VIBRATIONS OF COUPLED BEAM SYSTEMS
WITH NON-LINEAR INTERACTIONS

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ABSTRACT

This study has been concerned with complex forms of forced oscillatory behaviour in mechanical systems consisting of two coupled beams, arising from non-linear interactions between several modes of vibration. The system in particular comprises a horizontally mounted main cantilever beam capable of vertical in-plane bending motion and an auxiliary beam coupled vertically at the free end of the main beam. This coupled beam is capable of both out-of-plane bending and torsional motion. A finite degree-of-freedom model was formulated to represent the motion of the system. Perturbation analysis of the system equations revealed that if a special condition of internal resonance exists between three system natural frequencies, then a monofrequent excitation of the second bending mode of the main beam in forced resonance can lead to a response involving a combination of bending and torsion of the coupled beam. This form of behaviour is an example of autoparametric resonance. By considering an additional degree-of-freedom in the model, it was found that a second internal resonance condition could be realized simultaneously with the first. This explained the occurrence of a more complicated response of the system, involving simultaneous motion in four modes, namely the first and second bending modes of the main beam, and the fundamental bending and torsional modes of the coupled beam. The first order analysis permitted the investigation of stationary system responses under appropriate detuning conditions. The effects of system damping and external excitation amplitude on the system response were studied. Experimental investigations were conducted for the resonant cases considered. For the case of the single internal resonance, very reasonable agreement was obtained between measured responses and predicted stationary responses of the system.
In the case of the multiple internal resonance, numerical results for the stationary solutions were obtained only for the special case of the undamped system and only a qualitative comparison was possible with the experimental results.
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PRINCIPAL NOTATION

BIBLIOGRAPHY
APPENDIX I: Kinematics of the secondary beam
APPENDIX II: Velocity of the secondary beam mass
APPENDIX III: Deflection form function integrals
APPENDIX IV: Elements of matrix [C]
APPENDIX V: Listing of computer program for obtaining numerical solutions to the four mode system without damping

ACKNOWLEDGEMENTS
CHAPTER 1

Introduction

1.1 General Remarks

The theory of linear vibrations is well documented and represents the motion of physical systems at its simplest level. However under certain conditions, the linear theory becomes invalid and is unable to describe the system behaviour accurately. For instance, some systems can have regions of multi-valued solutions such that the response depends on the initial excitation conditions. This can lead to a discontinuous change in response amplitude manifested as a jump phenomenon. Under forced vibration, other systems display non-synchronous responses, where the frequency of response is some rational fraction or integral multiple of the excitation frequency. Depending on the frequency relationship, the responses are categorized as superharmonic or subharmonic resonances. In multi-degree-of-freedom systems, it is possible for more than one mode of motion to respond, giving rise to combination resonances. Other special systems possess limit cycles resulting in a non-steady response where the amplitude varies with time. It is therefore necessary to refine the mathematical model to include the relevant non-linear effects which are inherent in the system. However, the resulting non-linear equations are generally not amenable to exact analytical solution. In order to obtain any information about the system response some approximate analytical technique will then be required.

The problem under study belongs to the category of auto-parametrically excited systems. This type of vibratory motion is generally distinguished by the presence of product terms in the equations of motion, in the form $\ddot{Y}X$, where the acceleration of the $Y$-
co-ordinate is said to excite the X-co-ordinate parametrically. This phenomenon is closely related to parametric vibrations where instead, the X-co-ordinate is excited by an explicit time-dependent function of the form, $\cos (\omega t) X$.

The behaviour of parametrically excited systems is governed by differential equations with time-dependent coefficients which are generally called Mathieu-Hill equations. The feature of such systems is that the response is dependent upon the combination of the system parameters. In effect, the stability of the system represented in a two-dimensional parameter space is delineated into zones of stable and unstable responses.

1.2 Description of the problem

The mechanical system consists of two beams of uniform rectangular cross-section coupled at right angles as depicted in Figure 1. The main beam AB, referred to as the primary beam, is clamped at one end to a fixed support while the free end is coupled to a smaller beam BC, referred to as the secondary beam.

External excitation is applied to the primary beam at point D and the coupled configuration is free to vibrate in the plane of the paper. This displacement shall be referred to as in-plane or planar motion. The secondary beam is capable of torsional motion and lateral displacement in a direction perpendicular to the in-plane motion, that is, out-of-plane of the paper. These displacements shall be collectively referred to as out-of-plane or nonplanar motion.

A movable mass $m_0$ is attached to the secondary beam to enable a variation of the effective beam inertia and stiffness properties and hence an alteration of the system natural frequencies.

This model arrangement was used by Roberts (1) to study the simple
FIGURE 1 COUPLED BEAM SYSTEM MODEL

(a) Schematic Diagram (Side View)

(b) Planar Displacement, $q_1$

(c) Nonplanar Displacements, $\phi_0$, $u_0$
autoparametric vibration problem with $\omega_1 = 2 \omega_B$, under random excitation; where $\omega_1$ and $\omega_B$ are the fundamental bending natural frequencies of the planar and non-planar motion respectively. Complex instabilities were reported for excitation at frequencies much higher than the fundamental planar bending frequency. Subsequent observations revealed that the presence of torsional motion of the coupled beam appeared to be a significant contributing factor.

With the present configuration, when the primary beam is excited at a frequency in the region of the second bending mode of the planar motion, the coupling point experiences predominantly rotational motion. The secondary beam is then subjected to angular acceleration which acts as a load in the plane of greatest rigidity of the coupled beam. Hence the connection between the second planar bending mode and the coupled beam combined torsion and bending mode was established through the well known combination resonance of a thin beam undergoing parametric excitation.

According to linear theory, excitation at a frequency close to a system natural frequency leads to resonance of the planar motion. Due to the nature of the coupling, under appropriate conditions, non-planar motion of the secondary beam occurs as well. The nonplanar motion in turn reacts on the primary beam thus modifying the planar motion. This effect is referred to as autoparametric coupling and is particularly important under internal resonance conditions. This is when the natural frequencies $\omega_i$ of an N-degree-of-freedom system are commensurable, satisfying a linear relationship of the form,

$$\sum_{i=1}^{N} K_i \omega_i = 0$$

where $K_i$ are integers.
1.3 Literature review

A beam of narrow rectangular cross-section, when subjected to loads in the plane of greatest rigidity can become unstable under certain conditions. The static case is associated with the lateral buckling load and the loss of stability occurs by the transverse bending and torsional displacement of the beam in a direction perpendicular to the plane of the load. This static instability phenomena has been recently studied by Hodges and Peters (2) for the case of the cantilever beam under an end load.

Under dynamical loading, the equivalent problem then becomes one associated with parametric instability as the external periodic excitation appears as an explicit time-varying coefficient within the equations of motion. This instability is manifested as lateral bending-torsional vibrations. Bolotin (3) has treated the principal parametric resonance in a simply-supported beam subjected to periodic end moments, and in a cantilever with an end mass subjected to a periodic load at the free end. More recently, the parametric combination resonance of a cantilever beam undergoing support motion has been considered by Dugundji and Mukhopadhyay (4) and Dokumaci (5).

The fundamental source of parametric vibration study is the text by Bolotin (3). More recent work has been compiled by Evan-Iwanowski (6), Nayfeh and Mook (7) and the series of survey papers by Ibrahim and Barr (8, 9, 10, 11).

There are a number of analytical techniques available for treating non-linear vibration problems. Bolotin (3) uses a Fourier series method of solution whereby the temporal equations of motion are transformed into a system of homogeneous algebraic equations with respect to the Fourier coefficients. The determinant of this system
of equations is then set to zero to provide the equations for the transition curves between stable and unstable regions.

A majority of vibration problems can be posed mathematically as weakly non-linear systems,

\[ \ddot{x} + \omega^2 x = \varepsilon f(x,\dot{x},t) \]

where \( f(x,\dot{x},t) \) is in general a non-linear function with \( \varepsilon \) as a small parameter such that when \( \varepsilon = 0 \), that is in the absence of non-linearities, the system has a known exact solution. For such systems there exists a class of perturbation methods for deriving approximate solutions.

The text by Bogoliubov and Mitropolsky (12) describes the foundation of the asymptotic method which is used by Evan-Iwanowski (6) in analyzing a range of non-linear resonances. Nayfeh (13) gives a comprehensive account of perturbation methods in general. Among these techniques is the method of multiple scales used extensively by Nayfeh and Mook (7) in treating a variety of non-linear oscillation problems. Another useful method is a technique due to Struble (14), which is a combination of the method of averaging and the asymptotic method.

The work of Bolotin (3) deals mainly with the region of principal primary parametric resonance when the excitation frequency is approximately twice the system natural frequency. In extending the theoretical analysis to systems with multi-degree-of-freedom, Hsu (15, 16) adapted Struble's technique to establish the importance of combination resonances, this is,

\[ m_i \Omega = \sum_{i=1}^{N} k_i \omega_i \]
where $\Omega$ is the excitation frequency; $\omega_i$ are the system natural frequencies; $m_i$ and $k_i$ are integers; $N$ is the number of degrees-of-freedom. Later Yamamoto and Saito (17) and Mettler (18, 19) also examined combination resonances in various physical systems using the method of averaging.

Among more recent work in the general analysis of parametrically excited systems, Szemplinska-Stupnicka (20) extended the method of harmonic balance to determine the region of combination resonances by assuming a two term harmonic solution on the stability boundaries. Zajaczkowski (21, 22), Kotera (23) and Takahashi (24, 25, 26) utilized Liapunov-Floquet theory for the numerical evaluation of the characteristic exponents which were then inspected directly for stability. Noah and Hopkins (27) generalized Hill's method to multi-degree-of-freedom systems and proved the convergence of the infinite determinant. The stability of the solutions was then derived from calculation of the roots of the characteristic equation. Hu (28) and Papastavridis (29, 30) applied variational principles to determine the stability transition zones. Midha and Badlani (31) carried out a numerical study of the non-periodic response of the Mathieu-Hill equation.

The Mathieu-Hill equation is of intrinsic interest in many diverse fields of science and engineering. It also arises naturally when considering the stability of stationary solutions in non-linear systems. Some of the recent studies propose alternative schemes of computing approximate stability charts for the equation. Sinha, Chou and Denman (32) approximated the periodic coefficients of the equation by constant, linear and quadratic functions of time to derive the transition curves while Asner (33) used an iterative approach
based on Newton's method. Pedersen (34) considered the damped Mathieu-Hill equation using the Bubnov-Galerkin method. The Hill's equation with three independent parameters was studied by Stanisic (35) using Struble's method while Takahashi (36) used the method of harmonic balance.

The linear analysis of parametrically excited systems predicts unbounded response within the unstable regions. The presence of linear damping only modifies the zones of instability. In physical systems however, the response remains finite and this can be accounted for by including non-linear factors inherent in the system which become significant during large amplitude motions. In this aspect, Hsu (37, 38) considered the effects due to quadratic damping and cubic stiffness non-linearity in a single Mathieu equation. Mukhopadhyay (39) extended the analysis to two coupled Mathieu equations taking into account quadratic damping. Recently Tezak, Nayfeh and Mook (40) presented a general analysis of multi-degree-of-freedom systems with repeated natural frequencies having applications in aerodynamic flutter instabilities.

Parametric instabilities manifests itself in a variety of physical applications. The classical problem is the pinned-pinned beam excited by a periodic axial load. In some recent work, Sato, Saito and Otomi (41) considered the effect of a concentrated mass on a parametrically excited horizontal beam. Saito and Otomi (42) extended the analysis to viscoelastic beams with viscoelastic end supports. Carlson, Lo and Briley (43) discussed the effects of tension and initial curvature on the secondary resonances of a bar under parametric axial excitation.

Besides the application of external loading, a beam can also be parametrically excited by varying its length periodically. This
situation was analyzed theoretically by Zajaczkowski et al \((44, 45, 46, 47)\) in beam and plate elements. Tagata \((48)\) conducted a theoretical and experimental analysis of higher order resonances in a vibrating string. Eisinger and Merchant \((49, 50)\) utilized the effects of parametric vibrations in an application of a clamped-clamped beam as an inertia sensor. Ostiguy and Evan-Iwanowski \((51)\) studied the influence of aspect ratio on the dynamic response of parametrically excited plates while Tani and Nakamura \((52, 53)\) examined parametric resonance in annular plates under multiple loads. Thomas and Abbas \((54)\) noted the possibility of parametrically induced instabilities in off-shore structures. Unger and Brull \((55)\) studied both principal and combination resonances in rotating shafts.

Parametric instabilities can also occur in liquid-structure elements such as boilers, nuclear reactors, heat exchangers and steam generators. In such situations, pipes conveying fluid may be subjected to fluctuations in the fluid flow which act as a parametric load on the pipe itself. Paidoussis, Issid and Tsui \((56, 57)\) presented theoretical and experimental work on such a system and established the occurrence of principal parametric resonance regions. Noah and Hopkins \((58)\) considered the effect of support flexibility on a clamped pipe-fluid system and reported both principal and combination type resonances. In another application Khandelwal and Nigam \((59)\) examined the principal primary and secondary instability regions of a flexible container with liquid under vertical periodic excitation.

The progress in autoparametrically excited systems is less well developed. Minorsky \((60)\) is credited with the introduction of the term 'autoparametric' for describing quadratic inertia coupling in a two-degree-of-freedom system. The interesting feature of auto-
parametrically excited systems is the condition of internal resonance. This arises when \( \omega_i \), the linear natural frequencies of a N-degree-of-freedom system approximately satisfy the linear relationship,

\[
\sum_{i=1}^{N} k_i \omega_i = 0
\]

where \( k_i \) are integers.

Internal resonance can also occur within the context of parametric vibrations and this has been investigated by Asmis and Tso (61), Tso and Asmis (62), and Tezak, Mook and Nayfeh (63) for a two-degree-of-freedom system with cubic non-linearities.

The classical example of an autoparametrically excited system is the elastic pendulum which is capable of both longitudinal extension and swinging pendulum motion. This system has been extensively investigated by Kane and Kahn (64), Tsel'man (65), Van der Burgh (66, 67, 68), Srinivasan and Sankar (69), Roth and Kane (70) and recently by Breitenberger and Mueller (71). They showed that complex behaviour arises in the vicinity of exact internal resonance when the natural frequency of longitudinal motion is approximately twice the natural frequency of pendulum motion. A continuous exchange of energy occurs between the coupled motions resulting in amplitude modulated responses. Sevin (72) and Struble and Heinbockel (73, 74) have also studied a related phenomena in a beam-pendulum system.

Ryland and Meirovitch (75) extended the analysis of the elastic pendulum to include viscous damping and external oscillation of the pendulum support, while Crespo da Silva (76) considered the effect of a constant spin rate of the pendulum support on stabilizing the pendulum motion.

The effect of internal resonance in systems with cubic non-
linearities has also been investigated. These systems have applications in vibrations of beams and circular discs taking into account geometric non-linearity, and vibration isolating suspensions. Henry and Tobias (77, 78) and Henry (79) examined the response of an autonomous conservative two-degree-of-freedom system. Gilchrist (80) discussed the general analysis of such systems by the Krylov-Bogoliubov-Mitropolsky asymptotic method. Efstathiades (81) treated a two-degree-of-freedom system under external forcing and considered both harmonic and subharmonic responses of the externally excited mode. Two mode response in beam elements under harmonic excitation was studied by Nayfeh, Mook and Lobitz (82) for beams of variable thickness, and Nayfeh, Mook and Sridhar (83) for clamped-simply supported beams. Sridhar, Nayfeh and Mook (84) extended the analysis for hinged-clamped beams to include superharmonic, subharmonic and combination resonances. Sridhar, Mook and Nayfeh (85, 86) examined multi-mode responses in forced vibrations of circular plates while Lobitz, Nayfeh and Mook (87) considered elliptic plates.

The theoretical analysis of the effects of autoparametric resonance in non-autonomous systems was initially conducted by Sethna (88) for a two-degree-of-freedom system with one natural frequency twice the other. Recently, Yamamoto et al (89, 90) extended the analysis to include both quadratic and cubic non-linearities, while Yamamoto, Yasuda and Nagasaka (91) studied a system with cubic and fourth order non-linearities and with natural frequencies in the ratio 2:3. Sethna and Bajaj (92) presented an analytical method to determine amplitude modulated motions in a two-degree-of-freedom system with quadratic non-linearities.

In recent physical applications, Haxton and Barr (93), and
Haxton (94) examined a two mode interaction response in an auto-parametric vibration absorber. Barr and Nelson (95) considered a coupled beam system with internal resonance of the type $\omega_1 = 2\omega_j$, as well as a combination type, that is, $\omega_i = \omega_j + \omega_k$, involving three modes. Nayfeh, Mook and Marshall (96) analyzed a two mode interaction behaviour in ships involving a non-linear coupling between pitching and rolling motion. Mook, Marshall and Nayfeh (97) extended the analysis to include both superharmonic and subharmonic resonances. In another application, Ibrahim and Barr (98, 99) investigated a structure-liquid system exhibiting both two and three mode interactions. The combination type internal resonance has also been observed in an excited piezo-electric crystal by Yen and Kronauer (100) and in a spinning mutating plate by Klahs and Ginsberg (101). Hatwal, Mallik and Ghosh (102) has recently examined a two-degree-of-freedom auto-parametric system and determined a second order approximation solution.

Extending the analysis to include additional degrees of freedom, it is possible to generate multiple internal resonance conditions. Barr (103) describes the possibility of cascading interactions whereby one or more modes involved in one internal resonance condition also satisfy another separate internal resonance condition. This effect can propagate to involve multi-modal responses and in this manner, modes not directly excited by the external disturbance can still respond significantly through the non-linear resonance coupling. Ibrahim (104, 105) has studied such multiple internal resonances in a structure-liquid system while Barr and Ashworth (106) examined multi-modal interactions in a fuselage-tail plane assembly model.
1.4 Scope of investigation

The coupled beam system considered is represented as a finite degree-of-freedom model. Using a kinetic analogue method due to Kirchoff, the system kinematics is developed to establish the coupling between the planar and nonplanar motion. The generalized equations of motion are derived up to second order by the method of Lagrange. These equations are then transformed to normal co-ordinates by assuming a two mode Galerkin approximation for the planar motion.

The resulting system of non-linear ordinary coupled equations are analyzed by the method of multiple scales. Two sets of resonance conditions are studied. The first case considers external forced resonance of the second bending mode of the main beam, $\omega = \omega_2$, in the presence of a single internal resonance condition, $\omega_2 = \omega_B + \omega_T$. The second case extends to include a second internal resonance condition, $\omega_1 = 2\omega_B$. $\Omega$ is the frequency of external excitation; $\omega_1$ and $\omega_2$ are the natural frequencies of the first and second modes of the planar motion; $\omega_B$ and $\omega_T$ are natural frequencies of the fundamental bending and torsional modes of the coupled secondary beam.

Stationary solutions to the first order approximation are obtained for the resonance cases considered. The effects of system damping, detuning and force amplitude on the response amplitudes are studied. Comparison with the theoretical predictions are provided from measurements of the experimental model response.
2.1 Kinematics

Consider the coupled beam system in Figure 2.1, which consists of a primary beam OB of length L and a secondary beam of length k, coupled perpendicularly at the common point B. This system is fixed at end O to a clamp support.

A cartesian co-ordinate system X-Y-Z with origin at O is taken as the inertial frame of reference, with the positive Y-axis directed along the undeformed elastic axis of the primary beam. The unit base vectors \( \hat{I}, \hat{J}, \hat{K} \) are directed along the X, Y and Z axes respectively. The primary beam is constrained to move in the Y-Z plane.

A second cartesian co-ordinate system x-y-z with origin at B is defined for the secondary beam. The positive z-axis is directed along the undeformed elastic axis of the secondary beam and the set of unit base vectors \( \hat{x}, \hat{y}, \hat{z} \) are directed along the x, y, z axes respectively. The secondary beam is deemed capable of any general displacement relative to the x-y-z axes.

The component beams comprising the coupled system will be modelled on Euler-Bernoulli beam theory. Effects due to shear deformation and cross-section warping will be neglected. The coupled beam will be treated as having a single discrete mass.

An external dynamic excitation is considered to act on the primary beam in the form of a discrete force at some location as shown in Figure 2.1. Alternative forms of excitation such as a distributed dynamic force or a prescribed motion of the primary beam support could be substituted.
The planar motion of the primary beam relative to the XYZ axes will be specified by the co-ordinates \( q_i \), \( i = 1, 2, \ldots, n \), with \( n \) being the number of degrees of freedom. In particular, \( q_1 \) and \( q_2 \) represent the transverse and rotational displacement of the coupling point respectively. Furthermore, the axial motion of the primary beam is neglected.

The motion of the secondary beam relative to the xyz axes will be specified by the three Cartesian co-ordinates \( u_0, v_0, \) and \( w_0 \), and \( \phi_0 \), the angle of twist of the discrete mass about the elastic axis.

Let \( \vec{R} \) represent the position vector of the coupling point referred to the stationary XYZ axes and \( \vec{r} \) be the position vector of the secondary beam mass \( m_0 \), referred to the moving xyz axes, as depicted in Figure 2.2.

For some arbitrary deformation, the position vectors are given as,

\[
\vec{R} = L \vec{J} + q_1 \vec{K} \quad (2.1.1a)
\]

\[
\vec{r} = u_0 \vec{I} + v_0 \vec{J} + (z + w_0) \vec{K} \quad (2.1.1b)
\]

2.1.1 Constraints

Two equations of constraint are now imposed on the motion of the secondary beam. The first constraint rests on the assumption of inextensibility of the beam elastic axis. This gives rise to a non-linear coupling between the transverse bending and longitudinal displacements. The derivation of this constraint is described in Appendix I and follows from equations (I.3) in the appendix,

\[
w_0 = \int_0^z \frac{1}{2} \left[ (u')^2 + (v')^2 \right] \, dz \quad (2.1.2)
\]
FIGURE 2.1 CO-ORDINATE SYSTEMS OF COUPLED BEAM SYSTEM

FIGURE 2.2 ABSOLUTE DISPLACEMENT VECTOR OF $m_o$
where \( u, v \) are the transverse displacement functions of the elastic axis in the principal directions, and \( (\ )' = \partial / \partial z \).

The second constraint is based on the beam cross-section being thin and slender such that the flexural rigidity in one principal plane is very much greater than the other. The principal curvatures \( \kappa_1, \kappa_2 \) and the torsion \( \tau \) to second order have been derived in Appendix I and follows from equations (I.14 - I.16) in the appendix, these being:

\[
\kappa_1 = u'' \phi - v'' \quad (2.1.3)
\]

\[
\kappa_2 = u'' + v'' \phi \quad (2.1.4)
\]

\[
\tau = \phi'' + u'' v' \quad (2.1.5)
\]

By assuming that the curvature \( \kappa_1 \) in the plane of greatest rigidity is negligible, equation (2.1.3) provides a relationship between the transverse displacements \( u, v \) and the angle of twist \( \phi \).

\[
v'' = u'' \phi \quad (2.1.6)
\]

The deflection at the free end being given by the double integral,

\[
\nu_0 = \int_0^\xi \int_0^\zeta u'' \phi \ dz \ d\zeta \quad (2.1.7)
\]

2.2 Equations of motion

The equations of motion of the coupled beam system will be derived by the method of Lagrange. \( (\cdot)' = d / dt \).

The kinetic energy of the primary beam is given as,
where $m_i$ is the discretized mass associated with the $q_i^{th}$ co-ordinate. The kinetic energy of the secondary beam is given by,

$$T_2 = \frac{1}{2} m_0 \dot{R}_0 \cdot \dot{R}_0$$

$$+ \frac{1}{2} I \Phi_0^2$$

where $(\dot{R}_0 + \dot{R})$ is the absolute velocity of the discrete mass $m_0$ of the secondary beam, $I$ is the polar moment of inertia of $m_0$. The evaluation of the dot product term in (2.2.2) is given in Appendix II. The rotational kinetic energy associated with the bending motion has been neglected in comparison with the translational kinetic energy.

The strain energy of the primary beam is given as,

$$V_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} k_{ij} q_i q_j$$

where $k_{ij}$ are the stiffness coefficients of the primary beam.

The strain energy of the secondary beam is given as,

$$V_2 = \int_{0}^{L} \left( \frac{1}{2} EI_{yy} \kappa_2^2 + \frac{1}{2} GJ \tau^2 \right) dz$$

where $EI_{yy}$ and $GJ$ are the flexural and torsional rigidities of the secondary beam respectively.

The virtual work done by the external force $F$ applied at the $q_k^{th}$ co-ordinate is,

$$\delta W = F \delta q_k$$
The strain energy of the secondary beam given by (2.2.4) in general depends on the deformation of the beam. As the problem under consideration involves only the fundamental modes of bending and torsional vibrations of the secondary beam, the bending mode can be approximately represented by the static deflection curve of a cantilever under an end load, while the torsional mode is assumed to be a linear function of the beam length. Thus the displacement functions will be assumed to be separable in time and space and take the form,

\[ u = f(z) u_0(t) \]  
\[ \phi = g(z) \phi_0(t) \]

with

\[ f(z) = \frac{3}{2} \frac{z^2}{\ell^2} - \frac{1}{2} \frac{z^3}{\ell^3} \]  
\[ g(z) = \frac{z}{\ell} \]

where \( u(t) \), \( \phi_0(t) \) are the normalized displacements of the free end of the secondary beam.

The Lagrangean function \( L \) for the coupled beam system to third order can be written as,

\[ L = (T_1 - V_1) + (T_2 - V_2) \]

\[ = \sum_{i=1}^{n} \frac{1}{2} m_i \ddot{q}_i^2 \]
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} k_{ij} q_i q_j \]

\[ + \frac{1}{2} m_0 \left[ u_o^2 + \dot{q}_1^2 + \dot{q}_2^2 + \ddot{\xi}^2 \right] \]

\[ - 2 \dot{q}_1 \dot{q}_2 \ddot{\xi} \]

\[ - 2 B_3 \dot{q}_1 u_0 \dot{u}_0 \]

\[ - 2 B_4 \dot{q}_2 \ddot{\xi} \left( \dot{u}_0 \phi_0 + u_0 \ddot{\phi}_0 \right) \]

\[ + \frac{1}{2} I \dddot{\phi}_0 - \frac{1}{2} B_1 u_0^2 - \frac{1}{2} B_2 \ddot{\phi}_0^2 \]  \hspace{1cm} (2.2.8)

where \( B_i \), \( i = 1, \ldots, 4 \) are specific integrals of the assumed deflection form functions (2.2.6) and (2.2.7) and are given in Appendix III.

The Lagrange equations associated with the co-ordinates \( u_0 \) and \( \phi_0 \) gives the equations of nonplanar motion as,

\[ m_0 \dddot{u}_0 + B_1 u_0 = m_0 B_3 \dddot{q}_1 u_0 \]

\[ + m_0 B_4 \dddot{q}_2 \phi_0 \]  \hspace{1cm} (2.2.9)

\[ I \dddot{\phi}_0 + B_2 \dddot{\phi}_0 = m_0 B_4 \dddot{q}_2 u_0 \]  \hspace{1cm} (2.2.10)

The term \( \dddot{q}_2 \phi_0 \) in (2.2.9) and \( \dddot{q}_2 u_0 \) in (2.2.10) represents the coupled parametric excitation of the nonplanar motion by purely rotational motion of the coupling point. The term \( \dddot{q}_1 u_0 \) in (2.2.9) represents the direct parametric excitation of the coupled beam bending motion by
the linear displacement of the coupling point.

The Lagrange equations for the co-ordinates $q_i$, $i=1, 2, \ldots, n$, gives the equations of planar motion as:

\[
[m] \{\ddot{q}\} + [k] \{q\} = \{N\}
\]  

(2.2.11)

where $[m]$ and $[k]$ are the mass and stiffness matrices respectively, with $\{\ddot{q}\}$ and $\{q\}$ denoting vectors of acceleration of $q$ and of displacement $q$. $\{N\}$ is a vector consisting of generalized forces as well as non-linear inertia forces which arise from the interactions at the coupling point of the beam system. In particular,

\[
N_1 = m_0 B_3 (\ddot{u}_0 u_0 + \ddot{u}_0^2) \quad (2.2.12a)
\]

\[
N_2 = m_0 B_4 \kappa (\ddot{u}_0 \phi_0 + 2 \dot{u}_0 \phi_0 + u_0 \dot{\phi}_0) \quad (2.2.12b)
\]

\[
N_k = P_0 \cos \Omega t \quad (2.2.12c)
\]

where $P_0 \cos \Omega t$ is the prescribed external periodic excitation acting at the $q_k$-th co-ordinate. The term $N_1$ is the reaction force on the secondary beam due to the secondary beam bending motion. The term $N_2$ represents the reaction moment on the primary beam due to the coupled beam bending-torsional motion.

The planar motion is now assumed to be represented by a two mode Galerkin approximation such that,

\[
q_i = \psi_{i1} \xi_1 + \psi_{i2} \xi_2 \quad (2.2.13)
\]

for $i = 1, \ldots, n$

$\psi_{ij}$ is the $i$-th element of the $j$-th eigenvector of the linearized problem associated with the undamped free vibration of the planar
motion. \( \xi_1 \) and \( \xi_2 \) are the first and second bending modes of the planar motion. Substituting (2.2.13) into the planar equations of motion (2.2.11) and using the orthogonality properties for the eigenvectors yields,

\[
M_j \dddot{\xi}_j + K_j \dddot{\xi}_j = \psi_{1j} m_0 B_3 (\dddot{u}_0 \ u_0 + \dddot{u}_0^2)
+ \psi_{2j} m_0 B_4 \ell (\dddot{u}_0 \ \phi_0 + 2 \dddot{u}_0 \ \phi_0 + \dddot{u}_0 \ \phi_0)
+ \psi_{kj} P_0 \cos \Omega t
\] (2.2.14)

for \( j = 1, 2 \).

where \( M_j \) and \( K_j \) are the generalized mass and stiffness of the \( j \)th mode.

Classical linear modal damping is now assumed and the equations (2.2.9 - 2.2.11) are now expressed in the following non-dimensional form,

\[
\dddot{Y}_1 + \omega_1^2 Y_1 = \epsilon \left[ \rho \ u_{11} \ (\dddot{X}_1 \ X_1 + \dddot{X}_1^2)
+ \nu_{21} \ (\dddot{X}_1 \ X_2 + 2 \dddot{X}_1 \ X_2 + \dddot{X}_1 \ X_2)
- 2 \eta_1 \ \omega_1 \ \dot{Y}_1 + \rho_1 \ \cos \Omega t \right]
\] (2.2.15)

\[
\dddot{Y}_2 + \omega_2^2 Y_2 = \epsilon \left[ \rho \ u_{21} \ (\dddot{X}_1 \ X_1 + \dddot{X}_1^2)
+ \nu_{22} \ (\dddot{X}_1 \ X_2 + 2 \dddot{X}_1 \ X_2 + \dddot{X}_1 \ X_2)
- 2 \eta_2 \ \omega_2 \ \dot{Y}_2 + \rho_2 \ \cos \Omega t \right]
\] (2.2.16)
\[ \ddot{x}_1 + \omega^2_B x_1 = \epsilon \left[ \rho \kappa_{11} \ddot{y}_1 x_1 + \rho \kappa_{12} \ddot{y}_2 x_1 + \kappa_{21} \ddot{y}_1 x_2 + \kappa_{22} \ddot{y}_2 x_2 - 2 \eta_B \omega_B \dot{x}_1 \right] \tag{2.2.17} \]

\[ \ddot{x}_2 + \omega^2_T x_2 = \epsilon \left[ \kappa_{21} \ddot{y}_1 x_1 + \kappa_{22} \ddot{y}_2 x_1 - 2 \eta_T \omega_T \dot{x}_2 \right] \tag{2.2.18} \]

where

\[ y_1 = \frac{\psi_{r1} \xi_1}{L_0} \quad y_2 = \frac{\psi_{r2} \xi_2}{L_0} \]

\[ x_1 = \frac{u_0}{L_0} \quad x_2 = \frac{r \phi_0}{L_0} \]

\[ \epsilon = \frac{B_4 L_0}{r} \quad \rho = \frac{B_3 r}{B_4} \]

\[ \kappa_{11} = \frac{\psi_{11}}{\psi_{r1}} \quad \kappa_{12} = \frac{\psi_{12}}{\psi_{r2}} \quad \kappa_{21} = \frac{\psi_{21}}{\psi_{r1}} \quad \kappa_{22} = \frac{\psi_{22}}{\psi_{r2}} \]

\[ \epsilon \eta_B = \zeta_B \quad \epsilon \eta_T = \zeta_T \quad \epsilon \eta_1 = \xi_1 \quad \epsilon \eta_2 = \xi_2 \]

\[ \mu_{11} = \frac{\psi_{11} \psi_{r1} m_0}{M_1} \quad \mu_{12} = \frac{\psi_{12} \psi_{r2} m_0}{M_2} \]

\[ \mu_{21} = \frac{\psi_{21} \psi_{r1} m_0}{M_1 L_0} \quad \mu_{22} = \frac{\psi_{22} \psi_{r2} m_0}{M_2 L_0} \]
\[ \varepsilon p_1 = \frac{\psi k_1 \psi r_1 p_0}{m_1 l_0}, \quad \varepsilon p_2 = \frac{\psi k_2 \psi r_2 p_0}{m_2 l_0} \]

\( \psi r_j \varepsilon_j \) represents some suitable reference displacement point relating the physical motion to the modal amplitudes.

\( L_0 \) is a characteristic beam dimension used to normalize the displacements.

In anticipation of subsequent analysis, it is expedient at this stage to neglect certain non-resonant terms in the system equations to avoid unnecessary algebra. This is justifiable since only a first order approximation solution is contemplated. Two cases of resonance conditions are considered.

i) \( \Omega = \omega_2, \omega_2 = \omega_B + \omega_T \)

This case considers the interaction of three modes, the second planar bending mode \( Y_2 \) with natural frequency \( \omega_2 \) is in forced resonance at external excitation frequency \( \Omega \). The two nonplanar modes \( X_1 \) and \( X_2 \) with natural frequencies \( \omega_B \) and \( \omega_T \) respectively satisfy an internal resonance relationship with \( Y_2 \). The equation (2.2.15) representing the first planar mode \( Y_1 \) can as such be discarded along with any other terms containing \( Y_1 \) in the remaining equations (2.2.16 - 2.2.18). Furthermore if \( \omega_2 \neq 2 \omega_B \), the terms involving \( \ddot{Y}_2 X_1 \) representing the parametric excitation of \( X_1 \) by \( Y_2 \) and the term \( (\dddot{X}_1 X_1 + \dddot{X}_1) \) in equations (2.2.17, 2.2.18) can also be neglected. This reduces to the following set of three mode system equations,

\[ \dddot{Y}_2 + \omega_B^2 Y_2 = \varepsilon \left[ \mu_{22} (\dddot{X}_1 X_2 + 2 \dot{X}_1 X_2 + X_1 \dddot{X}_2) \right. \]

\[ \left. - 2 n_2 \omega_2 \dot{Y}_2 + p_2 \cos \omega t \right] \quad (2.2.19) \]
\[ \ddot{X}_1 + \omega_B^2 X_1 = \varepsilon \left[ \kappa_{22} \ddot{Y}_2 X_2 - 2 \eta_B \omega_B \dot{X}_1 \right] \quad (2.2.20) \]

\[ \ddot{X}_2 + \omega_T^2 X_2 = \varepsilon \left[ \kappa_{22} \ddot{Y}_2 X_1 - 2 \eta_T \omega_T \dot{X}_2 \right] \quad (2.2.21) \]

The form of interaction can be seen from the above equations. The second planar mode \( Y_2 \) is initially excited into forced resonance. The response of \( Y_2 \) then parametrically excites both the nonplanar modes \( X_1 \) and \( X_2 \) to respond in a combination resonance. This growth of the nonplanar modes generates a reaction moment on the primary system modifying the planar motion and thus completing the auto-parametric coupling.

\section*{ii) \( \Omega = \omega_2, \omega_2 = \omega_B + \omega_T, 2 \omega_B = \omega_1 \)}

This second case involves the interaction of four modes. Three of these modes respond as described in the previous section 2.2 (i). In addition, one of the nonplanar modes, \( X_1 \), satisfies a separate internal resonance relationship with the first planar bending mode \( Y_1 \) with natural frequency \( \omega_1 \). Since only \( Y_2 \) is in forced resonance, the excitation term \( P \cos \Omega t \) in (2.2.15) can be reasonably neglected. If \( \omega_1 \neq \omega_B + \omega_T \), the non-linear inertia term \( \dddot{X}_1 X_2 + 2 \dddot{X}_1 \dot{X}_2 + X_1 \dddot{X}_2 \) in (2.2.15) can be discarded as well. The remaining three equations (2.2.16 - 2.2.18) assume the same form as the three mode equations (2.2.19 - 2.2.21) except that an additional term involving \( Y_1 X_1 \), representing the parametric excitation of \( X_1 \) by \( Y_1 \), needs to be re-inserted into (2.2.20). This then gives the four mode system equations as,

\[ \dddot{Y}_1 + \omega_1^2 Y_1 = \varepsilon \left[ \rho \nu_{11} (X_1 X_1 + X_1^2) - 2 \eta_1 \omega_1 \dot{Y}_1 \right] \quad (2.2.22) \]
\[ \ddot{Y}_2 + \omega_n^2 Y_2 = \epsilon \left[ \mu_{22} \left( \ddot{X}_1 X_2 + 2 \dot{X}_1 \dot{X}_2 + X_1 \ddot{X}_2 \right) - 2 \eta_2 \omega_n Y_2 + P_2 \cos \Omega t \right] \]  
(2.2.23)

\[ \ddot{X}_1 + \omega_B^2 X_1 = \epsilon \left[ \rho \kappa_{11} \ddot{Y}_1 X_1 + \kappa_{22} \ddot{Y}_2 X_2 - 2 \eta_B \omega_B \dot{X}_1 \right] \]  
(2.2.24)

\[ \ddot{X}_2 + \omega_T^2 X_2 = \epsilon \left[ \kappa_{22} \ddot{Y}_2 X_1 - 2 \eta_T \omega_T \dot{X}_2 \right]. \]  
(2.2.25)

The response initially follows the sequence described in the previous section, 2.2 (i). In addition, the parametric excitation of \( X_1 \) by \( Y_2 \) produces an inertial force represented by \( (\dddot{X}_1 X_1 + \dddot{X}_2) \) in (2.2.23). This force acts essentially as an external disturbance to the first planar bending mode \( Y_1 \) thus causing \( Y_1 \) to respond. The growth of \( Y_1 \) in turn parametrically excites \( X_1 \). In this way, all four modes are coupled through the resonance conditions producing a range of complicated responses to the external excitation.
CHAPTER 3

Theoretical Analysis

3.1 Method of solution

The physical problem has been formulated in terms of a set of coupled non-linear ordinary differential equations (2.2.15 - 2.2.18). Exact solutions to such equations are not known. To proceed, it is necessary to resort to an approximate analytical technique.

For weakly non-linear systems of the type,

\[ \ddot{x} + \omega^2 x = \varepsilon f(x,\dot{x},t) \]  

(3.1.1)

with \( \varepsilon \) being a small parameter governing the strength of the non-linearity, there are a number of perturbation schemes for deriving approximate solutions. The solution process used in this study is the method of multiple scales (Nayfeh, ref. 13), which has been widely applied to a number of non-linear oscillatory problems (Nayfeh and Mook, ref. 7).

The procedure begins with the introduction of new time scales according to

\[ T_n = \varepsilon^n t \quad n = 0,1,2, ... \]  

(3.1.2)

such that successive \( T_n \) for increasing \( n \), corresponds to progressively "slower" time scales compared to ordinary time \( t \).

These different "slow" time scales are treated as independent variables and the number of time scales introduced depends in general on the order of approximation to which the solution is required. The solution to (3.1.1) is then formally expressed as an asymptotic series in terms of the small parameter \( \varepsilon \),

\[ x(t;\varepsilon) = x_0(T_0,T_1,...) + \varepsilon x_1(T_0,T_1,...) + ... \]  

(3.1.3)
where \( x_n \) are generally functions of the different time scales to be determined.

The derivatives with respect to \( t \) are then transformed as,

\[
\frac{d}{dt} = \frac{dT_0}{dt} \frac{\partial}{\partial T_0} + \frac{dT_1}{dt} \frac{\partial}{\partial T_1} + \ldots
\]

\[= D_0 + \epsilon D_1 + \ldots \quad (3.1.4 \text{ a})\]

\[
\frac{d^2}{dt^2} = D_0^2 + 2 \epsilon D_0 D_1 + \ldots \quad (3.1.4 \text{ b})
\]

where \( D_0 = \frac{\partial}{\partial T_0} \), \( D_1 = \frac{\partial}{\partial T_1} \)

Substituting (3.1.3) into (3.1.1) and equating coefficients of equal powers of \( \epsilon \), gives the following sequence of equations to first order,

\[
\epsilon^0: D_0^2 x_0 + \omega^2 x_0 = 0 \quad (3.1.5)
\]

\[
\epsilon: D_0^2 x_1 + \omega^2 x_1 = -2 D_1 D_0 x_0 - f(x_0, D_0 x_0, T_0) \quad (3.1.6)
\]

The solution to (3.1.5) gives the generating solution \( x_0 \), which is then used in (3.1.6). To ensure a uniformly valid expansion of the assumed solution (3.1.3), the non-homogeneous terms of (3.1.6) are inspected for secularity. Terms are deemed to be secular depending upon the resonance relationship being satisfied. In order that (3.1.3) produces a bounded solution, it is necessary to eliminate these secular terms by equating them to zero. This provides the
solvability conditions for (3.1.6), which can then be used to
determine the amplitude and phase response.

By considering additional time scales, the above process can be
repeated to achieve a higher order approximate solution. Generally,
the first order solution is sufficient to yield the essential
information about the system behaviour but in some instances, it is
necessary to consider a higher order approximation. This tends to
become very laborious.

3.2 Analysis of the system equations

It is convenient to begin the analysis with the four mode system
equations. As discussed previously, the three mode system is
contained as a subset of the four mode system. Hence the three mode
system response can be considered as a special case of the four mode
system.

The equations representing the four mode system follows from
equations (2.2.22 - 2.2.25)

\[ \dddot{Y}_1 + \omega_1^2 Y_1 = \varepsilon \left[ \rho \mu_1 \left( \dddot{X}_1 + \dddot{X}_2 \right) + 2 \eta_1 \omega_1 \dot{Y}_1 \right] \] (3.2.1 a)

\[ \dddot{Y}_2 + \omega_2^2 Y_2 = \varepsilon \left[ \mu_2 \left( \dddot{X}_1 + 2 \dddot{X}_2 + \dddot{X}_1 + \dddot{X}_2 \right) - 2 \eta_2 \omega_2 \dot{Y}_2 \right. \]
+ \( P \cos \Omega t \) \] (3.2.1 b)

\[ \dddot{X}_1 + \omega_B^2 X_1 = \varepsilon \left[ \rho \kappa_1 \dddot{Y}_1 X_1 + \kappa_2 \dddot{Y}_2 X_2 - 2 \eta_B \omega_B \dot{X}_1 \right] \] (3.2.1 c)

\[ \dddot{X}_2 + \omega_T^2 X_2 = \varepsilon \left[ \kappa_2 \dddot{Y}_2 X_1 - 2 \eta_T \omega_T \dot{X}_2 \right] \] (3.2.1 d)

with the following abbreviations,
\[ \mu_1 = \mu_{11}, \quad \mu_2 = \mu_{22} \]
\[ \kappa_1 = \kappa_{11}, \quad \kappa_2 = \kappa_{22} \]
\[ p = p_2 \]

The solution process will be conducted to the first order approximation. Thus only two time scales will be required. These are defined as,
\[ T_0 = t, \quad T_1 = \varepsilon t \]

The solutions to (3.2.1) are assumed to be represented by the following expansions,
\[ Y_1(t; \varepsilon) = Y_{10}(T_0, T_1) + \varepsilon Y_{11}(T_0, T_1) + O(\varepsilon^2) \quad (3.2.2 \text{ a}) \]
\[ Y_2(t; \varepsilon) = Y_{20}(T_0, T_1) + \varepsilon Y_{21}(T_0, T_1) + O(\varepsilon^2) \quad (3.2.2 \text{ b}) \]
\[ X_1(t; \varepsilon) = X_{10}(T_0, T_1) + \varepsilon X_{11}(T_0, T_1) + O(\varepsilon^2) \quad (3.2.2 \text{ c}) \]
\[ X_2(t; \varepsilon) = X_{20}(T_0, T_1) + \varepsilon X_{21}(T_0, T_1) + O(\varepsilon^2) \quad (3.2.2 \text{ d}) \]

Substituting (3.2.2) into (3.2.1), using (3.1.4), and equating the coefficients of equal powers of \( \varepsilon \) gives the following sequence of equations,
\[ \varepsilon^0 : D_0^2 Y_{10} + \omega_1^2 Y_{10} = 0 \quad (3.2.3 \text{ a}) \]
\[ D_0^2 Y_{20} + \omega_2^2 Y_{20} = 0 \quad (3.2.3 \text{ b}) \]
\[ D_0^2 X_{10} + \omega_B^2 X_{10} = 0 \quad (3.2.3 \text{ c}) \]
\[ D_0^2 X_{20} + \omega_T^2 X_{20} = 0 \]  \hspace{1cm} (3.2.3 d)

\[ \varepsilon : D_0^2 Y_{11} + \omega_1^2 Y_{11} = -2 D_1 D_0 Y_{10} - 2 n_1 \omega_1 D_0 Y_{10} \]

\[ + \rho \mu_1 [X_{10} (D_0^2 X_{10}) + (D_0 X_{10})^2] \]  \hspace{1cm} (3.2.4 a)

\[ D_0^2 Y_{21} + \omega_2^2 Y_{21} = -2 D_1 D_0 Y_{20} - 2 n_2 \omega_2 D_0 Y_{20} \]

\[ + \mu_2 [(D_0^2 X_{10}) X_{20} + 2 (D_0 X_{10}) (D_0 X_{20}) \]

\[ + X_{10} (D_0^2 X_{20})] \]

\[ + \rho \cos \Omega t \]  \hspace{1cm} (3.2.4 b)

\[ D_0^2 X_{11} + \omega_B^2 X_{11} = -2 D_1 D_0 X_{10} - 2 \eta_B \omega_B D_0 X_{10} \]

\[ + \rho \kappa_1 X_{10} (D_0^2 Y_{10}) \]

\[ + \kappa_2 X_{20} (D_0^2 Y_{20}) \]  \hspace{1cm} (3.2.4 c)

\[ D_0^2 X_{21} + \omega_T^2 X_{21} = -2 D_1 D_0 X_{20} - 2 \eta_T \omega_T D_0 X_{20} \]

\[ + \kappa_2 X_{10} (D_0^2 Y_{20}) \]  \hspace{1cm} (3.2.4 d)

The solutions to the zero order equations (3.2.3) in complex form are,

\[ Y_{10} = B_1 (T_1) \exp (i \omega_1 T_0) + \overline{B}_1 (T_1) \exp (-i \omega_1 T_0) \]  \hspace{1cm} (3.2.5 a)
\[ Y_{20} = B_2(T_1) \exp(i \omega_2 T_0) + \overline{B}_2(T_1) \exp(-i \omega_2 T_0) \quad (3.2.5 \text{ b}) \]

\[ X_{10} = A_1(T_1) \exp(i \omega_B T_0) + \overline{A}_1(T_1) \exp(-i \omega_B T_0) \quad (3.2.5 \text{ c}) \]

\[ X_{20} = A_2(T_1) \exp(i \omega_T T_0) + \overline{A}_2(T_1) \exp(-i \omega_T T_0) \quad (3.2.5 \text{ d}) \]

Substituting the zero order solutions (3.2.5) into (3.2.4) gives,

\[ D^2 Y_{11} + \omega^2 Y_{11} = -2i \omega_1 (D_1 B_1 + n_1 \omega_1 B_1) \exp(i \omega_1 T_0) \]

\[ - 2 \rho \nu_1 \omega_B^2 A_1^2 \exp(i 2 \omega_B T_0) + cc \quad (3.2.6 \text{ a}) \]

\[ D^2 Y_{21} + \omega^2 Y_{21} = -2i \omega_2 (D_1 B_2 + n_2 \omega_2 B_2) \exp(i \omega_2 T_0) \]

\[ - \mu_2 (\omega_B + \omega_T)^2 A_1 A_2 \exp \{i (\omega_B + \omega_T) T_0 \} \]

\[ - \mu_2 (\omega_B - \omega_T)^2 \overline{A}_1 A_2 \exp \{i (\omega_T - \omega_B) T_0 \} \]

\[ + \frac{1}{2} \rho \exp(i \Omega T_0) + cc \quad (3.2.6 \text{ b}) \]

\[ D^2 X_{11} + \omega_B^2 X_{11} = -2i \omega_B (D_1 A_1 + n_B \omega_B A_1) \exp(i \omega_B T_0) \]

\[ - \rho \kappa_1 \omega_1^2 A_1 B_1 \exp \{i (\omega_B + \omega_1) T_0 \} \]

\[ - \rho \kappa_1 \omega_1^2 \overline{A}_1 B_1 \exp \{i (\omega_1 - \omega_B) T_0 \} \]

\[ - \kappa_2 \omega_2^2 A_2 B_2 \exp \{i (\omega_T + \omega_2) T_0 \} \]

\[ - \kappa_2 \omega_2^2 \overline{A}_2 B_2 \exp \{i (\omega_2 - \omega_T) T_0 \} + cc \quad (3.2.6 \text{ c}) \]
\[ D_0^2 x_{21} + \omega_T^2 x_{21} = -2 i \omega_T (D_1 A_2 + n_1 \omega_T A_2) \exp (i \omega_T T_0) \]

\[ - \kappa_2 \omega_2 A_1 B_2 \exp \{i (\omega_B + \omega_2) T_0 \} \]

\[ - \kappa_2 \omega_2 A_1 B_2 \exp \{i (\omega_2 - \omega_B) T_0 \} + \text{cc} \] (3.2.6 d)

where \text{cc} denotes the complex conjugates of the preceding terms.

### 3.2.1 Ordinary forced resonance

Before considering the effect of internal resonance, it is convenient at this juncture to establish the ordinary linear forced resonance of the system. An external detuning parameter is defined as,

\[ \Omega = \omega_2 + \varepsilon_\sigma \] (3.2.7)

The solvability conditions from (3.2.4) are;

\[ B_1' + n_1 \omega_1 B_1 = 0 \] (3.2.8 a)

\[ B_2' + n_2 \omega_2 B_2 = \frac{P}{4 i \omega_2} \exp (i \sigma T_1) \] (3.2.8 b)

\[ A_1' + n_B \omega_B A_1 = 0 \] (3.2.8 c)

\[ A_2' + n_T \omega_T A_2 = 0 \] (3.2.8 d)

with \((\cdot)' = D_1\).

The solutions to the above equations (3.2.8) are

\[ B_1 = \frac{1}{2} b_1 \exp (- n_1 \omega_1 T_1) \] (3.2.9 a)
\[
B_2 = \frac{1}{2} b_2 \exp (-\eta_2 \omega_2 T_1) + \frac{P \exp (i \sigma T_1)}{4 i \omega_2 (\eta_2 \omega_2 + i \sigma)} \tag{3.2.9 b}
\]

\[
A_1 = \frac{1}{2} a_1 \exp (-\eta_B \omega_B T_1) \tag{3.2.9 c}
\]

\[
A_2 = \frac{1}{2} a_2 \exp (-\eta_T \omega_T T_1) \tag{3.2.9 d}
\]

The first terms on the right-hand sides of (3.2.9) represents the free vibration response of the corresponding mode. For a system with positive damping, these terms decay to zero as \( t \to \infty \). This leaves the linear forced vibration response of the externally excited mode \( b_2 \) as the stationary response,

\[
Y_2 = \frac{P}{2 \omega_2 \sqrt{(\eta_2 \omega_2)^2 + \sigma^2}} \sin (\Omega t - \theta) \tag{3.2.10}
\]

where \( \theta = \arctan (\sigma / \eta_2 \omega_2) \).

3.3 Three mode interaction

This case considers the response in the presence of an internal resonance condition. Two detuning parameters are defined as

\[
\Omega = \omega_2 + \varepsilon \sigma \tag{3.3.1 a}
\]

\[
\omega_2 = \omega_B + \omega_T + \varepsilon \sigma_1 \tag{3.3.1 b}
\]

The solvability conditions from (3.2.6) are
\[ 2 \, i \, \omega_2 \left( B_2^i + n_2 \, \omega_2 \, B_2 \right) + \mu_2 \left( \omega_B + \omega_T \right)^2 A_1 \, A_2 \exp \left(-i \, \sigma_1 \, T_1 \right) \]

\[ - \frac{1}{2} \, P \exp \left(i \, \sigma \, T_1 \right) = 0 \quad (3.3.2 \, a) \]

\[ 2 \, i \, \omega_B \left( A_1^i + n_B \, \omega_B \, A_1 \right) + \kappa_2 \, \omega_2^2 \, \overline{A}_2 \, B_2 \exp \left(i \, \sigma \, T_1 \right) = 0 \quad (3.3.2 \, b) \]

\[ 2 \, i \, \omega_T \left( A_2^i + n_T \, \omega_T \, A_2 \right) + \kappa_2 \, \omega_2^2 \, A_1 \, B_2 \exp \left(i \, \sigma \, T_1 \right) = 0 \quad (3.3.2 \, c) \]

The amplitude variables are expressed in polar form as

\[ B_2 = \frac{i}{2} \, b_2 \exp \left(i \, \beta_2 \right) \quad (3.3.3 \, a) \]

\[ A_1 = \frac{i}{2} \, a_1 \exp \left(i \, \alpha_1 \right) \quad (3.3.3 \, b) \]

\[ A_2 = \frac{i}{2} \, a_2 \exp \left(i \, \alpha_2 \right) \quad (3.3.3 \, c) \]

Using (3.3.3) in (3.3.2) and equating the real and imaginary parts separately to zero gives,

\[ \omega_2 \, b_2^i + n_2 \omega_2^2 \, b_2 - \frac{i}{2} \, \mu_2 \left( \omega_B + \omega_T \right)^2 \, a_1 \, a_2 \sin \left( \sigma_1 \, T_1 - \alpha_1 - \alpha_2 + \beta_2 \right) \]

\[ - \frac{1}{2} \, P \sin \left( \sigma \, T_1 - \beta_2 \right) = 0 \quad (3.3.4 \, a) \]

\[ \omega_2 \, b_2 \, \beta_2^i - \frac{i}{2} \, \mu_2 \left( \omega_B + \omega_T \right)^2 \, a_1 \, a_2 \cos \left( \sigma_1 \, T_1 - \alpha_1 - \alpha_2 + \beta_2 \right) \]

\[ + \frac{1}{2} \, P \cos \left( \sigma \, T_1 - \beta_2 \right) = 0 \quad (3.3.4 \, b) \]
\[ \omega_B a_1' + \eta_B \omega_B^2 a_1 + \frac{1}{2} \kappa_2 \omega_2^2 a_2 b_2 \sin (\sigma_1 T_1 - \alpha_1 - \alpha_2 + \beta_2) = 0 \] (3.3.4 c)

\[ \omega_B a_1 a_1' - \frac{1}{2} \kappa_2 \omega_2^2 a_2 b_2 \cos (\sigma_1 T_1 - \alpha_1 - \alpha_2 + \beta_2) = 0 \] (3.3.4 d)

\[ \omega_T a_2' + \eta_T \omega_T^2 a_2 + \frac{1}{2} \kappa_2 \omega_2^2 a_1 b_2 \sin (\sigma_1 T_1 - \alpha_1 - \alpha_1 + \beta_2) = 0 \] (3.3.4 e)

\[ \omega_T a_2 a_2' - \frac{1}{2} \kappa_2 \omega_2^2 a_1 b_2 \cos (\sigma_1 T_1 - \alpha_1 - \alpha_2 + \beta_2) = 0 \] (3.3.4 f)

Equations (3.3.4) can be transformed into an autonomous system by introducing two new variables,

\[ \gamma_1 = \sigma T_1 - \beta_2 \] (3.3.5 a)

\[ \gamma_2 = \sigma_1 T_1 - (\alpha_1 + \alpha_2) + \beta_2 \] (3.3.5 b)

From equations (3.3.4) and using (3.3.5),

\[ \omega_2 b_2' = -\eta_2 \omega_2^2 b_2 + \frac{1}{2} \mu_2 (\omega_B + \omega_T)^2 a_1 a_2 \sin \gamma_2 \\
+ \frac{1}{2} P \sin \gamma_1 \] (3.3.6 a)

\[ \omega_2 b_2 \gamma_1' = \omega_2 \sigma b_2 - \frac{1}{2} \mu_2 (\omega_B + \omega_T)^2 a_1 a_2 \cos \gamma_2 + \frac{1}{2} P \cos \gamma_1 \] (3.3.6 b)

\[ \omega_B a_1' = -\eta_B \omega_B^2 a_1 - \frac{1}{2} \kappa_2 \omega_2^2 a_2 b_2 \sin \gamma_2 \] (3.3.6 c)
\[ \omega_T a_2' = -n_T \omega_T^2 a_2 - \frac{1}{4} \kappa_2 \omega_2^2 a_1 b_2 \sin \gamma_2 \quad (3.3.6 \text{ d}) \]

\[ \omega_B \omega_T a_1 a_2 (\gamma_1' + \gamma_2') = \omega_B \omega_T (\sigma + \sigma_1) a_1 a_2 - \frac{1}{4} \kappa_2 \omega_2^2 (\omega_B a_1^2 + \omega_T a_2^2) b_2 \cos \gamma_2 \quad (3.3.6 \text{ e}) \]

The character of the solutions are described by these equations (3.3.6). Of interest are the stationary responses which correspond to the singular points of this set of autonomous equations, obtained by setting the left-hand sides to zero. Although the emphasis of the present study is on stationary solutions, this does not exclude the possible existence of non-stationary responses. Such responses have been observed in experimental tests. Hence setting equations (3.3.6) to zero gives rise to two possibilities,

i) \( a_1, a_2 = 0 ; b_2 \neq 0 \)

This reduces to the case of the linear forced resonance of the externally excited mode \( b_2 \) and has already been discussed in section 3.2.1.

ii) \( a_1, a_2, b_2 \neq 0 \)

This second possibility is considered for two cases. The first case concerns the undamped system response. By setting the damping factors to zero gives the response amplitudes as,

\[ b_2 = \pm 2 \sqrt{\frac{\omega_B \omega_T}{\kappa_2 \omega_2^2}} (\sigma + \sigma_1) \quad (3.3.7 \text{ a}) \]

\[ a_2 = \sqrt{\frac{\omega_B}{\omega_T}} a_1 \quad (3.3.7 \text{ b}) \]
For the general case of a damped system, the following response amplitudes are obtained,

\[ b_2 = 4 \sqrt{\frac{\eta_B \omega_B^2 \eta_T \omega_T^2}{\kappa_2 \omega_2}} \sqrt{1 + \frac{(\sigma + \sigma_1)^2}{(\eta_B \omega_B + \eta_T \omega_T)^2}} \]  
\[ a_2 = \frac{\sqrt{\eta_B \omega_B^2}}{\eta_T \omega_T} a_1 \]  
\[ a_1^2 = \frac{2}{\mu_2 (\omega_B + \omega_T)^2} \sqrt{\frac{\eta_T \omega_T^2}{\eta_B \omega_B^2}} \left[ -L_1^2 \pm \sqrt{p^2 - L_2^2} \right] \]

with

\[ L_1 = \frac{8 \sqrt{\eta_B \omega_B^2 \eta_T \omega_T^2}}{\kappa_2 (\eta_B \omega_B + \eta_T \omega_T) \omega_2} \left[ \eta_2 \omega_2 (\eta_B \omega_B + \eta_T \omega_T) - \sigma (\sigma + \sigma_1) \right] \]

\[ L_2 = \frac{8 \sqrt{\eta_B \omega_B^2 \eta_T \omega_T^2}}{\kappa_2 (\eta_B \omega_B + \eta_T \omega_T) \omega_2} \left[ (\eta_B \omega_B + \eta_T \omega_T) \sigma + \eta_2 \omega_2 (\sigma + \sigma_1) \right] \]
3.3.1 Comments on the theoretical response amplitudes

The overall effect of damping on the response is to produce finite amplitude values within a finite range of external detuning. The undamped response given by (3.3.7) predicts amplitude levels that exist continuously irrespective of external detuning. From here on, all discussion pertains to the response of the damped system.

i) The response of the externally excited mode \( b_2 \), is given by (3.3.8a) and shows some interesting patterns of behaviour. The amplitude level is notably independent of external excitation level. This results in a saturation phenomenon which is also a characteristic feature of autoparametrically excited systems with quadratic nonlinearities satisfying the simpler internal resonance relationship, \( \omega_1 = 2 \omega_2 \), when excited at the higher natural frequency. The response is also unaffected by primary system damping. There is a simple relationship for locating the point of minimum response. The term,

\[
1 + \frac{(\sigma + \sigma_1)^2}{(m_B \omega_B + m_T \omega_T)^2} \geq 1
\]

so that the minimum always occurs when \( \sigma + \sigma_1 = 0 \) or \( \sigma = -\sigma_1 \). This implies the point of minimum response is determined solely by the amount of internal detuning.

ii) The response of the coupled beam modes \( a_1 \) and \( a_2 \) are given by (3.3.8b) and (3.3.8c). Equation (3.3.8b) indicates that the ratio of their response values \( a_2/a_1 \), is governed by two ratios, these being, the ratio of the coupled system damping \( \sqrt{m_B/m_T} \), and the ratio of the coupled system natural frequencies \( \omega_B/\omega_T \). This result reflects a principal feature of combination resonances in
parametrically excited systems.

The roots of equation (3.3.8 c) determine the absolute amplitude values of $a_1$, and indicate that there are prerequisites for the existence of real positive solutions.

(a) For real solutions, $P > L_2$ must always hold.

(b) For two real solutions, $L_1 < 0$ and $P^2 < L_1^2 + L_2^2$.

(c) For one real solution, $P^2 > L_1^2 + L_2^2$.

When there are no real solutions to (3.3.8 c) then the response is given by the linear forced resonance solution of $b_2$, (3.2.10).

These conditions governing the existence of real solutions can be interpreted in $P - \sigma$ parameter space (Figs. 3.0.1 - 3.0.4). These graphs show the locus curves of the above conditions. The curve marked (i) represents $P = L_2$ while, the curve marked (ii) represents $P = \sqrt{L_1^2 + L_2^2}$. The regions of demarcation in parameter space indicating non-trivial solutions are as follows,

I : one real non-trivial solution,

II : two real non-trivial solutions,

III : no solutions.

The dashed vertical lines determine the value of the external detuning parameter $\sigma$ at which $|L_1| = 0$. The response region bounded by these dashed lines can only have at most, one real non-trivial solution.

Under exact internal resonance conditions (Fig. 3.0.1), these regions are located symmetrically about the $\sigma = 0$ axis. By introducing a finite amount of internal detuning, $|\sigma_1| > 0$ (Figs. 3.0.2, 3.0.3), the locus curves are shifted along the $\sigma$-axis. The effect of increasing internal detuning is to raise the minimum threshold value of external excitation required to produce autoparametric resonance. Also the curve $P = \sqrt{L_1^2 + L_2^2}$ loses its symmetry about the $\sigma = 0$ axis, giving rise to an asymmetric amplitude
frequency response curve, characteristic of an internally detuned system.

When primary system damping $n_2$ is decreased, a notable qualitative change in behaviour occurs (Fig. 3.0.4). The part of the curve (ii) that lies in between the vertical dashed lines, recedes along the $P$-axis. This creates an interesting change in the response region. For values of force amplitude, $P_B > P > P_A$, that lie within this region, two disjointed regions of non-trivial responses exist and are separated by a region of no response.

For the undamped system, a similar interpretation follows from equation (3.3.7 c). The condition for the existence of two non-trivial solutions is given as, $P = |L_3|$ where

$$L_3 = 4 \sqrt{\frac{\omega_B \omega_T}{\kappa_2 \omega_2}} \sigma (\sigma + \sigma_1)$$

For a system with exact internal resonance (Fig. 3.0.5) the regions of demarcation are,

I : one non-trivial solution,

II : two non-trivial solutions.

The curve marked (i) represents the condition $P = |L_3|$. This graph shows the existence of at least one non-trivial solution over the entire $P - \sigma$ parameter space. However, for a system with a finite amount of internal detuning, $|\sigma_1| > 0$ (Fig. 3.0.6), the locus curve (i) is shifted along the $\sigma$-axis, and an additional region of no non-trivial solution (marked III) appears.

The condition $P^2 = L_1^2 + L_2^2$ has a special significance as this represents the locus of the points of the lower vertical tangency of $a_1$ and $a_2$. It can be demonstrated that the points of intersection of the non-linear response of $b_2$ (eqn. 3.3.8) with the linear
resonance solution of $b_2$ (eqn. 3.2.10) does in fact reduce to the above condition, $P_2^2 = L_1^2 + L_2^2$. This result is independent of internal detuning with the implication that the transition to autoparametric resonance occurs only by a jump response of the parametrically excited modes $a_1$ and $a_2$. Hence the only discontinuous finite change in amplitude response experienced by $b_2$ is when the autoparametric instability ceases, this being at the point of upper vertical tangency of $a_1$ and $a_2$.

This behaviour is in contrast with a previous investigation by Haxton (94), who considered a two-degree-of-freedom autoparametric system. In that study, using Struble's method, it was indicated that 'the points of entry' (that is, the onset of autoparametric resonance) from the linear forced solution of the externally excited mode to the non-linear response, only coincided with the point of lower vertical tangency of the parametrically excited mode for the system with exact internal resonance.

Such a difference in qualitative prediction illustrates an inherent characteristic of the method of multiple scales which can be traced to the definition of the detuning parameter. With Struble's technique, the detuning expression is taken to be the difference of squares of the frequencies, i.e. $\Omega^2 - \omega^2$, whereas the method of multiple scales just takes the difference in frequency, i.e. $\Omega - \omega$. The former definition may be considered 'exact' since this is the form in which detuning is expressed in the exact solution of the forced vibration of a linear system. The solution by the method of multiple time scales for a linear vibrating system gives an amplitude curve which is symmetrical about the point of peak response. When compared to the exact solution, the multiple scale solution predicts lower amplitudes at excitation frequencies less than the resonant
FIGURE 3.0.1
REGIONS OF DEMARCATION IN PARAMETER SPACE INDICATING STATIONARY NONTRIVIAL SOLUTIONS
I: ONE SOLUTION, II: TWO SOLUTIONS
III: NO SOLUTION, $\sigma_1 = 0, \eta_b = \eta_t = 0.06, \eta_2 = 0.3$
FIGURE 3.0.2
REGIONS OF DEMARCATION IN PARAMETER SPACE
INDICATING STATIONARY NONTRIVIAL SOLUTIONS
I: ONE SOLUTION, II: TWO SOLUTIONS
III: NO SOLUTION, \( \sigma_1 = 10, \eta_s = \eta_t = 0.06, \eta_2 = 0.3 \)
FIGURE 3.0.3
REGIONS OF DEMARCATION IN PARAMETER SPACE
INDICATING STATIONARY NONTRIVIAL SOLUTIONS
I: ONE SOLUTION, II: TWO SOLUTIONS
III: NO SOLUTION, $\sigma_1 = 30$, $\eta_b = \eta_t = 0.06$, $\eta_2 = 0.3$
FIGURE 3.0.4
REGIONS OF DEMARCATION IN PARAMETER SPACE
INDICATING STATIONARY NONTRIVIAL SOLUTIONS
I: ONE SOLUTION, II: TWO SOLUTIONS
III: NO SOLUTION, $\sigma_1 = 30$, $\eta_B = \eta_T = 0.06$, $\eta_2 = 0.1$
FIGURE 3.0.5
REGIONS OF DEMARCATION IN PARAMETER SPACE
INDICATING STATIONARY NONTRIVIAL SOLUTIONS
I: ONE SOLUTION, II: TWO SOLUTIONS
\( \sigma_1 = 0, \eta_b = \eta_\tau = \eta_2 = 0 \)
FIGURE 3.0.6
REGIONS OF DEMARCATION IN PARAMETER SPACE
INDICATING STATIONARY NONTRIVIAL SOLUTIONS
I: ONE SOLUTION, II: TWO SOLUTIONS
III: NO SOLUTION, $\sigma_1 = 10$, $\eta_0 = \eta_1 = \eta_2 = 0$
frequency but conversely higher amplitude values at excitation frequencies higher than the resonant frequency.

Thus, the multiple scale method can only be a good approximation for sufficiently small values of detuning. It is for this reason that that method is unable to predict the qualitative changes in response in an internally detuned system as is possible with Struble's method.

3.3.2 Discussion of the theoretical response graphs

Typical three mode interaction response curves are given in Figures 3.1.1 - 3.1.4. The response of the system can be traced by following the response curves as both the external detuning parameter, $\sigma$, and external force amplitude, $P$, are varied independently.

i) Varying external detuning (Figures 3.1.1 - 3.1.3)

Force amplitude is held constant and the case of a system with exact internal resonance ($\sigma_1 = 0$) is considered first. Starting with excitation at point A (Fig. 3.1.1), the externally excited mode $b_2$ is initially in forced resonance with no response for either coupled system modes. As external detuning $\sigma$ is increased and reaches point B, the zero solution of the parametrically excited modes, $a_2$ and $a_1$ (Figs. 3.1.2, 3.1.3) become unstable, and a jump in response occurs from B to B'. At the same time the linear forced response of $b_2$ is also no longer stable with the response following the non-linear response curve BFC. It is noted that point B is exactly coincident with both the linear forced resonance curve and the non-linear response curve of $b_2$. With increasing external detuning, the response curves of all three modes decrease to a minimum value at $\sigma = 0$, beyond which the response increases again until point C, where real solutions to $a_2$ and $a_1$ cease to exist. Here a downward jump in response occurs with $b_2$ returning to the linear forced response and
the $a_1$ and $a_2$ modes returning to the zero solution. For an internally tuned system, the response curves are symmetric about the $\sigma = 0$ axis. Therefore, a reversal of the response pattern occurs with initial excitation frequency at point E and decreasing external detuning. The response of $b$ follows EFBGA while, $a_1$ and $a_2$ follows EFF'G'A.

Introducing a finite amount of internal detuning causes an asymmetry of the response curves as shown by NJMK for $b_2$ (Fig. 3.1.1) and JNJ'M'KM for $a_2$ and $a_1$ (Figs. 3.1.2, 3.1.3). The response sequence follows in a similar order as indicated by the arrows.

ii) Varying external force amplitude (Figures 3.1.4 - 3.1.6)

Here external detuning is held constant. Starting at point A (Fig. 3.1.4), and increasing the force amplitude $P$, only $b_2$ responds according to its linear resonance solution. On reaching point B, the zero solutions of $a_2$ and $a_1$ (Figs. 3.1.5, 3.1.6) are unstable and a jump in response occurs from B to B'. The linear response of the $b_2$ mode BH, is unstable and the response follows BC instead, while $a_2$ and $a_1$ follows B'C'. At points C and C', if the force amplitude is decreased, the response retraces the BC and B'C' curves and continue to points D and D', beyond which there are no real solutions for $a_2$ and $a_1$. A downward jump occurs with $b_2$ resuming the linear forced response while $a_2$ and $a_1$ return to their zero response solution. The saturation phenomenon of the non-linear $b_2$ mode response as discussed earlier is clearly identifiable in Figure 3.1.4.

Under exact external resonance, the response changes qualitatively. The threshold force amplitude at which autoparametric resonance begins is reduced. There is now no discontinuous change in $a_2$ and $a_1$, but instead the response follows a gradual increase in level when $P$ is increased according to FG (Figs. 3.1.4). At point G when P is decreased, the modes retrace the same paths identically.
FIGURE 3.1.1
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 15, \( \eta_b = \eta_1 = 0.05, \eta_2 = 0.2 \)
FIGURE 3.1.2
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 15, $\eta_8 = \eta_I = 0.05$, $\eta_2 = 0.2$
FIGURE 3.1.3
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 15, \( \eta_b = \eta_t = 0.05 \), \( \eta_2 = 0.2 \)
FIGURE 3.1.4
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma_1 = 0$, $\eta_b = 0.1$, $\eta_r = 0.1$, $\eta_2 = 0.05$
FIGURE 3.1.5
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma_1 = 0, \ \eta_B = 0.1, \ \eta_I = 0.1, \ \eta_2 = 0.05 \)
FIGURE 3.1.6
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma_1 = 0$, $\eta_B = 0.1$, $\eta_T = 0.1$, $\eta_2 = 0.05$
3.3.3 Effect of parameter variation on system response

The effect of varying system parameters on the response amplitudes is now discussed with reference to Figures 3.2.1 - 3.14.3. The response amplitudes of the three modes are each drawn separately on individual graphs for a variation of a particular parameter while the remaining parameters are held constant. The stable solutions are shown as continuous lines while the unstable solutions are shown as broken lines. The unstable regions of the linear forced resonance solution of $b_2$ has been omitted.

i) Response amplitude against external detuning; exact internal resonance

a) Varying force amplitude, $P$ (Figs. 3.2.1 - 3.2.3)

As previously noted, the non-linear response of $b_2$ is independent of $P$ and only the curve corresponding to $P = 20$ is shown. Increasing $P$ widens the region of autoparametric resonance and increases the overall amplitude levels of $a_2$ and $a_1$.

b) Varying internal detuning, $\sigma_1$ (Figs. 3.2.4 - 3.2.6)

The linear forced solution is necessarily independent of internal detuning and only the stable region corresponding to $\sigma_1 = 30$ is shown. Increasing $\sigma_1$ brings about,

1. a displacement of the resonance region along the $\sigma$-axis, in particular the point of minimum response of $b_2$ is shifted by $-\sigma_1$, as discussed in section 3.3.1 (i),
2. a general reduction of the width of the resonance region,
3. a loss of symmetry of the response of $a_2$ and $a_1$.
4. a lowering in the overall amplitudes of $a_2$ and $a_1$.

Response curves IV, V (Figs. 3.2.5, 3.2.6) are shown as two discontinuous branches of non-trivial solutions separated by a region of zero response. This possibility was discussed in section 3.3.1 (i).
c) Varying secondary system damping, $n_B$ and $n_T$ (Figs. 3.3.1 - 3.3.3)

Only the stable linear forced response of $b_2$ for $n_B = 0.01$ is shown. The effect of varying either damping parameter, $n_B$ or $n_T$, is qualitively identical. Here, $n_B$ is varied. Increasing $n_B$ leads to,

1. a reduction in the region of resonance,
2. an increase in the overall non-linear response of $b_2$,
3. a decrease in the overall amplitude of $a_1$, but increases that of $a_2$.

This reciprocal effect on $a_2$ and $a_1$ is a consequence of equation (3.3.8 b), that is,

$$\frac{a_2}{a_1} = \sqrt{\frac{n_B \omega_B^2}{n_T \omega_T^2}}$$


d) Varying primary system damping, $n_2$ (Figs. 3.4.1 - 3.4.3)

The non-linear response of $b_2$ is also independent of $n_2$, only the response corresponding to $n_2 = 0.05$ is shown. Increasing $n_2$

1. reduces the linear forced response of $b_2$ and hence reduces the region of autoparametric resonance.
2. reduces the overall amplitudes of both $a_2$ and $a_1$.

ii) Response amplitude against external detuning; internally detuned system

The response curves for a detuned system are presented in Figs. 3.5.1 - 3.8.3 in an identical format. The effect of varying system parameters $p$, $n_B$, $n_T$, $n_2$ are indicated. The overall effects are similar to the internally tuned system except here there is an asymmetry of the response curves as seen from the difference in maximum levels of the two 'overhanging' branches of $a_2$ and $a_1$, which for an internally tuned system are identical.
iii) **Response amplitude against force amplitude, P; exact external and internal resonance**

a) Varying secondary system damping, $\eta_B$, $\eta_T$ (Figs. 3.9.1 - 3.10.3)

Only the stable linear response of $b_2$ corresponding to $\eta_B = 0.01$ are shown. Increasing $\eta_B$ (Figs. 3.9.1 - 3.9.3) leads to an increase in the threshold value of $P$ at which autoparametric resonance begins. The finite response of $a_2$ and $a_1$ are noted to be single valued over the $P$-axis, so no jump phenomenon is expected. It is seen that the amplitude levels of $a_2$ increase with $\eta_B$ while $a_1$ decreases. The converse occurs when $\eta_T$ is increased (Figs. 3.10.2 - 3.10.3).

b) Varying primary system damping, $\eta_2$ (Figs. 3.11.1 - 3.11.3)

Again the non-linear response of $b_2$ is independent of $\eta_2$ and only the stable region corresponding to $\eta_2 = 0.05$ is shown. The effect of increasing $\eta_2$ also increases the threshold value of $P$ at which autoparametric resonance begins. The overall response of both $a_2$ and $a_1$ decrease as $\eta_2$ is increased.

iv) **Response amplitude against force amplitude, P; externally detuned; exact internal resonance** (Figs. 3.12.1 - 3.14.3)

The effect of external detuning is to produce regions of multi-valued responses in $a_2$ and $a_1$ resulting in discontinuous changes in amplitude levels. This causes the characteristic jump phenomenon associated with the point of vertical tangencies of the 'overhanging' branch in the response curves. The effects of varying system damping parameters $\eta_B$, $\eta_T$, $\eta_2$ are qualitatively similar to the system in exact external resonance.
FIGURE 3.2.1
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$\sigma_1 = 0$, $\eta_3 = \eta_4 = \eta_5 = 0.1$
FIGURE 3.2.2
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\( \sigma_1 = 0 \), \( \eta_b = \eta_r = \eta_2 = 0.1 \)
FIGURE 3.2.3
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$\sigma_1 = 0 \ , \ \eta_8 = \eta_1 = \eta_2 = 0.1$
FIGURE 3.2.4
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\( P = 10, \eta_B = \eta_T = \eta_2 = 0.1 \)
FIGURE 3.2.5
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 10 , η_b = η_r = η_2 = 0.1
FIGURE 3.2.6
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 10, \eta_b = \eta_t = \eta_z = 0.1$
FIGURE 3.3.1
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\( P = 4, \sigma_1 = 0, \eta_T = \eta_2 = 0.1 \)
FIGURE 3.3.2
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 4, \sigma_1 = 0, \eta_1 = \eta_2 = 0.1$
FIGURE 3.3.3
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 4, \alpha_1 = 0, \eta_1 = \eta_2 = 0.1$
Figure 3.4.1
Theoretical Three Mode System Response
Planar Amplitude against External Detuning
$P = 10, \sigma_i = 0, \eta_b = \eta_T = 0.1$
\[ a_2 \]

\[ \begin{align*}
\eta_2 & : 0.05 \\
\text{I} & : 0.2 \\
\text{II} & : 0.4 \\
\text{III} & : 0.6 \\
\text{IV} & : 0.8 \\
\end{align*} \]

**FIGURE 3.4.2**
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 10, \( \sigma_i = 0 \), \( \eta_b = \eta_t = 0.1 \)
FIGURE 3.4.3
THEORETICAL THREE MODE SYSTEM RESPONSE NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 10, \sigma_i = 0, \eta_b = \eta_T = 0.1$
FIGURE 3.5.1
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$\sigma_1 = 30$, $\eta_b = \eta_t = 0.06$, $\eta_2 = 0.1$
FIGURE 3.5.2
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
σ₁ = 30 , \( \eta_0 = \eta_t = 0.06 \), \( \eta_2 = 0.1 \)
FIGURE 3.5.3
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\( \sigma_1 = 30 \), \( \eta_b = \eta_1 = 0.06 \), \( \eta_2 = 0.1 \)
FIGURE 3.6.1
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 20, \( \sigma_1 = 10 \), \( \eta_1 = 0.05 \), \( \eta_2 = 0.4 \)
FIGURE 3.6.2
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\( P = 20 \), \( \sigma_1 = 10 \), \( \eta_1 = 0.05 \), \( \eta_2 = 0.4 \)
FIGURE 3.6.3
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 20, \( \sigma_i = 10 \), \( \eta_1 = 0.05 \), \( \eta_2 = 0.4 \)
FIGURE 3.7.1
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\(P = 20, \sigma_1 = 10, \eta_b = 0.05, \eta_2 = 0.4\)
FIGURE 3.7.2
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 20, \( \sigma_i = 10 \), \( \eta_b = 0.05 \), \( \eta_2 = 0.4 \)
FIGURE 3.7.3
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 20 , \sigma_1 = 10 , \eta_b = 0.05 , \eta_2 = 0.4
Figure 3.8.1
Theoretical three mode system response
Planar amplitude against external detuning

\( \eta_2 \)

I : 0.05
II : 0.2
III : 0.4
IV : 0.6
V : 0.8

P = 20 , \( \sigma_i = 10 \) , \( \eta_b = \eta_t = 0.05 \)
FIGURE 3.8.2
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 20 , $\sigma_i = 10$ , $\eta_a = \eta_t = 0.05$
FIGURE 3.8.3
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\( P = 20 \), \( \sigma_1 = 10 \), \( \eta_b = \eta_1 = 0.05 \)
FIGURE 3.9.1
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma = \sigma_1 = 0, \eta_T = 0.1, \eta_2 = 0.7$
FIGURE 3.9.2
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma = \sigma_1 = 0$, $\eta_1 = 0.1$, $\eta_2 = 0.7$
FIGURE 3.9.3
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\[ \sigma = \sigma_1 = 0, \quad \eta_1 = 0.1, \quad \eta_2 = 0.7 \]
FIGURE 3.10.1
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = \sigma_1 = 0, \eta_b = 0.1, \eta_2 = 0.7 \)
FIGURE 3.10.2
THEORETICAL THREE MODE SYSTEM RESPONSE NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = \sigma_1 = 0, \ \eta_b = 0.1, \ \eta_2 = 0.7 \)
FIGURE 3.10.3
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = \sigma_1 = 0 \), \( \eta_3 = 0.1 \), \( \eta_2 = 0.7 \)
FIGURE 3.11.1
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = \sigma_t = 0 , \ \eta_b = \eta_t = 0.1 \)
FIGURE 3.11.2
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = \sigma_t = 0 \), \( \eta_b = \eta_t = 0.1 \)
FIGURE 3.11.3
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = \sigma_i = 0, \quad \eta_b = \eta_t = 0.1 \)
FIGURE 3.12.1
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 30 \), \( \sigma_i = 0 \), \( \eta_T = 0.1 \), \( \eta_2 = 0.05 \)
FIGURE 3.12.2
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 30, \ \sigma_i = 0, \ \eta_1 = 0.1, \ \eta_2 = 0.05 \)
FIGURE 3.12.3
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 30, \ \sigma_i = 0, \ \eta_1 = 0.1, \ \eta_2 = 0.05 \)
FIGURE 3.13.1
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma = 20$, $\sigma_1 = 0$, $\eta_8 = 0.1$, $\eta_2 = 0.05$
FIGURE 3.13.2
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 20 \), \( \sigma_1 = 0 \), \( \eta_b = 0.1 \), \( \eta_2 = 0.05 \)
FIGURE 3.13.3
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
σ = 20 , σ₁ = 0 , η₈ = 0.1 , η₂ = 0.05
FIGURE 3.14.1
THEORETICAL THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\[ \sigma = 20, \ \sigma_r = 0, \ \eta_s = \eta_r = 0.1 \]
Figure 3.14.2
Theoretical three mode system response
Nonplanar amplitude against force amplitude

$\sigma = 20$, $\alpha = 0$, $\eta_b = \eta = 0.1$
FIGURE 3.14.3
THEORETICAL THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 20 , \sigma_i = 0 , \eta_b = \eta_t = 0.1 \)
3.4 Stability of solutions

The stability of the stationary solutions can be determined by adding an infinitesimal disturbance to the solutions,

\[ a_1 = a_{10} + \delta a_1, \quad a_2 = a_{20} + \delta a_2 \]
\[ b_2 = b_{20} + \delta b_2 \]
\[ \gamma_1 = \gamma_{10} + \delta \gamma_1, \quad \gamma_2 = \gamma_{20} + \delta \gamma_2 \]  \hspace{1cm} (3.4.1)

where the terms on the right-hand side are the steady-state solutions and the perturbations respectively. After substituting (3.4.1) into (3.3.6) and retaining only the linear terms in the perturbations, the following set of coupled, first order, linear differential equations with constant coefficients are obtained

\[ \{\delta V\}' = [C] \{\delta V\} \]  \hspace{1cm} (3.4.2)

where \( \{\delta V\} = [\delta a_1 \ \delta a_2 \ \delta b_2 \ \delta \gamma_1 \ \delta \gamma_2]^T \) and the elements of the matrix \([C]\) are given in Appendix IV. The solutions to this set of equations have the general form,

\[ \delta (\tau) = K \exp(\lambda T_1) \]  \hspace{1cm} (3.4.3)

with \( K \) and \( \lambda \) being constants.

Using (3.4.3) in (3.4.2), the equations governing the perturbations are transformed into an eigenvalue problem for \( \lambda \). The stability of the stationary solutions are then determined by the sign of the real part of any of the eigenvalues, with positive indicating instability. No attempt was made to obtain an analytical solution by evaluating the characteristic polynomial due to the relatively high order of the determinant. Instead, the eigenvalues were computed numerically for sets of sample solutions.
3.5 Four mode interaction

This case follows as an extension of the three mode interaction by admitting a second internal resonance condition. This second internal resonance arises from the non-linear inertial effect of the coupled beam bending motion. The non-linear inertial force provides a vertical disturbance load acting at the free end of the primary beam and is conducive to exciting the fundamental bending mode of the primary beam. Three detuning parameters are now defined as,

\[ \Omega = \omega_2 + \epsilon \sigma \]  

\[ \omega_2 = \omega_B + \omega_T + \epsilon \sigma_1 \]  

\[ \omega_1 = 2 \omega_B + \epsilon \sigma_2 \]  

The solvability conditions from (3.2.6) are,

\[ i \omega_1 (D_1 B_1 + n_1 \omega_1 B_1) + \rho \mu_1 \omega_B^2 A_1 \exp (-i \sigma_2 T_1) = 0 \]  

\[ 2 i \omega_2 (D_1 B_2 + n_2 \omega_2 B_2) + \nu_2 (\omega_B + \omega_T)^2 A_1 A_2 \exp (-i \sigma T_1) \]

\[ - \frac{i}{2} P \exp (i \sigma T_1) = 0 \]  

\[ 2 i \omega_B (D_1 A_1 + \eta_B \omega_B A_1) + \rho \kappa_1 \omega_1^2 A_1 B_1 \exp (i \sigma_2 T_1) \]

\[ - \kappa_2 \omega_2^2 A_2 B_2 \exp (i \sigma T_1) = 0 \]  

\[ 2 i \omega_T (D_1 A_2 + \eta_T \omega_T A_2) + \kappa_2 \omega_2^2 A_1 B_2 \exp (i \sigma T_1) = 0 \]  

Rewriting the amplitude variables in polar form as,
\[ B_n = \frac{1}{2} b_n \exp(i \beta_n) \quad (3.5.3a) \]

\[ A_n = \frac{1}{2} a_n \exp(i \alpha_n) \quad (3.5.3b) \]

where \( n = 1, 2 \).

Substituting (3.5.3) into (3.5.2) and equating the real and imaginary parts separately to zero gives,

\[ \omega_1 b_1^i + \omega_2^2 b_1 - \frac{1}{2} \rho \nu_1 \omega_B^2 a_1^2 \sin (\sigma_1 T_1 + \beta_1 - 2 \alpha_1) = 0 \quad (3.5.4a) \]

\[ \omega_1 b_1^i \beta_1^i - \frac{1}{2} \rho \nu_1 \omega_B^2 a_1^2 \cos (\sigma_1 T_1 + \beta_1 - 2 \alpha_1) = 0 \quad (3.5.4b) \]

\[ \omega_2 b_2^i + \nu_2 \omega_2^2 b_2 - \frac{1}{4} \nu_2 (\omega_B + \omega_T)^2 a_1 a_2 \sin (\sigma_1 T_1 + \beta_1 - 2 \alpha_1) \]
\[ \quad - \frac{1}{2} \rho \sin (\sigma T_1 - \beta_2) = 0 \quad (3.5.4c) \]

\[ \omega_2 b_2^i \beta_2^i - \frac{1}{4} \nu_2 (\omega_B + \omega_T)^2 a_1 a_2 \cos (\sigma_1 T_1 + \beta_1 - 2 \alpha_1) \]
\[ \quad + \frac{1}{2} \rho \cos (\sigma T_1 - \beta_2) = 0 \quad (3.5.4d) \]

\[ \omega_B a_1^i + \omega_B^2 a_1 + \frac{1}{2} \rho \kappa_1 \omega_B^2 a_1 b_1 \sin (\sigma_2 T_1 + \beta_1 - 2 \alpha_1) \]
\[ \quad + \frac{1}{4} \kappa_2 \omega_2^2 a_2 b_2 \sin (\sigma_1 T_1 + \beta_2 - \alpha_1 - \alpha_2) = 0 \quad (3.5.4e) \]

\[ \omega_B a_1^i \alpha_1^i - \frac{1}{2} \rho \kappa_1 \omega_1^2 a_1 b_1 \cos (\sigma_2 T_1 + \beta_1 - 2 \alpha_1) \]
\[ \quad - \frac{1}{2} \kappa_2 \omega_2^2 a_2 b_2 \cos (\sigma_1 T_1 + \beta_2 - \alpha_1 - \alpha_2) = 0 \quad (3.5.4f) \]
\[
\omega_T \frac{a_2}{\omega_T} + \eta_T \omega_T^2 \frac{a_2}{\omega_T} + \frac{1}{2} \kappa_2 \omega_T^2 \frac{a_1}{\omega_T} \frac{b_2}{\omega_T} \sin (\sigma_1 T_1 + \beta_2 - \alpha_1 - \alpha_2) = 0
\]  
(3.5.4 g)

\[
\omega_T a_2 \frac{a_2}{\omega_T} - \frac{1}{2} \kappa_2 \omega_T^2 a_1 b_2 \cos (\sigma_1 T_1 + \beta_2 - \alpha_1 - \alpha_2) = 0
\]  
(3.5.4 h)

These equations are now transformed into an autonomous system by introducing three new variables,

\[\gamma_1 = \sigma_1 T_1 - \beta_2\]  
(3.5.5 a)

\[\gamma_2 = \sigma_1 T_1 + \beta_2 - \alpha_1 - \alpha_2\]  
(3.5.5 b)

\[\gamma_3 = \sigma_2 T_1 + \beta_1 - 2 \alpha_1\]  
(3.5.5 c)

From (3.5.4) and using (3.5.5)

\[
\omega_1 b_1' = - \eta_1 \omega_1^2 b_1 + \frac{1}{2} \rho \mu_1 \omega_B^2 a_1 \sin \gamma_3
\]  
(3.5.6 a)

\[
\omega_B \omega_1 a_1 b_1 \gamma_3' = \omega_B \omega_1 \sigma_2 a_1 b_1 - \frac{1}{2} \rho \kappa_1 \omega_B^3 a_1 b_1^2 \cos \gamma_3
\]  
(3.5.6 b)

\[
\omega_2 b_2' = - \eta_2 \omega_2^2 b_2 + \frac{1}{2} \mu_2 (\omega_B + \omega_T)^2 a_1 a_2 \sin \gamma_2
\]  
(3.5.6 c)

\[
\omega_2 b_2 \gamma_1' = \omega_2 \sigma b_2 - \frac{1}{2} \mu_2 (\omega_B + \omega_T)^2 a_1 a_2 \cos \gamma_2
\]  
(3.5.6 d)
Stationary solutions are again determined by setting the left-hand sides of (3.5.6) to zero. An analytical solution to the resulting system of non-linear algebraic equations is not possible. A numerical solution to these equations (3.5.6) was conducted using a Fortran library subroutine (C05NAF) based on a hybrid method due to Powell. This approach was unsuccessful as the routine did not produce convergence to solutions of interest.

### 3.5.1 Undamped system response

Since no progress was achieved for the general case of the damped system, the special case for an undamped system was then considered. Setting \( \eta_B, \eta_T, \eta_2 \) and \( \eta_1 \) to zero in (3.5.6) reduces to the following set of equations,

\[
a_1^2 + \frac{\kappa_1 \omega_1^3}{\mu_1 \omega_B^3} b_1^2 - \frac{\omega_T}{\omega_B} a_2^2 = 0 \quad (3.5.7 \text{ a})
\]
\[ \omega_b \omega_1 \sigma_2 a_1 b_1 + \frac{1}{2} \rho \kappa_1 \omega_1^3 a_1 b_1^2 + \frac{1}{2} \kappa_2 \omega_2^2 \omega_1 a_2 b_1 b_2 \]
\[ = 0 \quad (3.5.7 \text{ b}) \]

\[ \omega_b \omega_T (\sigma + \sigma_1) a_1 a_2 + \frac{1}{2} \rho \kappa_1 \omega_T^3 a_1 a_2 b_1 \]
\[ = 0 \quad (3.5.7 \text{ c}) \]

\[ \omega_2 \sigma b_2 + \frac{1}{2} \mu_2 (\omega_b + \omega_T)^2 a_1 a_2 + \frac{1}{2} \rho \]
\[ = 0 \quad (3.5.7 \text{ d}) \]

A closed form solution to these equations is again not possible. However, numerical solution of these equations (3.5.7) proved successful. The results are presented in Figures 3.16.1 - 3.20.4.

3.5.2 Discussion of the theoretical response graphs

Figures 3.16.1 - 3.16.4 shows the modal response of \( b_2, a_2, a_1 \) and \( b_1 \) respectively for an internally resonant system. Figures 3.16.1 - 3.16.3 each contain two separate response curves labelled I and II. Figure 3.16.1 has an additional curve, III. The following discussion concerns curves I and III.

The system behaviour can be traced by varying external detuning \( \sigma \), for a fixed value of force amplitude. Starting with excitation at point A (Fig. 3.16.1), only \( b_2 \) responds. As \( \sigma \) is increased, \( b_2 \) follows the linear resonance curve III till B at which point the zero solution of the coupled system modes \( a_2 \) and \( a_1 \) became unstable and an upward jump occurs simultaneously from B to C for \( a_2, a_1 \) and \( b_1 \) (Figs. 3.16.2 - 3.16.4). On the other hand, \( b_2 \) jumps downward to C on curve I. As detuning is increased further, both \( b_2 \) and \( a_1 \) experience a suppression effect decreasing to a minimum at \( \sigma = 0 \), while \( a_2 \) and \( b_1 \) become resonant increasing to a maximum. Increasing detuning beyond \( \sigma = 0 \),
both $b_2$ and $a_1$ begin to increase, while $a_2$ and $b_1$ initially decrease rapidly over a short range of $\sigma$ after which both modes increase again. In the absence of damping all four modes will continue to increase indefinitely with increasing external detuning. The response curves are symmetrical about the $\sigma = 0$ axis. Thus, with initial excitation at $E$ and decreasing detuning $\sigma$, all the modes will trace the path EFGCH in an identical response sequence.

### 3.5.3 Effect of the second internal resonance condition

Curve II (Figs. 3.16.1 - 3.16.3) is the three mode interaction response, that is, in the absence of the internal resonance coupling between $a_1$ and $b_1$. These curves clearly indicate that the stability limits of the zero solution of $a_2$ and $a_1$ are identical for both cases. This is plausible from a physical viewpoint since the postulated mechanism by which $b_1$ is excited is only through $a_1$. Thus until $a_1$ achieves a finite response, $b_1$ will remain dormant. Therefore, the resonance coupling between $a_1$ and $b_1$ does not influence the onset of the zero solution response instability region. However, the coupling comes into effect only when the non-trivial response of $a_1$ commences. This is reflected in the difference in amplitude levels of the $b_2$, $a_2$ and $a_1$ modes when compared between the three mode and four mode systems. The amplitudes of $b_2$ and $a_2$ are notably reduced as a result of the resonance coupling between $a_1$ and $b_1$. This then indicates a transfer of energy from the two modes with higher frequencies ($b_2$, $a_2$) to the remaining modes ($b_1$, $a_1$) of lower frequencies. It is also noted that the lower branches of $b_1$ are horizontally tangent to the $\sigma$-axis in contrast to $a_2$ and $a_1$ whose lower branches retain the vertical tangency as in the three mode system response.
FIGURE 3.16.1
THEORETICAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING

$P = 5$, $\sigma_1 = \sigma_2 = 0$, $\eta_3 = \eta_4 = \eta_2 = \eta_1 = 0$
FIGURE 3.16.2
THEORETICAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\( P = 5, \sigma_1 = \sigma_2 = 0, \eta_8 = \eta_7 = \eta_2 = \eta_1 = 0 \)
Figure 3.16.3
Theoretical four mode system response
Nonplanar amplitude against external detuning
$P = 5,$ $\sigma_1 = \sigma_2 = 0,$ $\eta_8 = \eta_7 = \eta_2 = \eta_1 = 0$
FIGURE 3.16.4
THEORETICAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 5, \sigma_1 = \sigma_2 = 0, \eta_B = \eta_T = \eta_2 = \eta_4 = 0
3.5.4 Effect of parameter variation on undamped system response

The effects of force amplitude $P$, internal detuning parameters $\sigma_1$ and $\sigma_2$, are now discussed. The response curves for each mode are compared for a zero and a non-zero value of the parameter being varied. The continuous lines indicate stable solutions while broken lines indicate unstable solutions. All the response curves are shown as functions of external detuning. In the ensuing discussion, "region of resonance" refers in particular to the unstable regions of the trivial solutions of $a_2$, $a_1$ and $b_1$.

i) Varying force amplitude $P$, exact internal resonance, $\sigma_1 = \sigma_2 = 0$

Increasing $P$ generally widens the region of resonance. There is no indication of any saturation effect of $b_2$ (Fig. 3.17.1) as was with the case of the three mode system response. The overall amplitudes of $a_2$, $a_1$ and $b_1$ (Figs. 3.17.2 - 3.17.4) increase with $P$.

ii) Varying internal detuning parameter $\sigma_1$ ($\sigma_2 = 0$)

As is with the three mode system, the main effect of $\sigma_1$ is to shift the region of resonance along the $\sigma$-axis. Here too the minimum response point of $b_2$ (Fig. 3.18.1) is displaced by $-\sigma_1$ along the $\sigma$-axis. This gives rise to an asymmetric shape of the response of $a_2$, $a_1$ and $b_1$ (Figs. 3.18.2 - 3.18.4). The effect of negative detuning is a mirror image to that with positive detuning.

iii) Varying internal detuning parameter $\sigma_2$ ($\sigma_1 = 0$)

Increasing $\sigma_2$ leads to:

(a) a reduction of $b_2$, particularly at higher values of external detuning (Fig. 3.19.1).

(b) a reduction of $a_2$, whose stable response appears to be almost flat indicating possible saturation (Fig. 3.19.2). The resonant response at $\sigma = 0$ that exists for $\sigma_2 = 0$, has been reduced to almost
FIGURE 3.17.1
THEORETICAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\( \sigma_1 = \sigma_2 = 0, \ \eta_8 = \eta_T = \eta_2 = \eta_1 = 0 \)
Figure 3.17.2
Theoretical Four Mode System Response
Nonplanar Amplitude against External Detuning
$\sigma_1 = \sigma_2 = 0$, $\eta_b = \eta_1 = \eta_2 = \eta_1 = 0$
FIGURE 3.17.3
THEORETICAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$\sigma_1 = \sigma_2 = 0$, $\eta_8 = \eta_1 = \eta_2 = \eta_3 = 0$
FIGURE 3.17.4
THEORETICAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$\sigma_1 = \sigma_2 = 0$, $\eta_b = \eta_f = \eta_2 = \eta_1 = 0$
FIGURE 3.18.1
THEORETICAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 10$, $\sigma_2 = 0$, $\eta_b = \eta_T = \eta_2 = \eta_1 = 0$
FIGURE 3.18.2
THEORETICAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 10$ , $\sigma_2 = 0$ , $\eta_b = \eta_t = \eta_2 = \eta_1 = 0$
FIGURE 3.18.3
THEORETICAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 10, \sigma_2 = 0, \eta_b = \eta_t = \eta_2 = \eta_4 = 0$
FIGURE 3.18.4
THEORETICAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 10$, $\sigma_2 = 0$, $\eta_b = \eta_r = \eta_2 = \eta_t = 0$
FIGURE 3.19.1
THEORETICAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 10, \( \sigma_1 = 0 \), \( \eta_5 = \eta_I = \eta_2 = \eta_I = 0 \)
FIGURE 3.19.2
THEORETICAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 10, σ₁ = 0, η₈ = η₉ = η₂ = η₁ = 0
FIGURE 3.19.3
THEORETICAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 10, \sigma_1 = 0, \eta_B = \eta_I = \eta_2 = \eta_1 = 0$
FIGURE 3.19.4
THEORETICAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 10, $\sigma_1 = 0$, $\eta_b = \eta_f = \eta_2 = \eta_3 = 0$
FIGURE 3.19.5
THEORETICAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 10, \sigma_1 = 0, \eta_b = \eta_t = \eta_2 = \eta_1 = 0$
FIGURE 3.20.1
THEORETICAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 5, \( \sigma_1 = \sigma_2 = -5 \), \( \eta_8 = \eta_T = \eta_2 = \eta_1 = 0 \)
FIGURE 3.20.2
THEORETICAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 5$, $\sigma_1 = \sigma_2 = -5$, $\eta_b = \eta_f = \eta_2 = \eta_3 = 0$
FIGURE 3.20.3
THEORETICAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 5, σ₁ = σ₂ = -5, η₈ = η₁ = η₂ = η₃ = 0
FIGURE 3.20.4
THEORETICAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\( P = 5, \sigma_1 = \sigma_2 = -5, \eta_b = \eta_1 = \eta_2 = \eta_1 = 0 \)
negligible for $\sigma_2 = 20$.

(c) an increase of the overall response of $a_1$ (Fig. 3.19.3). The abrupt suppression in response at $\sigma = 0$ that exists for $\sigma_2 = 0$, has been replaced by a more gradual decrease to a minimum point for $\sigma_2 = 20$.

(d) a considerable decrease of $b_1$. The response for the $\sigma_2 = 0$ and $\sigma_2 = 20$ are drawn on separate graphs (Figs. 3.19.4 - 3.19.5) due to the large disparity in response values. The resonant response at $\sigma = 0$ for $\sigma_2 = 0$ has changed to a point of minimum response instead, indicating a suppression effect as a result of detuning.

iv) Generally detuned system $\sigma_1 \neq 0, \sigma_2 \neq 0$

The combined effects of finite detuning for $\sigma_1 = \sigma_2 = -5$ are shown in Figures 3.20.1 - 3.20.4. The two detuning parameters are evidently exerting their individual effects. $\sigma_1$ is displacing the region of resonance along the $\sigma$-axis with the resulting asymmetric response curves for $a_2$, $q_1$ and $b_1$. $\sigma_2$ on the other hand is causing a saturation effect on the response of $b_2$ and $a_2$ at higher values of external detuning. The considerable reduction in the response of $b_1$ is noted when compared to the system with exact internal resonance ($\sigma_1 = \sigma_2 = 0$, Fig. 3.16.4).

3.5.5 Stability considerations

The stability of the solutions for the four mode system without damping is inferred from the simpler three mode system response. From other previous studies of systems with two degrees-of-freedom, it is generally known that the region of unstable trivial solutions is bounded by the points of lower vertical tangency of the response curves. For the non-trivial solutions, the unstable responses are associated with the lower branches of the response curves, which lie between the upper and lower points of vertical tangency of the response curves.
CHAPTER 4

Experimental Investigation

4.1 Apparatus

The mechanical model used for experimental tests consisted principally of two steel beams of narrow rectangular cross-section, connected perpendicularly to one another. The main primary beam of dimensions 390mm x 25.4mm x 3.19mm has one end clamped horizontally in a steel block which in turn was secured to a concrete worktable. The motion of the primary beam was considered to be in the vertical plane. The secondary beam of dimensions 154mm x 19mm x 0.47mm was attached to the free end of the primary beam by means of two right angle support brackets. The secondary beam was arranged such that the bending displacement occurs out-of-the-plane of the primary beam motion. The secondary beam was also capable of torsional motion.

An adjustable mass was attached to the secondary beam to allow detuning between the system natural frequencies. External excitation was provided by an electrodynamic vibrator connected to the primary beam at a distance of 80mm from the clamped end. The vibrator connection had high axial stiffness but low flexural stiffness.

Primary beam motion was monitored by means of an accelerometer attached to the beam at the point coincident with the point of application of the external excitation. The nonplanar motion of the secondary beam consisting of both bending and torsional displacements, was monitored by strain gauges mounted near the clamped edge of the beam.

The gauge used for measuring bending displacement was initially calibrated by mounting the secondary beam on a horizontally vibrating base and exciting it into forced resonance. The tip deflection was
then determined by a suitably attached accelerometer. The gauge for measuring torsional displacement was also calibrated in a similar fashion where instead, the primary beam was mounted on a rotatable base such that the axis of rotation coincided with the elastic axis of the beam. By subjecting the base to suitable periodic rotation, the secondary beam was thus excited in forced torsional resonance. The angular displacement of the beam tip was then measured visually by a travelling microscope.

4.1.1 System parameters

It was necessary to determine the linear natural frequencies of the participating modes of motion in the absence of interactions for subsequent comparison with analysis.

The frequency of the nonplanar bending mode was determined conveniently by observing the frequency spectrum of the secondary beam in free vibration. For the nonplanar torsional mode, the secondary beam was again mounted on the rotatable surface and subjected to periodic rotational excitation. The amplitude response was however notably non-linear with a hardening-spring type behaviour. The linear natural frequency was determined indirectly by recording the upper branch "jump-down" points for varying excitation levels. For lightly damped systems, this "jump-down" point can be considered for all practical purposes to be almost coincident with the "backbone" curve. The equation of this "backbone" curve is given by the well known undamped free oscillation relationship between amplitude and frequency for the Duffing oscillator;

\[ n^2 = \omega^2 + \frac{3}{4} \nu A^2 \]
where  
\[ \Omega : \text{frequency of response} \]
\[ \omega : \text{linear natural frequency} \]
\[ \nu : \text{coefficient of non-linearity} \]
\[ A : \text{amplitude of response} \]

From a plot of \( \Omega^2 \) against \( A^2 \), the linear torsional frequency can then be determined by interpolation.

The mechanical impedance of the exciting vibration generator was considered to be part of the planar system linear parameters, and was coupled to the system during the measurements of natural frequencies and damping factors. To determine the planar bending natural frequencies, it was necessary to restrain the secondary beam from responding. This was achieved by positioning stops on both sides of the secondary beam. The first planar bending mode natural frequency was determined from a spectrum analysis of the beam system in free vibration. The second planar bending mode natural frequency was obtained from the resonant peak amplitude response frequency of the beam system under forced vibration. For lightly damped systems, the peak amplitude response frequency can be considered not to differ significantly from the linear natural frequency.

The form of dissipation assumed for the system was classical linear viscous damping. For the nonplanar fundamental bending and torsional modes, and the planar fundamental bending mode, the damping factors were measured directly from the free vibration decay curves. In the case of the planar second bending mode, the half-power bandwidth method was employed to obtain the damping factor.

4.1.2 Test procedure

The mechanical beam model was excited by an electrodynamic vibrator powered by a special amplifier unit. This amplifier unit
had feedback current control and ensured that a constant current level and hence a constant force was supplied to the vibrator armature independent of the motion of the beam. The vibrator force level was calibrated dynamically by recording the response of the coupled beam in forced vibration in the absence of internal resonance, for varying force amplitudes.

The power amplifier was driven from a frequency function generator which had a frequency sweep facility. During a sweep, the excitation frequency can be controlled to pause at selected frequencies. The system response was then recorded by increasing and decreasing frequency through the region of forced resonance of the beam system. Testing was also conducted by keeping the excitation frequency constant and varying the excitation amplitude level in a stepwise manner.

The transducer signals from the torsion gauge and the accelerometer measuring the primary beam response, were monitored on a Hewlett Packard HP 3582A real time frequency spectrum analyzer which was able to display a continuously updated spectrum of each of the input signals. The response waveforms were generally not of a steady sinusoidal nature and it was decided to record the absolute values of the main harmonic component in each signal as being representative of the experimental model response. This is justifiable as the main harmonic component was reasonably stationary over the range of excitation considered and the other harmonics were of smaller magnitude in comparison. The use of the spectrum analyzer under such circumstances was particularly convenient as the frequency content of complicated waveforms were immediately identifiable.
4.2 Experimental results

The experimental evidence is presented in two parts. The first part (Figs. 4.1.1 - 4.14.3) shows the three mode system response, while the second part (Figs. 4.16.1 - 4.31.3) deals with the four mode system response. The results relate the effect of varying detuning and force amplitude on the system response. No attempt was made to vary the system damping and the damping factors quoted are representative values decided from the pre-test response decay trials.

4.2.1 Three mode system response

Preliminary comparison of response curves based on theoretical system natural frequencies showed a major qualitative discrepancy from the measured values. This was attributed to the sensitivity of the detuning parameters to small variations in absolute frequency values. For example, an uncertainty of 0.5 Hz in frequency value results in a change of about 6 units of internal detuning. These differences between experimental and theoretical frequencies are shown in Fig. 4.0. Such differences are inevitable as a result of an approximate modelling of the physical system. Subsequently, in order to obtain a realistic comparison between measurements and predictions, the theoretical response curves were calculated on the basis of experimentally determined natural frequencies. The experimental results are presented in Figures 4.1.1 - 4.14.3. A set of results consists of three successive figures, one for each mode. The theoretical curves are shown as continuous lines representing the stable responses and dashed lines for unstable responses.

i) Amplitude response against external detuning (Figs. 4.1.1 - 4.7.3)

The results are for two values of the internal detuning parameter $\sigma_1$, at different values of force amplitude $P$. The relevant parameters...
FIGURE 4.0 SYSTEM NATURAL FREQUENCIES AS A FUNCTION OF MASS POSITION $\zeta$
($\zeta$: DISTANCE OF SECONDARY BEAM MASS $m_o$ FROM COUPLING POINT)
for the figures are summarized below.

Figs. 4.1.1 - 4.1.3 : $P = 3.8, \sigma_1 = 5.3$
Figs. 4.2.1 - 4.2.3 : $P = 7.6, \sigma_1 = 5.3$
Figs. 4.3.1 - 4.3.3 : $P = 15.1, \sigma_1 = 5.3$
Figs. 4.4.1 - 4.4.3 : $P = 22.7, \sigma_1 = 5.3$
Figs. 4.5.1 - 4.5.3 : $P = 12.8, \sigma_1 = 4.0$
Figs. 4.6.1 - 4.6.3 : $P = 19.2, \sigma_1 = 4.0$
Figs. 4.7.1 - 4.7.3 : $P = 25.6, \sigma_1 = 4.0$

where $\sigma_1$, the internal detuning parameter is defined as,

$\omega_2 = \omega_B + \omega_T + \epsilon \sigma_1$,

and $P$ is the force amplitude.

The experimental results are shown as discrete points and no distinction is made between measured values for increasing and decreasing external detuning parameter $\sigma$. A typical test run starts with negative detuning and progresses through the autoparametric resonance region to some positive value of $\sigma$ for which the instability has ceased. The onset of the resonance region is shown by the downward 'jump' arrows for $b_2$ and upward 'jump' arrows for $a_2$ and $a_1$. $b_2$ is the externally excited second planar bending mode while $a_2$ and $a_1$ are the parametrically excited nonplanar torsion and bending modes. The point at which autoparametric resonance stops is shown by the downward 'jump' arrows for $b_2$, $a_2$ and $a_1$.

There is reasonable agreement, both qualitatively and quantitatively, between measured values and predicted responses. Generally the theoretical curves of $b_2$ are higher than the experimental values, while the nonplanar modes $a_2$ and $a_1$ show the measured response to be higher than the predicted values. The region of autoparametric resonance measured in the model tends to be wider than the predicted regions. However, the measured points of
instability of the zero solution of $a_2$ and $a_1$ are generally narrower than the corresponding theoretical points.

For excitation at higher values of $P$, Figs. 4.3.1 - 4.4.3, 4.7.1 - 4.7.3, an additional instability feature was observed in the experimental model. This region is shown as dashed lines joining the experimental points and occurs only on the right-hand branch of $b_2$, $a_2$ and $a_1$ for increasing external detuning. Instead of the expected downward jump in the response of $a_2$ and $a_1$, the observed waveform of $a_2$ and $b_2$ becomes unsteady and shows a beating effect. This qualitative change in behaviour is shown by waveform graphs and accompanying frequency spectra plots of the response (Figs. 4.7.4 - 4.7.9). A typical response of a sample experimental point (Figs. 4.7.4 - 4.7.5) corresponds to the test conditions of Figs. 4.4.1 - 4.4.3. Although the response of $b_2$ does show beating, the frequency spectrum indicates the extra lower harmonic component to be practically negligible compared to the main harmonic at approximately $\omega_2$. The main harmonic of each response is identified as the peak located in the neighbourhood of the linear natural frequencies marked by the white triangles on the frequency axis.

Keeping $P$ constant and increasing $\sigma$ to a point on the right-hand branch, there is a pronounced beating effect present in the waveform trace of $a_2$ (Fig. 4.7.6). The corresponding frequency spectrum of $a_2$ (Fig. 4.7.7) shows side bands centered around the main harmonic at $\omega_T$. Here too, the extra harmonics present in both $a_2$ and $b_2$ appears negligible. However when $P$ was increased, keeping $\sigma$ constant, the waveform traces of $a_2$ and $b_2$ (Fig. 4.7.8) became notably irregular. The corresponding frequency spectrum (Fig. 4.7.9) shows several harmonic components in both $a_2$ and $b_2$. There is also considerable reduction of the main harmonics at $\omega_T$ and $\omega_2$ for $a_2$ and $b_2$. 
respectively. These extra harmonics are now no longer negligible but are of comparable magnitude. These non-stationary effects are not predicted by the present analysis. It has not been the object of this investigation to study multi-harmonic responses but they are documented here to illustrate possible additional instability phenomena. No explanation is offered as to the cause of this instability.

ii) Amplitude response against force amplitude

Figures 4.9.1 - 4.14.3 show the system response for constant values of the external detuning parameter $\sigma$ and the internal detuning parameter $\sigma_1$, with force amplitude $P$ increased to some upper limit and then decreasing in a stepwise manner. The relevant parameters for the figures are summarized below.

- Figs. 4.9.1 - 4.9.3: $\sigma = 2.7$, $\sigma_1 = 5.3$
- Figs. 4.10.1 - 4.10.3: $\sigma = 5.3$, $\sigma_1 = 5.3$
- Figs. 4.11.1 - 4.11.3: $\sigma = 10.6$, $\sigma_1 = 5.3$
- Figs. 4.12.1 - 4.12.3: $\sigma = 9.3$, $\sigma_1 = 4.0$
- Figs. 4.13.1 - 4.13.3: $\sigma = -4.0$, $\sigma_1 = 4.0$
- Figs. 4.14.1 - 4.14.3: $\sigma = 22.6$, $\sigma_1 = 4.0$

where $\sigma$, the external detuning parameter is defined as,

$$\Omega = \omega_2 + \varepsilon \sigma,$$

and $\sigma_1$, the internal detuning parameter is defined as,

$$\omega_2 = \omega_B + \omega_T + \varepsilon \sigma_1.$$  

The results are again shown as discrete points with no distinction made between increasing and decreasing force amplitude $P$. A test run starts with a small value of $P$ and is increased stepwise till the critical value when autoparametric resonance begins. $P$ is further increased to some upper value, after which it was decreased to zero. The threshold value of $P$ at which
autoparametric resonance begins is shown by upward 'jump' arrows for $a_2$ and $a_1$, and a downward 'jump' arrow for $b_2$. For the case of $P$ decreasing from its upper limit, the point at which resonance stops is shown by the downward 'jump' arrows for $b_2$, $a_2$ and $a_1$. In some tests (Figs. 4.10.1, 4.12.1) the change in the response of $b_2$ at the onset of autoparametric resonance was not discernable so that no 'jump' phenomenon was visible. In other tests (Figs. 4.9.1, 4.13.1) the resumption of the linear response of $b_2$ when autoparametric resonance stops, was also not distinctly noticeable and no 'jump' phenomenon was observed either.

The agreement between theory and experiments is reasonable. The non-linear response of $b_2$ does not show the predicted saturation effect as there is generally a small increase in $b_2$ with $P$ (Figs. 4.9.1, 4.10.1, 4.11.1). This deviation is more noticeable in Fig. 4.12.1 where the non-linear response of $b_2$ is seen to increase approximately linearly with $P$. The measured values of $a_2$ and $a_1$ are generally higher than the predicted values and this discrepancy increases with $P$. The calculated values of the threshold value of $P$ for the onset of autoparametric resonance are in all cases lower than the observed values.

It was noted in the previous section 4.2.1 (i), that multi-harmonic responses were observed in a region of external detuning corresponding to the right-hand branch of $a_2$ and $a_1$. Similar responses have also been recorded for a fixed value of the external detuning parameter $\sigma$ with varying force amplitude $P$ (Figs. 4.14.1 - 4.13.3). In addition to the jump response associated with autoparametric resonance, secondary jump responses were observed after the onset of autoparametric resonance, and these points are indicated by the double headed arrows.
FIGURE 4.1.1  THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 3.8, \( \alpha_1 = 5.3 \), \( \eta_b = 0.04 \), \( \eta_r = 0.03 \), \( \eta_2 = 0.25 \)

- Theory; \( \square \): Experiment
FIGURE 4.1.2  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 3.8, σ₁ = 5.3, η₈ = 0.04, η₁ = 0.03, η₂ = 0.25
THEORY, ——— : STABLE ; ———— : UNSTABLE
□ : EXPERIMENT
FIGURE 4.1.3  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 3.8, \( \sigma_i = 5.3 \), \( \eta_s = 0.04 \), \( \eta_i = 0.03 \), \( \eta_2 = 0.25 \)
THEORY, \______\ : STABLE ; \_________\ : UNSTABLE
\o\ : EXPERIMENT
FIGURE 4.2.1 THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\( P = 7.6, \sigma_1 = 5.3, \eta_0 = 0.04, \eta_1 = 0.03, \eta_2 = 0.25 \)

______ : THEORY; □ : EXPERIMENT
FIGURE 4.2.2  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 7.6, \( \sigma_1 = 5.3 \), \( \eta_\beta = 0.04 \), \( \eta_\tau = 0.03 \), \( \eta_2 = 0.25 \)

THEORY, \( \square \) : STABLE ; \( \triangle \) : UNSTABLE
\( \bigcirc \) : EXPERIMENT
FIGURE 4.2.3  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 7.6, σ₁ = 5.3, η₁ = 0.04, η₂ = 0.03, η₂ = 0.25
THEORY, _______ : STABLE ; ............ : UNSTABLE
○ : EXPERIMENT
FIGURE 4.3.1 THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 15.1, σ₁ = 5.3, η₁ = 0.04, η₂ = 0.03, η₂ = 0.25
——— : THEORY; □ : EXPERIMENT
□———□ : MULTI-HARMONIC RESPONSE
FIGURE 4.3.2  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 15.1, a₁ = 5.3, η₂ = 0.04, η₁ = 0.03, η₂ = 0.25
THEORY, ______ : STABLE ; ........ : UNSTABLE
□ : EXPERIMENT; □--□--□ : MULTI-HARMONIC RESPONSE
FIGURE 4.3.3 THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 15.1, $\sigma_t = 5.3$, $\eta_b = 0.04$, $\eta_t = 0.03$, $\eta_2 = 0.25$
THEORY, _____ : STABLE ; .......... : UNSTABLE
 o : EXPERIMENT; o----------o : MULTI-HARMONIC RESPONSE
FIGURE 4.4.1  THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 22.7, \sigma_1 = 5.3, \eta_b = 0.04, \eta_I = 0.03, \eta_2 = 0.25$

--- : THEORY;  ■ : EXPERIMENT
■---------■ : MULTI-HARMONIC RESPONSE
FIGURE 4.4.2
THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 22.7, $\sigma_1 = 5.3$, $\eta_\theta = 0.04$, $\eta_\tau = 0.03$, $\eta_z = 0.25$
THEORY, _____: STABLE; ........: UNSTABLE
□: EXPERIMENT; ▲:------▲: MULTI-HARMONIC RESPONSE
FIGURE 4.4.3  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 22.7$, $\sigma_1 = 5.3$, $\eta_8 = 0.04$, $\eta_1 = 0.03$, $\eta_2 = 0.25$
THEORY, $\cdots$: STABLE; $\cdots\cdots$: UNSTABLE
$\circ$: EXPERIMENT; $\circ\cdots\cdots\circ$: MULTI-HARMONIC RESPONSE
FIGURE 4.5.1  THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 12.8, \sigma_t = 4.0, \eta_b = 0.04, \eta_T = 0.03, \eta_2 = 0.25$

--- : THEORY; □ : EXPERIMENT
FIGURE 4.5.2 THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING

$p = 12.8, \sigma_1 = 4.0, \eta_8 = 0.04, \eta_1 = 0.03, \eta_2 = 0.25$

THEORY, \( \quad \); STABLE ; \( \quad \); UNSTABLE

\( \square \); EXPERIMENT
FIGURE 4.5.3  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 12.8, σ₁ = 4.0, η₈ = 0.04, η₁ = 0.03, η₂ = 0.25
THEORY, ______ : STABLE ; ........... : UNSTABLE
○ : EXPERIMENT
FIGURE 4.6.1 THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\( P = 19.2, \ \sigma_i = 4.0, \ \eta_\text{b} = 0.04, \ \eta_\text{T} = 0.03, \ \eta_2 = 0.25 \)

- Theory; \( \square \): Experiment
FIGURE 4.6.2
THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 19.2$, $\sigma_1 = 4.0$, $\eta_8 = 0.04$, $\eta_t = 0.03$, $\eta_2 = 0.25$
THEORY, ______ : STABLE ; ............ : UNSTABLE
☐ : EXPERIMENT
FIGURE 4.6.3  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 19.2, \sigma_1 = 4.0, \eta_3 = 0.04, \eta_1 = 0.03, \eta_2 = 0.25$
THEORY, ______ : STABLE ; .......... : UNSTABLE
o : EXPERIMENT
FIGURE 4.7.1  THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 25.6, σ₁ = 4.0, η₈ = 0.04, η₁ = 0.03, η₂ = 0.25
———: THEORY; □: EXPERIMENT
□□□□□: MULTI-HARMONIC RESPONSE
FIGURE 4.7.2  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 25.6, \( \sigma_1 = 4.0 \), \( \eta_3 = 0.04 \), \( \eta_r = 0.03 \), \( \eta_2 = 0.25 \)
THEORY, \( \cdots \cdots \) : STABLE ; \( \cdots \cdots \cdots \) : UNSTABLE
\( \square \) : EXPERIMENT; \( \cdots \cdots \cdots \cdots \cdots \cdots \) : MULTI-HARMONIC RESPONSE
FIGURE 4.7.3  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 25.6, σ₁ = 4.0, η₈ = 0.04, η₁ = 0.03, η₂ = 0.25
THEORY, ______ : STABLE ; ........ : UNSTABLE
○ : EXPERIMENT; .........○ : MULTI-HARMONIC RESPONSE
FIGURE 4.7.4
EXPERIMENTAL TIME RESPONSE
THREE MODE SYSTEM INTERACTION
$P = 22.7, \sigma = -1.3, \sigma_i = 5.3$
FIGURE 4.7.5
EXPERIMENTAL RESPONSE FREQUENCY SPECTRA
THREE MODE SYSTEM INTERACTION

$P = 22.7$, $\sigma = -1.3$, $\sigma_i = 5.3$
FIGURE 4.7.6
EXPERIMENTAL TIME RESPONSE
THREE MODE SYSTEM INTERACTION
\[ P = 22.7 \quad \sigma = 16.0 \quad \sigma_i = 5.3 \]
FIGURE 4.7.7
EXPERIMENTAL RESPONSE FREQUENCY SPECTRA
THREE MODE SYSTEM INTERACTION
P = 22.7 , σ = 16.0 , σ₁ = 5.3
FIGURE 4.7.8
EXPERIMENTAL TIME RESPONSE
THREE MODE SYSTEM INTERACTION
$P = 45.4 \ , \ \sigma = 16.0 \ , \ \sigma_1 = 5.3$
FIGURE 4.7.9
EXPERIMENTAL RESPONSE FREQUENCY SPECTRA
THREE MODE SYSTEM INTERACTION
P = 45.4, \( \sigma = 16.0 \), \( \sigma_i = 5.3 \)
FIGURE 4.9.1 THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 2.7, \sigma_1 = 5.3, \eta_b = 0.04, \eta_i = 0.03, \eta_2 = 0.22 \)

- : THEORY; □ : EXPERIMENT
FIGURE 4.9.2  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 2.7, \sigma_1 = 5.3, \eta_b = 0.04, \eta_1 = 0.03, \eta_2 = 0.22 \)
THEORY, ______ : STABLE ; .......... : UNSTABLE
\( \square \) : EXPERIMENT
FIGURE 4.9.3  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma = 2.7, \sigma_1 = 5.3, \eta_3 = 0.04, \eta_1 = 0.03, \eta_2 = 0.22$
THEORY, ______ : STABLE ; .......... : UNSTABLE
o: EXPERIMENT
FIGURE 4.10.1 THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 5.3, \sigma_1 = 5.3, \eta_b = 0.04, \eta_f = 0.03, \eta_2 = 0.22 \)

- : THEORY; \( \square \) : EXPERIMENT
FIGURE 4.10.2 THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 5.3, \sigma_i = 5.3, \eta_b = 0.04, \eta_i = 0.03, \eta_2 = 0.22 \)

THEORY, _______ : STABLE ; ............ : UNSTABLE
\( \square \) : EXPERIMENT
FIGURE 4.10.3  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 5.3, \sigma_1 = 5.3, \eta_B = 0.04, \eta_T = 0.03, \eta_2 = 0.22 \)
THEORY, ----- : STABLE ; ........ : UNSTABLE
\( \circ : \) EXPERIMENT
FIGURE 4.11.1 THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma = 10.6$, $\sigma_i = 5.3$, $\eta_b = 0.04$, $\eta_t = 0.03$, $\eta_2 = 0.22$

- : THEORY; □ : EXPERIMENT
FIGURE 4.11.2 THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 10.6, \sigma_i = 5.3, \eta_a = 0.04, \eta_i = 0.03, \eta_2 = 0.22 \)
THEORY, _____ : STABLE ; ........ : UNSTABLE
\( \square \) : EXPERIMENT
FIGURE 4.11.3  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\[ \sigma = 10.6, \ \sigma_1 = 5.3, \ \eta_b = 0.04, \ \eta_1 = 0.03, \ \eta_2 = 0.22 \]
THEORY, \[ \text{\_\_\_: STABLE; \_\_\_\_: UNSTABLE} \]
\[ \text{o: EXPERIMENT} \]
FIGURE 4.12.1 THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
σ = 9.3, σ₁ = 4.0, η₂ = 0.04, η₁ = 0.03, η₂ = 0.22
___ : THEORY; □ : EXPERIMENT
FIGURE 4.12.2  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma = 9.3$, $\sigma_i = 4.0$, $\eta_b = 0.04$, $\eta_r = 0.03$, $\eta_2 = 0.22$

THEORY, _____ : STABLE ; .......... : UNSTABLE
\( \square \) : EXPERIMENT
FIGURE 4.12.3  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 9.3, \sigma_1 = 4.0, \eta_0 = 0.04, \eta_1 = 0.03, \eta_2 = 0.22 \)

\( \circ \): EXPERIMENT

\( \theta \): THEORY, \( \ldots \ldots \): STABLE, \( \ldots \ldots \ldots \ldots \): UNSTABLE
FIGURE 4.13.1 THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = -4.0, \ \sigma_1 = 4.0, \ \eta_b = 0.04, \ \eta_t = 0.03, \ \eta_2 = 0.22 \)

- : THEORY; o : EXPERIMENT
FIGURE 4.13.2  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = -4.0, \sigma_1 = 4.0, \eta_3 = 0.04, \eta_1 = 0.03, \eta_2 = 0.22 \)
THEORY, _______ : STABLE ; ............ : UNSTABLE
\( \square \) : EXPERIMENT
FIGURE 4.13.3 THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = -4.0, \sigma_1 = 4.0, \eta_B = 0.04, \eta_T = 0.03, \eta_2 = 0.22 \)
THEORY, _______ : STABLE ; .......... : UNSTABLE
o : EXPERIMENT
FIGURE 4.14.1 THREE MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 22.6, \sigma_1 = 4.0, \eta_b = 0.04, \eta_I = 0.03, \eta_2 = 0.22 \)

---- : THEORY; □ : EXPERIMENT
□□□□□ : MULTI-HARMONIC RESPONSE
FIGURE 4.14.2 THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 22.6, \sigma_1 = 4.0, \eta_0 = 0.04, \eta_T = 0.03, \eta_2 = 0.22 \)

THEORY, \( \square \): STABLE; \( \triangle \): UNSTABLE
\( \square \): EXPERIMENT; \( \square \): MULTI-HARMONIC RESPONSE
FIGURE 4.14.3  THREE MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma = 22.6$, $\sigma_i = 4.0$, $\eta_b = 0.04$, $\eta_i = 0.03$, $\eta_3 = 0.22$

THEORY, _____ : STABLE ; --------- : UNSTABLE
o : EXPERIMENT; o--------o : MULTI-HARMONIC RESPONSE
4.2.2 Four mode system response

There is no direct theoretical comparison for the experimental four mode system response since only solutions to the special case of a system without damping have been obtained. Hence the following discussion is based only on experimental observations. The results are divided into two parts. Figures 4.16.1 - 4.22.3 show the system response as a function of the external detuning parameter $\sigma$, while Figures 4.23.1 - 4.31.3 are graphs of system response against force amplitude $P$. The data points are presented in the same manner as the three mode system experimental results with the 'jump' arrows indicating the limits of the region of autoparametric resonance. The system parameters for the figures are summarized as follows,

Fig. 4.16.1 - 4.16.3 : $P = 16.5$, $\sigma_1 = -22.6$, $\sigma_2 = 0$
Fig. 4.17.1 - 4.17.3 : $P = 33$, $\sigma_1 = -22.6$, $\sigma_2 = 0$
Fig. 4.18.1 - 4.18.3 : $P = 66$, $\sigma_1 = -22.6$, $\sigma_2 = 0$
Fig. 4.19.1 - 4.19.3 : $P = 15.1$, $\sigma_1 = -18.1$, $\sigma_2 = 6.4$
Fig. 4.20.1 - 4.20.3 : $P = 30.2$, $\sigma_1 = -18.1$, $\sigma_2 = 6.4$
Fig. 4.21.1 - 4.21.3 : $P = 18.1$, $\sigma_1 = -28.6$, $\sigma_2 = -8.2$
Fig. 4.22.1 - 4.22.3 : $P = 36.2$, $\sigma_1 = -28.6$, $\sigma_2 = -8.2$
Fig. 4.23.1 - 4.23.3 : $\sigma = 3.9$, $\sigma_1 = -22.6$, $\sigma_2 = 0$
Fig. 4.24.1 - 4.24.3 : $\sigma = 13.2$, $\sigma_1 = -22.6$, $\sigma_2 = 0$
Fig. 4.25.1 - 4.25.3 : $\sigma = -8.4$, $\sigma_1 = -22.6$, $\sigma_2 = 0$
Fig. 4.26.1 - 4.26.3 : $\sigma = 0.1$, $\sigma_1 = -18.1$, $\sigma_2 = 6.4$
Fig. 4.27.1 - 4.27.3 : $\sigma = -13.9$, $\sigma_1 = -18.1$, $\sigma_2 = 6.4$
Fig. 4.28.1 - 4.28.3 : $\sigma = 12.9$, $\sigma_1 = -18.1$, $\sigma_2 = 6.4$
Fig. 4.29.1 - 4.29.3 : $\sigma = 8.9$, $\sigma_1 = -28.6$, $\sigma_2 = -8.2$
Fig. 4.30.1 - 4.30.3 : $\sigma = -1.0$, $\sigma_1 = -28.6$, $\sigma_2 = -8.2$
Fig. 4.31.1 - 4.31.3 : $\sigma = -14.8$, $\sigma_1 = -28.6$, $\sigma_2 = -8.2$
where the internal detuning parameters $\sigma_1$ and $\sigma_2$ are defined as,

$$\omega_2 = \omega_B + \omega_T + \epsilon \sigma_1, \quad \omega_1 = 2 \omega_B + \epsilon \sigma_2,$$

and the external detuning parameter $\sigma$ is defined as,

$$\Omega = \omega_2 + \epsilon \sigma.$$

i) Response amplitude against external detuning

The response of the externally excited planar bending mode $b_2$ appears qualitatively similar to the three mode system response with the characteristic suppression effect of $b_2$ in the vicinity of exact external resonance. The width of the region of autoparametric resonance increases with force amplitude $P$. There is also a corresponding increase in the overall amplitudes of $b_2$, $a_2$, $a_1$ and $b_1$. $a_2$ and $a_1$ are the parametrically excited nonplanar torsion and bending modes, and $b$ is the planar fundamental bending mode. There is no clear general pattern in the response of $a_2$, $a_1$ and $b_1$, but a notable difference exists between the case when there is an exact internal resonance, $\sigma_2 = 0$, (i.e., $\omega_1 = 2 \omega_B$) Figs. 4.17.2, 4.18.2. The irregularity in the response pattern of $a_2$, $a_1$ and $b_1$ is amplified especially within the region bounded by the upward 'jump' arrows. The point of minimum response of $a_2$ and $a_1$ no longer coincides. In fact, $a_2$ and $a_1$ respond in opposition, i.e. when $a_2$ experiences a 'peak' response, $a_1$ is at a minimum point. Within the same region (Figs. 4.17.3, 4.18.3) $b_1$ experiences two 'resonant' peak responses. The first peak occurs just after the left-hand side upward 'jump' arrow and coincides with the peak response of $a_1$ (Figs. 4.17.2, 4.18.2) while the second peak coincides with the peak response of $a_2$. When there is a finite amount of internal detuning, $\sigma_2 \neq 0$, (i.e., $\omega_1 \neq 2 \omega_2$) the response of $a_2$ and $a_1$ (Figs. 4.19.2, 4.20.2, 4.21.2) and $b_1$ (Figs. 4.19.3, 4.20.3, 4.21.3) are seen to vary less abruptly with the forementioned peak responses being less distinguishable.
There is strictly no justification for comparing the theoretical solutions of the undamped system with the experimental measurements of a system with finite damping. Nevertheless, the role of damping can be gauged from the behaviour of other oscillatory systems whose general solutions are available for both cases of zero and finite damping. On this basis, the effect of damping in the present system can be expected to limit the region of non-zero response of the internally resonant modes to within a finite region of external detuning. This will give rise to the downward 'jump' response at the point when autoparametric resonance ceases as excitation frequency is changed.

One feature predicted by the numerical solutions when $\sigma_2 = 0$, (i.e. $\omega_1 = 2 \omega_2$), is the 'resonant' peak response of $b_1$ which coincides with the suppression of $b_2$. The corresponding experimental response (Figs. 4.17.3, 4.18.3) indicate the presence of two peak responses instead of one. Here again, this difference could possibly be due to damping, but however it is noted that peak response disappears when $\sigma_2 \neq 0$, (i.e. $\omega_1 \neq 2 \omega_2$), Figs. 4.19.3, 4.20.3, 4.21.3, 4.22.3. This is confirmed by the theoretical solution that the 'resonant' peak response at $\sigma = 0$ when $\sigma_2 = 0$ (Fig. 4.19.4) no longer exists when $\sigma_2 \neq 0$ (Fig. 4.19.5).

ii) Response amplitude against force amplitude

As with the three mode system response, starting with a small value of $P$, only $b_2$ responds according to the linear forced resonance solution. In all cases there is a threshold value of $P$ at which autoparametric resonance begins. This point is indicated by the upward 'jump' arrows in the response of $a_2$, $a_1$ and $b_1$. The linear response of $b_2$ is generally above the non-linear response, except in Fig. 4.28.1, so that a downward 'jump' response is observed. With
increasing P, the response of \( a_2, a_1 \) and \( b_1 \) increases. The response of \( b_2 \) also generally increases with P but in some cases (Figs. 4.26.1, 4.27.1, 4.30.1, 4.31.1) \( b_2 \) decreases.

If P is then decreased from some upper value, \( a_2, a_1 \) and \( b_1 \) decrease retracing the same path. In all cases, P has to be decreased to a value below the threshold value at which autoparametric resonance begins, before the resonance stops and this point is indicated by the downward 'jump' arrow in the graphs of \( a_2, a_1 \) and \( b_1 \). In most cases, the response of \( b_2 \) also retrace the same path when P is decreased, but in some tests (Figs. 4.26.1, 4.29.1, 4.30.1) a hysteresis effect is observed as \( b_2 \) takes a different path for decreasing P.

Waveform traces and the corresponding frequency spectra plots for typical responses are shown in Figures 4.32.1 - 4.32.6. The internal detuning conditions for these figures correspond to Figs. 4.17.1 - 4.17.3. Fig. 4.32.1 shows a typical response for a small value of external detuning parameter \( \sigma \). The waveform trace of the planar motion \( (b_1, b_2) \) indicates the presence of more than one harmonic and is confirmed by the frequency spectra plot (Fig. 4.32.2) which show two harmonic components located at \( \omega_1 \) and \( \omega_2 \), these being the linear natural frequencies of the planar bending modes \( b_1 \) and \( b_2 \) respectively. The response of the nonplanar bending and torsional modes, \( a_1 \) and \( a_2 \), are considered to be stationary as seen by their individual frequency spectra (Fig. 4.32.2) which show their responses to be mono-harmonic of frequency close to their individual linear natural frequencies \( \omega_B \) and \( \omega_T \). When the force amplitude P was increased, the other system parameters being kept constant, the waveform trace of the planar motion \( (b_1, b_2) \), Fig. 4.32.3, became irregular but their frequency spectra (Fig. 4.32.4) show the response
to be predominantly mono-harmonic at $\omega_1$. If now the external detuning parameter $\sigma$ is increased to a higher value, corresponding to the right-hand overhanging branch, (Fig. 4.32.5) $a_2$ takes on a pronounced beating response. The frequency spectra of $a_2$ (Fig. 4.32.6) show a main harmonic located at approximately $\omega_T$, the linear natural frequency of the nonplanar torsion mode $a_2$, with two side bands centred at approximately $\omega_T - 2\omega_B$ and $\omega_T + 2\omega_B$. $\omega_B$ is the linear natural frequency of the nonplanar bending mode $a_1$. 
FIGURE 4.16.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 16.5 , \( \sigma_1 = -22.6 \), \( \sigma_2 = 0.0 \)
\( \eta_\theta = 0.040 \), \( \eta_T = 0.027 \), \( \eta_2 = 0.239 \), \( \eta_1 = 0.126 \)
FIGURE 4.16.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST EXTERNAL DETUNING
P = 16.5, $\sigma_1 = -22.6$, $\sigma_2 = 0.0$
$\eta_b = 0.040$, $\eta_1 = 0.027$, $\eta_2 = 0.239$, $\eta_1 = 0.126$
FIGURE 4.16.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 16.5, σ₁ = -22.6, σ₂ = 0.0
η₈ = 0.040, η₁ = 0.027, η₂ = 0.239, η₁ = 0.126
FIGURE 4.17.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 33.0$, $\sigma_1 = -22.6$, $\sigma_2 = 0.0$
$\eta_\theta = 0.040$, $\eta_t = 0.027$, $\eta_2 = 0.239$, $\eta_1 = 0.126$
FIGURE 4.17.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST EXTERNAL DETUNING
P = 33.0, \(\sigma_1 = -22.6\), \(\sigma_2 = 0.0\)
\(\eta_b = 0.040\), \(\eta_t = 0.027\), \(\eta_2 = 0.239\), \(\eta_1 = 0.126\)
FIGURE 4.17.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\( P = 33.0 , \sigma_1 = -22.6 , \sigma_2 = 0.0 \)
\( \eta_B = 0.040 , \eta_1 = 0.027 , \eta_2 = 0.239 , \eta_1 = 0.126 \)
FIGURE 4.18.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 66.0 , \( \sigma_1 = -22.6 \), \( \sigma_2 = 0.0 \)
\( \eta_b = 0.040 \), \( \eta_T = 0.027 \), \( \eta_2 = 0.239 \), \( \eta_1 = 0.126 \)
FIGURE 4.18.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST EXTERNAL DETUNING
P = 66.0, \( \sigma_1 = -22.6 \), \( \sigma_2 = 0.0 \)
\( \eta_b = 0.040 \), \( \eta_t = 0.027 \), \( \eta_2 = 0.239 \), \( \eta_1 = 0.126 \)
FIGURE 4.18.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 66.0, $\sigma_1 = -22.6$, $\sigma_2 = 0.0$
$\eta_b = 0.040$, $\eta_1 = 0.027$, $\eta_2 = 0.239$, $\eta_1 = 0.126$
FIGURE 4.19.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 15.1$, $\sigma_1 = -18.1$, $\sigma_2 = 6.4$
$\eta_0 = 0.040$, $\eta_1 = 0.027$, $\eta_2 = 0.239$, $\eta_1 = 0.126$
FIGURE 4.19.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST EXTERNAL DETUNING
P = 15.1, $\sigma_1 = -18.1$, $\sigma_2 = 6.4$
$\eta_b = 0.040$, $\eta_f = 0.027$, $\eta_2 = 0.239$, $\eta_1 = 0.126
FIGURE 4.19.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
\( P = 15.1 \), \( \sigma_1 = -18.1 \), \( \sigma_2 = 6.4 \)
\( \eta_b = 0.040 \), \( \eta_1 = 0.027 \), \( \eta_2 = 0.239 \), \( \eta_1 = 0.126 \)
FIGURE 4.20.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 30.2 , $\sigma_1 = -18.1$ , $\sigma_2 = 6.4$
$\eta_b = 0.040$ , $\eta_1 = 0.027$ , $\eta_2 = 0.239$ , $\eta_i = 0.126$
FIGURE 4.20.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST EXTERNAL DETUNING
P = 30.2 , σ₁ = -18.1 , σ₂ = 6.4
η₃ = 0.040 , η₁ = 0.027 , η₂ = 0.239 , η₄ = 0.126
FIGURE 4.20.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 30.2, \( \sigma_1 = -18.1 \), \( \sigma_2 = 6.4 \)
\( \eta_b = 0.040 \), \( \eta_1 = 0.027 \), \( \eta_2 = 0.239 \), \( \eta_1 = 0.126 \)
FIGURE 4.21.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING

$P = 18.1, \sigma_1 = -28.6, \sigma_2 = -8.2$
$\eta_b = 0.040, \eta_f = 0.027, \eta_2 = 0.239, \eta_1 = 0.126$
FIGURE 4.21.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST EXTERNAL DETUNING
\( P = 18.1 \), \( \sigma_1 = -28.6 \), \( \sigma_2 = -8.2 \)
\( \eta_b = 0.040 \), \( \eta_1 = 0.027 \), \( \eta_2 = 0.239 \), \( \eta_i = 0.126 \)
FIGURE 4.21.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 18.1 , $\sigma_1 = -28.6$ , $\sigma_2 = -8.2$
$\eta_b = 0.040$ , $\eta_f = 0.027$ , $\eta_2 = 0.239$ , $\eta_l = 0.126$
FIGURE 4.22.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
$P = 36.2, \sigma_1 = -28.6, \sigma_2 = -8.2$
$\eta_b = 0.040, \eta_1 = 0.027, \eta_2 = 0.239, \eta_r = 0.126$
FIGURE 4.22.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST EXTERNAL DETUNING
$P = 36.2$, $\sigma_1 = -28.6$, $\sigma_2 = -8.2$
$\eta_b = 0.040$, $\eta_f = 0.027$, $\eta_2 = 0.239$, $\eta_1 = 0.126$
FIGURE 4.22.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST EXTERNAL DETUNING
P = 36.2, \( \sigma_1 = -28.6 \), \( \sigma_2 = -8.2 \)
\( \eta_B = 0.040 \), \( \eta_T = 0.027 \), \( \eta_2 = 0.239 \), \( \eta_1 = 0.126 \)
FIGURE 4.23.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 3.9 \), \( \sigma_1 = -22.6 \), \( \sigma_2 = 0.0 \)
\( \eta_3 = 0.040 \), \( \eta_1 = 0.027 \), \( \eta_2 = 0.239 \), \( \eta_1 = 0.126 \)
FIGURE 4.23.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST FORCE AMPLITUDE

\[ \sigma = 3.9, \ \sigma_1 = -22.6, \ \sigma_2 = 0.0 \]

\[ \eta_b = 0.040, \ \eta_1 = 0.027, \ \eta_2 = 0.239, \ \eta_1 = 0.126 \]
FIGURE 4.23.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\[ \sigma = 3.9 \quad \sigma_1 = -22.6 \quad \sigma_2 = 0.0 \]
\[ \eta_b = 0.040 \quad \eta_1 = 0.027 \quad \eta_2 = 0.239 \quad \eta_1 = 0.126 \]
FIGURE 4.24.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma = 13.2$, $\sigma_1 = -22.6$, $\sigma_2 = 0.0$
$\eta_b = 0.040$, $\eta_f = 0.027$, $\eta_2 = 0.233$, $\eta_1 = 0.126$
FIGURE 4.24.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST FORCE AMPLITUDE

\[ \sigma = 13.2, \ \sigma_1 = -22.6, \ \sigma_2 = 0.0 \]

\[ \eta_8 = 0.040, \ \eta_1 = 0.027, \ \eta_2 = 0.239, \ \eta_1 = 0.126 \]
FIGURE 4.24.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = 13.2 \), \( \sigma_1 = -22.6 \), \( \sigma_2 = 0.0 \)
\( \eta_b = 0.040 \), \( \eta_f = 0.027 \), \( \eta_2 = 0.239 \), \( \eta_1 = 0.126 \)
FIGURE 4.25.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE

\[ \sigma = -8.4, \ \sigma_1 = -22.6, \ \sigma_2 = 0.0 \]
\[ \eta_B = 0.040, \ \eta_1 = 0.027, \ \eta_2 = 0.239, \ \eta_4 = 0.126 \]
FIGURE 4.25.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST FORCE AMPLITUDE
\[ \sigma = -8.4, \sigma_1 = -22.6, \sigma_2 = 0.0 \]
\[ \eta_8 = 0.040, \eta_1 = 0.027, \eta_2 = 0.239, \eta_1 = 0.126 \]
FIGURE 4.25.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE

$\sigma = -8.4$, $\sigma_1 = -22.6$, $\sigma_2 = 0.0$

$\eta_0 = 0.040$, $\eta_1 = 0.027$, $\eta_2 = 0.239$, $\eta_3 = 0.126$
FIGURE 4.26.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE

\[ \sigma = 0.1, \sigma_1 = -18.1, \sigma_2 = 6.4 \]
\[ \eta_b = 0.040, \eta_1 = 0.027, \eta_2 = 0.239, \eta_3 = 0.126 \]
FIGURE 4.26.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST FORCE AMPLITUDE
\( \sigma = 0.1 \), \( \sigma_1 = -18.1 \), \( \sigma_2 = 6.4 \)
\( \eta_5 = 0.040 \), \( \eta_1 = 0.027 \), \( \eta_2 = 0.239 \), \( \eta_1 = 0.126 \)
FIGURE 4.26.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
σ = 0.1, σ_1 = -18.1, σ_2 = 6.4
η_B = 0.040, η_T = 0.027, η_2 = 0.239, η_1 = 0.126
FIGURE 4.27.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\[ \sigma = -13.9, \sigma_1 = -18.1, \sigma_2 = 6.4 \]
\[ \eta_0 = 0.040, \eta_1 = 0.027, \eta_2 = 0.239, \eta_3 = 0.126 \]
FIGURE 4.27.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST FORCE AMPLITUDE
$\sigma = -13.9$, $\sigma_1 = -18.1$, $\sigma_2 = 6.4$
$\eta_b = 0.040$, $\eta_1 = 0.027$, $\eta_2 = 0.239$, $\eta_1 = 0.126$
FIGURE 4.27.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\[ \sigma = -13.9, \ \sigma_1 = -18.1, \ \sigma_2 = 6.4 \]
\[ \eta_b = 0.040, \ \eta_1 = 0.027, \ \eta_2 = 0.239, \ \eta_1 = 0.126 \]
FIGURE 4.28.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma = 12.9$, $\sigma_1 = -18.1$, $\sigma_2 = 6.4$
$\eta_b = 0.040$, $\eta_f = 0.027$, $\eta_i = 0.239$, $\eta_1 = 0.126$
FIGURE 4.28.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST FORCE AMPLITUDE
\( \sigma = 12.9, \sigma_1 = -18.1, \sigma_2 = 6.4 \)
\( \eta_0 = 0.040, \eta_1 = 0.027, \eta_2 = 0.239, \eta_1 = 0.126 \)
FIGURE 4.28.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma = 12.9$, $\sigma_1 = -18.1$, $\sigma_2 = 6.4$
$\eta_B = 0.040$, $\eta_T = 0.027$, $\eta_2 = 0.239$, $\eta_1 = 0.126$
FIGURE 4.29.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma = 8.9$, $\sigma_1 = -28.6$, $\sigma_2 = -8.2$
$\eta_b = 0.040$, $\eta_1 = 0.027$, $\eta_2 = 0.239$, $\eta_1 = 0.126$
FIGURE 4.29.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST FORCE AMPLITUDE
\[ \sigma = 8.9, \sigma_1 = -28.6, \sigma_2 = -8.2 \]
\[ \eta_b = 0.040, \eta_f = 0.027, \eta_2 = 0.239, \eta_1 = 0.126 \]
FIGURE 4.29.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma = 8.9, \sigma_1 = -28.6, \sigma_2 = -8.2$
$\eta_b = 0.040, \eta_T = 0.027, \eta_2 = 0.239, \eta_1 = 0.126$
FIGURE 4.30.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = -1.0 \), \( \sigma_1 = -28.6 \), \( \sigma_2 = -8.2 \)
\( \eta_b = 0.040 \), \( \eta_t = 0.027 \), \( \eta_2 = 0.239 \), \( \eta_1 = 0.126 \)
FIGURE 4.30.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST FORCE AMPLITUDE
σ = -1.0, σ_1 = -28.6, σ_2 = -8.2
η_0 = 0.040, η_1 = 0.027, η_2 = 0.239, η_3 = 0.126
FIGURE 4.30.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
$\sigma = -1.0, \sigma_1 = -28.6, \sigma_2 = -8.2$
$\eta_b = 0.040, \eta_1 = 0.027, \eta_2 = 0.239, \eta_1 = 0.126$
FIGURE 4.31.1
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = -14.8 \), \( \sigma_1 = -28.6 \), \( \sigma_2 = -8.2 \)
\( \eta_0 = 0.040 \), \( \eta_1 = 0.027 \), \( \eta_2 = 0.239 \), \( \eta_1 = 0.126 \)
FIGURE 4.31.2
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
NONPLANAR AMPLITUDES AGAINST FORCE AMPLITUDE

\[ \sigma = -14.8, \quad \sigma_1 = -28.6, \quad \sigma_2 = -8.2 \]

\[ \eta_b = 0.040, \quad \eta_T = 0.027, \quad \eta_2 = 0.239, \quad \eta_1 = 0.126 \]
FIGURE 4.31.3
EXPERIMENTAL FOUR MODE SYSTEM RESPONSE
PLANAR AMPLITUDE AGAINST FORCE AMPLITUDE
\( \sigma = -14.8, \sigma_1 = -28.6, \sigma_2 = -8.2 \)
\( \eta_b = 0.040, \eta_r = 0.027, \eta_2 = 0.239, \eta_1 = 0.126 \)
FIGURE 4.32.1
EXPERIMENTAL TIME RESPONSE
FOUR MODE SYSTEM INTERACTION
\[ P = 33.0, \sigma = -8.0, \sigma_1 = -22.6, \sigma_2 = 0 \]
FIGURE 4.32.2
EXPERIMENTAL RESPONSE FREQUENCY SPECTRA
FOUR MODE SYSTEM INTERACTION
P = 33.0 , $\sigma = -8.0$ , $\sigma_1 = -22.6$ , $\sigma_2 = 0$
FIGURE 4.32.3
EXPERIMENTAL TIME RESPONSE
FOUR MODE SYSTEM INTERACTION
P = 132.0, \sigma = -8.0, \sigma_1 = -22.6, \sigma_2 = 0
FIGURE 4.32.4
EXPERIMENTAL RESPONSE FREQUENCY SPECTRA
FOUR MODE SYSTEM INTERACTION
P = 132.0 , σ = -8.0 , σ₁ = -22.6 , σ₂ = 0
FIGURE 4.32.5
EXPERIMENTAL TIME RESPONSE
FOUR MODE SYSTEM INTERACTION
\( P = 132.0 \), \( \sigma = 31.9 \), \( \sigma_1 = -22.6 \), \( \sigma_2 = 0 \)
FIGURE 4.32.6
EXPERIMENTAL RESPONSE FREQUENCY SPECTRA
FOUR MODE SYSTEM INTERACTION
P = 132.0, σ = 31.9, σ₁ = -22.6, σ₂ = 0
CHAPTER 5

Conclusions

5.1 General discussion

This study has examined some complex oscillatory instabilities in coupled beam systems. This particular configuration consisted of a main primary beam and a coupled secondary beam. The system kinematics were treated via the classical Euler-Kirchoff theory of rods. Two kinematical constraints were imposed on the motion of the secondary beam. The first constraint assumed an inextensibility of the beam elastic axis. The second constraint was based on the beam being thin and slender such that the principal curvature in the plane of greatest rigidity was assumed negligible. The resulting kinematical expressions provided a non-linear relationship between the combination bending-torsional motion of the secondary beam and the bending motion of the primary beam. To second order, this relationship is equivalent to the intuitive approach of considering the principal curvature in the plane of greatest rigidity as a vector projection of the curvature in the less rigid plane. However, this formal development of the system kinematics allows a methodical inclusion of higher order terms in a consistent manner.

The system equations of motion were derived by the method of Lagrange. The form of interaction between the coupled motions were clearly identifiable as a form typical of autoparametric instability. A two mode Galerkin approximation was assumed for the primary beam motion, while the coupled beam was considered as a two degree-of-freedom system. The resulting set of four coupled ordinary non-linear second order differential equations were then analyzed by the method of multiple scales. Two separate cases of resonance
conditions were studied. The first case considered the external resonance of the primary beam second bending mode together with an internal resonance between the primary beam motion and the secondary beam combination bending-torsional modes. The second case extended the previous case by including an additional internal resonance between the primary beam first bending mode and the secondary beam fundamental bending mode. First order approximation stationary solutions were determined for both resonance cases.

For the first resonance case, the response amplitudes were expressible in analytical form. The multiple scale solution predicted that the points of intersection of the linear resonance response curve of the externally excited mode with the non-linear response curve were always coincident with the points of lower vertical tangencies of the parametrically excited modes. In constrast, a previous study of a two-degree-of-freedom system, the solution by Struble's method predicted this coincidence of response curves only for an exactly internally resonant system. This difference in predicted behaviour is attributed to the form of detuning parameter used in the multiple scale method which is taken as the difference in frequencies whereas Struble's method expressed detuning as the difference of the squares of the frequencies.

For the second resonance case involving four modes, a closed form solution was not possible. A numerical solution of the system of non-linear equations representing the stationary responses of the four mode system was attempted. This was unsuccessful as the numerical algorithm used did not produce convergence to solutions of interest. Instead numerical solutions were obtained only for the special case of the system without damping. These results indicated
that the behaviour of the four mode system was qualitatively similar to the three mode system. In particular, the stability limits of the zero solution of the parametrically excited modes were identical for both cases. A quantitative change in response was also predicted where energy was transferred from the two modes of higher frequencies to the two remaining modes of lower frequencies.

Experimental tests were conducted on a mechanical model. For the three mode system there was reasonable agreement between measurements and predicted values. The predicted saturation effect of the externally excited mode when force amplitude is increased, was not observed in the experimental model. There was instead a general increase of the amplitude level of the externally excited mode with increasing force amplitude. An additional region of non-steady response was observed in the experimental model. This effect was confined to a region of external detuning corresponding to the right-hand overhanging branch of the response curve. This form of instability was not predicted in the present analysis.

For the four mode system, no direct comparison of the experimental results was possible. However, the experimental results did indicate some qualitative similarity with the three mode system responses. The suppression effect of the externally excited mode in the region of external resonance was particularly notable.

This present study and other previous studies of dynamical systems with internal resonance have been restricted to examining stationary responses. This has been primarily dictated by the difficulties in treating non-stationary responses. Numerical
simulation and experimental observations of such systems did however indicate the presence of non-stationary responses. These responses were generally reported to be amplitude modulated and appear to be almost periodic in nature.

5.2 Conclusions

In conclusion, this study has

i) formally established a physical basis for modelling non-linear interactions in a particular configuration of coupled beam systems.

ii) obtained first order stationary solutions to the case of the system with a single internal resonance between the second planar bending mode and the combination bending-torsional modes of the coupled beam. These results showed very reasonable agreement with measurements of the experimental model response.

iii) considered the first order stationary responses of the system with multiple internal resonances involving the planar first and second bending modes, and the combination bending-torsional modes of the coupled beam. Numerical solutions were obtained for the special case of the undamped system. These results provided only qualitative comparison with the measured responses of the experimental model.

5.3 Suggestions for further work

There is scope for further work to obtain numerical solutions to the four mode interaction problem taking into account damping. In addition, the mathematical modelling of the coupled beam system can be extended to include additional degrees of freedom. This creates the possibility of internal resonances between higher modes of motion. The system kinematics can also be developed to include higher order terms hence allowing the analysis of secondary
resonances. A recent theoretical study by Sethna and Bajaj (92) has presented an analytical method for determining critical system parameters at which non-stationary responses occur. This forementioned paper may provide the necessary impetus in advancing present knowledge into the general area of non-stationary response analysis.
**PRINCIPAL NOTATION**

$q_i$  generalized co-ordinates specifying displacement of primary beam.

$m_0$  secondary beam discrete mass.

$\lambda$  distance of $m_0$ from the coupling point of the primary and secondary beams.

$r$  radius of gyration of $m_0$ about secondary beam elastic axis.

$u_0, v_0$  generalized co-ordinates specifying bending displacements of $m_0$.

$w_0$  generalized co-ordinate specifying axial displacement of $m_0$.

$\phi_0$  generalized co-ordinate specifying angle of twist of $m_0$ about the secondary beam elastic axis.

$\Omega$  external excitation frequency.

$\omega_1, \omega_2$  linear natural frequencies of the planar first and second bending modes respectively.

$\omega_B, \omega_T$  linear natural frequencies of the nonplanar first bending and first torsional modes respectively.

$\varepsilon$  perturbation parameter.

$O(\varepsilon^n)$  symbol denoting terms of $n^{th}$ order and above.

$\sigma$  external detuning parameter, defined as

$$\Omega = \omega_2 + \varepsilon \sigma$$

$\sigma_1$  internal detuning parameter, defined as

$$\omega_2 = \omega_B + \omega_T + \varepsilon \sigma_1$$

$\sigma_2$  internal detuning parameter, defined as

$$\omega_1 = 2 \omega_B + \varepsilon \sigma_2$$
\[ b_1, b_2 \] non-dimensionalized amplitude of the planar first and second bending modes.

\[ a_1, a_2 \] non-dimensionalized amplitude of the nonplanar fundamental bending mode and torsional mode respectively.

The following symbols are defined on page 23 - 24.

\( \kappa_1, \kappa_2 \) principal curvatures of secondary beam elastic axis (Chapter 2, Appendix I)

OR

abbreviation for \( \kappa_{11}, \kappa_{22} \) respectively denoting ratio of elements of eigenvectors (Chapter 3).

\( \mu_1, \mu_2 \) abbreviation for \( \mu_{11}, \mu_{22} \) respectively denoting generalized mass ratios.

\( p \) ratio of coefficient of non-linearities.

\( P \) abbreviation for generalized force amplitude \( P_2 \).
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Appendix I: Kinematics of the secondary beam

The development of the following analysis is based on the classical Euler-Kirchoff theory of rods (107) and has been applied recently to evaluate the static lateral buckling load of a thin slender cantilever beam under a concentrated end load (2). This approach has also been extended to the dynamical analysis of rotor blades (108, 109) and beam elements (110, 111, 112).

Consider an unstressed beam of uniform cross-section with one end fixed at 0 and lying along the OZ axis (Fig. 1.1). The orthogonal OXYZ axes is taken to be a 'space-fixed' reference frame with the unit base vectors $\hat{X}, \hat{Y}, \hat{Z}$ corresponding to the X, Y, Z directions respectively.

Let a beam element be located initially at point P at a distance z from the origin 0. After some arbitrary deformation, the element is displaced to P' by the amount $u$, $v$ and $w$ in the X, Y and Z directions respectively. The displaced element is defined with an orthogonal axes system, $\hat{x}-\hat{y}-\hat{z}$, such that the $\hat{x}-\hat{y}$ plane always lie in the plane of the beam cross-section with the $\hat{x}$ and $\hat{y}$ axes along the principal directions, while the positive $\hat{z}$-axis is normal to the cross-section in the direction of increasing beam length. This is a 'body-fixed' reference system attached to the beam element cross-section with the unit base vectors $\hat{X}, \hat{Y}, \hat{Z}$ corresponding to the $\hat{x}, \hat{y}, \hat{z}$ directions respectively. The axes system so defined is termed the 'principal torsion-flexure axes' of the beam at that point.

The position vector of the displaced beam element at P' can be expressed as,

$$\mathbf{R} = u \hat{X} + v \hat{Y} + (z + w) \hat{Z}$$

(I.1)
FIGURE I.1 GEOMETRY OF DISPLACED ELEMENT
The unit tangent vector at \( P' \) is then given as,

\[
\frac{\partial \mathbf{R}}{\partial s} = \frac{\partial u}{\partial s} \mathbf{T} + \frac{\partial v}{\partial s} \mathbf{J} + \frac{\partial (z + w)}{\partial s} \mathbf{R} \tag{1.2 a}
\]

where \( s \) is the curvilinear co-ordinate with origin at 0 and defines the scalar distance of \( P' \) along the deformed elastic axis. Taking the following dot product,

\[
\frac{\partial \mathbf{R}}{\partial s} \cdot \frac{\partial \mathbf{R}}{\partial s} = \left( \frac{\partial u}{\partial s} \right)^2 + \left( \frac{\partial v}{\partial s} \right)^2 + \left[ \frac{\partial (z + w)}{\partial s} \right]^2 \tag{1.2 b}
\]

gives

\[
\frac{\partial (z + w)}{\partial s} = \sqrt{1 - \left( \frac{\partial u}{\partial s} \right)^2 \left( \frac{\partial v}{\partial s} \right)^2} \tag{1.2 c}
\]

or

\[
\frac{\partial s}{\partial z} = \sqrt{(1 + \frac{\partial w}{\partial z})^2 + (\frac{\partial v}{\partial z})^2 + (\frac{\partial u}{\partial z})^2} \tag{1.2 d}
\]

Hence to second order \( \frac{\partial}{\partial s} = \frac{\partial}{\partial z} \). From (1.2 c),

\[
\frac{\partial w}{\partial z} = -\frac{1}{2} \left[ \left( \frac{\partial u}{\partial z} \right)^2 + (\frac{\partial v}{\partial z})^2 \right] \tag{1.3}
\]

In addition to the linear displacements, the principal torsion-flexure axes at each beam element cross-section will also experience different rotations with respect to the elastic axis. This change
in orientation of the beam element can be described by three successive rotations about some suitably defined axes, these rotations being the Euler angles. Associated with this change in orientation is a rate of rotation vector, \( \vec{\omega} \), which is the resultant of the vectors describing the rate of change of the Euler angles about these predefined axes.

This is demonstrated in Fig. 1.2 which depicts the displaced beam element \( ds \) located relative to the space-fixed OXYZ axes by the angles \( \alpha, \beta \) and \( \phi \).

The first angle \( \alpha \), is about the OY axis and takes the XYZ axes to an intermediate \( x'y'z' \) axes with the unit vectors \( \vec{T}', \vec{J}', \vec{K}' \) such that,

\[
\begin{bmatrix}
\vec{T}' \\
\vec{J}' \\
\vec{K}'
\end{bmatrix} =
\begin{bmatrix}
\cos \alpha & 0 & -\sin \alpha \\
0 & 1 & 0 \\
\sin \alpha & 0 & \cos \alpha
\end{bmatrix}
\begin{bmatrix}
\vec{T} \\
\vec{J} \\
\vec{K}
\end{bmatrix}
\]  
(I.4)

The second angle \( \beta \), is about the ox' axis and takes the \( x'y'z' \) axes to the \( x'' y'' z'' \) axes with the unit vectors \( \vec{T}'', \vec{J}'', \vec{K}'' \), such that,

\[
\begin{bmatrix}
\vec{T}'' \\
\vec{J}'' \\
\vec{K}''
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \beta & -\sin \beta \\
0 & \sin \beta & \cos \beta
\end{bmatrix}
\begin{bmatrix}
\vec{T}' \\
\vec{J}' \\
\vec{K}'
\end{bmatrix}
\]  
(I.5)

The final angle \( \phi \) is about the oz'' axis and takes the \( x'' y'' z'' \) axes to the 'body-fixed' \( \hat{x'y'z'} \) axes such that,

\[
\begin{bmatrix}
\vec{T} \\
\vec{J} \\
\vec{K}
\end{bmatrix} =
\begin{bmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\vec{T}'' \\
\vec{J}'' \\
\vec{K}''
\end{bmatrix}
\]  
(I.6)
FIGURE 1.2 EULER ANGLES IN RELATION TO DISPLACEMENT VARIABLES
The rate of rotation vector can then be written as,

\[
\bar{\omega} = \alpha' \bar{J} - \beta' \bar{T} + \phi' \bar{K}
\]

\[
= (\alpha' \sin \phi \cos \beta - \beta' \cos \phi) \bar{T}
\]

\[
+ (\alpha' \cos \phi \cos \beta - \beta' \sin \phi) \bar{J}
\]

\[
+ (\alpha' \sin \beta + \phi') \bar{K}
\]

(I.7)

where \((\ )' = \frac{\partial}{\partial s} (\ ))

Any general displacement of the beam element is then determined by the three deflection components and the three Euler angles. However two of the angles are related to the derivatives of the deflection variables as seen from Fig. I.2. Since to second order \( \frac{\partial}{\partial s} = \frac{\partial}{\partial z} \), the derivatives \( \alpha' / \alpha s \) can now be replaced by \( \alpha' / \alpha z \). For some arbitrary displacements \( v'ds \) and \( u'ds \), the angles \( \alpha \) and \( \beta \) can be expressed as,

\[
\sin \alpha = \frac{u'}{\sqrt{1 - (v')^2}} \tag{I.8 a}
\]

\[
\cos \alpha = \frac{\sqrt{1 - (v')^2 - (u')^2}}{\sqrt{1 - (v')^2}} \tag{I.8 b}
\]

\[
\sin \beta = v' \tag{I.8 c}
\]

\[
\cos \beta = \sqrt{1 - (v')^2} \tag{I.8 d}
\]

From these equations, the derivatives of the angles can be obtained as,
\[ \alpha' = \frac{u''}{\{1 - (v')^2\}} + \frac{u' v' v''}{\sqrt{1 - (v')^2 - (u')^2}} \]  (I.9)

\[ \beta' = \frac{v''}{\sqrt{1 - (v')^2}} \]  (I.10)

The remaining angle \( \phi \) is then identified as the angle of twist of the beam elastic axis. \((\lambda') = \partial / \partial z, (\lambda'') = \partial^2 / \partial z^2\).

The rate of rotation vector from (I.7) in component form is written as,

\[ \omega_1 = \left[ \frac{u'' + u' v' v''}{\{1 - (v')^2\}} \right] \frac{\sqrt{1 - (v')^2}}{\sqrt{1 - (v')^2 - (u')^2}} \sin \phi \]

\[ - \frac{v''}{1 - (v')^2} \cos \phi \]  (I.11)

\[ \omega_2 = \left[ \frac{u'' + u' v' v''}{\{1 - (v')^2\}} \right] \frac{\sqrt{1 - (v')^2}}{\sqrt{1 - (v')^2 - (u')^2}} \cos \phi \]

\[ - \frac{v''}{\sqrt{1 - (v')^2}} \sin \phi \]  (I.12)

\[ \omega_3 = \left[ \frac{u'' v' + u' (v')^2 v''}{\{1 - (v')^2\}} \right] \frac{\sqrt{1 - (v')^2}}{\sqrt{1 - (v')^2 - (u')^2}} \]

\[ + \phi' \]  (I.13)

These components are associated with the principal bending curvatures.
\( \kappa_1 \) and \( \kappa_2 \), and the torsion \( \tau \) of the deformed elastic axis. To second order these quantities can be written as,

\[
\kappa_1 = u'' \phi - v'' \tag{1.14}
\]

\[
\kappa_2 = u'' + v'' \phi \tag{1.15}
\]

\[
\tau = \phi' + u'' v' \tag{1.16}
\]
Appendix II: Velocity of secondary beam mass

The resultant position vector of mass $m_0$ of the secondary beam is given from (2.2.1) as

$$\mathbf{P} = \mathbf{R} + \mathbf{\rho} \quad (II.1)$$

$\mathbf{R}$ is the position vector of the coupling point, and is

$$\mathbf{R} = L \mathbf{J} + q_1 \mathbf{K} \quad (II.2)$$

$\mathbf{\rho}$ is the position vector of the mass $m_0$ relative to the coupling point, and is

$$\mathbf{\rho} = u_0 \mathbf{I} + v_0 \mathbf{J} + (\lambda + w_0) \mathbf{K} \quad (II.3)$$

The absolute velocity is then given by,

$$\dot{\mathbf{P}} = \dot{\mathbf{R}} + \dot{\mathbf{\rho}} \quad (II.4)$$

with

$$\dot{\mathbf{\rho}} = (\mathbf{\rho})_r + \hat{\omega} \times \mathbf{\rho} \quad (II.5)$$

where

$(\mathbf{\rho})_r$: velocity vector of the mass $m_0$ relative to the coupling point.

$(\hat{\omega})$: absolute rotation rate vector of the secondary beam.

The moving frame base vectors are related to the inertial frame base vectors by the transformation,

$$\begin{bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_2 & \sin q_2 \\ 0 & -\sin q_2 & \cos q_2 \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{bmatrix} \quad (II.6)$$
where $q_2$ is the angle of rotation of the moving frame. For small angular rotations,

$$
\begin{bmatrix}
T \\
J \\
K
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & q_2 \\
0 & -q_2 & 1
\end{bmatrix}
\begin{bmatrix}
T \\
J \\
K
\end{bmatrix}
$$

Thus the absolute velocity vector is then written as,

$$
\dot{\mathbf{p}} = \dot{\mathbf{R}} + \dot{\mathbf{p}} = \dot{\mathbf{0}} + \dot{\mathbf{\Gamma}}
$$

$$
+ \left[ \dot{\mathbf{q}}_0 - q_2 (\mathbf{e} + \mathbf{w}_0) - q_2 (\dot{\mathbf{w}}_0 + \dot{\mathbf{q}}_2 \mathbf{v}_0) \right] \mathbf{J}
$$

$$
+ \left[ \dot{\mathbf{q}}_1 + q_2 \left( \dot{\mathbf{v}}_0 - \dot{\mathbf{q}}_2 (\mathbf{e} + \mathbf{w}_0) \right) + (\dot{\mathbf{w}}_0 + \dot{\mathbf{q}}_2 \mathbf{v}_0) \right] \mathbf{K}
$$

Taking the dot product of (II.8) with itself gives,

$$
\dot{\mathbf{p}} \cdot \dot{\mathbf{p}} = \dot{\mathbf{q}}_0^2 + \dot{\mathbf{q}}_0^2 + \dot{\mathbf{q}}_2^2 (\mathbf{e} + \mathbf{w}_0)^2 + q_2^2 (\dot{\mathbf{w}}_0 + \dot{\mathbf{q}}_2 \mathbf{v}_0)^2
$$

$$
- 2 \dot{\mathbf{q}}_0 \dot{\mathbf{q}}_2 (\mathbf{e} + \mathbf{w}_0) - 2 \dot{\mathbf{q}}_0 q_2 (\dot{\mathbf{w}}_0 + \dot{\mathbf{q}}_2 \mathbf{v}_0)
$$

$$
+ 2 q_2 (\mathbf{e} + \mathbf{w}_0) (\dot{\mathbf{w}}_0 + \dot{\mathbf{q}}_2 \mathbf{v}_0)
$$

$$
+ \dot{\mathbf{q}}_1^2 + q_2^2 \left[ \dot{\mathbf{v}}_0 - \dot{\mathbf{q}}_2 (\mathbf{e} + \mathbf{w}_0) \right]^2 + (\dot{\mathbf{w}}_0 + \dot{\mathbf{q}}_2 \mathbf{v}_0)^2
$$

$$
+ 2 \dot{\mathbf{q}}_1 + q_2 \left[ \dot{\mathbf{v}}_0 - \dot{\mathbf{q}}_2 (\mathbf{e} + \mathbf{w}_0) \right] + 2 \dot{\mathbf{q}}_1 (\dot{\mathbf{w}}_0 + \dot{\mathbf{q}}_2 \mathbf{v}_0)
$$

$$
+ 2 q_2 \left[ \dot{\mathbf{v}}_0 - \dot{\mathbf{q}}_2 (\mathbf{e} + \mathbf{w}_0) \right] (\dot{\mathbf{w}}_0 + \dot{\mathbf{q}}_2 \mathbf{v}_0)
$$
\[= \dot{u}_0^2 + \dot{v}_0^2 + \dot{q}_2^2 \ell^2 - 2 \ddot{v}_0 \ddot{q}_2 \ell \]

\[+ \dot{q}_1^2 + \dot{w}_0^2 + 2 \dot{q}_1 \dot{w}_0 + 2 \dot{q}_2^2 \ell w_0 \]

\[- 2 \ddot{q}_2 w_0 v_0 - 2 q_2 \ddot{w}_0 \ddot{v}_0 + 2 q_2 \ddot{q}_2 \ell w_0 \]

\[+ 2 \ddot{w}_0 \ddot{q}_2 v_0 + 2 \dddot{q}_1 q_2 \dddot{v}_0 - 2 \dddot{q}_1 q_2 \dddot{q}_2 \ell \]

\[+ 2 \dddot{q}_1 q_2 v_0 + 2 \dddot{q}_2 \dddot{v}_0 w_0 - 2 q_2 \dddot{q}_2 \ell \dddot{w}_0 \]

\[+ 0 (\varepsilon^4) \quad (\text{II.9}) \]
Appendix III : Deflection form function integrals

The assumed deflection form functions for the secondary beam are given as,

\[ u = f(z) u_0(t) \]  \hspace{1cm} (III.1)

\[ \phi = g(z) \phi_0(t) \]  \hspace{1cm} (III.2)

with

\[ f(z) = \left[ \frac{3}{2} \frac{z^2}{\lambda^2} - \frac{1}{2} \frac{z^3}{\lambda^3} \right] \]  \hspace{1cm} (III.3)

\[ g(z) = \frac{z}{\lambda} \]  \hspace{1cm} (III.4)

From the constraint of negligible curvature \( \kappa_1 = 0 \),

\[ v'' = u'' \phi \]  \hspace{1cm} (III.5)

\[ v_0 = \int_0^1 \int_0^\zeta u'' \phi \, dz \, d\zeta \]

\[ = B_4 u_0 \phi_0 \]  \hspace{1cm} (III.6)

where

\[ B_4 = \int_0^1 \int_0^\zeta f''(z) g(z) \, dz \, d\zeta \]

\[ = \frac{1}{4} \]  \hspace{1cm} (III.7)

The kinematical constraint relating axial to transverse motion is given as,
\[
w_0 = \int_0^l \frac{1}{2} \left[ (u')^2 + (v')^2 \right] \, dz \quad (\text{III.8})
\]

From (\text{III.5}),

\[
v_0 = \int_0^l u'' \phi \, dz \quad (\text{III.9})
\]

then to second order, the term \(v'\) in (\text{III.8}) can be neglected giving,

\[
w_0 = \int_0^l \frac{1}{2} (u')^2 \, dz
\]

\[
= \frac{1}{2} B_3 u_0^2 \quad (\text{III.10})
\]

where

\[
B_3 = \int_0^l \left[ f'(z) \right]^2 \, dz
\]

\[
= \frac{6}{5l} \quad (\text{III.11})
\]

The strain energy expression of the secondary beam is given as,

\[
V_{2,} = \int_0^l \left( \frac{1}{2} E I_{yy} \kappa_2^2 + \frac{1}{2} G J \tau^2 \right) \, dz \quad (\text{III.12})
\]

The principal curvature \(\kappa_2\) and torsion \(\tau\) are given as

\[
\kappa_2 = u'' + v'' \phi \quad (\text{III.13})
\]

\[
\tau = \phi' + u'' v' \quad (\text{III.14})
\]

To second order,
\[ \kappa_2 = u'' \quad \text{(III.15)} \]

\[ \tau = \phi' \quad \text{(III.16)} \]

then

\[ V_2 = \int_0^l \left[ \frac{1}{2} EI_{yy} (u'')^2 + \frac{1}{2} GJ (\phi')^2 \right] \, dz \]

\[ = \frac{1}{2} B_1 u_0^2 + \frac{1}{2} B_2 \phi_0^2 \quad \text{(III.17)} \]

where

\[ B_1 = EI_{yy} \int_0^l \left[ f''(z) \right]^2 \, dz \]

\[ = 3 \frac{EI_{yy}}{l^3} \quad \text{(III.18)} \]

\[ B_2 = GJ \int_0^l \left[ g'(z) \right]^2 \, dz \]

\[ = \frac{GJ}{l} \quad \text{(III.19)} \]
The elements of the matrix $[C]$ governing the stability of the stationary solutions are given as $C_{ij}$, ($i^{th}$ row and $j^{th}$ column)

\[
C_{11} = -\eta_B \omega_B
\]
\[
C_{12} = \sqrt{\eta_B \eta_T \omega_T}
\]
\[
C_{13} = \eta_B \omega_B a_{10}/b_{20}
\]
\[
C_{14} = 0
\]
\[
C_{15} = -\eta_B \omega_B K_2 a_{10}
\]
\[
C_{21} = \sqrt{\eta_B \eta_T \omega_B}
\]
\[
C_{22} = -\eta_T \omega_T
\]
\[
C_{23} = \sqrt{\eta_B \eta_T \omega_B a_{10}/b_{20}}
\]
\[
C_{24} = 0
\]
\[
C_{25} = -\sqrt{\eta_B \eta_T \omega_B K_2 a_{10}}
\]
\[
C_{31} = -\eta_B \omega_B^2 K_1 a_{10}/b_{20}
\]
\[
C_{32} = -\sqrt{\eta_B \eta_T \omega_T K_1 a_{10}/b_{20}}
\]
\[ C_{33} = - \eta_2 \omega_2 \]

\[ C_{34} = - \sigma b_{20} + \eta_B \omega_B^2 K_1 K_2 \frac{a_{10}^2}{b_{20}} \]

\[ C_{35} = \eta_B \omega_B^2 K_1 K_2 \frac{a_{10}^2}{b_{20}} \]

\[ C_{41} = - \eta_B \omega_B^2 K_1 K_2 \frac{a_{10}}{b_{20}} \]

\[ C_{42} = - \sqrt{\eta_B \eta_T \omega_B \omega_T} K_1 K_2 \frac{a_{10}}{b_{20}} \]

\[ C_{43} = \sigma / b_{20} \]

\[ C_{44} = - \eta_2 \omega_2 + \eta_B \omega_B^2 K_1 \frac{a_{10}^2}{b_{20}} \]

\[ C_{45} = - \eta_B \omega_B^2 K_1 \frac{a_{10}^2}{b_{20}} \]

\[ C_{51} = - c_{41} - (\eta_T \omega_T - \eta_B \omega_B) \frac{K_2}{a_{10}} \]

\[ C_{52} = - c_{42} - \left( \frac{n_B \omega_B}{n_T \omega_T} - \frac{\eta_T \omega_T}{\eta_B \omega_B} \right) \sqrt{\eta_T \omega_T} \frac{K_2}{\sqrt{n_B \omega_B} \frac{a_{10}}{a_{10}}} \]

\[ C_{53} = - (2 \sigma + \sigma_1) / b_{20} \]

\[ C_{54} = - c_{44} \]

\[ C_{55} = - c_{45} - (\eta_B \omega_B + \eta_T \omega_T) \]

where

\[ K_1 = \mu_2 \frac{(\omega_B + \omega_T)^2}{\kappa_2 \omega_B^3} \]

\[ K_2 = \frac{(\sigma + \sigma_1)}{(n_B \omega_B + n_T \omega_T)} \]
Appendix V: Listing of computer program for obtaining numerical solutions to the four mode system without damping

IMPLICIT REAL*8 (A-H,O-Z)  
INTEGER*4 IDIRECT(3)  
REAL*8 KAPPA1,KAPPA2,MU1,MU2,GAMMA(3,2)  
REAL*8 X(4),R(4),X0(4),SOLN(5,4),AJINV(4,4),W(70)  
REAL*8 VECTOR(11074),EXETUNING(110),Y(4),VGAMMA(8,3)  
COMMON /BLOCK1/CF(13),SCALE(4)  
EXTERNAL COEFF,RESIII,MONIT  
DATA NSIGMA0, NRANGESIGMA0, SIGMA0INT/0,50,1/  
DATA SM3LR,SM3UR,DSM3/-50.00,50.00,0.20/  
DATA NRESULTS, IDIRECT / 5,1,-1,0/  
DATA W1,W2,WT,10,0.50,0.5,45/  
DATA RH0,MU1,MU2/0.121E-2,0.361E-2,321E-2/  
DATA KAPPA1, KAPPA2/14.00,5.83D0/  
DATA A10,A20,B10,B20/0.312,0,4110,0,1D-1,0,1B-1/  
DATA GAMMA/1,11,-1,-1,-1/  
DATA A1MIN,A2MIN/0.410,0.0110/  
DATA NTA,BETTA/4,4,0,2D-7/  
DATA STEFX,MAXCAL,IPRINT/550,2001,3000/  
IW = N*(5+2*NN)  
FTOL = .10.0*X02AAF(RR)  
DO 12000 IGAM=1,2  
DO 12000 JGAM=1,2  
DO 12000 KGAM=1,2  
DO 30000 NN=1,2  
NDIRECT = IDIRECT(NN)  
X0(1) = A10  
X0(2) = A20  
X0(3) = B10  
X0(4) = B20  
WRITE (NRESULTS,6000)  
DO 15000 I=1,N  
SCALE(I) = 1/X0(I)  
15000 X0(I) = X0(I)*SCALE(I)  
DO 15500 I=1,5  
DO 15500 J=1,4  
15500 SOLN(I,J) = 0.0D0  
LL = 1  
SIGMA0 = FLOAT(NRANGESIGMA0)*NDIRECT  
SM3INT = - SIGMA0INT*NDIRECT  
18000 CONTINUE  
CALL COEFF (&W,WT,W1,W2,MU1,MU2,KAPPA1,KAPPA2,  
* RH0,P2,SIGMA2,SIGMA0,SIGMA1)  
CF(4) = CF(4)*GAMMA(3,KGAM)  
CF(5) = CF(5)*GAMMA(2,JGAM)  
CF(6) = CF(6)*GAMMA(3,KGAM)  
CF(8) = CF(8)*GAMMA(3,KGAM)  
CF(9) = CF(9)*GAMMA(2,JGAM)  
CF(10) = CF(10)*GAMMA(2,JGAM)  
CF(12) = CF(12)*GAMMA(2,JGAM)  
CF(13) = CF(13)*GAMMA(1,IGAM)
DO 16000 II=1,N
16000 X(II) = X0(II)
28000 IFL = 1
    CALL COSMCAF(N,X,R,F,AJINV,IA,W,IW,FTOL,DELTA,
* STEPMX,RESID,MONT,IPRINT,MAXCAL,IFL)
    EXDETUNING(LL) = SIGMA0
    DO 10000 II=1,4
10000 Y(II) = X(II)/SCALE(II)
    IF (Y(1) .GE. A1MIN.AND.Y(2) .GE. A2MIN.AND.Y(3) .GT. 0.0.DO.
* AND.Y(4) .GT. 0.0.DO.AND.IFL.EQ.0) GOTO 29000
    IF (SOLN(1,1) .GE. A1MIN.AND.SOLN(5,1) .GE. A1MIN) GOTO 23000
    GOTO 25000
23000 DO 24000 II=1,N
    SCALE(II) = 1/SOLN(1,II)
24000 X0(II) = SOLN(1,II)*SCALE(II)
29000 110 27000 J=194
    SOLN(1,J) = Y(J)
    DO 27000 1=1,4
27000 SOLN(I+1,J) = SOLN(I,J)
    DO 31000 11=1,4
31000 SCALE(II) = 1/Y(II)
    DO 25000 II=1,4
    SCALE(II) = 1/Y(II)
31000 X0(II) = Y(II)*SCALE(II)
25000 CONTINUE
00 11000 II=1,N
    VECDOR(LL,II) = Y(II)
    WRITE (NRESULTS,6100) (GAMMA(1,IGAM),GAMMA(2,JGAM),
* GAMMA(3,KGAM),EXDETUNING(LL),(VECTOR(LL,JJ),JJ=1,N),F)
    LL = LL + 1
    SIGMA0 = SIGMA0 + SM3INT
    IF (NDIRECT .LT. 0.AND.SIGMA0 .LE. 50.0.DO) GOTO 18000
    IF (NDIRECT .GT. 0.AND.SIGMA0 .GE. -50.0.DO) GOTO 18000
30000 CONTINUE
12000 CONTINUE
STOP
6000 FORMAT (/,' GM1 GM2 GM3 SIGMA0',6X,'A1',10X,'A2',10X,
* 'B1',10X,'B2',6X,'SUM OF SQUARES')
6100 FORMAT (1X,3(I2,2X),I6,4(D12,4),D14,4)
END

SUBROUTINE MONIT(N,XC,RC,FC,NCALL)
INTEGER N,NCALL
REAL*8 FC,XC(N),RC(N)
WRITE (8,9000) NCALL,FC,(XC(I),I=1,N)
RETURN
9000 FORMAT (I5,4X,D12,4,1X,4(3X,D12,4))
END
SUBROUTINE COEFF (WB, WT, W1, W2, MU1, MU2, KAPPA1, KAPPA2,  
* RHO, P2, SIGMA2, SIGMA0, SIGMA1)
IMPLICIT REAL*8 (A-H, O-Z)
REAL*8 MU1, MU2, KAPPA1, KAPPA2, GAMMA(3, 2)
COMMON /BLOCK1/CF(13), SCALE(4)
CF(1) = KAPPA1*W1*W1*W1/(2*MU1*WB*WB*WB)
CF(2) = WT/WB
CF(3) = WB*W1*SIGMA2
CF(4) = RHO*KAPPA1*W1*W1*W1/2
CF(5) = KAPPA2*W2*W2*W1/2
CF(6) = RHO*MU1*WB*WB*WB/2
CF(7) = WB*WT*(SIGMA0+SIGMA1)
CF(8) = RHO*KAPPA1*W1*W1*WT/4
CF(9) = KAPPA2*W2*W2*WT/4
CF(10) = KAPPA2*W2*W2*WB/4
CF(11) = W2*SIGMA0
CF(12) = MU2*(WB+WT)*(WB+WT)/4
CF(13) = P2/2
CF(1) = CF(1)/(SCALE(3)*SCALE(3))
CF(2) = CF(2)/(SCALE(2)*SCALE(2))
CF(3) = CF(3)/(SCALE(1)*SCALE(3))
CF(4) = CF(4)/(SCALE(1)*SCALE(3)*SCALE(3))
CF(5) = CF(5)/(SCALE(2)*SCALE(3)*SCALE(4))
CF(6) = CF(6)/(SCALE(1)*SCALE(1)*SCALE(4))
CF(7) = CF(7)/(SCALE(1)*SCALE(2))
CF(8) = CF(8)/(SCALE(1)*SCALE(2)*SCALE(3))
CF(9) = CF(9)/(SCALE(2)*SCALE(2)*SCALE(4))
CF(10) = CF(10)/(SCALE(1)*SCALE(1)*SCALE(4))
CF(11) = CF(11)/SCALE(4)
CF(12) = CF(12)/(SCALE(1)*SCALE(2))
RETURN
END

SUBROUTINE RESID(N, XC, RC)
IMPLICIT REAL*8 (A-H, O-Z)
INTEGER N
REAL*8 XC(N), RC(N)
COMMON /BLOCK1/CF(13), SCALE(4)
XC1S = XC(1)*XC(1)
XC2S = XC(2)*XC(2)
XC3S = XC(3)*XC(3)
RC(1) = XC1S + CF(1)*XC3S - CF(2)*XC2S
RC(2) = CF(3)*XC1S*XC(1) - CF(4)*XC1S*XC(1)
* - CF(5)*XC(2)*XC(3)*XC(4) + CF(6)*XC1S*XC(1)
RC(3) = CF(7)*XC1S*XC(2) - CF(8)*XC1S*XC(2)*XC(3)
* - CF(9)*XC2S*XC(4) - CF(10)*XC1S*XC(4)
RC(4) = CF(11)*XC(4) - CF(12)*XC1S*XC(2) + CF(13)
RETURN
END
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I hereby declare that this thesis has been composed by myself and that the work is my own.

S.L. BUX
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