticipated that some flexure of the base plate might

\[ \begin{bmatrix} P \\ Q \\ R \\ \delta_T \end{bmatrix} = T \begin{bmatrix} -2/A - 4\sqrt{2}/a \\ 1/A^2 + 4/A + 8\sqrt{2}/a \\ 2/A + 4\sqrt{2}/a \\ -\sqrt{2}/a \end{bmatrix} \begin{bmatrix} -1/A - 2\sqrt{2}/a & \sqrt{2}/2a & -E/L \end{bmatrix}^{-1} \begin{bmatrix} -6/A - 4\sqrt{2}/a \\ 2/A + 4\sqrt{2}/a \\ 1/A + 2\sqrt{2}/a \\ -\sqrt{2}/a \end{bmatrix} \]

As \( A^0 \to \infty \), \( 1/A^0 \to 0 \), and \( Q, R \to 0 \), \( P \to T/2 \)
Occur, thereby lowering the natural frequency of the system as a whole it was decided to increase the lower bound design frequency of the tower (alone) from 2-c/s, to approximately 3-c/s.

The force diagram for a 'shear' panel subjected to a lateral load, 'P', is shown in fig. B4.

Equating the external work done by the load, P, with the internal rise in strain energy (Castigliano's Theorem), the lateral end deflection is,

\[ \delta_p = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P} \left[ \frac{P^2 \sec^2 \theta \cdot \cosec \theta \cdot L}{2AE} \right] + \frac{\partial}{\partial P} \left[ \frac{P^2 \tan^2 \theta \cdot h}{2AE} \right] + \left( 2P \tan \theta \right)^2 \cdot \frac{h}{2AE} + \left( 3P \tan \theta \right)^2 \cdot \frac{h}{2AE} + \ldots \left( 2NP \right)^2 \tan^2 \theta \cdot \frac{h}{2AE} \]

and the stiffness is,

\[ k = \frac{P}{\delta_p} = \frac{E}{L} \left[ 1/a \cdot \cosec \theta \cdot \sec^2 \theta + 1/3A \cdot \tan^2 \theta \cdot (2N + 1)(4N + 1) \right] \]

(where N is the number of bracing stages).

Consider the application of a lateral load, W, to the top of the tower. If each panel is considered as acting separately, it is clear that the four panels FE, ED, AB, and BC provide all the resistance to deflection (the panels FA and DC contributing
practically nothing) — see fig. B5.

The inclination of the panels to the main load direction, however, means that only a component of their stiffness will be brought into play. For the purposes of this preliminary design calculation it was regarded as sufficiently accurate to assume that the load, 'W', is resisted by the stiffness of the projected trusses $F'E', E'D', A'B'$, and $B'C'$. These projected trusses will be assumed to be composed from the same members as the original truss $^1$, except that the bracing angle, $\theta$, will have increased (to $\phi$) in the proportion $\tan \phi/\tan \theta = 2\sqrt{3}$.

The overall stiffness of the tower is then,

$$K = 4K^1 = 4E/L \left[1/a. \sec \phi. \cosec \phi + 1/3a.(2N+1)(4N+1) \tan^2 \phi\right]$$

In an attempt to comply with all the various aerodynamic, dynamic and other criteria discussed in CHAPTER 2 it was decided that all members be circular, $\theta = 30^0$ and that the slenderness ratio be approximately 15, whence $N = 23$. More detailed

$^1$In fact the equivalent bracing is shorter, but since the bracing contribution in the stiffness equation is very small the difference is ignored.
specifications are:

**Bracing:** 3/8" dia. circular solid, mild steel; c/s. area, $a = 0.307$ sq.in.

**Uprights:** 1 11/32" o.s. dia., 8 s.w.g. circular hollow, HFW, mild steel tubing; $A = 0.595$ sq.in.

**Top plate:** 34" dia., circular plate, thickness 1/8" (mild steel).

$L = 30' 8" = \text{total height of tower}$

$N = 22, \text{corresponding to 23 bracing 'units'}$

Substitution into the stiffness expression developed above gives:

$$K = 300 \text{ lb/in.}$$

Drawing information from the final design - see fig.  , and assuming the weight of steel to be 0.283 lb/in., the total weight of the tower, $m$, and that of the top plate, $M$, were computed as follows:

(i) WEIGHT OF TOWER. The tower is composed of:

- **Uprights:** 184.0' x 1 11/32" dia. 8 s.w.g. tubing @ 1.98 lb/ft. ..................($m_1$)

- **External Bracing:** 358.6' x 3/8" dia. circular rod @ 1.042 lb/ft. ..................($m_2$)

- **Internal Bracing:** 90.0' x 3/32" dia. circular rod @ 1.042 lb/ft. ..................($m_3$)

- **Welding:** allowance of 5 lb. ...............($m_4$)

$$m_g = m_1 + m_2 + m_3 + m_4 = 364.3 + 373.6 + 93.8 + 5 = 836.7 \text{ lb wt.} = (m \times g)$$

(ii) WEIGHT OF TOP PLATE. This is composed of:

- **Top plate:** 34" dia. x 1/8" thickness @ 0.283 lb/in. .... ($M_1$)

- **Plate Supports:** 6 x 6" dia. x 3/8" thickness @ 0.283 lb/in. ($M_2$)

$$M_g = M_1 + M_2 = 128.5 + 18 = 146.5 \text{ lb wt.} = (M \times g)$$

For the stiffness, $K$, computed above, the masses, $m$ and $M$, the natural frequency of the tower, taking $g = 386 \text{ in/sec}^2$, will be
\[ n_0 = \frac{1}{2\pi} \sqrt{\frac{300.386}{\sqrt{146.5 + 0.23856\cdot7}}} = 2.87 \text{ c/s} \]

That is, the first mode natural frequency of the tower as designed should be higher than 2.87 c/s. This low-bound value has purposely been taken higher than the desired minimum of 2 c/s, in order to allow for any spring action of the base plate tending to reduce the overall stiffness of the system and, hence, its natural frequency.
APPENDIX C.

Photographs of Certain Structural Features of the Lattice Tower
APPENDIX D(i).

Unsymmetrical Bending - The Deflection Ellipse
Unsymmetrical Bending - The Deflection Ellipse

Consider the subjection of a beam of arbitrary cross-sectional shape to a pure bending moment $M$, about the X-axis of an arbitrarily defined co-ordinate system (axes: $X, Y, Z$, with $Z$ parallel to the running of the beam) whose origin, $O$, is any point on the as yet unknown line, PP, above which all the 'fibres' will be in tension and below which all the 'fibres' will be in compression. (See fig. D1.)

![Diagram of beam and coordinate system](image)

**FIG D1**

In general a small element of area $\delta A$ will bend about both the $X$- and the $Y$-axis, such that the total strain is,

$$\epsilon = \frac{x\theta}{\rho_x} + \frac{y\phi}{\rho_y} = \frac{x}{\rho_x} + \frac{y}{\rho_y}$$

where $\rho_x, \rho_y$ are the radii of curvature about the $X$- and $Y$-axes

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$^1$I.e., there are no forces inducing shear; it is further assumed that plane sections remain plane after bending.
respectively.

If the elastic range of the material is not exceeded, the stress experienced by the small element is,

$$\sigma = E\varepsilon = E\left[\frac{x}{\rho_y} + \frac{y}{\rho_x}\right]$$

where $E$ is the elastic modulus of the material.

The sum of all the constituent elemental stresses must clearly balance the external conditions; this means that,

$$\int_A \sigma_x \, dA = 0, \quad \text{for pure bending (i.e., with no external axial loading acting on the beam)}$$

$$\int_A \sigma_y \, dA = M$$

and

$$\int_A \sigma_x \, dA = 0$$

satisfying the presence of the external couple, $M$, about the $X$-axis.

Substituting $\sigma$ into the above equations gives

$$\frac{E}{\rho_y} x_c + \frac{E}{\rho_x} y_c = 0 \quad \text{(i)}$$

$$\frac{E}{\rho_y} I_{yx} + \frac{E}{\rho_x} I_x = M \quad \text{(ii)}$$

$$\frac{E}{\rho_y} I_y + \frac{E}{\rho_x} I_{xy} = 0 \quad \text{(iii)}$$

where $x_c = \int_A x \, dA$, $y_c = \int_A y \, dA$, $I_x = \int_A y^2 \, dA$, $I_y = \int_A x^2 \, dA$, $I_{xy} = \int_A xy \, dA$.

It is clear that $x_c$ and $y_c$ represent the co-ordinate distances from the origin, 0, to the centroid of the section. It can be argued that the only solution to equation (i) is $x_c = 0$, $y_c = 0$; which means that the neutral axis must pass through the centroid.
Substitution of the radii of curvature, \( \rho_x \) and \( \rho_y \) from (ii) and (iii), into the stress equation, gives the bending stress at any point \((x,y)\),

\[
\sigma = M \frac{[I_y y - I_{yx} x]}{[I_x x - I_{xy}^2]}
\]

Along the neutral axis, \( \sigma = 0 \), whereupon,

\[
y = \frac{I_{xy}}{I_y} x = \tan \beta x \quad \text{(say)}
\]

In other words, the beam will bend about a neutral axis defined as the straight line which subtends an angle of \( \tan^{-1} \left( \frac{I_{xy}}{I_y} \right) \) with the axis about which the external moment is applied. In general, then, the overall deflection will not be coincident with the direction of the applied loading. The only practical exception occurs when \( I_{xy} = 0 \), whence \( y = 0 \), the X-axis about which the moment is applied becomes the neutral axis. In this case no unsymmetrical bending action takes place and \( \sigma = M y / I_x \), the normal expression for symmetrical bending is applicable.
For any unsymmetrical section the area moments and products of inertia (about any two mutually perpendicular axis) vary according to a Mohr circle type of relationship (see fig. 2D.). There are two points in the Mohr circle separated by $\beta = 90^\circ$ (corresponding to two mutually perpendicular axes in the unsymmetrical section) at which the products of inertia are zero. These two points also correspond to the positions of maximum and minimum moment of inertia (the principal moments of inertia).

The existence of these two mutually perpendicular axes\(^1\) about which symmetrical bending takes place in an unsymmetrical section greatly facilitates the analysis since the loading may then be resolved along these axes thereby producing symmetrical deflections which may be vectorially summed to obtain the resultant unsymmetrical deflection. As an example of this consider the subjection of a cantilever of arbitrary section to an end load, $P$, acting at an angle $\phi$ to one of the axes of principal moment of inertia (see fig. 3D).

\[ \text{stiffness} = \frac{3EI_{yy}}{2P1^3} \]

For an unsymmetrical section there will always be two, and only two such axes; for a section showing the same symmetry about two or more equiangular axes (e.g. square, equilateral triangle, hexagon, circle, etc.) the product of inertia is always zero and symmetrical bending takes place in all directions.
The deflections about the respective principal axes are:

\[ \delta_x = P \cos \phi \cdot \frac{1^3}{3EI_{yy}} \]

\[ \delta_y = P \sin \phi \cdot \frac{1^3}{3EI_{xx}} \]

(whence the resultant \((\delta_x^2 + \delta_y^2)^{\frac{1}{2}}\) and its direction \(\beta = \tan^{-1}\left[\frac{I_{xx}}{I_{yy}} \cdot \tan \phi\right]\) may be found. Eliminating \(\phi\), viz,

\[ \cos^2 \phi + \sin^2 \phi = 1 = \delta_x^2 / \left[\frac{P1^3}{3EI_{yy}}\right]^2 + \delta_y^2 / \left[\frac{P1^3}{3EI_{xx}}\right]^2 \]

shows that the locus of the resulting deflection is an ellipse - or, more explicitly, that rotation of the load, \(P\) (through \(\phi = 2\pi\)) results in the deflection of the end of the beam following an elliptical orbit. This is shown in fig. D3(ii).

An equivalent mass-spring system - with the same statical properties and deflection ellipse - is shown in fig. D3(iii).

The stiffness of the springs which are situated along the axes of principal moment of inertia is made equal to the stiffnesses of the beam about the principal axes. Any system showing a polar variation in stiffness may be so represented.

Once the deflection ellipse has been obtained it is possible by means of a simple geometrical construction to establish the direction of application of the load causing a particular deflection.
The deflection OA (see fig. D4) consists of two components $\delta x$ and $\delta y$.

Draw an arc of radius $\delta y$ about $O$.

Draw the arc of radius $OB$ to meet the axis XX in $D$.

The slope of the tangent to the arc $\delta y$ drawn from $D$, $DE$, represents the direction of the applied loading, $P$, producing the deflection $OA$ (since $P \cdot \sin \phi = \delta y / \left[ \frac{1}{3} / 3EI_{xx} \right]$).
APPENDIX D(ii).

Dynamical Consequence of Unsymmetrical Bending - Free Vibration
APPENDIX D(ii)

Dynamical Consequence of Unsymmetrical Bending - Free Vibration

It was shown in APPENDIX D(i) that the non-dynamic behaviour of a cantilever or unsymmetrical section or of any other non-symmetrical system following an elliptical deflection law could be represented by the system shown below (fig. D5).

Application of a quasi-steady force, P, to the central mass, M, along any one of the axes XX or YY will mobilise the resistance of the (spring) stiffness along that axis only - the resistance along the mutually perpendicular axis will not be brought into play. Clearly, the position is not altered if an impulse is applied to the mass along the axis XX (say). The free vibration resulting from the impulse will constrain itself to the XX axis.

For a deflection $x$ from the neutral position, it follows that for dynamic equilibrium,

$$M \ddot{x} + K_x x + C_x \dot{x} = 0 \quad \text{.........(i)}$$

The damping is assumed viscous and proportional to the velocity ($C_x \dot{x}$). It is assumed that all vibration occurs within the
linear elastic range. The general solution of the equation (i) is

\[ x = \exp(-\beta_x t)(c_1 \sin \omega_x^' t + c_2 \cos \omega_x^' t) \] ...............(ii)

where \( \beta_x = c_x / 2M \), \( \omega_x^' = \sqrt{(\omega_x^2 - \beta_x^2)} \), \( \omega_x = \sqrt{K_x / M} \).

\( \beta_x \) is a direct measure of the damping present along the XX axis, \( \omega_x^' \) is the natural frequency (in radians) of the damped system, and \( \omega_x \) is the natural frequency of the system without damping.

If the mass is given an initial displacement \( x_0 \) and velocity \( \dot{x}_0 \), the displacement of the mass at any subsequent time, \( t \), is from equation (ii),

\[ x = \exp(-\beta_x t)[(\dot{x}_0 + \beta x_0) / \omega_x^' \sin \omega_x^' t + x_0 \cos \omega_x^' t] \] ...............(iii)

The response to an initial displacement \( x_0 \) is shown in fig. D6.

If the damping is so large that \( \beta_x = \omega_x \) then \( \omega_x^' \rightarrow 0 \), and the equation of motion, (iii), is no longer periodic, viz,

\[ x = \exp(-\beta_x t)[\dot{x}_0 t + x_0(1 + \omega_x^' t)] \] ...............(iv)

The response for an initial displacement \( x_0 \) is shown in fig. D7 with the mass slowly creeping back to its neutral position.
The amount of damping contributing to this state of affairs ($\beta_x = \omega_x$) is the critical damping ($c_{crx}$) and is used as a reference point for all other values of damping usually expressed in percentage form, viz,

$$\frac{c}{c_{cr}} = \% \text{ critical damping} = \frac{C_x}{c_{crx}} = \frac{\beta_x}{\omega_x}$$

In the above analysis the natural frequencies of the system with and without damping have been treated as distinct quantities. In fact, the two are very nearly equal, for values of damping less than 10% critical and will be treated as such from here on. For practical measurements of damping it is usual to obtain a record of the decaying motion (such as that shown in fig. D6) from which the logarithmic decrement, or the ratio of the ordinates of two consecutive peaks is measured. Since the difference between two consecutive peaks may be small, the decrement over several cycles may be measured. This is related to the critical damping in the following way:

For a system with 10% critical damping, $c/c_{cr} = 10/100 = 0.1 = \beta_x/\omega_x$

whence,

$$\omega_x' = \sqrt{\frac{\omega_x^2}{\beta_x^2}} = \left(0.99 \omega_x^2\right)^{\frac{1}{2}} = 0.995 \omega_x$$
Typical ordinates separated by n cycles (of period, T) are,

$$x_1 = x_0 \sqrt{1 + \left(\frac{\beta_x}{\omega_x}\right)^2} \exp\left[-\beta_x t\right]$$
$$x_2 = x_0 \sqrt{1 + \left(\frac{\beta_x}{\omega_x}\right)^2} \exp\left[-\beta_x (t + nT)\right]$$

whence, the log decrement is,

$$\log_e(x_1/x_2) = \log_e(\exp[\beta_x nT]) = \beta_x nT$$

and $$(c/c_{cr}) = \frac{\beta_x}{\omega_x} = \frac{1}{2\pi n} \log_e(x_1/x_2)$$

The brief mathematical description given above of the oscillatory and damping characteristics of the free vibration of what amounts to a single-degree-of-freedom system operating along the XX-axis is also perfectly valid for the vibration of the mass along the YY-axis. The question arises: what if an impulse is imparted to the mass along the two different axes at the same time? Since the motion along the two axes are completely independent, the free vibration will develop as before along each axis. But since the mass cannot be in two places at once it performs a complicated gyration, known as a Lissajous pattern, simultaneously satisfying the sinusoidal vibrations of both principal axes. The tower system is, in fact, equivalent to two completely independent single-degree-of-freedom systems at right
angles to each other and as such there can be no transfer of vibrational energy from one system to the other.

It is to be noted that the two natural (first mode) frequencies of the tower system do not form a low integer ratio and for this reason a sketch of the Lissajous pattern applicable will not be attempted (not even for a zero phase difference between the two constituent motions)!
APPENDIX E.

Difficulties Encountered in Open-air Preliminary Measurements
APPENDIX E.

Difficulties Encountered in Open-air Preliminary Measurements

The difficulties of experimentation which arise in open-air non-controlled conditions are considerable. In order to perform the measurements described in 3.2.1. (i.e., determining the deflection ellipse of the tower) two predominant weather conditions imposed themselves. These were that there should be

(i) dry, rainless conditions,
(ii) little or no wind.

Having thus restricted the available time to (what turned out to be) remarkably few days, it was found that a further curtailment was necessary due to the disturbing influence of heat radiation from the sun. It was found that even under apparently stable cloudless meteorological conditions the tower was subject to deflections (as measured by the dial gauges without any imposed external load) of sufficient magnitude and rate of change to preclude any accurate experimentation. These variations increased with the presence of intermittent cloud - a typical variation of both deflection and temperature (as measured by a thermometer exposed to direct sunlight) is given in fig. El). It was observed that these parasitic tower deflections showed a definite correspondence with the position of the sun, which would seem to suggest expansion of the exposed face of the tower. However, it was thought that exposure of the dial gauges, the supporting frame and the aluminium cladding from which the frame was cantilevered was also partly responsible. It was found that overcast rainless and windless weather was most favourable for the purposes of obtaining the experimental data required. Other minor hindrances included the formation of condensation on the dial gauge shafts.
Variation of dial gauge 'zero' compared with that in exposed thermometer
preventing their free running; parasitic vibrations transmitted via the base of the tower and the frame supporting the dial gauges; and the disadvantages associated with a restricted working space.
APPENDIX F.

Theoretical Analysis of Various Statical and Dynamical Deflection Modes Adopted by Tower
Theoretical Analysis of Various Statical and Dynamical Deflection Modes Adopted by Tower

Introduction

In general the tower structure will adopt three basic types of deflection profile within the confines of the experiment to be performed, producing, for a given deflection at the free-end of the structure, different deflections at the point of measurement. Two of these profiles arise from static or quasi-static loading, and can be briefly described as

(i) the deflection profile produced by the application of a lateral free-end load, $P$ - see fig. Fl(i); and

(ii) the deflection profile arising from the distributed loading, $W$, consisting of the quasi-steady component of the free-wind pressure - see fig. Fl(ii).

(iii) The third basic profile is that arising from dynamic wind forces and consists of the free vibration of the system about its natural frequency - see fig. Fl(iii).

Since the system is to be calibrated by noting the deflection, $a_p$, for a given load, $P$ (as in fig. Fl(i)) it is important to know what the relationship between $P$ and $W$ is for deflections at the point of measurement arising from subjection of the tower to loading, $W$. The problems involved in establishing this and other relationships is complicated by the fact that the normal cantilever modes of deflection is associated with a linear mode arising from flexure of the base plate. It is necessary to establish what proportion the linear mode occupies within the total deflection profile. Each case will be considered in turn.
CASE 1: Deflection Profile of Tower System Subject to Lateral Load at Free-end

The two constituent modes are shown superimposed in fig. F2—one mode is linear, arising from flexure of the base plate; the other is the normal cantilever mode.

For a given load, \( P \), the total deflections at heights \( Z_1 \) and \( Z_2 \) are \( A_p \) and \( a_p \) respectively. Since the constituent displacements, \( A_{pB}, A_{pT} \) and \( a_{pB}, a_{pT} \) cannot be measured by direct experimentation\(^1\) occurring simultaneously, it is not possible to isolate either one of the modes.

\(^1\)Occurring simultaneously, it is not possible to isolate either one of the modes.
expressions for \( A_{pB}, A_{pT} \) or \( a_{pB}, a_{pT} \) in terms of the directly measurable deflections, \( A_p, a_p \) must be determined theoretically.

By definition,

\[
A_p = A_{pB} + A_{pT} \quad \text{..................(i)}
\]

\[
a_p = a_{pB} + a_{pT} \quad \text{..................(ii)}
\]

For the linear mode,

\[
A_{pB} = a_{pB} \cdot L/Z_1
\]

For the cantilever mode,

\[
a_{pT} = \left[3Z_2^2/2Z_1^3(Z_1 - Z_2/3)A_{pT}\right]
\]

whence,

\[
A_{pT} = K a_{pT}
\]

Substitution into (i) gives,

\[
A_p = a_{pB}Z_1/Z_2 + a_{pT}K \quad \text{..................(iii)}
\]

Solving equations (i) and (ii) gives,

\[
a_{pT} = (A_p - a_pZ_1/Z_2)/(K - Z_1/Z_2)
\]

\[
a_{pB} = (a_p - A_pK)/(K - Z_1/Z_2)
\]

and, knowing \( a_{pT}, a_{pB}, A_{pT} = K a_{pT} \quad \text{..................(iv)} \)

\[
A_{pB} = Z_1/Z_2 \cdot a_{pB} \quad \text{..................(v)}
\]

where \( K = 2Z_1^3/3Z_2^2(Z_1 - Z_2/3) \)

Expressions (iv), (v) enable the deflection per unit load ellipse to be divided into the contribution from each of the two types of flexure (base and beam) involved.

Once the relative magnitudes of the constituent, free-end deflections, (for unit load) \((A_{pB})'\) and \((A_{pT})'\) are known it is possible by varying \( Z_2 \) to write down the equation of the deflection mode, viz,

\[
Y_p = Z_1 Z_2 \left[ (A_{pB})' + (A_{pT})' \cdot 3Z_1^2/2Z_2^2 \cdot (Z_1 - Z_2/3) \right]
\]

CASE 2: Deflection Profile of Tower System Subject to Quasi-Steady Wind Loading

The following analysis is based on the assumption that the \( (A_{pB})' = (A_{pB})_p, \text{(in}/\text{lb}) \text{ etc.} \)
quasi-steady or mean wind profile varies according to the velocity power law

\[ V_z = k \cdot z^n \]

where \( k \) and \( n \) are constants. The profile of the resulting force distribution will be of the form

\[ p_z \propto V_z^2 \text{ or } p_z = k \cdot z^{2n} \]

where \( K \) is assumed constant. \(^1\)

The force profile and associated tower deflections are sketched in fig. F3.

The total wind load, \( W \), is,

\[ W = \int_0^{Z_1} p_z \cdot dz = k \cdot Z_1^{2n+1}/(2n+1) \tag{1} \]

The bending moment at any point \( z \) of the tower is found by summing the moments of all the elementary wind force contributions above \( z \), such as \( p_x \cdot \delta x \cdot (x - z) \), viz,

\[ M_z = \int_z^{Z_1} p_x \cdot (x - z) \cdot dx = K \cdot \int_z^{Z_1} x^{2n}(x - z) \cdot dx \]

Evaluating the integral and substituting for \( K \) from (i),

\[ M_z = W/2(n + 1) \cdot [(2n + 1)Z_1 - (2n + 2)z + z(z/Z_1)^{2n+1}] \tag{ii} \]

\(^1\) \( K \) consists of \( 'k' \) and \( \frac{1}{2} \rho CA' \); \( \rho \) being the density of air; \( C \), the drag coefficient; \( A \), the projected area of the structure.
The moment exerted on the base plate (producing the linear mode)
is
\[ M_{z=0} = WZ_1(2n + 1)/(2n + 2) \]

This is equivalent to the application of a lateral top load,
\[ P_W = W(2n + 1)/(2n + 2), \]
so that the free-end deflection of the linear mode for the loading, \( W \), is,
\[ Y_{WBZ_1} = (A_{PB})^tP_W = (A_{PB})^tW.(2n+1)/(2n+2) \]
and the deflection at any point \( z \) is,
\[ Y_{WBZ_1} = z/Z_1 \cdot Y_{Nz_1} = (A_{PB})^t(2n + 1)/(2n + 2) \cdot W \cdot z/Z_1 ...(iii) \]

In order to find the deflection of the fixed-free cantilever mode, use is made of the fact that
\[ EI \cdot \frac{d^2y}{dz^2} = -M_z = -W/2(n+1) \left[ (2n + 1)Z_1 - (2n + 2)z + z(z/Z_1)^{2n+1} \right] \]
The complete deflection profile for a total wind load, \( W \), is then
\[ Y_{Wz} = Y_{WBZ} + Y_{WTZ} = W \left[ (A_{PB})^t \left( \frac{2n + 1}{2n + 2} \right) Z_1 + \frac{3(A_{PP})^t}{2(n+1)} \left( \frac{2n + 1}{2} \right) \left( \frac{Z_1}{z} \right)^2 \right] \]
\[ - \frac{1}{3}(n+1)\left( \frac{Z_1}{z} \right)^3 + \frac{1}{(2n+3)(2n+4)} \left( \frac{Z_1}{z} \right)^{2n+4} \]

It must be remembered that the deflection at \( z = Z_2 \), \( Y_{WZ_2} \) will be calibrated in terms of the lateral top load, \( P \) (discussed in CASE 1). Hence it is possible, for the same deflection in the two cases, \((viz, Y_{WZ_2} = Y_{PZ_2})\) to calculate the value of \( W \) in terms of \( P \).

For a calibration load, \( P \), the deflection at \( z = Z_2 \) (CASE 1) is,
\[ Y_{PZ_2} = Pz/Z_1 \left[ (A_{PB})^t + (A_{PP})^t \cdot 3z/Z_1^2(z_1 - z_2/3) \right] \]
For an effective wind load, \( W \), the deflection at \( z = Z_2 \) (CASE 2) is,
\[ Y_{WZ_2} = W \left[ (A_{PB})^t(2n+1)/Z_1 + \frac{3(A_{PP})^t}{2(n+1)} \left( \frac{2n+1}{2} \right) \left( \frac{Z_2}{Z_1} \right)^2 - \frac{1}{3}(n+1)\left( \frac{Z_2}{Z_1} \right)^{3/2} \right] \]
\[ + \frac{1}{(2n+3)(2n+4)} \left( \frac{Z_2}{Z_1} \right)^{2n+4} \]
Putting $y_{PZ_2} = y_{WZ_2}$ and dividing the two expressions, gives

$$
\frac{Z_2}{Z_1} \left[ (APB)^I + \frac{3Z_2}{2Z_1}(Z_1 - \frac{Z_2}{3}) \right]
$$

$$
\frac{W}{P} = \frac{\left[ (APB)^I \frac{(2n+1)Z_2}{Z_1} \right] + \frac{3(2n+1)}{2(2n+2)} \left( \frac{Z_2}{Z_1} \right)^2 - \frac{(n+1)}{3} \left( \frac{Z_2}{Z_1} \right)^3}{(2n+3)(2n+4)} \left( \frac{Z_2}{Z_1} \right)^{2n+4}
$$

Assuming that the aerodynamic drag coefficient, $C$, is constant over the height of the tower, $Z_1$, and does not vary with mean wind speed, the effective pressure (per unit area) exerted on the tower resulting from normal incidence of the wind, is

$$
p_z/D = \frac{1}{2}\rho V_z^2 \cdot C
$$

where $D$ is the lateral dimension of the tower, $V_z$ is the quasi-steady wind velocity, $\rho$ the air density and $C$ the drag coefficient.

But by previous definition,

$$
p_z = K_z^{2n}
$$

and $W = \text{total wind load} = K_z^{2n+1}/(2n+1)$

whence, $\frac{1}{2}\rho V_z^{2n} \cdot C \cdot D = K_z^{2n} = W(2n+1)/Z_1$

and $C = \frac{2\lambda \rho P(2n+1)}{\rho V_z^{2n} \cdot A}$

(whence $A = D.Z_1$ is the area enclosed by the perimeter of the tower in plane elevation).

The above expression for the drag coefficient depends on knowledge of the quasi-steady wind velocity, $V_{Z_1}$, at the top of the structure; if, however, the wind velocity, $V_{Z_3}$, at height $Z_3$ is that which is measured, it follows from the power law profile
variation that,
\[ V_{Z_1} = V_{Z_3} \left(\frac{Z_1}{Z_3}\right)^n \]
whence,
\[ C = \frac{2\lambda P (2n+1) Z_3^{2n}}{\rho V_{Z_3}^2 Z_1^{2n}} \]

For a particular profile (characterized by the power index, n) \( \lambda \) is obtained from equation (iv) (or the curve in CHAPTER 4). P and \( V_{Z_3} \) are direct observations whence C is obtained.
CASE 3: Mode Shape of Structure in Free Vibration

The mode shape of the tower considered as a fixed-free cantilever is first determined. From the moment exerted on the base by an arbitrary vibration of the tower (considered as a fixed-free cantilever), the proportional contribution from flexure of the base to the total vibrational mode is then determined.

\[ M_z + \alpha M_z v_z + \gamma v_z P(t) \]

\[ p(t, z) \]

\[ M \]

\[ M_0 \]

\[ \delta z \]

\[ \delta z \]

\[ M_z \]

\[ p(t, z) \]

\[ (\text{UDL}) \]

\[ m \]

\[ y \]

\[ z \]

FIG F4

The fixed-free cantilever is shown in fig. F4, where m is the total, uniformly distributed mass of the cantilever itself, M the mass of the end load and \( p(t, z) \) the applied loading intensity which varies with both time and position. Consider the dynamic equilibrium of an element, \( \delta z \), of the cantilever. The load intensity on the element is

\[ w = p(t, z) - m/z_1 \frac{d^2 y}{dt^2} \] .......................... (i)

If the flexural rigidity of the cantilever is \( EI = K \cdot z_1^{3/3} \), then the moment, loading and deflection are related by

\[ \frac{\partial^2 M_z}{\partial z^2} = -KZ_1^3/\gamma z_1 \cdot \frac{\partial^4 y}{\partial z^4} = \frac{\partial^4 y}{\partial z^4} \]

and (i) becomes

\[ KZ_1^3/3 \cdot \frac{\partial^4 y}{\partial z^4} + m/z_1 \cdot \frac{\partial^2 y}{\partial t^2} = p(t, z) \]  ........ (ii)

Where \( K \) is the lateral stiffness of the cantilever with respect to its free end.
where \( y \) is the deflection at time, \( t \), at any point, \( z \), along the cantilever. In free vibration, \( p(t, z) = 0 \) and \( y_n(t, z) \) are considered as the combination of a characteristic shape or mode, \( \phi_n(z) \), and a time function, \( f_n(t) \), such that for the \( n \)th mode
\[
y_n(t, z) = f_n(t) \phi_n(z)
\]
Substitution into (ii) gives
\[
K \frac{3}{3} \frac{\partial^4}{\partial z^4}(\phi_n(z)) + \frac{m}{Z_1} \phi_n(z) \frac{\partial^2}{\partial t^2}(f_n(t)) = 0
\]
on separation of the variables it is found that the solution consists of two equations:
\[
\frac{\partial^2}{\partial t^2}(f_n(t)) + \omega_n^2 f_n(t) = 0
\]
and
\[
\frac{\partial^4}{\partial z^4}(\phi_n(z)) - \frac{m}{Z_1} \frac{3}{3} \frac{K \phi_n(z)}{= 0}
\]
The solution of the first equation which is of the form \( f_n(t) = \cos(\omega_n t + \theta) \) shows that the motion is harmonic with natural frequency, \( \omega_n \). The solution of the second equation is
\[
\phi_n(z) = A \cos(\omega_n z) + B \cos(\omega_n z) + C \sin(\omega_n z) + D \sinh(\omega_n z)
\]
Where \( \omega = (3. \frac{m}{Z_1} \frac{\omega_n^2}{k})^{1/4} \) and \( A, B, C, D \) are constants dependent on the boundary conditions. In the case of the structure being considered the boundary conditions for a maximum deflection, viz, \( y_n(t, z)_{\text{max}} = \phi_n(z), (\cos(\omega_n t + \theta) = 1) \) are indicated in fig. F5.

\[\text{fig. F5}\]

\( I_M \) is the mass moment of inertia of the mass, \( M \), about its centroidal axis perpendicular to the paper.
From the boundary conditions at the base (viz, \( \phi(z = 0) = \dot{\phi}(z = 0) = 0 \)), \( A = C = 0 \), the end conditions at \( z = Z_1 \) are somewhat complicated by the presence of the load, \( M \). It is found that the reaction on the cantilever consists of both a shear and a rotational thrust (the first due to the linear displacement of the mass, the second to the slight rotation of the mass, \( M \), during vibration of the cantilever). Since

\[
y(t, z) = \phi_n(z) \cos(\omega_n t + \theta) \quad \text{(iii)}
\]

then by differentiation, for a maximum displacement,

\[
(\partial^2 y / \partial t^2)_{\text{max}} = -\omega_n^2 \cdot \phi_n(z)
\]

\[
\left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial y}{\partial z} \right) \right]_{\text{max}} = -\omega_n^2 \cdot \frac{\partial}{\partial z} (\phi_n(z))
\]

whence the end conditions become, \((K/3, Z_1^3 = EI)\)

\[
K/3, Z_1^3 \cdot (\partial^2 \phi / \partial z^2)_{Z_1} = -M \cdot \omega_n^2 \cdot \phi(Z_1)
\]

\[
K/3, Z_1^3 \cdot (\partial^2 \phi / \partial z^2)_{Z_1} = -I_m \cdot \omega_n^2 \cdot (\partial \phi / \partial z)_{Z_1}
\]

Substitution of \( \phi \) into the above equations means that the \((n)\)th mode shape may be written as,

\[
\phi_n(z) = B_1 \left[ \frac{D}{B} (\sinh az - \sin az) + (\cosh az - \cos az) \right] \quad \text{.........(iv)}
\]

where

\[
(\partial^2 \phi / \partial z^2)_{Z_1} = \frac{\omega_n^2}{K/3} \cdot (\partial^2 \phi / \partial z^2)_{Z_1} = -M \cdot \omega_n^2 \cdot \phi(Z_1)
\]

\[
K/3, Z_1^3 \cdot (\partial^2 \phi / \partial z^2)_{Z_1} = -I_m \cdot \omega_n^2 \cdot (\partial \phi / \partial z)_{Z_1}
\]

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\]

\[
K/3, Z_1^3 \cdot (\partial^2 \phi / \partial z^2)_{Z_1} = -I_m \cdot \omega_n^2 \cdot (\partial \phi / \partial z)_{Z_1}
\]

in which \( \omega_n = (a A_1^4 \cdot K/3m)^{1/2} \) and where \( a \) is the \((n)\)th root of the equation:

\[
(I_m / m, Z_1^4 - 1) + [(m, Z_1^4 - 1) / m, Z_1^4 - 1] \sin a az - \cos a az - [(m, Z_1^4 - 1) / m, Z_1^4 - 1] \sin a az = 0
\]

\[
\text{.........(vi)}
\]
Having obtained the lowest root, \( a \), the frequency of the first mode of vibration of the cantilever (considered fixed-free) is,

\[
\omega_1 = (a^4Z_1^4K/3m)^{1/2}
\]  

(vii)

Clearly if in any situation the first mode alone is likely to develop, it simplifies matters to imagine the fixed-free cantilever as an equivalent one-degree-of-freedom mass-spring system — see fig. F6.

Let the natural frequency of the equivalent mass-spring system be

\[
\omega_1 = \left[ K/(M + \gamma m) \right]^{1/2}
\]  

(viii)

Equating with expression (vii),

\[
\omega_1 = \left[ K/(M + \gamma m) \right]^{1/2} = (a^4Z_1^4K/3m)^{1/2}
\]

whence,

\[
\gamma = 3/(aZ_1^4) - M/m
\]  

(ix)

The equivalence of the two systems is such that any lateral loading, \( P \), dynamic or otherwise, applied as shown in fig. F6, (to the mass, \( M \), in the case of the cantilever, and to the mass, \((M + \gamma m)\), in the mass-spring system) will produce the same amplitudes and frequencies of vibration at the point of load application. In the case of the tower structure, however, flexure of the supporting base plate introduces a factor which has not been considered above. The system then becomes that which is shown in fig. F7.
$K_B$ and $K_T$ are the stiffnesses of the base and tower respectively referred to the top of the tower system (at height $Z_1$). The equivalent mass-spring system, now includes two springs and the resultant stiffness of the combination becomes, $(K_T + K_B)/(K_T K_B)$ such that the frequency of the system (equal to that of the tower system) is now

$$\omega_{B,T} = \left[ \frac{(K_T + K_B)/(K_T K_B)}{(M + \gamma m)} \right]^{1/2}$$

In the same way that any in-line force on the mass of the equivalent mass-spring system is transmitted simultaneously to both springs, thereby maintaining dynamic equilibrium, so, in the tower system, flexure of the base plate will occur in simultaneous response to the moment transmitted via the main body of the tower in free vibration.

Assume that the tower deflects according to the modal shape derived above, viz,

$$y = \phi(z) = B_1 \left[ (D/B)(\sinh az - \sin az) + \cosh az - \cos az \right]$$

where $B_1$ is an arbitrary amplitude. The bending moment at any point $(z)$ will be,

$$M_z = -EI \frac{\partial^2 \phi}{\partial z^2} = -K_T Z_1^2 \frac{\partial^2 \phi}{\partial z^2}$$

whence,

$$M_{z=0} = -K_T Z_1^3 \frac{\partial^2 \phi}{\partial z^2} \left[ \frac{D}{B}(\sinh az + \sin az) + \cosh az + \cos az \right]_{z=0}$$

$$= -\frac{2}{3} K_T B_1 Z_1^3 a^2$$
The deflection at the top of the tower due to flexure of the base plate alone will be,

\[ y(Z_1)_B = \frac{M}{2\pi Z_1} \]

The deflection of the tower due to flexure of the base plate at height, \( z \), is,

\[ y(z)_B = y(Z_1)_B \cdot \frac{z}{Z_1} = \frac{2}{3} \cdot \frac{K_T}{K_B} \cdot \frac{Z_1}{z} \cdot a^2 \cdot B_1 z \]

The total mode shape is therefore,

\[ \varphi(z) = (y_T + y_B) = B_1 \left[ \frac{2}{3} \cdot \frac{K_T}{K_B} \cdot \frac{Z_1}{z} \cdot a^2 \cdot z + \frac{D}{B} \right] \left[ \sinh z - \sin z \right] + \cosh z - \cos z \]
APPENDIX G.

(i) Forced Vibrations in a Single-degree-of-freedom System

(ii) Frictional Damping
(i) Forced Vibrations in a Single-degree-of-freedom System

Consider the system to be subjected to a time varying load as shown in fig. G1)

\[ p(t') = P_1 f(t') \]

If the mass of the system is initially at rest and is suddenly subjected to a constant force \( P_1 \) for a short duration of time, \( \delta t' \), it will undergo an initial acceleration, \( \ddot{x} = \frac{P_1}{M_o} \).

(Little spring or damping resistance will have developed if \( \delta t' \) is small). The acceleration can be assumed constant and the velocity at time \( t' + \delta t' \) is,

\[ \dot{x} = \ddot{x} t' = \frac{P_1}{M_o} \delta t' = \frac{i}{M_o} \]

where 'i' is defined as the applied impulse. It follows that, in general, an increment of velocity, \( P_1 f(t') \delta t/M_o \) is superimposed on the system at rest: it was shown in APPENDIX D(ii) that the displacement at a subsequent time, \( \delta t \), due to an initial velocity impart \( x_o \) was,

\[ x = e^{-\beta t} \left[ (\dot{x}_o + \beta x_o)/\omega_x \right] \sin \omega_x t + x_o \cos \omega_x t \]

If \( x_o = 0 \), then the contribution from the impulse 'i' is,

\[ x_i = P_1 f(t') \delta t'/M_o \omega_x \exp [-\beta(t - t')] \sin \omega_x (t - t') \]
If the system is assumed linear the displacement due to a sum of elemental impulses between 'o' and 't' will be,

\[ x = \int_{t_0}^{t} \frac{P_1 f(t')}{M_0 \omega_x} \cdot \exp \left[ -\beta(t - t') \right] \cdot \sin \omega_x'(t - t') dt' \]

Taking into account any initial conditions (displacement, \( x_0 \), or velocity, \( \dot{x}_0 \)) and writing \( x_{st} = \frac{P_1}{K} = \frac{P_1}{(\omega_x')^2 M_0} \) for the static deflection due to \( P_1 \), the total response may be written

\[ x = \exp(-\beta t) \left[ \frac{x_0 + \beta x_0}{\omega_x'} \cdot \sin \omega_x't + x_0 \cdot \cos \omega_x't \right] \]

\[ + x_{st} \cdot \omega_x' \int_0^t f(t') \cdot \exp \left[ -\beta(t - t') \right] \cdot \sin \omega_x'(t - t') dt' \]

It was shown in APPENDIX A that any random signal varying with time, \( f(t) \), could be broken down into an equivalent set of infinitely close sinusoidal waves. Let a typical component in this case be represented as

\[ \left[ P_1 f(t) \right]_{\xi} = P_1 \sin \xi t \]

The response from such a typical component, assuming \( \dot{x}_0 = x_0 = 0 \), is,

\[ x_\xi = x_{st} \cdot \omega_x' \int_0^t \sin \xi t \cdot \sin \omega_x'(t - t') \cdot \exp \left[ -\beta(t - t') \right] dt' \]

ie.,

\[ x_\xi = \left[ \exp(-\beta t) \left[ 1 - \xi^2 / (\omega_x')^2 \right] \cdot 2(\beta / \omega_x')^2 \cdot \sin \omega_x't + 2(\beta / \omega_x')^2 \cdot \cos \omega_x't \right] \cdot x_{st} \cdot \xi / \omega_x' \]

\[ + x_{st} \left[ 1 - (\xi / \omega_x')^2 \right] \sin \xi t \cdot 2(\beta \xi / (\omega_x')^2) \cdot \cos \xi t \]

\[ = \left[ 1 - (\xi / \omega_x')^2 \right]^2 + 4 \left[ (\beta \xi / (\omega_x')^2) \right]^2 \]

The first term of the above expression allows for the build up of
the motion of the system from rest when initially subjected to the sinusoidal loading. After a few cycles of vibration this term may be ignored.

It is found convenient to introduce the concept of the dynamic load factor (or magnification factor, mechanical admittance, etc.), which is defined as the ratio of the maximum dynamic displacement to the deflection which would have resulted from static application of the load (i.e., $P_1$, in $P_1f(t)$). Ignoring the transient part of (ii),

$$X = \frac{(x_x)^{\text{max}}}{x_{st}} = \frac{1}{\left[1 - \frac{\xi^2}{(\omega_x')^2} + 4(\beta \xi / (\omega_x')^2)^2\right]^{\frac{1}{2}}}$$

$$X_{\text{max}} = \omega_x' / 2\beta \quad (\text{when}, \quad \xi = \omega_x')$$

The dynamic load factor is shown in fig. G2 in terms of $(\xi / \omega_x')$.

**FIG G2**

Use is made at a later stage of the area under the square of the mechanical admittance curve ($X^2$). Consider a pass-band $'2p/\omega'$
centred on the natural frequency of the system.

Put,

$$\frac{\xi}{\omega_x'} = (1 + \frac{p}{\omega}),$$

such that,

$$\left[1 - \left(\frac{\xi}{\omega_x'}\right)^2\right]^2 = 4\frac{p^2}{(\omega_x')^2}$$

and

$$4\left(\frac{\beta}{\omega_x'}\right)^2 = 4\beta^2/\omega_x'^2$$

Then,

$$\chi^2 = \frac{1}{\left[4\frac{p^2}{(\omega_x')^2} + 4\beta^2/\omega_x'^2\right]} = \frac{(\omega_x')^2}{4(p^2 + \beta^2)}$$

and

$$\int_0^\infty \chi^2\frac{\xi}{\omega_x'}d\left(\frac{\xi}{\omega_x'}\right) = \frac{1}{\omega_x'}\int_0^\infty \left(\frac{\omega_x'}{4(p^2 + \beta^2)}\right)d\left(\frac{\xi}{\omega_x'}\right) = \frac{(\omega_x')^2}{4\beta}\left[\tan^{-1}\frac{1}{\beta}\right]_0^\infty = \frac{\pi \omega}{4\beta}$$

(ii) Frictional Damping

The effect of Coulomb or frictional damping on the motion of a one-degree-of-freedom system — in addition to viscous damping — is considered here. The magnitude of the frictional force, F, is constant but changes sign with every half cycle of vibration so as to oppose the motion. Consider a one-degree-of-freedom system given an initial displacement, \(x_0\). From equations (i) and (ii) (APPENDIX G(i)), the motion for the first half cycle is given by:

$$x = \exp(-\beta t)\left[\beta x_0/\omega_x' \cdot \sin \omega_x' t + x_0 \cdot \cos \omega_x' t\right]$$

$$+ \frac{F}{K}\left[1 - \exp(-\beta t)\left[\cos \omega_x' t + \beta/\omega_x' \cdot \sin \omega_x' t\right]\right]$$

\((0 < t < \pi/\omega_x')\)

The first negative peak is given by putting \(t = \pi/\omega_x'\)
\[ x_{N1} = -x_0 \exp(-\beta \pi / \omega_x') + F/K 1 + \exp(-\beta \pi / \omega_x') \]

The first positive peak is obtained in a similar way, using \( x_{N1} \) as the initial displacement and reversing the sign \( F \), viz,

\[ x_{P1} = (x_0 - F/K) \exp(-2\pi \beta / \omega_x') - 2F/K \exp(-\pi \beta / \omega_x') - F/K \]

The \( n \)th negative and positive peaks can be shown to be:

\[ x_{Pn} = \exp(-2n\pi \beta / \omega_x')[x_0 - F/K] - 2F/K \exp(-(2n-1)\pi \beta / \omega_x') - \ldots - 2F/K \exp(-\pi \beta / \omega_x') - F/K \]

\[ x_{Nn} = -\exp(-(2n-1)\pi \beta / \omega_x')[x_0 - F/K] + 2F/K \exp(-(2n-2)\pi \beta / \omega_x') + \ldots + 2F/K \exp(-\pi \beta / \omega_x') + F/K \]

On addition, this gives,

\[ (x_{Pn} + x_{Nn}) = \exp(-(2n-1)\pi \beta / \omega_x') \left[ \exp(-\pi \beta / \omega_x')(x_0-F/K)-(x_0+F/K) \right] \]

Similarly, for the \( m \)th negative and positive peaks,

\[ (x_{Pm} + x_{Nm}) = \exp(-(2m-1)\pi \beta / \omega_x') \left[ \exp(-\pi \beta / \omega_x')(x_0-F/K)-(x_0+F/K) \right] \]

such that,

\[ (x_{Pn} + x_{Nn})/(x_{Pm} + x_{Nm}) = \exp(-2\pi \beta / \omega_x')(m - n) \]
or,

\[ \beta = \omega_x' / 2\pi (m - n) \cdot \log e \left[ \frac{(X_{Pn} + X_{Nn})}{(X_{Pm} + X_{Nm})} \right] \]

Thus from a decay trace of the free vibration of the system it is still possible by measuring the amplitudes \(X_{Pn}, X_{Nn}, X_{Pm}\) and \(X_{Nm}\) to obtain the relative contribution of the viscous damping. Since \(X_{Nn}\) and \(X_{Nm}\) will represent negative quantities, the sums \((X_{Pn} + X_{Nn}), (X_{Pm} + X_{Nm})\) will be for low damping, exceedingly small quantities. Measurement of \(X_{Nm}\), etc. directly, will therefore not be very practicable. For this reason the slope of the decay envelope is usually measured at the required point.

\[ \frac{(X_{Nn})_T}{T} = \frac{(X_{Pn} + X_{Nn})/\eta/\omega_x'}{1/2 \text{ cycle}} \]

Fig. G3 shows that \((X_{n})_T / T = \frac{(X_{Pn} + X_{Nn})/\eta/\omega_x'}{1/2 \text{ cycle}}\) such that, for the same length \(T\) at two different points

\[ \beta = f_x/(n - m) \cdot \log e \left[ \frac{(X_n)_T}{(X_m)_T} \right] \]

Eliminating \(y_0\) from the expressions for two successive peak
displacements, \( x_{pn} \) and \( x_{n(n+1)} \) (say) gives the following expression for \( F \),

\[
F = K \left[ x_{N(n+1)} + \exp(-\pi \beta / \omega x') x_{pn} \right] / (1 + \exp(-\pi \beta / \omega x'))
\]

Again, in order to discriminate between \( x_{N(n+1)} \) and \( x_{pn} \), use is made of the slope of the decay trace (which may be fairly accurately measured), viz,

\[
|x_{N(n+1)}| = |x_{pn} + (x_n)_T / 2N_T|
\]
APPENDIX H.

The Highest Maxima of a Random Function
APPENDIX H.

The Highest Maxima of a Random Function

Consider a function $f(t)$ representing a signal fluctuating randomly with time (which could be either a record of wind velocity or structural response) - see fig. H1.

$$f(t) = \phi_1 = \sum_{n=1}^{\infty} c_n \cos(\alpha_n t + e_n)$$

FIG H1:

The function, $f(t)$, ($= \phi_1$, say) will include numerous stationary values which themselves will obey some probability distribution. While each peak may be regarded as a maximum derivation from the mean, some of these peaks will be larger than others. This Appendix deals with estimating the largest 'likely' peak for a given length of time. The analysis is carried out in two basic parts; firstly, the probability distribution of all the peaks is obtained and then the probability distribution of the largest peaks is established.

Consider a small interval, $\delta t$, of the signal $f(t)$ in fig. H1. It is desired to know whether the following conditions are met:

(i) that the signal passes through a particular value $f(t)$ (ie., $\phi_1$),
(ii) that it will have a stationary value at that point,
\( \dot{f}(t) = 0 \) (ie., \( \dot{\phi} = 0 \)),

(iii) that the stationary value is positive, \( \ddot{f}(t) \leq 0 \) (ie., \( \ddot{\phi} \leq 0 \)).

In other words, what is the probability that all three conditions are satisfied simultaneously? Or,

\[
p(f, \dot{f}, \ddot{f}) = p(\phi_1, \phi_2, \phi_3) \quad \text{(say)}
\]

where, \( \dot{f} = \phi_2 = 0 \)

\( \ddot{f} = \phi_3 \leq 0 \).

In probability theory, it is usual to express the above statement in the form of a joint probability function. For instance, the joint probability function of three random variables \( X, Y, Z \) (or the distribution function of a trivariate random vector with components \( X, Y, Z \)) may be,

\[
F(x, y, z) = P\left(\begin{array}{c}
X \leq x \\
Y \leq y \\
Z \leq z
\end{array}\right)
\]

(ie., the probability that the inequalities \( X \leq x, Y \leq y, Z \leq z \) are held simultaneously).

The conditions of probability may be further restricted, viz,

\[
F(x, y, z) = P\left(\begin{array}{c}
d \leq X \leq \beta \\
d \leq Y \leq \gamma \\
\phi \leq Z \leq \psi
\end{array}\right)
\]

This may be illustrated geometrically - see fig. H2. \( F(x, y, z) \) is the probability of a specified event occurring within the shaded prism. By varying the position of the prism the
distribution throughout the entire field is obtained. By
reducing the size of the prism to elementary proportions, the
joint probability density is defined, viz,
\[ f(x, y, z) = \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \lim_{\Delta z \to 0} \frac{P(x \leq X < x + \Delta x)}{\Delta x} \cdot \frac{P(y \leq Y < y + \Delta y)}{\Delta y} \cdot \frac{P(z \leq Z < z + \Delta z)}{\Delta z} \]

This can be thought of as representing the probability of ob-
taining a specified event at a particular point \((x, y, z)\) in the
field. It follows that the elementary probability - i.e., the
total probability of a specified event occurring within the
elementary volume \((\delta x \delta y \delta z)\) - is, \(f(x, y, z) \cdot \delta x \delta y \delta z\). The
total probability of a random event \(S\) represented by co-ordinates
\((X, Y, Z)\) being found in an arbitrary region \(B\) of the field is,
\[ P(S \in B) = \iiint_B f(x, y, z) \cdot dx \cdot dy \cdot dz \]
Clearly, \(p(\phi_1, \phi_2, \phi_3)\) as defined earlier represents the joint
probability density of obtaining a specified event defined by
\(\phi_1, \phi_2, \phi_3\). The total probability of obtaining a specified
event in the elementary region defined by \(\delta \phi_1 \delta \phi_2 \delta \phi_3\) is therefore,
\[ p(\phi_1, \phi_2, \phi_3) \cdot \delta \phi_1 \delta \phi_2 \delta \phi_3 \]
More specifically, the total probability of obtaining a peak at
a specific value of \(\phi_1\) in the time interval \(\delta t\) (applying the con-
dition that \(\phi_2 = 0\) and \(\phi_3 = 0\) for a maximum value) is,
\[ \int_{-\infty}^{0} p(\phi_1, 0, \phi_3) \cdot \delta \phi_1 \delta \phi_2 \, d\phi_3 \]
The total probability of getting a maximum in the interval for
any value of \(\phi_1\), i.e., \(+\infty < \phi_1 < -\infty\), is
\[ \int_{-\infty}^{0} \left[ \int_{-\infty}^{0} p(\phi_1, 0, \phi_3) \cdot \delta \phi_2 \, d\phi_3 \right] \, d\phi_1 \]
These various quantities can be thought of in the same way as
shown in fig. H2 (i.e., in a three-dimensional probability field)
- see fig. H3.
Expression (ii) will not be unity - i.e., there will be no certainty of obtaining a maximum in the interval $\delta t$. The relative probability of obtaining a maximum at $\phi_1$ (as given by (i)) to the total probability of obtaining a maximum at all in the interval $\delta t$ (as given by (ii)) will give the probability density of all maxima to be found along the signal, viz,

$$p(\phi_1) = \frac{\int_0^{\infty} p(\phi_1, 0, \phi_3) \delta \phi_1 \delta \phi_2 \delta \phi_3 \, d\phi_3}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\phi_1, 0, \phi_3) \delta \phi_2 \delta \phi_3 \, d\phi_3 \, d\phi_1} \quad \text{.........(iii)}$$

While the above expression gives the probability distribution of peaks in a signal, it does not give any information as to the frequency of peaks. The number of peaks is clearly dependent on

---

1 This step is easily understood from a simple example: if for instance, the probability of a green apple falling off a tree is 0.1 in every half hour, while the probability of any apple falling is 0.4, then, the probability that a fallen apple is green is $0.1/0.4 = \frac{1}{4}$. 
the rate of change of slope of the signal \( f(t) \); imagine a small increase in \( f(t) \) from \( \phi_1 \) to \( \delta \phi_1 \), during the interval \( \delta t \), by definition,

\[
\frac{\Delta f(t)}{\Delta t} = \frac{f(t + \delta t) - f(t)}{\delta t}
\]

or

\[
\phi_2 \cdot \delta t = (\phi_2 + \delta \phi_2)
\]

whence,

\[
\delta \phi_2 = \phi_2 \cdot \delta t
\]

Expression (ii) becomes, on substitution for \( \delta \phi_2 \),

\[
p(\phi_1, \phi_3) \cdot |\phi_3| \cdot \delta t \cdot \delta \phi_3 \cdot \delta \phi_1
\]

The **mean** frequency of peaks or maxima in the interval \( \delta t \), and consequently along the length of the signal in general, is therefore

\[
N_1 = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\phi_1, \phi_3) \cdot |\phi_3| \cdot \delta t \cdot \delta \phi_3 \cdot \delta \phi_1}{\delta t}
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\phi_1, \phi_3) \cdot |\phi_3| \cdot \delta \phi_3 \cdot \delta \phi_1 \quad \ldots \ldots \quad (iv)
\]

(iii) may now be written:

\[
p(\phi_1) = \frac{1}{N_1} \int_{-\infty}^{\infty} p(\phi_1, \phi_3) \cdot |\phi_3| \cdot \delta \phi_3 \quad \ldots \ldots \quad (v)
\]

It now remains to find an expression for \( p(\phi_1, \phi_2, \phi_3) \), the joint probability density function of a trivariate random vector. It is worthwhile to consider first of all the significance of moments as used in probability theory.

In mechanics most dynamic and static properties of a body possessing mass and size can be deduced from a knowledge of the overall centre of mass and moments of mass inertia of that body. In precisely the same way in probability theory, it is possible

\[1\text{This again is easy to imagine: if the probability of any apple falling is } 0.4 \text{ in half an hour, the mean frequency of apples falling is } 0.4/0.5 \text{-hour or 8 apples every 10 hours.} \]
to characterize probability density functions by their moments.

Consider a uni-variate probability density distribution, \( f(x) \), of a random variable \( X \) (see fig. H4(i))

\[
\begin{align*}
\text{FIG H4 (i)} \\
\text{FIG H4 (ii)}
\end{align*}
\]

The \( K \)th moment is, \( a_K = \int_{-\infty}^{\infty} x^K f(x) \, dx \). In particular, when \( K = 1 \), the mean value is defined, viz,

\[
M[X] = \int_{-\infty}^{\infty} x f(x) \, dx = m_X \text{ (say)}
\]

With \( K = 2 \) and higher, the expression for \( a_K \) becomes clumsy and so central moments (ie., moments referred about the mean) are used. The first order central moment is evidently equal to zero. The second order central moment is known as the variance (or dispersion, mean square, etc.). For a uni-dimensional random vector, the variance is

\[
D[X] = k_x \text{ (say)} = M[(X - m_x)^2] = \int_{-\infty}^{\infty} (x - m_x)^2 f(x) \, dx
\]

With several random variables together (as in joint probability distributions) the idea of 'mixed' moments emerges. For instance, in a two-dimensional random vector there will be three possible ways of writing the second order central moments (see fig. H4(ii)), viz,

\[
\begin{align*}
\text{variances:} \\
k_{xx} &= M[(X - m_x)^2] = \int_{-\infty}^{\infty} (x - m_x)^2 f(x, y) \, dx \\
k_{yy} &= M[(Y - m_y)^2] = \int_{-\infty}^{\infty} (y - m_y)^2 f(x, y) \, dy
\end{align*}
\]
co-variances:

\[ k_{xy} = k_{yx} = \mathbb{E}[(Y - \mu_y)(X - \mu_x)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y - \mu_y)(x - \mu_x)f(x, y)\,dx\,dy \]

It is usual to group the second-order central moments together into a single quantity by constructing a variance-co-variance matrix, viz,

\[
K_2 = \begin{bmatrix}
    k_{xx} & k_{xy} \\
    k_{yx} & k_{yy}
\end{bmatrix}
\]

In the case of a three-dimensional random vector a similar matrix may be constructed:

\[
K_3 = \begin{bmatrix}
    k_{xx} & k_{xy} & k_{xz} \\
    k_{yx} & k_{yy} & k_{yz} \\
    k_{zx} & k_{zy} & k_{zz}
\end{bmatrix}
\]

It is to be noted that these matrices are always symmetrical, since \( k_{\mu v} = k_{v \mu} \) \((\mu, \nu = x, y, z, \ldots)\).

In the same way as described above it is possible to characterize the joint probability distribution \( p(\phi_1, \phi_2, \phi_3) \) in terms of its mean and variance-co-variance matrix. It was shown in APPENDIX A that the random function \( f(t) \) could be expressed as the sum of an infinite number of sinusoidal components, viz,

\[
\phi_1 = f(t) = \sum_{n=1}^{\infty} c_n \cdot \cos(\sigma_n t + \epsilon_n)
\]

whence,

\[
\phi_2 = \ddot{f}(t) = -\sum_{n=1}^{\infty} c_n \cdot \sigma_n \cdot \sin(\sigma_n t + \epsilon_n) \quad \text{(vi)}
\]

\[
\phi_3 = \dddot{f}(t) = -\sum_{n=1}^{\infty} c_n \sigma_n^2 \cdot \cos(\sigma_n t + \epsilon_n)
\]

Since the components of \( \phi_1, \phi_2, \phi_3 \) are sinusoidal it follows that the mean of each vector is zero, or,

\[
\overline{\phi_1} = \overline{\phi_2} = \overline{\phi_3} = 0
\]

This means that the mean of the function \( p(\phi_1, \phi_2, \phi_3) \) is zero.

The variance-co-variance matrix is, as shown above for a tri-
variance vector,

\[
K = \begin{bmatrix}
\phi_1^2 & \phi_1\phi_2 & \phi_1\phi_3 \\
\phi_2\phi_1 & \phi_2^2 & \phi_2\phi_3 \\
\phi_3\phi_1 & \phi_3\phi_2 & \phi_3^2
\end{bmatrix}
\]

Evaluation of the co-variances and variances after substitution from (vi) gives that,

\[
\begin{align*}
\phi_1\phi_2 &= \phi_2\phi_1 = \phi_2\phi_3 = \phi_3\phi_2 = 0 \\
\phi_1\phi_3 &= \phi_3\phi_1 = -\sum_{n}^{\infty} \sigma_n^2 c_n^2 / 2 \\
(\phi_1^2) &= \sum_{n}^{\infty} \frac{1}{2} c_n^2 \\
(\phi_2^2) &= \sum_{n}^{\infty} \sigma_n^2 c_n^2 / 2 \\
(\phi_3^2) &= \sum_{n}^{\infty} \sigma_n^4 c_n^2 / 2
\end{align*}
\]

It will be remembered that the nth moment of the energy density spectrum, \( E(f) \), drawn up from the signal \( f(t) \) as shown in Appendix A was,

\[
m_n = \sum_{n}^{\infty} \sigma_n^2 c_n^2 / 2 = \int_{0}^{\infty} E(f) \cdot f^n \cdot df
\]

in other words, the variance-co-variance matrix may be written,

\[
K = \begin{bmatrix}
m_0 & 0 & -m_2 \\
0 & m_2 & 0 \\
-m_2 & 0 & m_4
\end{bmatrix}
\]

where \( m_0 \) is the area of the energy density spectrum, \( m_2 \) and \( m_4 \) its second and fourth moments respectively.

The Central Limit Theorem (Ljapunov) states that the distribution of the sum of \( N \) independent random vectors approaches a normal law as \( N \to \infty \). Assuming that the amplitudes, \( c_n \), of the constituent waves are random in nature, it follows that for both wind and structural response, \( \phi_1 \) follows a normal law.
Clearly, if \( \phi_1 \) is normal then so is \( \phi_2 \) and \( \phi_3 \). Since the central limit theorem is also valid in several dimensions it follows that \( p(\phi_1, \phi_2, \phi_3) \) is also normal. Distribution laws are not usually deducible; merely stated. It has been found that the multivariate normal distribution law takes the form:

\[
f(x_1, x_2, \ldots, x_n) = \sqrt{|C|/\pi^n} \exp \left[ -\frac{1}{2} \sum_{p,q=1}^{n} c_{pq} (x_p - a_p)(x_q - a_q) \right]
\]

where \( C \) is a matrix with coefficients \( c_{pq} \) and \( |C| \) is the determinant of \( C \). It may be shown\(^2\) that the above law can be expressed in terms of its characteristic moments - i.e., its mean and variance-co-variance matrix as obtained above, viz,

\[
f(x_1, x_2, \ldots, x_n) = 1/\sqrt{2^n \pi^n |K|} \exp \left[ -\frac{1}{2} (K^{-1}u, u) \right]
\]

where \( K \) is the variance-co-variance matrix, \( u_p = x_p - a_p, a_p = \text{mean}, p = 1, 2, \ldots, n \). For a three-dimensional vector,

\[
f(x_1, x_2, x_3) = 1/\sqrt{2^3 \pi^3 |K|} \exp \left[ -\frac{1}{2} (K^{-1}u, u) \right]
\]

Substitution of \( (x_1, x_2, x_3) = (\phi_1, \phi_2, \phi_3), a_p = \phi_p = 0 \ (p = 1, 2, 3) \),

\[
K = \begin{bmatrix}
m_0 & 0 & -m_2 \\
0 & m_2 & 0 \\
-m_2 & 0 & m_4
\end{bmatrix}
\]

and \( |K| = m_2 (m_0 \cdot m_4 - m_2^2) = m_2 \Delta \) (say)

into this equation gives,

\[
p(\phi_1, \phi_2, \phi_3) = 1/\sqrt{2\pi^2 m_2 \Delta} \exp \left[ -\frac{1}{2} \phi^2/m_2 - \frac{\phi_1^2 m_4 + 2\phi_2 \phi_3 + \phi_3^2}{2\Delta} \right]
\]

It will be remembered that the distribution of maxima or peaks is (equation (iii)),

\[
p(\phi_1) = \frac{\int_{-\infty}^{0} p(\phi_1, 0, \phi_3) \delta \phi_1 \delta \phi_2 \delta \phi_3 \, d\phi_1 \, d\phi_2 \, d\phi_3}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\phi_1, 0, \phi_3) \delta \phi_2 \delta \phi_3 \, d\phi_2 \, d\phi_3}
\]

\[= \frac{\delta N_1}{N_1}\] (say)

\(^1\)Cf. uni-variate law: \( f(x) = h/\sqrt{\pi} \exp \left[ -h^2 (x - a)^2 \right] \)

Evaluating, first of all, the numerator by substitution for $p(\phi_1, \phi_2, \phi_3)$ with $\phi_1 = 0$ gives,

$$
\delta N_1 = \frac{1}{\sqrt{(2\pi)^3(m_2 \Delta)}} \int_{-\infty}^{0} \exp \left[ \frac{(\phi_1 m_4 + 2m_2 \phi_1 \phi_3 + m_2 \phi_3^2)}{2 \Delta} \right] \delta \phi_1 \, d\phi_3
$$

Completing the square (in terms of $\phi_3$) and putting

$$
x = \sqrt{\frac{m_0}{\Delta} \cdot (\phi_3 + \frac{m_2}{m_0} \phi_1)},
$$

$$
\delta N_1 = \frac{1}{\sqrt{(2\pi)^3(m_2 \Delta)}} \exp \left[ \frac{(m_4 - m_2^2/m_0)}{2 \Delta} \right] \int_{-\infty}^{0} \exp \left( -\frac{1}{2} x^2 \right) \left[ \frac{\Delta}{m_2} x - \frac{m_2}{m_0} \phi_1 \right] \delta \phi_1
$$

$$
\phi = x = -\infty
$$

Putting $\eta = \phi_1 / \sqrt{m_0}$ and $\delta = \sqrt{\Delta / m_2}$ such that $\eta / \delta = \phi_1 \cdot m_2 / \sqrt{m_0 \Delta}$,

$$
\delta N_1 = \frac{1}{m_0 \sqrt{(2\pi)^3(m_2 / \Delta)}} \exp \left( -\frac{1}{2} \eta^2 \right) \int_{-\infty}^{0} \exp \left( -\frac{1}{2} \eta^2 / \delta^2 \right) + \eta / \delta \exp \left( -\frac{1}{2} \eta^2 \right) \delta \phi_1
$$

$$
\int_{-\eta / \delta}^{\infty} \exp \left( -\frac{1}{2} x^2 \right) \, dx = \frac{\sqrt{\pi}}{\delta} \int_{0}^{\infty} \exp \left( -\frac{1}{2} t^2 \right) \, dt.
$$

The denominator of the 'peak' probability distribution expression, $N_1$, may now be calculated:

$$
N_1 = \frac{1}{m_0} \int_{-\infty}^{\infty} \delta N_1
$$

$$
= \frac{1}{\sqrt{m_0}} \int_{-\infty}^{\infty} \frac{1}{m_0 \sqrt{(2\pi)^3(m_2 / \Delta)}} \exp \left( -\frac{1}{2} \eta^2 \right) \exp \left( -\frac{1}{2} \eta^2 / \delta^2 \right) + \eta / \delta \exp \left( -\frac{1}{2} \eta^2 \right) \, d\eta
$$

which on successive integration by parts gives (eventually),

$$
N_1 = \frac{1}{2\pi} \sqrt{m_4 / m_2} = \text{mean frequency of maxima}.
$$

Instead of expressing the distribution of 'peaks' in terms of $\phi_1$, it is found convenient to use the quantity $\eta = \phi_1 / \sqrt{m_0}$ (where $\sqrt{m_0}$ is
the RMS of the parent population \( f(t) \). This means that the areas of the distributions \( p(\phi) \) and \( p(\eta) \) will be different; it is clear, however, that the total probabilities must remain the same, viz,

\[
\int_{-\infty}^{\infty} p(\phi_1) \, d\phi_1 = \int_{-\infty}^{\infty} p(\phi_1/\sqrt{m_o}) \, d(\phi_1/\sqrt{m_o})
\]

or,

\[
p(\phi_1/\sqrt{m_o}) = p(\eta) = \sqrt{m_o} \cdot p(\phi_1) = \frac{\Delta}{N_1}
\]

Putting \( \epsilon^2 = \Delta/m_o \cdot m_4 \) and noting that \( \eta/\delta = \eta(1 - \epsilon^2)^{1/2}/\epsilon \), the above expression becomes,

\[
p(\eta) = \frac{\eta}{\sqrt{2\pi}} \left[ \epsilon \exp\left(-\frac{\eta^2}{2\epsilon^2}\right) + \eta (1 - \epsilon^2)^{1/2} \exp\left(-\frac{\eta^2}{2}\right) \right] \exp\left(-\frac{\eta^2}{2}\right) \, d\eta
\]

\( p(\eta) \) is the probability density distribution of the maxima or peak values in the random signal \( f(t) \), and is seen to be dependent on two variables, \( \eta \) and \( \epsilon \). The dependence on \( \eta \) is fairly straightforward; for large departures from the mean (i.e., \( \eta \) large) the probability of obtaining a peak decreases. The significance of the quantity \( \epsilon \) is, by definition, related to the area and moments of the energy density spectrum of the parent signal \( f(t) \). It can, in fact, be shown that \( \epsilon \) is a measure of the extent to which the maxima of the signal obey, on the one hand, a Gaussian or normal law (\( \epsilon = 1 \)) or, on the other, a Rayleigh law (\( \epsilon = 0 \)).

This is probably best understood from fig. H5. In general it may be stated that when a random signal is tied to a particular frequency (such as the natural frequency of a cantilever system) the

\[1\] It must be remembered that \( \eta \) is the standardized or reduced variate \( (\eta = f(t)/\sqrt{m_o}) \).
FIG H5  Probability distribution of peaks (maxima) for different types of fluctuations
maxima will all be concentrated above the neutral position (forming a Rayleigh distribution). When a random selection of frequencies are involved, there will be as many negative maxima as positive ones. This of course is reflected in the shape of the energy spectra.

From the expression for \( p(\eta) \) it is possible to calculate the probability of obtaining a peak in excess of a given \( \eta \), viz,

\[
P(N > \eta) = q(\eta) = \int_{\eta}^{\infty} p(\eta) \, d\eta \quad \text{(shaded area in fig. H5)}
\]

which on substitution of the expression for \( p(\eta) \) found above gives,

\[
q(\eta) = \frac{1}{\sqrt{2\pi}} \int_{\eta}^{\infty} \left[ \exp\left(-\frac{1}{2} \eta^2 / \epsilon^2 \right) + \sqrt{(1-\epsilon^2)} \eta \exp\left(-\frac{1}{2} \eta^2 \right) \right] \, \exp\left(-\frac{1}{2} x^2 \right) \, dx \, d\eta.
\]

After successive integration by parts this becomes,

\[
q(\eta) = \frac{1}{\sqrt{2\pi}} \left[ \int_{\eta/\epsilon}^{\infty} \exp\left(-\frac{1}{2} y^2 \right) \, dy + \sqrt{(1-\epsilon^2)} \eta \exp\left(-\frac{1}{2} \eta^2 \right) \right] \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} x^2 \right) \, dx.
\]

The proportion of negative maxima (i.e., maxima below the mean of the signal \( f(t) \)) is

\[
r = \int_{-\infty}^{0} p(\eta) \, d\eta = \int_{0}^{\infty} p(\eta) \, d\eta - \int_{-\infty}^{0} p(\eta) \, d\eta
\]

\[= 1 - q(0) = 1 - \frac{1}{\sqrt{2\pi}} \left[ \int_{0}^{\infty} \exp\left(-\frac{1}{2} y^2 \right) \, dy + \sqrt{(1-\epsilon^2)} \eta \exp\left(-\frac{1}{2} \eta^2 \right) \right] \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} x^2 \right) \, dx
\]

\[r = 1 - \frac{1}{\epsilon} - \frac{1}{2} (1 - \epsilon^2)^{\frac{1}{2}} = \frac{1}{2} \left[ 1 - (1 - \epsilon^2)^{\frac{1}{2}} \right]
\]

This is an important result; showing that it is possible to estimate the value of '\( \epsilon \)' by counting the number of negative maxima found in the parent signal, \( f(t) \). The relationship between \( r \) and \( \epsilon \) is shown graphically in fig. H6.

\(^{1}\)It may be shown by a simple geometrical argument that if \( N_o \) is the mean frequency of zero up-crossings, then

\[N_o = N_1 (1 - 2r).
\]

Since \( N_1 = \frac{2\pi}{\sqrt{m_4/m_2}} \) and \( r = \frac{1}{2} \left[ 1 - (1 - \epsilon^2)^{\frac{1}{2}} \right] \) where \((1 - \epsilon^2)^{\frac{1}{2}} = \sqrt{m_2/m_0 \cdot m_4} \), it follows that:

\[N_o = \frac{1}{2\pi} \cdot \sqrt{m_2/m_0} \]
FIG H6 Relationship between percentage of negative maxima and the quantity 'ε'
For high values of $x$,
\[
\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} x^2\right) \cdot dx = \exp\left(-\frac{1}{2} x^2\right) \left(1/x + 1/x^3 + \ldots\right) \to 0
\]

For high values of $f(t)$, $(\equiv \phi_1)$, $\eta/\varepsilon = \phi_1/\varepsilon \sqrt{m_0}$ is also large, such that
\[
q(\eta_{\text{large}}) \to \frac{1}{\sqrt{2\pi}} \left[0 + (1 - \varepsilon^2)^{\frac{1}{2}} \exp\left(-\frac{1}{2} \eta^2 \sqrt{2\pi}\right)\right] = \sqrt{1 - \varepsilon^2} \exp\left(-\frac{1}{2} \eta^2\right).
\]

Having obtained an expression for the distribution of maxima it is now possible to proceed with the determination of the highest likely peak in a given time. Consider a sample of the signal $f(t)$ consisting of $N$ maxima in a time $T$. Let the highest maximum be $\eta$. The probability of getting $\eta$ is $p(\eta)$ as found previously. The cumulative probability of getting a maximum greater than $\eta$ is $q(\eta)$, while the probability of getting a maximum less than $\eta$ is $1 - q(\eta)$. So that in a sample of $N$ maxima the probability of getting the highest maximum is
\[
p(\eta_{\text{max}}) = NC_1\left[1 - q(\eta)\right]^{N-1} \left[p(\eta)\right]^1 = N \left[1 - q(\eta)\right]^{N-1} p(\eta)
\]

Since $q(\eta) = \int_{\eta}^{\infty} p(\eta) \, d\eta$ it follows that $dq(\eta)/d\eta = \left[p(\eta)\right]_{\eta}^{\infty} = p(\infty) - p(\eta) = -p(\eta)$.

Whence, $p(\eta_{\text{max}}) \, d\eta = -N \left[1 - q(\eta)\right]^{N-1} d\eta(\eta) = d \left[1 - q(\eta)\right]^N$.

$p(\eta_{\text{max}})$ is the probability density distribution of the highest maxima, $\eta_{\text{max}}$. It is important to have a clear understanding of what this means. Consider a large number of samples all of roughly the same length or composed of the same number of maxima taken from the parent population $f(t)$ (see fig. H7). $p(\eta_{\text{max}})$ is the distribution constructed from the highest maxima in each sample; in practice a frequency histogram would be drawn up.

It is intuitively evident that the larger the samples taken,
the greater will be the chance of higher maximum peaks occurring, so shifting the distribution \( p(\eta_{\text{max}}) \) further along the random variate axis, \( \eta \).

\[ n \text{ samples of } N \text{ peaks} \]

**FIG H7**

The highest 'likely' peak expected in any sample of a given length is clearly the mean of the distribution \( p(\eta_{\text{max}}) \) - i.e., \( \bar{\eta}_{\text{max}} \). This mean is found by taking moments in the usual way, viz,

\[ \bar{\eta}_{\text{max}} = \int_{-\infty}^{\infty} \eta \cdot p(\eta_{\text{max}}) \cdot d\eta = \int_{-\infty}^{\infty} \eta \cdot d(1 - q(\eta))^N \cdot d\eta \]

Integrating by parts, this gives,

\[ \bar{\eta}_{\text{max}} = -\left[ 1 - q(\eta) \right]^N \cdot d\eta + \int_{-\infty}^{0} \left[ 1 - (1 - q(\eta))^N \right] d\eta \]

The first term representing the area of \( p(\eta_{\text{max}}) \) in the negative quadrant is clearly negligible. It was shown earlier that for high \( \eta \),

\[ q(\eta) = \sqrt{(1 - \varepsilon^2)} \cdot \exp(-\frac{1}{2} \eta^2) \]
Substituting for \( q(\eta) \) and putting:
\[
\lambda = \frac{1}{2} \eta^2, \\
\lambda_0 = \log_e \left(1 - \varepsilon^2 \right)^{\frac{1}{2}} N
\]
and
\[
\lambda' = \lambda - \lambda_0
\]
(noticing that:
\[
\exp(-\lambda) = \exp(\lambda_0 + \lambda') = \exp(-\lambda')/ \left[ (1 - \varepsilon^2)^{\frac{1}{2}} N\right] 
\]

it follows that:
\[
\bar{\eta}_{\text{max}} = \frac{1}{\sqrt{2}} \int_{\lambda' = -\lambda_0, \lambda = \infty}^{\lambda = \lambda_0} \left[ 1 - \exp(-\lambda')/N \right]^{N/2} \left( \lambda_0 + \lambda' \right)^{-1/2} d\lambda'
\]
Substituting the approximation, \( \exp(-\exp(-\lambda')) \approx \left[ 1 - \exp(-\lambda')/N \right]^{N} \) (valid for large \( N \)) \( \bar{\eta}_{\text{max}} \) becomes
\[
\bar{\eta}_{\text{max}} = \frac{1}{\sqrt{2}} \int_{\lambda' = -\lambda_0, \lambda = \infty}^{\lambda = \lambda_0} \left( \lambda_0 + \lambda' \right)^{1/2} \exp(-\lambda' + \exp(-\lambda')) d\lambda'
\]
Putting \( z = \exp(-\lambda') \) and noting that when
\[
\lambda' = \infty, \ z = 0 \\
\lambda' = -\lambda_0, \ z = \exp(\lambda_0) \quad \text{(if \( \lambda_0, \ N \) large)}
\]
\[
\bar{\eta}_{\text{max}} = -\sqrt{2} \int_{-1 \leq \log_e z \leq \lambda_0} \left[ 1 - \frac{1}{2} \left( \log_e z \right)/\lambda_0 - \frac{1}{8} \left( \log_e z \right)^2/\lambda_0^2 \right] \exp(-z) dz
\]
\[
= -\sqrt{2} \lambda_0 \left[ \exp(-z) - \frac{z}{\lambda_0} \log_e z \exp(-z) - \frac{z^2}{8 \lambda_0^2} \log_e z \exp(-z) \right]_0^{-1}
\]
Neglecting the terms in \( 1/\lambda_0^2, \ 1/\lambda_0^3, \text{etc.} \) (which \( \to 0 \) for high values of \( N \)) and remembering that \( \lambda_0 = \log_e \left( (1 - \varepsilon^2)^{\frac{1}{2}} N \right) \)
\[
1 \ \exp(-\exp(-\lambda')) = \left( 1 - \exp(-\lambda')/N \right)^N \Rightarrow -\exp(-\lambda')
\]
\[
= N \log(1 - \exp(-\lambda')/N)
\]
\[
= N \left[ -\exp(-\lambda')/N - \frac{1}{2} \exp(-\lambda'^2)/N^2 + ... \right]
\]
\[
\Rightarrow -\exp(-\lambda') \quad \text{(if \( N \) large)}
\]
\[
\eta_{\text{max}} = \sqrt{2} \left[ \frac{1}{2} \log_2 \left( 1 + \epsilon^2 \right) N \right] + \gamma \left[ \log_2 \left( 1 + \epsilon^2 \right) \right]^{-\frac{1}{2}}
\]

where \( \gamma = -\int_0^\infty \frac{\log z \exp(-z)}{z} \, dz = \text{Euler's constant} = 0.5772. \)

Instead of expressing \( \eta_{\text{max}} \) in terms of the number of maxima \( N \) taken in each sample it may be preferable to consider a sample duration, \( T \).

It was shown earlier that the mean frequency of maxima in a random signal was

\[
N_1 = \frac{1}{2\pi} \sqrt{\frac{m_4}{m_2}} \quad \text{(radians/sec.)}
\]

Hence in time \( T \) the total number of maxima will be

\[
N = N_1 T / 2\pi = \frac{1}{4\pi^2} \sqrt{\frac{m_4}{m_2}} T
\]

Putting \( \nu = \frac{1}{4\pi^2} \sqrt{\frac{m_2}{m_0}} \) and remembering that \( (1 - \epsilon^2)^{\frac{1}{2}} = \sqrt{\frac{m_2}{m_0}} m_4 \)

the expression for \( \eta_{\text{max}} \) can be written

\[
\eta_{\text{max}} = \sqrt{2} \left[ \log_2 (\nu T)^{\frac{1}{2}} + \frac{1}{2} \gamma \left( \log_2 (\nu T)^{-\frac{1}{2}} \right) \right]
\]
If the parent signal corresponds to a pure Rayleigh distribution (a system vibrating exclusively about its resonant frequency, say) then $\epsilon = 0$ and $N$, the number of maxima = $nT = N_1T/2\pi = n_0T$ and

$$\eta_{\text{max}} = \sqrt{2\left[(\log_e(n_0T))^{\frac{1}{2}} + \frac{1}{2} \gamma(\log_e(n_0T))^{-\frac{1}{2}}\right]}$$

It will be noticed that the above formula (ie., $Ba$ or $Bb$) is only valid for numerical values of $(\sqrt{1 - \epsilon^2}N)$ (or $\nu T$) substantially greater than unity; in other words, the samples taken at random from the parent population $f(t)$ must be fairly large. It will also be noticed that the formula is not valid when the population of the maxima (ie., $p(\eta)$) approaches a Gaussian or normal distribution ($\epsilon \rightarrow 1$) - a condition which is clearly approached in a signal $f(t)$ consisting of wind speed, for instance.

The complete theory of this particular case will not be given here; it has been shown by Fisher and Tippett (1928)\(^1\) that when $\epsilon \rightarrow 1$, the mean value of $\eta_{\text{max}}$ is, approximately,

$$\bar{\eta}_{\text{max}} = m + \gamma m/(1 + m^2)$$

where 'm' is the mode of the distribution of $\eta_{\text{max}}$, given, implicitly, by

$$\sqrt{(2\pi)m}\cdot\exp(\frac{1}{2}m^2) = N$$

or

$$\log_e m + \frac{1}{2}m^2 = \log_e N/\sqrt{2\pi}$$

whence if $m$ is relatively large,

$$\bar{\eta}_{\text{max}} = \sqrt{2\left[\log_e(N/\sqrt{2\pi})\right]^{\frac{1}{2}}}$$

This expression can only be used, however, for sample lengths with

\(^1\)See REF: Proceedings of Cambridge Philosophical Society Vol. XXIV (pp. 180-190). (REF 29.)
FIG H9  Highest likely maxima in samples of N maxima
a thousand or more maxima\(^1\). For samples with \(N < 1000\), the highest likely maximum may be obtained from fig. H9 (Tippett 1925)\(^1\).

The structural response as measured in the tower-structure is characterized by a high-frequency 'ripple' - corresponding to the natural (first mode) frequency - on the remaining waves. This means that the number of maxima, \(N\), in a given sample would correspond very closely to the number of cycles of the 'ripple' in that sample. Assuming the tower structure to behave in the manner of a single-degree-of-freedom system it will not respond to frequencies higher than that of resonance - ie., no higher frequencies will be superimposed on the 'ripple'.

![Diagram of +ve and -ve peaks and zero up-crossing](image)

\[N = \frac{s}{(1 - 2r)} = \frac{s}{\sqrt{(1 - c^2)}}\]

Consider two hypothetical extreme cases (shown in fig. H10(i) and (ii)).

**CASE (i): small ripple on a large random signal** (distribution of peaks \(\rightarrow\) Gaussian). A typical example might be: samples of

length 330 sec. with a ripple of 3.3 c/s. whence N = 1000.

Assume that on average only 50 zero up-crossings are measured in each sample such that the proportion of negative peaks is,

\[ r = \frac{1}{2}(1 - s/N) = \frac{1}{2}(1 - 50/1000) = 47.5\% \]

and,

\[ \epsilon = \left[ 1 - (1 - 2r)^2 \right]^{\frac{1}{2}} = 0.99875 \]

The highest likely peak is

\[ \eta_{\text{max}} = \sqrt{2} \left[ \log_e(1 - \epsilon^2)\frac{3}{2}N \right]^{\frac{1}{2}} + \frac{1}{2} \gamma \left[ \log_e(1 - \epsilon^2)\frac{3}{2}N \right]^{-\frac{1}{2}} \]

\[ = \sqrt{2} \left[ \log_e(50)\frac{3}{2} + \frac{1}{2} \cdot 0.5772 \cdot (\log_e 50)^{-\frac{1}{2}} \right] \cdot 3.0 \]

(This value is for a population having unit RMS and in any given case must be multiplied by the actual value of this quantity.)

CASE (ii): large ripple on a small random signal (Rayleigh distribution of peaks). In this case, \( r = 0 \) and \( \epsilon = 0 \) whence,

\[ \eta_{\text{max}} = \sqrt{2} \left[ \log_e 1000\frac{3}{2} + \frac{1}{2} \cdot 0.5772 \cdot (\log_e 1000)^{-\frac{1}{2}} \right] \cdot 3.8 \]

It is seen that for the same RMS and sample length in both of the above cases, the highest likely peak in the signal with the Rayleigh distribution is about 25% greater than that with the distribution of its peaks approaching a Gaussian 'peak' distribution.

The two cases above represent somewhat extreme examples; unless the damping in a structure is so great that practically no resonant motion occurs, the response of most structures exposed to the wind will fall into some intermediate category.

It can be seen that to a rough approximation the highest 'likely' peak in a system showing the response (described above) may be calculated by assuming a Rayleigh distribution (\( \epsilon = 0 \)); this

\[ ^{1}\text{Alternatively, this could be obtained directly from the sample peaks.} \]
will always be an overestimation.

Consider now a record of wind speed (fig. H11). There is no apparent upper limit to the frequencies of the constituent waves. This is reflected in the energy spectrum which does not have a sharp cut-off. Clearly, the higher moments of the spectrum will tend towards infinity. This means that in a finite sample there will be an infinite number of maxima. Since, however, the energy level at higher frequencies becomes very small, it is permissible to adopt a cut-off frequency, $n_0$, above which the spectral energy density is assumed to be zero. The moments, $m_0, m_2, m_4$, of the energy spectrum are then clearly defined. Assuming that the distribution of the maxima in the record is normal,

$$ \bar{\eta}_{\text{max}} = \sqrt{2 \left[ \frac{\log_e(N/\sqrt{2\pi})}{3} \right]} \quad \text{for } N > 1000 $$

(see fig. for $N \leq 1000$)

$N$ is finite for a particular length, $T$, of the signal, since,

$$ N = \frac{1}{4\pi^2} \cdot \sqrt{m_4/m_2} \cdot T \quad \text{where } m_4, m_2 \text{ are finite} $$

It is clear that unless anemometric instrumentation with a sharp frequency cut-off (at $n_0$) is used to obtain the record of wind speed, $N$ will not be determinable from simple counting of maxima. It will in fact be necessary to obtain the moment $m_4$ and $m_2$ from the energy spectrum characterizing the given wind record.
APPENDIX I.

Influence of an Auxiliary Mass System
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Influence of an Auxiliary Mass System

Consider the system shown in fig. II(a) which consists of a main or primary single-degree-of-freedom undamped system (consisting of a spring of stiffness, \( k_p \), and a mass, \( m_p \)) to which is coupled an auxiliary system (consisting of a spring of stiffness, \( k_a \), mass, \( m_a \), and viscous damping, \( c_a \)).

\[ \begin{align*}
\text{PRIMARY SYSTEM} & \quad \text{AUXILIARY SYSTEM} \\
\begin{array}{c}
\text{P. Cos } \omega t \\
\text{X}_0.\cos \omega t \\
\text{X}_r.\cos \omega t
\end{array} & \begin{array}{c}
\text{C}_a \text{X}_r \cos \omega t \\
\text{C}_a \text{X}_r \sin \omega t
\end{array}
\end{align*} \]

Assume initially that the mass, \( m_p \), vibrates sinusoidally with the motion, \( x_0.\cos(\omega t) \), due to some exciting force, \( P.\cos(\omega t) \), applied directly to it. Consider the effect of this motion on the auxiliary system: for dynamic equilibrium of the mass, \( m_a \), the dynamic forces imposed upon it must balance, so that if \( x_r.\cos(\omega t) \) is the motion of \( m_a \) relative to \( m_p \),
\[-k_a x_r \cos(\omega t) - c_a x_r \omega \sin(\omega t) = -m_a (x_o + x_r) \omega^2 \cos(\omega t)\]

whence,

\[x_r = m_a x_o \omega^2 \cos(\omega t) / \left[ k_a \cos(\omega t) + c_a \omega \sin(\omega t) - m_a \omega^2 \cos(\omega t) \right] \]

The reaction which the auxiliary system exerts on the primary mass, is, clearly,

\[F \cos(\omega t) = x_r (k_a \cos(\omega t) + c_a \omega \sin(\omega t)) \]

It is convenient to think of the force exerted on the primary system as an equivalent mass, \( m_{eq} \), additional to the primary mass, such that,

\[F = m_{eq} \omega^2 x_o \]

whence,

\[m_{eq} = \left[ k_a \cos(\omega t) + c_a \omega \sin(\omega t) / \left[ k_a \cos(\omega t) + c_a \omega \sin(\omega t) - m_a \omega^2 \cos(\omega t) \right] \right. \]

It was shown in APPENDIX G(i) that the maximum amplitude of forced vibration of a single-degree-of-freedom system was, in dimensionless form,

\[\chi = \left( x_t \right)_{\text{max}} / x_{st} = 1 / \left[ (1 - \xi^2 / \omega_x^2)^2 + 4(\beta \xi / \omega_x^2)^2 \right]^{1/2} \]

In the case being considered, the damping of the primary system is assumed to be zero, such that, with \( \beta = 0, \xi = \omega \),

\[\omega_x = \sqrt{k_p / (m_p + m_{eq})} \] (the natural frequency of the system) and \( x_{st} = p / k_p \), the non-dimensional amplitude of the primary system, using the above expression, becomes,

\[x_o / x_{st} = 1 / \left[ 1 - (m_p + m_{eq}) \omega^2 / k_p \right] \]
Substituting for the value of $m_{eq}$, found above,

$$x_0/x_{st} = \frac{1}{\left[1 - \frac{\omega^2}{k_p}\left[m_p + m_a \frac{\frac{k_a\cos \omega t + c_a\sin \omega t}{k_a\cos \omega t + c_a\omega \sin \omega t - m_a\omega^2\cos \omega t}}\right]\right]}$$

Putting the damping ratio, $\beta = \frac{c_a}{2\sqrt{k_a}\cdot m_a} = c_a/c_{ca}$, ($c_{ca}$ = critical damping) and operating on the above expression to eliminate the dependence on phase the maximum amplitude becomes,

$$\frac{x_0}{x_{st}} = \sqrt{\left[\frac{\left(1 - (\frac{\omega}{\omega_a})^2\right)^2 + (\frac{2\beta \omega}{\omega_a})^2}{\left(1 - (\frac{\omega}{\omega_a})^2\right)\left(1 - (\frac{\omega}{\omega_p})^2\right) - \frac{(\omega_p)^2 m_a}{m_p}}\right]^2 + \frac{2\beta \omega}{\omega_a} \left[1 - (\frac{\omega}{\omega_p})^2 - (\frac{\omega}{\omega_p})^2 \frac{m_a}{m_p}\right]^2}$$

............................... (i)

Equation (i) is best interpreted by considering firstly, the two extreme cases of damping: in which the damping in the auxiliary system is zero (CASE 1) and in which the damping is infinite (CASE 2). The more general situation in which there is a finite amount of damping in the auxiliary system is then dealt with (CASE 3).

CASE 1. Zero damping in auxiliary system.

With $\beta \rightarrow 0$, equation (i) becomes,

$$x_0/x_{st} = \frac{\left(1 - (\omega/\omega_a)^2\right)}{\left[\left(1 - (\omega/\omega_a)^2\right)\left(1 - (\omega/\omega_p)^2\right) - (\omega/\omega_p)^2 m_a/m_p\right]}$$

............................... (ii)

If the frequency of the auxiliary system is turned to the forcing frequency (ie., $\omega = \omega_a$),
In other words the motion of the primary system, at the forcing frequency, is eliminated entirely. In particular, if the forcing frequency is in tune with the resonant frequency of the primary system, the resonant motion will be eliminated. What is important to remember with this sort of auxiliary system, however, is that though it enforces a node point at the resonant frequency of the primary system, the composite system so formed can resonate by virtue of the auxiliary system introducing an extra degree-of-freedom at two other frequencies, the positions of which can be determined as follows:

For resonance in the primary system, \( x_0 / x_{st} \to \infty \), such that the denominator in equation (ii),

\[
\left[ (1 - (\omega / \omega_a)^2)(1 - (\omega / \omega_p)^2) - (\omega / \omega_p)^2 m_a / m_p \right] \to 0
\]

The resonant frequencies will be the roots of this equation, viz,

\[
\omega_n^2 = \frac{1}{2} \left[ (\omega_a^2 (1+m_a/m_p)+\omega_p^2)^\pm \sqrt[\omega_a^2 (1+m_a/m_p) - \omega_p^2]^2 + \omega_p^2 \omega_a^2 m_a/m_p \right]
\]

If \( \omega_1 \) and \( \omega_2 \) are the roots of the above equation, the maximum response of the composite system (i.e., primary system coupled with a tuned auxiliary system) for a range of forcing frequencies applied to the primary mass, will be similar to that sketched in fig. I2(b). Fig. I2(a) shows the response of the primary system alone. The auxiliary system described above is usually called a 'dynamic vibration absorber' and is clearly only of use in practice in reducing resonance arising from a constant forcing frequency; for a wide range of forcing frequencies (such as found in any random forcing function — eg., wind loading) resonant
amplification would occur at $\omega_1$ and $\omega_2$.

\begin{align*}
\omega_p &= \sqrt{\frac{k_p}{m_p}}
\end{align*}

**FIG 1.2**

(a)

**CASE 2. Infinite damping in auxiliary system.**

This case is of academic interest only, defining the 'locked frequency'. The mass of the primary system is merely augmented by that of the auxiliary system and the natural frequency of the composite system (the 'locked frequency') will be accordingly lower than that of the primary system - see fig. 1.3(b).

**FIG 1.3**

(a)
CASE 3. Damping, $\beta$, in auxiliary system.

From the two extreme cases briefly discussed above a response showing transition between the resonant 'locked frequency' peak and the two separate resonant peaks in the case of the dynamic absorber might be expected for an intermediate value of damping. Plotting equation (i) for various damping ratios and a particular mass ratio, $m_a/m_p$, shows this to be indeed the case. Fig. I4 shows the response of the primary system for a mass ratio, $m_a/m_p = 0.05$ and various values of damping in the auxiliary system. The natural frequency of the auxiliary system is tuned to that of the primary system. (Cases 1 and 2 are also shown, in dotted lines.)

![Diagram](image)

**FIG I.4**

Brief study of fig. I4 suggests the possibility of adjusting the damping, mass and frequency of the auxiliary system for a minimum response in the primary system for a wide range of excitation frequencies.

1Fig. I4 is reproduced from *Shock and Vibration Handbook* (REF 19), chapter 6, page 9.
It will be noticed that all the curves pass through two fixed points, A and B, irrespective of the value of the damping ratio, \( \beta \). This follows from equation (i) which is of the form,

\[
\frac{1}{(x_0/x_{st})^2} = \frac{f_1 + \beta^2 f_2}{(f_3 + \beta^2 f_4)} = \frac{f_2}{f_4} = \frac{f_1}{f_3} \quad \ldots \quad (iii)
\]

where, it will be noted, \( f_n (n = 1, 2, 3, 4) \) are functions of \( \omega, \omega_a, \omega_p, m_a \) and \( m_p \). It follows that,

\[
\frac{(2\omega/\omega_a)^2}{[1 - (\omega/\omega_a)^2]^2} = \frac{(2\omega/\omega_a[1-(\omega/\omega_p)^2 - m_a/m_p(\omega/\omega_p)^2])]^2}{[(1-(\omega/\omega_a)^2)(1-(\omega/\omega_p)^2) - (\omega/\omega_p)^2 m_a/m_p]^2}
\]

The only non-trivial solution of this equation yields,

\[
\omega^4[1 + m_a/2m_p] - \omega^2[\omega_p^2 + \omega_a^2 + \omega_a^2 m_a/m_p] + (\omega_a \omega_p)^2 = 0 \quad \ldots \quad (iv)
\]

The two positive roots of this equation, \( \omega_A, \omega_B \) correspond to the points A and B. The amplitude at A or B is, from (iii),

\[
\frac{(x_0/x_{st})_A}{(x_0/x_{st})_B} = \sqrt{f_1/f_3} = 1/\sqrt{1 - (\omega/\omega_p)^2 - m_a/m_p(\omega/\omega_p)^2} \quad \ldots \quad (v)
\]

where \( \omega = \omega_A \) or \( \omega_B \).

It can be shown by varying the frequency of the auxiliary system, \( \omega_a \), that the minimum response for a given damping ratio, \( \beta \), occurs when the amplitudes of the fixed points are equal, viz,

\[
\frac{(x_0/x_{st})_A}{(x_0/x_{st})_B} = \frac{x_0}{x_{st}}
\]

which, from (v), gives,

\[
\omega_A^2 + \omega_B^2 = 2 \omega_p^2/(1 + m_a/m_p)
\]

\[
\frac{f_2}{f_4} = \frac{f_1}{f_3} \Rightarrow \frac{f_1}{f_2} + \beta^2 = \frac{f_3}{f_4} + \beta^2 \Rightarrow (f_1 + f_2 \beta^2)/(f_3 + f_4 \beta^2) = f_2/f_4 \quad \text{etc.}
\]
But, from (iv),

\[ (\omega_A^2 + \omega_B^2) = (\omega_p^2 + \omega_a^2 + \omega_a^2 m_a/m_p)/(1 + \frac{1}{2}m_a/m_p) \]

From the last two equations, it follows that the natural frequency of the auxiliary system resulting in the response of the primary system being a minimum is,

\[ (\omega_a)_{opt} = \omega_p/(1 + m_a/m_p) \]

Substitution of \((\omega_a)_{opt}\) into equation (iv) yields,

\[ \omega^4 - \omega^2 \left[2/(1 + m_a/m_p)\right] \omega_p^2 + 2 \omega_p^4/(2 + m_a/m_p)(1 + m_a/m_p)^2 = 0 \]

whence the abscissae of the fixed points are,

\[ \omega_A, B = 1/(1 + m_a/m_p). \left[1 \pm \sqrt{m_a/m_p/(2 + m_a/m_p)}\right] \omega_p \ldots \ldots (vi) \]

Substitution of \(\omega_A, B\) into equation (v) enables the amplitude of the fixed points to be determined, viz,

\[ \left(x_0/x_{st}\right)_{A, B} = \sqrt{1 + 2m_p/m_a} \]

The damping ratio, \(\beta\), can also be chosen for a minimum response in the primary system. In fact, \(\beta\) can be chosen such that the fixed points represent very nearly the maximum response in the system. The exact mathematical analysis of this is somewhat tedious; it is normal to make some close approximation.

First of all the maxima in fig. 14 are found in the normal way by differentiating equation (i) with respect to \(\omega\) and equating to zero. Putting \(\omega\) equal to the value of the fixed points \(\omega_A\) and \(\omega_B\) (equation (vi)) the damping \(\beta\) is obtained first of all
for obtaining maximum response at the fixed point A and then, secondly, for obtaining the maximum response at B. The mean of the two values of \( \beta \) obtained gives the optimum damping ratio, viz,

\[
\beta_{\text{opt}} = \sqrt{\frac{3m_a}{m_p}} \frac{8}{8(1 + m_a/m_p)^3}
\]

Consider the case in which the auxiliary mass is coupled to the primary system by damping only. Clearly, the natural frequency of the auxiliary system \( \omega_a \to 0 \). Also, \( \beta = \frac{c_a}{c_{ca}} \to \infty \).

(Since \( c_{ca} = 2\sqrt{k_a m_a} = 0 \) with \( k_a = 0 \).) It can be shown, however, that the product \( (\beta, \omega_a) \) is finite and represents the damping in the system. Equation (i) may be written,

\[
\frac{x_0}{x_{st}} = \sqrt{\frac{\omega^2 + 4(\beta \cdot \omega_a)^2}{\omega^2 - 4(\beta \omega_a)^2 \left[ \omega_p^2 - \omega^2(1 + \frac{m_a}{m_p}) \right]^2}} \cdot \omega_p^2 \quad \ldots (vii)
\]

In the same way as before the fixed points may be found from the equality between the corresponding terms in the numerator and denominator, viz,

\[
\frac{\omega^2}{\omega^2(\omega_p^2 - \omega^2)^2} = \frac{4/4}{\omega_p^2 - \omega^2(1 + \frac{m_a}{m_p})^2}
\]

whence, the fixed points are defined:

\[
\omega^2 = 0
\]

\[
\omega^2 = 2/(2 + \frac{m_a}{m_p}) \cdot \omega_p
\]

The amplitude of the primary mass at the fixed point defined by \( \omega_B = \sqrt{2/(2 + \frac{m_a}{m_p})} \cdot \omega_p \) is (substituting into (vii)),

\[
\frac{x_0}{x_{st}} = \frac{(2 + \frac{m_a}{m_p})/(m_a/m_p)}{(m_a/m_p)}
\]

In the same way as before the damping can be made such that the
response at the fixed point is the maximum response in the system. By differentiating equation (vii) with respect to $\omega$, equating to zero, and substituting for the value of $\omega$ at the fixed point B, the optimum damping is obtained, viz,

$$ (\beta \omega_a)_{opt} = \sqrt{1/2(2 + m_a/m_p)(1 + m_a/m_p)} $$
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SESSION 3.

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