In this thesis, we discuss some of the problems associated with the breaking of the symmetry groups SU(3) and chiral SU(3) \( \otimes \) SU(3). The first chapter is an introduction to the use of symmetry groups in high-energy physics. We show how the groups SU(2) and SU(3) are introduced from the point of view of conserved quantum numbers and their relation to the invariance of the system under certain unitary transformations. The extension to chiral SU(3) \( \otimes \) SU(3) is made by considering the simplest group which can be generated by the charges associated with the electromagnetic and weak hadronic currents, and the success of PCAC and low energy theorems is shown to indicate that the group can be regarded as an approximate symmetry group of the strong interactions.

Chapter II is devoted to a critical review of the asymptotic symmetry scheme proposed by Oneda and Matsuda for the groups SU(2) and SU(3). Their assumption, that matrix elements of the (time dependent) charge \( V^K \) are effectively not renormalised in the infinite three-momentum frame, is claimed to be too restrictive. We show that the "good" results, e.g. the Gell-Mann Okubo formula, which they derive from equal time commutators (ETCs) of the form \( \left[ \v^{K^+}, A^{K^+} \right] = 0 \) can be obtained without their strong assumption, because of a special cancellation mechanism; however, the poorer results, e.g. the \( \Sigma^0 - \Lambda^0 \) degeneracy, obtained from ETCs like \( \left[ \v^{K^+}, A^{K^+} \right] = 0 \), arise only when their assumption is made.

In Chapter III, we extend their formalism to include non-vanishing ETCs such as \( \left[ V^{K^+}, V^{K^-} \right] \); using the latter, but without invoking their symmetry-breaking assumption, we re-derive the result that it is the one-particle matrix elements of the different parts of the Hamiltonian.
which are octet dominated, and not the Hamiltonians themselves. A corollary to this calculation is another well-known result, viz. that the octet parts of the medium-strong, electromagnetic and parity conserving non-leptonic weak Hamiltonians appear to belong to the same octet.

A restricted form of the SU(3) σ-Model is employed in Chapter IV to investigate the interplay between the symmetry-breaking of the Hamiltonian and of the vacuum, i.e. between intrinsic and spontaneous breakdown, with particular reference to the masses of the \( O^+ \) and \( O^- \) mesons and the decay rates of the scalars into pairs of pseudoscalars. Various combinations of the symmetry-breaking terms are tried, but although reasonable agreement can be obtained for the mass spectra, the predictions for the decay rates are generally much poorer.

In Chapter V, we try to relate some of the problems raised in the earlier chapters. In particular, we consider some of the implications of breaking the SU(3) symmetry by a term belonging to the \((1, 8) \oplus (8, 1)\) representation of \(SU(3) \otimes SU(3)\), as well as by a \((3, \overline{3}) \oplus (\overline{3}, 3)\) term, which is the more popular method at the present time. Some of the complications which arise with a \((1, 8)\) term are pointed out, but it is suggested that a more detailed study of the effects, possibly using hard meson calculations, might well throw some interesting light on some of the present problems of chiral symmetry-breaking.
SOME ASPECTS OF INTERNAL SYMMETRY BREAKING

IN HIGH-ENERGY PHYSICS

Thesis

Submitted by

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CHAPTER I

INTERNAL SYMMETRY GROUPS IN HIGH-ENERGY PHYSICS

I. Introduction

The use of symmetry groups to classify the properties of the large number of particles now observed in high energy physics has proved very fruitful. A great deal of work has been done on this since Heisenberg first suggested that the proton and neutron could be assigned to the basic spinor representation of SU(2); today, much interest centres on the chiral SU(3) \times SU(3) group, under which the strong interactions are assumed to be approximately invariant. As the search for larger symmetry groups has progressed, the problems of how to break the new symmetries have increased. However, this apparently paradoxical process has enabled physicists to restore more than a modicum of order to a situation which could otherwise have become quite chaotic as the number of new "particles" discovered by experimentalists rapidly increased.

In this chapter, we begin by pointing out how the idea of symmetry groups arises in high energy physics, and then proceed to outline the development of internal symmetry groups from the original SU(2) I-spin group up to chiral SU(3) \times SU(3).

II. General Considerations

The forces operating among the "elementary" particles appear to fall into four separate categories, namely strong, electromagnetic weak and gravitational. These are characterised by varying strengths
of interaction, and this is manifested in the widely different
decay rates and cross-sections for each force; because the
gravitational coupling is so much smaller than all of the others,
we neglect this interaction henceforth. At this stage, we can
make our first classification of the particles, according to the
interactions in which they participate. The photon relates to
everything else through electromagnetic forces only, while the
leptons undergo both electromagnetic and weak interactions; the
remaining particles, the hadrons, interact via all three forces,
although the strong interaction effects usually mask those of the
other forces. Hence, the information provided by the hadrons is
mainly about the strong forces although there are cases when we
can extract some details about weak and electromagnetic forces
as well.

The next point is that certain processes are apparently
forbidden by some of the interactions but not by others: to
"explain" this, we assign appropriate quantum numbers to the
particles involved and demand overall conservation of these quantum
numbers after some types of interaction, but not necessarily after
others. This idea of conserved quantum numbers can be related to
invariance of the system under symmetry transformations in the
following way.

All of the information about the effects of a particular
interaction is contained in the S-matrix. If we apply an infinitesi-
mal unitary transformation to the initial and final states \( |i\rangle \) and
\( |f\rangle \) respectively,

\[
|i\rangle \rightarrow |i'\rangle = U |i\rangle, \quad \langle f | \rightarrow \langle f' | = \langle f | U^+, \]

where $U = 1 + i c a a^*$, and the $G_a$ are the (hermitian) generators of the transformation, then invariance of the system implies that

$$
\langle f | S | i \rangle \rightarrow \langle f' | S | i' \rangle
$$

$$
= \langle f | U^* S U | i \rangle
$$

$$
= \langle f | S | i \rangle.
$$

\therefore \quad [s, u] = 0 \quad \text{and} \quad [s, G_a] = 0.

Some of the $G_a$ are diagonal operators whose eigenvalues $g$ are the quantum numbers associated with the group; thus, if we sandwich the second commutator, for such a diagonal $G_a$, between the initial and final states, we obtain

$$(g_i - g_f) \langle f | S | i \rangle = 0.$$ 

For the process to occur, $\langle f | S | i \rangle$ does not vanish, so we must have $g_i = g_f$. Hence, if the interaction, and so the $S$-matrix, is invariant under a certain group of transformations, the quantum numbers associated with that group are conserved.

Now, we can separate quantum numbers into two classes, according to the type of transformation involved; there are space time numbers, such as spin and parity, and so-called internal quantum numbers, e.g. electric charge, baryon number. In this thesis, we are concerned only with the second class.

We have indicated how invariance groups arise for the different interactions among the particles; the question now is - what information can these groups give us concerning the particles? Firstly,
we can find irreducible representations (IRs) of each group, and then try to assign particles to these IRs; in this way, we obtain multiplets which have the same mass, spin, parity etc. Secondly, we can relate different coupling constants to each other; for example, if we write an SU(2) invariant Lagrangian for the coupling of the pions to the nucleons, we have, in standard notation,

\[ \mathcal{L} = i g \bar{N} \gamma_5 L \cdot A N, \]

and this gives us the relations

\[ \sqrt{2} g_{\pi \pi^0} = -\sqrt{2} g_{\eta \eta^0} = g_{\pi n^0} = g_{\eta n^0}. \]

Of course, less strong interactions often break this invariance so that the multiplets are not exactly degenerate and the coupling constants differ somewhat from their symmetric values. Hence, an important problem is to relate the observed symmetry-breaking effects to some symmetry-breaking mechanism.

Now, the strong interactions conserve I-spin and hypercharge, i.e. they are invariant under SU(2) \( \otimes \) U(1). We can certainly classify particles into IRs of this group, but such multiplets are not very large, rarely containing more than three particles. In fact, for the coupling of the eight pseudoscalar mesons to the eight \( \frac{1}{2}^+ \) baryons, we have eight coupling constants which are all independent. Hence, a larger symmetry group would be very useful, even although this would not be an exact invariance group of the strong interactions. The problem then separates into two parts:
firstly, we must search for suitable larger groups and secondly, having found them, we must look for a symmetry breaking mechanism which allows us to fit the pattern of mass splittings, differences in coupling constants, etc.

At the present time, the strong interactions are believed to be approximately invariant under chiral SU(3) ⊗ SU(3); since the connection with SU(2) ⊗ U(1) is not an obvious one, we sketch the development from SU(2) in the next sections.

III. SU(2) and SU(3)

The discovery of the neutron in 1932 pointed to the existence of two forces previously unknown to physicists, i.e. the strong and weak nuclear forces, and this essentially marked the beginning of elementary particle physics as we know it today. At the same time, and not inappropriately, the proton-neutron system gave rise to the use of symmetry groups as a convenient formalism for describing elementary particle systems.

The charge independence of nuclear forces indicated that in a world without electromagnetic forces, the proton and neutron might be identical; in that case, their small mass-difference could be caused by different electromagnetic self-interactions for the two particles. This prompted Heisenberg to suggest the use of the basic spinor representation of the unitary unimodular group SU(2) to describe the nucleon doublet, N = (p,n); then, following Yukawa's postulate of the existence of mesons to mediate the strong nuclear forces, Kemmer was led by the charge independence
of these forces to propose that the mesons belonged to an $SU(2)$ triplet, and that the field theory Lagrangian describing the strong interaction should be an $SU(2)$ singlet, i.e. the strong forces were invariant under $SU(2)$ transformations. The symmetry was broken by electromagnetic forces which removed the degeneracy of the various multiplets. New particles were assigned to appropriate multiplets, and the scheme appeared to work pretty well.

However, one problem was introduced by the behaviour of some of these new particles, which were copiously produced but which took a very much longer time to decay. This was eventually "explained" by giving each particle a new, additive quantum number, strangeness ($S$), and then demanding that $S$ was conserved in the strong interactions, i.e. in the production process, but was not conserved by the weak interactions, thus allowing a slow decay to occur. In this way, the invariance group of the strong interactions was increased to $SU(2) \otimes U(1)$. Gell-Mann and Nishijima and Nakano then obtained an interesting relation between the known "internal" quantum numbers, viz. $Q = I_3 + \frac{1}{2}(B + S)$, where $Q$ is the electric charge, $I_3$ is the third component of $I$-spin, and $B$ is the baryon number. ($B$, like the two lepton numbers, is apparently conserved by all interactions; in this respect, it can always be "factored out", and because of this, we shall ignore it in future). However, as pointed out earlier, with the increasing number of particles, multiplets of $SU(2) \otimes U(1)$ were hardly adequate for classifying all of these particles. But the main problem about introducing a larger symmetry group was that a new symmetry breaking mechanism was also required since the strong
interactions did not seem to conserve any other quantum numbers. The question was - which was the most appropriate new group? Arguing by analogy with SU(2) where the nucleon \((p,n)\) doublet was used as the basic spinor representation of a group generated by the three \(2 \times 2\) traceless Hermitian matrices \(\gamma_i\), various people \(^6,^7\) took \((p, n, \Lambda)\) as the basic triplet for a group generated by the eight independent \(3 \times 3\) traceless Hermitian matrices \(\lambda_i\) - this new group was SU(3). Regarding mesons as bound states of appropriate baryon-antibaryon pairs, it was found that they ought to fit into an octet; furthermore, if the mass splitting operator was taken as the eighth component of an octet of operators, in order to preserve SU(2) \(\otimes\) U(1) as an exact symmetry, then the correct pattern for the mass splittings was found \(^8\). The baryons themselves proved to be something of a problem until Gell-Mann \(^7\) suggested that they, too, belong to an octet, and that the basic triplet states did not correspond to any known particles. Thereafter, many successful predictions were made using SU(3) as an approximate symmetry group.\(^1\)

IV. Chiral SU(3) \(\otimes\) SU(3)

So far, only the symmetry group for the strong interactions has been considered, from the point of view of conserved quantum numbers; the electromagnetic and weak interactions have been neglected since they generally break this symmetry. However, these two interactions can also give rise to symmetry group considerations from a rather different viewpoint, i.e. that of currents.

In Lagrangian field theory, it is known that, for each group
of transformations on the particle fields we can derive a current corresponding to each generator of the group. The integrated time components of these currents, the "charges", generate the algebra of the group by means of the equal time commutators (ETCs), even when these charges are time dependent, i.e. when the currents are not conserved.

Gell-Mann\(^7\) suggested that the hadron system was characterised by the algebra generated by the charges of the electromagnetic and weak hadronic currents, since it is through these currents that the hadrons take part in the corresponding interactions with the photon and leptons. In general, the ETCs of the different parts of these charges produce new operators, the ETCs of the latter with each other and with the original charges yield still more operators, etc. etc., until the algebraic system is eventually closed. All of these operators (including the original charges) can be expressed as linear combinations of a set (possible infinite) of N operators \(R^i(t)\) which satisfy the ETCs

\[
\left[ R^i(t), R^j(t) \right] = C^{ijk} R^k(t),
\]

where the \(C^{ijk}\) are the appropriate structure constants. Then the infinitesimal unitary operators \(U = 1 + i\varepsilon^{aR}(t)\) generate an N parameter continuous group of unitary transformations which leave the \(C^{ijk}\) invariant, i.e. the \(C^{ijk}\) characterise the system.

In order to have some indication about a suitable algebra generated by the electromagnetic and weak hadron currents, Gell-Mann extended the well-tried principle of universality in the weak interactions. Previously, in order to explain the near equality of the
decay constants for $\mu$ and $\beta$ decay, Feynman and Gell-Mann had put forward the CVC hypothesis, that the $\Delta Y = 0$ part of the weak hadronic vector current was one of the components of the $I$-spin current, and hence was conserved. This last point meant that the matrix element of the current for hadron states was renormalised by strong interactions: in that case, for the weak interaction Hamiltonian in the usual current-current form,

$$\mathcal{H}_\omega = \frac{G}{\sqrt{2}} \left[ J_\mu^\text{lepton} J_\mu^\text{hadron} \right]_+,$$

then the effective coupling constants for the leptonic and semi-leptonic decays would be equal. Thus, the ETCs involving one part of the hadron current were already known.

The universality which Gell-Mann now invoked was that of the strengths of the lepton and hadron parts of the weak current, as determined by some suitable non-linear relation for each part, such as was provided by ETCs. The lepton current was known to be of the form

$$J^\alpha(\text{leptons}) = \bar{\nu}_e \gamma^\alpha (1 + \gamma_5) e + \bar{\nu}_\mu \gamma^\alpha (1 + \gamma_5) \mu,$$

where $\bar{\nu}_e$, $e$ etc. are the usual Dirac spinors for the corresponding particles; in fact, $(\bar{\nu}_e, e)$ and $(\bar{\nu}_\mu, \mu)$ form two separate doublets in some internal lepton space. Now, $J^\alpha$ splits into a vector part $v^\alpha$ and an axial vector part $a^\alpha$, i.e. into components characterised by parity, a quantum number conserved by the strong interactions. The doublet notation allows us to write both $v^\alpha$ and $a^\alpha$ as the charged components of two $SU(2)$ currents $v^1_\alpha$ and $a^1_\alpha$. 
(i = 1, 2, 3). The ETCs of the corresponding charges $v^i$ and $a^i$ are easily shown to be

$$[v^i, v^j] = i\varepsilon^{ijk} v^k,$$
$$[v^i, a^j] = i\varepsilon^{ijk} a^k,$$
$$[a^i, a^j] = i\varepsilon^{ijk} v^k,$$

i.e., the algebra closes in the simplest way possible. The combinations $\frac{1}{2}(v^i + a^i)$ and $\frac{1}{2}(v^i - a^i)$ generate independent SU(2) groups, i.e. we have chiral SU(2) ⊗ SU(2), and the weak lepton current is a component of the "left-handed" SU(2) current. When the electromagnetic lepton current is taken as a linear combination of the third component of this "left-handed" SU(2) group, together with an SU(2) scalar part, we find that the electromagnetic and weak leptonic currents generate the algebra of SU(2) ⊗ U(1).

Gell-Mann's hypothesis was that the total hadronic weak and electromagnetic currents should also generate an SU(2) ⊗ U(1) algebra. Again using quantum numbers conserved in the strong interactions, this time parity and hypercharge $Y$, in order to split up the weak currents, we find vector and axial currents, as in the lepton case, and also each of these has a $\Delta Y = 0$ part and a $\Delta Y = 1$, $\Delta Y/\Delta Q = 1$ part. In the CVC hypothesis, the $\Delta Y = 0$ part of the vector current was taken as the I-spin current: furthermore, it was known that the three I-spin (or SU(2)) currents can be taken as the first three components of an SU(3) octet of currents. Thus by assuming (1) that both the vector and axial vector parts of the weak hadronic current belong to (separate) SU(3) octets, (2) that the total leptonic weak current is as given above, and (3) that the total weak hadronic current generates the same algebra
as the leptonic current, then the algebra of chiral $SU(3) \otimes SU(3)$ is generated by the vector and axial charges $V^i$ and $A^i$ ($i = 1, \ldots, 8$). This is the simplest algebra which can be generated by these two octets; the ETCs are

$$\left[ V^i, V^j \right] = if^{ijk} V^k, \quad \left[ V^i, A^j \right] = if^{ijk} A^k,$$

$$\left[ A^i, A^j \right] = if^{ijk} V^k.$$

Thus, Current Algebra was introduced to high energy physics: it was a very elegant way of linking the three different interactions, viz. taking the electromagnetic and weak hadron currents to generate an algebra which was characteristic of the strong interactions. In fact, local generalisations of the ETCs have been made, i.e. with one or both of the charges replaced by the appropriate time component of the current; however, this involves even stronger assumptions, and we shall not discuss Current Algebra further here, but mention the excellent reviews by Adler and Dashen and by Renner.

When Current Algebra was formulated, chiral $SU(3) \otimes SU(3)$ was probably not considered as even an approximate symmetry group for the strong interactions, but only as a set of dynamical conditions from which relations between form factors and coupling constants could be derived. For one thing, there did not appear to be any approximate $SU(3) \otimes SU(3)$ multiplets of particles, whereas there was plenty of evidence for approximate $SU(3)$ multiplets. However, Dashen has pointed out that an exact symmetry does not always have to manifest itself in degenerate multiplets. In the context of the $SU(2) \otimes SU(2)$ $\sigma$-Model, which involves the pseudo-scalar I-spin triplet $\pi$ and the scalar I-spin singlet $\sigma$, he
shows that, depending on the sign of the interaction term in the Lagrangian, the symmetry can be realised (1) by having the $\pi$ and $\sigma$ degenerate or (2) by having massless pions and a massive $\sigma$. This second alternative is often referred to as spontaneous breakdown of the symmetry since it corresponds to the invariance group $G'$ of the vacuum being smaller than $G$, the symmetry group of the Hamiltonian; in this case, some of the (time independent) generators of $G$ do not annihilate the vacuum but transform it into another massless state with the same quantum numbers as the generator involved. For example, we can have

$$\langle 0 | A_{\mu}^i | \pi^j(p) \rangle = i f_1 \delta^{ij} p_{\mu}, \quad (i, j = 1, 2, 3).$$

Then

$$\langle 0 | \partial_\mu A_{\mu}^i | \pi^j(p) \rangle = f_1 m_1^2 \delta^{ij},$$

and since $\partial_\mu A_{\mu}^i = 0$, then $m_1^2 = 0$ as well, provided that the weak decay constant $f_1$ does not vanish.

The chiral SU(3) $\otimes$ SU(3) symmetry can be realised either by an octet of $\pi^-$ mesons and another octet of $\pi^+$ mesons, degenerate with the first, or else by an octet of massless pseudoscalars, with the scalar octet having some quite distinct mass. In practice, the $\pi^-$ octet is very much lighter than all of the other hadrons, mesons or baryons, and so it is quite possible for the pseudoscalars to be approximate Goldstone Bosons.

Another piece of supporting evidence, according to Dashen, is the set of successful predictions based on PCAC and soft meson theorems\(^2\): these are equivalent to pole dominance of the form factors of the axial current divergence by the $\pi^-$ mesons. As an
illustration, consider the chiral $SU(2) \times SU(2)$ symmetry to be broken by a term of order $O(\epsilon)$, so that $\partial_{\mu} A_{\mu}^1$ is $O(\epsilon)$, $(i = 1, 2, 3)$. Taking matrix elements of $\partial_{\mu} A_{\mu}^1$ between two nucleon states, we have

$$\langle N | \partial_{\mu} A_{\mu}^1 | N \rangle = (\bar{u} \gamma^1 \gamma_5 u) \cdot d(q^2),$$

where we can write $d(q^2)$ in the form

$$d(q^2) = \frac{r}{q^2 - m_\pi^2} + \int_{M^2}^{\infty} \frac{\rho(m^2)}{q^2 - m^2} \, dm^2;$$

$r$ is the residue of the pion pole while the integral gives the contribution of higher intermediate states, with the threshold at $M^2 > (3m_\pi^2)$. Since $\partial_{\mu} A_{\mu}^1$ is $O(\epsilon)$, then so are $r$ and $\rho(m^2)$. For $q^2 \ll m_\pi^2$, the second term is $O(\epsilon)$ while the first is also $O(\epsilon)$ if the symmetry is realised through degenerate multiplets, i.e. if $m_\pi^2$ is not $O(\epsilon)$; on the other hand, if spontaneous breakdown occurs, $m_\pi^2$ is also $O(\epsilon)$ so that the first term is $O(1)$, and clearly dominates $d(q^2)$ for low values of $q^2$. Hence, PCAC is valid only when the chiral symmetry is realised through massless pions (or a massless $0^-$ octet, for $SU(3) \times SU(3)$). Since the predictions based on PCAC are in reasonably good agreement with experiment, and would presumably be exact in the symmetry limit when the mesons are massless, it appears that the strong interactions are indeed almost invariant under chiral $SU(3) \times SU(3)$.

Now that we have shown why $SU(3) \times SU(3)$ can be regarded...
as a good symmetry of the strong interactions, the problem confronting us is the symmetry breaking mechanism. The standard references for this at present are the papers of Gell-Mann, Oakes and Renner, (GOR)\textsuperscript{10} and of Dashen\textsuperscript{3}. In the first paper, the chiral symmetry is broken to SU(3) by a term $u_\omega$ which transforms according to the $(3, \overline{3}) \oplus (\overline{3}, 3)$ representation of SU(3) $\times$ SU(3), while the SU(3) symmetry is broken to SU(2) $\times$ U(1) by the term $c u_3$ where $u_3$ belongs to the same representation as $u_\omega$. The parameter $c$ is not small (if it were, then SU(3) would be a very much better symmetry of the Hamiltonian, $H$, than chiral SU(3) $\times$ SU(3)), but is taken to be approximately $\sqrt{2}$, corresponding to $H$ being approximately SU(2) $\times$ SU(2) invariant. One effect of this is that the three pions are still almost Goldstone Bosons, even when the remaining $0^-$ mesons have acquired mass, and the experimental situation certainly bears this out. Although SU(2) $\times$ SU(2) is thus a much better symmetry of the Hamiltonian than SU(3), the latter symmetry group has not disappeared since the vacuum is assumed to be approximately SU(3) invariant, corresponding to an SU(3) pattern for the spectra of the particles; again, this agrees reasonably well with experiment.

However, despite the successes of the scheme, some problems still remain, notably the apparently poor prediction for the parameter $\delta$ appearing in $K_\Delta$ decay data\textsuperscript{11}; more details of this particular problem are given in Chapter V, where we discuss some of the implications of the GOR assumptions.

It will be very interesting to find out whether there is a relatively simple explanation for these few poorer predictions, or whether they will lead us to a radically new approach to symmetries and their breakdown.
CHAPTER II

CRITICAL REMARKS ON THE ASYMPOTIC SYMMETRY SCHEME OF ONEDA AND MATSUDA

I. Introduction

This chapter consists of a critical discussion of the assumptions contained in the asymptotic symmetry scheme defined in a recent series of papers by Oneda and Matsuda. The symmetry groups involved here are simply $SU(3)$ and $SU(2)$, not the larger chiral $SU(3) \times SU(3)$ or $SU(2) \times SU(2)$ groups.

Several distinct aspects of the scheme must be noted immediately. The first, the distinguishing feature of the scheme, is the special assumption that, even in broken $SU(3)$ symmetry, matrix elements of the $SU(3)$ "stepping up/down" operator $\sqrt{K}$ between physical states whose three-momentum is infinite, are effectively not renormalized, at least when compared with other symmetry breaking effects such as mass splittings within an $SU(3)$ multiplet. The second point is not unique to this scheme, but has been utilised on other occasions: this is the technique of using the so-called "exotic" equal time commutators (ETCs), e.g.

$$\left[ \sqrt{K}^+, \sqrt{K}^+ \right] = 0 , \quad (2.1)$$

involving the time derivative of the "stepping up" charge together with some other charge, in order to derive relations between various quantities (e.g. masses or coupling constants) even when the symmetry is broken. Finally, the use of the infinite momentum frame is again not new, although Oneda and Matsuda stress that their
non-renormalisation assumption is claimed to hold only in this particular frame. The "critical remarks" of the title are addressed to the first-mentioned point, i.e. the special non-renormalisation assumption.

At this stage, it seems appropriate to give the original motivation for the following critical analysis. Briefly, some of the mass formulae arising from the model are rather poor; in particular, the $\Sigma^0$ and $\Lambda^0$ are mass degenerate, even in broken $SU(2)$ symmetry, while some quite general intermultiplet mass formulae are predicted which are not in very close agreement with the experimental data; however, the same scheme also produces several well-established relations involving coupling constants and others involving masses, the most notable being the Gell-Mann Okubo (GMO) mass formula. The simultaneous prediction of such "good" and "not-so-good" relations, apparently depending on the same critical assumption, was rather intriguing, and suggested that perhaps the assumption was not really necessary in certain cases, i.e. those leading to the 'good' results. In fact, this is exactly what we claim to prove in this chapter.

Our starting point is the prediction that $\Sigma^2 = \Lambda^2$, where, e.g., $\Lambda^2$ denotes the squared mass of the $\Lambda^0$ particle. Despite the fact that this degeneracy has been obtained in other symmetry breaking schemes, the present author does not regard it as a particularly successful prediction, although Oneda and Matsuda, not unnaturally, take a somewhat different viewpoint. One feature which this result has in common with the other "unsuccessful" mass sum rules, the intermultiplet ones, is the type of ETC from which they are all derived, viz.
Oneda and Matsuda suggest that since results derived from the ETC

\[
\left[ \nu K^+, A^+ \right] = 0 \quad (2.2)
\]

are certainly more accurate, then the form of symmetry-breaking Hamiltonian required for the validity of eq. (2.2) is rather too restrictive. In fact, consider the following form for \( \mathcal{H}' \), the Hamiltonian density:

\[
\mathcal{H}'(x) = \alpha \mathcal{B}(x) + \alpha \delta_{ij} J_{\mu i}^i(x) J_{\mu j}^j(x), \quad (2.4)
\]

where \( \mathcal{B} \) is the usual scalar density belonging to the \((3, \bar{3}) \oplus (\bar{3}, 3)\) representation of \( SU(3) \otimes SU(3) \),

\[
J_{\mu i}^i J_{\mu j}^j = V_{\mu i}^i V_{\mu j}^j + \gamma A_{\mu i}^i A_{\mu j}^j, \quad (2.5)
\]

where \( V_{\mu i}^i \) and \( A_{\mu i}^i \) are the usual vector and axial vector currents, and \( \alpha, \beta, \gamma \) are arbitrary parameters.

Then ETCs like that in eq. (2.2) (Group B ETCs in the notation of Oneda and Matsuda) vanish only when \( \gamma \) assumes the value unity, whereas ETCs of Group A, e.g. that in eq. (2.3), vanish for arbitrary values of \( \gamma \), as well as of \( \alpha \) and \( \beta \). However, at least for the corresponding ETCs in broken \( SU(2) \) symmetry, Oneda and Matsuda are prepared to accept the meson mass sum rules derived from Group B ETCs, although they reject the baryon results, which are very poor indeed. But by doing this, they appear to raise some kind of
distinction between baryons and mesons, a situation which is not particularly satisfactory.

A different attitude would be to regard the assumptions of the asymptotic symmetry scheme as too restrictive. Initial support for this viewpoint comes from the fact that the sum rules derived from all of the ETCs except those of the form $[v^k, A^i] = 0$ can be obtained immediately by using the normal formal counting approach, since the sum over intermediate states is cut off at exactly the same point as when the stronger assumptions are applied. For ETCs involving $A^i$, all of the terms in the series are formally of order $O(\varepsilon)$; in that case, if we are to distinguish between the predictions arising from the ETC $[v^K, A^K] = 0$ and those from $[v^K, A^K] = 0$, using only the formal counting procedure, then we must show that some kind of cancellation takes place between the off-diagonal terms arising from the former ETC, but not among those from the latter.

In this chapter, we show that such a cancellation does occur, multiplet by multiplet, when $V$ and $A$ carry the same $SU(3)$ quantum number, since for each multiplet which contributes off-diagonal terms, exactly the same intermediate states appear on each side of the commutator. However, when $V$ and $A$ have different quantum numbers, the states coming from one particular multiplet are different on the two sides of the commutator; in some cases, a term appears on one side only, which shows immediately that this type of cancellation cannot arise. Of course, there is nothing to prevent cancellation among the contributions from different multiplets, but this cannot be proved unless we consider a larger symmetry group.

In which the $SU(3)$ multiplets are actually sub-multiplets of some $\langle a|v^K|b\rangle$ is off-diagonal when $\langle a|/|b\rangle$ belong to different irreducible representations of $SU(3)$ in the exact symmetry limit.
larger multiplet. Thus, all of the relations involving coupling constants, together with the Gell-Mann Okubo (GMO) and equal-spacing decuplet mass formulas, can be derived without making the asymptotic symmetry assumption that off-diagonal matrix elements of $V^K$ are effectively of a higher order in the symmetry-breaking parameter $\varepsilon$ than $O(\varepsilon)$; but the more questionable results such as the $\Sigma - \Lambda$ degeneracy and the intermultiplet mass sum rules require the stronger assumption. Accordingly, it appears that such a condition is too strong, and possibly unnecessary.

In the next section, we summarise the main assumptions of the asymptotic symmetry scheme, together with the supporting arguments, while Section III contains a critical discussion of one or two of these arguments. The results of the scheme are mentioned in Section IV, where we show in some detail how the cancellation occurs for singlet contributions to the GMO formula, but how it cannot occur in the derivation of the $\Sigma - \Lambda$ mass difference; we also show how it fails to appear for higher decuplet states when we try to derive an intermultiplet formula connecting octet and decuplet baryons. We conclude that section by giving a model independent argument to show that intermultiplet formulas cannot be derived between $SU(3)$ multiplets when the largest symmetry group involved is $SU(3)$ itself. Finally, in the Appendix, we show how the cancellation occurs for intermediate off-diagonal decuplet and octet states in the derivation of the GMO formula.

II. The Asymptotic Symmetry Scheme

Before outlining the assumptions of the scheme, we shall describe the formalism involved in deriving the different sum rules: in the
rest of this chapter, we shall deal only with SU(3); also, all matrix elements are to be considered in the infinite momentum frame.

Basically, the sum rules are obtained by sandwiching ETCs of the form \[ [V^K, Q^J] = i^{Kij} Q^J 2 \] (for coupling constants) and \[ [V^K, Q^J] = 0 \] (for masses) between the appropriate single-particle states and truncating the sum over intermediate states with the aid of the postulates of the asymptotic symmetry scheme. In the above ETCs, \( Q \) may be either a vector or axial vector charge; \( i, j \) and \( k \) are SU(3) indices, with \( i^{Kij} \) the appropriate structure constant; \( V^K \) is the time derivative of the strangeness-changing vector charge.

When the symmetry is broken by a term \( \varepsilon H' \) in the Hamiltonian, \( V^K \) becomes time dependent, the mass degeneracy within each SU(3) multiplet is removed, and the new physical states no longer belong to irreducible representations (IRs) of SU(3) but contain small \( O(\varepsilon) \) admixtures of states belonging to other IRs. Thus, matrix elements of \( V^K \) between two physical states no longer assume their symmetric values, but are renormalised: in particular, off-diagonal terms are at most \( O(\varepsilon) \).

To see the effect of working in the infinite momentum frame, consider the following matrix element of

\[
V^K = \int d^3x (V_0^h(0,x) + i V_0^5(0,x))
\]

in this limit.

\[
\langle \pi^+(p')|V^K|\bar{K}^0(p)\rangle = (2\pi)^3 \delta^3(p-p') \left[ (p_0 + p'_0) F_+(q^2) + (p_0 - p'_0) F_-(q^2) \right] \]

\[
= 2p_0 (2\pi)^3 \delta^3(p-p') F_+(0), \text{ as } |p| \to \infty. \quad (2.9)
\]

Now, in the exact SU(3) limit, \( m_{\pi^+} = m_{K^0} \), so that \( p'_0 = p_0 \), and
\[ \langle \pi^+(p')|V^K|\pi^+(p) \rangle = 2p_0(2\pi)^3 \delta^3(p-p') F_+^S(0), \quad (2.10) \]

where \( F_+^S(0) = 1 \) is the exact SU(3) symmetry value of \( F_+(0) \).

Thus, by working in the infinite momentum frame, we may ignore \( F_-(q^2) \) and consider only \( F_+(q^2) \); furthermore, we have \( q_\mu = 0 \), just as in the exact symmetry limit.

Now take matrix elements of the ETC
\[ [V^K^+, V^K^-] = V^3 + \sqrt{3} V^8 \quad (2.11) \]

between states at infinite momentum, and sum over a complete set of intermediate states.

\[ \langle \pi^+|V^K^+|\pi^+ \rangle + \sum_n \langle \pi^+|V^K^+|n \rangle \langle n|V^K^-|\pi^+ \rangle \]
\[ - \sum_m \langle \pi^+|V^K^-|m \rangle \langle m|V^K^+|\pi^+ \rangle = \langle \pi^+(p')|V^3 + \sqrt{3} V^8|\pi^+(p) \rangle, \quad (2.11) \]

as \( |p| \to \infty \).

Since the elements \( \langle \pi^+|V^K^+|n \rangle \) are off-diagonal, they are formally at most \( O(\varepsilon) \). Hence, omitting a common factor of \( 2E_+(2\pi)^3 \delta^3(p-p') \), we obtain the relation
\[ [F_+(0)]^2 + O(\varepsilon^2) = 1 = [F_+^S(0)]^2, \quad (2.12) \]
i.e. in the infinite momentum limit, matrix elements of \( V^K \) between states belonging to the same SU(3) multiplet are renormalised by a term which is at most \( O(\varepsilon^2) \) (Ademollo-Gatto Theorem)\(^{20}\).

It is at this point that Oneda and Matsuda impose their strong conditions on the \( V^K \) matrix elements, but only in this infinite
momentum frame. Formally, all off-diagonal matrix elements of $\gamma^K$ are $O(\varepsilon)$ when the states involved have the same $I$-spin and hypercharge. However, in the asymptotic symmetry approximation, only those states are retained which have the same space-time quantum numbers, and which are so close to each other in mass as to make mixing an important problem. (e.g. $\gamma^0 - X^0$ and $\omega^0 - \rho^0$ mixing). The argument for neglecting states with different space-time quantum numbers is that $\gamma^K$ is essentially a scalar operator, thus causing a "momentum-barrier" to be set up; also, more distant states with the same $J^{PC}$ are neglected since their contributions are damped by the large mass differences involved.

As an example of how mixing is treated, consider the $\gamma^0 - X^0$ case.

$$\gamma^0(p) = \pi_8 \cos \theta + \pi_9 \sin \theta,$$

$$X^0(p) = -\pi_8 \sin \theta + \pi_9 \cos \theta,$$

where $\gamma^0(p), X^0(p)$ represent the creation operators of the physical states, $\pi_8$ and $\pi_9$ represent the corresponding operators for the symmetric states, $\theta$ is the mixing angle, and the equations are assumed valid when $|p| \to \infty$. Then

$$\langle \gamma^0(p) | V^K^- | K^+(p') \rangle = 2E_* (2\pi)^3 \delta^3(p-p') G_+(0) \cos \theta,$$

$$\langle X^0(p) | V^K^- | K^+(p') \rangle = 2E_* (2\pi)^3 \delta^3(p-p') G_+(0)(-\sin \theta),$$

where $G_+(0)$ is the appropriate form factor. Use of eq. (2.11) again, this time between $K^+$ states, shows that $G_+(0)$ also differs from its symmetric value by a term of order $O(\varepsilon^2)$. 
Now, since the off-diagonal terms, formally of order \( O(\epsilon) \), are effectively of a higher order than this because of the extra damping effect, the renormalisation of the diagonal form factors is effectively of a higher order than \( O(\epsilon^2) \). The basic assumption of the asymptotic symmetry scheme is that the renormalisation of the form factors of \( V^K \) at \( q^2 = 0 \) is small, and negligible compared with other SU(3) breaking effects, e.g. mass differences. 

"... We neglect all the non-diagonal elements \( \langle b | V^K | A \rangle \) of the vector charge \( V^K \) (except for cases when there is a mixing problem for the states under consideration, ...) only in the infinite-momentum limit; ..." \(^{16}\)

These, as far as the present author is concerned, are the basic assumptions of the asymptotic symmetry scheme, together with the motivating ideas.

III. Some Criticisms

In this section, we take a rather critical look at the above assumptions and arguments. The first comment to be made is that the decisions as to which mixing to neglect and which to include seem somewhat arbitrary, particularly when the lack of mixing between low-lying and higher lying multiplets is subsequently given as the main reason for the poorer predictions of the intermultiplet mass formulas, at least, for the baryons. In addition, the argument used for neglecting mixing with states of different space-time quantum numbers does not seem to be valid. To begin with, it is not \( V^K \) which does the mixing, but \( \mathcal{H}' \), the symmetry-breaking part of the
Hamiltonian. This is essentially a scalar operator, and it is interesting to consider one or two examples of matrix elements involving states with different spin and parity, in the infinite-momentum limit.

Firstly, compare the case of two $\frac{1}{2}^+$ baryons with that of one $\frac{1}{2}^+$ and one $\frac{1}{2}^-$ baryon. In the infinite momentum limit, the respective matrix elements are

$$\bar{u}_1(p, r) u_2(p, s) = (M_1 + M_2) \delta_{rs}, \quad (2.17)$$

and

$$\bar{u}_1(p, r) \gamma_5 u_2(p, s) = (M_1 - M_2) h \delta_{rs}, \quad (2.18)$$

where $h$ is the helicity of the states, and $r, s$ denote the spins. Clearly, except when $M_1 \approx M_2$, these elements are of comparable magnitude, i.e. the second is not highly damped when $|p| \to \infty$. In fact, in the rest frame, the corresponding values are $2(M_1 M_2)^{\frac{1}{2}} \delta_{rs}$ and zero.

For the mesons, certain couplings are forbidden in any frame, e.g. $\langle 0^- | H^* | 0^+ \rangle = 0$, and $\langle 0^- | H^* | 1^- \rangle = 0$. However, for those matrix elements which are allowed, the infinite momentum limit does not appear to reduce the magnitude of the off-diagonal elements. Consider

$$\langle 0^- (p) | H^* | 1^+(k, r) \rangle = g(q^2) \gamma^\mu(k, r) p_\mu = g(q^2) \gamma^0(k, r) (p^0 - k^0) \delta_{rs}, \quad (2.19)$$

where $\gamma^\mu(k, r)$ is the polarisation vector for the $1^+$ state, and we have used the conditions $k^\mu \gamma^\mu(k) = 0$, $p = k$, and the fact that
only the third polarisation vector has a non-zero time component.

When \( |p| \to \infty \), the rhs of eq. (2.19) tends to the value
\[
g(0) \delta_{rs} \left( m_0^2 - m_1^2 \right) / 2m_1,
\]
where \( m_0 \) and \( m_1 \) are the masses of the \( 0^- \) and \( 1^+ \) states respectively. Moreover, for \( p = 0 \), the rhs of eq. (2.19) vanishes, since \( \gamma^0(k, 3) = |k|/m_1 \).

Thus, the argument that, in the infinite momentum frame, the off-diagonal matrix elements of \( V^K \) are effectively smaller than \( O(\varepsilon) \) does not appear to be well justified.

Another criticism is that it is rather unsatisfactory to have widely differing degrees of success between predictions for baryons and those for mesons, when the same ETC is used in the derivations. It seems even more questionable to reject the results for the baryons while accepting those for the mesons, for no better reason than that the former results are very poor while the latter may perhaps be better; such an a posteriori distinction between baryons and mesons is not a welcome feature in a model which ought to treat both types of particle on the same footing. When such a situation appears to arise, as it does for the intermultiplet formulas in broken SU(2), it seems advisable to check whether such differences actually do exist.

For the SU(3) case, the following equation is predicted from
\[
\left[ V^K \circ, A^\pi \right] = 0;
\]
\[
\Omega^{-2} - \Sigma^{-2} = -\Sigma^0 - \sum^0 - \Lambda^0 - \Lambda^0.
\]
(2.20)

Experimentally, the factors (in GeV\(^2\)) are 0.46; 0.31; 0.48. The second factor certainly does not fit, since the \( \Sigma^0 \) and \( \Lambda^0 \) are not degenerate; hence, the baryon predictions are not spectacularly good.
For the corresponding meson case, the formula is

\[ K_a - \pi_a = \text{constant}, \quad (2.21) \]

where \( a \) denotes the particular octet involved. Although this is claimed to be reasonably successful, the following data indicate that this success is only partial.

<table>
<thead>
<tr>
<th>Multiplet</th>
<th>Formula</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0^-</td>
<td>( K^2 - \pi^2 )</td>
<td>( 0.227 \pm 0.003 )</td>
</tr>
<tr>
<td>1^-</td>
<td>( K^2 - \rho^2 )</td>
<td>( 0.21 \pm 0.14 )</td>
</tr>
<tr>
<td>1^+</td>
<td>( K_A^2 - A_1^2 )</td>
<td>( 0.40 \pm 0.18 )</td>
</tr>
<tr>
<td>2^+</td>
<td>( K_{\pi^2}^2 - A_2^2 )</td>
<td>( 0.35 \pm 0.16 )</td>
</tr>
<tr>
<td>0^+</td>
<td>( \kappa^2 - \delta^2 )</td>
<td>( 0.23 )</td>
</tr>
<tr>
<td></td>
<td>( \kappa^2 - \pi_N^2 )</td>
<td>( 0.13 )</td>
</tr>
</tbody>
</table>

Since the members of the \( 0^+ \) octet are still not well established, the results from this multiplet should not really be used as evidence.

Thus, for the \( SU(3) \) intermultiplet formulas, neither set of predictions is very good, so that the mesons and baryons are back on the same footing.

For \( SU(2) \), the results from the ETC \( \mathcal{E}_{\pi^+, K^+} = 0 \) are very poor for the baryons, where we have the equations

\[ \sum -2 - \sum 0^2 = \triangle -2 - \triangle 0^2 = \sum -2 - \wedge 0^2. \quad (2.23) \]

Experimentally, the outside factors are \( 0.012 \text{ GeV}^2 \) and \( 0.189 \text{ GeV}^2 \).

As for the mesons, we obtain

\[ (K_a^-)^2 - (K_a^+)^2 = \delta_a = \text{constant}. \quad (2.24) \]
This time, the values of $\delta_a$ are known for only two octets, viz.

$$
0^{-+}: \delta_a \approx 0.004 \text{ GeV}^2 ;
1^{--}: \delta_a \approx 0.01 \pm 0.007 \text{ GeV}^2.
$$

(2.25)

Although these are not inconsistent, it is hardly fair to claim this as a great success. Thus, once again, the clash between baryon and meson predictions is not nearly as bad as it first appears.

However, since these paradoxes are avoided by accepting less success for the mesons rather than increased success for the baryons, the basic problem of poor predictions arises. But, as mentioned in the first part of this section, there does not appear to be strong backing for the assumptions of the scheme, so that these poorer results may possibly disappear. In the next section, we show in detail how to obtain the "better" results without these assumptions, leaving the above "poorer" predictions as the only necessary consequences of the asymptotic symmetry scheme.

IV. Derivation of Results

We begin this section by comparing the derivation of the various results in the asymptotic symmetry scheme with that using the normal formal counting procedure. The results may be separated into two classes, depending on whether $V^K$ itself is involved or its time derivative $\dot{V}^K$.

In the first category, since the only terms involved are matrix elements of $V^K$ and those of some other charge, the only formally
0(ε) factors are off-diagonal elements of $V^K$. Hence, working to leading order in $ε$ (here, $O(1)$), we can truncate the sum over intermediate states at exactly the same point as Oneda and Matsuda do when they ignore all off-diagonal $V^K$ matrix elements, i.e. there is no difference between the results of an ordinary broken symmetry theory (in the infinite momentum frame) and the asymptotic symmetry scheme.

When $V^K$ appears, the situation is affected by mass difference terms, with those corresponding to diagonal matrix elements being $O(ε)$ whereas all other mass differences are independent of $ε$ (apart from small corrections). Nevertheless, for an ETC of the form $[V^K, V^1] = 0$ between physical states belonging (in the symmetry limit) to the same IR, terms involving intermediate states from the same IR as the outside states are of order $O(ε)$ (from the mass difference), while all other intermediate states give an $O(ε^2)$ contribution, since both matrix elements are off-diagonal. Thus, once again, the extra assumptions are unnecessary for deriving intra-multiplet mass sum rules such as the GMO formula.

However, complications arise for ETCs such as $[V^K, A^1] = 0$. Consider the general case where the ETC is sandwiched between states $|a\rangle$ and $|b\rangle$ which belong, in the exact symmetry limit, to different IRs of SU(3). Intermediate states fall into two categories, viz. diagonal ones $|a'\rangle$, $|b'\rangle$, and off-diagonal ones, $|c\rangle$, $|d\rangle$. In more detail, we obtain the equation

$$\langle a|V^K|a'\rangle \langle a'|A^1|b\rangle (m_a^2 - m_a^2) + \sum_c \langle a|V^K|c\rangle \langle c|A^1|b\rangle (m_c^2 - m_a^2)$$

$$= \langle a|A^1|b'\rangle \langle b'|V^K|b\rangle (m_b^2 - m_b^2) + \sum_d \langle a|A^1|d\rangle \langle d|V^K|b\rangle (m_b^2 - m_d^2).$$

(2.26)
Here, the first terms on each side of eq. (2.26) are $O(\varepsilon)$ because the mass differences involve states in the same multiplet, while all of the remaining terms are also $O(\varepsilon)$, this time because the $V^K$ matrix elements are off-diagonal.

Now, in the asymptotic symmetry scheme, each term in the sums over $c, d$ is neglected, and simple sum rules are obtained from the remaining terms: such sum rules give intermultiplet formulas when $|a\rangle$ and $|b\rangle$ belong to different multiplets, and intramultiplet formulas when they belong to the same multiplet. The latter results coincide with those derived from ETCs of the form $[V^K, V^l] = 0$ and since, in that case, the derivation of these results did not involve the asymptotic symmetry assumptions, this agreement is taken as some kind of justification for these assumptions. It then follows that the intermultiplet formulas are readily accepted.

On the other hand, if it can be shown that special circumstances allow cancellation between the leading (i.e. $O(\varepsilon)$) off-diagonal terms in eq. (2.26) when $|a\rangle$, $|b\rangle$ belong to the same multiplet, so that the already-derived intramultiplet sum rules also arise in the usual formal counting method, then we are still without any justification for applying the stronger assumptions to the case of intermultiplet sum rules: in that case, a comparison of the predictions made from the two different approaches is quite interesting. In fact, we believe that such special circumstances do arise, and prove this below for the case of the GMO formula.

First of all, we must try to relate the $O(\varepsilon)$ off-diagonal $V^K$ matrix elements; this is certainly possible for such elements as $\langle A^i | V^K | B^j \rangle$ and $\langle C^i | V^K | D^j \rangle$ where, e.g. $|B^j\rangle$ and $|D^j\rangle$ refer to
different I-spin sub-multiplets in the $j^{th}$ SU(3) multiplet. However, we cannot relate terms like $\langle A^i | V^K | B^j \rangle$ and $\langle C^i | V^K | D^\ell \rangle$ to each other, since $j$ and $\ell$ refer to different SU(3) multiplets. More explicitly, when the symmetry breaking part of the Hamiltonian, $\varepsilon \mathcal{H}'$, is "switched on", the originally degenerate states within an SU(3) multiplet are separated into distinct I-spin sub-multiplets; each one of these then acquires small $O(\varepsilon)$ admixtures of corresponding sub-multiplets from other SU(3) multiplets. In the usual perturbation expansion approach, the physical state $|A^i\rangle$ belonging (mainly) to the $i^{th}$ SU(3) multiplet may be expressed to $O(\varepsilon)$ as

$$|A^i\rangle \approx |A_o^i\rangle + a^{ij}_A |A_o^j\rangle,$$

$$\langle A^i | \approx \langle A_o^i | + (a^{ij}_A)^* \langle A_o^j |,$$  \hspace{1cm} (2.27)

with

$$a^{ij}_A \approx \frac{\langle A^j_o | \varepsilon \mathcal{H}' | A^i_o \rangle}{E^i_o - E^j_o},$$  \hspace{1cm} (2.28)

where $|A_o^i\rangle$ is an exact symmetry state, and $E^i_o$ is the original energy eigenvalue of the $i^{th}$ multiplet.

The orthogonality condition for the physical states yields the condition $(a^{ij}_A)^* = - a^{ji}_A$; also the fact that matrix elements of $V^K$, the constant SU(2) generators, are not renormalised as long as SU(2) is an exact symmetry, allows us to show that $a^{ij}_A$ is constant within each I-spin sub-multiplet. In order to relate $a^{ij}_A$ to $a^{ij}_B$ where $A, B$ refer to different I-spin sub-multiplets within the same SU(3) multiplet, we must know the SU(3) transformation properties of $\mathcal{H}'$. Taking $\mathcal{H}'$ as the eighth component of an octet for simplicity, we can easily derive the following results, which shall prove useful later on.
a) For two distinct octets, denoted by \( a \) and \( b \), we have

\[
a^N_{ab} = 3f^ab - d^ab; \quad a_{-1} = -3f^ab - d^ab; \quad a^\Sigma_0 = -a^\Lambda_0 = 2d^ab,
\]

where \( f^ab \), \( d^ab \) are proportional to the corresponding reduced matrix elements.

b) For an octet, \( a \), and decuplet \( b \),

\[
a^\Sigma_{ab} = a^\Xi_{ab}; \quad \text{no other sub-multiplets can mix.} \quad (2.30)
\]

c) For two distinct decuplets,

\[
a^\Xi_{ab} = 2a^\Xi_{-1} = -2a^\Lambda_{-1}; \quad a^\Xi_{1,1} = 0. \quad (2.31)
\]

A. Gell-Mann Okubo Formula

Let us now return to the derivation of the GMO formula. The leading \( O(e) \) terms of the ETC

\[
\left< p \left| \left[ V^{K^+}_K, A^{K+}_K \right] \right| \Xi^- \right> = 0 \quad (2.32)
\]

fall into distinct groups depending on the multiplet to which the intermediate states belong. Off-diagonal terms can belong to \( 1, 10, 10, 27 \) and higher \( \delta \) multiplets, but we shall consider only the \( 1, \delta \) and \( 10 \) contributions here.

Firstly, however, the diagonal terms arising from \( \Sigma^0 \) and \( \Lambda^0 \) intermediate states yield the expression

\[
2F \left[ 2(N^2 + \Xi^2) - (3 \Lambda^2 + \Sigma^2) \right], \quad (2.33)
\]
where $F$ is the reduced matrix element $\langle \bar{8} \parallel A \parallel \bar{8} \rangle$ for antisymmetric (f) coupling, and e.g. $\Lambda^2$ denotes the squared mass of the $\Lambda$ particle.

Next, using the Wigner-Eckhart theorem, the singlet $\Lambda'$ gives the terms

$$
\langle p | \mathcal{V}_K | \Lambda' \rangle \langle \Lambda' | A^+ | \Xi^- \rangle (\Lambda'^2 - p^2) - \langle p | A^+ | \Lambda' \rangle \langle \Lambda' | \mathcal{V}_K | \Xi^- \rangle (\Xi^2 - \Lambda'^2)
$$

$$
= \frac{\sqrt{3}}{4} a_{\Lambda}^{1,8} g_{\Lambda}^{1,8} (\Lambda'^2 - p^2) + \sqrt{3} a_{\Lambda}^{8,1} g_{\Lambda}^{8,1} (\Xi^2 - \Lambda'^2),
$$

(2.34)

where $g_{\Lambda}^{1,8}$ is the reduced matrix element $\langle 1 \parallel A \parallel \bar{8} \rangle$, and $a_{\Lambda}^{1,8}$ is the $O(\epsilon)$ mixing parameter.

Now, each of the terms in (2.34) is only $O(\epsilon)$, and since we are attempting to show that the expression (2.33) is, in fact, $O(\epsilon^2)$, it would be very helpful to know that the particular combination in (2.34) is also $O(\epsilon^2)$. To this end, consider the ETC

$$
\langle \Xi^- | \left[ \mathcal{V}_K^-, A^+ \right] | p \rangle = 0,
$$

(2.35)

which is simply minus the hermitean conjugate of the first ETC. Once again, the diagonal terms yield the expression (2.33), with exactly the same sign. However, the $\Lambda'$ contribution becomes

$$
- \frac{\sqrt{3}}{4} a_{\Lambda}^{1,8} g_{\Lambda}^{1,8} (\Lambda'^2 - \Xi^2) - \sqrt{3} a_{\Lambda}^{8,1} g_{\Lambda}^{8,1} (p^2 - \Lambda'^2).
$$

(2.36)

If we add the diagonal and $\Lambda'$ off-diagonal contributions to eqs. (2.32) and (2.35), we obtain
\[ 2(N^2 + \Xi^2) - (\Sigma^2 + 3\Lambda^2) + \frac{\sqrt{3}}{4} \alpha_8^{1,8} \Phi_8^{1,8}(\Xi^2 - \mu^2) \]
\[ + \sqrt{\frac{3}{2}} \alpha_8^{8,1} \Phi_8^{8,1}(\Xi^2 - \mu^2) + \beta, \ldots \text{ terms} \]
\[ = 0(\varepsilon^2). \] \tag{2.37}

This time, since \((\Xi^2 - \mu^2)\) is \(O(\varepsilon)\), as are \(\alpha_8^{1,8}\) and \(a_8^{8,1}\), the second and third terms of eq. (2.37) are \(O(\varepsilon^2)\). In the Appendix, we show that exactly the same happens for the octet and decuplet off-diagonal contributions. Accordingly, we can rearrange eq. (2.37) to yield the desired result, viz.

\[ 2(N^2 + \Xi^2) - (\Sigma^2 + 3\Lambda^2) = 0(\varepsilon^2). \] \tag{2.38}

In similar fashion, we can obtain the equal spacing rule for the decuplet, viz.

\[ \Delta^2 - Y_1^2 = Y_1^2 - \Xi^2 = \Xi^2 - \Omega^2. \] \tag{2.39}

Thus, we have proved that the GMO and equal spacing formulas can be derived from the "exotic" ETCs \([V_{K^+}^+, A_{K^+}] = 0\) and \([V_{K^-}^-, A_{K^-}] = 0\) using only the formal perturbation counting technique; the special assumption of the asymptotic symmetry scheme, that matrix elements of \(V_K\) are effectively not renormalised in the infinite momentum frame, is unnecessary.

B. \(\Sigma - \Lambda\) Mass Difference

Now consider the \(\Sigma - \Lambda\) mass difference, as derived from the pair of equations
\[ \langle \mathbf{p} | \left[ \mathbf{V}_{K^+}, A_{\pi^+} \right] \mathbf{\Sigma}^{-} \rangle = 0, \quad (2.40) \]
\[ \langle \Sigma^{-} | \left[ \mathbf{V}_{K^+}, A_{\pi^+} \right] | \mathbf{p} \rangle = 0. \quad (2.41) \]

This time, the cancellation of the leading off-diagonal terms, multiplet by multiplet, is not at all obvious. In fact, the singlet contribution to eq. (2.40) consists of only one term,
\[ \frac{\sqrt{3}}{4} a_{\lambda} g_{1,8} (\Lambda' - p^2), \quad \text{and this is clearly only of order } O(c). \]
The corresponding contribution to eq. (2.41) is \[ \frac{\sqrt{3}}{2} a_{\lambda} g_{1,8} (\Lambda' - p^2), \]
whereas the diagonal term in each case is \[ D(\mathbf{\Sigma}^{-} - \mathbf{\Lambda}^{-}), \]
where \( D \) is the symmetric reduced matrix element \( \langle \mathbf{8} | A \mathbf{8} \rangle \). The previously successful procedure of adding together the individual equations, here (2.40) and (2.41), yields the equation
\[ 2D(\mathbf{\Sigma}^{-} - \mathbf{\Lambda}^{-}) + \left( \frac{\sqrt{3}}{4} a_{\lambda} g_{1,8} + \frac{\sqrt{3}}{2} a_{\lambda} g_{1,8} \right) (\Lambda' - p^2) + \text{etc. states} = 0(c^2). \quad (2.42) \]

However, a comparison of eqs. (2.37) and (2.42) reveals an important difference. Whereas in eq. (2.37), the \( \Lambda' \) contribution is clearly \( O(c^2) \) since it consists of an \( O(c) \) mixing parameter \( a \) multiplied by an \( O(c) \) mass difference, in eq. (2.42), the mass factor is not \( O(c) \) since the two masses involved come from different \( SU(3) \) multiplets.

In this case, if we want the \( \Lambda' \) contribution to be \( O(c^2) \), then we require the following condition,
\[ \frac{\sqrt{3}}{4} a_{\lambda} g_{1,8} + \frac{\sqrt{3}}{2} a_{\lambda} g_{1,8} = 0(c^2). \quad (2.43) \]

If this condition is valid, along with the corresponding ones for
all the other off-diagonal terms, then we deduce from eq. (2.42) that
\[ \Sigma^2 - \Lambda^2 = 0(\varepsilon^2), \]  
(2.44)
i.e. \( \Sigma \) and \( \Lambda \) are degenerate, because the off-diagonal contributions to eq. (2.42) cancel multiplet by multiplet. But substitution of eq. (2.44) in either eq. (2.40) or eq. (2.41) implies that the leading (i.e. \( 0(\varepsilon) \)) off-diagonal contributions there must also cancel. However, they cannot do this multiplet by multiplet as in eq. (2.42), since e.g. the \( \Lambda' \) contribution consists of a single term. Thus, the cancellation mechanism in eqs. (2.40) and (2.41) is different from that in eq. (2.42), and must involve all of the off-diagonal terms simultaneously.

At this stage, it is only fair to point out that although the GMO relation was derived from the sum of eqs. (2.32) and (2.35), it was not derived from either equation individually. In fact, use of eqs. (2.43) and (2.34) shows that the \( \Lambda' \) contribution to eq. (2.32) is
\[ \frac{\sqrt{3}}{4} g_{\Lambda}^{-1,8} g_{\Lambda'}^{-1,8} (2 \Lambda'^2 - p^2 - \Xi^2) + 0(\varepsilon^2) (\Xi^2 - \Lambda'^2), \]
and this is clearly only \( 0(\varepsilon) \). Similar results are obtained for the other off-diagonal multiplets. Thus, the particular condition (eq. (2.43)) which permits the cancellation of the leading \( \Lambda' \) contributions to the \( \Sigma - \Lambda \) mass difference (and consequently allows the \( \Sigma \) and \( \Lambda \) to be degenerate) prevents the same form of cancellation, multiplet by multiplet, in the individual equation for the GMO formula, i.e. just as in the \( \Sigma - \Lambda \) case, the cancellation of the leading off-diagonal terms (which must occur, because of eq. (2.38)) does not occur separately in each multiplet, but only via
some overall mechanism. Such a situation is not very satisfactory.

On the other hand, if eq. (2.43) is not valid, then $\Sigma^2 - \Lambda^2$ is not $O(\epsilon^2)$, and there is no necessity for the overall cancellation in eqs. (2.40) and (2.41) (unless a similar mechanism is in operation in eq. (2.42)): thus, consistency has been restored between the individual equations (2.40), (2.41) and the sum equation (2.42). Can we achieve the same consistency for the equations involved in the GMO derivation?

To answer this, we must find the condition for each multiplet's leading contributions to cancel separately in eqs. (2.32) and (2.35), since this is exactly what happens in the sum of the two equations. This condition is

$$\frac{\sqrt{3}}{4} a_x^1,8 g^1,8 - \frac{\sqrt{3}}{2} a_x^8,1 g^8,1 = O(\epsilon^2), \quad (2.45)$$

and clearly, eqs. (2.45) and (2.43) are incompatible except in the trivial case.

Thus, if we accept the conditions defined by eq. (2.45) and the corresponding equations for the other off-diagonal terms, then the leading off-diagonal contributions to the GMO formula always cancel, multiplet by multiplet, whereas they cannot possibly do so for the $\Sigma - \Lambda$ mass difference. Such consistency between the individual equations and the corresponding sums, in both cases, seems much more attractive than the inconsistencies arising when we demand that the $\Sigma$ and $\Lambda$ should be degenerate, and that this degeneracy should be realised in exactly the same way as the GMO formula is derived.
To conclude this section, we emphasise that the GMO formula can be derived independently of the condition defined by eq. (2.45), whereas the $\Sigma - \Lambda$ degeneracy arises only if eq. (2.43) is correct. The above discussion shows that there is much more self-consistency among the equations if we accept eq. (2.45) in preference to eq. (2.43), i.e. if we do not demand the degeneracy of the $\Sigma$ and $\Lambda$. The basis of the difference between the GMO and $\Sigma - \Lambda$ cases is the "symmetry" of the $V$ and $A$ indices in the ETCs used in the derivations; in eqs. (2.32) and (2.35), exactly the same intermediate states contribute to both sides of the equation, since $V$ and $A$ have the same SU(3) quantum numbers, whereas in eqs. (2.40) and (2.41), the lack of this symmetry means that different states appear, and any possibility of cancellation is thereby reduced.

C. Intermultiplet Mass Formulas

When we attempt to derive intermultiplet formulas by sandwiching ETCs like $\left[ V^+_K, A^+_K \right]$ between a pair of states belonging to different multiplets, the lack of symmetry, this time between the states, again inhibits cancellation.

As an illustrative example, consider the equations

$$\langle p | \left[ V^+_K, A^+_K \right] | \Xi^- \rangle = 0 \quad \text{and} \quad (2.46)$$
$$\langle \Xi^- | \left[ V^-_K, A^-_K \right] | \Delta^+ \rangle = 0 . \quad (2.47)$$

The diagonal terms in eqs. (2.46) and (2.47) are, respectively,
\[ \frac{1}{\sqrt{30}} g^{8,10}(\Sigma^2 - 3\lambda^2 + 2p^2) + \frac{2}{\sqrt{30}} g^{8,10}(\Xi^* - \Xi^0) \]

\[ \approx \frac{2}{\sqrt{30}} g^{8,10}\left[(\Sigma^2 - \Xi^2) - (\Xi^0 - \Xi^2)\right] \] ; \hspace{1cm} (2.48)

and

\[ \frac{2}{\sqrt{30}} g^{8,10}\left[(\Sigma^2 - \Xi^2) - (\Delta^2 - \Xi^0)\right] \] , \hspace{1cm} (2.49)

where \( g^{8,10} \equiv \langle 8 \mid A \mid 10 \rangle \), and we have used the GMO formula to simplify the octet terms.

The contributions from another decuplet state are, respectively

\[ \langle p|\vec{v}^K|\gamma_1^0\rangle\langle\gamma_1^0|\vec{v}^K|\Xi^2\rangle - \langle p|\vec{v}^K|\gamma_1^0\rangle\langle\gamma_1^0|\vec{v}^K|\Xi^*\rangle \]

\[ \approx - c^{10',8}_{\Xi} H^{10',10}(\gamma_1^2 - p^2) + \frac{2}{\sqrt{30}} c^{10,10'}_{\Xi} g^{8,10'}(\Xi^2 - \gamma_1^2) , \hspace{1cm} (2.50) \]

and

\[ - c^{10',8}_{\Xi} H^{10',10}(\gamma_1^2 - \Xi^2) + \frac{2}{\sqrt{30}} c^{10,10'}_{\Xi} g^{8,10'}(\Delta^2 - \gamma_1^2) , \hspace{1cm} (2.51) \]

where \( H^{10',10} \equiv \langle 10' \mid A \mid 10 \rangle \), and \( g^{8,10'} \equiv \langle 8 \mid A \mid 10' \rangle \).

Now, inspection of eq. (2.50) indicates that because masses from three separate multiplets are involved, we cannot possibly arrange any cancellation of the leading terms, as we did in eq. (2.31). Furthermore, there is no group theoretical connection between the reduced matrix elements \( H^{10',10} \) and \( g^{8,10'} \), so that such an arrangement is even more unlikely. Consequently, we are unable to derive an intermultiplet equal-spacing formula,
\[
\sum^2 - \Xi^2 \approx Y_1^2 - \Xi^*^2 \quad . \tag{2.52}
\]

However, subtraction of the appropriate terms of eqs. (2.46) and (2.47) gives the equation
\[
\frac{2}{\sqrt{30}} \, g^{8,10} \left[ (\Delta^2 - Y_1^2) - (Y_1^2 - \Xi^2) \right] - a^{10',8}_{\Xi} h^{10',10}(\Xi^2 - p^2)
\]
\[
+ \frac{2}{\sqrt{30}} \, a^{10,10'}_{\Xi} g^{8,10'}(\Xi^*^2 - \Delta^2) + \text{octet contributions} = 0(\varepsilon^2). \tag{2.53}
\]

Here, the two \(10'\) terms are separately \(0(\varepsilon^2)\); similar results for the octet terms allow us to re-derive the equal-spacing rule internally for the decuplet. But addition of the equations does not yield a useful intermultiplet formula, confirming the result of our inspection of eq. (2.50).

The use of ETCs like
\[
\langle p|\left[\bar{V}^+, V^+\right]|\Xi^-^- \rangle = 0 \quad . \tag{2.54}
\]
also fails to produce intermultiplet rules. In more detail, we note that the leading terms here are
\[
\langle p|V^+|Y_1^0\rangle \langle Y_1^0|V^+|\Xi^-^- \rangle - \langle p|V^+|\Sigma^0\rangle \langle \Sigma^0|V^+|\Xi^-^- \rangle - \Sigma^0 \to \Lambda^0 = 0(\varepsilon^2). \tag{2.55}
\]
\[
\therefore \quad a^{10,8}_{\Xi}(Y_1^2 - p^2) + \frac{1}{2}(a^{10,8}_{\Xi} - 2a^{10,8}_{\Xi})(\Xi^*^2 - \Sigma^2) + \frac{3}{2} \, a^{10,8}_{\Xi}(\Xi^*^2 - \Lambda^2)
\]
\[
= \frac{1}{2} \, a^{10,8}_{\Xi} \left[ (2p^2 + \Sigma^2 - 3\Lambda^2) + 2(\Xi^*^2 - Y_1^2) \right] \]
\[
\approx a^{10,8}_{\Xi} \left[ (\Xi^2 - \Xi^2) - (Y_1^2 - \Xi^*^2) \right] = 0(\varepsilon^2), \quad (2.56)
\]
where we use the condition \( a_{\Sigma}^{10,8} = a_{\Xi}^{10,8} \) from eq. (2.30), and also the GMO formula. But eq. (2.56) tells us nothing new, as \( (\Sigma^2 - \Xi^2) \), \( (y_1^2 - \Xi^2) \) and \( a_{\Xi}^{10,8} \) are each \( O(\varepsilon) \). Other examples confirm that \([V^k, V^l] = 0\) ETGs do not produce new results when sandwiched between states from different multiplets.

Hence, useful intermultiplet mass sum rules cannot be derived from \([V, V]\) ETGs, nor from those \([V, A]\) ETGs which yielded 'good' results for intramultiplet formulas; in the latter case, this is because of the lack of symmetry between the reduced matrix elements and also between the masses. As for the ETGs which were unsuccessful before, e.g. \([V^k, A^{\pi^+}] = 0\), further difficulties arise when they are placed between off-diagonal states. Accordingly, we are forced to the conclusion that intermultiplet mass sum rules cannot be derived by these methods when no special assumptions are made regarding the off-diagonal matrix elements of \( V^k \).

In order to obtain the results of the asymptotic symmetry scheme, we simply neglect all terms involving the mixing parameters \( a_i \) from eq. (2.34) onwards. In this case, both the GMO formula and the equal-spacing rule (eqs. (2.38), (2.39)) are derived immediately; however, \( \Sigma^0 \) and \( \Lambda^0 \) are now obviously degenerate (all off-diagonal contributions in eq. (2.42) vanish), and eq. (2.46) yields the result

\[
(\Sigma^2 - \Xi^2) - (y_1^2 - \Xi^2) = O(\varepsilon^2),
\]

which is not well satisfied experimentally since the two factors involved are \(- 0.32 \text{ GeV}^2\) and \(- 0.46 \text{ GeV}^2\).

So far, only the baryons have appeared in our equations. However, exactly the same arguments apply to the mesons, although slightly
more care must be taken with the matrix elements to ensure that the
couplings are not forbidden by considerations such as $G$ invariance.
Nevertheless, the same broad conclusions are drawn, that intermutil-
plet mass formulas can be derived only under the stronger assumptions
of the asymptotic symmetry scheme. As pointed out earlier in Section
III, the results are not exceptionally good either.

D. Concluding Remarks

We have shown that it is possible to derive the usual broken
symmetry sum rules using the purely formal counting technique, but
that the poorer predictions, e.g. the $\Sigma^0 - \Lambda^0$ degeneracy and
intermultiplet mass formulas cannot be so derived. In fact, one
ought not to be surprised at this failure to obtain any effective
intermultiplet formulas when only group theoretical arguments are
involved. Exact $SU(3)$ symmetry does not relate the mean masses of
different multiplets to each other, so we should not expect any
direct correspondence between the mass differences arising in one
multiplet and the differences in any other multiplet.

In more detail, when the $SU(3)$ symmetry is broken by a term
$\epsilon O^8$ in the Hamiltonian, where $\epsilon$ is a small parameter, and $O^8$
denotes the eighth component of an octet operator $O^8$, the $(mass)^2$
of the $i$th state belonging to the $\mu$ IR is

$$m_i^2(\mu) = \langle \psi_\mu^i | \mathcal{H}_o + \epsilon O^8 | \psi_\mu^i \rangle$$

$$= \langle m^2(\mu) \rangle + \epsilon C_{\mu 8}^1 \langle \mu || O^8 || \mu \rangle,$$  \hspace{1cm} (2.58)

where $\mathcal{H}_o$ is invariant under $SU(3)$, $\langle m^2(\mu) \rangle$ is the mean $(mass)^2$
of the \( \mu \) IR, \( C \) is the appropriate SU(3) Clebath-Gordon coefficient, and \( \langle \mu | 0_\delta | \mu \rangle \) is the reduced matrix element of the octet operator \( 0_\delta \) between \( \mu \) states. Since the value of \( \langle \mu | 0_\delta | \mu \rangle \) is determined by dynamical considerations and not by group theory, we cannot relate the reduced matrix elements for different IRs without a knowledge of the dynamics, and so cannot estimate a priori the magnitude of the mass splittings within the different multiplets.

Hence, since only the relatively poorer results of the asymptotic symmetry scheme require the use of the stronger assumptions concerning the non-renormalisation of \( v^K \) matrix elements, it does not seem too unreasonable to conclude that the assumptions of the scheme are too strong, and perhaps unnecessary. However, to end on a slightly brighter note, the techniques of the scheme, i.e. the use of ETCs and the infinite momentum frame, certainly are useful, especially with regard to the question of the basic and effective symmetry breaking parts of the Hamiltonian.

Appendix

In this appendix, we show the cancellation mechanism for the leading (i.e. order \( O(\epsilon) \)) contributions for higher octet and decuplet intermediate states in the derivation of the GMO formula.

The decuplet contributions to \( \langle p | [v^K, A^K] | \Xi^- \rangle \) are
\[
\langle p | v^K^+ | \lambda_1^0 \rangle \langle \lambda_1^0 | \Lambda K^+ | \Xi^- \rangle - \langle p | \Lambda K^+ | \lambda_1^0 \rangle \langle \lambda_1^0 | v^K^- | \Xi^- \rangle
\]
\[
\approx a_{\Xi}^{10,8} (\frac{1}{\sqrt{2}})(-\frac{1}{2/3} g^{10,8}(Y_1^2 - p^2) - (a_{\Xi}^{8,10}(\sqrt{2}) - a_{\Xi}^{8,10}(\frac{1}{\sqrt{2}})) \times
\]
\[
(- \frac{1}{\sqrt{15}} g^{8,10})(\Xi^2 - Y_1^2)
\]
\[
= (\sqrt{6}/12) a_{\Xi}^{10,8} g^{10,8}(Y_1^2 - p^2) + (\sqrt{30}/30) a_{\Xi}^{8,10} g^{8,10}(\Xi^2 - Y_1^2). \quad (A1)
\]

We have used the fact that \( a_{\Xi}^{8,10} = a_{\Xi}^{8,10} \).

The corresponding terms for the ETC \( \langle \Xi^- | \left[ v^K^-, \Lambda K^- \right] | p \rangle \) are
\[
\langle \Xi^- | v^K^- \rangle \langle \lambda_1^0 | \Lambda K^- | p \rangle - \langle \Xi^- | \Lambda K^- \rangle \langle \lambda_1^0 | v^K^- | p \rangle
\]
\[
\approx (a_{\Xi}^{10,8} \frac{1}{\sqrt{2}} - a_{\Xi}^{10,8} \sqrt{2}) \times (1/2\sqrt{3}) g^{10,8}(Y_1^2 - \Xi^2) - (a_{\Xi}^{8,10})(-\frac{1}{\sqrt{2}}) \times
\]
\[
(1/\sqrt{15}) g^{8,10}(p^2 - Y_1^2)
\]
\[
= - (\sqrt{6}/12) a_{\Xi}^{10,8} g^{10,8}(Y_1^2 - \Xi^2) - (\sqrt{30}/30) a_{\Xi}^{8,10} g^{8,10}(p^2 - Y_1^2). \quad (A2)
\]

The sum of these terms is
\[
(\sqrt{6}/12) a_{\Xi}^{10,8} g^{10,8}(\Xi^2 - p^2) + (\sqrt{30}/30) a_{\Xi}^{8,10} g^{8,10}(\Xi^2 - p^2). \quad (A3)
\]

Clearly, both parts of this expression are of order \( O(\varepsilon^2) \), as required.

For the octet intermediate states, the situation is complicated by three factors, viz. the presence of two separate intermediate
states $\Sigma_j^0$ and $\Lambda_j^0$, the two types of coupling allowed, $D^{ij}$ and $F^{ij}$, and (thirdly) the fact that all of the states can mix, and with parameters $a^{ij}$ which have two components, $d^{ij}$ and $f^{ij}$. (The indices $i, j$ refer to the two different octets. To simplify the notation we shall define $a^{ij} = a$, and $a^{ji} = a^T$, and similarly for the other terms.)

From $\langle p_1 \left| v_{K^+}, A_{K^+} \right| \equiv_1 \rangle = 0$, the $\Sigma_j^0$ and $\Lambda_j^0$ terms are

$$-\frac{1}{2}(a_{\Sigma}^T - a_{\Lambda}^T)(D + F)(\Sigma_j^2 - p_1^2) + \frac{1}{2}(a_{\Lambda}^T - a_{\Sigma}^T)(D^T - 3F^T)(\Lambda_j^2 - p_1^2)$$

$$+ \frac{1}{2}(a_{\Lambda} - a_{\Sigma})(D - F)(\equiv_2^1 - \Sigma_j^2) - \frac{1}{2}(a_{\Lambda} - a_{\Sigma})(D + 3F)(\equiv_1^2 - \Lambda_j^2). \quad (A4)$$

From $\langle \equiv_1^2 \left| v_{K^+}, A_{K^-} \right| p_1 \rangle = 0$, the corresponding terms are

$$\frac{1}{2}(a_{\Sigma}^T - a_{\Lambda}^T)(D^T - F^T)(\Sigma_j^2 - \equiv_1^2) - \frac{1}{2}(a_{\Lambda}^T - a_{\Sigma}^T)(D^T + 3F^T)(\Lambda_j^2 - \equiv_1^2)$$

$$- \frac{1}{2}(a_{\Lambda} - a_{\Sigma})(D + F)(p_1^2 - \Sigma_j^2) + \frac{1}{2}(a_{\Lambda} - a_{\Sigma})(D - 3F)(p_1^2 - \Lambda_j^2). \quad (A5)$$

Adding eqs. (A4) and (A5), making use of eqs. (2.29) to relate the different $a$'s, and neglecting terms such as

$$(a_{\Sigma} D - a_{\Sigma}^T D^T)(\equiv_1^2 - p_1^2)$$

which are obviously of order $O(\varepsilon^2)$, we arrive at the following expression:
\[
(\alpha_\Lambda F - \alpha_\Lambda F^T)(2 \sum_j - p_1^2 - \frac{\epsilon^2}{2}) + 3(\alpha_\Lambda F - \alpha_\Lambda F^T)(2 \sum_j - p_1^2 - \frac{\epsilon^2}{2})
\]

\[
- (\alpha_N F - \alpha_N F^T)(\sum_j - p_1^2 + 4p_1^2) - (\alpha_\Xi F - \alpha_\Xi F^T)(\sum_j - 2p_1^2 - \frac{\epsilon^2}{2})
\]

\[
= 2(dF - dF^T) \left[2(\sum_j - 3p_1^2 + p_1^2 + \frac{\epsilon^2}{2}) + (\sum_j - 3p_1^2 - 2p_1^2 - \frac{\epsilon^2}{2})\right]
\]

\[
+ 3(fF - fF^T) \left[4p_1^2 - \frac{\epsilon^2}{2}\right]
\]

\[
= 6(dF - dF^T)(\sum_j - p_1^2) + 12(fF - fF^T)(p_1^2 - \frac{\epsilon^2}{2})
\]

\[
= 0(\epsilon^2) .
\]

Hence, the leading terms from the off-diagonal singlet, octet, and decuplet intermediate states separately cancel. The proofs for the anti-decuplet and 27-plet follow similar lines.
CHAPTER III

AN EXTENSION OF THE EQUAL-TIME COMMUTATOR FORMALISM

I. Introduction

In the previous chapter, the strong assumption of the Asymptotic Symmetry Scheme — that matrix elements of $V^K$ are effectively not renormalised, even in broken SU(3) — was criticised as being unnecessarily restrictive. However, the techniques employed in deriving the mass formulae are themselves quite useful; these are (1) the use of equal time commutators (ETCs) involving the time derivative of $V^K$, the SU(3) "stepping up" operator, and (2) the use of the infinite momentum frame. In the present chapter, we extend the commutator formalism to include non-vanishing ETCs. Although this has the disadvantage that a specific form for the Hamiltonian $H$ must be assumed, we can turn this explicit appearance of $H$ to good account by investigating some of its dynamical properties in addition to the usual group theory relations.

The main result which we obtain is a more carefully worded statement of the phenomenon of octet dominance than we could make from considerations involving only vanishing ETCs; it is single-particle matrix elements of the effective SU(3) symmetry-breaking Hamiltonian which are octet dominated, and not the Hamiltonian itself. A subsidiary result is that the $d/f$ ratios for the couplings of the $1/2^+$ octet baryons to the medium strong Hamiltonian and the parity conserving part of the non-leptonic weak Hamiltonian are approximately equal.
These are certainly not new results\textsuperscript{22-24}, but their derivation here is somewhat different from usual in that we use the effective Hamiltonian as a Quantum Mechanical operator to calculate the mass splittings in the octet rather than employ a Quantum Field Theory perturbation expansion involving the basic interaction Hamiltonian.

We begin with a few comments on the ETC formalism and the use of the rest and infinite momentum frames, as applied to the derivation of mass formulae. Next, we consider the differences between the approach involving vanishing ETCs and that employing non-vanishing commutators, and show how the latter method leads to a more correct formulation of octet dominance. Finally, we discuss briefly the hypothesis that the octet parts of the strong, electromagnetic and parity conserving non-leptonic weak Hamiltonians all belong to the same octet.

II. Basic Formalism

The essential point of the ETC formalism is that the one-particle states are treated as eigenstates of the effective Hamiltonian, with the energy eigenvalues involving the physical masses of the corresponding particles.

As an illustration, consider the matrix element of $\not{v}^K$ between $\langle n \mid$ and $\mid \Sigma \rangle$ states.

\begin{align*}
-1 \langle n(p', r) \mid \not{v}^K \mid \Sigma^- (p, s) \rangle &= \langle n \mid [H, v^K] \mid \Sigma^- \rangle \\
&= (E_n - E_\Sigma) \int d^3 x \langle n \mid v^K_o (x, 0) \mid \Sigma^- \rangle \\
&= (E_n - E_\Sigma) \cdot (2\pi)^3 \delta^3 (p - p') \cdot \bar{u}_\Sigma (p, s) \left[ F_+ (q^2) \gamma_0 + F_- (q^2) \sigma_{\mu} q^\mu \right] u_n (p', r) \\
&= (2\pi)^3 \delta^3 (p - p') \cdot (E_n - E_\Sigma) (\bar{u}_\Sigma \gamma_0 u_n) F_+ (q^2). \\
\end{align*} (3.1)
In the rest frame, \( q^2 = (M_n - M_\Sigma)^2 \), i.e. \( q^2 \) is of order \( O(\varepsilon^2) \), so that

\[
F_+(q^2) = F_+(0) (1 + O(\varepsilon^2)).
\]

Then the leading term on the rhs of eq. (3.1) (apart from the factor \( (2\pi)^3 \delta^3(p-p') \)) is

\[
(M_n - M_\Sigma) \cdot 2(M_n M_\Sigma)^{1/2} F_+(0).
\]

At this point, we could assume that the linear mass difference term is \( O(\varepsilon) \) so that we could then write \( 2(M_n M_\Sigma)^{1/2} \approx 2M_\Sigma \), where \( M_\Sigma \) is the mean mass of the octet: this leads to linear mass formulae. Alternatively, we could obtain rather awkward mass formulae containing fractional powers if we insist on retaining all mass factors explicitly.

In the infinite momentum frame, on the other hand, \( q^2 \rightarrow 0 \) and

\[
(E_n - E_\Sigma)^\nu_n \gamma_\nu \ll_L \rightarrow \frac{1}{2} \left[ 1 + \frac{M_n^2}{2E_n^2} - 1 - \frac{M_\Sigma^2}{2E_\Sigma^2} + O(1/E_n^4) \right] x 2 |E_n| (1 + O(1/E_n^2))
\]

\[
\rightarrow M_n^2 - M_\Sigma^2.
\]

Thus, when \( |E_n| \rightarrow \infty \), the only mass term is \( (M_n^2 - M_\Sigma^2) \) so that quadratic mass formulae appear to be the most obvious; however, once again, we could easily obtain linear formulae.

Similar considerations apply to the \( 0^- \) mesons, except that in both special frames, the leading terms involve \( (\pi^2 - K^2) \) as the only mass factor.
The above discussion has been given simply to show that the formalism used here can lead easily to either linear or quadratic mass formulae; in practice, the difference between the two types is negligible, apart from the case of the exceptionally low mass 0\(^{-}\) mesons.

III. Octet Dominance

We come now to the differences arising when we choose a non-vanishing ETC to derive mass relations instead of using a vanishing commutator. The situation is probably simplest to explain if we look at a particular ETC, e.g. \([\mathbf{\mathcal{K}}^{+}, \mathbf{\mathcal{V}}^{+}] = 0\). When this is sandwiched between \(\langle \mathbf{p} \rangle\) and \(\langle \Xi \rangle\) states at infinite three-momentum, we can immediately derive the quadratic Gell-Mann - Okubo (GMO) formula for the \(\frac{1}{2}^{+}\) baryon octet, viz.

\[
2(N^2 + \Xi^2) - (\Sigma^2 + 3\Lambda^2) = 0(\epsilon^2),
\]

where, e.g. \(N^2 - \Sigma^2 = 0(\epsilon)\),

and \(\epsilon\) is the small symmetry breaking parameter.

This is an example of a vanishing ETC: however, the rhs of the commutator equation disappears only when certain assumptions are made concerning the Hamiltonian; in this case, if \(\mathcal{H}\) belongs only to an octet, then the ETC vanishes. It appears that this is simply the assumption of octet dominance, an observed phenomenon in different mass spectra, and so we are reasonably well justified in using it as input information.

In general, we can break the SU(3) symmetry of the Hamiltonian down to SU(2) \(\oplus\) U(1) by adding on a term which transforms as a
linear combination of the $I = 0, Y = 0$ members of different
SU(3) multiplets (with the exception of the singlet, of course).
When we calculate $\chi^{K+}$ from the ETC $i[H, V^{K+}]$, the different
multiplet components of $H$ are transformed into the corresponding
strange members of their own multiplets; this follows immediately
by definition since the terms are assigned to the different multi-
plets according to their transformation properties with the SU(3)
generators:

$$\left[V^{K+}, O_1(Y; I, I_3)\right] = k_+ O_1(Y+1; I^{+\frac{1}{2}}, I_3^{+\frac{1}{2}}) + k_- O_1(Y+1; I^{-\frac{1}{2}}, I_3^{+\frac{1}{2}}),$$

(3.3)

where the coefficients $k_{\pm}$ are defined in the usual way in
terms of $Y, I, I_3$ and $(\lambda, \mu)$, with $(\lambda, \mu)$ specifying the
particular representation.

Next, when we evaluate the 'exotic' ETC $i[V^{K+}, V^{K+}]$, the
various $O_1^{K+}$ are transformed into the corresponding $O_1^{K_{++}}$, where
$O_1^{K+}$ and $O_1^{K_{++}}$ denote the operators with $Y = 1, I = I_3 = \frac{1}{2}$ and
$Y = 2, I = I_3 = 1$ respectively. In certain cases, e.g. in the
octet, $O_1^{K_{++}}$ does not exist, so that the particular multiplet in
question does not contribute to the exotic ETC. This can be seen
most easily if we work in terms of V-spin, since $V^{K+}$ is the
V-spin stepping down operator. In the octet, $O_1^{K_{++}} = O_8 (Y = 0; I = I_3 = 0)$ is a linear combination of a $V = 0$ and a $V = 1,$
$V_3 = 0$ term, so that only one operation by $V^{K+}$ is possible: in
the 27plet, however, we have also a $V = 2, V_3 = 0$ term, so that
the double operation is possible here.

Thus, in general,

$$\left[V^{K+}, V^{K+}\right] = \sum_i C_1^{K_{++}},$$

(3.4)
where \( i \) runs over the various "symmetric" multiplets excluding the singlet and octet. The vanishing of this ETC means that the symmetry-breaking Hamiltonian contains only an octet term.

However, this is all the information we can gain about the Hamiltonian when the ETC vanishes. On the other hand, if we take different SU(3) quantum numbers on the vector charges so that, even for only an octet term in \( H \), the exotic ETC does not vanish, then we can relate the mass differences to the one particle matrix elements of the Hamiltonian.

As an illustration, consider the following form for \( H_{\text{ms}} \), the SU(3) symmetry-breaking part of the Hamiltonian density:

\[
H_{\text{ms}} = 0_8^{\gamma} + 0_{27}^{\gamma}. \tag{3.5}
\]

Then an "exotic" ETC which contains an octet term is

\[
- i \left[ \hat{\mathbf{V}}^+ \mathbf{K}^- \right] = \int d^3 \mathbf{x} \left[ \frac{3}{2} 0_{27}^0 (\mathbf{x}, 0) + 2 0_{27}^0 (\mathbf{x}, 0) \right]. \tag{3.6}
\]

where \( 0_{27}^0 = 0_4 (Y = 1; I = \frac{1}{2}, I_3 = -\frac{1}{2}), \quad I = 8, 27. \)

Sandwiching both sides of eq. (3.6) between single-particle \( \langle n \mid \) and \( \mid \Sigma^0 \rangle \) states and letting \( |\mathbf{p}| \to \infty \), we obtain for the lhs,

\[
-i \langle n \mid \left[ \hat{\mathbf{V}}^+ \mathbf{K}^- \right] \mid \Sigma^0 \rangle \approx -i \langle n \mid \mathbf{v}^+ \mathbf{K}^- \mid \Sigma^- \rangle \langle \Sigma^- \mid \mathbf{v}^- \mathbf{K}^- \mid \Sigma^0 \rangle + i \langle n \mid \mathbf{v}^- \mid \mathbf{p} \rangle \langle \mathbf{p} \mid \mathbf{v}^+ \mathbf{K}^- \mid \Sigma^0 \rangle
\]

\[
= \left[ \sqrt{2} \left( \Sigma^{-2} - n^2 \right) - \frac{1}{\sqrt{2}} \left( \Sigma^{02} - p^2 \right) \right] (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')
\]

\[
\approx \frac{1}{\sqrt{2}} \left( \Sigma^{-2} - n^2 \right) (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}'), \tag{3.7}
\]
where we assume $\Sigma^2 = \Sigma^0$, $n^2 = p^2 = N^2$, since $SU(2)$ is still a good symmetry.

Also, using the Wigner-Eckhart theorem to express the rhs in a more useful form,

$$\langle n | \text{rhs} | \Sigma^0 \rangle = \frac{1}{2\sqrt{2}} \left[ (-\frac{3}{\sqrt{5}} D + F) + \frac{8\sqrt{5}}{45} G \right]. (2\pi)^{3/2} (p \cdot p'),$$

(3.8)

where $G = \langle B_8 \parallel 027 \parallel B_8 \rangle$ is the reduced matrix element for a 27-plet operator between two octet baryon states and $D, F$ are the corresponding elements for an octet operator.

Thus,

$$\Sigma^2 - N^2 = \frac{1}{4} \left[ -\frac{3}{\sqrt{5}} D + F + \frac{8\sqrt{5}}{45} G \right].$$

(3.9)

In similar fashion, we obtain the following relations:

$$\Lambda^2 - N^2 = \frac{1}{4} \left[ \frac{1}{\sqrt{5}} D + F + \frac{8\sqrt{5}}{15} G \right],$$

(3.10)

$$\Xi^2 - \Sigma^2 = \frac{1}{4} \left[ \frac{3}{\sqrt{5}} D + F - \frac{8\sqrt{5}}{45} G \right],$$

(3.11)

$$\Xi^2 - \Lambda^2 = \frac{1}{4} \left[ -\frac{1}{\sqrt{5}} D + F - \frac{8\sqrt{5}}{45} G \right].$$

(3.12)

From these relations, we can find numerical values for $F$ and $G$ immediately:

$$F = \Xi^2 - N^2 \approx 0.856 \text{ GeV}^2,$$

(3.13)

$$-\frac{8\sqrt{5}}{9} G = 2(\Xi^2 + N^2) - (\Sigma^2 + 3\Lambda^2) \approx 0.078 \text{ GeV}^2,$$

(3.14)

where we have used the average values of the squared masses for each I-spin multiplet.$^{21}$ $D$ can be found from any of the individual
equations now, and the appropriate numerical factors may be taken thus:

\[ F = 0.856, \quad 3D/\sqrt{5} = -0.243; \quad 8\sqrt{5} 0/15 = -0.047. \quad (3.15) \]

It is clear that the 27 contribution is very much smaller than that of the octet, i.e., we have a form of octet dominance. However, what we have demonstrated by the above procedure is that one particle matrix elements of the effective Hamiltonian are octet dominated in that the couplings of the baryons to the octet part of \( \mathcal{H} \) are appreciably greater than those to the 27 part. This version is certainly not equivalent to the statement that the Hamiltonian itself is octet dominated.

In fact, the last form of octet dominance is manifestly not true, for the following reason.\textsuperscript{24} We separate the (medium strong) Hamiltonian density into the octet part \( \mathcal{H}_{\text{MS}}^{(8)} \) and the non-octet part \( \mathcal{H}_{\text{MS}}^{'} \), and define the divergence of the vector current \( v_{K^+}^{\mu} \) in the usual manner, viz.

\[ \partial_{\mu}v_{K^+}^{\mu}(x) = i \left[ \mathcal{H}(x), v_{K^+}^{\mu} \right]. \quad (3.16) \]

Next, we consider S-matrix elements of this divergence: the difference between the present case and that involving single particle matrix elements is that the four-momenta of the initial and final states are equal now, not just the space components of the momenta. Accordingly, the lhs of eq. (3.16) has vanishing S-matrix elements, leading to the condition

\[ \langle f | \left[ \mathcal{H}_{\text{MS}}^{(8)}, v_{K^+}^{\mu} \right] | i \rangle = -\langle f | \left[ \mathcal{H}_{\text{MS}}^{'}, v_{K^+}^{\mu} \right] | i \rangle, \quad (3.17) \]
where \( |i\rangle \) and \( \langle f| \) are the initial and final states respectively. Obviously, we cannot have octet dominance of \( H_{ms} \) by itself, but only in certain couplings of \( H_{ms} \), e.g. to single particle states.

Thus, by extending the commutator formalism to include those ETCs which generally do not vanish, we have rederived this more restricted form of octet dominance. When vanishing ETCs are used, the stronger (and incorrect) statement of the phenomenon has already been employed implicitly: this does not have any appreciable effect on the mass formulae although considerable confusion can arise in the more general situation.

### IV. Similarity of the Octet Parts of \( H_{ms} \) and \( H_{PC} \)

A second useful piece of information obtained from the above analysis is the \( d/f \) ratio for the coupling of the octet part of \( H_{ms} \) to the \( \frac{1}{2}^+ \) octet baryons. This ratio is characteristic of the particular octet to which \( H_{ms} \) belongs, since other octet operators will, in general, couple to these same states with different values for both \( d \) and \( f \); the difference in magnitude of the two \( f \) values, for example, simply reflects the different strengths of the couplings, and this can be factored out into a coupling constant: but the ratio of \( d \) to \( f \) is another independent quantity, and varies from one octet operator to another.

In the present case, we have found \( d/f = (3D/\sqrt{5})/F \approx -0.3 \), a value which closely resembles that for the corresponding couplings of the parity-conserving non-leptonic weak Hamiltonian \( H_{PC} \). This last result is calculated from the rates for the \( s \)-wave non-leptonic hyperon decays in the following way.\(^2\), \(^{26}\)
The first order S-matrix element $\mathcal{M}$ for the s-wave decay $\Lambda^0 \rightarrow p\pi^-$ is given by

$$\mathcal{M} = \langle p(k_2) \pi^-(k_1) | \mathcal{H}^{\text{PC}}_\omega | \Lambda^0(k) \rangle . \quad (3.18)$$

The standard current-algebra procedure involving PCAC and soft pions gives

$$\mathcal{M} \approx - i (\sqrt{2} f_\pi)^{-1} \langle n(k_2) | \mathcal{H}^{\text{PC}}_\omega | \Lambda^0(k) \rangle , \quad (3.19)$$

where $f_\pi (\approx 130 \text{ MeV})$ is the pion decay constant. An alternative definition of the s- and p-wave amplitudes $A, B$ respectively is

$$\mathcal{M} \equiv i \bar{u}_{p(k_2)} (A + BY_5) u_{\Lambda^0}(k) . \quad (3.20)$$

Therefore,

$$\langle n | \mathcal{H}^{\text{PC}}_\omega | \Lambda^0 \rangle \approx - \sqrt{2} f_\pi \bar{u}_p u_{\Lambda^0} A(\Lambda^0)$$

$$\rightarrow - \sqrt{2} f_\pi (M_p + M_\Lambda) \cdot A(\Lambda^0) \text{ as } |k| \rightarrow \infty . \quad (3.21)$$

Similarly, we can relate $\langle n | \mathcal{H}^{\text{PC}}_\omega | \Sigma^- \rangle$ to $A(\Sigma^-)$ and $\langle p | \mathcal{H}^{\text{PC}}_\omega | \Sigma^+ \rangle$ to $A(\Sigma^+)$, where, in the standard notation for the amplitudes $A$, the upper index refers to the charge of the decaying hyperon and the lower index to the charge of the pion.

Now, the strangeness-changing part of $\mathcal{H}^{\text{PC}}_\omega$, which is usually given in the form$^{26}$

$$\mathcal{H}^{\text{PC}}_\omega(x)(\Delta S = \pm 1) = \frac{g}{\sqrt{2}} \sin \theta \cos \theta \left[ \{v^+_{\mu}(x), v^-_{\mu}(x)\} + \{v^+_{\mu}, v^-_{\mu}\} + v_{\mu} \rightarrow A_{\mu} \right] , \quad (3.22)$$

can be expressed as a combination of octet and 27-plet operators
\( T_8 \) and \( T_{27} \):

\[
\mathcal{H}_w^{\text{PC}}(\Delta S = +1) = 3 \, T_8(Y = 1; I = \frac{1}{2}, I_3 = -\frac{1}{2}) + T_{27}(Y = 1; I = \frac{1}{2}, I_3 = -\frac{1}{2}) + \sqrt{5} \, T_{27}(Y = 1; I = \frac{3}{2}, I_3 = -\frac{1}{2}). \tag{3.23}
\]

Thus, we can express the above single particle matrix elements of \( \mathcal{H}_w^{\text{PC}} \) in terms of the reduced matrix elements \( \mathcal{D}, \mathcal{J}, \) and \( \mathcal{F} \): the only real change from the corresponding expression for \( \mathcal{H}_{\text{ms}} \) is that the presence of an \( I = \frac{3}{2} \) \( 27 \) operator, in addition to the \( I = \frac{1}{2} \) one, causes alterations in the coefficients of the reduced matrix element \( \mathcal{F} \) for the \( 27 \) operator.

\[
\langle n | \mathcal{H}_w^{\text{PC}} | \Sigma^0 \rangle = \frac{\sqrt{2}}{2}(-\frac{3}{\sqrt{5}} \mathcal{D} + \mathcal{J}) - \frac{\sqrt{10}}{5} \mathcal{F} = +(12.70 \pm 0.34) \times 10^{-2} \text{MeV}^2 \tag{3.24}
\]

\[
\langle n | \mathcal{H}_w^{\text{PC}} | \Lambda^0 \rangle = -\frac{3}{2}(\frac{1}{\sqrt{5}} \mathcal{D} + \mathcal{J}) - \frac{\sqrt{30}}{15} \mathcal{F} = -(14.32 \pm 0.33) \times 10^{-2} \text{MeV}^2, \tag{3.25}
\]

\[
\langle p | \mathcal{H}_w^{\text{PC}} | \Sigma^+ \rangle = -\frac{3}{2}(-\frac{3}{\sqrt{5}} \mathcal{D} + \mathcal{J}) - \frac{4 \sqrt{5}}{15} \mathcal{F} = -(19.4 \pm 3.5) \times 10^{-2} \text{MeV}^2. \tag{3.26}
\]

A separate experimental result \( ^{27} \) for the decay \( \Sigma^+ \to n \Sigma^+ \) is

\[
\langle p | \mathcal{H}_w^{\text{PC}} | \Sigma^+ \rangle + \sqrt{2} \langle n | \mathcal{H}_w^{\text{PC}} | \Sigma^0 \rangle = -\frac{2 \sqrt{5}}{3} \mathcal{F} = -(2.25 + 3.6) \times 10^{-2} \text{MeV}^2. \tag{3.27}
\]

These experimental data \( ^{21} \) do not place very tight restrictions on the values of \( \mathcal{D}, \mathcal{J} \) and \( \mathcal{F} \), but they are certainly consistent with a value for the d/f ratio in the range \(-0.3\) to \(-0.4\).
Furthermore, Hefft and Stech\(^ {28} \) have shown that the \( \delta/f \) ratio for the electromagnetic Hamiltonian is approximately \( 0.45 \). This similarity between the \( \delta/f \) ratio for the different parts of the Hamiltonian has given rise to the rather elegant hypothesis that the octet parts of the medium strong, electromagnetic and parity-conserving non-leptonic weak Hamiltonians all belong to the same octet.\(^ {24} \)

Further evidence for this comes from a comparison of the ratio of the couplings of \( \mathcal{H}_{\text{ms}} \) and \( \mathcal{H}_{\omega}^{\text{PC}} \) to the mesons with the corresponding ratio involving the baryons. So far, we have neglected the relative scales of the couplings, as we have been dealing only with the baryons.

The present author has employed the commutator formalism involving non-vanishing ETCS and a particular symmetry breaking Hamiltonian,

\[
\mathcal{H}_{\text{ms}} = \varepsilon \delta_{ij} \left( v_i^\mu v_j^\mu + A_i^\mu A_j^\mu \right), \tag{3.28}
\]

to relate the mass splittings in the \( \frac{1}{2}^+ \) baryon octet to the different s-wave non-leptonic hyperon decay rates. This is straightforward since \( \mathcal{H}_{\text{ms}} \) in eq. (3.28) is proportional to the eighth component of the octet whose sixth component is simply the octet part of \( \mathcal{H}_{\omega}^{\text{PC}} \). The procedure given in ref. (29) (and very similar to that outlined in the present chapter) can immediately be extended to the \( 0^- \) mesons, where we have the relation

\[
\pi^2 - \xi^2 = \frac{\varepsilon}{\delta} \frac{1}{\sqrt{6}} \left[ \langle \pi^0 | \mathcal{H}_{\omega}^{\text{PC}} | \pi^0 \rangle - \frac{\sqrt{3}}{4} \langle k^+ | \mathcal{H}_{\omega}^{\text{PC}} | \pi^+ \rangle \right]. \tag{3.29}
\]
with \( \varphi = \frac{G}{2\sqrt{2}} \sin \theta \cos \theta \), \( \theta \) being the Cabibbo angle \(^{30}\) and 
G the Fermi constant. If we make the assumption that the 27-plet contribution to the matrix elements can be neglected, then the relation simplifies to

\[
\pi^2 - k^2 = \frac{5\sqrt{3}}{12} \varphi \left< \pi^+ | \mathcal{H}^{PC} | K^+ \right>
\]  
(3.30)

We have already estimated \(^{29}\) that \( \varepsilon/\varphi \) lies in the region \((4.0 - 5.6) \times 10^6\), while a value of \(4.4 \times 10^6\) yielded reasonably good predictions for \( \Xi^2 \) and \( \Xi^- \). If we use this particular estimate of \( \varepsilon/\varphi \) together with the value of \( \left< \pi^+ | \mathcal{H}^{PC}_{\theta} | K^+ \right> \) calculated from the decay \( K^0 \rightarrow \pi^+\pi^- \), then we obtain a prediction for \( \pi^2 - k^2 \) of around \(0.23 \text{ GeV}^2\), which is in pretty good agreement with experiment.

Such a result is most easily explained by the hypothesis that there is only one octet operator, with the strengths of the three reactions corresponding to the different coupling constants \( \varepsilon, \epsilon \) and \( G \). However, perhaps a note of caution should be added here. The fact that octet dominance always appears to occur for single-particle matrix elements of the various parts of the Hamiltonian does not follow from the hypothesis. As was shown earlier, the non-octet part of \( \mathcal{H}_{ms} \) plays an essential part in S-matrix elements, so that it is only one particle matrix elements of \( \mathcal{H}_{ms} \), and not \( \mathcal{H}_{ms} \) itself, which are octet dominated. But for \( \mathcal{H}^{PC}_{\theta} \), the \( \Delta I = 1/2 \) rule appears to be valid for the non-leptonic decays: although this does not necessarily imply octet dominance.

\(^{\text{a}}\) In ref. \((29)\), a factor of \(1/\sqrt{2}\) was incorrectly omitted from the rhs of eq. \((19)\). When this is included, the numerical values of the matrix elements are increased by a factor of \(\sqrt{2}\), while the resulting values of \( \varepsilon/\varphi \) are reduced by the same factor.
of the $S$-matrix elements, the $\Delta I = \frac{1}{2}$ rule is certainly consistent with it, so that we could possibly have a more general form of octet dominance for $H_{\omega}^{PC}$ than for $H_{\text{ms}}$.

Finally, the introduction of this hypothesis raises some interesting questions about the more general transformation properties of $H_{\text{ms}}$ and $H_{\omega}^{PC}$. At the present time, $H_{\text{ms}}$ is usually taken as $u_8$, the quark density which belongs to the $(3, \overline{3}) + (\overline{3}, 3)$ representation of the larger group $SU(3) \otimes SU(3)$. However, the most usual form for $H_{\omega}^{PC}$ is the current-current model $^{31}$ which transforms according to the $(1, 8) + (8, 1)$ representation of $SU(3) \otimes SU(3)$. Investigations have been made into the possibility of $H_{\omega}^{PC} \sim u_6$, $^{32}$ with reasonable success for the $s$-wave non-leptonic decays: but serious difficulties arise for the $p$-wave amplitudes. $^{31}$ Furthermore, the connection between the leptonic and non-leptonic weak interaction Hamiltonians is lost for this last form for $H_{\omega}^{PC}$. One alternative is to consider the effects of a $(1, 8) + (8, 1)$ term in $H_{\text{ms}}$. In Chapter V, the question of $(3, \overline{3}) + (\overline{3}, 3)$ and $(1, 8) + (8, 1)$ terms in $H_{\text{ms}}$ is considered in some detail.
I. Introduction

In this chapter, the effects of symmetry breaking, both intrinsic and spontaneous, are considered for the masses and widths of the spinless mesons in the context of the SU(3) σ Model. Although the σ-model was originally formulated to investigate relationships between currents and fields, e.g. the PCAC relation, and as such incorporated various types of fields, we shall restrict ourselves in this chapter to the eighteen scalar and pseudoscalar fields which belong to the \((3, \overline{3}) \oplus (\overline{3}, 3)\) representation of \(SU(3) \otimes SU(3)\). All of the corresponding particles have not yet been definitely established experimentally, but there appears to be sufficient evidence to warrant an investigation into the variation produced in the predictions for the mass spectra, decay widths, and weak decay constants when the form of the symmetry breaking is altered.

The approach which we shall employ is a Classical Field Theory one, but as far as the masses and coupling constants are concerned, we expect the Tree Graph Approximation in Quantum Field Theory to give exactly the same results; this has been shown by B.W. Lee for the SU(2) σ-Model, and there seems to be no reason why similar arguments as he uses there should not also apply to the SU(3) model.

The basic argument is as follows. Consider an SU(2) \(\otimes\) SU(2) invariant Lagrangian constructed out of the pseudoscalar I-vector fields \(\pi_i^1 (i = 1, 2, 3)\) and the scalar I-scalar field \(\sigma\), with all combinations normal ordered. The addition of a term linear in the \(\sigma\) field
breaks the $\text{SU}(2) \otimes \text{SU}(2)$ symmetry, and leads to the usual PCAC relation.

Now, if we assume that the vacuum expectation value of the $\sigma$ field does not vanish, then the corresponding $\sigma$-particle can disappear into the vacuum. Because of this, it is possible to construct Feynman diagrams containing any number of four point $\sigma$ vertices without introducing any closed loops; such diagrams are termed "Tree Graphs". In addition, since we have $\sigma^2\pi_1\pi_1$ vertices, we can have similar "trees" emanating from $\pi$ lines as well as from $\sigma$ lines. Thus, when we do a perturbation calculation in Quantum Field Theory and confine ourselves to this "Tree Graph Approximation" (T.G.A.), we find that the $\pi$ and $\sigma$ masses are "renormalised": also, two three-point vertices $\sigma^3$ and $\sigma\pi_1\pi_1$ appear in addition to the previous two four-point vertices.

If we define a new field $S \equiv \sigma - \langle 0 | \sigma | 0 \rangle$, substitute for $\sigma$ in the original Lagrangian, forget about the normal ordering, and demand that no terms linear in $S$ should be present, we find that the new expressions for the masses of the $\pi$ and $\sigma$ coincide exactly with those obtained above in the T.G.A. Furthermore, there are now two three-point vertices with exactly the same coupling constants as those above.

Hence, this Classical Field Theory procedure of neglecting the normal ordering and solving for the stability point of the Lagrangian (no linear terms!) leads to exactly the same results as the T.G.A. in Quantum Field Theory. Although the analysis
is more complicated for the SU(3) $\sigma$-Model, there is no difference in principle so that we expect all our results to coincide with those of the T.G.A.

We conclude this section with a brief word on nomenclature. The $0^-$ nonet consists of the usual $\pi$, $K$ and $\eta$ states, together with the $X(958)$ (sometimes called $\eta'$); the corresponding $0^+$ states are denoted by $\delta(966)$, $\kappa(1080)$, $\eta'(1060)$ (alternatively called $\eta_{0^+}$ or $\delta^*$) and $\sigma(720)$ (alternatively $\varepsilon$). At the present time, the $\eta'$, $\sigma$ and $\delta$ seem to be reasonably well established, although there may be another $0^+$ I-vector state very near the $\delta$, viz. the $X_n(1016)$. As for the $\kappa$, although it has been removed from the main Rosenfeld Table, there is some evidence for such a state in the range 1080 - 1260 MeV.

II. Basic Lagrangian Model

A very good introduction to the basic formalism is given by Schecter and Ueda, and we shall follow their outline reasonably closely. However, the original motivation for the present work came from papers published by the Pisa group, in particular by Cicogna, and we shall tend to follow his philosophy in matters such as restricting the form of the Lagrangian.

The model involves the nonets of scalar fields $u_i$ and pseudoscalar fields $v_i$ ($i = 0, 1, \ldots, 8$) which are sometimes more conveniently expressed in the matrix notation

$$u_\alpha^\beta = \frac{1}{\sqrt{2}} u_i(\lambda_i)_\alpha^\beta; \quad v_\alpha^\beta = \frac{1}{\sqrt{2}} v_i(\lambda_i)_\alpha^\beta; \quad a, b = 1, 2, 3$$

$$(w_\alpha^+)^\beta = u_\alpha^\beta \pm i v_\alpha^\beta, \quad (4.1)$$
where the $\lambda_i$ are the usual Gell-Mann matrices. The combinations $W^+$ and $W^-$ transform under chiral $SU(3) \otimes SU(3)$ according to the $(3, \overline{3})$ and $(\overline{3}, 3)$ representations respectively. For example, under the "left-hand" transformation

$$x_\alpha \rightarrow \Lambda_\alpha^\beta x_\beta,$$

$$ (W^+)_\alpha^\beta \rightarrow \Lambda_\alpha^\gamma (W^+)_\gamma^\beta ; \quad (W^-)_\alpha^\beta \rightarrow (\Lambda_\beta^\gamma)^* (W^-)_\alpha^\gamma. \quad (4.2)$$

These transformations are, in fact, those of the $U(3) \otimes U(3)$ group, unless we impose the extra condition

$$\det A = 1, \quad \det B = 1, \quad (4.3)$$

where $B$ is the corresponding transformation matrix for the "right-handed" group. In this case, we have $SU(3) \otimes SU(3)$. Other transformation properties of $W$ are

(i) hermiticity : $$(W^+)_\alpha^\beta = (W^-)_\beta^\alpha$$

(ii) parity : $$W^+_{\alpha} (x, t) \rightarrow W^-_{\alpha} (-x, t). \quad (4.4)$$

The main idea is to write down a Lagrangian density $\mathcal{L}$ in terms of invariants constructed out of the fields so as to preserve parity and charge conjugation; and then to add on a term $\mathcal{L}'$, which breaks the chiral $SU(3) \otimes SU(3)$ symmetry down as far as $SU(2) \otimes U(1)$, i.e. we still maintain conservation of I-spin and hypercharge.

Next, using a purely classical field theory approach, we search for the stability points of $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}'$, by solving the equations
where \( \langle \rangle_o \) implies that the quantity inside the brackets is evaluated at the stability point. When these points coincide with the "origin", i.e. when \( \langle W^+ \rangle_o = 0 \), then we do not have spontaneous symmetry breaking; in quantum field theory language, all of the fields have vanishing vacuum expectation values, since the equilibrium point corresponds to the vacuum or ground state of the system. However, the situation is more interesting when the equilibrium point is shifted from the origin. In order to have the vacuum invariant under parity, I-spin and hypercharge transformations, the most general type of stability point corresponds to the conditions

\[
\langle (W^+)_{1} \rangle_o = \langle (W^+)_{2} \rangle_o = x ; \quad \langle (W^+)_{3} \rangle_o = z.
\] (4.6)

Thereafter, we expand \( \mathcal{L} \) about this point in terms of the "normal coordinates", with the coefficients of the quadratic terms representing the corresponding squared masses (which are positive from the condition for stable equilibrium). In addition, we use the third derivative of \( \mathcal{L} \) as the appropriate three-particle coupling constant, from which the decay rates of a scalar into two pseudoscalars can subsequently be calculated.

A. Construction of the Lagrangian

The problem now confronting us is the explicit form of \( \mathcal{L} \). Clearly, since we want \( \mathcal{L} \) to be \( SU(3) \otimes SU(3) \) symmetric, it
is very convenient to consider it as some function of all the independent $SU(3) \otimes SU(3)$ invariants which can be constructed out of the given fields. These are four in number and may be chosen as

\[
\begin{align*}
I_1 & = \text{Tr}(W^+ W^-), \\
I_2 & = \text{Tr}(W^+ W^- W^+ W^-), \\
I_3 & = \text{Tr}(W^+ W^- W^+ W^-), \\
I_4 & = 6(\det W^+ + \det W^-).
\end{align*}
\]

However, at this point, we impose the condition that the Lagrangian shall not contain interaction terms involving powers of the fields greater than four, since such models are not renormalisable. This immediately prevents the inclusion of $I_3$ and also requires $\mathcal{L}_0$ to be a linear combination of $I_1$, $I_2$, $I_2$, and $I_4$. The effect of this is an appreciable simplification of the resulting mass equations obtained by Schechter and Ueda.

Next, we must choose $\mathcal{L}_1$, that part of $\mathcal{L}$ which breaks the symmetry down from $SU(3) \otimes SU(3)$ to $SU(2) \otimes U(1)$. Schechter and Ueda are guided by the success of the Gell-Mann, Oakes, Renner (GOR) model for symmetry breaking, viz. a simple linear term $u_0 + c u_3^*$ involving two terms which belong to the same $(3, \bar{3}) + (\bar{3}, 3)$ representation of $SU(3) \otimes SU(3)$. When $c = 0$, $\mathcal{L}_1$ is $SU(3)$ invariant, whereas $c = -\sqrt{2}$ corresponds to $SU(2) \otimes SU(2)$ symmetry; GOR show that there is strong evidence that $c \approx -\sqrt{2}$. On the other hand, Cicogna considers this term (with $c = -\sqrt{2}$) in combination with a bilinear term
\[ I_+ = \frac{1}{2\sqrt{2}} \varepsilon_{cd} \left[ (W^+)_a (W^+)_b d + (W^-)_a (W^-)_b d \right] (a, b = 1, 2). \]

(4.8)

\( I_+ \) is another \( SU(2) \otimes SU(2) \) invariant, constructed out of the non-strange fields, but it is the only invariant which transforms in the same way as \( u_0 = \sqrt{2} u_8 \).

At this stage, it is appropriate to make some comments on Cicogna's approach. He considers that \( SU(2) \otimes SU(2) \) is a pretty good symmetry group for the Lagrangian (or Hamiltonian), following GOR again, and accordingly fits the \( u_1 \) and \( v_1 \) into representations of this group. \( (w^-)_a^b (a, b = 1, 2) \) comprise the reducible \( (2, \overline{2}) \oplus (\overline{2}, 2) \) representation, \( (w^+)_3^a \) are two singlets while \( (w^+)_3^3 = (w^-)_3^a \) and \( (w^+)_3^a \) are two strange doublets. Thus, the eighteen fields lie in three separate subspaces of \( SU(2) \otimes SU(2) \). With this classification, and also the decomposition of the \( (2, \overline{2}) \oplus (\overline{2}, 2) \) representation into the two quadruplets \( u \equiv (u_1, u_2, u_3), \ v' \equiv \frac{1}{\sqrt{3}} (\sqrt{2} v_0 + v_8) = \sqrt{2} v_3' \) and \( v, u', \) we see that the \( SU(2) \otimes SU(2) \) symmetry cannot be exact for the following reasons. If we identify these fields with the corresponding \( 0^+ \) particle states, then there is large "ideal" nonet mixing between \( \eta \) and \( X \) (and also \( \eta' \) and \( \sigma \)); furthermore, the pions are exactly mass degenerate with \( u', \) one of the non-strange \( I\)-singlets \( \sigma \) or \( \eta' \), as are the kaons with the kappa mesons. Such predictions are certainly not borne out by the experimental data.

However, by writing \( \zeta \) as a linear combination of the \( SU(3) \otimes SU(3) \) invariants \( I_1, I_2 \) and \( I_4 \) (to take account
of the SU(3) symmetry) together with $I_+$ and $(u_0 - \sqrt{2} u_8)$, and allowing spontaneous symmetry breaking to occur, Cicogna manages to obtain a remarkably good fit to the whole $0^+$ and $0^-$ mass spectrum, with only the mass of the $\delta$ as a relatively poor prediction. The inclusion of the bilinear term $I_+$ in $\mathcal{L}$ allows Cicogna to maintain exact SU(3) invariance of the vacuum, even when the Lagrangian symmetry is SU(2) $\otimes$ SU(2), whereas the presence of only the linear term $(u_0 - \sqrt{2} u_8)$ necessitates a comparable amount of spontaneous breaking.

B. General Equations

We follow the general approach of Schechter and Ueda here, but with the appropriate modifications mentioned above. Firstly, we write $\mathcal{L}$ in the form

$$\mathcal{L} = -\frac{1}{2} \text{Tr} \left[ (\partial_\mu u)^2 + (\partial_\mu \nu)^2 \right] + \frac{1}{2} \mu^2 I_1 + \rho I_1^2 + \lambda I_2 + \nu I_4$$

$$+ 2g_0 U_3 + 2g_0 U_3^3 + b I_+ . \quad (4.9)$$

Next, using eqs. (4.5) and (4.6), we obtain the two constraint equations

$$\mu^2 x + 4\lambda x^3 + 4\gamma x(2x^2 + z^2) + 6vzx + 2bx + 2g_0 = 0 \quad (4.10)$$

$$\mu^2 z + 4\lambda z^3 + 4\gamma z(2x^2 + z^2) + 6vz^2 + 2g + 2g_0 = 0 \quad (4.11)$$

The corresponding (mass)$^2$ matrices

$$\left( m^2 \right)^{ac}_{bd} = \frac{\partial^2 \mathcal{L}}{\partial V_a \partial V_c}$$

and
\[(\sum^2)_{bd}^{ac} \equiv \left\langle \frac{\delta^2 L}{\partial v_a^b \partial v_c^d} \right\rangle_o \text{ are}
\]
\[
(\prod^2)_{bd}^{ac} = \left[ \mu^2 + 4\rho (2x^2 + z^2) \right] \delta_b^a \delta_d^c + 4\lambda \left[ \delta_d^a \left\langle (U^2)_b^c \right\rangle_o + \delta_c^a \left\langle (U^2)_d^b \right\rangle_o \right]
\]
\[
+ \delta_b^c \left\langle (U^2)_d^a \right\rangle_o - \left\langle U_d^a \right\rangle_o \left\langle U_b^c \right\rangle_o \right] + 6v \epsilon_{ace} \epsilon_{bdf} \left\langle U_e^f \right\rangle_o + \left\langle \frac{\delta^2 L}{\partial v_a^b \partial v_d^c} \right\rangle_o,
\]
\[(4.12)\]
\[
(\sum^2)_{bd}^{ac} = \left[ \mu^2 + 4\rho (2x^2 + z^2) \right] \delta_b^a \delta_d^c + 4\lambda \left[ \delta_d^a \left\langle (U^2)_b^c \right\rangle_o + \delta_c^a \left\langle (U^2)_d^b \right\rangle_o \right]
\]
\[
+ \left\langle U_d^a \right\rangle_o \left\langle U_b^c \right\rangle_o ] + 6v \epsilon_{ace} \epsilon_{bdf} \left\langle U_e^f \right\rangle_o + \left\langle \frac{\delta^2 L}{\partial v_a^b \partial v_d^c} \right\rangle_o.
\]
\[(4.13)\]

Finally, the coupling constants corresponding to the coupling of two pseudoscalars to one scalar,

\[
c(b^c d^f) \equiv \left\langle \frac{\delta^3 L}{\partial v_d^a \partial v_c^e} \right\rangle_o , \text{ are}
\]
\[
c(b^c d^f) = 4\lambda \left[ \delta_a^d \left( \delta_c^f \left\langle U_e^b \right\rangle_o + \delta_c^f \left\langle U_e^f \right\rangle_o \right) + \delta_b^d \left( \delta_c^f \left\langle U_d^b \right\rangle_o + \delta_c^f \left\langle U_d^f \right\rangle_o \right) \right]
\]
\[
- (\delta_a^f \delta_d^e \left\langle U_c^b \right\rangle_o + \delta_c^f \delta_e^b \left\langle U_d^a \right\rangle_o )
\]
\[(4.14)\]
\[
+ 8 \rho \delta_a^d \delta_c^b \left\langle U_e^f \right\rangle_o - 6v \epsilon_{ace} \epsilon_{bdf}.
\]

It should be pointed out here that the \(C(ab^c \gamma)\) defined above are not always exactly equal to the usual coupling constants \(g_{a\beta \gamma}\), but occasionally contain extra factors of 2. For example, for
where \( \vec{x}, \vec{\delta} \) are \( I \) vectors, and \( \sigma, \gamma \) are \( I \) scalars, then

\[
C(\pi_a \pi_b | \sigma) = 2\delta_{ab} g_{\sigma \pi \pi}; \quad C(\gamma_a \gamma_b | \delta) = \delta_{ab} g_{\gamma \gamma}. \tag{4.16}
\]

Next, the \( \gamma - \chi \) and \( \gamma' - \sigma \) mixing has to be taken into account. For

\[
\gamma = \gamma_0 \cos \theta + \chi_0 \sin \theta,
\]

\[
\chi = -\gamma_0 \sin \theta + \chi_0 \cos \theta, \tag{4.17}
\]

then

\[
C(\delta \gamma \pi) = C(\delta \gamma_0 \pi) \cos \theta + C(\delta \chi_0 \pi) \sin \theta. \tag{4.18}
\]

For completeness, we give the actual algebraic expressions for the decay widths; in each case, \( C \) is the appropriate third derivative:

\[
\Gamma(\delta \to \eta \pi) = \frac{c^2}{4\pi} \cdot \frac{p_\pi}{2\delta^2} = C^2(\delta \eta \pi). P(\delta \eta \pi),
\]

\[
\Gamma(\delta \to \bar{K}K) = \frac{c^2}{4\pi} \cdot \frac{p_K}{\delta^2},
\]

\[
\Gamma(\gamma' \to \pi \pi) = \frac{c^2}{4\pi} \cdot \frac{3p_\pi}{4\eta'^2}, \quad \Gamma(\sigma \to \pi \pi) = \frac{c^2}{4\pi} \cdot \frac{3p_\pi}{4\sigma_\pi^2}, \tag{4.19}
\]

\[
\Gamma(\gamma' \to \bar{K}K) = \frac{c^2}{4\pi} \cdot \frac{p_K}{\eta'^2}. \tag{4.19}
\]

In these equations, \( p_a \) is the magnitude of the three-momentum of one of the decay products \( a \). In our calculations, we use the
values of $C$ arising from the model under consideration; however, for the kinematic factors $F$, we use the numerical values corresponding to the experimental data appearing in the Rosenfeld Tables of August, 1970, even when the predicted and experimental values of a particular mass do not coincide. In actual fact, apart from the factor $F(\delta KK)$, the variations produced in the different $F$'s are rather small; by using the same value of $F$ in each model, we gain a better idea of how the widths are changing.

Finally, we can use the PCAC and PCVC relations to obtain algebraic expressions for the weak decay constants $f_\pi$, $f_K$ and $f_\kappa$, which are defined as follows:

$$\langle 0 | \delta^{\mu} A_{\mu}^{i} | \pi^j \rangle = f_1 m_1^2 \delta^{ij}, \quad (i,j = 1, \ldots, 7) \quad (4.20)$$

$$\langle 0 | \delta^{\mu} V_{\mu}^{4-i5} | \chi^{4+i5} \rangle = +i m_\pi^2 f_\pi. \quad (4.21)$$

$$f_i \equiv f_\pi \quad (i = 1, 2, 3) \quad \text{and} \quad f_i \equiv f_K \quad (i = 4, 5, 6, 7).$$

We can relate the current divergences to the (symmetry breaking part of the) Lagrangian through the relation

$$\delta^{\mu} A_{\mu}^{i} = i [\mathcal{H}, A^{i}] = i [A_{\mu}^{i}, \mathcal{L}] \quad (4.22)$$

where $A_{\mu}^{i}$ is the charge corresponding to the current $A^{i}$, i.e. $A^{i}(t) = \int d^3 \mathbf{x} \ A_{\phi}^{i}(t, \mathbf{x})$.

Explicitly, we have
\[ \partial_{\mu} A_{\mu} = \sqrt{2} \cdot 2g_0 v^3 = 2(b' - b)(u'v^3 - v'u^3), \quad (4.23) \]

\[ \partial_{\mu} A_{\mu}^{l+15} = \sqrt{2} \cdot (2g_0 + g)v^{l+15} + b \left[ (u' - u^3)v^{l+15} - u^{l+12}v^{6+17} \right] \]

\[ - b' \left[ (v' - v^3)u^{l+15} - v^{l+12}u^{6+17} \right], \quad (4.24) \]

\[ \partial_{\mu} v^{l+15} = -\sqrt{2} g_0 u^{l+15} + b \left[ (u' - u^3)u^{l+15} - u^{l+12}u^{6+17} \right] \]

\[ - b' \left[ (v' - v^3)v^{l+15} - v^{l+12}v^{6+17} \right], \quad (4.25) \]

where the parameter \( b' \equiv ab \), and \( a \) appears when the \( SU(2)_x \otimes SU(2) \) invariant \( I_+ \equiv (u'^2 - u^2) - (v'^2 - v^2) \) is modified to \( I'_+ \), where

\[ I'_+ \equiv (u'^2 - u^2) - a(v'^2 - v^2). \quad (4.26) \]

The reason for this modification is explained later in Section IV(C); suffice it to say that the pion (mass) \( \pi^2 \) is given by the expression

\[ \pi^2 = -2g_0x + 2(b' - b). \quad (4.27) \]

In the above expressions for the divergences, there are both linear and bilinear terms in the fields; however, if we make the replacement

\[ u' \rightarrow S' = u' - \langle u' \rangle_0 = u' - \sqrt{2}x \quad (4.28) \]

in the Lagrangian, so that \( \langle S' \rangle_0 = 0 \), then we can conveniently group the linear and bilinear terms in \( \partial_{\mu} A_{\mu} \) as follows:
\[ \partial_{\mu} A^3_\mu = -\sqrt{2} x \left[ -\frac{2g_2}{x} + 2(b' - b) \right] v^3 + 2(b' - b)(v'u^3 - s'v^3) \]

\[ = -\sqrt{2} x \left[ \pi^2 \right] v^3 + \text{bilinear term}. \quad (4.29) \]

Thus, in the approximation where the contributions of the bilinear terms are neglected, we can obtain the useful relation

\[ F_{\pi} = -\sqrt{2} x. \quad (4.30) \]

Similarly, use of the constraint equations (4.10), (4.11), together with the mass formulae (which appear later in eqs. (4.54), (4.55)), allows us to derive the other relations

\[ \frac{F}{F_{\pi}} = \frac{\omega + 1}{2\omega}; \quad \frac{F}{F_{\pi}} = \frac{1 - \omega}{2\omega}, \quad (4.31) \]

where \( \omega = \frac{x}{z} \) is a measure of the amount of spontaneous symmetry breaking involved.

III. Review of the Models of Schechter and Ueda, and Cicogna

Having given most of the general equations required for calculations, we shall now give a brief summary of the work done by Schechter and Ueda, and by Cicogna.

When \( \mathcal{L}_1 = 0 \), i.e. when the Lagrangian is \( \text{SU}(3) \otimes \text{SU}(3) \) symmetric, Schechter and Ueda investigate the six solutions arising when different relations exist between \( x \) and \( z \). For example,
x = z = 0 corresponds to an SU(3) \( \otimes \) SU(3) invariant vacuum, i.e. there is no spontaneous breaking; for \( x = 0, z \neq 0 \), we have an SU(2) \( \otimes \) SU(2) spectrum, while \( x = z \neq 0 \) occurs when the vacuum is SU(3) invariant. Although massless particles (i.e. Goldstone Bosons) are predicted, this is not always a serious problem since the introduction of a suitable \( \mathcal{L}_1 \) allows the states to acquire mass. However, if we assume that this last effect is relatively small, i.e. of the order of the squared masses of the \( 0^- \) octet (approximately 0.25 GeV\(^2\)), and if we also accept the experimental evidence that all of the \( 0^+ \) octet states have masses in the region of 1 GeV, then the prediction of the \( \kappa \) as a Goldstone Boson in four of these solutions is rather unsatisfactory. A fifth solution in which there is no spontaneous breakdown at all is also not very attractive, since all eighteen states are then degenerate. Hence, it seems sensible to concentrate our attention on the solution where \( x = z (\omega = 1) \) for \( \mathcal{L}_1 = 0 \), i.e. where the complete octet of \( 0^- \) mesons are Goldstone Bosons, the \( 0^+ \) octet is a massive degenerate multiplet, and the two singlets are in general distinct.

When we allow \( \mathcal{L}_1 \) to become non-zero, the effect of Cicogna's bilinear term \( b \mathcal{L}_1 \) is immediately obvious: it is still possible to maintain exact SU(3) invariance of the vacuum, provided \( g = bx \). In order to illustrate just how simple Cicogna's solution is, we give the mass relations which arise on elimination of the parameters \( \lambda x^2 \) etc. from the mass formulae. One relevant point should be mentioned here: Cicogna does not consider a term \( \rho \mathcal{L}_1^2 \) in \( \mathcal{L}_0 \), nor a term \( 2g_0 v_a^a \); the omission of this second term means that \( \mathcal{L} \) is
exactly $\text{SU}(2) \otimes \text{SU}(2)$ invariant, and that the pions are still massless. Below, we include the second term, although $\rho I_1^2$ is still omitted.

Cicogna's results are very conveniently summarised in the following equations:

$$3\gamma^2 = 4K^2 - \pi^2; \quad 3\gamma'^2 = 4\kappa^2 - \delta^2, \quad (14.32)$$

$$\gamma^2 \chi = \frac{\sqrt{2}}{3}(K^2 - \pi^2); \quad \gamma^2 \sigma = \frac{\sqrt{2}}{3}(\kappa^2 - \delta^2), \quad (14.33)$$

$$\eta^2 - \chi^2 = \pm (\eta'^2 - \sigma^2), \quad (14.34)$$

$$\eta^2 + \chi^2 + \eta'^2 + \sigma^2 = 2(\kappa^2 + K^2), \quad (14.35)$$

$$\kappa^2 - \delta^2 = -(K^2 - \pi^2), \quad (14.36)$$

$$\tan \theta = \frac{3\gamma - 4K^2 + \pi^2}{\sqrt{2}(K^2 - \pi^2)} = \tan \theta', \quad (14.37)$$

Clearly, both octets satisfy Gell-Mann - Okubo formulae before the $\eta - \chi$ and $\eta'^2 - \sigma^2$ systems are diagonalised. Also, the mixing angles $\theta$ and $\theta'$ are equal here, although the relative positions of the singlet and octet eighth components are reversed on going from the $0^-$ to the $0^+$ nonet. To make comparisons with experiment

---

In both the text and in equations, we denote the squared mass of particle $\beta$ by $\beta^2$; when mixing occurs, e.g. between $\gamma$ and $X$, we denote the unmixed squared masses by $\gamma^2_0$ and $X^2_0$, while the off-diagonal (mass) matrix element is $\gamma^2_0 X$. Finally, we use $K^2 = 0.246 \text{ GeV}^2$, and $\pi^2 = 0.019 \text{ GeV}^2$, these values being averages of the different values in the appropriate I-spin multiplet. Also, we take $f_\pi \approx 130 \text{ MeV}$. 

we can use the $\pi$, $K$, $X$ and $\gamma^+$ masses as input. The first three imply that $\eta_2^2 = 0.301 (0.301)$ (in units of GeV$^2$), which is in exact agreement with the experimental value (given in parenthesis). Use of $\eta_1^2$, together with eqs. (4.34) - (4.36) in turn gives $\sigma^2 = 0.514 (0.518)$, $\kappa^2 = 1.185 (1.166)$ and $\delta^2 = 1.412 (0.933)$. The agreement between the predictions and experimental data is remarkably good, the only exception being the $\delta$ mass.

The simple explanation of this one discrepancy comes from the fact that when we use a $(3, \overline{3}) + (\overline{3}, 3)$ symmetry breaking model, as we have done here, we expect the $0^+$ and $0^-$ spectra to be mirror images of each other, since the relevant parts of the Hamiltonian couple with opposite signs to the two multiplets. This corresponds well to the experimental situation, with the notable exception of the $\delta$, which ought here to be the heaviest scalar and not the lightest. If we had used a $(1,8) + (8, 1)$ symmetry breaking term, then the two spectra would be "parallel", i.e. the $\delta$ would be predicted to be the lightest scalar.

Considering the simplicity of the approach, i.e. the linear combination of invariants in $\mathcal{L}_0$ and the presence of only two of the eleven $SU(2) \times SU(2)$ invariants which can be constructed out of the eighteen fields (albeit the only two which transform like the GOR symmetry breaking term), then the agreement with experiment is quite amazing.

IV. Modifications to Cicogna's Model

At the time of receiving Cicogna's paper, the present author was just concluding a very similar investigation, using the same Lagrangian with the addition of $\rho I_2$, but without the $bI_2$ term.
As has been pointed out, the absence of $b_{1+}$ means that the vacuum is no longer exactly SU(3) invariant. Accordingly, this approach involved doing a perturbation expansion about the original solution, with $g$ as the small expansion parameter. The results were rather disappointing and led to some serious problems regarding the "bare" masses of some of the states when certain masses were used as input. The problem particle was the $\sigma$: for $\sigma^2 < 0.75$ GeV$^2$, the bare squared mass $\sigma^2_0$ was negative, and even for $\sigma^2 = 1$ GeV$^2$, the bare masses of the $0^+$ states were so small as to cast grave doubts on the validity of a perturbation expansion involving $g$. In these circumstances, the excellent fit by Cicogna of all the masses except that of the $\delta$ was extremely interesting, and prompted several lines of further investigation. The remainder of this chapter is devoted to a description of the subsequent work and a summary of the results obtained.

A. The Basic Problem

The most obvious question concerns the $\delta$: given that the experimental data are reasonably accurate, is there any relatively simple way in which the $\delta$ mass can be brought down from its high value of around 1200 MeV, but without spoiling the excellent fit to all the other states? Cicogna, who kept the pion massless, suggested that the mechanism whereby the pion acquired mass could perhaps also reduce the $\delta$ mass by approximately the correct amount. If this is so, then certainly we require a more complicated way of giving the pion non-zero mass than simply by adding a linear term
to the Hamiltonian, as we have explicitly done in the present situation.

However, it seems extremely unlikely that we can alter the Hamiltonian by adding on various terms whose only effect is to change the $\delta$ mass by the correct amount: such terms will most probably alter the other (previously excellent) results at the same time. If a simple alternative solution does exist, it is possibly quite different from Cicogna's, e.g. a combination of spontaneous breaking of the SU(3) vacuum symmetry and a different form for $\mathcal{L}$. Of course, any of these solutions will probably remove the most attractive features of Cicogna's approach, i.e. its simplicity and elegance, with the vacuum exactly SU(3) invariant and the Hamiltonian exactly $SU(2) \times SU(2)$ symmetric.

A further check on this simple model, and on any others which are tried, is clearly desirable. The easiest appears to be a calculation of the various decay widths of the scalars into pairs of pseudoscalars, using the third derivatives of $\mathcal{L}$ (at the stability point) as the appropriate coupling constants. In fact, this particular set of calculations in Cicogna's model is not nearly as successful as that for the masses, with the $\delta$ partial widths appreciably larger than those suggested by experiments. Thus, the motivation for investigating modifications of the symmetry breaking mechanism is increased: in addition, the predictions of each model can be checked at more points.
B. The Underlying Structure of Cicogna's Model

Before discussing such modifications, it is interesting to investigate the structure of the terms $b I_1^+ + 2 g_0 U_3^3$ which break the $SU(3) \otimes SU(3)$ symmetry of $\mathcal{L}$ down to $SU(2) \otimes SU(2) \otimes U(1)$, and allow such a good fit to the experimental data when combined with the spontaneous breakdown of the vacuum symmetry to $SU(3)$.

When this spontaneous breaking occurs on its own (i.e. $\mathcal{L}$ is still $SU(3) \otimes SU(3)$ symmetric), the originally degenerate masses split into separate $SU(3)$ multiplets, as shown in Fig. 1: the $0^-$ octet are Goldstone Bosons ($m^2 = 0$) while the other masses satisfy the relation

$$\sigma_0^2 + \lambda_0^2 = \kappa_0^2 \quad \text{(4.38)}$$

On the other hand, the term $b I_1^+$ in $\mathcal{L}_1$ has the explicit form in terms of the fields

$$I_+ = (u'^2 + v'^2) - (v^2 + u^2) \quad \text{(4.39)}$$

where $u, v$ are the $0^+$ and $0^-$ I-vectors $(\delta, \chi)$ and

$$u^\prime = \frac{1}{\sqrt{3}} (\sqrt{2} u_0 + u_3)$$

is the I-singlet $\bar{\sigma}$ obtained by "ideal" nonet mixing of the pure octet and singlet components. The effect of this term on the completely symmetric
configuration is to separate the \((\delta, \bar{x})\) and \((\bar{x}, \bar{o})\) quadruplets equally above and below the unaffected \(K, \kappa, \gamma\) and \(\gamma\) states (where \(-\) denotes ideally nonet mixed states), as shown in Fig. II.

Now consider the result of combining the two symmetry breaking effects. For the \(\gamma - X\) system, it is simplest to work in terms of the unmixed states \(\gamma_0\) and \(x_0\), related to \(\gamma\) and \(\bar{x}\) by the equations

\[
\gamma = \sqrt{\frac{2}{3}} \gamma_0 - \frac{1}{\sqrt{3}} x_0 = \frac{1}{\sqrt{3}} (\sqrt{2} u_8 - u_0),
\]

(4.40)

\[
\bar{x} = \frac{1}{\sqrt{3}} \gamma_0 + \frac{2}{\sqrt{3}} x_0 = \frac{1}{\sqrt{3}} (u_8 + \sqrt{2} u_0).
\]

The corresponding equations for the squared masses are

\[
\gamma^2 + \bar{x}^2 = \gamma_0^2 + x_0^2;
\]

(4.41)

\[
(\gamma^2 - \bar{x}^2)^2 = (\gamma_0^2 - x_0^2)^2 + 4(\gamma_0 x_0)^2,
\]

(4.42)

\[
\tan \theta = \frac{\gamma^2 - \gamma_0^2}{\gamma_0 x_0} = -\frac{1}{\sqrt{2}}.
\]

(4.43)

From eq. (4.42), we see that the separation of the eigenvalues \(\gamma^2\) and \(\bar{x}^2\) is greater than that of the diagonal elements \(\gamma_0^2\) and \(x_0^2\):
this situation is represented in Fig. III.

Now, when we take the spontaneous breaking into account, i.e. we allow \( X \) to be distinct from the 0\(^{-}\) octet, then

\[
X_0^2 \rightarrow X_1^2 = X_0^2 + \delta x^2,
\]

where \( \delta x^2 \) is related to the separation of the 0\(^{-}\) octet and singlet. With \( \gamma_0^2 \) and \( \gamma_0 x_0 \) unchanged, the eigenvalues \( \gamma^2 \) and \( x^2 \) of the new system now differ from those of the old. We have

\[
\gamma^2 + x^2 = \gamma_0^2 + x_1^2 = \gamma^2 + \bar{x}^2 + \delta x^2, \tag{4.49}
\]

and

\[
(x^2 - \gamma^2)^2 = (x_0^2 + \delta x^2 - \gamma_0^2)^2 + 4(x_0 x_0)^2 - 2 \delta x^2 (x_0^2 - \gamma_0^2) + (\delta x^2)^2
\]

\[
< (x^2 - \gamma^2)^2 + 2 \delta x^2 (x^2 - \gamma^2) + (\delta x^2)^2
\]

\[
= (x^2 - \gamma^2 + \delta x^2)^2, \tag{4.50}
\]

where we use eq. (4.42) to obtain the inequality \( x_0^2 - \gamma_0^2 < x^2 - \gamma^2 \).

Thus, \( x^2 - \gamma^2 < \bar{x}^2 - \gamma^2 \), since \( x^2 - \gamma^2 > 0 \). (4.51)

Then \( \gamma^2 \) is related to \( \bar{\gamma} \) by

\[
\gamma^2 = \frac{1}{2} \left[ (x^2 + \gamma^2) - (x^2 - \gamma^2) \right]
\]

\[
> \frac{1}{2} \left[ (x^2 + \gamma^2 + \delta x^2) - (\bar{x}^2 - \gamma^2 + \delta x^2) \right]
\]

\[
= \bar{\gamma}. \tag{4.52}
\]
In other words, because of the spontaneous breakdown effect, which makes X distinct from (and more massive than) the 0\(^{-}\) octet, we are able to show that \(x^2 > \eta^2 > K^2 > \kappa^2\), i.e. we have the correct shape for the 0\(^{-}\) nonet. Furthermore, the two independent parameters, corresponding to \(K^2 - \kappa^2 (= -2b)\) and the basic octet-singlet splitting induced by the spontaneous breaking, combine to relate the other two mass differences \(x^2 - \eta^2\) and \(\eta^2 - K^2\) to each other. It is extremely interesting that these two parameters allow all three mass differences to be fitted with an error of less than 1%.

A similar argument shows that \(\sigma^2 < \eta^2 < \kappa^2 < \delta^2\), which corresponds to the experimental situation except that \(\delta^2\) is at the wrong end of the spectrum. Finally, the symmetry of \(I_+\) together with eq. (4.38) allow us to show that the scales of the 0\(^{+}\) and 0\(^{-}\) octets are the same. Again, this is not very different from experiment. The predicted spectra are illustrated in Fig. IV.

C. Possible Modifications

We now return to the question of alternative solutions. Since a simple linear symmetry breaking term does not lead to satisfactory results when \(\mathcal{L}\) is SU(2) \(\otimes\) SU(2) invariant, whereas the combination of one linear and one bilinear term (constructed only out of the non-strange fields) gives very good predictions, an attempt at breaking the symmetry of \(\mathcal{L}\) even further to SU(2) \(\otimes\) U(1) in the same way is not unattractive. Accordingly, as well as adding a term \(g_U a^a\) to \(\mathcal{L}\), we also alter the SU(2) \(\otimes\) SU(2) invariant
I+ "internally" by changing

\[ I_+ = (u'^2 - u^2) - (v'^2 - v^2) \] to

\[ I'_+ = (u'^2 - u^2) - \alpha (v'^2 - v^2), \]

with \( \alpha \neq 1 \) in general. The effect of this is to allow the relative splitting in the \( 0^- \) octet to differ from that in the \( 0^+ \) octet, as shown in Fig. V.

Algebraically, the expressions for \( \pi^2, K^2, \gamma_o^2, \gamma_o X_o^2, X_o^2 \) and \( \kappa^2 \) have to be replaced by \( b' \equiv ab \).

\[ \frac{K, \kappa}{\eta, \eta'} \]

while \( \delta^2, \gamma_o'^2, \gamma_o'^2 \) and \( \sigma_o^2 \) each have a different additional term proportional to \( (\alpha - 1) \).

Thus, if we use the pseudoscalar masses, together with \( \kappa^2 \), as input, then exactly the same values for the basic parameters \( \lambda z^2, \nu z \) appear; the main difference is that we must use one extra scalar, e.g. \( \delta^2 \), in order to fix \( b' \).

So far, we have confined our attention to the case of an exactly \( SU(3) \) invariant vacuum. However, other solutions can exist with the vacuum symmetry broken to \( SU(2) \otimes U(1) \), corresponding to \( x \neq z \), i.e. \( \omega \neq 1 \). Such solutions are not quite as simple to treat as those with \( \omega = 1 \), but since they yield another free parameter, they offer the possibility of obtaining a better overall fit to the data, i.e. giving the \( 0^+ \) spectrum the correct order, and predicting
the decay rates more accurately.

One problem which now arises is the following. Do we treat this further spontaneous breakdown as a small perturbation of the previous solution, perhaps related to the small intrinsic breaking of $SU(2) \otimes SU(2)$ in $\mathcal{L}$? Or is the total spontaneous breaking the dominant effect, with the intrinsic breaking from $SU(3) \otimes SU(3)$ to $SU(2) \otimes SU(2)$ a small perturbation? Or should we treat our basic solution as that with exact $SU(3) \otimes SU(3)$ invariance of $\mathcal{L}$, and $SU(3)$ invariance of the vacuum; then the further spontaneous breaking from $SU(3)$ to $SU(2)$ is closely linked with the intrinsic breakdown to $SU(2) \otimes SU(2)$?

If our aim is to try to fit the $\delta$ into the $0^+$ spectrum properly, i.e. to have $\kappa^2 - \delta^2 \approx K^2 - \pi^2$, then the last course ought to be followed, for the following reason. In Cicogna's solution, $\kappa^2 - \delta^2 = -(K^2 - \pi^2) = 2b$; if we are to overcome this by an effect involving $(x - z)$, then this effect must be of the same magnitude as $-2b \approx K^2$. This rules out relating $(x - z)$ to the breaking of $SU(2) \otimes SU(2)$, i.e. to effects of $O(\kappa^2)$. As for treating $b_{I+}$ as a perturbation of a solution where $x \neq z$, we have already seen that such a solution involves the $\kappa$ as a Goldstone Boson, along with the $\pi$, $K$ and $\eta$, and this situation is not an attractive one. Thus, we must consider the last-mentioned solution.

Finally, we can consider the effect of introducing the term $\rho I_1^2$ into $\mathcal{L}_0$. This maintains the $SU(3) \otimes SU(3)$ symmetry, but means that $\mathcal{L}_0$ is now the most general renormalisable function of the invariants.
In the remainder of this chapter, we begin by listing the general formulae for the masses and coupling constants, together with a few useful relations derived from the formulae. We then give an outline of the method of calculation used each time one or more of the parameters assumes a special value (e.g. \( \omega = 1 \), \( \rho = 0 \) etc.). The actual numerical results are given in Table I, and are grouped according to the particular solution involved. Finally, we discuss the results, solution by solution, and close with a few comments on possible explanations for the rather poor fits obtained.

V. The Alternative Solutions

A. General Formulae

First of all, we give the algebraic expressions for the masses and coupling constants in the general case.

\[
\begin{align*}
\pi^2 &= -2gz/x + 2(b'-b) \\
K^2 &= \pi^2 - 2b' + (4\lambda z - 6\nu)(z - x) \\
\gamma_o^2 &= \pi^2 - \frac{8}{3}b' + \frac{1}{3} \left[ 2\lambda(z+x) - 6\nu \right](z-x) \\
\frac{1}{\sqrt{2}} \gamma_o X_o &= -\frac{2}{3}b' - \frac{1}{3} \left[ 4\lambda(z+x) + 6\nu \right](z-x) \\
X_o^2 &= \pi^2 - \frac{10}{3}b' + \frac{1}{3} \left[ \lambda(z+x) + 6\nu \right](z-x) - 18\nu z. \\
\delta^2 &= \pi^2 - 4b' + 3\lambda z^2 - 12\nu z + 2(b' - b) \\
\kappa^2 &= \pi^2 - 2b' + (4\lambda z - 6\nu)(z + x),
\end{align*}
\]
\[
\gamma_0 = \gamma + \frac{1}{3}b' - 2\nu (b'' - b) + \frac{16}{3}p(x-z)^2,
\]
\[
\frac{1}{\sqrt{2}}\sigma_0 = \frac{2}{3}b' - \left[ 4\lambda(z + x) - 2\nu \right](z - x) - \frac{2}{3}(b'' - b) + \frac{8}{3}p(x-z)(2x+z),
\]
\[
\sigma_0 = \gamma_0 - \frac{1}{2}b' + 4\lambda(z^2 + x^2) + 2\nu(4x-z) - \frac{1}{3}(b'' - b) + \frac{8}{3}p(2x+z)^2.
\]
\[
 f_{\pi} = \sqrt{2} x ; \quad f_{\rho} = \frac{1 + \omega}{2\omega} ; \quad \frac{f_{\rho}}{f_{\pi}} = \frac{1 - \omega}{2\omega}.
\]
\[
-r_{\pi}^C(\delta\gamma_0) = \frac{1}{\sqrt{3}}(3\lambda x^2 - 12\nu x),
\]
\[
-r_{\pi}^C(\delta\nu_0) = \sqrt{2} \frac{3}{2}(8\lambda x^2 + 6\nu x),
\]
\[
-r_{\pi}^C(\delta\phi_0) = (3\lambda x^2 - 6\nu x - 4\lambda xz),
\]
\[
-r_{\pi}^C(\gamma_0\kappa) = \frac{1}{\sqrt{3}}(8\lambda x^2 - 12\nu x + 16\rho x(x - z)),
\]
\[
-r_{\pi}^C(\sigma_0\kappa) = \sqrt{2} \frac{3}{2}(8\lambda x^2 + 6\nu x + 8\rho x(2x + z)),
\]
\[
-r_{\pi}^C(\gamma_0\kappa) = \frac{1}{\sqrt{3}}[16\lambda x^2 + 6\nu x - 20\lambda xz + 16\rho x(x - z)],
\]
\[
-r_{\pi}^C(\sigma_0\kappa) = \sqrt{2} \frac{3}{2}[4\lambda x^2 + 6\nu x + 4\lambda xz + 8\rho x(2x + z)].
\]

One interesting feature is that \(\rho\) appears only in the \(\gamma' - \sigma\) system, which means that the excellent fit to the \(0^-\) spectrum is unaffected, whereas variations in \(\rho\) allow modification to the \(0^+\) masses when either \(\gamma_0^2\) or \(\sigma^2\) is used as input.

In the general case, we cannot obtain a simple relation involving
only the masses, i.e. with all of the parameters eliminated. However, it is useful to have an explicit expression for these parameters so that when, for example, the coupling constants are found to be too large, the necessary modifications to the appropriate masses can be seen more easily.

\[ 24\lambda x^2 + 16\rho(2x^2+z^2) = 2(\gamma^2+\delta^2) - 3(\chi^2+K^2) \quad (0.926) \quad (4.58) \]

\[ 24\lambda z^2 - 8\rho(2x^2+z^2) = \left(\gamma^2 + \delta^2 + 6(\chi^2+K^2) \right) \]

\[ -3(\gamma^2+\chi^2+\pi^2) \quad (2.180) \quad (4.59) \]

\[ 2(b'-b) - 8\rho(2x^2+z^2) = -\left(\gamma^2 + \chi^2 + \delta^2 + 6(\chi^2+K^2) \right) \]

\[ 2(\delta') + 2(\chi^2+K^2) \quad (-0.042) \quad (4.60) \]

\[ 2(b'-b) - 8\lambda(z^2-x^2) = \delta^2 + \pi^2 - (\chi^2+K^2) \quad (0.460) \quad (4.61) \]

The numerical values computed from the experimental masses are given in brackets.

In the following subsections, we shall outline the method of calculation used for each solution.

B. \( \rho = 0; \omega \neq 1. \)

The relations below are particularly useful for obtaining a fourth order equation for \( \omega \) when the four \( \Omega^- \) masses and \( \chi^2 \) are used as input.

\[ 8\lambda z x - 12x = \chi^2 - K^2 \quad (4.62) \]

\[ -4b' + (8\lambda z^2 - 12x z) = \chi^2 + K^2 - 2\pi^2 \quad (4.63) \]
\[(2 + \omega^2)8\lambda z^2 = 3(\chi^2 + K^2) - 2(\gamma^2 + X^2 + \pi^2), \quad (4.64)\]

\[(1 - x^2)^2 = (\gamma_0^2 - x_0^2)^2 + 4(\gamma_0 x_0)^2, \quad (4.65)\]

\[\gamma_0^2 - x_0^2 = -\frac{1}{\sqrt{2}} \gamma_0 x_0 + 18 \nu x, \quad (4.66)\]

\[6 \frac{1}{\sqrt{2}} \gamma_0 x_0 = 3\chi^2 + \left[ \frac{5K^2}{2} - 2(\gamma_0^2 + X^2 + 2\pi^2) \right] + \frac{\omega - \nu}{\omega + 2} \left[ 3\chi^2 + \left( \frac{3K^2}{2} - 2(\gamma_0^2 + X^2 + 2\pi^2) \right) \right] \]

\[= 3\chi^2 - 1.282 + \frac{\omega - \nu}{\omega + 2} \left( 3\chi^2 - 0.499 \right), \quad (4.67)\]

Using values of \(\chi^2\) around 1.1 GeV\(^2\), the two real roots of the equation for \(\omega\) are \(\omega \approx 1\) and \(\omega \approx 0.7\), the exact solutions depending on the precise value of \(\chi^2\) involved: once \(\omega\) has been fixed, substitution into eqs. (4.62) - (4.64) yields \(8\lambda z^2, 8\lambda z x, 8\lambda x^2, -12\nu x, -12\nu z\) and \(-2\nu^2\), while the last parameter, \(-2\nu\), is fixed by the particular value of \(\delta^2\) chosen.

At this stage, the masses of \(\chi^1\) and \(\sigma\) can be found, together with both mixing angles \(\theta\) and \(\theta^\prime\), and all of the coupling constants, i.e. we have predictions for the masses and widths. Although the basic coupling constants do not involve \(b\), i.e. do not depend on the value of \(\delta^2\), the physical constants, e.g. \(C(\chi^1\pi\pi)\) and \(C(\sigma\pi\pi)\), involve the mixing angles; since the sign of \(\theta^\prime\) can be quite important in deciding whether e.g. \(C(\sigma\pi\pi)\) is very much smaller than \(C(\sigma_0\pi\pi)\) because of cancellation between the two 'bare' terms, the chosen value of \(\delta^2\) is clearly important when it comes
to fitting some of the widths. Hence, for each $\kappa^2$ used, we have given $\delta^2$ several different values, all within about 15% of the experimental value of 0.933 GeV$^2$; the corresponding numbers for $\gamma^2$ and $\sigma^2$ have been calculated, together with the mixing angles and (less frequently) the widths. Because $\Gamma(N_N \to K\bar{K})$ does not involve any mixing angle while the $\gamma - \pi$ mixing angle $\theta$ does not depend on $\delta^2$, i.e. $\Gamma(\delta \to \gamma \pi)$ is independent of $\delta^2$, both of the widths remain fixed once $\kappa^2$ has been chosen. As $\Gamma(\delta \to \gamma \pi)$ is one of the "problem" predictions and thus serves as an indication of the degree of success of a particular solution, the whole series of widths has been calculated only for the value of $\delta^2$ leading to the best mass predictions.

C. $\rho = 0$; $\omega = 1$

When we apply the constraint $\omega = 1$, a great simplification occurs in many of the formulae, irrespective of whether $\alpha$ assumes the value unity or not. Some of the more useful relations are listed below.

\begin{align*}
(\gamma^2 + x^2 + \kappa^2) + (\gamma^2 + \sigma^2 + \delta^2) &= 3(\kappa^2 + K^2), \quad (4.68) \\
3\gamma_0^2 &= 4k^2 - \kappa^2, \quad 3\gamma_1^2 = 4\kappa^2 - \delta^2, \quad (4.69) \\
\frac{1}{\sqrt{2}}\gamma_0 x_0 &= \frac{1}{3}(K^2 - \kappa^2), \quad \frac{1}{\sqrt{2}}\gamma_0 \sigma_0 = \frac{1}{3}(\kappa^2 - \delta^2), \quad (4.70) \\
\gamma_0^2 - x_0^2 &= -\frac{1}{3}(K^2 - \kappa^2) - (\gamma^2 + x^2 + \kappa^2 - 3K^2), \quad (4.71) \\
\gamma_0^2 - \sigma_0^2 &= -\frac{1}{3}(\kappa^2 - \delta^2) + (\gamma^2 + x^2 + \kappa^2 - 3K^2), \quad (4.72)
\end{align*}
\begin{align*}
\kappa^2 - \kappa^2 &= -2b', \quad \kappa^2 - \delta^2 = +2b, \quad (4.73) \\
\kappa^2 - K^2 &= 8\lambda x^2 - 12\nu x, \quad (4.74) \\
-18\nu x &= \gamma^2 + x^2 + \pi^2 - 3K^2, \quad (4.75) \\
C(\delta \gamma_0 \pi) &= C(\gamma_0^\prime \pi \pi) = \frac{2}{\sqrt{3}} C(\delta \pi K) = -2C(\gamma_0^\prime K K) \\
&= -\frac{1}{\sqrt{3} f_\kappa} (\kappa^2 - K^2), \quad (4.76) \\
C(\delta \pi_0 \pi) &= C(\sigma_0 \pi \pi) = C(\sigma_0 \pi K) = -\frac{\sqrt{2}}{\sqrt{3} f_\pi} \left[ \kappa^2 - \left( \gamma^2 + x^2 + \pi^2 - 2K^2 \right) \right]. \quad (4.77)
\end{align*}

In this case, the use of three of the $0^-$ masses allows us to predict the fourth with excellent accuracy; the knowledge of two of the $0^+$ masses permits us to predict the remaining two, the most convenient pairs being $(\gamma^*, \sigma)$ and $(\kappa, \delta)$. If we use $\gamma^*^2$ and $\sigma^2$ as input, we find two solutions. The first practically coincides with Cicogna's solution while the second yields the unacceptable result $\delta^2 = 0.235$ GeV$^2$. It thus appears to be more convenient to use $\kappa^2$ and $\delta^2$ as input, and to vary the combination to obtain the best fit to $\gamma^*^2$ and $\sigma^2$.

D. $\rho \neq 0$

Since the only terms among the masses which actually contain a $\rho$ factor are $\gamma_0^2$, $\gamma_0^\prime \sigma_0$ and $\sigma_0^2$, then the methods employed for the previous solutions will also work here, to a large extent; the main difference occurs at the end of the calculation when one of
\( \gamma' \) or \( \sigma' \) has to be used as input to evaluate \( \rho \gamma' \). However, such solutions have exactly the same values of \( \lambda z^2 \), \( \gamma z \) etc. as in the cases where \( \rho = 0 \), so that these "new" solutions are hardly different from the "old" ones. Because of this, some calculations are also made with \( \gamma' \) and \( \sigma' \) as input (along with the 0 masses). The purpose is to fit both \( \gamma' \) and \( \sigma' \), which are not accurately predicted in any solution except Ciganova's, and then to use the extra freedom granted by the parameter \( \rho \) to try to obtain a completely different solution, i.e. one in which \( \lambda z^2 \), \( \gamma z \) etc. take on quite different values from these previously obtained.

One improvement which is hoped for is in the width predictions, particularly \( \Gamma(\delta \to \gamma \pi) \) and \( \Gamma(\pi_N \to K\bar{K}) \), since the appropriate coupling constants are independent of \( \rho \).

Briefly, a new variable \( Y \) is defined by \( Y = 4 \lambda z^2 (1 - \omega) \), so that

\[
-18 \nu x = \left( \gamma^2 + \chi^2 + \pi^2 - 3K^2 \right) + \left( 2 - \omega \right) Y .
\]

The value of \( (\gamma^2 - \chi^2) \) is used to produce an equation quadratic in both \( \omega \) and \( Y \), and this can be solved to give \( \lambda z^2 \) as a function of \( \omega \). A "trial and error" procedure is then employed: for each value of \( \omega \), \( \lambda z^2 \) can be found, while the sum \( \gamma'^2 + \sigma'^2 \) allows us to determine \( \rho z^2 \); finally, \( \gamma'^2 - \sigma'^2 \) can be calculated, and the value of \( \omega \) varied until the correct value of \( \gamma'^2 - \sigma'^2 \) is produced. Thereafter, the calculation of \( \kappa^2 \) and \( \delta^2 \) is straightforward, as is that of the widths.

The numerical results obtained in the different models described above are given in Table I.
<table>
<thead>
<tr>
<th>Model</th>
<th>$\chi^2$</th>
<th>$\delta^2$</th>
<th>$\eta^2$</th>
<th>$\sigma^2$</th>
<th>$\theta$</th>
<th>$\phi^2$</th>
<th>$\psi^2$</th>
<th>$R_{\eta}$</th>
<th>$R_{x}/R_{x}$</th>
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<td>Experiment</td>
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<td></td>
</tr>
<tr>
<td>$\phi = 0$:</td>
<td>1.166</td>
<td>0.933</td>
<td>1.130</td>
<td>0.518</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega = \alpha = 0$:</td>
<td>1.188</td>
<td>1.145</td>
<td>1.131</td>
<td>0.517</td>
<td>$-10^\circ 55'$</td>
<td>$-10^\circ 55'$</td>
<td>170</td>
<td>110</td>
<td>475</td>
</tr>
<tr>
<td>$\omega = \alpha = 1$:</td>
<td>0.691</td>
<td>0.918</td>
<td>0.635</td>
<td>0.021</td>
<td>$-10^\circ 55'$</td>
<td>$-10^\circ 55'$</td>
<td>50</td>
<td>25</td>
<td>0.01</td>
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<tr>
<td>$\phi = 0$:</td>
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</tr>
<tr>
<td>$\omega = 1, \alpha \neq 1$:</td>
<td>1.050</td>
<td>0.920</td>
<td>1.076</td>
<td>0.595</td>
<td>$-10^\circ 55'$</td>
<td>$+7^\circ 15'$</td>
<td>115</td>
<td>70</td>
<td>65</td>
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<tr>
<td>$\omega = 1$:</td>
<td>1.108</td>
<td>0.910</td>
<td>1.159</td>
<td>0.686</td>
<td>$-10^\circ 55'$</td>
<td>$-10^\circ 55'$</td>
<td>145</td>
<td>95</td>
<td>130</td>
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<tr>
<td>$\phi = 0$:</td>
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<tr>
<td>$\omega \neq 1$:</td>
<td>1.200</td>
<td>1.142</td>
<td>1.448</td>
<td>0.386</td>
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<td>$-10^\circ 25'$</td>
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<td>65</td>
<td>275</td>
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<tr>
<td>$\alpha = 1$:</td>
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</table>

Note: The table contains values for different models and parameters, including chi-squared ($\chi^2$), delta ($\delta^2$), eta ($\eta^2$), sigma ($\sigma^2$), theta ($\theta$), and phi ($\phi^2$) values, along with other parameters such as $R_{\eta}$ and $R_{x}/R_{x}$. The table provides a comparison of these values across different conditions.
| Model       | $\chi^2$ | $\delta^2$ | $\eta^2$ | $\sigma^2$ | $\phi$ | $\phi'$ | $\Gamma(\phi)$ | $\Gamma(\phi')$ | $\Gamma(\phi'\phi)$ | $\Gamma(\phi'k)$ | $\Gamma(\phi'\pi)$ | $\Gamma(\phi'\pi')$ | $\Gamma(\phi'k\pi)$ | $\Gamma(\phi'k\pi')$ | $\Gamma(\phi'k\pi\pi')$ | $\Gamma(\phi'k\pi\pi')$ | $R_\nu$ | $\frac{f_K}{f_K}$ |
|-------------|----------|------------|----------|------------|--------|--------|----------------|----------------|----------------------|----------------|----------------|----------------|----------------|----------------|----------------|------------------|------------------|--------|------------------|
| Experiment  | $1.166$  | $0.933$    | $1.130$  | $0.518$    |        |        |                |                |                      |                |                |                |                |                |                |                  |                  |        |                   |
| $\rho = 0$: |          |            |          |            |        |        |                |                |                      |                |                |                |                |                |                |                  |                  |        |                   |
| $\omega \neq 1$, $\alpha \neq 1$. | 1.050 | (0.834, 1.204, 0.295) | (1.197, 0.207) | (1.185, 0.117) | $-4^030'$ | $-14^045'$ | 115 | 40 | 30 | 180 | 120 | 300 | 1.5 | 1.25 |
| $\omega = a = 1$. |          |            |          |            |        |        |                |                |                      |                |                |                |                |                |                |                  |                  |        |                   |
| $\rho \neq 0$, $\omega = a = 1$. | 0.613 | (0.639, 1.130, 0.518) | (1.147, 0.367) | (1.442, 0.317) | $-10^055'$ | $-79^050'$ | 40 | 15 | 265 | 970 | 500 | 1470 | 1.9 | 1 |
| $\rho \neq 0$, $\omega = a = 1$. | 1.108 | (1.043, 1.130, 0.518) | (1.447, 0.367) | (1.717, 0.317) | $-10^055'$ | $0^035'$ | 145 | 95 | 75 | 400 | 50 | 450 | 8.0 | 1 |
| $\alpha = 1$. |          |            |          |            |        |        |                |                |                      |                |                |                |                |                |                |                  |                  |        |                   |
| $\rho \neq 0$, $\alpha = 1$. | 1.166 | (1.301, 1.130, 0.518) | (1.741, 0.317) | (1.741, 0.317) | $-10^055'$ | $-8^005'$ | 160 | 105 | 315 | 375 | 75 | 450 | 5.0 | 1 |
| $\rho \neq 0$, $\alpha = 1$. | 0.584 | 0.817 | 1.130 | 0.518 | $-10^025'$ | $71^015'$ | 35 | 15 | 45 | 1800 | 740 | 2540 | 2.4 | 0.77 |
Footnotes for Table I

a) The numerical values used for the kinematic factors \( F(\alpha\beta\gamma) \) appearing in the formulae for the decay widths, (see eqs. (4.19)), viz.

\[
\Gamma(\alpha \to \beta\gamma) = C^2(\alpha\beta\gamma) F(\alpha\beta\gamma),
\]

are given below in units of \( \text{GeV}^{-1} \):

\[
\begin{align*}
F(\delta\eta\pi) & = 0.01315, & F(\pi^+_N K^0) & = 0.00855, \\
F(\eta'\pi\pi) & = 0.0270, & F(\eta'K^0) & = 0.01335, \\
F(\sigma\pi\pi) & = 0.0382.
\end{align*}
\]

b) Because of the large uncertainties in the experimental values of the decay widths, all width calculations (in units of MeV) have been rounded off to the nearest 5 MeV.

c) \[
R_{\eta'} = \frac{\Gamma(\eta' \to \pi\pi)}{\Gamma(\eta' \to K\overline{K})}.
\]

d) As mentioned in the text, the experimental data are based largely on the Rosenfeld Tables of August, 1970. However, in the most recent edition of the Tables (April, 1971), several modifications to the previous results have been made.

(1) The \( \delta \) mass has returned to its original value of 962 MeV, but a new \( \pi^+_N \) has appeared at 975 MeV, with a width of \( 58 \pm 11 \) MeV. This last value is much more in line with the other \( 0^+ \) decay widths than that of the \( \delta \), which is less than 5 MeV.

(2) The situation for the \( \eta'(1060) \) has become less clear, with a possibility of two different particles here corresponding to the two distinct final states \( K\overline{K} \) (leading to a mass of 1070 MeV) and \( \pi^+\pi^- \) (1080 MeV). In addition, widely differing
values are quoted for the widths, ranging from 40 MeV to 360 MeV. The main table confines the total width to the region 150 - 300 MeV, and gives an upper limit for the branching ratio $R_{\eta}$ of 1.86. Using these numbers, we have written down approximate inequalities for the partial widths $\Gamma(\eta' \to \pi\pi)$ and $\Gamma(\eta' \to KK)$.

(3) The most recent data for the $\sigma$ particle give only a very broad resonance in the region 700 - 1000 MeV. However, earlier experiments indicated a resonance around 720 MeV, with a width of approximately 150 MeV, and these particular values have been used in the present work to judge the accuracy of the different predictions.
VI. Discussion of Results

In this section, we discuss the various solutions in turn, and try to explain why the results are not particularly satisfactory.

A. \( \rho = 0, \omega = a = 1 \). (Cicogna's Solution). This is the simplest model considered, with \( \mathcal{L}_0 \) a linear combination of the three independent \( SU(3) \otimes SU(3) \) invariants \( I_1, I_2 \) and \( I_4 \), while the \( SU(2) \otimes SU(2) \) symmetry of \( \mathcal{L}_1 \) is broken only by the linear term \( g_0 U_a^a \). Despite the lack of complication, a remarkable fit to the mass spectrum is obtained, with the exception of the \( \delta \) which is too high. The explanation for this last result is that in this model, the \( 0^- \) and \( 0^+ \) spectra ought to be mirror images of each other, with the same scale; apart from the \( \delta \) appearing as the lightest member instead of the heaviest, this corresponds closely to the experimental situation.

However, the width predictions are very much poorer: only \( \Gamma(\gamma' \to KK) \) appears to lie within the experimentally allowed range, while all of the others are too large. The reason why some of these widths are too large is that the 'bare' coupling constants \( C(\gamma_0) \) and \( C(\gamma_0^\pi\pi) \) are equal to each other and proportional to the factor \( (\kappa^2 - K^2) \); in addition, the two (equal) mixing angles are small and negative, so that \( |C(\delta\gamma\pi)| \) and \( |C(\gamma'\pi\pi)| \) are not very much smaller than the 'bare' values. Since the latter are too large for the value of \( \kappa^2 \) used, it is interesting to work in reverse, fit \( \Gamma(\delta\gamma\pi) \) approximately, and find out what value of \( \kappa^2 \) is consistent with this width. For \( \Gamma(\delta KK) = 25 \) MeV, we find \( \Gamma(\delta\gamma\pi) = 53 \) MeV, which is very good; but we also predict \( \kappa^2 = 0.691 \) GeV\(^2\), which is very low. And worse is in store: such...
a low value of $\kappa^2$ gives us the very low value of 1.574 GeV$^2$ for the combination $\gamma^2 + \sigma^2 + \delta^2$, whereas the experimental value of $\gamma^2 + \sigma^2$ alone is 1.648! In fact, explicit calculation gives $\gamma^2 = 0.635$, again about half as large as it should be, while $\sigma^2 = 0.021$, i.e. the $\sigma$ is practically degenerate with the pion!

Hence, it is impossible to have low $\delta$ widths and simultaneously obtain a good fit to the mass spectrum in this model.

B. $\rho = 0, \omega = 1, \alpha \neq 1$. The basic solution here is the same as that in (A), i.e. we use $\chi^2, K^2, \gamma^2$ and $\kappa^2$ to fix $8\lambda x^2, -12\nu x$, and $b'$, but we have one extra degree of freedom for the $0^+$ states, since we have altered the scales of the two octets by letting $\alpha$ vary.

We use $\delta^2$ as our extra input to fix $b$ (i.e. $\alpha$), and for each value of $\kappa^2$, several different values of $\delta^2$ are used to investigate the variation in $\gamma^2, \sigma^2, \theta'$ and the various widths, although the last mentioned vary only to the extent that $\theta'$ changes.

This time, although there is not the spectacularly accurate set of mass predictions of (A), we are able to obtain the correct form for the $0^+$ spectrum, and with errors of less than 10%. In addition, all of the widths decrease as we use lower values of $\kappa^2$ (as expected), but even for $\kappa^2 = 0.990$ GeV$^2$ (i.e. about 15% lower than the experimental value), $\Gamma(\delta \to \gamma \pi)$ is still 114 MeV, i.e. about double the value we would like it to be, and the other widths are also too large.

The same reasons apply as before: for $\omega = 1$ (even when $\rho \neq 0, \alpha \neq 1$), i.e. when the vacuum is exactly SU(3) invariant, we have the relation $C(\delta \gamma \pi) \propto (\kappa^2 - K^2)$ and $\theta \approx -11^0$ for
$\kappa^2 \approx 1.1 \text{ GeV}^2$, so that we just cannot fit both $\kappa^2$ and $\Gamma(\delta \rightarrow \gamma \pi)$.

C. $\rho = 0$, $\omega \neq 1$, $\alpha = 1$. Clearly, for $\rho = 0$, this type of solution (with $\omega \neq 1$) is our only hope of fitting $\kappa^2 \geq 1 \text{ GeV}^2$ and $\Gamma(\delta \rightarrow \gamma \pi) \approx 50 \text{ MeV}$. Unfortunately success still eludes us: although some of the other widths are slowly approaching better values, both $\Gamma(\delta \rightarrow \gamma \pi)$ and $\Gamma(\eta' \rightarrow \pi \pi)$ are still too high, by a factor of 2-3. In addition, the predictions for $\gamma^2$ and $\delta^2$ are rather poor.

D. $\rho = 0$, $\omega \neq 1$, $\alpha \neq 1$. Once again, the basic solution here is the same as that in (C), but with the extra freedom given by an unconstrained $\delta^2$. However, since the values of $\delta^2$ were one of the few better features of solution (C), the extra freedom is hardly necessary, and thus cannot really solve the main problems of the high widths.

The reason for this failure to fit the $0^+$ spectra at the same time as the decay $\Gamma(\delta \rightarrow \gamma \pi)$ becomes a little clearer if we consider eqs. (4.58) - (4.63), which express the various parameters in terms of the masses. With $\rho = 0$, and all of the experimental values being used for the masses, we find from eqs. (4.58) - (4.59) that $\omega^2 = 0.425$, i.e. $\omega = 0.651$, and $8\lambda z^2 = 0.727$. The important factor is $8\lambda z^2 - 12 \nu x = 0.754$.

Now, if we assume that the $\gamma - X$ mixing angle $\theta$ is small, then a value for $\Gamma(\delta \rightarrow \gamma \pi)$ of 50 MeV implies that $|C(\delta \gamma o\pi)| \approx 2 \text{ GeV}$. Hence, using $f_{\pi} = 130 \text{ MeV}$, we obtain the desired value of
Thus, neglecting the unwelcome case of a large mixing angle $\theta$, we see that even for $\omega \neq 1$, but for a value of $\omega$ deduced from the experimental values for the masses, the resulting value for $|C(\delta \gamma_0 \pi)|$ is still far too high above that necessary to fit $\Gamma(\delta \to \gamma \pi)$, although there is some improvement on the value obtained when $\omega = 1$. In fact, when we do a similar check on the other troublesome coupling constant, we find that for $\Gamma(\eta' \to \pi \pi) \approx 150$ MeV, a small value for $\theta'$ requires that $|C(\gamma' \pi \pi)| \approx 2$ GeV. This is encouraging in one respect, since for $\rho = 0$, $C(\delta \gamma_0 \pi) = C(\gamma' \pi \pi)$; but it also underlines the necessity for reducing the value of $8\lambda x^2 - 12 \gamma x$.

There are several ways in which this last feat may be achieved. Writing the relevant expression in the form

$$8\lambda x^2 - 12 \gamma x = 8\lambda x^2 (1 - \frac{1}{\omega}) + (\kappa^2 - K^2),$$

we see that a drastic reduction of $\kappa^2$ will work: but, as shown in Table I, a value of $\kappa^2 = 0.691$ GeV$^2$ is required for $\Gamma(\delta \to \kappa \kappa)$ = 25 MeV when $\omega = 1$, and such a low value of $\kappa^2$ leads to a completely unacceptable value of $\sigma^2$. Also, for the cases when $\omega \neq 1$, inspection of Table I shows that $\kappa^2$ would still have to be well below 1 GeV$^2$ before $\Gamma(\delta \to \gamma \pi)$ came near the experimental value. Alternatively, we could try to maintain $\kappa^2$ around 1 GeV$^2$, provided that $8\lambda x^2 (1 - \frac{1}{\omega})$ was large enough to cancel most of the term $(\kappa^2 - K^2)$. Since the experimental masses do not give a sufficiently large term, then we either increase $8\lambda x^2$ or decrease $\omega$. The first case could be realised by increasing $\delta^2$ and/or reducing $\kappa^2$, but the main effect of this is to let $\omega$ tend to unity, a
case which has already been shown to be unsatisfactory. If we keep $\delta \lambda z^2$ fixed, e.g. by keeping the combination $2\delta^2 - 3 \kappa^2$ constant, then $\omega$ falls if $\delta \lambda z^2$ rises, i.e. if the term $6\kappa^2 - \delta^2 = 2(3\kappa^2 - 2\delta^2) + 3\delta^2$ increases: thus, we require larger values of $\delta^2$ and smaller values of $\kappa^2$, with the combination $(3\kappa^2 - 2\delta^2)$ fixed. However, the indications from the solutions are that the required values of both $\kappa^2$ and $\delta^2$ would be well removed from the experimental region.

Thus, when $\rho = 0$, the solutions for $\omega = 1$ and $\omega \neq 1$ are both unable to fit the mass spectra and the two widths $\Gamma(\delta \to \eta \pi)$ and $\Gamma(\eta' \to \pi \pi)$ simultaneously.

E. $\rho \neq 0$. For the simplest case, with $\alpha = 1$ and $\omega = 1$, i.e. with SU(3) invariance of the vacuum and approximate SU(2) $\otimes$ SU(2) invariance of the Lagrangian, the use of $\eta'^2$ and $\sigma^2$ as input yields two solutions, one corresponding to a very small value of $\rho$, and thus practically coinciding with Cicogna's model, the other yielding a completely different solution. Instead of having $\delta \lambda z^2 \simeq 0.6$ GeV$^2$, as in the previous solutions, we now find $\delta \lambda z^2 = 0.033$, whereas $\delta \rho z^2 = 0.333$. Unfortunately, the $\kappa$ mass is very low, with $\kappa^2$ about half of the experimental value, while $\delta^2$ is also about 100% too small. However, as may be expected from a solution with such a low $\kappa^2$ value, the width $\Gamma(\delta \to \eta \pi) \approx 40$ MeV, which is by far the best prediction so far obtained. In addition, $\Gamma(\delta \to K\bar{K}) \approx 17$ MeV and $\Gamma(\sigma \to \pi \pi) \approx 267$ MeV, both of which are pretty good predictions. But the $\eta'$ widths are enormous, with $\Gamma(\eta' \to \pi \pi) = 970$ MeV, and so this particular solution fails badly at one or
two points, even though it succeeds well at others.

Another rather odd feature of this solution is the very large mixing angle $\theta'$ for the $\eta' - \sigma$ system; this comes about because $\eta_0'^2 = 0.537 \text{ GeV}^2$, which is far removed from the physical value of $\eta'^2 = 1.130 \text{ GeV}^2$. This large mixing angle partly explains why we can have $\Gamma(\delta \rightarrow \eta \pi)$ quite low for once, while $\Gamma(\eta' \rightarrow \pi \pi)$ takes a ridiculously high value; it is $C(\sigma_0 \pi \pi)$ which plays the dominant role in $C(\eta' \pi \pi)$. The second part of the explanation is that the coupling constants for $\eta_0'$ and $\sigma_0$ involve $\rho$ factors (which vanish in some cases for $\omega = 1$), whereas the $\delta$ couplings do not involve $\rho$; thus, for a solution with $|\lambda/\rho| < 1$, it is not surprising that the $\eta'$ constants are much larger than the $\delta$ ones.

When we allow $\alpha \neq 1$, we must use either $\chi^2$ or $\delta^2$ as our final input (mass)$^2$. If we take advantage of this to restore $\chi^2$ to a value much nearer the experimental one, then we find that one of the predicted values for $\delta^2$ is reasonably accurate; but we must remember that we have used three $0^-$ masses and three $0^+$ masses to predict the remaining two, so that our achievement is hardly surprising. What is also not surprising is the return to high values of the decay widths of the $\delta$; since we use $\chi^2 \approx 1.1 \text{ GeV}^2$ and the $0^-$ spectrum as the remaining input, the value of $\lambda z^2$, $v_z$ and $b'$ are exactly the same as those derived when $\rho = 0$. The only difference now is that we have an extra parameter, $\rho$, with which to arrange a good fit to the $\eta' - \sigma$ system. The whole discussion given earlier about the impossibility of obtaining a simultaneous fit to the mass spectra and the decay widths applies here, since the inclusion of $\rho$ has only a small
effect on the mass spectrum.

For $\rho \neq 0$, $\omega \neq 1$, but $\alpha = 1$, we meet a situation similar to that for $\rho \neq 0$ and $\omega = 1$; $\kappa^2$ and $\delta^2$ are again too low, particularly $\kappa^2$, while the $\delta$ widths are almost unchanged, and the $\eta'$ widths have become even worse. The one change is in the $\eta' - \sigma$ mixing angle, which has changed sign, although remaining numerically very large; this change of sign is partly responsible for the large increase in $\Gamma(\eta' \to \pi \pi)$ and the simultaneous decrease in $\Gamma(\rho \to \pi \pi)$, the other relevant factor being the increase in $|C(\eta_0 \to \pi \pi)|$ due to the $\rho$ term becoming operative.

In fact, the greater freedom allowed by having $\rho \neq 0$ can be seen from eqs. (14.58) – (14.60) again. As pointed out before, solutions with $\omega = 1$ are not welcome. Now, for $0 < \omega < 1$, eqs. (14.58), (14.59) are consistent with having $\lambda > 0$, $\rho < 0$, and $|\lambda/\rho| > 1$, i.e. the effect of $\rho \neq 0$ is rather small. On the other hand, for $|\lambda/\rho| < 1$, we require $\rho < 0$, $\lambda > 0$ and $\omega > 1$; this is the solution appearing in Table I, corresponding to $\omega = 1.848$, and is thus quite distinct from all of the previous solutions. However, from the simple relation

$$f_K/f_\pi = \frac{\omega + 1}{2\omega},$$

we see that solutions with $\omega > 1$ are not particularly desirable, since the currently accepted value of the $\theta$s is approximately 1.28, corresponding to $\omega = 0.64$: for $\omega = 1.848$, we have $f_K/f_\pi = 0.77$.

Thus, the "completely different" solutions involving $\rho$ are not as successful as was hoped. Certainly the value for $\Gamma(\rho \to \eta \pi)$ is easily the best one obtained from any solution, and falls into the range at present indicated by experiment; but this is more than balanced by the ridiculous values obtained for $\Gamma(\eta' \to \pi \pi)$. 
The poor predictions for $\chi^2$ provide further evidence that these solutions are not very attractive, while the ratio of $f_N/f_\pi$ indicates that we should confine our attention to those solutions with $\omega \leq 1$, i.e. the original solutions with $\rho = 0$.

VII. Conclusions

In the light of the above disappointing results, it seems worthwhile to attempt to pick out the weakest points of the present approach.

Perhaps the most obvious approximation which we have made is to confine our attention to the Tree Graph Approximation (always assuming that the T.G.A. is exactly equivalent to this Classical Field Theory Approach). Higher order corrections from loop diagrams may well alter the results significantly. Another serious restriction is the absence of any other type of particle in the model, e.g. spin $1/2$, spin 1, etc. Although the presence of such states may not affect the results directly, it may well have an indirect influence on the decay rates, for example, through unitarity conditions. And there is always the criticism of a field theory model that once the Lagrangian has been specified, the dynamics of the system under consideration are fully determined; such a criticism is probably more relevant to the calculations of the decay rates than to those of the masses, and this may well explain why more accurate width predictions for certain states proved extremely elusive, even when some variations in the mass spectra were allowed.

It has been suggested that another source of error in our
calculation of the decay rates is our use of the third derivatives of the Lagrangian, evaluated at the stability point, as the appropriate coupling constants. The proposal is that these derivatives actually correspond to the coupling constants extrapolated to zero four-momentum for the states involved. At the present time, we do not understand why this should be so. Furthermore, if this were so, then the effect should possibly be greatest for the decays \( \eta' \to K\overline{K} \), and \( \delta \to K\overline{K} \), and least for \( \sigma \to \pi\pi \); as it is, the predictions for all three decays are not nearly as bad (in general) as those for \( \delta \to \eta\pi \) and \( \eta' \to \pi\pi \), which are consistently too high, but which might be expected to be somewhat more accurate if the main errors lie in the extrapolation to zero momentum.

Thirdly, we have considered only \((3, \overline{3}) \oplus (\overline{3}, 3)\) symmetry breaking terms here. This is certainly in line with current ideas, although very recently, suggestions have been made that the \((1, 8) \oplus (8, 1)\) contributions may not be negligible. Such an argument contains some interesting implications which are of relevance to matters beyond those involving the \(\sigma\)-model. Accordingly, we shall close the present chapter at this point, and proceed to a short discussion of some more general aspects of symmetry breaking in the next chapter.
CHAPTER V

SOME DIFFERENCES BETWEEN $(3, \bar{3}) \oplus (\bar{3}, 3)$ AND $(1, 8) \oplus (8, 1)$ SYMMETRY-BREAKING TERMS.

1. Introduction

The purpose of this chapter is to try to relate some of the problems arising in earlier chapters to a common origin, namely, the transformation properties under chiral $SU(3) \times SU(3)$ of the medium strong $SU(3)$ symmetry-breaking part of the Hamiltonian.

In Chapter III, we mentioned the hypothesis that the octet parts of the strong, electromagnetic and parity conserving non-leptonic weak Hamiltonians all belong to the same octet, and that this raises problems concerning $H_{ms}$ and $H_{pc}$, since the generally accepted forms for the Hamiltonians are quite different. The first involves the scalar quark density $u_8$, which belongs to the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation of $SU(3) \times SU(3)$, whereas the second has the usual current-current form, with the octet part proportional to the expression $d^{i j l}(V^i_\mu V^j_\mu + A^i_\mu A^j_\mu)$, and this transforms according to the $(1, 8) \oplus (8, 1)$ representation.

In Chapter IV, symmetry breaking of the larger $SU(3) \times SU(3)$ group was investigated in terms of the $\sigma$-Model, but the only breaking terms considered were those transforming under the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation. A remarkably good fit to the whole $0^+$ and $0^-$ mass spectra was obtained using a very simple form for $H$ (or $\mathcal{L}$), with the one notable exception of the $\delta$ mass variations on the form of symmetry breaking, both in $H$ and in the
vacuum, failed to bring the δ into line without spoiling the other good results (as was expected, of course), and the suggestion was made that a \((1, 8) \oplus (8, 1)\) symmetry breaking term might possibly have a useful effect.

Further motivation for looking more closely at the relative importance of the \((1, 8)\) and \((3, \bar{3})\) terms (we abbreviate the notation when no confusion is likely to arise) comes from two main sources. The first is the problem of the \(\xi\) parameter in \(K_{43}\) decays\(^{11}\): according to the Gell-Mann, Oakes, Renner (GOR) symmetry breaking scheme, the value of \(\xi\) ought to be close to zero whereas recent experiments indicate a value around \(-0.65\). In fact, Arnowitt et al.\(^{42}\) claim that the effect of a \((1, 8)\) non-pole term in \(H\) allows a good fit to \(\xi\), on the basis of a hard pion calculation. Secondly, Cheng and Dashen\(^{43}\) have used \(\pi N\) phase shifts and fixed-\(t\) dispersion relations to calculate the nucleon matrix element of the "\(\sigma\)-commutator", and have found a value which is an order of magnitude larger than that predicted by the \((3, \bar{3})\) symmetry breaking model; this also indicates that \(SU(2) \otimes SU(2)\) breaking is comparable to \(SU(3)\) breaking.

Thus, it would appear to be an interesting and possibly quite useful exercise to consider the basic assumptions made in the GOR model, and also to investigate one or two situations where \((3, \bar{3})\) and \((1, 8)\) terms could have different effects; this is what we try to do in the present chapter. We begin by describing the different transformation properties of the \((1, 8)\) \(\oplus (8, 1)\) and \((3, \bar{3}) \oplus (\bar{3}, 3)\) states, and then show that a symmetry-breaking term belonging to the former representation couples to the \(0^+\) and \(0^-\) states with the same sign whereas
a $(3, 3)$ term couples with opposite signs, when this coupling is $SU(3) \otimes SU(3)$ invariant. Next, we consider the reasons why GOR conclude that the $(1, 8)$ terms have no effect, in first order, on the $0^-$ masses, and then show that this $(1, 8)$ term contributes only if the matrix elements are not constrained to be $SU(3)$ invariant, i.e. if we allow further spontaneous breaking from $SU(3)$ to $SU(2) \otimes U(1)$; even then, the effect appears only in the $K$ masses. Finally, we take a brief look at the $K_{3\bar{3}}$ system and indicate that it is possible for a $(1, 8)$ term to have an influence on the form factors, and hence on $\zeta$, even when the vacuum is $SU(3)$ symmetric, provided that the variation of the coupling off-shell is not as smooth as possible but behaves like $p.k$, where $p$ and $k$ are the four-momenta of the two particles involved.

II. Transformation and Combination Properties

First of all, we look at the difference in transformation properties of the $(1, 8) \oplus (8, 1)$ and $(3, 3) \oplus (\bar{3}, \bar{3})$ representations.

Now, the "left-handed" generator $L^i = \frac{1}{2}(V^i + A^i)$ itself transforms as an octet operator under the left-handed group, and as a singlet under the right-handed one.

$$[\frac{1}{2}(V^i + A^i), L^j] = i\epsilon^{ijk}L^k; \quad [\frac{1}{2}(V^i - A^i), L^j] = 0. \quad (5.1)$$

Thus, $L^i$ belongs to the $(8, 1)$ representation of $SU(3) \otimes SU(3)$; similarly, $R^i = \frac{1}{2}(V^i - A^i)$ belongs to the $(1, 8)$ representation. However, $V^i$ and $A^i$ separately do not belong to a single representation of the chiral group,
but only to the combination \((1, 8) \oplus (8, 1)\); also, under the parity transformation,

\[
L^i(x, t) \leftrightarrow R^i(-x, t) .
\]

Later, we shall use the Lorentz scalars \(g^i\) and pseudoscalars \(h^1\) \((i = 1, \ldots, 8)\) which also belong to the \((1, 8) \oplus (8, 1)\) representation; their commutation relations with \(V^i\) and \(A^i\) are

\[
\begin{align*}
[V^i, g^j] &= i\epsilon^{ijk} g^k , \quad [A^i, g^j] = i\epsilon^{ijk} g^k , \\
[V^i, h^j] &= i\epsilon^{ijk} h^k , \quad [A^i, h^j] = i\epsilon^{ijk} g^k .
\end{align*}
\]

The other quantities which we shall deal with are the scalars \(u^i\) and pseudoscalars \(v^i\) \((i = 0, 1, \ldots, 8)\) which belong to the \((3, \bar{3}) \oplus (\bar{3}, 3)\) representation: the combinations

\[
\begin{align*}
\omega^+ &= \frac{1}{\sqrt{2}} (u + iv) \quad \text{and} \quad \omega^- = \frac{1}{\sqrt{2}} (u - iv)
\end{align*}
\]

belong to the individual representations \((3, \bar{3})\) and \((\bar{3}, 3)\) respectively.

The commutation relations of \(u^j\) and \(v^j\) with \(V^i\) and \(A^i\) are

\[
\begin{align*}
[V^i, u^j] &= i\epsilon^{ijk} u^k , \quad [A^i, u^j] = -i\epsilon^{ijk} v^k , \\
[V^i, v^j] &= i\epsilon^{ijk} v^k , \quad [A^i, v^j] = +i\epsilon^{ijk} u^k .
\end{align*}
\]

The relation between the structure constants \(\epsilon^{ijk}\) and \(\delta^{ijk}\) involved in the commutators and the \(\text{SU}(3) \otimes \text{SU}(3)\) representations of \(u^j, v^j\) etc. can be found by expressing the latter quantities in terms of the basic spinors \(\mathbf{b}\) and their conjugates \(\bar{\mathbf{b}}\) of the "diagonal" \(\text{SU}(3)\) group. We have

\[
\begin{align*}
u^j &\propto \bar{b} \gamma^j \lambda^0 \mathbf{b} , \quad v^j \propto -i\bar{b} \gamma^j \lambda_5 \mathbf{b} , \\
v_j &\propto \bar{b} \gamma^j \lambda^0 \gamma^\mu \mathbf{b} , \quad \lambda^j \propto \bar{b} \gamma^j \lambda^0 \gamma^\mu \gamma_5 \mathbf{b} .
\end{align*}
\]

(5.5)
where $\gamma_\mu$ and $\gamma_5$ are the usual Dirac matrices and the $\lambda^j$ are the Gell-Mann matrices. The basic "left" and "right" spinors, $\ell$ and $r$, of the chiral $SU(3)$ groups are related to $b$ by

$$\ell = \frac{1}{2} (1 + \gamma_5) b, \quad \bar{\ell} = \bar{b} \frac{1}{2} (1 - \gamma_5),$$
$$r = \frac{1}{2} (1 - \gamma_5) b, \quad \bar{r} = \bar{b} \frac{1}{2} (1 + \gamma_5).$$

(5.6)

If we introduce the matrix notation, e.g. $u_{a}^{b} = \frac{1}{\sqrt{2}} u_{\lambda}^{(\lambda_1)} b$, (where we sum over $i$), then we can easily find

$$(w^+)_{a}^{b} \propto \bar{r}^{b} \ell_{a} \quad \text{and} \quad (w^-)_{c}^{d} \propto \bar{\ell}^{d} r_{c},$$

(5.7)

which shows immediately that $w^+$ belongs to the $(3, \bar{3})$ representation, and $w^-$ to $(\bar{3}, 3)$. These relations will prove useful shortly.

Now we turn to the effects of breaking the $SU(3)$ symmetry in the Hamiltonian by linear terms $u_8$ and $g_8$. Since we can distinguish between $u_8$ and $g_8$ only when we consider their properties under the larger chiral $SU(3) \otimes SU(3)$ group, then we must look at situations where these properties play a part. In general, it is no good confining our attention to the $0^-$ spectrum, for example, since it is only the $SU(3)$ nature of $u_8$ and $g_8$ which is important here: we must consider the effects on the combined $0^+$ and $0^-$ spectra in order to detect $SU(3) \otimes SU(3)$ effects. However, one exception to this arises when we use PCAC and a soft meson treatment, since this introduces commutators involving the axial charges, and the $(1, 8)$ and $(3, \bar{3})$ terms yield different structure constants: this case will be considered in more detail later.
Now, when we break the symmetry of $H$ from $SU(3) \otimes SU(3)$, the simplest way is to add on a term $u_0$ from the $(3, \overline{3})$ representation (there is no $(1, 8)$ counterpart, of course). At one time, it was thought that this was the dominant symmetry-breaking effect, and that the subsequent breakdown to $SU(2) \otimes U(1)$ was much smaller. More recently, however, the proposal was made that a better symmetry for $H$ was chiral $SU(2) \otimes SU(2)$, with the breakdown from $SU(3) \otimes SU(3)$ achieved by the combination $u_0 - \sqrt{2}u_8$; this is particularly simple since both $u_0$ and $u_8$ fall into the same representation of $SU(3) \otimes SU(3)$. Thus, because there is a distinct possibility that the coefficient of $u_8$ may not be appreciably smaller than that of $u_0$, we look for $SU(3) \otimes SU(3)$ invariant couplings of $u_8$ and $g_8$ to the $0^+$ and $0^-$ fields, not just $SU(3)$ invariant couplings.

For the case of $g_8$, the problem confronting us is to find a suitable combination of two $(3, \overline{3}) \pm (\overline{3}, 3)$ terms which will combine with a $(1, 8) \pm (8, 1)$ term to yield a $(1, 1)$ factor. Clearly, the combination

$$(3, \overline{3}) \otimes (3, \overline{3}) = (6 + \overline{3}, \overline{3} + 3)$$

cannot possibly give the desired result, whereas the grouping

$$(3, \overline{3}) \otimes (\overline{3}, 3) = (1 + 8, 1 + 8)$$

looks very promising. In fact,

$$(1 + 8, 1 + 8) \otimes (1, 8)$$

yields $(1, 1)$ if we consider only the singlet part of the left-handed (lh) combination $(1 + 8)$, and also contract the two $8$ terms from the right-handed (rh) groups. In matrix notation,
we must combine $(w^+)_a^b$ with $(w^-)_c^d$ to yield a lh singlet
and a rh octet: the correct combination, using $(w^+)_a^b = \bar{r}^b \ell_a$
and $(w^-)_c^d = \bar{c}^d \bar{c}_c$, is

$$(\bar{r}^b \bar{r}_c - \bar{r}^a \bar{r}_c \delta^b_c) \bar{r}^b \ell_a \delta^d_c = \left[(w^+)_a^b (w^-)_c^d - \delta^b_c \text{Tr}(w^+ w^-) \right] \delta^d_a .$$

Next, we contract this with a $(1, 3)$ term, i.e.

$$Y^f_c = \frac{1}{\sqrt{2}} (G - \bar{H})^f_c \equiv \frac{1}{2} (g^i - h^i) (\lambda^i)^f_c ;$$

this gives us $Y^f_c (w^+)_a^b (w^-)_b^c$, the second term disappearing since

$$Y^f_c = 0 .$$

Finally, since we want to maintain invariance under
parity transformations, we must take the combination

$$\text{Tr}(Y w^+ w^- + X w^- w^+),$$

where $X$ belongs to the $(8, 1)$ representation. In the case under consideration, i.e. the couplings of $g_8$
to the $w_1$ fields, $X = Y$ and the $SU(3) \otimes SU(3) \otimes P$ term
becomes $\text{Tr} \left[ X (w^+ w^- + w^- w^+) \right] = \text{Tr} \left[ X (U^2 + V^2) \right]$. Thus, $g_8$
couples to the scalars and pseudoscalars with the same sign.

When we come to the scalar "field" $S_8$ (we temporarily use $S, P$
to distinguish between the particle fields $u, v$ and the densities
$S, P$), the combination of fields which we must now take is the
first one, i.e. $(6 \oplus \overline{3}, \overline{6} \oplus 3)$: then we couple this to
another $(3, \overline{3})$ term, using only the $\overline{3}$ and $3$ terms above, i.e.
the antisymmetric combinations from both the lh and rh groups.

The first combination of terms is

$$(w^+)_a^b (w^+)_c^d - (w^+)_a^c (w^+)_c^b - (w^+)_c^b (w^+)_a^d + (w^+)_c^d (w^+)_a^b$$

$$= \left[ \varepsilon^{ab \gamma} \varepsilon^{\lambda \mu \nu} (w^+)_a^\lambda (w^+)_a^\mu \right] \varepsilon^{\gamma \alpha \varepsilon} \varepsilon^{\nu \beta} .$$
Contracting this with the other \((3, \bar{3})\) term \(\frac{1}{\sqrt{2}}(S + iP) = T^+\) gives

\[ \varepsilon^{\alpha\beta\gamma} \varepsilon_{\lambda\mu\nu} (w^+, \lambda \gamma \gamma c) (T^+) \gamma. \]

Again, to maintain parity invariance, this must be taken together with the same combination involving \(w^-\) and \(T^-\). This time, the relevant feature is that we have \(\varepsilon^{\alpha\beta\gamma} \varepsilon_{\lambda\mu\nu} (U^+ - V^+ S) \gamma\), i.e. \(S_8\) couples to the \(0^+\) and \(0^-\) states with opposite signs.

In the context of the \(\sigma\)-Model, the effect of this is for a \(g_8\) term to produce 'parallel' \(0^+\) and \(0^-\) mass spectra, i.e. there is an exact correspondence between the I-spin submultiplets in each SU(3) octet: but for a \(u_8\) term, the two spectra are mirror images of each other. This second case was what we found explicitly in the \(\sigma\)-Model.

Another interesting difference between the effects of \(u_8\) and \(g_8\) is that \(u_8\) cannot contribute in first order to the mass splittings of terms transforming according to the \((1, 8) \oplus (8, 1)\) representation, since it is impossible to construct an SU(3) \(\times\) SU(3) invariant out of two sets of \((1, 8) \oplus (8, 1)\) fields and one \((3, \bar{3}) \oplus (\bar{3}, 3)\) set of fields. Since \(g_8\) certainly can contribute, and since the "normal" spin 1 mesons, i.e. \(J^{PC} = 1^{--}\) and \(1^{++}\), presumably belong (like the vector and axial vector currents) to the \((1, 8) \oplus (8, 1)\) representation, then the dominant mass differences should be caused by the \(g_8\) term. In fact, in 1964, Gell-Mann broke the SU(3) \(\times\) SU(3) symmetry with the \(u_8\) term, and then allowed \(g_8\) to break the SU(3) symmetry; however, in 1968, Gell-Mann, Oakes and Renner chose \(u_8\) to break SU(3), but with a coefficient such that the
Hamiltonian was approximately SU(2) \( \otimes \) SU(2) invariant. No mention of the effect on the spin 1 meson masses was mentioned there, and we are not aware of any recent discussion of this point.

III. Soft Meson Results

Let us now consider why GOR take the dominant part of the symmetry-breaking Hamiltonian to belong only to the \((3, \bar{3}) + (\bar{3}, 3)\) representation. They assume at the beginning of their paper that the simplest case to consider is that in which both the chiral breaking term and the SU(3) breaking factor belong to the same representation of SU(3) \( \otimes \) SU(3), i.e. \((3, \bar{3}) + (\bar{3}, 3)\). Later on, in fact, they show that this initial proposal is consistent with the other assumptions and approximations which they make.

The proof of this last point is as follows. Consider the one particle matrix element of \(g^j\) between two \(0^-\) states \(\langle P^i(p)|\) and \(|P^k(k)\rangle\); if matrix elements are assumed to be approximately SU(3) symmetric, then this particular one may be written as

\[
\langle P^i(p) | g^j | P^k(k) \rangle = \delta^{ijk} \gamma(t),
\]

where \(t = (k-p)^2\), \(i, j, k = 1, \ldots, 8\), and singlet-octet mixing is neglected. For \(g^j\) replaced by \(u^j\), we write

\[
\langle P^i(p) | u^j | P^k(k) \rangle = \alpha(t) \delta^{j0} \delta^{ik} + \beta(t) \delta^{ijk},
\]

with \(i, k = 1, \ldots, 8\) as before, and \(j = 0, 1, \ldots, 8\).

Strictly, to take account of off-shell behaviour, we should
allow $\alpha$, $\beta$ and $\gamma$ to be functions of $p^2$ and $k^2$ as well as of $t$. However, the usual smoothness assumption which is made in hard pion calculations\textsuperscript{45} is that the vertex functions are "as smooth as possible" when the pole terms have been explicitly removed. In practice, this means that we would approximate a scalar vertex like those above by the expression

$$
\frac{z^J}{t-M^2_J} \left( p^2, k^2, t \right) \approx \frac{z^J}{t-M^2_J} \left( \lambda_i^{ij} + O(p^2, k^2, t) \right),
$$

where $M^2_J$ is the mass of the scalar particle whose pole dominates $g^J$ or $u^J$, $z^J$ is the proportionality constant between the particle field and the scalar density, and $\lambda_i^0$ is some (as yet unknown) constant. This approximation is assumed valid for "small" variations of $p^2$, $k^2$ and $t$, e.g. $0 \leq t \leq K^2$, and this seems to be reasonably well justified by the various results obtained. Now, because the nearest scalar masses are around 1 GeV, which is well outside the region of $t$ variation, we can effectively neglect the $t$-dependence of these explicit pole terms. In fact, for a $(1, 8) \oplus (8, 1)$ term such as $g^J$, we do not expect any pole term since the presently known scalar mesons are believed to belong to the $(3, \overline{3}) \oplus (\overline{3}, 3)$ representation. Thus, it seems quite reasonable to assume that $\alpha$, $\beta$ and $\gamma$ should all be independent of $t$, as well as of $p^2$ and $k^2$, when we work in the region $0 \leq t \leq K^2$.

Now apply the usual PCAC and soft meson procedure to $p_1^i$ in eqs. (5.8) and (5.9). In the first equation (and neglecting any $t$ dependence of $\gamma$), we obtain

$$
\frac{1}{\pi} f^{ijk} \langle 0 \mid h^e \mid p^k(k) \rangle = \gamma d^{ijk}.
$$

(5.11)
Because of the antisymmetry between \( i \) and \( j \) on the lhs and the symmetry on the rhs, we deduce immediately that

\[
\langle 0 \left| H^\nu \right| F^k \rangle = 0 \quad \text{and} \quad \gamma = 0,
\]  
(5.12)

i.e. the \((1, 8) \oplus (8, 1)\) term does not couple to the \(0^-\) mesons under the present assumptions.

On the other hand, eq. (5.9) gives

\[
-\frac{1}{f_i} \delta^{ij} \langle 0 \left| v^\nu \right| F^k \rangle = a \delta^{ik} \delta^{j0} + \beta \delta^{ijk}.
\]  
(5.13)

From an analysis involving different combinations of the indices \( i, j \) and \( k \), GOR show that eq. (5.13) is consistent with

\[
a = 0 \quad ; \quad f_i = f \quad (i = 1, \ldots, 8) \]  
(5.14)

and \( \langle 0 \left| v^i \right| F^i \rangle = \beta f \).

Finally, the relation of the masses to \( a, \beta \) and \( \gamma \) is obtained by the same procedure:

\[
\langle F^i(p) \left| H^* \right| F^i(k) \rangle \rightarrow -\frac{1}{f_i} \langle 0 \left| [A^i, H] \right| F^i(k) \rangle
\]

\[
= \frac{1}{f_i} \langle 0 \left| \delta A^i \right| F^i(k) \rangle
\]  
(5.15)

where \( H^* = u_0 + cu_8 + g_8 \) is the symmetry breaking part of the Hamiltonian.

It follows at once that masses receive contributions only from the \((3, \bar{3}) \oplus (\bar{3}, 3)\) part of \( H^* \). This result follows
from (1) the assumption of SU(3) symmetry for matrix elements, 
(2) the SU(3) \( \otimes \) SU(3) transformation properties of \( u^i \) and 
\( g^j \) and (3) the smoothness assumptions for \( \alpha, \beta \) and \( \gamma \).

In fact, a closer examination reveals that this result still holds when the third assumption is dropped. This is because the exact expression for \( m_1^2 \) is

\[
m_1^2 = \alpha(0, m_1^2, m_1^2) + d_{ijl} \left[ \beta(0, m_1^2, m_1^2) + \gamma(0, m_1^2, m_1^2) \right].
\]

But we have used the same soft meson limit to derive eq. (5.11); in the exact form, this becomes

\[
\frac{1}{f_1} f^{ijl} \langle 0 \mid \hat{h}^l \mid p^k(k) \rangle = \gamma(0, k_1^2, k_2^2) d_{ijl},
\]

and once again, we find \( \gamma(0, k_1^2, k_2^2) \equiv 0 \). Similarly,

\[
\gamma(p_1^2, p_2^2, 0) \equiv 0.
\]

Hence, the \((1, 8) \otimes (8, 1)\) term in \( H \) does not contribute in first order perturbation theory to the \( 0^- \) masses if we demand SU(3) invariance for the matrix elements.

Now consider the effects of relaxing this condition; first of all, the number of parameters increases. For example,

\[
\langle p^i \mid g^j \mid p^k \rangle = \delta^{ij8} \gamma_1 \delta^{18} + \delta^{ik} \left[ \gamma_2 (\delta^{i1} + \delta^{i2} + \delta^{i3}) + \gamma_3 (\delta^{i4} + \ldots + \delta^{i7}) + \right. \\
+ (\delta^{j1} + \delta^{j2} + \delta^{j3}) \left[ \gamma_4 (\delta^{j8} + \delta^{j8} + \delta^{j8} + \delta^{j8}) + \gamma_5 \delta^{i1} \right] \\
+ (\delta^{j1} + \ldots + \delta^{j7}) \left[ \gamma_6 (\delta^{j8} + \delta^{j8} + \delta^{j8} + \delta^{j8}) + \gamma_7 \delta^{i1} \right],
\]

where the functions \( \gamma_i \) correspond to the coefficients of the seven
independent couplings of two pseudoscalars to one scalar when
the symmetry is only SU(2) \(\otimes\) U(1).

However, even with this greater freedom, we can show that
\(g_8\) contributes only to the strange meson masses. From eqs.
(5.11) and (5.15), we see that the \((1, 8) \oplus (8, 1)\) part of the
\(i\)-th mass is given by the relation

\[
\langle m^2 \rangle_i \equiv \frac{1}{f_i^2} \frac{1}{f_{18}} \langle 0 | h^8 | p_i(k) \rangle,
\]

and the rhs of eq. (5.19) vanishes for \(i = 1, 2, 3\) or 8.

In fact, we can obtain some simplification of eq. (5.18) by
performing an analysis similar to that of GOR, i.e. looking at
the equation for particular combinations of the indices. One
constraint which we can apply is the following. The \(h\) octet
has odd \(C\)-parity whereas the \(g\) octet and both \((3, \bar{3}) \oplus (\bar{3}, 3)\)
octets have even \(C\)-parity, so that the coupling of \(h^6\) to \(p^k\)
above must be antisymmetric; this follows since the second, fifth
and seventh members of each octet always have the opposite \(C-
parity from all of the other members.\(^{26}\) The results are (neglect-
ing any momentum dependence in the \(\gamma_i\)):

\[
\gamma_1 = \gamma_2 = 0 ; \quad \gamma_6 = -\sqrt{3} \gamma_4 ; \quad \gamma_3 = \frac{\sqrt{3}}{2} \gamma_5 ;
\]

\[
\sqrt{2} \langle 0 | h^K | K^+ \rangle = -\frac{\gamma_1}{\sqrt{3}} \gamma_3
\]

\[
\frac{f_K}{f_{\pi}} = -\frac{\sqrt{3}}{2} \frac{\gamma_7}{\gamma_3} = -\frac{\gamma_7}{\gamma_5}
\]

\[
\frac{f_\pi}{f_\pi} = \frac{3\gamma_7}{2\sqrt{3} \gamma_6 \gamma_7}
\]
Hence, there are actually only three independent parameters, e.g. $\gamma_3$, $\gamma_6$, and $\gamma_7$.

When $g^j$ is replaced by $u^j$, we can allow $j = 0$ as well. We can then replace all the $\gamma_i$ by $\beta_i$ in eq. (5.18), and add on the terms

$$\delta_{j0} \left[ a_1 \delta^{18} \delta^8 + \delta_{ik} \{ a_2 (\delta^{11} + \delta^{12} + \delta^{13}) + a_3 (\delta^{14} + \ldots + \delta^{17}) \} \right].$$

Another similar analysis allows us to relate the $a_1$ and some of the $\beta_i$ to the remainder, so that we can write the new expressions for the masses as

$$\pi^2 = (a_2 + c \beta_2) = (\sqrt{2} + c) \beta_2; \quad (5.24)$$

$$K^2 = a_3 + c \beta_3 + \gamma_3 = (-2\sqrt{2} + c) \beta_3 + \gamma_3; \quad (5.25)$$

$$\gamma^2 = a_1 + c \beta_1 = (-\sqrt{2} + c) \beta_1. \quad (5.26)$$

Unfortunately, we cannot derive any relations between $\beta_1$, $\beta_2$, and $\beta_3$ by the above methods, so that the masses of the three I-spin submultiplets are now unrelated. An important consequence of this is that $c$ is not now fixed by $\pi^2$ and $K^2$ alone: the value $c \approx -\sqrt{2}$ still corresponds to a low pion mass, but this time, the converse does not follow; a low value of $\pi^2$ (relative to $\gamma^2$) can be achieved by a small value of $\beta_2$ relative to $\beta_1$.

In fact, for $|\gamma_3/\beta_3| \ll 1$, we can rederive a result of Anvil and Deshpande. First of all, we apply the soft meson approach to both mesons in the equation for $u^j$ corresponding to eq. (5.18), and then, using particular values of the indices, we obtain
where $\sigma_i \equiv \langle 0 | u_1 | 0 \rangle$. Hence, we have

$$\frac{f_\pi^2}{f_K^2} \propto \frac{\sqrt{2+c}}{(-2\sqrt{2+c})} \cdot \frac{\sqrt{2}\sigma_0 + \sigma_8}{(-2\sqrt{2}\sigma_0 + \sigma_8)}, \quad (5.28)$$

i.e. either $c = -\sqrt{2}$ or $\sigma_8/\sigma_0 = -\sqrt{2}$ gives $\pi^2 = 0$ and $K^2 \neq 0$. Thus, it is not essential to put $c = -\sqrt{2}$ in order to make the pion massless; however, the alternative solution introduces the complication that the kappa meson has a very low mass. Fuller details of this, and related topics, can be found in the above reference.

To return to the present subject of $(1, 3)$ effects on the $0^-$ masses, we see from eqs. (5.24) - (5.26) that additional constraint equations are required to restore the predictive power of the model. All that we can say at present is that in first order, $g_8$ has no effect on the $0^-$ masses unless the vacuum is not SU(3) invariant, and even then, only the strange mesons are affected, and to an undetermined extent. But we have not shown that $g_8$ has no effect on any other masses; our approach applies only to the $0^-$ mesons since they are approximately Goldstone Bosons, and so the usual PCAC treatment can be used with reasonable confidence. For states such as the $0^+$ mesons, different techniques must be employed.
IV. $K_{\ell 3}$ Form Factors

Another quite different reason for investigating the possible effects of a $(1,8) \oplus (8,1)$ symmetry-breaking term in $H^+$ is the poor prediction for a particular relation involving the $K_{\ell 3}$ form factors $f_\pm(t)$ when the GOR model is used.\textsuperscript{11}

We define $f_\pm(t)$ by the equation

$$
\langle p^\mu(k) \mid V_\mu^\nu \mid p(k) \rangle = i f_\pm \left[ f_+(t)(k+p)^\mu + f_-(t)(k-p)^\mu \right],
$$

(5.29)

where $t = (k-p)^2$. Thus, for the $K_{\ell 3}$ decay, we have

$$
\sqrt{2} \langle \pi^0 \mid V_\mu^{K-} \mid K^+ \rangle = - \left[ f_+(t)(k+p)^\mu + f_-(t)(k-p)^\mu \right],
$$

(5.30)

and also

$$
-\sqrt{2} \langle \pi^0 \mid \epsilon^{\mu\nu} V_{\mu}^{K-} \mid K^+ \rangle = + \left[ f_+(t)(k^2-p^2) + f_-(t)(k-p)^2 \right]
$$

$$
= (k^2-p^2) \left[ f_+(t) + \frac{t}{k^2-p^2} f_-(t) \right].
$$

(5.31)

The usual parametrisation of the form factors is

$$
f_\pm(t) \approx f_\pm(0)(1 + \lambda_\pm \frac{t}{\pi^2}),
$$

(5.32)

$$
\xi(t) \equiv \frac{f_-(t)}{f_+(t)}.
$$

(5.33)

Experimentally, $\lambda_+$ lies in the region 0.015 to 0.045, $\lambda_- \approx 0$, while different sets of experiments give quite distinct values for $\xi(0)$; polarisation data indicate $\xi \lesssim -1$ while Dalitz plot analyses lead to $\xi \gtrsim 0$. The most recent data\textsuperscript{47} from both sources appear to be converging on the region $-0.5 \lesssim \xi \lesssim -1$, with $\xi \approx -0.65$ as a possible end point.

Now, to evaluate the lhs of eq. (5.31), we can substitute the explicit expression $-i \sqrt{2} (\epsilon^{\mu-15} + g^{\mu-15})$ for $\epsilon^{\mu\nu} V_{\mu}^{K-}$ into the matrix element, and then use the parametrisation previously
described in order to relate this to the masses. We find

\[ \sqrt{2} \left< \pi^0 | \frac{e^{\mu\nu} K^+}{\mu} | K^- \right> = \frac{\sqrt{2}}{2} [c_\gamma(t) + \gamma_7(t)] \]  

(5.34)

where both \( \beta_7 \) and \( \gamma_7 \) can be related to \( \beta_3 \) and \( \gamma_3 \), the particular parameters which appear in the mass formulae.

Earlier, we followed GOR in assuming (1) that the only variation of \( \alpha, \beta \) and \( \gamma \) was in \( t \), and not in \( p^2 \) and \( k^2 \), and (2) that even this variation in \( t \) was negligible in the region \( 0 \leq t \leq K^2 \). This gives reasonably consistent results when applied to the mass formulae; however, in the present case, we want to know the \( t \)-dependence of \( f_+(t) \) so we need to take the \( t \)-dependence of \( \gamma(t) \) and \( \beta(t) \) into account. To do this, requires hard meson calculation, e.g. using Ward Identities or phenomenological Lagrangians.

But the point which concerns us is this. The usual soft meson techniques and the GOR symmetry breaking model yield a value of \( \xi \approx 0 \), whereas the experimental value of approximately \( -0.65 \) is gaining more support now. One of several ways in which we could attempt to remedy the situation is to allow a \((1,8) \oplus (8,1)\) symmetry breaking term in \( \mathcal{H} \) : in fact, Arnowitt et al. have done a hard pion calculation using just such a non-pole \((1,8)\) term and have obtained a fit to \( \xi \) when the coupling to the \( 0^- \) mesons of this \((1,8)\) term is comparable in magnitude to that of the \( u_0 \) term. The question now is: can the \((1,8) \oplus (8,1)\) term affect the \( K_{13} \) form factors to such an extent without affecting the \( 0^- \) masses?

Previously, we showed that a \((1,8)\) term could not contribute
to the $0^-$ masses as long as we maintained SU(3) invariance of the matrix elements, and hence of the vacuum, since the relevant value of $\gamma$ was $\gamma(0, k^2, k^2) \equiv 0$. However, for a function of the form $\gamma(p^2, k^2, t) = p.k \gamma'(p^2, k^2, t)$, with $\gamma'$ a power expansion in the variables, the on-shell values of $\gamma$ would not be zero, and, in general, the coupling of $g_i^j$ to the $0^-$ mesons would not vanish. The main argument against such a form for $\gamma$ is that the function is certainly not "as smooth as possible", i.e. approximately constant, for small values of $p^2$, $k^2$ and $t$. Nevertheless, until a more detailed calculation is performed to investigate the momentum dependence of $\gamma$, we cannot rule out the conclusion that it is possible for such a $(1, 8)$ term to play an important role in the $K_{l3}$ form factors without contributing to the $0^-$ masses.

Of course, if the smooth off-shell extrapolation is reasonably well justified, then the effect of the $(1, 8)$ term on $\xi(0)$ may possibly be insignificant. In this case, we should have to relax the condition about matrix elements being SU(3) invariant, so that the kaon masses would now have a $(1, 8)$ contribution. Again, a detailed hard meson calculation is necessary to determine the variation with $t$, for example, but in the light of the success of the GOR scheme in most applications and the relative failure here, such an investigation seems very worthwhile.

V. Last Thoughts

Finally, it would be very interesting to discover the relative importance of $(3, \bar{3})$ and $(1, 8)$ terms in the mass spectra of
other particles besides the $0^-$ mesons. Apart from the obvious reason, i.e. obtaining a good fit in as many multiplets as possible, the question of which symmetry group, $SU(3)$ or chiral $SU(2) \times SU(2)$, is the better intermediate group between $SU(3) \times SU(3)$ and $SU(2) \times U(1)$ may be answered more definitely. At present, the evidence pointing to chiral $SU(2) \times SU(2)$ seems to be based largely on data for the $0^-$ mesons, since we can apply certain techniques such as PCAC and soft meson theorems only in this case.

However, the very fact which allows such procedures, i.e. that the $0^-$ mesons are almost Goldstone Bosons, may possibly make these mesons atypical of the hadrons in general; certain effects may occur here which do not arise anywhere else and we may be regarding these as typical of the hadrons when they are indeed exceptional. Because so many calculations involve the $0^-$ mesons, such circumstances are certainly not to be desired.

But in a subject in which theoreticians appear to be producing models (with varying decay rates, of course) almost as prolifically as experimentalists come up with new particles, it seems appropriate to end a thesis on a cautionary note such as this.
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