Measurement of Two-Phase Flow using Particle Image Velocimetry

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A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy to the University of Edinburgh 1997
Abstract

A theoretical and experimental study has been carried out into the separation of particle images of different sizes. Particular emphases has been placed on the analysis and assessment of factors which affect the data obtained from two-phase flow experiments. Statistical theory has been developed to account for the recorded intensity of side-scattered light from a particle due to: (a) the integration effects of the pixels in a CCD sensor, (b) the quantization effects of the electronics associated with CCD cameras and (c) the position of the particle in a scanning-beam PIV illumination system. The theory was found to agree well with experiment and indicates that for typical PIV experimental parameters, the variation of light recorded due to a particle’s random position in the laser sheet is an order of magnitude higher than the variation of light intensity due to the integration effects of the CCD sensor. A further experiment indicated that the variation of intensity due to the light scattering characteristics of quartz sediment particles is of the same order of magnitude as the variation due to the position of the particle in the laser sheet.

An automatic system was developed to locate and analyse the particle images of both phases. The primary difficulty for the system was found to be in separating the phases. Experiments were performed to determine statistical confidence levels for this purpose. The analysis method was developed and applied to experiments involving the measurement of two-phase (sediment/water) flow
over a rippled-sand bed. For this purpose, a flexible, convenient and robust PIV illumination system was built. The results obtained are discussed to indicate the validity of the method and show that it works well when applied to a practical two-phase flow experiment.
Acknowledgements

This project was funded through the EPSRC and their support is greatly appreciated.

I would like to thank my supervisor Clive Greated for allowing me to do this research and for his guidance and support. I would also like to thank the technicians, Frank Morris and Ronnie Proc, whose skill, expertise and feedback were invaluable.

Very grateful thanks go to Alistair Arnott, Jim Buick, Narumon Emarat, Jon Entwistle and Dave Forehand for both their technical help and friendship throughout. I would like to thank Roger Sayle for the same reasons during the earlier stages of this project.

Many thanks to Mike, Pete, Helen (Irish), Helen, Iain, Kim, Paul and Tom, whose friendship has kept me sane. Particular thanks goes to Kiki, for all the good times over the last couple of years.

Last, but by no means least, I would like to thank my family: my brothers and sisters for all the laughter and tears and the auld fella, for the faith he showed in his kids, his years of hard work and for being as good a role model as there is. Most of all I would like to thank my Mum, whose selfless devotion to her children gave them the opportunity to fulfill many of their ambitions at the expense of her own.
Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(John A. Cosgrove)
December 1997
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Chapter 1
Introduction

In the past three decades, fluid mechanics has increased in both the diversity and complexity of its application. The advent of sophisticated numerical techniques and the rapid advances in the hardware to implement these techniques has been, and continues to be, a major reason for this increase. A further and no less important reason is the significant advances in fluid flow measurement techniques.

The range and diversity of fluid flow applications necessitates a correspondingly large range of measurement techniques. However one relatively new technique, Particle Image Velocimetry (PIV), has become increasingly widespread in the last decade. This is mainly due to its ability to combine the accuracy of point measurements, such as hot-wire, or laser-doppler, anemometry with the spatial content of whole flow visualization techniques.

In PIV, a section of the flow to be measured is illuminated. There is often no interaction between the incident light and the fluid. Therefore minute particles, of the same density and which follow the flow, are added to reflect (scatter) the light so that the motion of the particles and hence the flow may be observed. An image of the motion, or a sequence of images with small separation times between them, is/are recorded. The velocity information in the illumination plane can then be extracted from this raw image data.
1.1 PIV applied to two-phase flows

The current work concentrates on the measurement of two-phase fluid flows using PIV and is mainly concerned with the measurement technique itself rather than interpreting the results of the measurements. A two-phase flow is simply a flow that consists of two different media such as a gas/liquid flow. In our case the measuring technique is applied to a water/sediment flow over a rippled-sand bed near a beach wall.

This work has environmental relevance, particularly to coastal erosion and marine life as the formation of the ripples is a phenomenon which occurs both naturally and as a consequence of man-made marine structures. For example, offshore of a beach toe, a series of sand bars often develops parallel to the shore line, over a distance which may be many miles from the shore. Such features can
be found in Lake Michigan, the Chesapeake Bay area, the Black Sea and various parts of the Mediterranean Sea. These bars remain very stable in virtually tideless seas, but may migrate in areas subject to larger tidal ranges [7]. An example of a man-made structure which produces these ripples is a coastal defense wall.

The ability to understand the basic mechanisms of the development of rippled-sand beds is thus of great interest and the currently implemented system for the measurement of two-phase flow serves as a useful tool in aiding this understanding. Figure 1.1 shows a recorded PIV image of this two-phase flow. The bottom third or so of the image is the rippled-sand bed and one can see two main ripple peaks. Close to the bed, over the peaks and in between the peaks, are bright relatively large sediment images. The sediment has been lifted from the bed into the flow by vortices formed as the water flows over the sand-bed ripple peaks. In the top half of the image, the large sediment particles are not present. However one can observe smaller particles, which are the images of pollen particles used to seed the water phase (for the purposes of measuring the water velocity).

1.2 Image identification

From the above brief description and from a visual inspection of figure 1.1, one is able to discern the bed boundary and areas of high sediment concentration relatively easily. This is due to the advanced human visual process. However attempting to mimic this process in an automated system is extremely difficult. For example, human vision can easily discern structural information in images that contain a large amount of extraneous detail, such as gradual variations in intensity. Human beings also have the ability to compare unknown visual occurrences with a lifetime of previously identified and classified visual situations. The visual information is stored in such a way that it is often possible to identify and
classify objects which may possess varying attributes in comparison with similar objects encountered previously. It accomplishes these feats in a manner that is not easily understood and the ability to assimilate data in this way is extremely difficult to incorporate into artificial vision systems. Currently, the nearest that computer-based vision systems can get to the human process is to attempt to mimic certain attributes, under certain limiting circumstances.

In this project, image processing routines are used initially to identify possible particle images. The problem being tackled here is generally simpler than many tasks in computer vision, e.g. a general such task may be to identify a car, regardless of angle of view, distance, colour, body shape, its surroundings and conditions of illumination. For the two-phase flow here, most of these problems are eliminated by defining very specific illumination and recording conditions.

1.3 Factors affecting image formation

In spite of the simplifications however, there are still several factors which affect the system's ability to determine which are sediment and which are pollen particles. These are described subsequently.

In general PIV applications, the images are recorded by a medium such as a conventional wet-film, or Charge Coupled Device (CCD), camera. It is pertinent
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to look at the effects of the sensor used, on the intensity of the image recorded. In particular, the integration effects of the pixels in the CCD array on the recording of the image need to be examined. The position of the particle in the illumination area is also a factor. In the analysis of this, it is reasonable to assume that a particle image in the image plane is of a Gaussian form and that the illuminating sheet is also Gaussian in cross section [4].

Once an image has been obtained, there are further hurdles to be overcome in its analysis, in order to extract the velocities of the water and sediment phases. The main problem is illustrated by a comparison of figures 1.2, 1.3 and 1.4. Figure 1.2 is an image of a single sediment particle in an 18 × 18 pixel subsection taken from a larger image of purely sediment particle images. The pixels are square and 9 μm in length. Figure 1.3 is an image of a single pollen particle image (with perhaps another image, which is less-bright, at the top left of the figure) in an 18 × 18 pixel subsection taken from a larger image consisting solely of pollen particles.

The sediment particle shown is one of the brighter sediment images from those recorded, and similarly the pollen particle shown is one of the brighter images of the pollen particles recorded. Both images are recorded with identical recording parameters. However there are clear differences between both particle images in terms of size and intensity. In this case, it is relatively easy to design an algorithm to separate these images. For example a suitable intensity threshold can be defined and if a particle image does not have a sufficient number of pixels above this intensity, it is rejected.

Now consider figure 1.4. The bright region of pixels in the top half of the figure appears to be of a similar intensity and size to the image of the pollen particle in figure 1.3. In fact, figure 1.4 is taken from the same PIV image as figure 1.2 and
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is thus an image of a sediment particle. Although the brighter sediment images are clearly different from the pollen images, the less-bright sediment images are not.

Clearly this presents difficulties for an automated process to separate the phases. This and the factors mentioned above, need to be investigated in order to develop such a system for the analysis of two-phase PIV images. This provides the raison d'être for the investigation which is the subject of this thesis. The research programme to achieve this was split into several stages, which are described briefly below.

1.4 Outline of thesis

In Chapter 2 statistical methods are derived relating to the general effects of randomly sampling a Gaussian function and the specific effects of sampling with a CCD sensor. Mathematical expressions for the quantisation effects of the frame grabber electronics, i.e. the analogue to digital (A/D) conversion process are also developed.

The factors which affect the intensity of light in a recorded image are investigated in Chapter 3. In addition theory relating to the formation of particle images is reviewed and three experiments are performed. The first is concerned with the integration effects of CCD pixels on the intensity amplitude of a recorded particle image, while the second deals with the effect on the recorded intensity due to a particle's random position in the laser light sheet. The application of the one-dimensional discrete sampling derivation (Chapter 2) to the random positioning of a particle in a laser light sheet is also detailed in this section. The final experiment investigates the effects of the light-scattering characteristics of quartz sediment particles on the image recorded.
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The main concern of Chapter 4 is with the design of an illumination system for the measurement of two-phase flow (sediment and water) over a rippled-sand bed. Experiments using this system were performed in conjunction with The School of Engineering at Aberdeen University. Some basic theory relating to this experiment is also given.

In Chapter 5 image processing techniques concerned with the segmentation of images are discussed and applied to the raw PIV data obtained from the experiments in Chapter 4. In addition, experiments are presented which involved recording images of sediment particles (without pollen particles present) and pollen particles (without sediment particles present). The results of these experiments are used to determine confidence levels for separating the different phases. From these and the above image processing routines, an algorithm is developed to locate particles. A method is outlined to produce velocity information from the separated phase data. The novelty of the analysis as a whole is that it is applied to images containing two-phase data.

Chapter 6 shows the results of applying the two-phase analysis method developed in the previous chapter to some of the raw data obtained from the experiments in Chapter 4. The results give a visual indication of the performance of the two-phase analysis method applied to a real flow situation. The results obtained are discussed to indicate that they agree sensibly with those expected. Following from this, the conclusions from the investigation and suggestions for future work are presented in the final chapter.
Chapter 2

Statistics relating to the spatial sampling of a Gaussian function

In this chapter statistics are derived relating to the general effects of spatially sampling a Gaussian (particle image) and the specific effects of sampling with a CCD sensor. After some introductory remarks and preliminaries, (notation, definitions and theorems), the derivations begin with an analysis of the effects of discretely sampling a one dimensional image. In section 3.6.2 it is shown that this derivation extends to the random sampling of a particle in a scanning Gaussian laser sheet with little extra effort. The one dimensional derivations are subsequently extended to two dimensions and following this, the integration effects of the pixels within the CCD sensor are accounted for. Mathematical expressions for the quantisation effects of the frame grabber electronics, i.e. the analogue to digital (A/D) conversion process are also given. Throughout the derivations, the particle image in the image plane is assumed to take a Gaussian form.

2.1 Introduction

The two main theoretical analyses, relating to the accuracy of the general technique of Particle Image Velocimetry, PIV, (see section 3.3 for a description of the
Chapter 2 — Statistics relating to the spatial sampling of a Gaussian function

technique) have been carried out by Adrian [4] and Westerweel [56]. In both these analyses first and second order statistics are derived for mathematical models of a PIV image. Second order statistics are easily related to the correlation method of image analysis and thus these statistics are used to determine the accuracy of image analysis by this method.

The present work is also concerned with the statistics of PIV images, however, from the point of view of particle phase separation rather than correlation analysis. Here phase separation refers to the process of isolating the particle images (in a PIV image) of a particular type from the images of other particle types. The current work is concerned with separating the particle images produced by pollen from the particle images produced by beach sediment. The implementation of the separation process is given in section 5.3. For our experiments, recorded particle images consist of only a few pixels, of the order $10^1$, and we are interested in determining how effective simple processing techniques such as the amplitude thresholding of an image, are at separating two phase particle flow.

The content of this chapter is mainly concerned with the derivation of statistics relating to the sampling of a Gaussian function. This function is chosen to mathematically model the particle image formed in the image plane as well as the beam profile produced by a laser. An analysis of the nature of recorded PIV data is not a trivial task as there are many components involved in PIV, each producing its own particular set of variables which contribute to the complexity and uncertainty of the data. For a discussion as to why a Gaussian image profile is chosen see section 3.3 while Goldstein [14] gives an account of Gaussian laser profiles in the context of fluid measurement. The application and use of the derived results are shown in sections 3.5 and 3.6.

Despite the current rise in popularity of the CCD sensor as a measurement
Chapter 2 — Statistics relating to the spatial sampling of a Gaussian function

device in PIV, much analysis is lacking in certain aspects of recording specific to the CCD. In the following sections we look at statistics relevant to the recording of a particle randomly placed in a Gaussian light beam profile. The approach taken involves setting up suitable mathematical experiments to describe these effects and the subsequent derivation of the relevant statistics.

2.2 Notation, definitions and theorems

We now introduce some notation and a set of definitions and theorems relating to probability theory and random processes. A detailed treatment of these topics can be found in [39].

2.2.1 Probability

We refer to an experiment as a set $S$ of elements of outcomes $\zeta$. Certain subsets of $S$ are called events. The space $S$ is the certain event and the empty set $\emptyset$, the impossible event. Two events $\alpha$ and $\beta$ are mutually exclusive if they have no common elements, i.e., if $\alpha \cap \beta = \emptyset$. We now assign to each event $\alpha$ a number $P(\alpha)$ which we call “the probability of the event $\alpha$”. This number is chosen so as to satisfy the following three axioms:

$$P(\alpha) \geq 0 \quad (2.1)$$

$$P(S) = 1 \quad (2.2)$$

$$P(\alpha + \beta) = P(\alpha) + P(\beta) \quad \text{if} \ \alpha \cap \beta = \emptyset \quad (2.3)$$

If the event $\alpha$ is expressed with braces denoting the elements of the event, then instead of $P(\{\})$, we write $P\{\}$. 

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2.2.2 Random variables

Definition: A real random variable $X$ is a real function whose domain is the space $S$, i.e., a process of assigning a real number $X(\zeta)$ to every outcome $\zeta$ of the experiment $E$, and such that:

1. The set $\{X \leq x\}$ is an event for any real number $x$

2. The probability of the events $\{X = +\infty\}$ and $\{X = -\infty\}$ equals zero:

$$P\{X = +\infty\} = P\{X = -\infty\} = 0 \quad (2.4)$$

Random variables, r.v., are denoted in boldface calligraphic uppercase $X, Y, Z, \ldots$ and may be subscripted to a function, $F$, in the following manner $F_X, F_Y, F_Z, \ldots$, the subscripts are still calligraphic uppercase but no longer boldface. In dealing with r.v., questions of the following form often arise: Given a real number $x$, what is the probability that the r.v. $X$ is less than or equal to $x$, or alternatively what is the probability of $X$ being greater than $x$; these probabilities are denoted as $P\{X \leq x\}$ and $P\{X > x\}$ respectively.

2.2.3 The distribution and density functions

Definition: The distribution function of the r.v. $X$ is the function

$$F_X(x) = P\{X \leq x\} \quad (2.5)$$

defined for any number $x$ from $-\infty$ to $\infty$. 
Chapter 2 — Statistics relating to the spatial sampling of a Gaussian function

Definition: The derivative
\[ f_X(x) = \frac{d}{dx} F_X(x) \]  \hspace{1cm} (2.6)
of the distribution function \( F_X(x) \) is called the density (function) of the r.v. \( X \)
it is also known as the frequency function.

The distribution functions of the r.v. \( X, Y \) and \( Z \) are denoted by \( F_X(x), F_Y(y) \) and \( F_Z(z) \) respectively. In this notation, the arguments \( x, y \) and \( z \) are real numbers ranging from \(-\infty\) to \(\infty\) and can be identified by any letter; \( i.e., \) we can write the distribution functions in the form \( F_X(w), F_Y(w) \) and \( F_Z(w) \) identifying the corresponding r.v. by the subscripts. For example, \( F_Z(w) \) is the probability that \( Z \) is less than or equal to the number \( w \). However, if there is no fear of ambiguity the subscripts are omitted and by using the notation \( F(x), F(y) \) and \( F(z) \) we shall mean the distribution functions of the r.v. \( X, Y \) and \( Z \) respectively. The distribution is sometimes defined, not by \( 2.6, \) but by
\[ F_X(x) = P\{X > x\} \]In parts of the analysis of the current work the \( P\{X > x\} \) is specifically derived. However we maintain the notation of equation \( 2.6 \) referring equivalently to the probability of \( X \) less than or equal to \( x, P\{X \leq x\}, \) and the distribution of \( x, F_X(x) \), while the \( P\{X > x\} \) is denoted by the single expression \( P\{X > x\} \).

The respective probabilities are related to the density function as follows
\[ P\{X > x\} = \int_{x}^{\infty} f_X(y) \, dy \]  \hspace{1cm} (2.7)
\[ P\{X \leq x\} = \int_{-\infty}^{x} f_X(y) \, dy \]  \hspace{1cm} (2.8)
where we have introduced \( y \) as a dummy variable of integration.
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2.2.4 Functions of random variables

We are given a real r.v \( X \) with domain the space \( S \) and range the set \( I_X \) of real numbers \( X(\zeta) \). We are also given a function \( g(x) \) specified by a formula. This function is defined for every experimental outcome in the following way: For a given \( \zeta \), \( X(\zeta) \) is a real number and

\[
g[X(\zeta)]
\]

is another number specified by \( X(\zeta) \) and \( g(x) \). This number is the value

\[
Y(\zeta) = g[X(\zeta)]
\]

of the r.v. \( Y \). Thus the domain of \( g(X) \) is the set \( S \) of all experimental outcomes, whereas the domain of \( g(x) \) is a set of real numbers. If \( g(x) \) satisfies certain conditions \(^1\) then the function so defined is a r.v.

The function \( F_Y(y) \)

\[
F_Y(y) = P\{Y \leq y\} = P\{g(X) \leq y\} \tag{2.9}
\]

is the distribution of the r.v. \( Y \) and can be determined in terms of the distribution \( F_X(x) \) or density \( f_X(x) \) in the following manner. Given a real number \( y \), we denote by \( I_y \) the set of all real numbers \( x \) such that \( g(x) \leq y \)

\[
x \in I_y \quad \text{iff} \quad g(x) \leq y \tag{2.10}
\]

If for a certain \( \zeta \) we have \( Y(\zeta) = g[X(\zeta)] \leq y \), then \( X(\zeta) \in I_y \) as can be seen from 2.10 and conversely, if \( X(\zeta) \in I_y \), then \( g[X(\zeta)] = Y(\zeta) \leq y \) from which follows

\[
F_Y(y) = P\{Y \leq y\} = P\{X \in I_y\} \tag{2.11}
\]

\(^1\)Namely: (1) The set \( I_y \) of all real numbers \( x \) such that \( g(x) \leq y \) should be a countable union or intersection of intervals for any \( y \); only then \( \{Y \leq y\} \) is an event. (2) The set of outcomes \( \zeta \) such that \( g[X(\zeta)] = +\infty \) or \( g[X(\zeta)] = -\infty \) should be an event with zero probability.
Thus to determine $F_Y(y)$ for a given $y$ we must find the set $I_y$ and the probability that $X$ is in $I_y$.

The density $f_{Y_y}$ of the r.v. $Z$ may be determined using the following theorem.

**Theorem 1**: To find $f_{Y_y}$ for a given $y$ we solve the equation

$$y = g(x)$$

for $x$ in terms of $y$. If $x_1, x_2, \ldots, x_n, \ldots$ are all its real roots,

$$y = g(x_1) = g(x_2) = \cdots = g(x_n) = \cdots$$

(2.12)

then

$$f_{Y_y}(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} + \cdots + \frac{f_X(x_n)}{|g'(x_n)|} + \cdots$$

(2.13)

where

$$g'(x) = \frac{dg(x)}{dx}$$

**Definition**: The joint distribution function of the r.v. $X$ and $Y$ is defined by

$$F_{X,Y}(x, y) = P\{X \leq x, Y \leq y\}$$

(2.14)

We now assume that $F(x, y)$ has partial derivative up to two. The quantity

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}$$

(2.15)

is known as the joint density function of the r.v. $X$ and $Y$.
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In order to determine the statistics of the two r.v. we consider a region $R$ of the $x, y$ plane and denote by

$$\{(X, Y) \in R\}$$

the event consisting of all outcomes $\zeta$ such that the point with coordinates $X(\zeta), Y(\zeta)$ is in the region $R$. The joint statistics of the r.v. $X$ and $Y$ are specified if the probability of the above event is known for any $R$ and we note

$$P\{(X, Y) \in R\} = \int \int_{R} f(x, y) \, dx \, dy \quad (2.16)$$

where the integration extends over the region $R$ of the plane.

**Definition:** The two r.v. $X$ and $Y$ are called independent if the events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent for any $x$ and $y$, that is, if

$$P\{X \leq x, Y \leq y\} = P\{X \leq x\}P\{Y \leq y\} \quad (2.17)$$

For the independent r.v. $X$ and $Y$

$$F_{X,Y}(x,y) = F_{X}(x)F_{Y}(y) \quad (2.18)$$

and if the densities exist

$$f_{X,Y}(x,y) = f_{X}(x)f_{Y}(y) \quad (2.19)$$

### 2.3 Sampling the image plane

One of the ways to model the effects of digital recording, Lourenco [30] and Bracewell [5], is to assume the image is sampled by the discrete pulse train $\mathbf{III}(x)$

$$\mathbf{III}(x) = \sum_{n=-\infty}^{\infty} \delta(x - nd)$$
where $\delta$ is the dirac delta function (impulse function), $d$ is the spacing between multiple impulses and $n$ is an integer which can be used to index the impulses. Thus an arbitrary function $f(x)$ sampled in the image plane can be represented as

$$\mathbf{III}(x)f(x) = \sum_{n=-\infty}^{\infty} f(nd)\delta(x - nd) = f_n \quad (2.20)$$

Figure 2.1 shows a continuous one dimensional Gaussian intensity profile discretely sampled at regular intervals and indicates the difference between the particle image in the image plane and the particle image after recording by a discrete sensor. Generally the domain and range of the image in the image plane are considered continuous whereas the sensor image, for most of this analysis has a discrete domain and continuous range. In further sections we look at quantisation effects and restrict the range to discrete values, corresponding to the the analogue to digital effects of the recording electronics.

Let us now look at how mathematical experiments have been defined to account for the discrete sampling of an image and the integration effects caused by the individual pixels of CCD sensors.
2.3.1 Mathematical experiments

We begin by assuming the image is sampled by a regular grid and that the image has a Gaussian form. To account for an image's random position on the grid, it seems natural to visualize this uncertainty by randomly "dropping" the Gaussian onto the grid. In fact, when defining mathematical experiments, the author found it easier to visualize dropping a grid onto a Gaussian rather than the equivalent vice versa and hence the introduction "Suppose we drop a grid onto a Gaussian" to most of the separate derivation sections.

Now had the analysis kept to the format of dropping a Gaussian onto a grid, then it would have been natural to choose the centre of the Gaussian as a reference point. The sample nearest the reference point could have been defined as the centre sample. An equivalent way of looking at the problem is to centre the Gaussian on the origin and randomly drop a grid onto it. The grid point lying in the interval $-\frac{d}{2} \leq x \leq \frac{d}{2}$, figure 2.1, is thus considered the central grid point and it is a r.v. which, for the one dimensional case, is denoted by $X_0$. Further it can be assumed that the same grid point $X_0$ always falls in the same interval. This does not affect the generality of the analysis however it simplifies the subscript notation and provides a more concrete way of visualizing the problem. We refer to $X_0$ as the central/centre most grid point, central/centre grid point or zeroth grid point and refer to other grid points by their corresponding subscript value. $X_n$ is the $n^{th}$ grid point and for $n$ positive it is referred to as the $n^{th}$ positive grid point and for $n$ negative it is referred to as the $n^{th}$ negative grid point.

Each grid point has a corresponding sample value, which we also refer to more simply as sample, and is denoted, in the one dimensional case, by $Y$. The same subscript notation applies to the samples as to the grid points. Thus $X_0$ has a corresponding sample $Y_0$ referred to as the zeroth sample and $X_n$'s corresponding
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sample, \( Y_n \), is the \( n^{th} \) sample and so on. For a regular grid the value of the samples will depend on the position at which the particle image is dropped onto the grid. Since we assume this position is a random variable then the sample value, which is a function of position, is also a random variable.

\[ X_0 \]

\[ \begin{array}{c}
\text{Figure 2.2: Minimum and maximum samples for the centre grid point}
\end{array} \]

For the discrete sampling case figure 2.2 gives an indication of the effects of dropping the grid point \( X_0 \) randomly in the interval \(-\frac{d}{2} \leq x \leq \frac{d}{2}\). If the grid point falls on the origin then we obtain a maximum central sample value. If the grid point randomly falls at \( x = \frac{d}{2} \) then we obtain a minimum central value and so \( Y_0 \) is confined to a specific range. The other samples are similarly confined to their respective specific ranges and in appendices A.1 and A.2 the distributions of these samples, that is, the \( n^{th} \) positive side sample \( n > 0 \) and the \( n^{th} \) negative side sample, \( n < 0 \), are derived. We now look at the distribution of the central most sample.

2.4 Discrete Sampling

We now proceed with the business of deriving statistics relating to the sampling of a Gaussian intensity profile (particle image). We first consider discretely sampling the image plane in the one dimensional case and then extend the analysis to two
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dimensions.

2.4.1 One Dimensional Gaussian Model

In this section we derive the distribution $P\{Y_0 > y\}$ and begin by mathematically
defining our sampling grid (discrete sensor).

2.4.1.1 The grid

Suppose instead of dropping a Gaussian onto a grid we equivalently drop a grid
to a Gaussian. Let the Gaussian be centred at $x = 0$ i.e.

$$I(x) = Ae^{-ax^2}$$  \hfill (2.21)

and let the grid be uniform with spacing $d$

$$X_i = id + X_0, \quad \forall i \in \mathbb{Z}$$  \hfill (2.22)

where $X_0$ is a uniformly distributed r.v.. We have thus formed a sequence of
r.v., $\ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots$ Clearly these r.v. are all uniformly (but
not identically) distributed and are not independent. In fact if any one of the
r.v. is assigned a specific value then the remaining values of the sequence are
completely defined. The sequence may thus be viewed as a (very regular) discrete
stochastic process. We restrict our attentions to deriving the statistics of the r.v.,
$Y_i = I(X_i)$, either in the form of a density or one of the distributions $P\{Y > y\}$,
$P\{Y \leq y\}$. Again these r.v. are not independent, but are completely specified
by assigning a value to any one r.v. in the sequence.

2.4.1.2 The central most sample

Let us sample $I(x)$ at the points $X_i$ and let $Y$ be the maximum of the discretely
sampled data. Then $Y$ is also a r.v. and we seek the probability that $Y > y$,
where $y$ is some amplitude threshold i.e. $P\{Y > y\}$. Now suppose without loss
of generality that \( x_0 \) is defined in the interval \(-\frac{d}{2} \leq x \leq \frac{d}{2}\). Furthermore the value of the Gaussian at this point will be \( y_0 \). Consider Figure 2.3 with a view to finding the \( P\{y_0 > y\}\). Now from the graph

\[
y_0 \geq I(-d/2) = I(d/2) = A e^{-ad^2/4}\tag{2.23}
\]

Let \( y^* = A e^{-ad^2/4} \) and therefore

\[
P\{y_0 > y\} = 1 \quad \text{for } y < y^*.
\]

(2.24)

Also from the graph \( y_0 \leq A \) and therefore

\[
P\{y_0 > y\} = 0 \quad \text{for } y > A. \tag{2.25}
\]

Now consider the case of \( y^* \leq y \leq A \). From the graph for \( y_0 > y \), \( x_0 \) must satisfy \(-l < x_0 < l\). Hence

\[
P\{y_0 > y\} = \frac{\text{length of the inner interval}}{\text{length of outer interval}} = \frac{2l}{d}
\]

but \( y = A e^{-ad^2} \) and on rearranging terms we have

\[
l = \sqrt{\frac{1}{a} \ln \left( \frac{A}{y} \right)} \quad \text{since } l > 0
\]
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Hence

\[ P\{Y_0 > y\} = \begin{cases} 
1 & \text{if } y < y^* \\
\frac{2}{d\sqrt{a}}\sqrt{\ln\left(\frac{A}{y}\right)} & \text{if } y^* \leq y \leq A \\
0 & \text{if } y > A 
\end{cases} \quad (2.26) \]

where

\[ y^* = Ae^{-ad^2/4} \quad A, a, d \in \mathbb{R} \quad A < 0, a < 0, d < 0 \]

The derivations of the distributions for the general positive and negative samples are similar to that of the central sample and are given in appendices A.1 and A.2 respectively.

2.4.2 Extension to two dimensions

Let us now generalise to two dimensions. Suppose we drop a uniform rectangular grid onto a Gaussian. Let the Gaussian be centred at \( \bar{x} = 0 \), or equivalently, \( (x, y) = (0, 0) \), so that

\[ I(x) = Ae^{-a(x^2)} \quad (2.27) \]

or

\[ I(x, y) = Ae^{-a(x^2+y^2)} \quad (2.28) \]

The uniform rectangular grid is defined to be

\[ X_{i,j} = (X_0 + id_x, Y_0 + jd_y) \quad \forall \ i, j \in \mathbb{Z} \quad (2.29) \]

where \( d_x \) and \( d_y \) are the grid spacings in the \( x \) and \( y \) directions respectively and \( X_0 \) and \( Y_0 \) are r.v., with \( X_0 \) uniformly distributed in the interval \(-\frac{d_x}{2} \leq x \leq \frac{d_x}{2}\) and \( Y_0 \) uniformly distributed in the interval \(-\frac{d_y}{2} \leq y \leq \frac{d_y}{2}\). Now let us sample \( I(x, y) \) at the points \( X_{i,j} \) and let \( Z_{0,0} \) be the maximum of the discretely sampled
Figure 2.4: Surface plot of two dimensional Gaussian intensity profile enclosed within a rectangular grid $-d_x/2 \leq x \leq d_x/2, \ -d_y/2 \leq y \leq d_y/2$
data. Then $Z_{0,0}$ is also a r.v. and we seek the probability that $Z_{0,0}$ is greater than $z$, $P\{Z_{0,0} > z\}$, where $z$ is some amplitude threshold.

Now only one of the sample points, say $X_{k,l}$, will fall inside the rectangle,

$$\frac{-d_x}{2} \leq x \leq \frac{d_x}{2}, \quad \frac{-d_y}{2} \leq y \leq \frac{d_y}{2}$$

(2.30)

unless it falls on an edge of this rectangle, in which case another sample point will lie in the rectangle, or it falls at a vertex, in which case another three points will lie in the rectangle. Furthermore the value of the Gaussian at this point (these points) will be $Z_{0,0}$.

Now without loss of generality we can assume that $d_x \geq d_y$. Consider the graph in Figure 2.4 with a view to finding the $P\{Z_{0,0} > z\}$.

Now from the graph

$$Z_{0,0} \geq I(d_x/2, -d_y/2) = I(d_x/2, d_y/2)$$

$$= I(-d_x/2, d_y/2) = I(-d_x/2, -d_y/2)$$

$$= Ae^{-a(d_x^2 + d_y^2)/4}$$

(2.31)
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Let

\[ Z_1^* = Ae^{-a(dx^2 + dy^2)} \]

therefore

\[ P\{Z_{0,0} > z\} = 1 \quad \text{for } z < Z_1^* \quad (2.32) \]

Also from the graph, \( Z_{0,0} \leq A \) and therefore

\[ P\{Z_{0,0} > z\} = 0 \quad \text{for } z > A \quad (2.33) \]

Let us define

\[ Z_3^* = I(0, -d_y/2) = I(0, d_y/2) = Ae^{-ad_y^2/4} \quad (2.34) \]

Now consider the case of \( Z_3^* < z < A \) as in Fig. 2.5. It is obvious that for \( Z_{0,0} > z \), \( X_{k,l} \) must lie inside the shaded circle. Hence

\[ P\{Z_{0,0} > z\} = \frac{\text{area of circle}}{\text{area of rectangle}} \quad (2.35) \]

where the rectangle is defined by the bounds in 2.30. But \( z = Ae^{-ar^2} \) and on rearranging terms we obtain

\[ r^2 = \frac{1}{a} \ln \left( \frac{A}{z} \right) \quad (2.36) \]

where \( r = \sqrt{x^2 + y^2} > 0 \) and we note the area of the circle is \( \pi r^2 = (\pi/a) \ln (A/z) \).

Therefore

\[ P\{Z_{0,0} > z\} = \frac{\pi}{ad_x d_y} \ln \left( \frac{A}{z} \right) \quad \text{for } Z_3^* \leq z \leq A \quad (2.37) \]

Let us define

\[ Z_2^* = I(-d_x/2, 0) = I(d_x/2, 0) = Ae^{-ad_x^2/4} \quad (2.38) \]
Figure 2.6: View of Gaussian, Figure 2.4, from above. The shaded area indicates the region $Z_2^* \leq z \leq Z_3^*$

Figure 2.7: Geometrical construction for the case of $Z_2^* \leq z \leq Z_3^*$
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Now consider the case of $Z_2^* \leq z \leq Z_3^*$ and again look at the graph of our two dimensional Gaussian (Fig. 2.4) from above as shown in Fig. 2.6. We can see that for $Z_{0,0} > z$, $X_{k,l}$ must lie inside the shaded region. Hence

$$P\{Z_{0,0} > z\} = \frac{\text{area of shaded region}}{\text{area of rectangle}} \quad (2.39)$$

From figure 2.7 we can see

$$\text{area of shaded region} = 4(\text{area of triangle + area of sector}) \quad (2.40)$$

The area of the triangle is given by

$$\frac{1}{2} \text{base x height} = \frac{1}{2} \frac{d_y}{2} \sqrt{\frac{1}{2} \ln \left(\frac{A}{z}\right) - \frac{d_y^2}{4}} \quad (2.41)$$

while the area of the sector is

$$\frac{\theta_2 r^2}{2} = \frac{\theta_2}{2a} \ln \left(\frac{A}{z}\right) \quad (2.42)$$

where

$$\theta_2 = \sin^{-1} \frac{d_y}{2\sqrt{1/2 \ln \left(\frac{A}{z}\right)}} \quad (2.43)$$

Hence from equations 2.39, 2.40, 2.41 and 2.42

$$P\{Z_{0,0} > z\} = \frac{1}{d_z} \sqrt{\frac{1}{a} \ln \left(\frac{A}{z}\right) - \frac{d_y^2}{4}}$$

$$+ \frac{2\theta_2}{2\pi d_x d_y} \ln \left(\frac{A}{z}\right) \quad \text{for } Z_2^* < z < Z_3^* \quad (2.44)$$

Now finally consider the case of $Z_1^* \leq z \leq Z_2^*$ and we look at the graph in Figure 2.8. Again it is obvious that for $Z_{0,0} > z$, $X_{k,l}$ must lie inside the shaded region. Hence

$$P\{Z_{0,0} > z\} = \frac{\text{area of shaded region}}{\text{area of rectangle}} \quad (2.45)$$
Figure 2.8: View of Gaussian, Figure 2.4, from above. The shaded area indicates the region \( Z_1^* \leq z \leq Z_2^* \)

With reference to Figure 2.9 this shaded area, \( A_s \), is given by

\[
A_s = 4 \left( \text{area of triangle 1} + \text{area of triangle 2} + \text{area of sector} \right) \tag{2.46}
\]

The individual areas of triangle 1, triangle 2 and the sector are given by

\[
\begin{align*}
\text{Area of triangle 1} &= 1/2 \text{ base } \times \text{ height} \\
&= (1/2) \frac{dy}{2} \sqrt{\frac{1}{a} \ln \left( \frac{A}{z} \right) - \frac{d_y^2}{4}} \tag{2.47}
\\
\text{Area of triangle 2} &= 1/2 \text{ base } \times \text{ height} \\
&= (1/2) \frac{dx}{2} \sqrt{\frac{1}{a} \ln \left( \frac{A}{z} \right) - \frac{d_x^2}{4}} \tag{2.48}
\\
\text{Area of sector} &= \frac{\theta_1 r^2}{2} \\
&= \frac{\theta_1}{2a} \ln \left( \frac{A}{z} \right) \tag{2.49}
\end{align*}
\]
Figure 2.9: Geometrical construction for the case of \( Z_1^* \leq z \leq Z_2^* \)

where

\[
\theta_1 = \frac{\pi}{2} - \alpha - \beta \\
= \frac{\pi}{2} - \cos^{-1} \frac{d_y}{2 \sqrt{\frac{1}{a} \ln \left( \frac{A}{z} \right)}} - \cos^{-1} \frac{d_x}{2 \sqrt{\frac{1}{a} \ln \left( \frac{A}{z} \right)}}
\]

Hence from equations 2.45 to 2.49

\[
P\{Z_{0,0} > z\} = \frac{1}{d_x} \sqrt{\frac{1}{a} \ln \left( \frac{A}{z} \right)} - \frac{d_y^2}{4} + \frac{1}{d_y} \sqrt{\frac{1}{a} \ln \left( \frac{A}{z} \right)} - \frac{d_x^2}{4} + \frac{2\theta_1}{ad_x d_y \ln \left( \frac{A}{z} \right)} \\
\text{for } Z_1^* < z < Z_2^*
\]

Putting equations 2.32, 2.33, 2.37, 2.44 and 2.50 together we obtain
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\[
P\{Y_n > y\} = \begin{cases} 
1 & \text{if } Z < Z_1^* \\
\frac{1}{d_x} \sqrt{\frac{1}{a} \ln\left(\frac{A}{z}\right)} \frac{d_y^2}{4} & \text{if } Z_1^* \leq z \leq Z_2^* \\
+ \frac{1}{d_y} \sqrt{\frac{1}{a} \ln\left(\frac{A}{z}\right)} - \frac{d_x^2}{4} & \text{if } Z_2^* \leq z \leq Z_3^* \\
\frac{2\theta_1}{\pi} + \frac{ad_x d_y \ln(A/z)}{2} & \text{if } Z_3^* \leq z \leq A \\
0 & \text{if } z > A
\end{cases}
\]  

(2.50)

where

\[
Z_1^* = Ae^{-a(d_x^2 + d_y^2)/4} \\
Z_2^* = Ae^{-ad_x^2/4} \\
Z_3^* = Ae^{-ad_y^2/4}
\]

\[
\theta_1 = \frac{\pi}{2} - \cos^{-1} \left( \frac{d_y}{2\sqrt{\frac{1}{a} \ln \left( \frac{A}{z} \right)}} \right) - \cos^{-1} \left( \frac{d_x}{2\sqrt{\frac{1}{a} \ln \left( \frac{A}{z} \right)}} \right) \\
\theta_2 = \sin^{-1} \left( \frac{d_y}{2\sqrt{\frac{1}{a} \ln \left( \frac{A}{z} \right)}} \right)
\]

If in fact \( d_y > d_x \) then just swap the roles of \( d_x \) and \( d_y \)

### 2.5 Pixel sampling

We now look at the integration effects of a CCD caused as pixels sample the image formed in the image plane.
2.5.1 One dimensional case

Suppose we drop a grid onto a Gaussian centred at \( x = 0 \), i.e.

\[
I(x) = Ae^{-ax^2}
\]

and let us assume the grid to be uniform with spacing \( d \)

\[
X_i = id + X_0, \quad \forall i \in \mathbb{Z}
\]

where \( X_0 \) is a uniformly distributed r.v..

Now at each grid point \( X_i \) let us integrate \( I(x) \) between \((2i-1)d < x < (2i+1)d/2\)
to give, say \( Y_i \), that is, we form a r.v. \( Y_i \) of the form

\[
Y_i = \int_{X_i-d/2}^{X_i+d/2} I(x) \, dx
\]

(2.51)

Here \( Y_i \) corresponds to the pixel reading of the pixel centred at \( X_i \) and of width \( d \). Let \( Y \) be the maximum of the \( Y_i \)'s, i.e.

\[
Y = \max_i Y_i
\]

Then \( Y \) is also a r.v. and we seek the probability, \( P\{Y > y\} \), that \( Y \) is greater than \( y \), where \( y \) is some amplitude threshold. Now only one of the sample points, say \( X_j \), will fall in the interval \(-d/2 < x < d/2\), unless it falls at \( x = -d/2 \), in which case another sample point will fall at \( x = d/2 \), and we will have two sample points in this interval. Furthermore

\[
Y_j = \max_i Y_i
\]

2.5.1.1 The central most sample

Now suppose without loss of generality that \( X_0 \) is defined in the interval \(-d/2 < x < d/2\).

Furthermore the value of the Gaussian at this point will be \( Y_0 \), that is

\[
Y_0 = \int_{X_0-d/2}^{X_0+d/2} I(x) \, dx = \int_{X_0-d/2}^{X_0+d/2} Ae^{-ax^2} \, dx
\]

(2.52)
Consider a Gaussian density of the form $e^{-x^2}$. Let us define the error function, \( \text{erf}(x) \) as

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt
\]  

(2.53)

where we have introduced \( t \) as a dummy variable of integration. Now consider

\[
\int_{x_1}^{x_2} e^{-ax^2} dx
\]

If we let \( t = \sqrt{ax} \) and \( \therefore dx = dt/\sqrt{a} \), then when \( x = x_1, \) \( t = \sqrt{ax_1} \), and when \( x = x_2, \) \( t = \sqrt{ax_2} \).

Therefore

\[
\int_{x_1}^{x_2} e^{-ax^2} dx = \frac{1}{\sqrt{a}} \int_{\sqrt{ax_1}}^{\sqrt{ax_2}} e^{-t^2} dt
\]

\[
= \frac{1}{2\sqrt{a}} \left( \frac{2}{\sqrt{\pi}} \int_{\sqrt{ax_1}}^{\sqrt{ax_2}} e^{-t^2} dt \right)
\]

\[
= \frac{1}{2\sqrt{a}} \left( \text{erf}(\sqrt{ax_2}) - \text{erf}(\sqrt{ax_1}) \right)
\]  

(2.54)

Now from equation 2.52 we may write

\[
\mathcal{Y}_0 = A \int_{\mathcal{X}_0-d/2}^{\mathcal{X}_0+d/2} e^{-ax^2} dx
\]

and it is obvious from equation 2.54 that

\[
\mathcal{Y}_0 = \frac{A}{2} \sqrt{\frac{\pi}{a}} \left( \text{erf}(\sqrt{a}(\mathcal{X}_0 + d/2)) - \text{erf}(\sqrt{a}(\mathcal{X}_0 - d/2)) \right)
\]  

(2.55)

Now consider

\[
g(x) \overset{\text{def}}{=} \frac{A}{2} \sqrt{\frac{\pi}{a}} \left( \text{erf}(\sqrt{a}(x + d/2)) - \text{erf}(\sqrt{a}(x - d/2)) \right)
\]  

(2.56)

This can be visualized as the function formed from the integration effects of moving a one dimensional pixel, centred on \( x \) and of width \( d \), across the Gaussian
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image in the image plane, i.e. the convolution of a Gaussian with a top hat (pixel). There was a temptation to refer to this as the sensor image. However this name has been reserved for the intensity recorded by the CCD sensor of an image in the image plane, i.e a realized intensity on a CCD sensor. Instead, equation 2.56 is referred to as the pixel function and it can be thought of, for the purposes of the present analysis, as lying between the image and sensor planes and is used in the following way. Suppose we randomly drop a point onto this function. Since it explicitly takes into account the integration effects of a one dimensional pixel, randomly dropping a point onto the pixel function corresponds to randomly dropping a pixel onto the Gaussian image in the image plane.

Figure 2.10 shows the form of the function and if we compare it to the Gaussian intensity profile in Figure 2.3 we see that both functions have two important similarities; they are both even and have two real solutions defined in the ranges $[0, \infty] \text{ and } [0, -\infty]$. From the above discussion and with reference to figure 2.10 it is obvious that we can derive the probability of $\gamma_0$ being greater than some threshold value $y$, $P\{\gamma_0 > y\}$, in a fashion similar to the derivation of the discrete sampling counterpart. In other words, the experiment defined to account for the effects of integration in the image plane can be accounted for by discrete sampling experiments of the pixel function. We now proceed to derive the $P\{\gamma_0 > y\}$.

With reference to figure 2.10, $\gamma_0$ is greater than or equal to $y$ when $-\frac{d}{2} \leq x \leq \frac{d}{2}$. Therefore

$$\gamma_0 \geq \frac{A}{2} \sqrt{\frac{\pi}{a}} \text{erf}(\sqrt{ad}) \quad (2.57)$$

since $\text{erf}(0) = 0$ and $\text{erf}(-x) = -\text{erf}(x)$. Let

$$y_1^* = \frac{A}{2} \sqrt{\frac{\pi}{a}} \text{erf}(\sqrt{ad}) \quad (2.58)$$

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and therefore

\[ P\{y_0 > y\} = 1 \quad \text{for } y_1 < y \] (2.59)

Also from figure 2.10, \( y_0 \) is greater than or equal to \( y \) when \( x = 0 \) and therefore

\[ y_0 \leq A\sqrt{\pi / a} \text{ erf}(\frac{\sqrt{a}d}{2}) \] (2.60)

since \( \text{erf}(-x) = -\text{erf}(x) \). Let

\[ y_2^* = A\sqrt{\pi / a} \text{ erf}(\frac{\sqrt{a}d}{2}) \] (2.61)

and therefore

\[ P\{y_0 > y\} = 0 \quad \text{for } y > y_2^* \] (2.62)
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Now consider the case of $y_1^* \leq y \leq y_2^*$. From the graph for $Y > y$, $x$ must satisfy $-l < x < l$. Hence it is obvious that

$$P\{Y > y\} = \frac{\text{length of inner interval}}{\text{length of outer interval}} = \frac{2l}{d}$$

(2.63)

Considering equation 2.56 we note $g$ maps $[0, d/2]$ one to one and onto $[y_1^*, y_2^*]$ and so $g$ has an inverse function $g^{-1}$ which maps $[y_1^*, y_2^*]$ onto $[0, d/2]$ and

$$l = g^{-1}(y)$$

(2.64)

Unfortunately there is no closed form expression for $g^{-1}(y)$ and so we use numerics. For a given $y$ and a set of parameter values $\lambda$, $\alpha$, and $d$ we can use Newton-Raphson iteration.

Equation 2.64 can be restated as $g(l) = y$ or $g(l) - y = 0$. Thus applying Newton-Raphson to this we obtain

$$l_{n+1} = l_n - \frac{g(l_n) - y}{g'(l_n)} \quad \text{for } n \in \mathbb{Z} \quad n = 0, 1, 2, 3, \ldots$$

(2.65)

where $l_0$ is an initial approximation to the $l$ we are trying to find. From the above equation we can see that we require $g'(l_n)$. Now we use the fact that

$$\frac{d}{dx} \text{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

from our definition of the error function, equation 2.53. and the derivative of $g(x)$, $g'(x)$, simplifies to

$$g'(x) = e^{-\frac{2}{4}(2x+d)^2} - e^{-\frac{2}{4}(2x-d)^2}$$

hence

$$g'(l_n) = e^{-\frac{2}{4}(2l_n+d)^2} - e^{-\frac{2}{4}(2l_n-d)^2}$$

Thus, once we have found an accurate approximate for $l$ using Newton-Raphson iteration, we can find

$$P\{Y_0 > y\} = \frac{2l}{d}$$
2.5.2 Effects of integration in two dimensions

Let us now generalise to two dimensions. Suppose we drop a uniform rectangular grid onto a Gaussian. Let the Gaussian be centred at \((x, y) = (0, 0)\), so that

\[
I(x, y) = A e^{-a(x^2 + y^2)}
\]

The uniform rectangular grid is defined to be

\[
\mathcal{X}_{i,j} = (\mathcal{X}_0 + id_x, \mathcal{Y}_0 + jd_y) \quad \forall \ i, j \in \mathbb{Z}
\]

where \(d_x\) and \(d_y\) are the grid spacings in the \(x\) and \(y\) directions respectively and \(\mathcal{X}_0\) and \(\mathcal{Y}_0\) are r.v., with \(\mathcal{X}_0\) uniformly distributed in the interval \(-\frac{d_x}{2} \leq x \leq \frac{d_x}{2}\) and \(\mathcal{Y}_0\) uniformly distributed in the interval \(-\frac{d_y}{2} \leq y \leq \frac{d_y}{2}\). Now at each grid point \(\mathcal{X}_{i,j}\) let us integrate \(I(x, y)\) between

\[
\frac{2i-1}{2}d_x \leq x \leq \frac{2i+1}{2}d_x, \quad \frac{2j-1}{2}d_y \leq y \leq \frac{2j+1}{2}d_y
\]

to give, say \(Z_{i,j}\), that is, we form a r.v. of the form

\[
Z_{i,j} = \int_{\mathcal{Y}_{j-d_y/2}}^{\mathcal{Y}_{j+d_y/2}} \int_{\mathcal{X}_{i-d_x/2}}^{\mathcal{X}_{i+d_x/2}} I(x, y) \, dx \, dy
\]

\[
= \int_{\mathcal{Y}_{j-d_y/2}}^{\mathcal{Y}_{j+d_y/2}} \int_{\mathcal{X}_{i-d_x/2}}^{\mathcal{X}_{i+d_x/2}} Ae^{-a(x^2 + y^2)} \, dx \, dy \quad (2.66)
\]

Here \(Z_{i,j}\) corresponds to the pixel reading of the pixel centred at \(\mathcal{X}_{i,j}\) and of area \(d_x \times d_y\). Now only one of the sample points, say \(\mathcal{X}_{k,l}\), will fall inside the rectangle,

\[
\frac{d_x}{2} \leq x \leq \frac{d_x}{2}, \quad \frac{d_y}{2} \leq y \leq \frac{d_y}{2}
\]

unless it falls on an edge of this rectangle, in which case another sample point will lie in the rectangle, or it falls at a vertex, in which case another three points will lie in the rectangle. The value of the Gaussian at this point (these points) will be \(Z_{k,l}\), and furthermore

\[
Z_{k,l} = \max_{i,j} Z_{i,j} \quad (2.67)
\]
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2.5.2.1 The central most sample

Now suppose without loss of generality that $X_{0,0}$ is defined in the interval

$$\frac{dx}{2} \leq x \leq \frac{dx}{2}, \quad \frac{dy}{2} \leq y \leq \frac{dy}{2}$$

Furthermore the value of the pixel at this point will be $Z_{0,0}$, that is

$$Z_{0,0} = \int_{y_0-dy/2}^{y_0+dy/2} \int_{x_0-dx/2}^{x_0+dx/2} I(x, y) \, dx \, dy$$

$$= \int_{y_0-dy/2}^{y_0+dy/2} \int_{x_0-dx/2}^{x_0+dx/2} A e^{-a(x^2+y^2)} \, dx \, dy \quad (2.68)$$

Now consider the real function

$$I(x_c, y_c) = A \int_{y_c-dy/2}^{y_c+dy/2} \int_{x_c-dx/2}^{x_c+dx/2} e^{-a(x^2+y^2)} \, dx \, dy$$

$$= A \int_{y_c-dy/2}^{y_c+dy/2} \int_{x_c-dx/2}^{x_c+dx/2} e^{-a(x^2)} e^{-a(y^2)} \, dx \, dy$$

$$= A \int_{y_c-dy/2}^{y_c+dy/2} e^{-a(y^2)} \, dy \int_{x_c-dx/2}^{x_c+dx/2} e^{-a(x^2)} \, dx$$

$$= \frac{A \pi}{4a} \left( \text{erf}(\sqrt{a}(x_c + dx/2)) - \text{erf}(\sqrt{a}(x_c - dx/2)) \right)$$

$$\times \left( \text{erf}(\sqrt{a}(y_c + dy/2)) - \text{erf}(\sqrt{a}(y_c - dy/2)) \right) \quad (2.69)$$

This can be visualized as the function formed from the integration effects of moving a two dimensional pixel, centred on $(x_c, y_c)$ and of area $dx \times dy$, across the Gaussian image in the image plane. Now from the above equation and 2.68 it is obvious that

$$Z_{0,0} = \frac{A \pi}{4a} \left( \text{erf}(\sqrt{a}(X_0 + dx/2)) - \text{erf}(\sqrt{a}(X_0 - dx/2)) \right)$$

$$\times \left( \text{erf}(\sqrt{a}(Y_0 + dy/2)) - \text{erf}(\sqrt{a}(Y_0 - dy/2)) \right) \quad (2.70)$$

Unfortunately this function is not axisymmetric and therefore we cannot use the derivation method of section 2.4.2. Rather numerical experiments are performed in section 3.5.2 by randomly sampling the above equation to generate the required statistics.
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2.6 Quantisation

An important aspect of the digital image representation is the quantisation process, also known as analog to digital conversion. Each spatial sample of the image must be represented by a discrete integer value.

If the r.v. $A$ is of continuous type but $g(x)$ is of a staircase form then the r.v. $B = g(A)$ is of discrete type. The effects of quantisation are accounted for using a staircase function, which we denote by $g_q(x)$, given by

$$g_q(x) = \text{ps} \quad \text{where } ps < x \leq (p+1)s$$

where $p$ is an integer and $s$ a given constant. Now the the r.v. $B = g(A)$ is of discrete type taking the values $ps$ with

$$P\{B = ps\} = P\{ps < A \leq (p+1)s\} = F_A[(p+1)s] - F_A(ps) \quad (2.71)$$

Its distribution function $F_B$ is a staircase function with jumps at the points $ps$. Now we note

$$P\{A \leq a\} = 1 - P\{A > a\}$$

or equivalently

$$F_A(a) = 1 - P\{A > a\}$$

in which case we may rewrite equation 2.71 as

$$P\{B = ps\} = 1 - P\{A > (p+1)s\} - \{1 - P\{A > (p)s\}\} = P\{A > (p)s\} - P\{A > (p+1)s\} \quad (2.72)$$

Thus we can determine the quantisation effects from probabilities we have already derived.
2.6.1 Discrete case

Consider a grid, uniform with spacing $d$ given by

\[ X_i = id + X_0, \quad \forall i \in \mathbb{Z} \quad (2.73) \]

where $X_0$ is a uniformly distributed r.v. and we further form the sequence of r.v $Y_i = I(X_i)$, as described in section 2.4.1.1. Without loss of generality we assume the r.v $X_0$ is uniformly distributed in the range $-\frac{d}{2} < x < \frac{d}{2}$ and we denote the value of the Gaussian at this point as $Y_0$. We now form the r.v

\[ Y_{qi} = g_q(Y_i) \quad i \in \mathbb{Z} \]

where $g_q$ is a staircase (quantisation) function. Now the distribution $P\{Y_0 > y\}$ for the central most sample is given by equation 2.26 and from equation 2.71 we can see that

\[ P\{Y_{q0} = ps\} = \left\{ \begin{array}{ll}
\frac{3}{a} \sqrt{\frac{A}{ps}} \left[ \sqrt{\ln(A/ps)} - \sqrt{\ln(A/(p+1)s)} \right] & \text{if } y^* \leq ps \leq A \\
0 & \text{elsewhere}
\end{array} \right. \quad (2.74) \]

where

\[ y^* = Ae^{-ad^2/4} \quad A, a, d \in \mathbb{R} \quad A < 0, a < 0, d < 0 \]

For the general positive $n^{th}$ sample with reference to equations A.7 and 2.71 we can see that

\[ P\{Y_{qn} = ps\} = \left\{ \begin{array}{ll}
\frac{1}{a} \sqrt{\frac{A}{ps}} \left[ \sqrt{\ln(A/ps)} - \sqrt{\ln(A/(p+1)s)} \right] & \text{if } y_{min} \leq ps \leq y_{max}^* \\
0 & \text{elsewhere}
\end{array} \right. \quad (2.75) \]

where

\[ y_{min}^* = Ae^{-a\left(\frac{(2n+1)d}{2}\right)^2} \]
\[ y_{max}^* = Ae^{-a\left(\frac{(2n-1)d}{2}\right)^2} \]

$n \in \mathbb{Z}, \quad n = 1, 2, 3, \ldots ,

A, a, d \in \mathbb{R} \quad A < 0, a < 0, d < 0$
For the general negative $n^{th}$ sample with reference to equations A.12 and 2.71 we can see that

$$P\{Y_{qn} = ps\} = \begin{cases} \frac{1}{d/a} \left[ \sqrt{\ln(A/ps)} - \sqrt{\ln(A/(p+1)s)} \right] & \text{if } y_{min}^* \leq ps \leq y_{max}^* \\ 0 & \text{elsewhere} \end{cases}$$

where

$$y_{min}^* = A e^{-a\left(\frac{(2n+1)d}{2}\right)^2}$$
$$y_{max}^* = A e^{-a\left(\frac{(2n)d}{2}\right)^2}$$

$n \in \mathbb{Z}$, $n = -1, -2, -3, \ldots$

$A, a, d \in \mathbb{R}$, $A < 0, a < 0, d < 0$

A similar analysis can be applied to the two dimensional discrete sampling cases. However for pixel sampling, no closed form expression is obtained for the $P\{Y > y\}$ (or $P\{Z > z\}$) and therefore equation 2.72 has to be numerically evaluated in these cases.

### 2.7 Discussion and summary

In section 2.2 a general method is given (Theorem 1) for determining the density of a one dimensional function of a r.v. However, rather than deriving the density, it is preferred to directly derive the distribution as this allows a more physical interpretation of the results.

For discrete sampling there is a further reason for computing the $P\{Y > y\}$, namely, that due to certain specifics of the problem, this quantity is more easily found directly, rather than deriving the density as an intermediate stage. These specifics are (a) the form of the particle image i.e. Gaussian, and (b) the simplicity of the uniform distribution of the sampling point.

Normally, the method for calculating the probability $P\{y_{min} < Y < y_{max}\}$ is to find the area under the density function between the points $y_{min}$ and $y_{max}$.

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However, here in the one-dimensional central most sample case \( P\{y_{\text{min}} < \mathcal{Y}_0 < y_{\text{max}} \} \) can simply be found as a ratio of two lengths.

The extension in the one dimensional case to general positive (appendix A.1) and negative samples (appendix A.2) follows the same methodology as in the central sample case, i.e. the simplicity of the Gaussian and uniform density allowed the \( P\{\mathcal{Y}_n > y\} \) to be computed as simple ratios. This methodology is further extended to the two dimensional discrete central most sample albeit with more complicated ratios. These are now of the form of ratios of areas of circles to rectangles and combinations of triangles and sectors to rectangles. This approach still allows relatively simple expressions for \( P\{Z_0 > z\} \) to be found.

Extending the geometrical analysis to the two dimensional general positive and negative sample points becomes cumbersome as can be seen from figure 2.11. As the circles become progressively larger they encompass more and more grid cells. The number and complexity of shapes, needed to determine the probability, increases as a contour encompasses more grid cells.

\[\text{Figure 2.11: Looking down on a Gaussian function centred on a rectangular grid. The dashed circles represent contours of decreasing } z \text{ with increasing radius as they come into contact with grid sides or corners}\]

As a result, theorem 1 is used in the initial stages of the evaluation of the
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$P\{Z_n > z\}$ utilizing the fact that the assumed particle image function is separable, that is

$$Ae^{-a(x^2+y^2)} = Ae^{-ax^2}e^{-ay^2}$$

Therefore the joint density is easily calculated since $e^{-ax^2}$ and $e^{-ay^2}$ are independent. The methodology described in section 2.2.4 is then used to determine the $P\{Z_n > z\}$ as shown in appendix A.3

In section 2.5 (pixel sampling) the integration effects of the CCD sensor are taken into account when sampling a Gaussian image in the image plane. It is shown that with the use of a suitable intermediate function, which is referred to as the pixel function, pixel sampling a Gaussian image in the image plane is equivalent to discretely sampling the pixel function. Thus the methodology of one dimensional discrete sampling can be used to compute the $P\{Y > y\}$ for the one dimensional pixel sampling case. This does not hold for the two dimensional case. Although the pixel function is derived, it is found not to be axisymmetric, a fact which is relied upon in the two dimensional discrete sampling derivation. Thus numerics must be used to compute this function. Finally, in section 2.6 quantisation effects are modelled using a staircase function.
Chapter 3

Recording particle images; review and experiment

In this chapter, theory relating to the formation of particle images, in the context of PIV applications, is reviewed and aspects of CCD sensors are outlined. Three experiments are performed concerned with the effects on the recorded intensity of a PIV image due to (a) the integration effects of CCD pixels, (b) the position of a particle in the laser sheet and (c) the light-scattering characteristics of particles.

3.1 Introduction

In Chapter 2, quantities referring to the statistical fluctuations caused by the sampling of a Gaussian function have been derived. The basic theory and simplifying assumptions used in deriving this Gaussian image are detailed in Section 3.3. As shown in Section 3.6.2, the discrete sampling derivations are also applicable when dealing with the random positioning of a particle in a PIV illumination system. This illumination is assumed to be a two-dimensional sheet in many PIV applications, since the length scales of the measurement region are often several orders of magnitude higher than the width of the beam (which is usually of the order $1 \times 10^{-3}\text{m}$). In the experiments detailed in this chapter, the $(x,y)$ plane is used to denote the plane corresponding to the large (normally assumed) two-dimensional...
measurement plane while the z-axis refers to the axis associated with the width of the laser beam.

In the first experiment, the variations in a particle's recorded intensity due to the integration effects of the pixels comprising the CCD sensor are investigated. For this, a particle is moved randomly in the (x, y) plane only of the illuminating sheet with the incident light on the particle kept constant. The second experiment is concerned with the effect on a particle's recorded intensity due to its random position in the z-axis (with fixed (x, y)) of the laser light sheet. The profile of the beam in this axis is assumed to be Gaussian. The final experiment, as in the first, again restricts itself to the (x, y) plane but in this case a relatively large number of particles are examined in an attempt to determine the variation caused by the particle's characteristics.

The raw data produced by all three experiments consists of graphics image files (specifically Portable GreyMaps PGMs) containing intensity information of the light scattered by the particle(s) In the first two experiments, the quantity used to quantify the amount of light recorded for each particle image is the particle's central pixel (as defined in section 2.3) intensity value. The reasons for choosing this quantity is treated with more detail in section 3.8. These central intensity values are then plotted as histograms and compared with the theoretical predictions of Chapter 2.

3.2 Particle Image Velocimetry

PIV is now a well established experimental technique for the measurement of fluid velocities and an indication of its widespread applicability and usage can be gained from a perusal of Adrian's particle velocimetry bibliography [3]. The technique is now also well documented and described, [2], [30], and [54], and
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correction
therefore only a brief description of PIV is given below.

The measurement of fluid velocity invariably involves the determination of the fluid displacement over a known interval of time. In a typical PIV set-up, a light source illuminates the flow, in which tracer particles have been added.

PIV relies on the ability to record and measure accurately the positions of these particles which follow the flow as a function of time. The velocity is then deduced from the relationship:

\[
\vec{u}_t = \frac{\vec{x}_{t+\delta t} - \vec{x}_t}{\delta t}
\]  

(3.1)

where \( \vec{x}_t \) is the position vector of the fluid tracer at time \( t \), \( \vec{x}_{t+\delta t} \) is its position at a small time \( \delta t \) later and \( \vec{u}_t \) is the approximation to the local fluid velocity at time \( t \).

![Figure 3.1: Typical PIV recording set-up using CCD camera](image)

A schematic diagram of the PIV imaging and recording systems for the current work are shown in figure 3.1, where side-scattered light from particles is focused onto a CCD sensor which records the intensity of the incoming light.
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This imaging process converts the scene information into an illumination pattern in the image plane. The manner in which the flow pattern is recorded is referred to as the recording mode whilst the type of sensor used (CCD in this case) is referred to as the recording medium. The camera's analogue output is fed into a frame grabber which converts this to a digital signal. Software then converts this signal to a suitable format for storage, typically a graphic file format (e.g. TIFF, BMP, PGM, etc.). This recorded pattern is subsequently analysed to extract the required velocity information [58].

Figure 3.2: Schematic showing the process of image recording and acquisition indicating the main components affecting the recorded image

Figure 3.2 shows a schematic representation of this process indicating the main components relevant to image formation and recording. One of the main concerns of the current investigation is with measuring the two particle phases separately. The formation of the particle's image in the image plane and its recording by CCD are thus of great interest and we now review and outline some basic theory
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relating to this.

3.3 Image formation

The most rigorous tool to describe the scattering of a beam by a spherical particle is the generalised Lorenz-Mie theory (GLMT) [24], [18]. GLMT codes can be linked with diffraction code to take into account the effect of lens arrangement [49], [50]. However an analysis such as this is beyond the scope of the present investigation. Furthermore, even if the method were adopted, the computation time is prohibitive, a fact which excludes its use for the extraction of particle size and location directly from recorded patterns. The approach also yields numerical rather than analytical results. Since we are interested in determining statistics relating to the sampling of the image formed by a CCD sensor, an analytical image description is preferable to numerical results. Finally, the approach assumes a spherical particle. It is difficult to determine whether seeding materials such as pollen can be sensibly approximated as spheres.

Rather than use Lorenz-Mie theory, we adopt the Point Spread Function (PSF) approach [40]. A derivation given by Goodman [16], forms the basis of this approach and this derivation is now briefly outlined followed by a description of the PSF approach applied to PIV.

3.3.1 The impulse response of the optical system

The analysis begins by considering the geometry of figure 3.3 containing a circular, positive and thin lens, of focal length $f$, which is assumed to be aberration-free. The illumination is monochromatic, implying that the imaging system is linear in complex field amplitude and the image formed is real in the sense that an actual distribution of intensity appears across a plane behind the lens. Since
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Figure 3.3: Geometry for image formation

the wave-propagation phenomenon is linear we may express this field \( U_i \) by the superposition integral:

\[
U_i(x_i, y_i) = \iiint h(x_i, y_i; x_o, y_o) U_o(x_o, y_o) \, dx_o \, dy_o ,
\]

where \( h \) is the field amplitude produced at coordinates \((x_i, y_i)\) by a unit amplitude point source applied at object coordinates \((x_o, y_o)\) and \( U_o \) is the complex field immediately behind the object. Note that boldface letters are used to describe complex quantities in this chapter.

With the further restriction that the object distance \( d_o \) and image distance \( d_i \) satisfy the geometrical lens law:

\[
\frac{1}{d_o} + \frac{1}{d_i} - \frac{1}{f} = 0 ,
\]

the impulse response \( h \), after certain simplifying approximations, of such a system is:

\[
h(x_i, y_i; x_o, y_o) \approx \frac{1}{\lambda^2 d_o d_i} \iiint P(x, y) 
\exp \left( -\frac{2\pi}{\lambda d_i} \left[ (x + M x_o) x + (y + M y_o) y \right] \right) \, dx \, dy ,
\]

where \( \lambda \) is the wavelength of the light, \( M \) is the image magnification defined as:

\[
M = \frac{d_i}{d_o}
\]
and $P$ is the pupil function given by:

$$P(x, y) = \begin{cases} 
1 & \text{inside the lens aperture} \\
0 & \text{otherwise.} 
\end{cases}$$

Thus the impulse response is given by the Fraunhofer diffraction pattern of the lens aperture, centered on image coordinates $(x_i = -Mx_o, y_i = -My_o)$. The lens aperture plays a significant role in the system response since by choosing $d_i$ to satisfy the lens law, we are examining the plane toward which the spherical wave leaving the lens is converging and it is the lens aperture that limits the extent of this wave. The Fraunhofer diffraction pattern for a circular lens and hence the impulse response of the optical system of figure 3.3, is given by

$$h(r) = \exp(jkd_i) \exp\left(\frac{k r^2}{2d_i}\right) \frac{k l^2}{j8d_i} \left[2\frac{J_1(klr/2d_i)}{klr/2d_i}\right].$$

(3.5)

Here $J_1$ is a Bessel function of the first kind order 1, $l$ is the diameter of the aperture, $r$ is the radius coordinate in the image plane and $k$ is the wavenumber:

$$k = 2\pi \left(\frac{\nu}{c}\right) = \frac{2\pi}{\lambda}$$

where $\nu$ is the optical frequency and $c$ is the speed of light.

Real radiation detectors respond to optical intensity rather than field amplitude and so the diffraction pattern is described as a distribution of the intensity $|h|^2$. Using equation 3.5, this is given by:

$$|h|^2(r) = \frac{k l^2}{8d_i} \left[2\frac{J_1(klr/2d_i)}{klr/2d_i}\right].$$

(3.6)

This function is better known as the Airy Function or Airy Pattern.

### 3.3.2 The relation between object and image

The image predicted by geometrical optics, $U_g$, for the configuration of figure 3.3 is given by:

$$U_g(x_i, y_i) = \frac{1}{M} U_o \left(\frac{-x_i}{M}, \frac{-y_i}{M}\right).$$

(3.7)
This image is an exact replica of the object, magnified and inverted in the image plane. When diffraction effects are included, this is no longer the case. With the change of variables:

\[
\begin{align*}
\dot{x} &= \frac{x}{\lambda d_i} \\
\dot{y} &= \frac{y}{\lambda d_i} \\
\dot{x}_o &= -M x_o \\
\dot{y}_o &= -M y_o,
\end{align*}
\]

(3.8) (3.9)
equation 3.4 may be rewritten as:

\[
h(x_i, y_i; \dot{x}_o, \dot{y}_o) = M \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P(\lambda d_i \dot{x}, \lambda d_i \dot{y}) \exp \left(-j2\pi \left[(x_i - \dot{x}_o) \dot{x} + (y_i - \dot{y}_o) \dot{y}\right]\right) d\dot{x} d\dot{y}
\]

(3.10)

and we see that \( h \) is now space invariant depending only on the coordinate differences \((x_i - \dot{x}_o, y_i - \dot{y}_o)\). Substituting

\[
\dot{h} = \frac{1}{M} h,
\]

the superposition integral, equation 3.2, becomes:

\[
U_i(x_i, y_i) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(x_i - \dot{x}_o, y_i - \dot{y}_o) \left[\frac{1}{M} U_o \left(\frac{-\dot{x}_i}{M}, \frac{-\dot{y}_i}{M}\right)\right] d\dot{x}_o d\dot{y}_o,
\]

(3.11)

which is the convolution of the diffraction-limited, amplitude impulse response \( \dot{h} \) with the image predicted by geometrical optics. Thus the actual image obtained is a smoothed version of the object due to the nonzero width of the impulse response.

### 3.3.3 PIV assumptions

In theoretical descriptions and investigations of PIV images, [4], [56], the Airy function is approximated by a Gaussian and the image formed is assumed to be the convolution of the geometrical optics prediction with this Gaussian. This is
similar to approaches taken in other fields, for example astronomy, where it is often referred to as the Point Spread Function approach.

For a particle diameter, $d_p$, Adrian [2], defines the following approximate formula for the image diameter $d_e$:

$$d_e \approx \sqrt{M^2d_p^2 + d_s^2}, \quad (3.12)$$

where $d_s$ is the diameter of the point response function of the diffraction-limited lens, measured at the first dark ring of the intensity distribution of the Airy disc given by:

$$d_s = 2.44(1 + M)f\# \lambda. \quad (3.13)$$

Here we have used $f\#$ to denote the f-number of the camera with:

$$f\# = \frac{f}{l}.$$

If the point response and geometrical optics functions are both Gaussian then equation 3.12 is exact.

For typical values of the parameters, say, $M = 1$, $f\# = 8$ and $\lambda = 0.6993\mu m$ and $d_s = 25\mu m$, $d_e$ is approximately independent of particle size for particle diameters less than about $10\mu m$. Conversely, for particle diameters greater than $50\mu m$, the image diameter is essentially $Md_p$ [2].

3.3.4 Recording the image

The depth-of-field of the lens is given by [2] as:

$$\delta z = 4(1 + M^{-1})^2 f\#^2 \lambda \quad (3.14)$$

and places a constraint on an imaging system.

For the case of $f\# = 8$ and $\lambda = 0.6993\mu m$, the depth of field is only 0.7mm. For a small depth-of-field it is appropriate to illuminate just that region over which
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the particles are in focus, whereas for imaging over deep regions, as might be the case in three-dimensional particle tracking, the requirement for large depth-of-field implies that $f^\#$ and hence the particle diameters must be large. For example, to increase the depth-of-field from 0.7mm to 11mm, $f^\#$ must be increased by a factor of four and $d_s$ exceeds 100μm.

3.4 Recording technology

Any given sensor has a characteristic spectral sensitivity, i.e. its response varies with the wavelength of the incident light. Thus its total response to light at any point can be expressed by an integral of the form:

$$\int E(\lambda)I(\lambda)\,d\lambda,$$

where $I(\lambda)$ is the light intensity and $E(\lambda)$ is the efficiency of the conversion and recording process, as a function of wavelength. The two most common types of image sensors are photographic film and CCD array.

3.4.1 Photographic film

The intensity pattern falling on the film gives rise to a pattern of variations in the optical properties of the film. The best pictures are achieved using the finest grain film possible, however the emulsions of such films are the least sensitive to light forcing a compromise in image quality versus the amount of illumination available. For an exposed film, the optical transmittance $t$, i.e. the fraction of light transmitted by the film, varies from point to point. The quantity $-\log t$ is called the optical density; a density close to zero corresponds to an almost perfect transmission while a very high density, values of 2 to 3, correspond to almost perfect opacity. For ordinary film and developing processes, the density
is roughly a linear function of the log of the amount of incident light. The slope of the line is called Gamma.

3.4.2 CCD Array

The light intensity at each point is integrated and measured over the area covered by each of the array elements positioned in the image plane, and converted to an electronic or electric signal. Thus the sensor provides an electrical or electronic analogue of the image intensity.

3.4.3 Comparison of technologies

The three primary advantages of digital recording over traditional wet-film photography are firstly, the increases in speed and efficiency of the recording, secondly the elimination of the errors associated with wet-film photography [43] (although digital recording introduces its own errors) and thirdly the fact that once the image is recorded there is no degradation of the data which is not the case in wet recording. Furthermore, the CCD has a dynamic range (ratio of maximum detectable light intensity to minimum detectable light intensity) which covers more than ten magnitudes, generally a greater dynamic range than emulsions. CCD’s also feature high quantum efficiency and excellent linearity.

However CCDs have certain drawbacks. For PIV applications the primary disadvantage is that current CCDs have much smaller photosensitive areas than photographic medium. The consequential lower resolution renders them less suitable for experiments requiring the imaging of relatively large areas. In spite of their small areas, CCD’s produce enormous quantities of data which must be stored and analysed. This requires substantial hardware facilities and specialised software to analyse the data. In addition there are further hardware problems: due to manufacturing imperfections, individual elements may have different sen-
sitivities. Also if the time interval between two successive reading-out actions (exposure time) is too long, receiving elements will saturate and even overflow to adjacent cells. This saturation effect will cause registration of maximum capacity, whatever the actual amount of light intensity is. If there is a hardware problem between the output register and a column of CCD pixels, the registration of the entire column will be false and one has a bad column. Registered intensity will not always completely flow to an adjacent pixel and some may be left behind (but not lost), this is referred to as a charge trap. Also the images produced require careful calibration to remove certain two-dimensional patterns imposed by the detectors themselves such as bias and flat field patterns which are position dependent. Finally the electronics associated with CCDs, such as the timing of column readout in an interlaced fashion or the quantisation of the CCD's analogue output may affect the data in an undesirable way.

3.5 Experiment: the integration effects of pixels

We now detail an experiment to look at the effect of pixel integration on the central pixel amplitude of a particle image. We begin with a description of the experimental procedure.

3.5.1 Procedure

A single glass bead, with a diameter of 105\(\mu\)m, is encased in an optically clear gel. The gel is formed from a mixture of two solutions (available from GE Silicones). The first component, RTV615A is a clear silicone liquid which cures at room temperature to a high-strength, optically-clear silicone rubber with the addition of the second component RTV615B which acts as the curing agent. The recommended mixing ratio of RTV615A:RTV615B is 10:1. These components
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<table>
<thead>
<tr>
<th>Temperature, °C (°F)</th>
<th>Cure Time*</th>
</tr>
</thead>
<tbody>
<tr>
<td>25(77)</td>
<td>6-7 days</td>
</tr>
<tr>
<td>65(149)</td>
<td>4 hrs.</td>
</tr>
<tr>
<td>100(212)</td>
<td>1 hr.</td>
</tr>
<tr>
<td>125(257)</td>
<td>45 min.</td>
</tr>
<tr>
<td>150(302)</td>
<td>15 min.</td>
</tr>
</tbody>
</table>

* Cure times are only approximate. The actual time is affected by the mass of the unit and the time required to reach the desired temperature.

Table 3.1: Cure times for RTV615 solution

are mixed in this ratio to form 550ml of gel solution and stirred thoroughly. The mixture is then poured into an open-topped glass container of dimensions 130mm x 90mm x 56mm (length x breadth x height) and the glass bead pushed into the solution, using an alcohol cleansed thin metal rod, to approximately the centre of the mixture's volume.

Mixing the two solutions has the undesirable effect of introducing a large number of air bubbles into the resultant liquid (as does placing the bead into the mixture, but to a much lesser extent). Therefore the liquid is placed into a refrigerator since, with reference to table 3.1, the gel sets slower at low temperatures and this gives the air bubbles, which are less dense than the gel, time to rise to the top of the solution whereupon they burst. The open top of the glass container is sealed using cling film to prevent any dust particles from settling on the gel. The glass bead sinks at only a small fraction of the rate at which the air bubbles rise. Thus the timing of refrigeration is not critical and the gel is taken from the fridge when the air bubbles are considered removed by visually inspecting the gel. After this, the glass container is placed on a level optical table and allowed to completely set over approximately five days, whereupon it is removed from the glass container. The glass bead is illuminated by the light produced by a 20mW Helium-Neon (He-Ne) laser. Both the gel and laser are mounted on a
base which is attached to a traversing mechanism as shown in figure 3.4. The traversing mechanism is similar to that shown in figure 3.10, with the translation mechanisms adjusted to allow two-degrees freedom of movement in the \((x, y)\) plane rather than the \((x, z)\) plane. The translation mechanisms allow movement to an accuracy of \(5\mu m\).

Two sets of 200 random numbers, corresponding to random \(x\) and \(y\) co-ordinates are generated on a computer. These numbers are used to traverse the gel in the \(x\) and \(y\) directions in a random fashion. After the gel is moved to each new location an image is recorded by triggering the shutter of a Kodak Megaplus ES1.0 camera manually. Four sets of the above experiment are performed. In each experiment, \(f_#\) is kept constant at 5.6 while the magnitude ratio of image to object is varied for the values 0.0520, 0.0611, 0.0694 and 0.1237.

### 3.5.2 Data analysis and results

In Section 2.5.2.1, the fluctuations in intensity recorded by a CCD, of a Gaussian image in the image plane, are considered. Specifically, the centre pixel’s intensity
Z_{0,0} is given by equation 2.70. In order to evaluate the \( P\{Z_{0,0} > z\} \) we can perform a numerical experiment whereby equation 2.70 is randomly sampled. Now consider a comparison between this numerical experiment and the experiment of Section 3.5.1 in order to validate equation 2.70.

Firstly, for the CCD array used in this experiment, the pixels are square and so \( d_x = d_y \) (\( = d \) say). In addition, suppose we choose the length scale to be \( d \), then from equation 2.69, the intensity output from a pixel centred at \( (x_c, y_c) \) will be:

\[
I(x_c, y_c) = C \left( \text{erf}(\sigma(x_c + 1/2)) - \text{erf}(\sigma(x_c - 1/2)) \right) \times \left( \text{erf}(\sigma(y_c + 1/2)) - \text{erf}(\sigma(y_c - 1/2)) \right),
\]

where \( \sigma = \sqrt{\alpha d} \) is a non-dimensional parameter and \( C \) is a constant not necessarily equal to \( \frac{4\pi}{4\alpha} \) because the intensity output from the pixel is proportional only to equation 2.69. The experimental details of recording particle images are outlined in Section 3.5.1. For each of these images the maximum intensity is found and a graph is built up of the maximum intensities found against the number of images that have that maximum intensity, \( i.e. \) a histogram of peak intensities is generated.

Following this, exactly the same experiment is performed numerically on equation 3.15. Now, it can be shown that if a grid of pixels is dropped randomly on a Gaussian function, then the pixel that gives the maximum output intensity is the pixel whose centre \( (x_c, y_c) \) satisfies \( -d_x/2 \leq x_c \leq d_x/2 \) and \( -d_y/2 \leq y_c \leq d_y/2 \), which, in our experiment, translates as \( -1/2 \leq x_c \leq 1/2 \) and \( -1/2 \leq y_c \leq 1/2 \). Therefore the numerical experiment involves randomly choosing various points \( (x_c, y_c) \) in the region \([-1/2, 1/2] \times [-1/2, 1/2]\) and for each point recording the value of \( I(x_c, y_c) \), which is obtained from equation 3.15. A graph is then built up of the \( I(x_c, y_c) \) generated against the number of \( (x_c, y_c) \) that give that \( I(x_c, y_c) \), \( i.e. \) we numerically generate a probability density func-
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This graph should be the same as the graph from the physical experiment and how close it is will give an indication of how well the present theory models the real situation.

Finally, in order to perform the numerical experiment, we need to obtain numerical values for $C$ and $\sigma$ in equation 3.15. These values are obtained from the physical experiment. In a way equation 3.15 is fitted to the real data and the fit compared. Suppose that in the physical experiment, sufficiently many images are taken such that in some images, the centre of the Gaussian will lie very close to the centre of a pixel. In these images, the maximum intensity found will correspond to the maximum intensity found over all the images. Let $I_{\text{max}}$ denote this absolute maximum intensity found, then $I_{\text{max}}$ will correspond to:

$$(x_c, y_c) = (0, 0)$$

in equation 3.15. That is:

$$I_{\text{max}} = C \left( \text{erf}(\sigma/2) - \text{erf}(-\sigma/2) \right) \times \left( \text{erf}(\sigma/2) - \text{erf}(-\sigma/2) \right),$$

which reduces to:

$$I_{\text{max}} = C (2\text{erf}(\sigma/2))^2 = 4C(\text{erf}(\sigma/2))^2. \quad (3.15)$$

This gives one equation in order to evaluate $C$ and $\sigma$. Therefore another is still needed and this is obtained in the following way. Consider the images whose maximum intensity equals $I_{\text{max}}$, then in these images the centre of the Gaussian lies at the centre of a pixel as shown in figure 3.5. Let the output from the pixels N, S, E and W be $I_N = I(0,1)$, $I_S = I(0,-1)$, $I_E = I(1,0)$ and $I_W = I(-1,0)$
respective. Then \( I_N = I_S = I_E = I_W \) (= \( I_{\text{adjacent}} \) say). That is,

\[
I_{\text{adjacent}} = C(\text{erf}(3\sigma/2) - \text{erf}(\sigma/2)) \times 2\text{erf}(\sigma/2)
\]
\[
= 2C(\text{erf}(3\sigma/2) - \text{erf}(\sigma/2)) \times \text{erf}(\sigma/2)
\] (3.16)

The best way to calculate \( I_{\text{adjacent}} \) is to take all the images whose maximum intensity is \( I_{\max} \) and for each image calculate

\[
\frac{I_N + I_S + I_E + I_W}{4}
\]

and then set \( I_{\text{adjacent}} \) to the average of this quantity over all such images. Equation 3.16 represents the second equation necessary to find \( C \) and \( \sigma \). By dividing equation 3.16 by equation 3.15 we obtain

\[
\frac{I_{\text{adjacent}}}{I_{\max}} = \frac{2C(\text{erf}(3\sigma/2) - \text{erf}(\sigma/2)) \times \text{erf}(\sigma/2)}{4C(\text{erf}(\sigma/2))^2}
\]
\[
= \frac{1}{2} \left( \frac{\text{erf}(3\sigma/2)}{\text{erf}(\sigma/2)} - 1 \right)
\] (3.17)

This is a nonlinear equation for \( \sigma \) which can be solved by Newton-Raphson iteration. That is, the following iterative scheme should converge to \( \sigma \) given a sufficiently good initial guess \( \sigma_0 \).
Figure 3.6: Histogram and theoretical distribution for camera magnification $M = 0.0520$

Figure 3.7: Histogram and theoretical distribution for camera magnification $M = 0.0611$
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Figure 3.8: Histogram and theoretical distribution for camera magnification $M = 0.0694$

Figure 3.9: Histogram for camera magnification $M = 0.1237$
Consider the equation:

\[ \sigma_{n+1} = \sigma_n - \frac{\left( \frac{1}{2} \left( \frac{\text{erf}(\frac{3\sigma}{2})}{\text{erf}(\frac{\sigma}{2})} - 1 \right) - \frac{L_{\text{adjacent}}}{L_{\text{max}}} \right)}{\frac{3}{\sqrt{\pi}} \text{erf} \left( \frac{\sigma_n}{2} \right) e^{-\frac{\sigma_n^2}{4}} - \frac{1}{\sqrt{\pi}} \text{erf} \left( \frac{3\sigma_n}{2} \right) e^{-\frac{3\sigma_n^2}{4}}} \]

for \( n = 0, 1, 2, \ldots \)

(3.18)

Here we have used the Newton-Raphson formula:

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

for finding the root of \( f(x) = 0 \), together with the fact that:

\[ \frac{d}{dx}(\text{erf}(x)) = \frac{2}{\sqrt{\pi}} e^{-x^2}. \]

Once \( \sigma \) has been found, \( C \) can be obtained from either equation 3.15 or equation 3.17.

Figures 3.6, 3.7 and 3.8 and 3.9 show the theoretical distributions and experimental histograms for the camera magnifications 0.0520, 0.0611 and 0.0694 respectively. No theoretical distribution is generated for the final magnification as quantisation effects are clearly dominating the result in this case. The values of \( \sigma, C \) and the standard deviations of the experimental and theoretical data, \( s_e \)
and \( s_t \), are shown in table 3.2. The standard deviation \( \bar{s} \) is defined as:

\[
\bar{s} = \sqrt{\frac{1}{N} \sum_{j=1}^{N} (I_j - \bar{I})^2}
\]  

(3.19)

Here \( I_j \) is the \( j^{th} \) sample of the data, \( N \) is the total number of data samples and \( \bar{I} \) is the average intensity defined as:

\[
\bar{I} = \frac{\sum_{j=1}^{N} I_j}{N}
\]  

(3.20)

3.6 Experiment: Effects of randomly positioning a particle in a scanning laser sheet

In the following section an experiment to investigate the effect on the intensity of a recorded image's centre pixel due to its random position in a scanning laser illumination system is detailed. After this, the data analysis and results section is outlined.

3.6.1 Procedure

The gel containing the glass bead is clamped to the traversing mechanism of figure 3.10 and illuminated with the system shown in figure 3.11. The mechanism is traversed randomly in the \( z \)-direction. At each location an image is recorded on a Kodak Megaplus ES1.0 camera, with \( f^\# \) set to 5.6 and using a magnification of 0.1237.
3.6.2 Data analysis and results

In section 2.4.1 the distribution $P\{Y_0 > y\}$ is derived relating to an experiment of randomly sampling a Gaussian of the form:

$$I(x) = Ae^{-ax^2}.$$  

The profile of the laser beam is assumed to be Gaussian [13]. Although the derivation of Section 2.4.1 is used to introduce the random effects associated with recording sensors, the mathematical results obtained can be used, with minor changes, to describe the effects of the scanning laser light sheet as follows. Consider a particle located randomly in the measurement volume and illuminated by the scanning beam system described in Section 3.5.1. With reference to figure 3.12, for a particle with a fixed $(x,y)$ coordinate underneath the beam it is assumed the laser scans horizontally (in the $x$ direction) across the particle as...
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Figure 3.11: Illumination and recording system
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Figure 3.12: Schematic of scanning laser beam
shown in figure 3.12(a). Figure 3.12(c) shows the $z$ profile of the beam for fixed $z$ and although the scan is finite it is convenient in mathematical terms to assume an infinite extent. The $z$ profile of the beam, figure 3.12(b), is assumed to be Gaussian but truncated in width. In the physical experiment, this truncation is achieved by moving the particle within a fixed $z$ range. As the beam scans in the $x$-direction, the cumulative intensity of the light side-scattered by the particle is recorded by the sensor. Assuming the profile of the beam is of the form:

$$Ae^{-a(x^2+z^2)}$$

and that the particle is a point, this cumulative intensity $I(x, z)$ can be expressed mathematically as an integral of the form:

$$I(x,z) = A \int_{-\infty}^{\infty} e^{-a(x^2+z^2)} \, dx = Ae^{-az^2} \int_{-\infty}^{\infty} e^{-ax^2} \, dx.$$  \hspace{1cm} (3.21)

Here

$$\int_{-\infty}^{\infty} e^{-ax^2} \, dx$$

is a constant and noting that $I$ is now a function of $z$ only, the integration effects of the scanning beam can be written as:

$$I(z) = A_k e^{-az^2}$$  \hspace{1cm} (3.22)

where:

$$A_k = \int_{-\infty}^{\infty} e^{-ax^2} \, dx.$$  

A comparison with equation 2.21 indicates these two equations, apart from notation differences, are the same. Now in the experiment the particle has been randomly positioned in the $z$ direction over a specific range, i.e. $z$ is now a random variable ($= Z$), and therefore we may rewrite equation 3.22 as

$$I(Z) = A_k e^{-aZ^2}$$  \hspace{1cm} (3.23)
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In section 2.4.1 the Gaussian function was sampled randomly over a fixed range, exactly as has been done in this section. The experiments are analogous and so we may directly obtain the distribution $P\{Z_0 > z\}$ from equation 2.26 as:

$$P\{Z > z\} = \begin{cases} 
1 & \text{if } z < z^* \\
\frac{2}{d\sqrt{a}} \sqrt{\ln \left( \frac{A}{z} \right)} & \text{if } z^* \leq z \leq A \\
0 & \text{if } z^* > A,
\end{cases} \quad (3.24)$$

where:

$$z^* = Ae^{-ad^2/4} \quad A, a, d \in \mathbb{R} \quad A < 0, a < 0, d < 0.$$

In the results (figure 3.13) we record a histogram rather than the above distribution. The density $f_Z$ is easily derived from the distribution using equation 2.6 as:

$$f_Z = \begin{cases} 
\frac{1}{d\sqrt{a}} \sqrt{\ln \left( \frac{A}{z} \right)} & \text{if } z^* \leq z \leq A \\
0 & \text{elsewhere}
\end{cases} \quad (3.25)$$

where

$$z^* = Ae^{-ad^2/4} \quad A, a, d \in \mathbb{R} \quad A < 0, a < 0, d < 0.$$

In order to fit a curve to experimental data we need values for $A_k$ and the ratio $d\sqrt{a}$. $A_k$ is taken as the intensity of the mode of the histogram. A convenient ratio of $d\sqrt{a} = 2$ (corresponding to the $1/e$ width of the Gaussian) is chosen. This is determined by moving the mechanism to the two $z$ coordinates corresponding to an intensity of 74 grey levels (the $1/e$ intensity in this case). These positions are marked, defining the region over which to move the mechanism.
Figure 3.13: Histogram generated from laser scan data and theoretical distribution

3.7 Experiment: particle characteristics

For this experiment 250ml of RTV615A and 25ml of compound RTV615B are mixed thoroughly together and poured into an open-topped glass tank. The mixture is allowed to harden as described in Section 3.5. An identical second solution is prepared and a small quantity of quartz sand (the same beach sand/sediment used in the experiment of Chapter 4) is thoroughly stirred into this second solution. The resulting mixture is then poured into the glass container which holds the first (now hardened) gel.

The relatively heavy sediment particles sink to the bottom of the second prepared mixture, since this is still liquid, where they settle on the top of the first hardened gel. Once the second preparation is also hardened, a final gel with a thin layer of sediment particles in a plane in the middle of the gel results. Three
Figure 3.14: Histogram generated from particle characteristic data

further sediment gels are made in this fashion.

The first gel is clamped to the traversing mechanism of figure 3.10 and illuminated with the system shown in figure 3.11 with the sediment particles carefully aligned in the \((x, y)\) plane in the centre of the pseudo laser sheet beam. An image is recorded with the camera's \(f^\#\) and magnification set to 5.6 and 0.1237 respectively. The gel is rotated 180° about its \(z\)-axis and another image is recorded. The gel is then rotated 180° about its \(y\)-axis and again 180° about its \(z\)-axis with an image recorded at each orientation. Thus four images in total are taken for the first gel. This procedure is repeated for the other three gels resulting in a total of 16 images. Figure 3.14 shows the histogram resulting from the experiment.
3.8 Discussion

Three experiments are performed concerned with the factors affecting the formation of particle images in the context of PIV applications. The first two experiments deal with the effects of the random location of a particle in the measurement volume. The first of these is concerned with the random location of a particle in the \((x, y)\) plane (with \(z\) fixed) of the measurement volume, causing a corresponding random location of the position of the recorded image on the CCD array. The second experiment deals with the random location in the \(z\) plane (with fixed \(x\) and \(y\)) of a particle, which affects the quantity of light it scatters, due to the intensity variation caused by the movement and profile of the scanning laser illumination system.

In both cases, a glass bead encased in an optically clear gel is randomly moved in the required direction(s) in the measurement volume. Gouesbet [17] states that the agreement between the Point Spread Function approach and generalized Lorenz-Mie theory, incorporating a finite degree of coherence in the illuminating Gaussian beam is very good for a spherical particle. This justifies the assumption of a Gaussian PSF. Thus, a smooth spherical glass bead is chosen as the measurement particle rather than say, pollen or sediment (quartz sand), as these have irregular geometries. Sediment particles are used in the third experiment, which is concerned with the effects of particle characteristics on the image formed, since such a particle forms one of the phases to be measured in the experiments described in Chapter 4.

A question arises as to which quantity should be used to measure the recorded image intensity. For example, the average intensity of the recorded image over a specific set of pixels can be used, as can a specified number of pixels deemed to comprise the recorded image (say those above a specific threshold). However,
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the central image sample, as defined in Chapter 2, is chosen for the following reasons. Of the individual pixels comprising a single particle image, the central sample contains the greatest intensity (grey-level) value and therefore should yield a better signal to noise (S/N) ratio than other pixels taken from the same particle image.

The noise associated with the CCD array, camera and frame-grabber electronics is calculated in the following manner. The camera is turned on and allowed to warm up for 30 minutes. Then 10 images are recorded without any light incident on the CCD array. After a period of 30 minutes a further 10 images are recorded. For the first 10 images captured, a 100 x 100 pixel region in each is selected (approximately corresponding to the area in which particle images appear) and the average light intensity and standard deviations are calculated. This yielded a noise level of 0.54 ± 0.32 grey levels for the first 10 images and a noise level of 0.69 ± 0.36 for the set of images taken after 30 minutes. The average noise level increased by 0.15 grey levels indicating a systematic error while the fluctuation about the mean increased by 0.04 grey levels. The camera is run for 30 minutes as this is the approximate maximum time over which a sequence of images are successively captured. This is to minimise the drift in laser power, specified as ±2.5% with respect to mean power (over 8 hours). The laser is allowed run for 30 minutes before a sequence of images are recorded as the maximum warm-up time is specified as 10 minutes to 95% power.

The electronic shutter on the camera is accurate to within ±0.5μs. The average shutter exposure time for the experiments is of the order 50ms and therefore the ratio of shutter error to average exposure time is of the order $10^{-5}$.

In the first experiment there are errors associated with the alignment of the plane of movement of the particle which should be parallel to the image plane
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(CCD sensor). Careful alignment of the mechanical assembly minimised this and suggested a movement in the z-axis, after the particle had been translated over its 200 positions, of the order 100µm. The distance of the object to the camera lens is of the order 10^{-1}m to 10^{0}m giving an error in the magnification of 10^{-4} to 10^{-5}.

Adrian [2] gives the average mean exposure for a particle image as:

\[
\bar{\varepsilon} = \frac{\lambda^2 W \int |\sigma_m|^2 d\Omega}{\pi^3 (M^2 d_p^2 + 2.44^2 (1 + M^2) f^2 \lambda^2) \Delta y_0 \Delta z_0}
\]  

(3.26)

where \(\sigma_m\) is the Mie scattering coefficient, \(W\) is the energy of the light pulse and the subscript in \(y_0\) and \(z_0\) denotes object plane coordinates. If the magnification is the only parameter varied then the error associated with this quantity is dependent on \(M^2\). This reduces the error associated with the magnification by a further order of magnitude.

Figures 3.6, 3.7, 3.8 and 3.9 show the histograms of the centre sample data generated from the pixel integration experiment. The experimental histograms are shown with their corresponding theoretical densities generated from the numerical experiment of randomly sampling the function described by equation 2.69, with quantisation effects taken into account using equation 2.72. Estimates of \(C\) (proportional to \(\frac{A^2}{4a}\)) and \(\sigma\), the non-dimensional parameter of image width to grid (CCD pixel) size are obtained as described in Section 3.5.2. Table 3.2 shows the values of \(\sigma\), \(C\), \(\bar{s}_e\) and \(\bar{s}_e\) for the three magnifications 0.0520, 0.0611, 0.0694.

The final magnification of 0.1237 does not produce a sensible value for the standard deviation of light intensity as quantisation effects are clearly dominating the result. Since this indicated that pixel integration effects are negligible (compared to quantisation), this magnification is used for the experiment into the random positioning of a particle in the laser beam.

The theoretical graph produced for a magnification of 0.0520 appears to over-
estimate the spread of intensities produced by the experimental data (Figures 3.6, 3.7, 3.8 and 3.9). This is due to an underestimation of the width of the assumed Gaussian image recorded. As the magnification increases, with a corresponding increase in the width of the recorded image, the problem of underestimation lessens. This may be due to the fact that the geometrical image prediction of the particle is not a perfect Gaussian. For example, if the image prediction due to geometrical-optics is a $\text{circ}$ function, then the convolution of this function with the point response of the optical system (assumed Gaussian) results in the intensity function:

$$I(r) = \left[ 1 - \text{circ} \left( \frac{r}{a} \right) \right] \ast \frac{8}{\pi w^2} \exp \left( \frac{-8r^2}{w^2} \right).$$  \hspace{1cm} (3.27)

Here $w$ is the width of the Gaussian and $r$ is the radial co-ordinate and the $\text{circ}$ function is defined as:

$$\text{circ} \left( \frac{r}{a} \right) = \begin{cases} 1 & \text{for } r \leq a \\ 0 & \text{otherwise.} \end{cases}$$

This function $I(r)$ is broader than its equivalent Gaussian close to the centre but with a steeper fall-off further from the centre. At low magnitudes this would be noticeable as the method of analysis (i.e. the method of estimating the values of $C'$ and $\sigma$ for the theoretical model) uses the four adjacent pixels to the central pixel. As the camera magnitude setting increases, so does the width of the recorded image and the estimation of $\sigma$ improves, i.e. the samples close to the central pixel can be better approximated by a Gaussian for a larger image.

With reference to the second experiment (Section 3.6) figure 3.13, shows the experimental histogram for the intensity data of the centre sample generated by moving a particle through the light sheet as well as the theoretical distribution described by equation 3.25. The standard deviations for the experimental and theoretical data are 52.34 and 51.95 grey levels respectively. The theoretical curve
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is found to underestimate the experimental minimum intensity. This is due to (a) the fact that at the 1/e point of the Gaussian, the gradient is relatively large causing a correspondingly relatively large fluctuation in the intensity recorded for a relatively small $z$ movement and (b) the noise associated with the CCD array, camera and frame grabber electronics. Also the experimental data do not show the same sharp maximum frequency as the theoretical graph. This is due to (a) the systematic error of the fluctuation in laser power (it appears to increase over the recording period) and (b) noise effects. However the general agreement appears good.

Finally figure 3.14 shows the histogram for the centre sample intensities for sediment particles confined to a plane ($z$ fixed) of the illumination sheet with a standard deviation of 72.35. The above experiments indicate that for typical PIV experimental arrangements, the effects of particle position in the $z$-axis of the illumination plane and the particle characteristics are an order of magnitude higher than the pixel integration effects.

From the physical nature of the problem it is sensible to assume that the light scattering characteristics of a sediment particle are independent of its position (but not orientation) in the beam. Therefore the joint density of the centre sample intensity $f(z, c)$ is given by:

$$f(z, c) = f(z)f(c), \quad (3.28)$$

where $f(z)$ is the density given by equation 3.25 and $f(c)$ is a theoretical density of the intensity due to the particles' light scattering characteristics. No theoretical density for $f(z)$ has been derived in the current work for the reasons stated in Section 3.3. Also the experimentally produced data does not give a good fit to any simple distribution, such as a uniform or normal distribution and therefore it is not convenient mathematically to work out an analytical expression for $f(z, c)$.
Chapter 4

Sediment transport near beach walls: measurement aspects, theory and experiment

The main concern of this chapter is with the design of a PIV illumination system and its use in experiments to measure two-phase flow (sediment and water) over a rippled sand bed. The experiments have been performed in conjunction with The School of Engineering at Aberdeen University. As a background to the experiments, some basic expressions taken from linear wave theory are given, as well as a descriptive outline of sediment movement under standing waves. Following this, the experimental facilities and the limitations that they imposed on the illumination system are described. The development of the illumination system to overcome these is then detailed. Lastly, the experimental technique is described along with a discussion and summary of the system.

4.1 Theory

In this section, basic linear wave theory is outlined including its application to standing waves. The forces acting on sand bed (sediment) particles under a standing wave and two basic types of sediment movement are also described. For a detailed treatment of mass transport in water waves see [29]. In addition, Carter
et al. [7] provide a similar analysis in the context of offshore sand bedforms. This is briefly described in section 4.1.2.2.

4.1.1 Linear wave theory

Figure 4.1 shows a cosinusoidal wave of amplitude $A$ and wavenumber $k$ in water of depth $h$. The free surface elevation above the still water level, $\eta$, for such a wave is given by:

$$\eta = A \cos(\omega t)$$

(4.1)

where $\omega$ is the radian wave frequency. It can be shown [27] using potential theory that this leads to the following expressions for the horizontal $u$ and vertical $v$ velocities under the wave:

$$u = \frac{Agk \cosh k(y + h)}{\omega \cosh kh} \cos(kx - \omega t), \quad (4.2)$$

$$v = \frac{Agk \sinh k(y + h)}{\omega \cosh kh} \sin(kx - \omega t), \quad (4.3)$$
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where $y$ is measured from the still water level. It is also possible to derive the dispersion formula:

$$\omega^2 = gk \tanh kh, \quad (4.4)$$

which relates the frequency $\omega$ and wavenumber $k$.

Now consider the motion caused by the reflection of waves by a plane vertical barrier located at $x = b$. The assumption of Rahman [44], that the outgoing reflective wave has the same amplitude as the incident wave, is used. In addition, the frequency and wave number of the reflected wave are taken to be identical with those of the incident wave. Rahman [44] shows that the surface elevation of the wave generated in these circumstances, $\eta_T$, is given by:

$$\eta_T = 2A \sin (kb - \omega t) \cos (kx - kb), \quad (4.5)$$

The velocity components of the standing wave are given by

$$u = \frac{2Agk \cosh k(y + h)}{\omega} \frac{\cosh kh}{k(y + h)} \cos (kb - \omega t) \sin (kx - kb), \quad (4.6)$$

$$v = \frac{-2Agk \sinh k(y + h)}{\omega} \frac{\cosh kh}{k(y + h)} \cos (kb - \omega t) \cos (kx - kb) \quad (4.7)$$

Equation 4.5 is the product of two terms, one independent of $x$ and the other independent of $t$. Thus, there are certain times when $\eta_T = 0$ for all $x$ and there are certain $x$ for which $\eta_T = 0$ for all time. These latter points are called the nodes of the system and are located by the condition:

$$\cos (kx - kb) = 0 \quad (4.8)$$

which on solving yields:

$$x = b + \frac{[(2n + 1)\pi]}{2k} \quad \text{for} \quad n = 0, 1, 2, 3 \ldots \quad (4.9)$$
Such a condition of stationary nodes defines a standing wave. It is convenient to take the origin of \( x \) at the barrier by setting \( b = 0 \). This simplifies equation 4.5 to:

\[
\eta_T = -2A \sin(\omega t) \cos(kx)
\]  

and on setting \( x = 0 \) we see that the reflection process preserves the phase of the incident waves.

### 4.1.2 Sediment Movement

We now look at the factors affecting the movement of sediment under standing waves.

#### 4.1.2.1 Forces acting on bed particles

We begin by considering a particle which is initially at rest on the bed. Raudkivi [46] categorises the forces acting on such a grain as gravity, drag, lift and volume. These are denoted by \( F_G \), \( F_D \), \( F_L \) and \( F_V \) respectively and are given by:

\[
\begin{align*}
F_G & = \frac{\pi}{6} d^3 \rho (s - 1), \\
F_D & = C_D \frac{\pi}{2} d \sqrt{u^2}, \\
F_L & = C_L \frac{\pi}{2} d u^2, \\
F_V & = \rho \frac{\pi}{6} d^3 \left( \frac{du_0}{dt} + C_M \frac{du}{dt} \right),
\end{align*}
\]

where \( d \) is the mean grain diameter, \( \rho \) is the density of water, \( s \) is the relative density of the sediment, \( u \) is the velocity of the water at the level of the particle, \( u_0 \) is the water velocity outside the boundary layer and \( C_D \), \( C_L \) and \( C_M \) are constants. The total disturbing horizontal force \( F \) is given by

\[
F = F_D + F_V,
\]
and Nielsen [37] considers the effect of the volume force term by deriving the ratio

$$\frac{F_{V, \text{max}}}{F_{D, \text{max}}} = \frac{\pi r}{6} d \sqrt{\frac{\omega}{\nu}}, \quad (4.16)$$

where $\nu$ is the viscosity of water. In Dingler's experiments [12] into sediment motion, the ratio of equation 4.16 is measured as approximately $2 \times 10^{-2}$ and Nielsen [37] concludes that "the influence of volume forces is unimportant for the onset of sediment motion under waves when the flow is laminar".

### 4.1.2.2 Mass transport at the boundary layer

Near the bottom of the bed, a boundary layer of thickness $\delta$ where:

$$\delta = \left( \frac{2\nu}{\omega} \right)^{\frac{1}{2}} \quad (4.17)$$

is formed. In this boundary layer, the velocities oscillate with the same frequency as described by linear wave theory, but with their magnitudes reduced by a factor which is dependent on the height above the bed and the boundary layer thickness [7]. The oscillating motion in the horizontal plane is greatest under nodes and smallest under anti-nodes. By considering the effects of Reynolds stresses and the second order pressure gradient imposed by the external flow field, a second order mass transport velocity can be derived [7] within the boundary layer. For a standing wave, this changes sign within the boundary layer and is directed from nodes to anti-nodes in the upper part of the boundary layer and from anti-nodes to nodes in the lower part. Thus two types of sediment transport can occur: (a) sand in suspension is transported from node to antinodes and (b) bed sand is transported from between a node and an anti-node toward the node. These types of transport are often referred to [23] as L-type and N-type respectively as shown in figure 4.2. A movability parameter:

$$\frac{u_{\text{max}} - u_{\text{crit}}}{w},$$

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L-Type: From node towards antinode
Relatively high movability parameter

N-Type: From between node and antinode towards node
Relatively low movability parameter

Figure 4.2: Standing wave

can be used to determine which of the two types of transport is likely to occur for a given set of conditions [53]. Here $u_{max}$ is the maximum bottom orbital velocity at the node, $u_{crit}$ is the critical velocity for sand particles and $w$ is the settling velocity (sediment fall velocity). The critical velocity is defined as the bottom orbital velocity at $l_r$, where $l_r$ is the measured distance from the anti-node to the start point at which bed scouring becomes appreciable [53].

For relatively low values of the movability parameter, the transport of beach material is likely to be N-type, while relatively high values will result in L-type transport. In the experiment described below, the initial movement of sediment is N-type. However, for a fully-mobile bed under a standing wave, ripples form very quickly and subsequently play an important role in the sand transport process. For the following experiments, a standing wave is used since the experiments simulate erosion in front of a sea wall.

4.2 Experimental arrangement

The experimental arrangement is now detailed, with particular emphasis placed on the design of an illumination system.
4.2.1 The wave flume

The two-phase sediment transport experiments were conducted in a wave flume at the Engineering Department of Aberdeen University, shown in figures 4.3 and 4.4. The flume dimensions are 20m in length, 0.45m in width with a working water depth at the wavemaker of 0.7m. The wavemaker itself is hinged at the base of the tank so that it moves in an arc. The flume has a model beach and seawall (barrier) at its end furthest from the wavemaker. The beach comprises a 1:20 sloping perspex panel, supported on an aluminium frame over most of the length of the flume, meeting a horizontal tray approximately 3m long. The tray, supported by a frame fixed to the tank, contains a sand bed 200mm deep, the top 100mm consisting of a well sorted sand with $D_{\text{mean}} = 0.32\text{mm}$. Waves generated by the paddle are reflected at the shore end by a vertical impermeable wall. This wall is hinged at its base allowing varying degrees of reflection, however for these experiments the wall was locked in a position normal to the direction of wave propagation. The mean water level depth in front of the wall is 150mm.
4.2.2 Development of illumination system

In PIV experiments involving wave flumes the most common method of illumination involves directing the illumination light (from a laser) through the bottom of the wave tank. However, this method of introducing the illumination proved unsuitable for the present experiments; the model beach and beach sand impede a light path directed in such a manner. Rather than attempt to re-engineer the model beach, the following novel approach is adopted with the light sheet directed downwards into the water from the top of the tank, by mounting a beam delivery system as shown in figure 4.4.

Introducing the laser light into the water from above brings with it the problems of reflections from the surface and refractions of the beam caused by the generated waves. These affect the PIV images produced in an undesirable way. To overcome this, a glass sheet is attached underneath the beam delivery system to act as a light guide as shown in figure 4.5. This carries the light through the wave surface, eliminating reflections and refraction effects. The following sec-
sections describe the constituent parts of the delivery system; detailed drawings of the mechanical assembly are included at Appendix B.

4.2.3 Possible light sources

A reference to Adrian [4] indicates the large variety of ways that illumination may be provided in a PIV experiment. The nature of the current investigation requires a powerful light source, e.g. a laser. PIV illumination systems which utilise lasers as their light source fall into three main categories; namely pulsed, expanded beam and scanning beam systems.

4.2.3.1 Pulsed lasers

Pulsed lasers, as the name might suggest, supply short but high powered bursts of laser light. A single Neodymium:Yttrium Aluminum Garnet (Nd:YAG) laser can emit single pulses at a rate of 10Hz, or by using Q-switching, emit a pair of pulses, each half the power of the single pulse, with a time interval between them of 40 - 100\(\mu s\). These time limits normally restrict this method of illumination to applications associated with high flow rates. This can be overcome using a pair of pulsed lasers, with the optics necessary to combine the beams co-axially and associated electronics to have one laser trigger the other after a variable
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time delay. However this latter approach often requires much effort and care in aligning the two beams together.

4.2.3.2 Expanded beam

A more flexible method of illumination can be achieved by expanding the light output from a laser. Typically a mechanical or electro-optical beam modulator chops the beam produced by a continuous wave laser into pulses. These pulses are fed into a cylindrical lens which expands the beam to form a pulsing sheet of laser light.

An important limitation of this method is that the modulation of the light sheet results in only a fraction of the available light being utilised to illuminate the flow. A further disadvantage, due to the Gaussian profile of laser beams, is the significant variation of light intensity across the sheet on expansion. This leads to problems of over-exposure in the middle of the flow field and loss of images due to under-exposure at the edges. Furthermore, since the exposure time of the recording plane is a function of the pulse duration, shorter pulse durations are required for faster flows leading to a requirement for increased laser power, a very important consideration to take into account when designing an illumination system.

4.2.3.3 Scanning beam

For the purposes of the present investigation, the illumination of the flow is provided by the scanning beam method, this being especially suited to PIV measurements of flow fields of this size and range of velocities [20]. The implementation of the scanning beam system in the current project is now briefly explained, followed by a more detailed explanation of the component parts in subsequent sections.

The light from a continuous wave 15W Argon Ion laser is directed into a
multimode fibre-optic cable, 30m in length, by an adjustable mounting attached to the beam exit of the laser. The output from the fibre is collimated using a telescope arrangement and the light directed onto an eight sided rotating mirror. This mirror, whose rotation speed can be adjusted very accurately in the range 12 - 250 rps, causes the beam to be swept repeatedly through an arc. This produces a pseudo light sheet in the flow field as shown in figure 4.6.

4.2.4 The laser and fibre-optic cable

The light is produced by an Argon Ion laser which has nine wavelengths. The multiple wavelengths are formed by transitions from several excited states of the an Argon atom [59]. The most predominant one has a wavelength of 488nm. Although it is possible to select a particular wavelength output, with an etalon, this is not desirable for this application because of the corresponding power reduction.

The multimode fibre-optic cable comprises a flexible quartz-fibre light guide, an input coupling lens to allow good coupling of the light guide to the laser and an output coupling lens to produce the beam parameters required. Laser power up to 12W can be transmitted in the spectral range 440-680nm, with a
transmission factor of 50% to 85% dependent on laser type and laser light guide length. Calibration measurements suggested this figure was approximately 73% for the current experimental set up. The cable sheathing is a low-smoke, zero-halogen plastic with Kevlar strain relief for high mechanical loads.

4.2.5 Optics

Figure 4.7 shows the lens mounting assembly including two holders (no. 06 5061)\(^1\) placed at either end, which together with four 300mm stainless steel rods (no. 06 1211) mechanically hold the system together. The fibre-optic cable is connected to the assembly by a mount attached to one of the holders. Two mounting plates and a beam steering mirror can be moved along the rods in between the two holders, allowing adjustment of the distance between the lens for beam collimation. The plate nearest the fibre-optic connection is a centre mounting assembly.

\(^1\)All lens mounting assembly components are manufactured by Spindler and Hoyer and component numbers are taken from the Spindler and Hoyer 1996 supply catalogue
Figure 4.8: Lens arrangements

plate (no. 06 5010) with an adjustable centering fixture allowing two degrees of freedom in the plane normal to the direction of the beam. This plate holds a plano-convex mounted lens (no. 06 3046) with a focal length of 80mm, clear lens diameter of 21.4mm and a mount outer diameter of 25mm while the mounting plate nearest the beam steering mirror holds a plano-concave lens (no. 06 3075) with a focal length of -16mm, clear lens diameter of 9mm and a mount outer diameter of 25mm. The beam steering mirror assembly (no. 06 3711) consists of an elliptical (22.4mm x 31.5mm) front surface plane mirror inclined at 45° in its mounting plate. Three thumbscrews allow alignment adjustment (±4°) and setting of the axial position.
It is possible and probably more common, to make a telescope arrangement with two convex lenses as shown in figure 4.8(a). However, this has two drawbacks, namely (a) the quite large physical separation of the lenses (equal to \( f_1 + f_2 \)) and (b) the fact that the beam is brought to a focus between the lenses. The beam at this point is at its most intense and consequently at its most dangerous. A further consequence of this second point, although not applying to this laser, is that the power may be so large at this point as to ionise the air, as has been observed with pulsed Nd:Yag lasers with a power of 200mJ/pulse. The lens arrangement thus opted for is shown in figure 4.8(b) and consists of a plano-convex lens followed by a plano-concave lens.

The reason for using plano-concave/convex lenses in laser optics is that they closely approach the best form for infinity and near-infinity conjugate ratios, with the preference focal plane being adjacent to the plano surface. In other words, to minimise spherical aberrations, one has to maintain equal angles of light incidence, which are as small as possible, on all lens surfaces. Plano-concave/convex lenses are the best suited for plane focused beams. Achromats, which are composites of two lenses may be better suited for this purpose, as the total ray bending can be shared over three or four surfaces instead of the normal two. However, the constituent lenses are cemented together and it is possible that the cement may be affected adversely by the high intensities of light. Therefore achromats are not chosen for this application.

4.2.6 Recording of Images

Images are recorded onto a cross-correlation (two image) camera which uses two CCD arrays of \((756 \times 468)\) pixels. The camera captures two successive images separated by an adjustable timing factor. The output of the camera is attached to a PCI-bus frame grabber and camera control card. This feeds into a Windows-
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**Figure 4.9:** Ripple formation
based image acquisition software package which stores the recorded images.

4.2.7 Procedure

In order to study two-phase flow over rippled-sand bedforms, the detailed flow behaviour for a single wave condition \((f = 0.9\text{Hz}, A = 25\text{mm})\) is examined. Four regions of interest, figure 4.9, denoted by A, B, C and D indicate the areas of the flow the camera is focused on. The anti-nodes are located to each side of the regions of scour. The distance from node to anti-node is approximately 1.5 times the distance from the node to the region of maximum scour. Initially the camera is focused on area A and starting with a smooth sand bed, the standing wave is generated \((t = 0)\) and at \(t = 4\) minutes a series of 21 image pairs are recorded. A further 21 image pairs are recorded at \(t = 10, 31\) and 60 minutes, giving a total of 84 image pairs for position A. Generation of the standing wave is stopped after the last \((t = 60\text{ minutes})\) images are recorded. During the 60 minutes of wave generation a distinct rippled sand bed forms and this sand bed is smoothed after the run, in preparation for the next run with the camera repositioned to view area B. The process of generating the wave and recording the images is repeated. Similarly at the end of this run, the above process is repeated with the camera positioned to view areas C and D in turn.

4.3 Discussion

In section 4.1, the basic theory relating to linear waves and sediment transport has been outlined. This gives some background to the sediment transport experiment of section 4.2 and is referenced when qualitatively discussing the results in chapter 6. The design and implementation of an illumination system used for measurements of sediment transport has also been described. Although the
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system has been designed specifically for this experiment, it has qualities which make it attractive for more general PIV use.

The use of lasers is often made problematic, restricted or even impossible by difficult working environment conditions. In many conventional set-ups, the laser beam is directed to the measurement area via sets of beam steering mirrors. This requires much effort from the experimenter to align the beam path correctly. For a fibre optic delivery system, beam alignment is concerned with achieving optimum transmission of light and the input coupling of the cable must be aligned in two axes to the laser. However a mechanical alignment unit (available from the cable manufacturer, Schott Faseroptik) allows very reliable and relatively efficient coupling.

For conventional illumination systems, if the physical location of the measurement area changes, the process of alignment must be repeated, with possibly a redesign of the optical path. This is not the case for the fibre-optic system which can be easily fed to any measurement area within reach of the cable.

Laser safety is important at all times, but more so when using a laser of such power (15W) as in this case. Normally this involves the construction of a casing enclosing the beam path which again, if the measurement area changes, may require a new construction. Since the fibre-optic cable is sheathed in the manufacturing process, this problem does not arise for the present system. In general, the current system is installed and maintained more quickly and easily. As an example, the system has been installed over a wave flume in The School of Engineering at Aberdeen University and in almost two years of operation the optics and light coupling have never required major readjustment. The system has also been used in several other tanks in the laboratory.

The design also has its limitations. The illumination system incorporating the
vertical sheet of glass piercing the water surface interferes, to a greater or lesser extent, with a flow being measured, depending on the thickness of the glass. The relative seriousness of the problem will depend on the magnitude of the velocities associated with the vortices generated by the glass compared with the velocities arising from waves and wave interaction with the bed in the absence of the glass.

For the two-phase experiments, a 6mm thick sheet of glass was used initially. However vortices were shed from the underside of the glass and migrated towards the bed. This effect was more pronounced with the glass located at the antinode because of the high velocities in the vertical direction past the bottom edge of the glass. Reducing the thickness of the glass to 4mm considerably lessened the adverse effects of the vortices produced in terms of size and migration towards the bed and reducing the glass thickness even further correspondingly lessened the vortex shedding.

The thickness of the fanned laser sheet, alignment considerations and mechanical strength place a limit on the minimum thickness of the glass. The thickness of the pseudo light sheet, 1-2mm, imposes a minimum thickness on the glass of 2mm, since noticeable diffraction effects occur if some of the light sheet crosses the edge of the glass. This minimum thickness is increased when alignment considerations are taken into account. Also a glass sheet of this thickness is less resistant to wave forces imposed on it than a thicker sheet. Taking all these factors into account, a glass thickness of 4mm is chosen as the best compromise.

Finally we mention that although the glass sheet interferes slightly with the flow, it has the advantage of straightening the fan of light produced from the rotating mirror due to the difference in the refractive index of the air and glass interface.
A flexible and robust PIV illumination system has been designed and successfully implemented in a two-phase, sediment transport experiment. The system provides a novel method of obtaining measurements to aid the understanding of sand-bed/water interaction. However, the processing of the data obtained from the above experiment is complicated by the fact that there are two phases in each image. The considerations thus needed for the processing are detailed in the next chapter.
Chapter 5

Analysing two-phase PIV image files

The main concern of this chapter is with the processing of raw PIV image files, i.e. graphics image files such as TIFF, PGM etc., into velocity vector maps. The novelty of the analysis is that it is applied to images containing two-phase data. Image processing techniques are applied to the raw data to determine their suitability at locating and segmenting individual particle images. Experiments are performed to determine the capability of specific particle image characteristics at separating phases. In this way confidence levels are obtained in the context of phase separation. Correlation methods are applied to the separated data to produce velocity information.

5.1 Review

In this section, two-phase flow analyses and image processing techniques, used for particle location in PIV applications, are reviewed. Literature on particle location often involves the study of flows in which Particle Tracking Velocimetry (PTV) is used to extract velocity data. In the current application this method of velocimetry proved unsuitable. Although the present study is concerned with two dimensional (2-d) recording, three dimensional (3-d) techniques have been
included in the review as many location techniques used for 2-d apply equally well to 3-d.

5.1.1 Two dimensional techniques

Observations of the motion of particles in fluids has been used for decades to qualitatively examine flows. Possibly the earliest recorded attempt to locate and track particles quantitatively was by Naylor and Frazier [36]. In their experiment, particle images were recorded on successive frames of a cinematic film. To interrogate the film, pins were used to mark each particle location on a piece of paper under the film using the images from several frames to form streaks. This was done for 80 frames. The results produced were similar in detail and quality to many current PIV results and emphasises the fact that much of the progress in the analysis technique is in the automation of the process rather than the fundamental method itself.

To this end, one of the more recent attempts to obtain quantitative data is by Dimotakis et al. [11] in an experiment developed from conventional particle streak photography. A 3W Argon-ion laser produced a thin sheet of laser light to illuminate the flow. The images were recorded on standard 35mm photographic film with an exposure time of 1/8 s. A single particle’s image appeared as a streak with one exposure per frame. Each negative was subsequently digitised, using 2500 scans consisting of 1024 elements. The resulting images were thresholded and displayed and the streaks located by manual inspection. The length of the streaks were taken as the distance between end points with the resulting velocity assigned to the centre point of the streak.

In [22] Imachi and Ohmi attempted a similar experiment, again utilising streaked particle images. The illumination was provided by several projector bulbs shining through a 5mm slit and the seeding material was aluminium pow-
Chapter 5 — Analysing two-phase PIV image files

der of diameter 2 to 7 μm. Similarly to Dimotakis et al. [11], the images were recorded on standard 35mm photographic film, however only the ends of the streaks were recorded during the digitisation process. The photographic images in this case were digitised to a resolution of 1500 by 1100 elements. The resulting grey-level image was thresholded to produce a binary image and this was subjected to a streak finding algorithm based on an 8 point adjacency method. The length and mid-points of feasible streaks were calculated to produce the required vectors.

Marko and Rimai [33] investigated unsteady flows within a purpose made glass cylinder engine. Light from an Argon-ion laser was coded by an acousto-optic modulator, with a single particle image appearing as a series of dots and dashes. The coding eliminated directional ambiguity. Images were recorded on a Vidicon camera, digitised and displayed. Tracks were recorded by manually identifying individual particles belonging to a streak and recording these regions by means of a movable cursor. Velocity information was obtained easily since the pulse duration and separation times were known.

Ciccone [9] used the beam from a 35mW He-Ne laser, chopped at regular intervals by a slotted disk rotating at 30Hz, to produce a pulsing light source. A 35mm camera with a shutter speed of 1/2s was used to record the images. A single particle produced approximately 10 streaks in a single image with each image containing approximately 60 particles. Each image was digitized onto a 512 x 512 grid with a 0-255 grey-scale and enhanced using a Butterworth bandpass filter. The filtered images were then cross-correlated with a circular template of the same magnitude as the streaks, the streaks giving a high response to the template, making them easier to identify. Segmentation of the cross-correlated image was performed using a column by column search of pixels satisfying certain heuristic
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conditions based on the linearity and size of a streak. Segmented particle streaks were linked to one another according to heuristic rules based on the distance and angle between streaks.

Shekarriz et al. [52] used 35mm film to record fluorescent particles illuminated by an Argon-ion laser in a study of tip-vortices. A Bragg cell produced a coded signal consisting of a streak and two pulses. The negatives produced were illuminated from behind by a diffuse white light and examined by a Vidicon camera. The camera was equipped with a microscopic objective lens which focused on a 3mm x 3mm area of the negative. Each of these illuminated small areas was recorded on successive frames of the camera. These frames were subsequently digitised onto a 512 x 512 grid with a 0-255 grey-scale. Segmentation was performed by thresholding and region growing. Regions thus formed containing less than ten pixels were considered noise, while regions containing ten pixels or more and with an aspect ratio greater than two were considered a streak. All other regions were considered pulses. Once a streak was located, its corresponding pulses were sought. The velocity of the particle was calculated by measuring the distance from the front of the short pulse to the front of the corresponding long pulse. If no pulse was found the streak was discarded.

Cui [10] presents a method for measuring two-phase flow using a dual-beam scanning laser system. The output beam from a multi-wavelength laser is separated using a colour splitter resulting in two beams with different colours, green (\(\lambda = 514.5\,nm\)) and blue (\(\lambda > 514.5\,nm\)). Reflected beams of each colour successively sweep through the flow field within the recording object plane where the velocity measurements are to be taken. The seeding particles in the flow field are exposed twice by the two different coloured beams within a short time interval. The two images with different colours are then collected by a lens, separated by a
dichroic mirror and are sent to two separate video cameras. In this way, separate images of the particles are recorded on separate frames at each exposure time. This technique permits the use of cross-correlation techniques and the problem of overlapping images that occurs in auto-correlation is avoided.

In [35], McCluskey et al. discuss the measurement of two-phase flows and present the results from two experiments. In the first, a particle-laden jet is injected into a turbulent air flow in a pipe. The particles in the jet were solid glass spheres with a mean diameter of 76\( \mu \text{m} \). The air seeding consisted of 1\( \mu \text{m} \) corn oil droplets. The air-particle flow was illuminated by a pulsing light sheet from two double-Q-switched Nd:YAG lasers. A cylindrical lens converted the train of 4 pulsing beams into a train of pulsing light sheets, which were then collimated and reflected into the flow field by a parabolic mirror. PIV images were recorded onto 35mm Kodak T-Max 100ASA film using a Nikon camera fitted with an iris shutter and a 50mm lens. The image:object magnification was 1:4.

In the second experiment, a seeded air flow above a 75mm square-sided fluidised bed was illuminated using a scanning beam technique. The bed consisted of solid glass spheres with diameters in the range 150\( \mu \text{m} \) to 325\( \mu \text{m} \), which were fluidised by air. The air phase was seeded by hollow glass spheres with a mean diameter of 8\( \mu \text{m} \). The flow field was photographed onto 35mm Kodak T-Max 100ASA film, at an image:object magnification of 2:5, using a Nikon camera fitted with an iris shutter and a 50mm lens. Particle images were selected based on a combination of criteria for intensity, size and shape. This enabled simultaneous measurements of the separate phases in both experiments.

In [38], Ohta et al. investigate an upward liquid flow containing a gas bubble in a rectangular channel. The test section was lighted through slits by two 650W
projector lamps and images recorded onto a high speed video camera. The liquid seeding consisted of Nylon-12 particles. The velocity of the liquid phase was obtained using binary cross-correlation between successive images. The bubble shape was obtained by thresholding the image.

Gui and Merzkirch [21] discuss an algorithm for analysing two-phase flow. For demonstration purposes the technique is applied to a flow induced in a water tank by large solid buoyant particles released to the water at the bottom of the tank. Separation relies on the fact that one phase is significantly larger than the other and velocimetry is achieved by particle tracking.

### 5.1.2 Three dimensional techniques

In [8] Chang et al. studied the motion of tracer particles, whose images were recorded in stereo on motion picture film, in a turbulent mixing flow. Initially, the recorded stereo images were digitised, to a resolution of 2048 x 2048, into their corresponding left and right (view) frames and then segmented by thresholding and by detecting groups of pixels in each row. Possible particle images were represented by combinations of these groups and were more conveniently described by the region's centroid, the number of pixels spanning the row and column directions and the average light intensity. The process was somewhat complicated by the fact that only two rows of the digitised image could be held in memory at a given time.

The frame to frame tracking of particles consisted of two steps. The first was to locate a feasible search area for a particular particle. The second involved finding the actual trajectory of that particle. Particles that appeared in the relevant search areas were linked to the particle that instigated the search if the respective angles and displacements between the particles did not differ by more than a specified amount. This information as well as the differences between
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particles' intensities and sizes were used to form a weighted cost function. The weights being determined empirically. In the case of more than one feasible particle being added to a trajectory, the particle yielding the highest score for the trajectory was added. Trajectories in each of the 2-d scenes were then matched to form a 3-d scene.

Chang et al. [8] also looked at the effects of threshold level on particle identification and tracking. The total number of: particles, the left and right frame trajectories and matching (3-d) trajectories were counted manually over four frames. These results were compared with those obtained by running the matching algorithm using four different levels of threshold. The errors involved in estimating particle centroids were examined by adding artificial pixels to the original data and noting the change of the position of the centroid.

Racca and Dewey [42] used a split view optical system to record two perpendicular views of a flow. An 800W quartz lamp was used as the source of illumination and the images were recorded on a 16mm high speed cine camera, capable of rates up to 5000 frames a second. Each frame was digitised to a resolution of 512 x 512 pixels with a grey-scale ranging from 0-255. The resulting image was thresholded separately and subjected to a Sobel operator (edge detector). Only those pixels that exceeded both the intensity and gradient thresholds were selected as particle tracer images. The segmented image was then represented in a binary format. Particle images which were selected were represented as region boundaries, however it was first necessary to perform an edge thinning algorithm so that only a thin line of pixels defined the outer boundary. This effect was achieved using a contour following procedure. Centroids of the contours were calculated and particles in each of the stereo views were matched yielding 3-d coordinates.
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The technique for following a tracer was based on the assumption that from one frame to the next, the velocity vector of that particle would not undergo major changes. Linear extrapolation of the displacement during one time step was used to approximate the path in the next step. Tracer locations in the next frame were scanned to seek unassigned tracers within a given range of the extrapolated position and if any were found, the trajectory was extended to the one nearest the centre of the range. Once a trajectory was assigned it could not be modified. Tracking was performed in a parallel fashion rather than following a single trajectory from start to end before considering another. As tracking progressed, trajectories with the largest number of tracers were examined first.

Adamczky and Rimai [1] attempted another technique based on an orthogonal viewing system using two video cameras placed perpendicularly to each other. Illumination was provided by modulating the light produced from an Argon-ion laser with tracer images appearing as a series of dots and dashes. Potential streaks were located by a thresholding procedure followed by a simple connectivity algorithm used to grow adjacent cells. Once a potential streak had been identified, its area, end points, centre of gravity and aspect ratio were calculated. If the potential streak's area was found to be outside specified limits, it was rejected. Also, a threshold was applied to the aspect ratios of potential streaks which eliminated all rounded, curved or intersecting streaks. Corresponding streaks in each of the stereo views were then matched and velocity information extracted.

In [45] Ramer and Shaffer apply tracking to multiple-pulse, particle image velocimetry data. Image segmentation was performed by thresholding and region growing, particles being represented by the corresponding regions formed. The particles were described in terms of the area and centroid of these regions, the centroid being computed as the average row and column co-ordinates delimiting
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the perimeters of the particles. Ramer and Schaffer stated that generally the uncertainties in the coordinates of the particle image centroids were the $\pm 1/2$ pixels.

Majumdar et al. [32] studied the motion of a continuous and disperse phase suspension subjected to a homogeneous simple shear. The disperse phase of the suspension consisted of 0.635cm diameter polymethylmethacrylate (PMMA) spherical particles. The processing of the images was complicated by colour degradation of the continuous phase and hence, the recorded image of the suspension with time. Furthermore, the degradation was not uniform spatially and so two different image analysis techniques were employed. If the effects of degradation were negligible, then image noise was removed by subtracting a calibration image (the flow without the disperse phase) from the image under inspection. Thresholding and edge detection were then employed to obtain a boundary representation of the particle.

As degradation effects became more pronounced over the sequence of images the noise appeared nonstationary and was not satisfactorily reduced by subtracting the calibration image. The edge detection algorithm, relying on first and second derivates of the image were affected adversely by the remaining noise and so particle centres were identified manually. After smoothing a region local to the particle, a suitable threshold was calculated based on an average of pixels local to the manually identified centroid.

In [25] Khaligi and Lee deal with multiple exposures on a single image. A particle's image consists of two short pulses (dots), followed by a long pulse (streak), then a short pulse at the end. This coding enables the flow direction to be determined and by fitting a curve to the streak, velocity information to be extracted. The image is formed by pulsing a first order diffracted beam produced
by an acousto-optic modulator (AOM) The pulsing is produced by modulating the AOM video frequency with a specified burst signal. The image processing task was to identify streaks and dots and the initial stage of the process was to subject the original image to four different gradient masks. The pixel by pixel maxima of these resulting images were found and stored in a new output image, i.e. this image contained the maximum positive gradient value of the four possible directions. Streaks in the image were located by identifying blobs of ridge points and dots were also identified as a subset of ridges.

Racca and Dewey [42] apply particle tracking to three dimensional flows. Simultaneous orthogonal views of a tracer seeded flow are recorded, by a single high speed cine camera, through a split field mirror system and subsequently converted to machine-readable form by a video digitiser. After suitable digital enhancement to separate the tracers from the contrasting background, the projections of the individual tracers in the two views are matched. From these, 3-d co-ordinates could be obtained and the tracers followed from frame to frame to compute the velocity vectors along the particle trajectories.

Maas et al. [31] details a three camera system for the determination of three-dimensional tracer particle coordinates and subsequent tracking. The recorded images were highpass filtered to remove non-uniformities of the background intensity level due to reflections and the non-uniform intensity profile of the light sheet. Particle images were segmented using thresholding and described by their centroids. Particular attention was given to identifying overlapping particles. A modified anisotropic thresholding operator searches for discontinuities in the grey values inside segmented blobs and if a discontinuity exceeding a certain empirically determined limit was detected, the blob was split.

Carosone and Cendese [6] apply artificial intelligence concepts to find a suit-
able intensity threshold in a PIV image for the purpose of segmenting particle image data from background noise. The histogram of the grey-levels in the image is assumed to approximate a Bigaussian probability density function and a Kohonen Self-Organising Map [26] is used to find an optimal threshold.

The map consists of one sensor in the input layer and two neurons in the Kohonen layer. The input sensor accepts the intensity values of the pixels in sequence from the raw image and as each pixel is sampled one of the two neurons is fired depending on whether the sampled value is closer to a to weight 1 (connecting the sensor to neuron 1) or weight 2 (connecting the sensor to neuron 2). The weights correspond to the background and foreground (particle data) intensity modes of the assumed Bigaussian density and are established in an automated fashion by sampling the image. Carosone and Cendese [6] report that only a few tens of pixels are needed to establish a stable behaviour of the network. Particle images were described by two features. The first was a circularity measure \( \text{crf} \) with

\[
\text{crf} = \frac{4\pi \text{ Area}}{p^2}
\]

and the second was a convexity measure \( \text{cnv} \) with

\[
\text{cnv} = \frac{\text{smallest enclosing convex } p}{p}
\]

where \( p \) was the spot diameter. In this way overlapping particles could be distinguished from non-overlapping particles. Carosone and Cendese [6] did not use neural networks in the tracking and linking of particles, however this has been investigated in [19] and [48].
5.2  Image processing applied to particle location

In this section image processing techniques are described and applied to the data produced from the experiments of Chapter 4 to evaluate their suitability at the task of particle identification.

5.2.1 Introduction

Image processing can be divided into two main categories, namely preprocessing and image analysis. Preprocessing operations include the manual (or automatic) defining of regions as well as image enhancement. The removal of noise or other unwanted components by filtering may also be placed in this category.

The second and generally more challenging category, image analysis, deals with the extraction of information from an image and this can be subdivided further into the three sub-categories of segmentation, representation and description, and recognition. In segmentation an image is subdivided into component parts. For example, the simple thresholding of an image divides it into two distinct sections. The level to which the subdivision is carried out and the subdivision methods used depend on the nature of the problem. The resulting aggregate of segmented pixels can now be represented and described in a form more suitable for further processing. Generally a region is represented by either its external characteristics (the boundary) or by its internal characteristics (the pixels comprising the region). The region is then described in terms of this representation. For example, a region represented as a boundary may be described by the length of the boundary, by specific widths spanning the boundary, or by the number of concavities of the boundary. Generally an external representation is chosen when the primary focus is on shape characteristics and an internal representation, when
the main focus is on properties such as colour and texture.

The final, and probably most difficult task in image analysis, is an attempt to endow the process with some understanding of the regions it has described. For example, consider an image of a car. An automated image processing system should be capable of recognising the component parts, such as tyres, doors and roof and furthermore be able to recognise that these items when joined together form a car. For the present purposes, recognition is the ability to distinguish between sediment and pollen particles in the flow and experiments are performed in Section 5.5 to address this problem.

5.2.2 The data

Many of the images used to illustrate certain image processing features are in fact not typically representative of the 336 image pairs taken in the experiments of Section 4.2. The images chosen generally contain a non-uniform background intensity caused by clouds of sediment particles being lifted off the sand bed. Of the data recorded only 5 - 10 % of the image pairs contain clouds of sediment large enough to cause this effect. However, these sections of images have been selected for illustration, since they represent some of the more challenging images in image processing terms. Furthermore, since the measurement of sediment velocities is one of the major objectives of this work, these clouds represent the regions where sediment velocities are obtained.

5.2.3 Point operations in the spatial domain

Spatial domain methods are procedures which operate directly on the pixels comprising an image. These pixels are referred to as the spatial domain of the image. An operation which takes an image \( I(x, y) \) into an output (transformed) image
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$I_T(x, y)$ may be expressed mathematically as:

$$I_T(x, y) = f[I(x, y)] \tag{5.1}$$

with $f$ the operator on $I$ defined over some neighbourhood, usually a square or rectangular region, of $(x, y)$. If the neighbourhood comprises a single pixel then $f$ becomes a grey-level transformation function often referred to as a point operation.

Point operations change the grey-scale of an image and allow the modification of the way in which the data fills the available range of grey-levels. They may be viewed as pixel-by-pixel copying operations, except that the grey-levels are modified in a predetermined way. In this case a more convenient method than equation 5.1 of expressing a point operation is by:

$$s = f(r)$$

where, for simplicity of notation, $r$ and $s$ are variables denoting the ranges of $I(x, y)$ and $I_T(x, y)$ respectively. The point operation is completely specified by the function $f$, which determines the mapping of input grey-level to output grey-level. Each pixel's grey-level depends only on the corresponding input pixel and this contrasts with local operations in which a neighbourhood of input pixels determines the grey-level of each output pixel.

5.2.3.1 Processing of the grey-scale histogram in an image

The two main techniques in this area are histogram flattening (equalisation) and histogram matching. The theory behind these methods [15] relies on estimating the probability density function of a process from the histogram of its image normalised to unit area. The purpose of equalisation is to transform an image's histogram so that it has a flat appearance, thus utilising the entire grey-scale in an even fashion.
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Figure 5.1: Original image

Figure 5.2: Image subject to histogram equalisation
Figure 5.2 shows the result of applying histogram equalisation to the raw image in figure 5.1. Although the method is totally unsuitable for preprocessing an image in the current application, it highlights the problem of non-uniform background intensity mentioned in Section 5.2.2. Histogram equalisation has the effect that most of the available grey-scale in the image is then occupied by the background. This confines the useful data (particle images) to only a small range of the grey-scale.

Histogram matching is also inapplicable to the present problem. It is a method of transforming one image so that its histogram matches that of another image or some specified functional form. It can be used, for example, to add, subtract, average etc., two similar images that have not been digitised in the same way.
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**Figure 5.4:** Cropped section of figure 5.1

**Figure 5.5:** Figure 5.4 thresholded at a grey-scale value of 226

**Figure 5.6:** Figure 5.4 thresholded at a grey-scale value of 178

**Figure 5.7:** Figure 5.4 thresholded at a grey-scale value of 120
5.2.3.2 Thresholding

Thresholding is one of the most important approaches to image segmentation. Consider an image, \( I(x, y) \) composed of light objects on a dark background, or vice-versa, in such a way that the image's grey-level histogram consists of two dominant modes, corresponding to the light objects and dark background. In this case a single threshold \( T \) can be used to extract objects from the background and any \((x, y)\) for which \( I(x, y) > T \) is called an object point, otherwise the point is called a background point.

Figure 5.3 shows five point transforms. The linear transformation of figure 5.3(a) is simply the identity operation which copies \( I(x, y) \) to \( I_T(x, y) \) and is the usual transform used to display an image. In figure 5.3(b), \( f(r) \) produces a two-level (binary) image and this transformation is the basis of thresholding techniques. Figures 5.3(c), 5.3(d) and 5.3(e) show piecewise linear transformations, which are referred to here as below thresholding, above thresholding and clipping respectively. Transforms (c) and (d) are used when the data below or above a certain
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The threshold is to be kept and are useful for displaying the effects of thresholding. The transform of figure 5.3(e) subtracts the threshold \( T \) from all the data values above or equal to \( T \) and sets all values below \( T \) to zero. This transform is used in the implemented particle location algorithm described in Section 5.3. The relevant equations for the transforms are given by:

\[
s_a = r
\]

\[
s_b = \begin{cases} 
0 & \text{if } r < T \\
1 & \text{if } r \geq T 
\end{cases}
\]

\[
s_c = \begin{cases} 
r & \text{if } r < T \\
0 & \text{if } r \geq T 
\end{cases}
\]

\[
s_d = \begin{cases} 
r & \text{if } r > T \\
0 & \text{if } r \leq T 
\end{cases}
\]

\[
s_e = \begin{cases} 
r - T & \text{if } r \geq T \\
0 & \text{if } r < T 
\end{cases}
\]

where \( T \) is the selected threshold.

Figures 5.5, 5.6, and 5.7 show the results of applying decreasing thresholds to the image of figure 5.4 and indicate the limits of applying global thresholding to this problem. It is difficult to choose a single threshold for the image; setting the threshold too high results in most of the information being lost, whilst decreasing the threshold to include less bright particles can cause larger, brighter particles to merge, as shown in figure 5.8.

5.2.3.3 Image enhancement by point processing

There are several other point processes, apart from thresholding and histogram processing, including contrast stretching, image inversion and dynamic range.
compression. For a detailed treatment of these topics see [47]. These operations are generally not effective for automated image segmentation. For example, inversion simply requires a new threshold to be calculated below which all data is considered valid and therefore thresholding is affected in only a trivial way when the above process is applied to an image. Segmentation techniques, such as those based on local intensity gradient (Section 5.2.5) are affected by contrast stretching and dynamic range compression operations, but in an unpredictable fashion and so these processes are not used in the present study.

5.2.4 Filters

Local processing allows a greater variety of transformations than point processing. In the spatial domain, the most usual way to implement local operations is to define a mask (also referred to as template, window or filter) and to convolve this mask with the image \( I(x, y) \). The general equation for the convolution of the two real functions \( f(x,y) \) and \( g(x,y) \), is given by:

\[
f(x,y) \ast g(x,y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\alpha, \beta)g(x - \alpha, y - \beta) \, d\alpha \, d\beta,
\]

(5.7)

with its discrete counterpart defined as [15]:

\[
f(x,y) \ast g(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)g(x - m, y - n).
\]

(5.8)

Here \( f(x,y) \) can be interpreted as referring to the image and \( g(x,y) \) to the mask, the values of the mask’s coefficients determining the nature of the process.

There are two main methods of implementing filters, either in the frequency domain or the spatial domain. In this section the latter method of implementation is chosen. The reasons for this choice of implementation are detailed in Section 5.6.
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From the numerous types of filters available zero-phase filters are chosen for this application. One beneficial characteristic of a zero-phase filter is that it produces relatively small modifications to the shape of the signal component in the passband region of the filter. These filters have an impulse response, in the spatial domain, which is symmetric with respect to the origin. A non-linear phase distorts the proper registration of different frequency components that make up the lines, curves, etc. of an image, from which the human visual system gleans so much information. For a more detailed discussion of zero-phase filters see [28].

Filters can be further subdivided under the categories of low-pass, high-pass, band-stop and band-pass filters. The relevant equations describing their impulse responses are:

\[
\begin{align*}
    h_{lp}(m, n) &= \frac{R}{2\pi \sqrt{m^2 + n^2}} J_1(R\sqrt{m^2 + n^2}) \\
    h_{hp}(m, n) &= \delta(m, n) - h_{lp}(m, n) \\
    h_{bp}(m, n) &= \frac{R_2}{2\pi \sqrt{m^2 + n^2}} J_1(R_2\sqrt{m^2 + n^2}) - \frac{R_1}{2\pi \sqrt{m^2 + n^2}} J_1(R_1\sqrt{m^2 + n^2}) \\
    h_{bs}(m, n) &= \delta(m, n) - h_{bp}(m, n)
\end{align*}
\]

respectively. Here \((m, n)\) is the two-dimensional domain (where \(m\) and \(n\) are integer numbers) of the filter, with \(R\) defining the cut-off frequency for the low-pass/high-pass filter and \(R_1\) and \(R_2\) determining the passband/bandstop range for \(h_{bp}/h_{bs}\): the impulse response is \(\delta(m, n)\).

Lowpass filters are used for blurring and noise reduction in image processing applications. The shape of the impulse response needed to implement a lowpass spatial filter generally results in the implemented filter having all positive coefficients (for practical image processing applications). Figure 5.9 shows a typical
mask used to implement a low pass filter while figure 5.11 shows the result of applying the filter to the image of figure 5.10. Figures 5.12 and 5.13 show the result of convolving the image, with the mask of the blurring filter increased in size to 5x5 and 7x7 respectively.

In the frequency domain the power of typical PIV images, as well as most images in general, is primarily concentrated in the lower end of the spectrum due to the high spatial correlation between neighbouring pixels in the real domain. Pollen and sediment particles occupy spatial extents somewhere in the range of 1x1 to 10x10 pixels and so we might expect a reasonable amount of their power to be located in regions of higher frequency. Note that the blurring of the bright particles with the background cloud intensity makes them less distinguishable. Thus linear smoothing filters such as that of figure 5.9 and non-linear, low-pass filters such as median and out-of-range filters, while reducing noise tend, to have adverse effects on the image. With a median filter a mask slides along the image, and the median intensity of the area covered by the mask becomes the output intensity of the pixel being processed, which is at the centre of the mask. An out of range filter is implemented in a similar fashion to a median filter, with the average of the window calculated rather than the median. If the central pixel differs from the average by a specified amount the pixel is set to the average value.

The impulse response needed for a highpass filter indicates that the filter
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Figure 5.10: Original image

Figure 5.11: Image after lowpass filtering with 3x3 mask

Figure 5.12: Image after lowpass filtering with 5x5 mask

Figure 5.13: Image after lowpass filtering with 7x7 mask
have positive coefficients near its centre and negative coefficients in the outer periphery. Figure 5.14 shows typical highpass spatial filters. Filter (a) is the classic implementation of the 3x3 sharpening filter; the sum of the coefficients is zero. Filters (b) and (c) increase the contrast between adjacent pixels but no longer sum to zero. Figures 5.16, 5.17 and 5.18 show the results of applying the filters (a) (b) and (c) respectively to the image of figure 5.15. These filters have the desirable effect of reducing the background intensity cloud around regions of high sediment concentration. However, they have the adverse effects of suppressing some of the larger sediment images while enhancing the pollen images.

Figures 5.20, 5.21 and 5.22 show the results of applying the bandpass filter described by equation 5.11 with the ratio $R_1/R_2$ set to 0.01, 0.02 and 0.1 respectively. Note the similarity between the 0.01 level bandpass filtered image and the highpass filtered images of figures 5.16 5.17 and 5.18. The differences in the bandpass ranges can be noted visually by observing the centre of the larger particles in the image. For the 0.01 and 0.02 level passband ranges some of the larger particles are visibly suppressed. At the 0.1 level this is no longer visually evident.

With regard to bandstop filters there is no obvious band of frequencies that are to be suppressed in this application and thus they are not examined. From the above discussion, bandpass filter are the best suited for this application.
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Figure 5.15: Original image

Figure 5.16: Image after highpass filtering with mask of figure 5.14 (a)

Figure 5.17: Image after highpass filtering with mask of figure 5.14 (b)

Figure 5.18: Image after highpass filtering with mask of figure 5.14 (c)
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Figure 5.19: Original image

Figure 5.20: Image after bandpass filtering with 1% passband range

Figure 5.21: Image after bandpass filtering with 2% passband range

Figure 5.22: Image after bandpass filtering with 10% passband range
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5.2.5 Edge detection

The purpose of edge detection is to identify areas of an image where large changes of intensity occur. These areas often occur at a boundary or contour where significant change occurs in some physical aspect of the image. Edge detection is often implemented using gradient or Laplace operators. Whilst gradient operators may be used for image enhancement, they are more commonly used as a first step in segmentation.

5.2.5.1 Gradient operators for edge detection

The gradient, $\nabla I$, of an image $I(x, y)$, is defined as

$$\nabla I(x, y) = \frac{\partial I(x, y)}{\partial x}i_x + \frac{\partial I(x, y)}{\partial y}i_y$$

(5.13)

$$= G_x + G_y$$

(5.14)

where $i_x$ and $i_y$ are unit vectors in the x and y directions respectively. An important quantity for edge detection is the magnitude of the gradient, $|\nabla I|$, referred to simply as the gradient (often denoted as $|\nabla I|$ in many image processing texts) with:

$$|\nabla I| = \sqrt{(G_x^2 + G_y^2)}.$$

A common practice is to approximate the gradient as:

$$|\nabla I| \approx |G_x| + |G_y|,$$

since this is much simpler to implement.

For a discrete image $I(m, n)$ the partial derivatives of equation 5.13 can be approximated by some form of finite difference scheme. Two such estimates of

$$\frac{\partial I(x, y)}{\partial x}i_x$$
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\begin{align*}
\Delta_P I(m,n) & \overset{\text{def}}{=} [I(m+1,n+1) - I(m-1,n+1)] + [I(m+1,n) - I(m-1,n)] \\
& + [I(m+1,n-1) - I(m-1,n-1)] \tag{5.15}
\end{align*}

\begin{align*}
\Delta_S I(m,n) & \overset{\text{def}}{=} [I(m+1,n+1) - I(m-1,n+1)] + 2[I(m+1,n) - I(m-1,n)] \\
& + [I(m+1,n-1) - I(m-1,n-1)] \tag{5.16}
\end{align*}

These differences are typically implemented by convolving their respective masks with the image. The mask on the left hand side of figure 5.23(a) is generated from equation 5.15 while the right hand side shows its corresponding y-derivative component and these are termed Prewitt operators. Figure 5.23(b) shows the masks of the Sobel operator, that on the left hand side is generated from equation 5.16. While finite difference schemes may be implemented in many ways, the above operators improve the reliability and continuity of the estimate of equation 5.13 and in particular the Sobel operator provides an addi-
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The Sobel and Prewitt operators detect edges with a bias towards the vertical and horizontal. For example, independently computing $G_x$ results in the detection of edges in the vertical direction. Other operators may be defined which have a bias in other directions. Figure 5.24 shows two such operators which perform similarly to the Sobel and Prewitt operators, but with a bias in the diagonal directions.

Figures 5.25 to 5.30 show the effect of applying the separate gradient masks to the image of figure 5.19. Note that in applying the masks separately the magnitude is not computed and therefore particles have a positive response to the mask on one side of a tracer and a negative response on the other side. The Sobel and Prewitt responses are similar for their corresponding horizontal and vertical masks. The images illustrate the edge enhancement preferences of the Sobel, Prewitt and diagonal operators.

Figure 5.31 shows the result of applying the Sobel operator to figure 5.15 and figures 5.32 to 5.34 show the results of applying successively decreasing thresholds to figure 5.31. This illustrates the problem of applying a global threshold towards segmentation. If the threshold is too high, only the boundaries of a few high intensity particles are selected and these boundaries are often incomplete.

\[
\begin{array}{ccc}
2 & 1 & 0 \\
1 & 0 & -1 \\
0 & -1 & -2 \\
\end{array}
\quad
\begin{array}{ccc}
0 & -1 & -2 \\
1 & 0 & -1 \\
2 & 1 & 0 \\
\end{array}
\]

**Figure 5.24:** Diagonal edge operators
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Figure 5.25: Horizontal Sobel mask applied to figure 5.19

Figure 5.26: Vertical Sobel mask applied to figure 5.19

Figure 5.27: Horizontal Prewitt operator applied to figure 5.19

Figure 5.28: Vertical Prewitt operator applied to figure 5.19
Selecting too low a threshold results in boundaries which are too thick and it is evident that further processing, such as edge thinning, is necessary before particles are segmented successfully.

While the Sobel operator provides a smoothing effect, it is typical in image processing applications to provide extra smoothing prior to implementing the gradient operator using a lowpass filter. However as we have shown previously, directly lowpass filtering the images is unsuitable for the present application.

### 5.2.5.2 Laplace operators for edge detection

The Laplacian $\nabla^2$ of an image $I(x, y)$ is defined as:

$$
\nabla^2 I(x, y) = \frac{\partial^2 I(x, y)}{\partial^2 x} + \frac{\partial^2 I(x, y)}{\partial^2 y}.
$$

Similarly to gradient operators, for a discrete image, this formula can be replaced by some form of differences (second-order) which are implemented as masks and
Figure 5.31: Image after Sobel operator applied

Figure 5.32: Grey-level threshold set at 197

Figure 5.33: Grey-level threshold set at 104

Figure 5.34: Grey-level threshold set at 60
convolved with the discrete image. The mask shown in figure 5.35(a), with its inverse in figure 5.35(b) are characteristic of the type used to implement Laplace operators. This is due to Marr and Hildreth [34] and is based on the concept of convolving an image with the Laplacian of a two-dimensional Gaussian function. Figure 5.36 shows again the original image and figures 5.37 and 5.38 show the result of applying the respective masks to it.

### 5.3 Implementation of particle location

Following on from the discussions of image processing methods, the actual implementation is now described.

#### 5.3.1 Introduction

In the gradient-based method, searching for edges corresponds to searching for regions where $|\nabla I|$ is above a certain threshold. Another way to determine whether the magnitude of the gradient is large is to look for regions where it has a local maximum or minimum, that is, wherever the second derivative has a zero crossing as shown in figure 5.39. Since zero crossing contours are boundaries between regions, they tend to be continuous thin lines and edge thinning algorithms are generally not needed. However, obtaining a second derivative is evidently very sensitive to noise and therefore a considerable number of false edges are generated. An improved method of edge detection, due to White and Rohrer [57], can
Figure 5.36: Original image

Figure 5.37: Laplacian operator of figure 5.35(a) applied to image

Figure 5.38: Laplacian operator of figure 5.35(b) applied to image
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Figure 5.39: Edge detection by derivative operators
be achieved by combining both the gradient and Laplace methods to produce a three-level image. A modified version of this is now implemented.

### 5.3.2 Segmentation into a three level image

The image is initially bandpass filtered with the filter described in Section 5.2.4 with $R_2 = 0.99 \times f_{\text{max}}$ and $R_1 = 0.89 \times f_{\text{max}}$, where $f_{\text{max}}$ is the maximum frequency of the image. The image is clipped with the noise threshold set at 6 grey-level units. This threshold is determined empirically by selecting 10 images. A relatively particle free $32 \times 32$ pixel area of each image is selected. The average noise level $\bar{\eta}$ and the standard deviation $\bar{s}$ are calculated with

\[
\bar{\eta} = \frac{\sum_{j=1}^{10} \sum_{i=1}^{32 \times 32} I_{ij}}{10 \times 32 \times 32}, \tag{5.18}
\]

where $I_{ij}$ is the intensity of the $i^{th}$ pixel in the $32 \times 32$ square region of the $j^{th}$ image. The standard deviation $\bar{\sigma}$ is calculated using the formula

\[
\bar{\sigma} = \sqrt{\frac{\sum_{j=1}^{10} \sum_{i=1}^{32 \times 32} (I_{ij} - \bar{\eta})^2}{10 \times 32 \times 32}}. \tag{5.19}
\]

The average and standard deviation were found to be 19.63 grey-levels and 3.71 grey-levels respectively. A minimum threshold $T_{\text{min}}$ is defined as:

\[
T_{\text{min}} = \bar{\eta} + 3\bar{\sigma} \tag{5.20}
\]

calculated as 31 grey-levels correct to two significant figures (the threshold can only take integer values).

Both the gradient and Laplacian are computed separately for the image, with the former computed using the Sobel operator and the latter using the mask of figure 5.35(a). Then a zero, plus or minus is assigned to each pixel according to
the following formula.

\[
s(x, y) = \begin{cases} 
0 & \text{if } |\nabla I| < T \\
+ & \text{if } |\nabla I| \geq T \text{ and } \nabla^2 I \geq 0 \\
- & \text{if } |\nabla I| \geq T \text{ and } \nabla^2 I \leq 0 
\end{cases}
\]  

(5.21)

where the 0, + and − represent three arbitrary, but distinct, grey-levels of the programmer’s choice and \( T \) is a threshold.

For an image \( s(x, y) \) all pixels that are not on an edge, determined by \( |\nabla I| \) less than \( T \), are labeled zero. Pixels in the vicinity of an edge are labelled + or −. For a light object on a dark background, equation 5.21 produces an image \( s(x, y) \) in which all pixels on an edge and on the dark side of the object are assigned +, while all the pixels in the vicinity of an edge and on the light side of the object are assigned −.

The information obtained by implementing this procedure is used to generate a segmented, binary image in which the 1’s correspond to objects of interest and the 0’s to background. The transition along horizontal or vertical scan lines from a dark object to a light background must be characterised by the occurrence of a − followed by a + in \( s(x, y) \). The interior of the object is composed of pixels that are labelled either 0 or +. Finally, the transition from a light object to a dark background must be characterised by the occurrence of a + followed by a − in \( s(x, y) \). Thus a horizontal or vertical scanline containing a section of an object has the following structure:

\[
(...)(-,+)(0 \text{ or } +)(+,-)(...) 
\]

where (...), represents any combination of +, − and 0. The innermost parentheses containing (0 or +) as above, are interpreted as object points and are labelled 1. All other pixels along the scan line are labelled 0, with the exception of any other sequence of (0 or +) preceded by (−, +) and followed by (+, −)
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However, this does not solve the problem entirely. In the present study, the gradient on either side of a particle image varies, resulting at times in the rising side of a particle being detected but not its corresponding falling side. This is overcome in a twofold fashion. Firstly, the gradient associated with the falling side is set to only 50% (determined through trial and error) of the threshold T. Secondly, if a falling side is not found after a specified number of pixels after detecting the rising side, the algorithm starts afresh at the pixel following the first rising side found.

5.3.3 Pixel aggregation

Pixels which have been selected by the above process still remain in matrix format in memory. So, although we can visually associate the image into individual particles when it is displayed on the screen, there is no internal representation in the computer memory that the pixels which comprise an individual particle belong together. This is needed if we are to examine individual particle images further. The algorithm which performs this final stage of segmentation is described below.

Particle images are stored in memory in table format. For example, consider an image with $N$ tracers. This results in a table from 1 to $N$. Associated with each table index are the matrix locations comprising the particle image. In order to form the table the image is scanned in horizontal rows from top to bottom. As each row is scanned, the start and end pixel locations of segments of bright pixels, $s(x, y) = 1$, are recorded with two scanned rows held in memory at any given time.

Two rows from a hypothetical image are shown in figure 5.40(a). The relationship between the bright segments of the current row and those of the previous
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(a)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Previous row</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Current row</td>
<td>1</td>
<td>2</td>
<td></td>
<td>3</td>
</tr>
</tbody>
</table>

Figure 5.40: Segmentation matrix (b) generated by two image rows (a)

row is determined by a connection matrix as shown in figure 5.40(b). If a current row segment \( i \) touches a previous row segment \( j \), a 1 is placed in the matrix. A value of 0 is placed at matrix entries where segments do not touch.

The connection matrix is then parsed row by row to link adjoining segments with one another. First the total number of 1’s (segments) in a row are counted. Whenever a 1 is found in a row, a check is made to establish whether it joins any segment(s) in the previous row. If it does not, then it is placed into the first unoccupied table index. If it does, a check is made as to how many links it makes. If it is only joined to one segment, it is placed with it, at that segment’s table entry. If it joins two or more segments from the previous row, as in segments 1 and 2 of the current row of figure 5.40(a), then they are amalgamated by placing all the segment entries into a single table index. The table index is chosen, as that corresponding to the first adjoining segment of the previous row. The table is updated accordingly, i.e. the table entries of the moved segments are freed. After the final row has been scanned all the locations of all particles images are stored.
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5.3.4 Description and Separation

Rather than use a particle image’s raw intensity values, a set of descriptors (referred to as a pattern) are used to more compactly represent it. Vectors provide convenient notation for the descriptors and thus

\[
\vec{x} = \begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
\]

is used to denote a particle image’s pattern with \( x_1, x_2, \ldots, x_n \) the individual descriptors chosen.

Using this notation each particle image can be visualized as a point in an n-dimensional descriptor space. For example, a particle might be described in terms of the x coordinate \((x_1)\) and y coordinate \((x_2)\) of its centroid as well as its average intensity \(I_{\text{avg}}(x_3)\), with the resultant 3 dimensional descriptor space as shown in figure 5.41.

The computational criteria for the descriptors are the usual ones i.e. they should be easily implemented, quickly executed and use relatively little memory. To this end the following four descriptors are defined. The first and simplest quantity to calculate is the number of pixels \(N\) comprising the particle image.
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The second quantity is the maximum intensity $I_{\text{max}}$ of the particle’s pixels where

$$I_{\text{max}} = \max_i(I_i) .$$

(5.22)

Here each pixel comprising the particle image is assigned an integer $i$ in the range $1$ to $N$ with no two pixels assigned the same integer, and $I_i$ denotes the intensity of the $i^{\text{th}}$ pixel. Two other descriptors based on intensity are used namely the average intensity $I_{\text{avg}}$

$$I_{\text{avg}} = \frac{\sum_{i=1}^{N} I_i}{N} ,$$

(5.23)

and the normalised average intensity

$$I_{\text{norm}} = \frac{I_{\text{avg}}}{I_{\text{max}}} .$$

(5.24)

The quality of the above descriptors at separating two phase flows are determined by calibration experiments which are performed in Section 5.5. The results of these experiments indicate that the two descriptors $N$ and $I_{\text{norm}}$, set to 3 and 0.611 respectively will result in an average of approximately only 1 pollen particle being detected for every 100 sediment particles detected. These descriptor values are used in the implemented particle location routine and the images resulting from applying these descriptor thresholds can now be processed further.

5.4 Particle velocimetry by digital correlation

The usual method of displaying the information produced by a PIV experiment is in the form of a velocity vector map. In this case, correlation analysis is used to produce the required vectors. Particle tracking was also applied to the problem, but produced very poor results. The particle tracking method implemented is described in Appendix C.
In this section a description of the digital correlation method is given. Following this, the method of correlation analysis implemented in the current work is detailed. For a detailed description of the digital cross correlation method applied to PIV see [58].

5.4.1 Description

In digital correlation methods an image or images are successively subsampled via an interrogation window. The window should encompass a region where the relative displacements of successive particle images are similar. Each subsample (window) is correlated, either with itself (auto-correlation) or with another window (cross-correlation). This results in maxima in the correlation plane at locations corresponding to the dominant inter-spacing between particle images. A suitable choice of experimental parameters results in the global maximum of the correlation plane being located at a position corresponding to the average spacing between corresponding particle images (in the case of autocorrelation the second maximum should yield the desired displacement). Therefore locating this maximum yields the displacement and hence velocity information required. Correlation may be implemented as auto-correlation or cross correlation. Autocorrelation is normally used for multiple exposures on a single frame of data (although cross correlation may also be used) while cross correlation is used to analyse a single exposure over multiple frames. The main advantages of the cross correlation approach over autocorrelation are: (a) directional ambiguity is removed and (b) a better signal to noise ratio is produced. The main disadvantages are: (a) the image acquisition system (camera) must be able to capture two images in quick succession. Digital cameras capable of this are currently limited to approximately 1000 x 1000 pixels and are mainly used when relatively small flow areas (0.25m x 0.25m) are under investigation. For higher resolutions autocorrelation
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is the preferred choice and (b) performing a cross correlation is computationally more expensive requiring 3 two-dimensional transforms compared to 2 for autocorrelation.

5.4.2 Method of correlation

The correlation of the two real functions, \( f(x_1, x_2) \) and \( g(x_1, x_2) \), is given by

\[
f(x_1, y_1) \ast g(x_1, y_1) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\alpha, \beta) g(x_1 + \alpha, y_1 + \beta) \, d\alpha \, d\beta,
\]

with its discrete counterpart defined as

\[
f(k_1, k_2) \ast g(k_1, k_2) = \sum_{n_2=-N_2}^{N_2-1} \sum_{n_1=-N_1}^{N_1-1} f(n_1, n_2) g(k_1 + n_1, k_2 + n_2).
\]

Correlation was initially implemented directly using equation 5.26, which requires \( O(N_1^2 \times N_2^2) \) operations but this evidently became computationally slow and inefficient as \( N_1 \) and \( N_2 \) became large. Fourier methods were thus chosen, providing a more economic means in terms of speed to achieve the correlation.

The real function \( h(x) \) has a Fourier transform \( H(f) \) and between the two the following relations exist:

\[
H(f) = \int_{-\infty}^{+\infty} h(x) e^{2\pi i f x} \, dx;
\]

\[
h(x) = \int_{-\infty}^{+\infty} H(f) e^{-2\pi i f x} \, df.
\]

and its discrete equivalent is given by:

\[
H_n = \sum_{k=0}^{N-1} h_k e^{2\pi i kn/N};
\]

\[
h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi i kn/N}.
\]
The two-dimensional Fourier transform pair are defined as:

\[
H(f_1, f_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(x_1, x_2) e^{2\pi i f_1 x_1} e^{2\pi i f_2 x_2} \, dx_1 \, dx_2 \; ;
\]  

\[
h(x_1, x_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} H(f_1, f_2) e^{-2\pi i f_1 x_1} e^{-2\pi i f_2 x_2} \, df_1 \, df_2 .
\]  

In the two variable case the discrete Fourier transform pair is:

\[
H(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{n_2=0}^{N_2-1} \sum_{n_1=0}^{N_1-1} H(n_1, n_2) e^{-2\pi i k_2 n_2 / N_2} e^{-2\pi i k_1 n_1 / N_1} ;
\]  

\[
h(k_1, k_2) = \sum_{n_2=0}^{N_2-1} \sum_{n_1=0}^{N_1-1} h(k_1, k_2) e^{2\pi i k_2 n_2 / N_2} e^{2\pi i k_1 n_1 / N_1} .
\]

The discrete Fourier transforms of equation 5.30 can be expressed in the separable forms:

\[
H(n_1, n_2) = \sum_{k_2=0}^{N_2-1} \left( \sum_{k_1=0}^{N_1-1} h(k_1, k_2) e^{2\pi i k_2 n_2 / N_2} \right) e^{2\pi i k_1 n_1 / N_1} ;
\]  

\[
h(k_1, k_2) = \frac{1}{N_1 N_2} \sum_{n_2=0}^{N_2-1} \left( \sum_{n_1=0}^{N_1-1} H(n_1, n_2) e^{-2\pi i k_2 n_2 / N_2} \right) e^{-2\pi i k_1 n_1 / N_1} .
\]

Taking the Fourier transform \( \mathcal{F}\{ \} \) of equation 5.25 yields:

\[
\mathcal{F}\{ f(x_1, x_2) \ast g(x_1, x_2) \} = F(f_1, f_2) G^*(f_1, f_2)
\]

i.e. correlation in the real domain domain is equivalent to multiplication in the Fourier domain. From equation 5.32, it can be seen that suitably multiplying the Fourier transforms of \( f(x_1, x_2) \) and \( g(x_1, x_2) \), and inverse Fourier transforming the result yields their correlation. However, implementing this using equations 5.30 will not result in a speed increase. Considering equation 5.28, the one dimensional discrete Fourier transform, requires a number of complex multiplications and additions proportional to \( N^2 \). That is for each of the \( N \) values of \( H_n \), expansion of the summation requires \( N \) complex multiplications of \( h_k \) by:

\[
e^{2\pi i kn / N}
\]
The exponential terms can be computed once and stored in a table for subsequent calculations and for this reason the multiplication in these terms is usually not considered a direct part of the implementation. Using the separability of the Discrete Fourier Transform (DFT) will require $O(N_1^2 \times N_2^2)$ operations for the two-dimensional implementation and thus no computational advantage over the direct correlation method is gained. However proper decomposition of equation 5.29 can make the number of multiplications and additions proportional to $N \log_2 N$ and $N - 1$ additions of the results. The decomposition procedure is called the Fast Fourier Transform (FFT) algorithm. The FFT algorithm in this implementation is taken from [41]

5.4.3 Implementation

Before the correlation algorithm is performed, the image pair are gridded, with the separation between successive grid points in the horizontal and vertical directions user selectable. Each of these grid points represents the centre of an area over which the correlation is to be performed. Again this area is user selectable taking on the possible values 8 x 8, 16 x 16, 32 x 32, 64 x 64 or 128 x 128.

FFTs are applied to the two interrogation regions which are then correlated in the Fourier domain and inverse transformed to produce their correlation in the real domain. Correlation peaks are located and the maximum of these peaks yields the displacement peak. The peak is fitted to a one-dimensional Gaussian in the $x$ and $y$ axes to obtain $x$ and $y$ coordinates to sub-pixel accuracy. Westerweel [56] describes this process in more detail.

The correlation method thus described is used on the sediment only image, resulting after the particle separation algorithm has been applied, to produce a sediment phase vector map. If the correlation analysis is applied to the raw PIV image then, from the nature of the experimental data, most of the vectors
produced will result from the water phase. This was determined qualitatively from a visual comparison of the images produced in the experiments of Chapter 4 with the images produced in the calibration experiment described in the next section. It was observed that appreciable sediment concentration appeared in relatively few of the images. However, to produce a vector map consisting of the water phase (pollen) only, it is still necessary to reject those areas of sediment concentration and to do this in an automatic fashion. That is, a quantifiable criterion is needed to reject regions of appreciable sediment concentration. This criterion must be implemented in the algorithm.

The signal component of a PIV image is derived from its covariance [56]. For the case of images recorded in cross-correlation mode the relevant expression for the ensemble cross covariance $R_{11}$ [56] is given by,

$$R_{11}(x', y' ; x'', y'') = \langle I'(x', y')I''(x'', y'') \rangle - \langle I'(x', y') \rangle \langle I''(x'', y'') \rangle$$

(5.33)

where $I'(x', y')$ and $I''(x'', y'')$ are images recorded at times $t'$ and $t''$ respectively and $\langle \rangle$ denotes the ensemble average. An appreciable quantity of sediment is defined, for current purposes, as that concentration of sediment yielding an ensemble cross-covariance greater than or equal to 0.12 the ensemble cross-covariance of the pollen images. These ensemble cross-covariances are estimated for the images taken in the experiments of Section 5.5. The sediment concentration c3 is found to yield an average cross-covariance of 0.12 times that of the average pollen cross-covariance (in fact, the concentration was adjusted by trial and error to produce this figure).

Now it remains to define a suitable criterion to reject these areas of appreciable sediment concentration. The particle location routine is applied to the 10 images of the c3 sediment concentration and the 10 pollen images with the descriptors $I_{\text{norm}} = 0.55$ and $N = 3$ used to separate the particles. The number of particles
located in each of these regions are counted and recorded. Thus interrogation regions can be accepted or rejected based on the number of particles they contain. A rejection level of two or more particles was found to be suitable i.e. an interrogation region was rejected if two or more particles were found in it. Using the above parameters it was found that 92.35% of the sediment interrogation regions were rejected with 8.32% of the pollen regions rejected. The rejection level of two or more particles produced the highest sediment:pollen rejection ratio.

5.5 Experiment: Separation of phases

In this chapter an algorithm to locate particles has been described. Using this algorithm, raw PIV images are segmented into individual aggregates of pixels, with each aggregate representing a particle image. In the two-phase flow experiment of Chapter 4, both pollen and sediment images are recorded on a single frame and so they must be separated if velocity data can be obtained for the separate phases.

In the following experiment pollen and sediment images are recorded separately. The algorithm is run on the PIV images produced, resulting in separate pollen and sediment particle data sets. These data sets are used to examine the performance of the image descriptors defined in Section 5.2.

5.5.1 Procedure

Pollen with a volume fraction of $1 \times 10^{-6}$ is mixed with water in a small glass tank of dimensions (130mm $\times$ 60mm $\times$ 86mm). The tank is placed underneath the illumination system described in Section 4.2.2 and an image recorded using the camera as described in Section 4.2, with $f\# = 4$ and a magnification of 0.0268. After the image is recorded, the water/pollen mixture is stirred and an image is
again recorded. This procedure is repeated 10 times resulting in 10 PIV images. The tank is emptied and thoroughly cleaned. Sediment with a volume fraction of approximately $1.2 \times 10^{-4}$ (c1) is mixed with a solution of glycerol and water in the tank (the glycerol is added progressively to increase the density of the solution until the sediment's settling velocity appears negligible). An image is recorded using the same illumination system, camera and camera settings as described above. After the image is recorded the water/glycerol/sediment mixture is stirred and an image is again recorded. This procedure is repeated 10 times resulting in a further 10 PIV images. The sediment concentrations are increased to volume fractions of approximately $5 \times 10^{-4}$ (c2) and $1 \times 10^{-3}$ (c3) respectively and the above procedure repeated for each concentration.

### 5.5.2 Data Analysis

A phase separation factor $\theta$ is defined with

$$\theta(\bullet; c) = \frac{\theta_p(\bullet)}{\theta_s(\bullet; c)} \quad (5.34)$$

where $\theta_p(\bullet)$ and $\theta_s(\bullet; c)$ are the cumulative distributions for the pollen and sediment particles respectively and $c$ is a concentration parameter corresponding to the three different sediment concentrations in the experiment. Here the notation $(\bullet)$ is used to denote the dependence of the various quantities on, as yet, unspecified variables. These variables are determined from the descriptor being used. For example, in the case of $I_{\text{max}}$, the variable is the intensity threshold $T$ (measured in grey-level units) and the cumulative distribution is formed in the following way. Starting at $T = 255$, which in this case is the maximum intensity for these grey-level images, all those particles with an intensity greater than or equal to this threshold are counted. The threshold value is decreased in unit decrements, from 255 to the minimum intensity and the maxima greater than or equal to the
specific threshold are counted and recorded. For each descriptor one distribution is generated for the pollen data and three distributions are generated for the sediment (corresponding to the three sediment concentrations). The phase separation factor $\theta$ can be obtained from these cumulative distributions using equation 5.34.

For the descriptor $I_{avg}$, the variable is also the threshold intensity while for $I_{norm}$ the variable is the dimensionless normalized intensity threshold. The descriptor $N$ (number of pixels comprising a particle) is also dimensionless.

The 10 images of the sediment at concentration $c_1$ and the 10 pollen images produce 100 sediment/image pairs. For each of the 100 image pairs $\theta_p(\bullet), \theta_s(\bullet; c_1)$ and $\theta(\bullet; c_1)$ are calculated. The 100 $\theta_p(\bullet)$ are summed to produce an average pollen distribution $\bar{\theta}_p(\bullet)$ with

$$\bar{\theta}_p(\bullet) = \frac{\sum_{i=1}^{100} \theta_{pi}(\bullet)}{100}, \quad (5.35)$$

where $\theta_{pi}$ denotes the $i^{th}$ pollen distribution. The 100 sediment distributions for $c_1$ are similarly averaged producing an average sediment distribution $\bar{\theta}_s(\bullet; c)$ given by

$$\bar{\theta}_s(\bullet; c) = \frac{\sum_{i=1}^{100} \theta_{si}(\bullet; c)}{100}, \quad (5.36)$$

with $\theta_{si}$ denoting the $i^{th}$ sediment distribution and $c = c_1$ in this case. Finally, the 100 $\theta(\bullet; c)$ distributions are averaged to give $\bar{\theta}(\bullet; c)$ defined as:

$$\bar{\theta}(\bullet; c) = \frac{\sum_{i=1}^{100} \theta_i(\bullet; c)}{100}, \quad (5.37)$$

where $\theta_i$ denotes the $i^{th}$ phase separation factor.

Figures 5.42, 5.44, 5.46 and 5.48 show the cumulative distributions for the
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Descriptors $N$, $I_{\text{max}}$, $I_{\text{avg}}$ and $I_{\text{norm}}$ respectively while figures 5.43, 5.45, 5.47 and 5.49 show their respective phase separation factors.

5.6 Discussion and summary

In this chapter point processes as well as filtering and edge detection masks are applied to two-phase data in order to locate particles. The filters in the current work have been implemented in the spatial domain. The reasons are firstly, to perform efficient Fourier transformations, the image dimensions must conform to an area of $2^m \times 2^n$ where $m$ and $n$ are positive integers. Our images are in fact 700x460 (although the CCD is a (756 x 468) array) and therefore require considerable zero padding (or loss of data) to conform to the required size. Both these options are undesirable. Secondly, it is difficult to justify the extra computational burden, both in time of execution and complexity of code, associated with implementing the filters in the Fourier domain for the current work. As has been shown, adequate results can be obtained by spatial methods. Fourier methods are used in Section 5.4 for correlation purposes.

Thresholding the gradient of an image and finding the zero crossing in the Laplacian of an image are two methods of edge detection. The first method works well with sharp intensity transitions while the latter works well where the edges are blurry. Both methods are sensitive to noise, particularly when the Laplacian is computed directly using the mask of figures 5.35. Both methods are combined to provide an improved method of particle segmentation. The individual pixels comprising a particle image are aggregated and stored in memory in a table format. This raw particle data is further processed into a more convenient vector representation (pattern) using intensity and size parameters (descriptors).

These descriptors are evaluated as to their effectiveness in separating the
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Figure 5.42: Cumulative distributions for pollen and sediment images for the descriptor $N$

Figure 5.43: Phase separation factor for pollen and sediment images for the descriptor $N$
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Figure 5.44: Cumulative distributions for pollen and sediment images for the descriptor $I_{max}$

Figure 5.45: Phase separation factor for pollen and sediment images for the descriptor $I_{max}$
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Figure 5.46: Cumulative distributions for pollen and sediment images for the descriptor $I_{avg}$

Figure 5.47: Phase separation factor for pollen and sediment images for the descriptor $I_{avg}$
Figure 5.48: Cumulative distributions for pollen and sediment images for the descriptor $I_{\text{norm}}$

Figure 5.49: Phase separation factor for pollen and sediment images for the descriptor $I_{\text{norm}}$
phases of a two-phase flow through experiment and by defining a quantity referred
to as the phase separation factor. The phase separation factor depends on the
particular implementation of the phase separation algorithm and the descriptor
being evaluated.

For separation purposes the requirement is to find a suitable boundary in
the descriptor space with particle patterns on one side of the boundary accepted
as, say sediment images and those on the other side of the boundary rejected.
The boundary in descriptor space could enclose a region (or regions) or have an
arbitrarily complex topography. For the current work, however, the decision as
to whether a particle image should be accepted or rejected is based on a single
specific threshold for each descriptor, resulting in very simple boundaries.

The resulting sediment images are analysed with cross-correlation methods,
implemented using the FFT. For the pollen phase, the raw images are analysed
and appreciable areas of sediment concentration are rejected from the analysis.
Chapter 6

Results

In this chapter, six selected PIV image pairs from the data produced in the experiment described in Chapter 4 and their corresponding vector maps are shown. The vector maps are produced using the analysis method described in Section 5.3. Two vector maps are produced for each PIV image pair, one corresponding to the sediment phase and the other to the water phase. The results give a visual indication of the performance of the two phase analysis method applied to a real flow situation.

6.1 Discussion

Figure 6.1 indicates the chronological development of the beach profile in the present study. During the very early stages of beach development, the primary mechanism of transport is N-type (Section 4.1) with sand oscillating back and forth on an initially flat sand bed and ripples quickly forming underneath the node. A greater quantity of sand appears to move in this vicinity, where the large oscillatory velocity makes rippling most prominent, with the largest ripple wavelength found underneath the node. Smaller ripples progressively form away from the node, with their wavelengths and heights decreasing as their distances from the node decrease, until they reach points on either side of the node where no
appreciable sand movement occurs. At $t = 10$ minutes, regions of scour become prominent between the node and anti-node. Sand is transported from this area towards the node resulting in a net increase of sand height above the mean level near the node and a corresponding net reduction between the node and anti-node. As time progresses, the regions of scour deepen while the net increase of sand around the node continues until an equilibrium bed is reached. At this stage there is a zero net transport of sand towards the node. Seaman and O’Donoghue [51] describe the major features of the profile of this bed.

In the experiment described in Chapter 4, a total of 336 image pairs containing sediment and pollen are taken. With reference to figure 4.9 the images taken at camera positions A and B tend to have a much lesser sediment content than the images recorded at positions C and D. Also images, taken at the time interval $t = 4$ minutes tend to have much less sediment being lifted into the flow, as the ripples on the sand bed haven’t developed fully. This leaves the images taken at the time intervals $t = 10$ minutes, $t = 31$ minutes and $t = 60$ minutes at camera positions C and D to be considered. Of these, six image pairs and their
corresponding vector maps taken at camera position C and with the standing wave generated for 31 minutes \((t = 31\text{ minutes})\) are chosen to indicate the main features of the flow. For each phase, the flow field obtained from an analysis of the pollen seeding is shown separately from the flow field calculated from the sediment which is transported from the bed by the wave action. A raw PIV image pair is shown after its corresponding vector maps (pollen and sediment phase) for comparison purposes.

At \(t = 31\) minutes and at position C bed ripples are well established, however the shape of the bed between successive image pairs can change slightly. The estimated mean position of the bed is superimposed on the vector maps acting as a reference position. The beach wall is situated to the left of the photographed area. The velocity due to the wave motion is given by equations 4.6 and 4.7 with

\[
t = 1674T + \Delta t
\]

where \(T\) is the wave period and \(\Delta t\) determines the phase. The time interval between successive image captures is known and therefore the relative phase between velocity vector maps can be determined. Since the capture process was initiated (triggered) manually, the initial phase of the wave is unknown, however the vector maps have been compared to those obtained by Seaman and O’Donoghue [51] to qualitatively determine estimates of the phase.

Figures 6.2 to 6.5 correspond to velocities occurring slightly after the maximum wave velocity. The beginning of the formation of a vortex, to the beach wall side of the ripple, can be seen in figure 6.3. There is a significant amount of sediment transported in this region. There is no evidence of this motion in the water phase. At this magnification and size of measurement area, there is little chance of obtaining water phase measurements within a region of high sediment concentration and therefore most of the water phase measurement is contained
Figure 6.2: Water vector map at $\Delta t = 6T/10$

Figure 6.3: Sediment vector map at $\Delta t = 6T/10$
Figure 6.4: First raw PIV image at $\Delta t = 6T/10$

Figure 6.5: Second raw PIV image at $\Delta t = 6T/10$
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Figure 6.6: Water vector map at $\Delta t = 7T/10$

Figure 6.7: Sediment vector map at $\Delta t = 7T/10$
Figure 6.8: First raw PIV image at $\Delta t = 7T/10$

Figure 6.9: Second raw PIV image at $\Delta t = 7T/10$
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Figure 6.10: Water vector map at $\Delta t = 8T/10$

Figure 6.11: Sediment vector map at $\Delta t = 8T/10$
Figure 6.12: First raw PIV image at $\Delta t = 8T/10$

Figure 6.13: Second raw PIV image at $\Delta t = 8T/10$
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Figure 6.14: Water vector map at \( \Delta t = 9T/10 \)

Figure 6.15: Sediment vector map at \( \Delta t = 9T/10 \)
Figure 6.16: First raw PIV image at $\Delta t = 9T/10$

Figure 6.17: Second raw PIV image at $\Delta t = 9T/10$
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Figure 6.18: Water vector map at \( \Delta t = 11T/10 \)

Figure 6.19: Sediment vector map at \( \Delta t = 11T/10 \)
Figure 6.20: First raw PIV image at $\Delta t = 11T/10$

Figure 6.21: Second raw PIV image at $\Delta t = 11T/10$
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Figure 6.22: Water vector map at $\Delta t = 12T/10$

Figure 6.23: Sediment vector map at $\Delta t = 12T/10$
Figure 6.24: First raw PIV image at $\Delta t = 12T/10$

Figure 6.25: Second raw PIV image at $\Delta t = 12T/10$
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from the mid mean water height to the water surface whereas most of the sediment measurements are contained near the bed, as indicated by the sediment velocity maps.

Figures 6.6 to 6.9 show the wave just prior to flow reversal. Here the vortex is fully established and can be seen in figures 6.6 and 6.7. In the area of the vortex, there is a considerable amount of sediment which has been lifted off the bed. This is seen in the raw data images of figures 6.8 and 6.9 and also in the velocities in figure 6.7.

Figures 6.10 to 6.13 show the wave just after flow reversal. Here the vortex is lifting off the bed. This is seen in both the sediment velocities and the pollen velocities. It is clear from figures 6.12 and 6.13 as well as their corresponding vector maps that as the vortex lifts off the bed it carries a considerable amount of sediment with it.

Figures 6.14 to 6.17 show the wave just before the maximum velocity, in the reversal direction, while figures 6.18 to 6.21 show the wave just after the maximum velocity. A small vortex is visible in figures 6.16 and 6.17 forming over a ripple and the concentration of sediment is largest in this region. In figures 6.18 and 6.19, the vortex has moved off the edge of the measurement volume and there is little evidence of its motion although there is some evidence of motion on the left hand side of the figures resulting from a vortex forming downstream of the ripple which is to the left of the measurement area. Figures 6.22 to 6.25 correspond to the velocity reversal. There is not much sediment motion visible although remnants of the vortex can be seen above the ripple.
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6.2 Conclusions

The two-phase analysis method described in Section 5.3 has been successfully applied to a real wave motion flow. The technique produces two separate flow maps corresponding to the water and sediment phases which enables the motion of the sediment relative to the water to be examined (although it has only been discussed qualitatively in this case). The vector maps produced by the two phase analysis software are of good quality in terms of vector content. The results produced agree sensibly with the fluid motion expected.
Chapter 7

Conclusions

7.1 General conclusions

- A theoretical and experimental investigation has been presented into the measurement of two-phase flows, with particular emphases placed on the analysis and assessment of factors which affect the data obtained from the experiments. The theory derived, the two-phase analysis method and the two-phase measurements are thought to be unique.

- An automated system has been developed to locate, separate and analyse the particle images produced by two-phase flow experiments.

- The automated analysis system has been applied to experiments involving the measurement of two-phase (sediment/water) flow over a rippled-sand bed. For this purpose, a flexible, convenient and robust PIV illumination system was built.

7.2 Specific conclusions

- The theoretical derivations for the discrete random sampling of a Gaussian function produce relatively simple analytical expressions for the one-dimensional centre sample and general sample points as well as for the
two-dimensional centre sample. The theoretical analysis for sampling using a CCD array produces results which must be evaluated numerically.

- The theoretical derivations were found to agree well with experiment and indicated that for typical PIV experimental parameters, the variation of light recorded due to a particle's random position in the laser sheet is an order of magnitude higher than the variation of light intensity due to the integration effects of the CCD sensor.

- It was found experimentally that the variation of intensity due to the light scattering characteristics of quartz sediment particles is of the same order of magnitude as the variation due to the position of the particle in the laser sheet.

- Experiments performed to determine statistical confidence levels for the separation of phases, indicate that the sediment phase can be separated. Pollen can be analysed in areas of low sediment concentration which are determined via calibration experiments.

- The results obtained from the experiment show that both the illumination and analysis systems work well when applied to a practical two-phase flow experiment.

### 7.3 Future work

- The current implementation of two-phase analysis uses software to separate the phases. A possible hardware implementation is as follows. Two distinct wavelengths of light produced from a single multi-wavelength laser [10] [55], or from two separate lasers are used to illuminate the volume. The wavelengths can correspond to say, red and green light. A scanning
beam illumination system can be used with the two beams of light combined coaxially to form an apparently single scanning beam. The separate phases of the flow are coloured correspondingly: red and green. Thus one phase scatters green light and the other red. A two camera system is used to record the flow with a red light filter placed in front of one camera and a green filter in front of the other. Thus the camera with the red filter should record mainly the light scattered from the red-phase particles and the camera with the green filter should record the green-phase particles. If cross-correlation images are to be recorded then a four camera system consisting of two of the above camera pairs can be used.

- There is the possibility of measuring the concentration levels of sediment (in this case) within a given image and for this two quantities are required, the volume of the flow being measured and the number of sediment particles this contains. It is a trivial task to count the number of particles located for a specific image and the length and height of the measurement region can be determined accurately. The problem is in determining the width of the measurement region and more specifically how this width relates to the number of particles detected by the algorithm. One approach to the problem would be to fix the algorithm's parameters to sensible values and to perform calibration experiments over a large number and range of sediment concentrations. If the numbers of particles found at each concentration do not fluctuate greatly and these numbers differ for each level, then this information could be used to estimate sediment concentration levels.
Appendix A

Statistics relating to PIV sampling

A.1 Positive side samples

With reference to figure A.1 consider the general positive sample point $X_n$, which falls in the interval

$$\frac{(2n-1)d}{2} \leq x < \frac{(2n+1)d}{2}, \quad n \in \mathbb{Z} \quad n = 1, 2, 3, \ldots ,$$

Now from the graph in figure A.2

$$\begin{array}{cccc}
\frac{d}{2} & \frac{3d}{2} & \frac{(2n-1)d}{2} & \frac{(2n+1)d}{2} \\
X_1 & & X_n & \\
\end{array}$$

**Figure A.1:** Positive sample points

$$\mathcal{Y}_n \geq I \left( \frac{(2n + 1)d}{2} \right) = Ae^{-a\left(\frac{(2n+1)d}{2}\right)^2} \quad (A.1)$$

Let $y_{min}^* = Ae^{-a\left(\frac{(2n+1)d}{2}\right)^2}$ and therefore

$$P\{\mathcal{Y}_n > y\} = 1 \quad \text{for } y < y_{min}^*. \quad (A.2)$$

Also from the graph

$$\mathcal{Y}_n \leq I \left( \frac{(2n - 1)d}{2} \right) = Ae^{-a\left(\frac{(2n-1)d}{2}\right)^2} \quad (A.3)$$
Appendix A — Statistics relating to PIV sampling

Let \( y_{\text{max}}^* = A e^{-a \left( \frac{(2n-1)d}{2} \right)^2} \) and therefore

\[
P\{ Y_n > y \} = 0 \quad \text{for} \quad y > y_{\text{max}}^*. \tag{A.4}
\]

Now consider the case of \( y_{\text{min}}^* \leq y \leq y_{\text{max}}^* \). From the graph for \( Y_n > y \), \( X_n \) must satisfy

\[
\frac{(2n - 1)d}{2} < X_n < \frac{(2n - 1)d}{2} + l
\]

Hence it is obvious that

\[
P\{ Y_n > y \} = \frac{\text{length of the inner interval}}{\text{length of outer interval}} = \frac{l}{d}
\]

but

\[
y = A e^{-a \left( \frac{(2n-1)d}{2} + l \right)^2} \tag{A.5}
\]

Rearranging terms yields

\[
l = \sqrt{\frac{1}{a} \ln \left( \frac{A}{y} \right) - \frac{(2n - 1)d}{2}} \quad \text{since} \quad l > 0 \tag{A.6}
\]
Appendix A — Statistics relating to PIV sampling

and hence

\[ P\{Y_n > y\} = \begin{cases} 
1 & \text{if } y < y_{\text{min}}^* \\
\frac{1}{d\sqrt{\pi}} \left( \ln\frac{A}{y} - \frac{(2n-1)d}{2} \right) & \text{if } y_{\text{min}}^* \leq y \leq y_{\text{max}}^* \\
0 & \text{if } y > y_{\text{max}}^* 
\end{cases} \]  

(A.7)

where

\[ y_{\text{min}}^* = Ae^{-a\left(\frac{(2n+1)d}{2}\right)^2} \]

\[ y_{\text{max}}^* = Ae^{-a\left(\frac{(2n-1)d}{2}\right)^2} \]

\[ n \in \mathbb{Z}, \quad n = 1, 2, 3, \ldots, \]

\[ A, a, d \in \mathbb{R} \quad A < 0, a < 0, d < 0 \]
Appendix A — Statistics relating to PIV sampling

A.2 Negative side samples

Consider the negative sample point $X_n$ which falls in the interval
\[
\frac{(2n-1)d}{2} \leq X_n \leq \frac{(2n+1)d}{2} \quad n \in \mathbb{Z} \quad n = 1, 2, 3, \ldots ,
\]
as shown in figure A.3. Now from the graph in figure A.4

\[
\frac{(2n-1)d}{2} \quad \frac{(2n+1)d}{2} \quad \frac{-3d}{2} \quad \frac{d}{2}
\]

**Figure A.3:** Negative sample points

\[
Y_n \geq I \left( \frac{(2n-1)d}{2} \right) = A e^{-\alpha \left( \frac{(2n-1)d}{2} \right)^2} \quad (A.8)
\]

Let $y_{\text{min}} = A e^{-\alpha \left( \frac{(2n-1)d}{2} \right)^2}$ and therefore

\[
P\{Y_n > y\} = 1 \quad \text{for} \quad y < y_{\text{min}}. \quad (A.9)
\]

Also from the graph

\[
Y_n \leq I \left( \frac{(2n+1)d}{2} \right) = A e^{-\alpha \left( \frac{(2n+1)d}{2} \right)^2} \quad (A.10)
\]

Let $y_{\text{max}} = A e^{-\alpha \left( \frac{(2n+1)d}{2} \right)^2}$ and therefore

\[
P\{Y_n > y\} = 0 \quad \text{for} \quad y > y_{\text{max}}. \quad (A.11)
\]

Now consider the case of $y_{\text{min}} \leq y \leq y_{\text{max}}$. From the graph for $Y_n > y$, $X_n$ must satisfy

\[
\frac{(2n+1)d}{2} - l < X_n < \frac{(2n+1)d}{2}
\]

Hence it is obvious that

\[
P\{Y_n > y\} = \frac{\text{length of the inner interval}}{\text{length of outer interval}} = \frac{l}{d}.
\]

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Figure A.4: Graph of general $n^{th}$ positive sample region of a Gaussian function

but

$$y = Ae^{-a\left(\frac{(2n+1)d}{2} - l\right)^2}$$

Rearranging terms yields

$$l = \sqrt{\frac{1}{a} \ln\left(\frac{A}{y}\right) + \frac{(2n+1)d}{2}} \quad \text{since } l > 0$$

and hence

$$P\{Y_n > y\} = \begin{cases} \frac{1}{d\sqrt{a}} \sqrt{\ln\left(\frac{A}{y}\right) + \frac{(2n+1)d}{2}} & \text{if } y < y_{min}^* \\ \frac{1}{d\sqrt{a}} \sqrt{\ln\left(\frac{A}{y}\right) + \frac{(2n+1)d}{2}} & \text{if } y_{min}^* \leq y \leq y_{max}^* \\ 0 & \text{if } y > y_{max}^* \end{cases} \quad (A.12)$$

where

$$y_{min}^* = Ae^{-a\left(\frac{(2n+1)d}{2}\right)^2}$$

$$y_{max}^* = Ae^{-a\left(\frac{(2n+1)d}{2}\right)^2}$$

$n \in \mathbb{Z}, \quad n = -1, -2, -3, \ldots,$

$$A, a, d \in \mathbb{R} \quad A < 0, a < 0, d < 0$$
A.3 The surrounding samples for the 2-d discrete sampling case

Now consider the \((m, n)^{th}\) sample point \(x_{m,n}\) which falls in the interval

\[
\frac{2(m-1)d}{2} < x < \frac{2(m+1)d}{2}, \quad \frac{2(n-1)d}{2} < y < \frac{2(n+1)d}{2}
\]

A geometrical interpretation is given in the discussion of Chapter 2. Reference to this indicates that it is difficult to tackle the problem using geometry, since we have to consider each sample point individually and the growth of terms in each increases as the modulus of the sample index increases. Therefore we adopt a different approach as follows.

We first consider the densities of the r.v. \(X_m\)

\[
X_m = X_0 + md, \quad m \in \mathbb{Z}, \, d \in \mathbb{R}
\] (A.13)

where \(X_0\) is an r.v. uniformly distributed in the interval \(-\frac{d}{2} \leq x \leq \frac{d}{2}\). Without loss of generality we consider an arbitrary but specific \(m^{th}\) density, say the \(k^{th}\) density, so that we have \(X_k = X_0 + kd\). The equation \(x = x_0 + kd\) has one real solution \(x_0 = x - kd\) for every \(x\). Now let \(g(x) = x_0 + kd\), in which case \(g'(x) = 1\) and we conclude from 2.13 that

\[
f_{X_k}(x_k) = f_{X_0}(x - kd)
\]

and therefore for the general \(m^{th}\) sample

\[
f_{X_m}(x_m) = \begin{cases} 
\frac{1}{d} & \text{if} \quad \frac{(2m-1)d}{2} < x < \frac{(2m+1)d}{2} \\
0 & \text{elsewhere}
\end{cases}
\] (A.14)

\(m \in \mathbb{Z}, \, m \neq 0\).

Now consider the sequence of r.v. \(U_m\)

\[
U_m = e^{-\alpha(X_m)^2}
\] (A.15)
Appendix A — Statistics relating to PIV sampling

If \( u < 0 \), then the equation \( u = e^{-ax^2} \) has no real solutions; hence

\[
f_{\mathcal{U}_m}(u) = 0 \quad \text{for } u < 0 \tag{A.16}
\]

If \( u > 0 \), then the equation \( u = e^{-ax^2} \) has two solutions

\[
x_1 = \sqrt{\frac{1}{a} \ln \left( \frac{1}{u} \right)} \]
\[
x_2 = -\sqrt{\frac{1}{a} \ln \left( \frac{1}{u} \right)}
\]

Now let \( g(x) = e^{-ax^2} \) and therefore \( g'(x) = -2axe^{-ax^2} \), in which case

\[
g'(x_1) = -2\sqrt{au} \sqrt{\ln \left( \frac{1}{u} \right)}
\]
\[
g'(x_2) = 2\sqrt{au} \sqrt{\ln \left( \frac{1}{u} \right)}
\]

From equation 2.13 it can be seen that for \( m < 0 \),

\[
f_{\mathcal{U}_m}(u) = \frac{f_{\mathcal{X}_m}(x_1)}{|g'(x_1)|}
\]
and for \( m > 0 \),

\[
f_{\mathcal{U}_m}(u) = \frac{f_{\mathcal{X}_m}(x_2)}{|g'(x_2)|}
\]
and in both cases we find

\[
f_{\mathcal{U}_m}(u) = \begin{cases} 
\frac{1}{2d \sqrt{au} \sqrt{\ln \left( \frac{1}{u} \right)}} & \text{for } e^{-a\left(2m-1\right)^2d^2} < u < e^{-a\left(2m+1\right)^2d^2} \\
0 & \text{elsewhere}
\end{cases}
\]

where \( m \in \mathbb{Z}, m \neq 0 \).

The r.v. \( \mathcal{V}_n \) is now formed such that

\[
\mathcal{V}_n = e^{-a(\mathcal{Y}_n)^2}
\]
and reasoning as in the derivation of the density \( f_{m,n}(u) \) we obtain

\[
f_{\nu_n}(v) = \begin{cases} 
\frac{1}{2d\sqrt{a\nu}\sqrt{\ln(\frac{1}{\nu})}} & \text{for } e^{-a \frac{(2n-1)^2d^2}{4}} < v < e^{-a \frac{(2n+1)^2d^2}{4}} \\
0 & \text{elsewhere}
\end{cases}
\]

(A.17)

where \( n \in \mathbb{Z}, n \neq 0 \)

Now consider the r.v. \( Z_{m,n} \)

\[
Z_{m,n} = U_m V_n = e^{-a(X_m)^2} e^{-a(Y_n)^2}
\]

(A.18)

The r.v. \( U_m \) and \( V_n \) are independent and from 2.19 the joint density \( f_{m,n}(uv) \) is given by

\[
f_{m,n}(uv) = f_{U_m}(u)f_{V_n}(v)
\]

(A.19)

where we have dropped the \( UV \) subscript in the joint density and simply retained the \( m, n \) subscript for compactness of notation and we can therefore write,

\[
f_{m,n}(uv) = \begin{cases} 
\frac{1}{4d^2 a u (\sqrt{\ln(\frac{1}{u})}) v (\sqrt{\ln(\frac{1}{v})})} & \text{for } e^{-a \frac{(2n-1)^2d^2}{4}} < v < e^{-a \frac{(2n+1)^2d^2}{4}} \\
0 & e^{-a \frac{(2m+1)^2d^2}{4}} < u < e^{-a \frac{(2m+1)^2d^2}{4}} 
\end{cases}
\]

In order to visualize the problem \( f(u, v) \) can be interpreted as surface mass density in the \( uv \) plane, the total mass in the \( uv \) plane being equal to 1. The probability that \( U(\zeta), V(\zeta) \) is a point in a region \( R \), equals the total mass in \( R \). This allows us to equate finding the \( P\{Z > z\} \) for a specific \( z \), say \( z^* \), with finding the mass density of \( f(u, v) \) defined by the region \( z > z^* \).

Figure A.5 shows the function \( z = uv \) for \( z = z^* \) and the region of the \( uv \) plane \( z > z^* \) and from figure A.6 it is obvious that for an arbitrary \( k^{th} \) density, \( f_k(u,v) \), \( k = (k_1,k_2) \), we have

\[
P\{Z_k > z\} = \iint_R f_k(u,v) \, du \, dv
\]
Appendix A — Statistics relating to PIV sampling

**Figure A.5:** Graph of the $(u, v)$ plane. The shaded area indicates the region $z > z^*$

**Figure A.6:** Region $R$ of the $(u, v)$ plane defined by $z = uv$ and the boundary of the rectangle $u_1 < u < u_2, v_1 < v < v_2$

**Figure A.7:** Region $R$ of the $(u, v)$ plane defined by $z > z_4^*$ and the boundary of the rectangle $u_1 < u < u_2, v_1 < v < v_2$
Appendix A — Statistics relating to PIV sampling

Figure A.8: Region $R$ of the $(u, v)$ plane defined by $z < z_1^*$ and the boundary of the rectangle $u_1 < u < u_2$, $v_1 < v < v_2$

Without loss of generality we continue to consider a specific $(m, n)^{th}$ density, say the $k^{th}$ density, and from figure A.7 it is obvious that for $z < z_1^*$

$$P\{Z_k > z\} = 1 \quad \text{for} \quad z < z_1^*$$  \hspace{1cm} (A.20)

and similarly from figure A.8 it can be seen that

$$P\{Z_k > z\} = 0 \quad \text{for} \quad z > z_4^*$$  \hspace{1cm} (A.21)

Now consider $z_1^* < z < z_4^*$

Figures A.9, A.10 and A.11 show the sequence of events for $z$ increasing. There are, in fact, two possible sequences depending on whether $z_2^* < z_3^*$ or $z_2^* > z_3^*$. We first look at $z_2^* < z_3^*$ and address the question of determining the limits of integration for each case.

For $z_1^* < z < z_2^*$, figure A.9, we see that

$$P\{Z_k > z\} = \int \int f_k(u,v) \ du \ dv$$

$$= \int \int f_k(u,v) \ du \ dv \ + \int \int f_k(u,v) \ du \ dv$$

$$= \int_{v_1}^{v_2} \int_{u_1}^{u_2} f_k(u,v) \ du \ dv \ + \int_{u_1}^{u_2} \int_{z/v}^{z/v} f_k(u,v) \ du \ dv$$  \hspace{1cm} (A.22)
Appendix A — Statistics relating to PIV sampling

Figure A.9: Region $R$ of the $(u, v)$ plane defined by $z < z_1^*, z_1^* < z < z_2^*$ and the boundary of the rectangle $u_1 < u < u_2$, $v_1 < v < v_2$

From A.10 where $z_2^* < z < z_3^*$ we have

$$P\{Z_k > z\} = \int_{R} \int f_k(u, v) \, du \, dv$$
$$= \int_{v_1}^{v_2} \int_{z/v}^{u_2} f_k(u, v) \, du \, dv$$

(A.23)

When $z_3^* < z < z_4^*$ we can see from figure A.11 that

$$P\{Z_k > z\} = \int_{R} \int f_k(u, v) \, du \, dv$$
$$= \int_{z_1/v}^{v_2} \int_{z_1/v}^{u_2} f_k(u, v) \, du \, dv$$

(A.24)

If in fact $z_2^* > z_3^*$, then the limits of integration for the cases shown in figures A.9 and A.11 remain the same and we only have to reconsider $z_2^* < z < z_3^*$ in which case, with reference to figure A.12 we have
Appendix A — Statistics relating to PIV sampling

Figure A.10: Region $R$ of the $(u,v)$ plane defined by $z_2^* < z < z_3^*$ and the boundary of the rectangle $u_1 < u < u_2$, $v_1 < v < v_2$

Figure A.11: Region $R$ of the $(u,v)$ plane defined by $z_2^* < z < z_3^*$ and the boundary of the rectangle $u_1 < u < u_2$, $v_1 < v < v_2$

\[ P\{Z_k > z\} = \int_R f_k(u,v) \, du \, dv \]
\[ = \int_R f_k(u,v) \, dv \, du \]
\[ = \int_{u_1}^{u_2} \int_{z/u}^{v_2} f_k(u,v) \, dv \, du \]

(A.25)
Appendix A — Statistics relating to PIV sampling

Figure A.12: Region $R$ of the $(u, v)$ plane defined by $z_2^* > z_3^*$, $z_2^* < z < z_3^*$ and the boundary of the rectangle $u_1 < u < u_2$, $v_1 < v < v_2$. 
Appendix A — Statistics relating to PIV sampling

A.4 Positive side samples for pixel sampling

The derivation of $P\{Y_n > y\}$ for the case of pixel sampling in the image plane, is equivalent to discrete sampling in the sensor plane for the general $n^{th}$ positive and negative sample cases. Thus the derivation of these cases proceeds in a similar fashion to Sections A.1 and A.2 with the obvious difference that the function presently under consideration is the sensor image given by equation 2.56. With reference to figure A.1 consider the general positive sample point $X_n$ which falls in the interval

$$\frac{(2n-1)d}{2} \leq x \leq \frac{(2n+1)d}{2}, \quad n \in \mathbb{Z} \quad \text{for } n = 1, 2, 3, \ldots,$$

Now from the graph in figure A.2

$$Y_n \geq I \left( \frac{(2n+1)d}{2} \right) = \frac{A}{2} \sqrt{\frac{\Pi}{a}} \left( \text{erf}((n+1)d) - \text{erf}(nd) \right)$$

(A.26)

Let

$$y_{\text{min}}^\ast = \frac{A}{2} \sqrt{\frac{\Pi}{a}} \left( \text{erf}((n+1)d) - \text{erf}(nd) \right)$$

and therefore

$$P\{Y_n > y\} = 1 \quad \text{for } y < y_{\text{min}}^\ast.$$  \hspace{1cm} (A.27)

Also from the graph

$$Y_n \leq I \left( \frac{(2n-1)d}{2} \right) = \frac{A}{2} \sqrt{\frac{\Pi}{a}} \left( \text{erf}(nd) - \text{erf}((n-1)d) \right)$$

(A.28)

Let

$$y_{\text{max}}^\ast = \frac{A}{2} \sqrt{\frac{\Pi}{a}} \left( \text{erf}(nd) - \text{erf}((n-1)d) \right)$$

and therefore

$$P\{Y_n > y\} = 0 \quad \text{for } y > y_{\text{max}}^\ast.$$ \hspace{1cm} (A.29)
Appendix A — Statistics relating to PIV sampling

Now consider the case of \( y_{\min} \leq y \leq y_{\max} \). From the graph for \( \mathcal{Y}_n > y \), \( \mathcal{X}_n \) must satisfy

\[
\frac{(2n - 1)d}{2} < \mathcal{X}_n < \frac{(2n - 1)d}{2} + l
\]

Hence it is obvious that

\[
P\{\mathcal{Y}_n > y\} = \frac{\text{length of the inner interval}}{\text{length of outer interval}} = \frac{l}{d}
\]

Considering equation 2.56 we note \( g \) maps \([\frac{(2n-1)d}{2}, \frac{(2n+1)d}{2}]\) one to one and onto \([y_{\max}; y_{\min}]\) and so \( g \) has an inverse function \( g^{-1} \) which maps \([y_{\max}, y_{\min}]\) onto \([\frac{(2n-1)d}{2}, \frac{(2n+1)d}{2}]\) and

\[
l = g^{-1}(y)
\]

Since there is no closed form expression for \( g^{-1}(y) \) we use the numerics as described in the previous section.
A.5 Negative side samples for pixel sampling

Consider the negative sample point $X_n$ which falls in the interval

$$\frac{(2n-1)d}{2} \leq X_n \leq \frac{(2n+1)d}{2}, \quad n \in \mathbb{Z}, \quad n = 1, 2, 3, \ldots,$$

as shown in figure A.3.

Now from the graph in figure A.4

$$\mathbf{Y}_n \geq I \left( \frac{(2n-1)d}{2} \right) = \frac{A}{2} \sqrt{\frac{\pi}{a}} (\text{erf}(nd) - \text{erf}((n-1)d))$$

(A.30)

Let

$$y^*_{\text{min}} = \frac{A}{2} \sqrt{\frac{\pi}{a}} (\text{erf}(nd) - \text{erf}((n-1)d))$$

and therefore

$$P\{\mathbf{Y}_n > y\} = 1 \quad \text{for } y < y^*_{\text{min}}.$$  
(A.31)

Also from the graph

$$\mathbf{Y}_n \leq I \left( \frac{(2n+1)d}{2} \right) = \frac{A}{2} \sqrt{\frac{\pi}{a}} (\text{erf}((n+1)d) - \text{erf}(nd))$$

(A.32)

Let

$$y^*_{\text{max}} = \frac{A}{2} \sqrt{\frac{\pi}{a}} (\text{erf}((n+1)d) - \text{erf}(nd))$$

and therefore

$$P\{\mathbf{Y}_n > y\} = 0 \quad \text{for } y > y^*_{\text{max}}.$$  
(A.33)

Now consider the case of $y^*_{\text{min}} \leq y \leq y^*_{\text{max}}$. From the graph for $\mathbf{Y}_n > y$, $X_n$ must satisfy

$$\frac{(2n+1)d}{2} - l < X_n < \frac{(2n+1)d}{2}$$

Hence it is obvious that

$$P\{\mathbf{Y}_n > y\} = \frac{\text{length of the inner interval}}{\text{length of outer interval}} = \frac{l}{d}.$$
Appendix A — Statistics relating to PIV sampling

Considering equation 2.56 we note $g$ maps $\left[\frac{(2n-1)d}{2}, \frac{(2n+1)d}{2}\right]$ one to one and onto $[y_{\text{min}}, y_{\text{max}}]$ and so $g$ has an inverse function $g^{-1}$ which maps $[y_{\text{min}}, y_{\text{max}}]$ onto $\left[\frac{(2n-1)d}{2}, \frac{(2n+1)d}{2}\right]$ and

$$l = g^{-1}(y)$$

Since there is no closed form expression for $g^{-1}(y)$ we use the numerics as described in the Section 2.5.1.1
Appendix B

Mechanical assembly for a PIV illumination system
Appendix B — Mechanical assembly for a PIV illumination system

Figure B.1: Frame plate
Appendix B — Mechanical assembly for a PIV illumination system

Figure B.2: Main plate for optical components
Appendix B — Mechanical assembly for a PIV illumination system

Figure B.3: Rotating mirror holder
Appendix B — Mechanical assembly for a PIV illumination system

Figure B.4: Axle holder plate
Appendix B — Mechanical assembly for a PIV illumination system

Figure B.5: Hinge plate

Hole A: dia. M5 clearance

4 holes type A
Vertical and horizontal pitch between lines of centres = 20.0 mm
Appendix C

Data analysis by PTV

There are two distinct aspects to PTV namely the location of particle images which has been described in Chapter 5 and the linking of these tracers which are now discussed.

C.1 Introduction

In digital correlation analysis the basic method of implementation is quite standard for most PIV applications (using Fourier methods as described in Section 5.4) with perhaps minor differences, for example as to the exact FFT algorithm used.

This is not the case for PTV. A single description of tracking methods is not possible, however, in Section 5.1 a review of tracking methods is given and it emphasizes the large variety of ways that the actual linking of particles can be achieved. We now describe the implementation of PTV used in the current work.

C.2 Choice of scheme

The main factors which influence the choice of linking scheme chosen are: 1) whether the information is recorded as 2-d or 3-d scenes, 2) the mode the data is recorded e.g. multiple exposure single frame, multiple frames with a single
Appendix C — Data analysis by PTV

exposure on each, 3) the density of particle images per frame 4) the amount of descriptor information that can be obtained from a descriptor and 5) computational factors such as speed and memory requirements.

Obtaining 3-d information using a stereo imaging system was unfeasible for the sediment experiments of Chapter 4. The concentration of sediment is far too high to obtain useful results. Also, implementing the laser illumination system of Section 4.2.2 to provide a 3-d sheet of laser light would be exceptionally difficult without affecting the flow regime in an undesirable way. Therefore 2-d images are recorded.

The mode of recording is a single exposure per frame with double images recorded on image pairs. This affected the linking algorithm the most. Normally tracking is performed over multiple frames which allows the use of characteristics of each trajectory formed to be validated using criteria such as linearity. Since particles are tracked over only two frames these criteria cannot be used in our case and so emphasis is placed on individual particle characteristics (descriptors).

C.3 Description of linking algorithm

Figure C.1(a) shows two hypothetical frames consisting of four particle pairs which are labelled AA' through to DD'. Consider two descriptors, $x_1$ and $x_2$ (say, the x coordinate and average image intensity) of these particles. We attempt to link, in turn, each particle in frame 1 with its candidate in frame 2 using the descriptor space. Consider the linking of particle A. Figure C.1(b) shows the relevant descriptor space formation. The distances from A, the linking particle, to B', C' and D', the candidate particles, are calculated and the candidate returning the lowest distance is assumed the matching particle and in this trivial example it is easy to see that A links with A'. This is the basis of the algorithm With
Figure C.1: (a) Two hypothetical frames and (b) descriptor space for linking particle A
Appendix C — Data analysis by PTV

$N$ descriptors, an $N$-dimensional space is formed and it is easier to formulate the problem with vector notation. The $i^{th}$ particle in the first frame is denoted as $\vec{x}_i$ and the $j^{th}$ particle in the second frame is denoted as $\vec{x}_j'$. The Euclidean distance, $d_i$,

$$d_i(\vec{x}) = ||\vec{x}_i - \vec{x}_j'|| \quad j = 1, 2, \ldots, M$$  \hspace{1cm} (C.1)

is used to determine the closeness of the $i^{th}$ particle in the first frame with each of the $M$ particles in the second frame. The pair returning the lowest score are considered matching. It is difficult to perform experiments to determine the best descriptors and so these were determined instead through trail and error. Five descriptors are used and these are: 1) the number of pixels comprising a particle image, 2) the maximum length of a particle, 3) the ratio of maximum length to maximum width, 4) the $x$ coordinate and 5) the $y$ coordinate. The first three of the above descriptors are referred to as shape descriptors.

An exhaustive search of all possible candidate tracers is computationally very expensive and increases the possibility of shape descriptors of incorrect particle pairs matching closely. Instead a feasible search volume about a linking particle within the descriptor space is introduced. This is implemented in a simple fashion by rejecting any candidate particles which fail to satisfy certain criteria. These criteria are chosen from observation of the data. For the $x$ and $y$ coordinates, particles outside a 20 pixel radius of the linking particle’s centroid are rejected. For the shape descriptors, those that vary more than $\pm 40\%$ of the linking particle’s corresponding shape descriptors are rejected. These rejection criteria reduce the number of candidate links enormously with, on average, approximately 4.3 candidate links per linking particle.

However, by examining the raw images and the vectors, incorrect vector matches are still produced by this method in regions of relatively high sediment
Appendix C — Data analysis by PTV

concentration. One of the main drawbacks of method of recording the data in PIV terms is that linking can only be performed over two frames and therefore many of the extra checking procedures used in other tracking algorithms, e.g. linearity, cannot be used.
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