COMPUTATIONAL STUDIES IN STELLAR DYNAMICS

Winston Lemay Sweatman

Abstract

Computational studies have been used in conjunction with theoretical approaches to investigate a number of problems in stellar dynamics. These problems have particular relevance to globular star clusters.

The investigations began in the area of three-body scattering (i.e. encounters between a binary and a single star), dealing especially with close triple encounters. A prediction was made, using two theoretical approaches, of the probability distribution for the energy of the binary at the end of an encounter, in cases where the energy is either very large or very small. Programs were written to run on the Edinburgh University mainframe computer to provide a numerical test of the theory.

To tackle larger problems an N-body code has been developed for the Edinburgh Concurrent Supercomputer, and its performance analysed. The analysis included a brief study of the optimum order of the algorithm used for this code. The program has been used to simulate Plummer model star clusters containing 1024 and 10048 stars. From the results of these simulations, investigations have been made into the problems of Lagrangian radii oscillations and core wandering. The latter is the motion of the densest part of the star cluster, whilst the former is to do with the movement of mass towards and away from this position. The approach involved a direct look at the variation in the coordinates, supplemented by the computation of autocorrelations and variances.
"Not from the stars do I my judgement pluck,
And yet methinks I have astronomy,
But not to tell of good or evil luck,
Of plagues, of dearths, or seasons' quality:
Nor can I fortune to brief minutes tell,
Pointing to each his thunder, rain, and wind,
Or say with princes if it shall go well
By oft predict that I in heaven find.
But from thine eyes my knowledge I derive,
And, constant stars, in them I read such art
As truth and beauty shall together thrive
If from thyself to store thou wouldst convert;
Or else of thee this I prognosticate:
Thy end is truth's and beauty's doom and date."

William Shakespeare: Sonnet XIV
Foreword

The inspiration to use computers and investigate topics in stellar dynamics came from my supervisor Dr. Douglas C. Heggie. He has remained interested, encouraging and enthusiastic throughout our relationship and with an "open door" policy has been readily available for consultation. I greatly appreciate his continuing help, advice and friendship.

I am grateful to the Science and Engineering Research Council for a research studentship, to the Department of Mathematics at Edinburgh University for a place in the department, to the Edinburgh Concurrent Supercomputer Project for computer time and to Heriot-Watt University Mathematics Department where I am currently a lecturer. I have had many useful discussions on mathematics, astronomy and computing. In particular I would like to thank Dr. Frank J. Śmiej a and Mr. Greg V. Wilson for their help with the Edinburgh Concurrent Supercomputer.

The work of Chapter II was part of a collaborative project with Dr. Heggie, part of which has been published elsewhere. We are grateful to Dr. S.J. Aarseth for the provision of a copy of the RX78 integration subroutine, and for helpful comments on a previous draft of that paper. Unless specific reference has been made to the work of others, the results described and derived here are original and my own.

I would like to express my thanks to my parents by dedicating this thesis to them.

Contents

Abstract i
Photograph of NGC 362 ii
Foreword iii

Chapter I Introduction 1

Chapter II Three-Body Scattering Near Triple Collision Or Expansion 6

1. Introduction 6

2. The collinear problem 8

2.1 Use of McGehee’s coordinates 9

2.1.1 Coordinate transformations 10

2.1.2 The zero-energy surface 13

2.1.3 Neighbourhood of stationary configurations 16

2.1.4 Binary binding energy after passing close to triple collision 20

2.1.5 Binary binding energy after passing close to triple expansion 21

2.2 Second approach 22

2.3 A comparison of the two approaches 26

2.4 Probability of ionisation in the proximity of triple expansion 29

2.4.1 From the first study 29

2.4.2 From the second study 30

2.5 Numerical integrations in one dimension 32

2.5.1 Numerical method 32

2.5.2 Comparison with the analytic approach 36

3. Extension to two and three dimensions 39
3.1 Eigenvalues: the Siegel exponents 40
3.2 Final binary binding energy for h=0 41
3.3 A comparison of the probabilities of forming very tight binaries near triple collision for the two configurations 43
3.4 A comparison of the probabilities of forming very loose binaries near triple expansion for the two configurations 46
3.5 Relative probabilities of ionisation for small positive h 47
3.6 Numerical integrations in two dimensions 49
3.6.1 Initial conditions 49
3.6.2 Comparison of numerical and analytic results 51
3.7 Summary 58
4. The connection problem 61
5. Discussion and conclusions 65
5.1 Comparison with results in the literature 65
5.2 Conclusions 68

Chapter III The Development Of An N-Body Code For The Edinburgh Concurrent Supercomputer 70
1. Introduction 70
2. Computing environment 72
3. The program 73
3.1 Overall structure 73
3.2 Communications 74
3.3 The integration algorithm in the main sequence of operation 76
3.3.1 Statistical validity and accuracy 78
3.3.2 Algorithm of the first program: the lockstep leapfrog algorithm 78
3.3.3 Ideas for algorithm development from the lockstep leapfrog program

3.3.4 Introducing individual variable timesteps

3.3.5 Changing to a higher order algorithm

3.3.6 New timestep

3.4 The development of structure from the lockstep leapfrog program

3.5 Timestep sorting on master

3.6 Initialisation

3.7 The duration of the integration

4. The performance of the program

4.1 Tests performed on the final program

4.2 Optimal order

5. Fixed-potential models

6. The simulations

Chapter IV Lagrangian Radii Oscillations

1. Introduction

2. Measurement from the simulations

3. Analysis of the results

3.1 Observed oscillations

3.2 A related quantity: the mass within a fixed radius

3.3 The autocorrelations of the Lagrangian radii time series

3.4 Autocorrelations of the masses in fixed radii

3.4.1 Exact autocovariance

3.4.2 Autocovariance estimated from a discrete finite time series
3.4.3 A comparison of the mass autocorrelation and corresponding Lagrangian radius autocorrelation at given radii in the fixed potential

3.5 Comparing the different models and different radii

3.5.1 Differences in frequency and variance for the various models

3.5.2 Relating the frequency and variance in the various models

4. A comparison with previous work by other authors

5. Conclusions

Chapter V Core Wandering

1. Introduction

2. Defining the centre of the core of a globular star cluster

3. Results from the simulations

3.1 Direct measurements

3.2 Autocorrelation and variance

4. Discussion and conclusions

4.1 A comparison with other author's results

4.2 Conclusions

Appendices

A1. The heapsort routine

A2. Generating initial conditions to fit a Plummer model

References
I

Introduction

"Twinkle, twinkle, little star,
How I wonder what you are!"

Jane Taylor: "The Star"

During the past thirty or so years computers have been applied to the study of stellar dynamics. The early work of von Hörner (1960, 1963) studied the evolution of systems with up to twenty-five bodies. Later came more sophisticated algorithms in the work of Aarseth (1985 and references therein) and Wielen (1967) that dealt with larger systems. Meanwhile, the power of computers has increased so that problems with thousands of bodies may now be tackled, though there is still some way to go before computation reaches a speed where the equations of motion of a realistic model of a globular star cluster can be integrated within the lifetime of the cluster (see e.g. Hut, Makino & McMillan (1988)). In this thesis computers are used to model larger systems that may show some of the properties of a globular star cluster, and smaller systems that represent a constituent part of such a cluster. It is the latter kind of system which shall be studied first.

First we shall outline the format of this thesis. It is divided into five chapters (including this introduction). Some chapters are subdivided into sections (§ 1, § 2, etc.) and subsections (e.g. § 1.1, § 3.4.1). References to subsections refer to the ones within the current chapter, unless explicitly stated otherwise. References to the work of other authors are given by name and year and are listed in alphabetical order after the appendices at the end of the entire dissertation.
Binary stars (bound pairs of stars) play a rôle in the mechanics of a cluster of stars (see e.g. Heggie(1975)). Three-body scattering, which is the name given to interactions between a single star and a binary, is a fundamental process in the evolution of star clusters. During such an interaction there may be a change in the energy of the binary, where the energy measures how quickly the two stars orbit about each other and how far apart they are. Energy absorbed or given out by the binary results in changes in the speed at which the binary and the single star move through the cluster. The importance of three-body scattering lies in the cumulative effect of many such encounters, which may drive the evolution of the much larger stellar system within which they occur. Statistical results on three-body scattering are necessary for realistic modelling of larger stellar systems.

Close triple encounters occur when all three stars in a three-body scattering pass at a distance which is small compared with the initial or final size of the binary. The research reported in Chapter II uses two theoretical approaches to study the outcome of such encounters. Numerical simulations are then used to support these studies. (They were performed upon the mainframe computer at Edinburgh University, using a computer program written in the computing language FORTRAN.)

The chapter begins with the one-dimensional collinear problem, looking at the orbits close to triple collision/expansion on the McGehee manifold (McGehee 1974). The time spent in the neighbourhood of triple collision/expansion determines the energy of the resulting binary, and asymptotic predictions can be made about the energy distribution.
Similar results can be obtained for the probability of ionisation. These results can also be found by examining the Siegel exponents (e.g. Siegel & Moser 1971) for motion in the vicinity of triple collision/expansion. They are illustrated by a series of numerical integrations.

The study is then extended to two and three dimensions. There are two possible configurations close to triple collision/expansion: the equilateral and the collinear configurations. As in one dimension, a series of analytic predictions are made and are compared with the results of numerical simulations. The predictions are also compared with numerical data that has already been presented in papers by Szebehely (1974), Alexander (1986), and Hut & Bahcall (1983).

In Chapter III the work progresses on to the study of larger systems. An N-Body code has been developed in the OCCAM programming language, to run on the Edinburgh Concurrent Supercomputer (E.C.S.), a far more powerful machine than the mainframe used for the studies of Chapter II. The E.C.S. is an array of fast computers which work in parallel, sharing between them the intensive numerical effort required. The program was based upon the traditional methods and algorithms given by Aarseth(1985) and Wielen(1967), but adapted to this new and developing computational environment. A detailed study showed how much more quickly the computations could be carried out with a parallel computer than with one containing only a single processor. Another code was written for the computer to simulate systems of stars moving in a fixed-potential; i.e. the stars move under a fixed field of force, rather than the force exerted by the moving stars themselves. These models act as a kind of
"control" for the other (self-consistent) simulations. Four simulations are described, based on either self-consistent or fixed potentials, and containing approximately either a thousand or ten thousand stars.

In Chapters IV and V, we use the results gathered from the simulations of Chapter III to study two features of star cluster models: oscillations of "Lagrangian radii" and "core wandering", respectively. These phenomena may be described loosely as follows. In clusters with dynamical equilibrium the stars are in ceaseless motion, but on average the clusters have almost a constant structure, just as the molecules of air in a room move at hundreds of metres per second, even though the air is almost at rest. Looked at closely, however, the clusters are seen to be oscillating slightly in various ways, and one of the main aims of the simulations was to study these motions. If one measures the radius of an imaginary sphere containing a fixed proportion of a particular cluster's mass (Lagrangian radius) it appears to perform regular oscillations (Heggie(1989)). Also, in previous simulations other authors (Makino & Sugimoto 1987, Heggie 1989) have observed surprisingly large oscillatory movements of the core of the system (i.e. the region with the highest concentration of stars). Some simplified simulations of stellar systems are forced to assume that the core is at rest or that the distribution is smooth and unchanging. However, movements of the core and radial oscillations could influence the motions of the stars and hence the dynamical evolution of the entire system. Therefore, we need to study these phenomena and to determine their physical nature, in order to assess their effect upon unconstrained systems. During the simulations two estimates for the position of the centre of the core and
measurements of the Lagrangian radii were recorded at intervals in order to follow their changes.
II

Three-Body Scattering Near Triple Collision Or Expansion

"Two's company, three's a crowd."

proverb

1. Introduction

Three-body scattering is one of the fundamental processes of stellar dynamics, especially in the study of star clusters. The reason for its importance is that interactions between binary stars and single stars (in three-body scattering events) can lead to changes in the energies of the participants, which in turn cause evolution of the entire system (see, for example, parts of the review by Elson et al. (1987)).

For purposes of application, then, what is needed is information on the energetics of three-body interactions, especially the distribution (for a random selection of encounter parameters) of the change in the binding energy of the binary. Numerical experiments (e.g. Hut & Bahcall 1983, Hut 1984) are an important source of information here, but there are several regimes in which analytic approximations can help. For example, the cases of very fast (impulsive) encounters, or tidal ones (in which the single star does not come particularly close to either component of the binary) can be analysed with some exactness. Such results often complement the numerical results, as they suggest analytic approximations for fitting to the numerical results, and also indicate how the numerical results may be extended. An attempt to marry these two approaches is given in Hut & Heggie (1991).

The purpose of the work presented in this chapter was to introduce
and exploit an analytic approximation which extends these results into a corner of parameter space which had not been adequately explored hitherto in this context, and to verify the results numerically. It concerns those encounters in which all three stars temporarily come very close to each other, in a suitable sense; that is, very close when compared with the initial, or final, semi-major axis of the binary. As will be seen, these are circumstances in which the binary can become very much more energetic, or its energy may be reduced to a very small fraction of its initial value.

Before proceeding, we must clarify what is meant by saying that this analytical method is hitherto largely unexplored in this context. In the context of atomic scattering it has been known for a long time, and the main example of its use is in the study of threshold behaviour (Wannier 1953), i.e. the energy-dependence of a cross-section (such as an ionisation cross-section) near the limiting energy at which the process first becomes energetically possible. Even in the gravitational context the study of close triple approach has a lengthy history. Many fundamental results will be found in the book by Siegel & Moser (1971), and the alternative approach pioneered by McGehee (1974) has attracted much attention. The former approach has been used to exhibit individual orbits for three point-masses near close triple approach (e.g. Marchal & Losco 1980), but little has been done to provide the statistical information which is of importance in stellar dynamics. That is the purpose of our study. Statistical questions are discussed by Grujić & Simonović (1988), but our results differ from theirs, for a reason to be explained in § 3 below.
For the purposes of this investigation, numerical simulations of three-body interactions have been carried out on the Edinburgh University mainframe computer. They model encounters in one and two dimensions. Their purpose is to support the analytic approaches and act as a partial verification. Nevertheless, the novelty of the analytic methods requires that they be described in some detail, even though the main emphasis of this thesis is numerical.

The chapter begins with a study, from several points of view, of the one-dimensional (collinear) case. This is entirely artificial, but is easily visualised, thanks to McGehee's method, and can be fairly completely described. In particular it is shown (§ 2) that the two approaches (McGehee's and that of Siegel & Moser) lead to the same conclusions, which are also partially verified by numerical calculations. Then § 3 extends the second method to the cases of two- and three-dimensional scattering, and compares the results with a new set of numerical results. The connection problem is studied in § 4, and § 5 draws some of the threads together, including other evidence available from the literature. A fuller account of this work with the extension to unequal masses appears in Heggie & Sweatman(1991).

2. The Collinear Problem

In this section the one-dimensional problem is studied by three approaches, two analytic and one numerical. We begin with the method which is easiest to visualise, being based on the change of variables introduced by McGehee for the study of triple collision. This method is particularly simple for dealing with systems in which the total energy
of the triple system (in the rest-frame of its centre of mass) is zero. It leads to predictions for the distribution of the energy of the binary remaining after the encounter, in cases where this energy is either very small or very large. Next, we introduce a method which is easier to extend to higher dimensions; it is based on power-series expansion of motions in the neighbourhood of triple collision and expansion. We show that it leads to the same results as the first method, and we also use it to compute the energy-dependence of the probability of ionisation (destruction) of the binary, when the total energy of the triple system is positive but small. Finally, some of the predictions are tested by comparison with numerical results.

2.1 Use Of McGehee's Coordinates

The basic idea behind this approach is to scale the triple system by its radius of inertia. Then, when the total energy of the triple system is zero, it turns out that the configuration of the system can be described by reference to a certain surface in a three-dimensional space. (This surface is formally the same as McGehee's triple collision manifold.) A point representing a given triple system moves up and across this surface, except at two places, which represent (respectively) the approach to exact triple collision or the expansion away from such a collision, with the middle particle exactly midway between the two outer ones (in the case of equal masses). A triple system which comes close to one or other of these configurations either contracts or expands by a very large factor. In one case a very tight binary emerges, and in the other the final binding energy of the binary is very small. The scale of the resulting binary depends on how long the
triple system has spent near triple collision or triple expansion, which in turn depends on how close it has come to one of the two special positions. For random initial conditions, the distribution of the distance of closest approach can be calculated, and so the form of the distribution of the energy of the emerging binary follows.

2.1.1 Coordinate Transformation

We consider for simplicity the case of equal masses, m, and work in the rest-frame of the centre of mass of the three stars. We also use units for which \( G = m = 1 \).

Let the three stars have coordinates \( X_1 < X_2 < X_3 \) in the rest frame of their centre of mass. Then their kinetic and potential energies are

\[
T = \frac{1}{2} \left( \dot{X}_1^2 + \dot{X}_2^2 + \dot{X}_3^2 \right)
\]

\[
U = -\frac{1}{(X_2 - X_1)} - \frac{1}{(X_3 - X_2)} - \frac{1}{(X_3 - X_1)}
\]

and the energy relation is

\[
h = T + U.
\]

Because the centre of mass is fixed at the origin, only two coordinates are independent, and McGehee's idea was to replace these by one variable describing the size of the system, and another defining its configuration. Let us choose coordinates \( r \) and \( s \) defined by

\[
r = X_1^2 + X_2^2 + X_3^2, \quad r > 0,
\]

\[
s = \frac{X_2}{r}.
\]

Then all three coordinates can be expressed as

\[
X_i = r G_i(s)
\]
for suitable functions $G_i$. From (3) and (5) it follows that

$$
3 \sum_{i=1}^{2} G_i^2 = 1 ,
$$

whence by differentiation,

$$
3 \sum_{i=1}^{2} G_i \frac{dG_i}{ds} = 0 .
$$

(We shall need these results shortly.)

It only remains to define a pair of scaled velocities, $v$ and $w$.

Differentiating (5) we get

$$
\dot{x}_i = \dot{r} G_i + r G'_i \dot{s} ,
$$

where the prime here denotes differentiation with respect to $s$. Now in close triple encounters, when $r$ is small, we expect velocities of order $r^{-1/2}$, and so we define

$$
v = r^{1/2} \dot{r} ,
$$

and

$$
w = b^{-1} r^{3/2} \dot{s} ,
$$

where

$$
b^{-2} = \sum_{i=1}^{3} (G'_i)^2 .
$$

Then (8) becomes

$$
\dot{x}_i = r^{-1/2} v G_i + r^{-1/2} b w G'_i .
$$

Now we must find equations of motion. Those for $r$ and $s$ follow from the definitions (9) and (10). Those for $v$ and $w$ require some dynamics. Substituting (12) in (1a) and using (6), (7) and (11), we obtain
\[ T = \frac{v^2 + w^2}{2r} \]

\[ = \frac{1}{2} \left( \dot{r}^2 + \left( \frac{\dot{s}}{b} \right)^2 \right), \]

using (9) and (10). We can obviously also define a scaled potential function

\[ V(s) = -r U. \quad (13) \]

Then (2) becomes

\[ \frac{v^2 + w^2}{2} - V(s) = r h, \quad (14) \]

and the Lagrangian is

\[ L = \frac{1}{2} \left( \dot{\dot{r}}^2 + \left( \frac{\dot{s}}{b} \right)^2 \right) + \frac{V(s)}{r}. \]

The Lagrange equation for \( r \) is

\[ \ddot{r} = -\frac{V}{r^2} + b^{-2} r \dot{s}^2 \]

whence by (9) and (10) we obtain

\[ \dot{v} = r^{-3/2} \left( \frac{1}{2} \dot{v}^2 + \frac{w^2}{b} - V \right). \quad (15) \]

By differentiating (14), substituting (9), (10), (15) and using (14) again, we also find that

\[ \dot{w} = r^{-3/2} \left( bV'(s) - \frac{1}{2} vw \right). \quad (16) \]

The introduction of the new variables is almost complete, but the factor of \( r^{-3/2} \) in (10), (15) and (16) suggests replacement of time by a new independent variable \( \tau \), which gives the following set of equations:

\[
\begin{align*}
\frac{dt}{d\tau} &= r^{3/2} \\
\frac{d\ln r}{d\tau} &= v \\
\frac{ds}{d\tau} &= bw \\
\frac{dv}{d\tau} &= \frac{1}{2} \dot{v}^2 + \frac{w^2}{b} - V(s) \\
\frac{dw}{d\tau} &= -\frac{1}{2} vw + bV'(s). 
\end{align*} \]
2.1.2 The Zero-Energy Surface

If \( r = 0 \), (14) forms a surface in \((s,v,w)\)-space; this surface is analogous to McGehee's triple collision manifold. (We have chosen a different definition of \( s \).) However, exactly the same surface arises if the total energy, \( h \), vanishes, and this is the interpretation of the surface which we exploit in this chapter.

Figure 1 illustrates the surface on which zero-energy motions take place, though it uses McGehee's original choice for the variable \( s \). However, both choices have important similarities. Thus \( s = 0 \) corresponds to the central particle being exactly midway between the two outer ones; the maximum value of \( s \) occurs when there is a collision between the central star and the particle on the right; and similarly the minimum value of \( s \) occurs when there is a collision between the central star and the particle on the left.

Motion is never downwards in this diagram, and is usually upwards, by (17d) and (14) with \( h = 0 \). In a scattering event in which the particle on the left approaches a binary consisting of the two particles on the right, the trajectory begins on the lower right-hand "trouser-leg" of Figure 1, spiralling around it as the components of the binary perform rectilinear near-Kepler motion. Eventually they emerge on the central part of the diagram, where all three particles are interacting, and then move up one of the two upper "sleeves", as one of the stars recedes from the other two, which once more form a binary. If the system moves up the left-hand sleeve, an exchange has taken place, the new binary consisting of the central star and the one on the left.
Figure 1: The main part of the zero-energy surface for the collinear 3-body problem, after McGehee (1974). Actually, regularisation of binary collisions is needed to transform the surface represented by equation (14), with $h = 0$, into the surface depicted here.

Note that this diagram gives only the configuration of the three bodies; it does not say anything directly about the scale of the triple system, which is determined by the variable $r$. If $r$ becomes very small then the emerging binary has very large binding energy; and if $r$ becomes large it has very small binding energy. The evolution of $r$ is determined by (17b), from which it is clear that the result depends on how long (in
scaled time \( \tau \) the system spends in the lower half of the diagram \((v < 0)\), relative to the time it spends in the upper half \((v > 0)\). If it spends a long time in the lower half, where \(r\) is decreasing, then the energy of the emerging binary may be very high.

It turns out that there are two places on this surface where a system can spend an arbitrarily long time; one in the lower half and one in the upper half. Indeed, there are equilibrium points on this diagram where a system can remain forever. If a system passes very close to one of these equilibria it will spend a very long time there, and the time it spends essentially determines the scale and energy of the emerging binary. Therefore, we now proceed to study these two equilibria and the motions in their vicinity.

The equilibria occur where \( \frac{ds}{d\tau} = \frac{dv}{d\tau} = \frac{dw}{d\tau} = 0 \) (treating \( s, v \) and \( w \) as functions of \( \tau \)), whence, by (17c-17e)

\[
\begin{align*}
  w & = 0 , \\
  V'(s) & = 0
\end{align*}
\]  

and

\[
  v^2 = 2V(s) .
\]  

(These satisfy (14) with \( h = 0 \).) Recall that \( V(s) \) is the scaled potential function defined in (13). In order to compute it, note that it is easy to prove from (3)-(5) and the condition \( X_1 + X_2 + X_3 = 0 \) that

\[
\begin{align*}
  G_1(s) & = -\frac{1}{2}s - \left( \frac{1}{2} - \frac{3}{4}s^2 \right)^{1/2} \\
  G_2(s) & = s \\
  G_3(s) & = -\frac{1}{2}s + \left( \frac{1}{2} - \frac{3}{4}s^2 \right)^{1/2}.
\end{align*}
\]
whence

\[ V(s) = \frac{5 - 12s^2}{(1 - 6s^2)(2 - 3s^2)^{1/2}} \quad . \quad (22) \]

The unique solutions of (18)-(20) occur at

\[
\begin{align*}
s & = 0 \\
w & = 0 \\
v & = -\sqrt[5]{(5/2)} , +\sqrt[5]{(5/2)} . \quad (23)
\end{align*}
\]

The solution with \( v > 0 \) corresponds to triple expansion, and that with \( v < 0 \) to triple collision, as can be seen from (17b). In both cases the central particle remains exactly midway between the outer two. Also, in both cases there is an essential singularity in the motion at the time when \( r=0 \). Therefore, for physically relevant motions, we study the orbits not at, but in the vicinity of, these exact solutions.

2.1.3 Neighbourhood Of Stationary Configurations

In the neighbourhood of triple collision/expansion \( v \) is a constant to first order in \( w \) and \( s \), by (14) (with \( h=0 \)) and (22). Also, it follows from (11) and (21) that \( b = \sqrt[2]{(2/3)} \) to the same order. Hence, linearisation of (17c) and (17e) gives

\[
\begin{align*}
\frac{ds}{d\tau} & = \sqrt[2]{(2/3)} w \\
\frac{dw}{d\tau} & = -\frac{1}{2} vw + \frac{1}{2} 2\sqrt[3]{3} s \\
\end{align*}
\]

where \( v \) is \(-\sqrt[2]{(5/2)}\) for triple collision, and \(+\sqrt[2]{(5/2)}\) for triple expansion.

These equations are of the general form

\[
\frac{dx}{d\tau} = Ax
\]
where
\[ x = (s, w) \quad ; \quad A = \begin{bmatrix} 0 & \sqrt{(2/3)} \\ \frac{29\sqrt{3}}{2} & -\frac{1}{2} v \end{bmatrix}. \]

The coordinate system can be changed so that the equations are of the form
\[ \frac{dy}{d\tau} = By \quad ; \quad y = (s', w') \quad , \quad B = \text{diag}(\mu_1, \mu_2) \]
and then the solution is
\[ s' = C_1 \exp(\mu_1 \tau) \quad , \quad w' = C_2 \exp(\mu_2 \tau) \]
the C's being constants.

The eigenvalues (\mu) are found to be real with opposite signs, as the stationary points are saddle points of the flow on the McGehee manifold. The orientation is chosen so that \mu_1 < 0, and \mu_2 > 0. The values taken by the eigenvalues are:

\[ \mu_1 = 2^{-7/4}(\sqrt{5} - \sqrt{237}) \]
\[ \mu_2 = 2^{-7/4}(\sqrt{5} + \sqrt{237}) \]
in the neighbourhood of triple collision, and

\[ \mu_1 = 2^{-7/4}(-\sqrt{5} - \sqrt{237}) \]
\[ \mu_2 = 2^{-7/4}(-\sqrt{5} + \sqrt{237}) \]
in the neighbourhood of triple expansion.

Also depicted in Figure 1 is an orbit passing close to one of the equilibrium points. It approaches along the direction of the s'-axis, and departs along the direction of the w'-axis. Any other orbit passing close to this equilibrium, and on the same side of it, will follow a very similar path, and the coordinates s, v and w will evolve in a very
similar way. But if the second orbit approaches the equilibrium very closely, it will spend a great deal of time there, and by \( (17\text{b}) \) the evolution of \( r \) will be very different. We therefore have to consider this evolution while the orbit is close to the equilibrium.

Consider a square neighbourhood, of side \( 2a \) in the new coordinates, of one of the stationary points (Figure 2). For convenience set \( \tau = 0 \) when \( s' = a \). Then:

\[
\begin{align*}
    s'(\tau) &= a \exp(\mu_1 \tau) \\
    w'(\tau) &= w'(0) \exp(\mu_2 \tau)
\end{align*}
\]

The orbit leaves this neighbourhood of the stationary point when \( w' = \pm a \), i.e. when:

\[
\tau = \frac{1}{\mu_2} \ln \left| \frac{a}{w'(0)} \right|
\]

The change in scale of the triple system in this time is given, as always, by

\[
\frac{d \ln r}{d\tau} = v,
\]

i.e. \((17\text{b})\). Since \( v \) is constant to the first order (in the neighbourhood of a stationary point), we have:

\[
\ln r = \text{constant} + v \tau.
\]

Thus on leaving the neighbourhood of the stationary point

\[
\ln r = \text{constant} + \frac{1}{\mu_2} v \ln \left| \frac{a}{w'(0)} \right|
\]

whence

\[
r = W \left| w'(0) \right|^k.
\]

Here \( W \) is a constant and

\[
k = -\frac{1}{\mu_2} v.
\]
Figure 2: A neighbourhood of an equilibrium point on the McGehee manifold. Normal coordinates are denoted by $s'$ and $w'$. The change in scale of a triple system depends mainly on how long it spends within the neighbourhood.

This is the essence of the calculation. It shows that, as we consider systems which enter the neighbourhood at smaller and smaller distances from the $s'$-axis, they spend longer and longer times in the vicinity of the rest point, and reach smaller and smaller scales (if the rest point corresponds to triple collision) or larger and larger scales.
(otherwise). Of course our expression for $r$ gives only the change in scale during passage near the point of triple collision/expansion. There are also changes outside the region. However, these changes are almost independent of $w'(0)$, and can be absorbed into the constant $W$.

2.1.4 Binary Binding Energy After Passing Close To Triple Collision

For $v = -\sqrt{5/2}$ i.e. triple collision, the eigenvalues $\mu$ are given by (24), and so (27) implies that

$$k = \frac{4\sqrt{5}}{(\sqrt{237} + \sqrt{5})}.$$ 

Now the true separation of the components of the new binary is either $X_3 - X_2$ or $X_2 - X_1$, depending on whether or not exchange has taken place. By (5) the separation can be written as $r(G_3(s) - G_2(s))$ or $r(G_2(s) - G_1(s))$. Since the values of $s$ are nearly identical for orbits passing very close to triple collision (or expansion), it is clear that the separation of the components, and therefore the semi-major axis of the new binary, is proportional to $r$. From (26) it follows that for the binding energy we obtain

$$E = D |w'(0)|^{-k},$$  \hspace{1cm}  (28)

where $D$ is a constant.

In order to determine the distribution of $E$ given some distribution of initial conditions, we must first determine the distribution of $w'(0)$. Suppose the initial energy of the binary is fixed, and that the total energy ($h$) of the three-body system vanishes. Suppose also we consider that the encounter starts with the third body at a fixed large
distance from the centre of mass of the binary. Then the only initial condition to be chosen is the phase, $\theta$, of the binary, which must be uniformly distributed over $[0, 2\pi]$. There will be some phase $\varphi$ where the motion ends up in triple collision, i.e. $w'(0) = 0$. Near this phase, it is clear that the relation between $(\theta - \varphi)$ and $w'(0)$ will be nearly linear. Hence, locally, the distribution of $w'(0)$ will be uniform.

Consider $w'(0)$ uniformly distributed in the range $[-a, a]$. Then by (28) the probability distribution function for the binding energy of the resultant binary is:

$$P(E > E_0) = P(|w'(0)| < (D / E_0)^{1/k})$$

where

$$\alpha = \frac{1}{k} = \frac{1}{4}(1 + \sqrt{(237/5)})$$.  

(29)

2.1.5 Binary Binding Energy After Passing Close To Triple Expansion

For $v = \sqrt{5}/2$, i.e. triple expansion, the eigenvalues $\mu$ are given by (25), and so (27) gives

$$k = -\frac{4\sqrt{5}}{(\sqrt{237} - \sqrt{5})}$$.  

Now if the total energy of the system, $h$, is 0, then as in the previous case of orbits passing close to triple collision:

$$E = D |w'(0)|^{-k}$$

(30)

where $D$ is a constant.

If $w'(0)$ is uniformly distributed in the range $[-a, a]$, then the
probability distribution function for the binding energy of the resultant binary is:

\[ P( E < E_0 ) \propto E_0^\beta \]

where

\[ \beta = \frac{1}{4} ( \sqrt{237/5} - 1 ) \]

(31)

2.2 Second Approach

The foregoing analysis centred on the behaviour of triple systems very close to triple collision or triple expansion, by using McGehee-like coordinates for the case \( h = 0 \). It is also quite easy to analyse the neighbourhood of these special orbits directly, in Cartesian coordinates. This also has the advantage that introduction of small but non-zero values of \( h \) can be made quite easily, whereas the McGehee approach loses some of its simplicity.

In this approach, then, the coordinates are treated as fractional power series in \( |t| \), the modulus of time. The exact solutions representing triple collision and triple expansion are found, and then small perturbations are introduced and the equations of motion are linearised. Here it is more convenient to use as coordinates the separations between the first and second, and second and third stars, i.e.

\[ x = X_2 - X_1 \text{ and } y = X_3 - X_2 \]
The equations of motion are:

\[
\begin{align*}
\frac{d^2 x}{dt^2} &= -\frac{2}{x^2} - \frac{1}{(x+y)^2} + \frac{1}{y^2} \\
\frac{d^2 y}{dt^2} &= \frac{1}{x^2} - \frac{1}{(x+y)^2} - \frac{2}{y^2}.
\end{align*}
\]  

There are exact solutions of the form:

\[
x = C |t|^{2/3}
\]

\[
y = C |t|^{2/3}
\]

provided that the constant \(C\) is chosen to be

\[
C = \left[\frac{45}{8}\right]^{1/3}.
\]

These solutions represent triple collision or triple expansion with the total energy, \(h\), equal to zero, for \(t < 0\) or \(t > 0\), respectively. Close triple encounters may be expressed as small perturbations from these solutions of the form:

\[
x = C |t|^{2/3} (1 + dx)
\]

\[
y = C |t|^{2/3} (1 + dy)
\]

The linearised equations for \(dx\) and \(dy\) form a system of fourth order, and it will be seen that this system has four linearly independent solutions proportional to \(|t|^\lambda\), where \(\lambda\) is one of four constants ("eigenvalues"), the Siegel exponents, which occur in pairs. From a somewhat more physical point of view, each pair corresponds to a degree of freedom of the system. In the case under consideration there are two degrees of freedom if the centre of mass is held fixed. For encounters
sufficiently close to triple collision/expansion, and appropriate ranges of time, the general solutions are:

$$
x = C|t|^{2/3}(1 + dx_1|t|^{\lambda_1} + dx_2|t|^{\lambda_2} + dx_3|t|^{\lambda_3} + dx_4|t|^{\lambda_4})
$$

$$
y = C|t|^{2/3}(1 + dy_1|t|^{\lambda_1} + dy_2|t|^{\lambda_2} + dy_3|t|^{\lambda_3} + dy_4|t|^{\lambda_4})
$$

where the $dx$'s and $dy$'s are constants. (Only four are arbitrary; the ratio $dx_1/dy_1$ is fixed for each $i$.)

In order to determine the four independent solutions, we seek a solution of the form:

$$
x = C|t|^{2/3}(1 + dx|t|^\lambda)
$$

$$
y = C|t|^{2/3}(1 + dy|t|^\lambda)
$$

Then the two members of (32) linearise to give:

$$
-14dy = (45\lambda^2 + 15\lambda - 44) dx
$$

$$
-14dx = (45\lambda^2 + 15\lambda - 44) dy
$$

Eliminating $dx$, $dy$, which cannot both be zero, gives an equation for the four possible eigenvalues, $\lambda$:

$$
(45\lambda^2 + 15\lambda - 44)^2 - 14^2 = 0
$$

whence

$$
\lambda = \frac{2}{3}, -1, -\frac{1}{6}(1 + (237/5)^{1/2}) \text{ or } -\frac{1}{6}(1 - (237/5)^{1/2}).
$$

Incidentally, it is easy to see from (38) that, in all solutions, $|dx| = |dy|$: for the first two solutions $dx$ and $dy$ have the same sign, for the last two they have opposite sign.
To interpret these eigenvalues, and the corresponding solutions, let us first compute the total energy, h, of the solution (37). One finds after linearisation that:

$$h = \frac{2}{3} C^2 |t|^{\lambda-2/3} (dx + dy) (\lambda + 1) .$$

(40)

Hence the only solution leading to a constant non-zero value of h is the solution corresponding to $\lambda = \frac{2}{3}$. Using this and the fact that $dx = dy$ we obtain from (40) the result

$$h = 5 \left[ \frac{5}{3} \right]^{2/3} dx .$$

(41)

(There are, of course, exact solutions of the equations of motion for triple collision or expansion in which the motion is hyperbolic or elliptic, in place of the parabolic solutions (33) that we have studied. These solutions correspond to $h > 0$ and $h < 0$ respectively, and linearisation with respect to $h$ leads to expressions of the form (37) with $\lambda = \frac{2}{3}$.)

Next, let us consider an exact solution of the form of (33), except that the three bodies coincide at $t=\epsilon$, where $\epsilon$ is small, instead of at $t=0$. Then:

$$x = C |t-\epsilon|^{2/3}$$

$$= C |t|^{2/3} (1 - \frac{2\epsilon}{3t})$$

for $|t| >> |\epsilon|$

approximately. Hence the eigenvalue $-1$ corresponds to such solutions, which arrive slightly early or late at collision/expansion.

The last two eigenvalues $\lambda$ are related to the eigenvalues $\mu$ arrived at in the previous approach, as is shown in the following section.
2.3 A Comparison Of The Two Approaches

In the first study (§ 2.1), it was found that the perturbation from the exact collision/expansion solution can be expressed approximately as:

\[ \delta q \propto \exp(\mu \tau) \]  

(42)

where \( \delta q = w' \) or \( s' \), and \( \mu = \mu_1 \) or \( \mu_2 \) (cf. § 2.1.3). In the second study the perturbation can be a constant multiple of \( |t|^\lambda \). But from equations (17a,b) we also have:

\[ \frac{dt}{d\tau} = r^{3/2} \quad \text{and} \quad \frac{d\ln r}{d\tau} = v, \]  

(43)

where \( v \) is a constant in the neighbourhoods of triple collision/expansion: \( v = -\sqrt{(5/2)}, +\sqrt{(5/2)} \) respectively, by (23).

Integration of (43b) immediately gives an expression for \( r(\tau) \), whence equation (43a) becomes:

\[ \frac{dt}{d\tau} \propto (\exp(v\tau))^{3/2}. \]

Hence

\[ |t|^\lambda \propto \exp(3v\lambda \tau/2), \]

for suitable choice of the origin of \( t \). For agreement with (42) we require that:

\[ \mu = \frac{3}{2} v \lambda \]

Substituting the values of \( v \) and \( \mu \) for triple collision/expansion, one finds in both cases that the required values of \( \lambda \) are:

\[ -\frac{1}{6}( 1 + \sqrt{(237/5)} ), -\frac{1}{6}( 1 - \sqrt{(237/5)} ) \]

respectively. These are precisely the last two eigenvalues found by the second method.

We now show how to compute the probability of obtaining a binary of given energy when the three bodies pass close to either triple expansion
or collision, using the second approach. This is a useful exercise, because it turns out to be easier to extend the second approach to non-zero energy, and to two and three dimensions. However, for the moment, we consider the case \( h = 0 \). From the results of § 2.2, the general solution for motion in the vicinity of close triple approach or expansion is given by

\[
\begin{align*}
    x &= C|t|^{2/3} \left( 1 + dx_1|t|^{\lambda_1} + dx_2|t|^{\lambda_2} + dx_3|t|^{-1} \right) \\
y &= C|t|^{2/3} \left( 1 - dx_1|t|^{\lambda_1} + dx_2|t|^{\lambda_2} + dx_3|t|^{-1} \right)
\end{align*}
\]

where \( C = (45/8)^{1/3} \), \( \lambda_1 = -\frac{1}{6}(1 + \sqrt{237/5}) \), and \( \lambda_2 = -\frac{1}{6}(1 - \sqrt{237/5}) \). (We have omitted the eigenfunction corresponding to non-zero energies.) We shall also suppose that the origin of time is chosen so that exact triple collision would occur at \( t = 0 \), and so the solution reduces to

\[
\begin{align*}
    x &= C|t|^{2/3} \left( 1 + dx_1|t|^{\lambda_1} + dx_2|t|^{\lambda_2} \right) \\
y &= C|t|^{2/3} \left( 1 - dx_1|t|^{\lambda_1} - dx_2|t|^{\lambda_2} \right)
\end{align*}
\]

It is clear that this form of solution is valid only if the perturbations are much smaller than unity. Now one of these terms is increasing, and the other is decreasing. As soon as the increasing term becomes of order unity, the triple system as a whole cannot contract or expand much further. Therefore the time at which this term becomes of order unity, i.e. \( t = -|dx_1|^{-1/\lambda_1} \) for triple collision, or \( t = |dx_2|^{-1/\lambda_2} \) for triple expansion, sets the scale, \( r \), of the triple system at the time when the binary begins to form, and indeed sets the scale and energy of this binary. (See § 3.2 for a discussion of the validity of this argument.) By substituting this expression for the time
into the leading term, we find that the scale is of order $|dx_1|^{-2/3\lambda_1}$
for triple collision, or $|dx_2|^{-2/3\lambda_2}$ for triple expansion. Hence, the
binding energy, $E$, of the binary is given, to order of magnitude, by an
expression of the form:

$$E \propto |dx_1|^{2/3\lambda_1}.$$ 

It only remains to determine the probability distribution of the dx's. However, it is clear that these vary approximately linearly with the
initial conditions, in the vicinity of initial conditions leading to
exact triple collision or expansion. Hence, it follows that the
probability distribution of the energy of the emerging binary is given
for large $E_0$ by:

$$P(E > E_0) = P(|dx_1| < \text{constant} \times |E_0|^{3\lambda_1/2})$$

$$\propto E_0^{-\alpha},$$

where $\alpha = -3\lambda_1/2 = \frac{1}{4}(1 + \sqrt{(237/5)})$, since such energies arise in the
neighbourhood of triple collision. Likewise

$$P(E < E_0) \propto E_0^\beta$$

for small $E_0$, where $\beta$ is $3\lambda_2/2 = \frac{1}{4}(\sqrt{(237/5)} - 1)$. These agree with the
results of the first approach (equations (29) and (31)).

At this point it is worth investigating the range of values of $h$ for
which the results are approximately valid. We consider for definiteness
the case of passage close to triple collision, and concentrate on the
use of Siegel exponents. The effect of a non-zero value of $h$ can be
neglected provided that it does not invalidate our stated condition for
the end of the contraction of the triple system. Therefore we require
that the perturbation corresponding to the non-zero value of $h$ should be
small at a time of order \(-|dx_1|^{-1/\lambda_1}\) (cf. the foregoing discussion).

From the discussion of equations (40) and (41) we therefore require
\[|h||dx_1|^{-2/3\lambda_1} \ll 1, \text{ i.e. } |h| \ll E,\]
where \(E\) is the final energy of the binary. A discussion of triple expansion proceeds along lines similar to §§ 2.4.2 and 3.5 below, and again leads to the condition \(|h| \ll E,\)
where \(E\) is the final binding energy of the binary.

2.4 Probability Of Ionisation In The Proximity Of Triple Expansion

For motions in the vicinity of close triple expansion, the binding energy of the final binary is very small. However, it can never become unbound so long as the total energy is held fixed at \(h=0\). Now we consider what happens when \(h\) is small but positive, so that disruption of the binary is energetically possible. Clearly, the system must pass very close to triple expansion, and so we can in principle adapt either of the two approaches to predict the relationship between \(h\) and the probability of totally disrupting or "ionising" the binary (producing three unbound stars).

2.4.1 From The First Study

In the case \(h = 0\), the minimum possible binding energy of the final binary \((E_{\min})\) is zero. When \(h\) is small and positive, the minimum possible binding energy is now negative (i.e. all three particles may recede without either pair forming a bound system), and will be of order \(-h\). (It is not easy to be more precise, because the energy must be shared between all three particles when no binary is formed.) Now we shall assume that when \(h > 0\) the probability distribution of \((E - E_{\min})\) has the same form as in the case \(h = 0\), at least for systems passing
sufficiently close to triple expansion and $h \ll 1$ in appropriate units.

Hence by (31) we have:

$$P(E - E_{\text{min}} < E_0) \propto E_0^\beta$$

where

$$\beta = \frac{1}{4}(\sqrt{(237/5)} - 1).$$

Therefore, the probability of ionisation ($E < 0$) is

$$P(E < 0) \propto (-E_{\text{min}})^\beta,$$

and since $E_{\text{min}}$ is of order $-h$ when $h$ is small, we deduce that the probability of ionisation varies as $h^\beta$.

### 2.4.2 From The Second Study

From the second study, near triple expansion there are approximate solutions of the form:

$$x = C|t|^{2/3}(1 + dx_1|t|^{\lambda_1} + dx_2|t|^{\lambda_2} + K h|t|^{2/3})$$

$$y = C|t|^{2/3}(1 + dy_1|t|^{\lambda_1} + dy_2|t|^{\lambda_2} + K h|t|^{2/3})$$

where

$$\lambda_1 = \frac{1}{6}(\sqrt{(237/5)} - 1)$$

$$\lambda_2 = \frac{1}{6}(-\sqrt{(237/5)} - 1)$$

and, as before, $C = \frac{45^{1/3}}{2}$ and $K = \frac{45^{1/3}}{25}$ (cf. (34), (36), (39) and (41)). Note that we have chosen the origin of time to be the instant at which triple expansion begins.

The above solution breaks down when the linearisation ceases to be valid. This occurs when one of the last three bracketed terms in the expression for $x$ or $y$ reaches a magnitude of order 1 (cf. § 3.2. We are dealing with triple expansion, $t > 0$, and so it will be one of the
positive powers of $|t|$, in this case $|t|^\lambda_1$ or $|t|^{2/3}$, that is involved).

If $h=0$, we have seen that the scale of the final energy is set by the time at which the term involving $\lambda_1$ is of order 1. If the term involving $h$ is still small at this time then the non-zero value of $h$ has a small effect on the system, and again a binary forms with a length scale similar to the case $h=0$. Now suppose that the term in $\lambda_1$ vanishes. Then the three stars form no binary, and the nearly parabolic recession of the stars changes into hyperbolic recession at a time when the term in $h$ becomes of order 1. (We assume $h > 0$.) Next, take the term in $\lambda_1$ to be non-zero but still small when the term in $h$ is of order 1. Then the motion of the three particles will still be one of hyperbolic recession, and no binary will form. Therefore, the occurrence of a binary depends upon which term becomes of order 1 first: if it is the term in $\lambda_1$ that first reaches a magnitude of order unity then a binary will form; however, if it is the term involving $h$ then total ionisation will take place.

From the foregoing discussion it follows that, for ionisation, we require

$$|dx_1| |t|^{\lambda_1} < 1$$

at a time when

$$|Kh| |t|^{2/3} = 1$$

and so

$$|dx_1| < |Kh|^{3\lambda_1/2}.$$

Therefore, if $dx_1$ is uniformly distributed on some small interval $(-a, a)$, the probability that the system is left totally ionised is of
order
\[ \frac{1}{a} |\mathcal{K}|^{3\lambda_1/2} \propto \hbar^\beta, \]
where
\[ \beta = \frac{3}{2} \lambda_1 = \frac{1}{4}(\sqrt{237/5} - 1) \]
This is exactly the result found in § 2.4.1, using the methods of the first study.

2.5 Numerical Integrations in One Dimension

As we have seen, study of motion in the vicinity of triple collapse and expansion leads to asymptotic results for certain scattering probabilities, viz. the distribution of the binding energy of the final binary, when this is very large or very small, and the probability of ionisation (or destruction) of the initial binary. The fact that certain of these probabilities can be obtained independently by two rather different approaches is some reassurance that the methods used are sound. Furthermore, it is possible to present the second treatment (which uses Siegel exponents) in a more rigorous fashion by the method of matched asymptotic expansions (cf. Waldvogel 1976 and § 3.2 below). Nevertheless, it is desirable at this stage to verify the correctness of the results independently, and the purpose of the present section is to do this by means of numerical scattering experiments.

2.5.1 Numerical Method

The equations of motion of the collinear (one-dimensional) three-body problem can be derived from the Hamiltonian
\[ H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{1}{x_2 - x_1} - \frac{1}{x_3 - x_2} - \frac{1}{x_3 - x_1}, \]
where the $p_i$ are the momenta conjugate to the coordinates $X_i$ introduced in § 2.1.1. However, because collisions are inevitable in one-dimensional scattering events, the resulting Hamiltonian equations are unsuitable for our purpose. Therefore the equations are regularised (for collisions between particles 1 and 2, and 2 and 3) by the following one-dimensional adaptation of the method of Aarseth and Zare (1974).

Defining, as before, the relative coordinates $x = X_2 - X_1$ and $y = X_3 - X_2$, we see easily that the equations of relative motion, (32), are derivable from the Hamiltonian

$$H' = p_x^2 + p_y^2 - p_x p_y - \frac{1}{x} - \frac{1}{y} - \frac{1}{x+y},$$

where $p_x$ and $p_y$ are momenta conjugate to $x$ and $y$, respectively. We now transform to coordinates $Q_1, Q_2$, where

$$Q_1^2 = x, \quad Q_2^2 = y,$$

introduce corresponding conjugate momenta $P_1, P_2$, and introduce a new time variable ($\tau$), unrelated to the variable introduced in § 2.1.1, by the equation

$$\frac{d\tau}{dt} = Q_1^2 Q_2^2. \quad (44)$$

The new equations of motion are Hamilton's equations for a new Hamiltonian $H^*$, given by

$$H^* = H^*(P_1, Q_1) = Q_1^2 Q_2^2 (H' - h_0'),$$

where $h_0'$ is the constant value of $H'$, and $H'$ is regarded as a function of the $P_i$ and $Q_i$. This gives the equations
\[
\begin{align*}
\frac{dP_1}{d\tau} &= \frac{1}{4} P_1 P_2 Q_2 - \frac{1}{2} P_2 Q_1 + \frac{2Q_1(Q_1^4 + 2Q_1^2 Q_2^2 + 2Q_2^4)}{(Q_1^2 + Q_2^2)^2} \\
\frac{dP_2}{d\tau} &= \frac{1}{4} P_1 P_2 Q_1 - \frac{1}{2} P_1 Q_2 + \frac{2Q_2(Q_2^4 + 2Q_1^2 Q_2^2 + 2Q_1^4)}{(Q_1^2 + Q_2^2)^2} \\
\frac{dQ_1}{d\tau} &= \frac{1}{2} P_1 Q_2^2 - \frac{1}{4} P_2 Q_1 Q_2 \\
\frac{dQ_2}{d\tau} &= \frac{1}{2} P_2 Q_1^2 - \frac{1}{4} P_1 Q_1 Q_2
\end{align*}
\]  

(45)

provided, as in the cases considered below, \( h_0' = 0 \). From given initial states, the selection of which is described below, the five equations (44), (45) were numerically integrated using a high order Runge-Kutta method (Fehlberg 1968).

The criterion used to establish the end of a triple encounter was that all the following conditions should be satisfied:

(a) a binary with positive binding energy had formed, either with stars 1 and 2 or stars 2 and 3 as components;

(b) the distance between the single star and the barycentre of the binary was at least 20 times the current semi-major axis of the binary; and

(c) the single star was moving away from the binary.

It is worth remarking that these criteria are imposed not only to ensure that the third body will escape to infinity, but also in order that the binary will have reached nearly its asymptotic binding energy. The condition on the ratio of distances could be greatly relaxed if it were necessary only to ensure escape (Marchal 1990, § 11.7.10.3).
The initial conditions were chosen to give rise to a scattering event in which star number 1 encounters a binary consisting of stars 2 and 3. Initially the binary was taken to be at apocentre, the separation of its components being 1 unit. Since the eccentricity of a one-dimensional binary is \( e = 1 \), its semi-major axis was \( a = \frac{1}{2} \) and its binding energy unity. The first ("free") star was started at a distance \( x_0 \) from the second star with a velocity in the direction of the binary, such that the total energy \( h_0 \) in the rest frame of the centre of mass of all three bodies was zero. It follows from these data that the speed of the first star relative to the centre of mass of the binary (at infinity) is \( \sqrt{3} \), and that the period of the binary is \( \frac{\pi}{2} \). The only free parameter now is the initial relative distance \( x = x_0 \) of the free star, and the full range of encounters can be sampled by allowing \( x_0 \) to vary over the amount by which \( x \) changes during the period of the binary. Also we require that \( x_0 \) is at least 20 times the initial value of \( a \), just as with the second condition for stopping (b). Therefore the initial value of \( x \) was allowed to range over the interval \([10, 13.5]\). At the end of each integration eight items were recorded: the initial value of \( x \), the energy of the final binary, the identities of its components, the time from the initial state to the final state, the total energy after the integration, and the separations of the stars in the final state.

At first the range of initial values of \( x \) was covered coarsely at equal intervals, to find roughly the two positions that lead to triple collision and triple expansion. Then further integrations at finer intervals about these points were performed. The positions of triple collision and expansion may be easily pinpointed when it is observed
that orbits on either side of the exact triple collision/expansion orbits on the McGehee manifold (corresponding to small positive or negative changes in $x_0$), follow paths up opposite sides of the manifold, and finally go up different "arms". The different "arms" correspond to different pairs of stars forming the final binary at the end of the encounter; hence, an interval of the domain of $x_0$ containing a triple collision or expansion orbit, will be distinguished by a change in the pair forming the final binary.

The results have been plotted as graphs of the binding energy of the final binary against $x_0$ (Figure 3). Note that both "spikes" should extend to infinity on the logarithmic vertical scale. The limit of the upper "spike" corresponds to an orbit leading to triple collision, and the other similarly corresponds to triple expansion. In the narrow range of initial conditions between the spikes there is no overall exchange of components in the binary. Curiously in this respect the three bodies behave like hard spheres, except in this range of initial conditions: one sphere striking two collinear spheres at rest will be brought to a halt; the third sphere will be sent off with all the momentum of the incomer.

2.5.2 Comparison With The Analytic Approach

In § 2.1.5, equation (30), for encounters in the neighbourhood of triple expansion, we obtained the relation
Figure 3: Energy of the final binary in one-dimensional scattering events. The initial conditions are described in detail in the text. Briefly, one body is approaching a binary of binding energy unity, and the total energy in the rest frame of the centre of mass of the three bodies is zero. The quantity $x_0$ is the initial distance between the approaching body and the nearer component of the binary, and initially the components of the binary are at apocentre.
Figure 4: Comparison between theoretical and numerical results for 1-dimensional scattering in the vicinity of triple approach. (The former is the straight line and the latter the curved line.) The ordinate is the energy of the final binary. The abscissa is determined by the initial position of the incoming single body, and $x_0 = C$ corresponds to conditions leading asymptotically to exact triple expansion.
\[ E = D | w'(0)|^{-k}, \quad (46) \]

where
\[ k = \frac{-4\sqrt{5}}{(\sqrt{237} - \sqrt{5})}, \]

between the coordinate \( w'(0) \) (which is zero for an encounter leading to exact triple expansion) and the final binding energy \( E \) of the emerging binary. In the context of the initial conditions discussed above, there will be a particular value of \( x_0 \), say \( x_0 = C \), which will lead asymptotically to exact triple expansion. Hence we may assume that \( w'(0) \) is approximately proportional to \( (x_0 - C) \) when this is small. Hence (46) leads immediately to a similar relation between \( E \) and \( (x_0 - C) \). To numerically verify this prediction a series of integrations were made at spacings of \( x_0 \) close to the value of \( C \) which leads to triple expansion (11.473934 approximately). Taking \( C = 11.473934 \), \( D = 4.56 \) and \( k = -0.6797 \) it was found that the results for the interval [11.4720, 11.4739] agreed with the formula \( E = D|x_0 - C|^{-k} \) to 1%, even though over this domain \( E \) ranges between 6.47E-2 and 4.20E-3. The asymptotic nature of this result is illustrated in Figure 4.

3. Extension To Two And Three Dimensions

Exact triple collision/expansion may take one of two configurations in two and three dimensions (cf. Siegel & Moser 1971): either the three stars are collinear or they are situated on the vertices of an equilateral triangle. These are denoted the collinear and equilateral configurations respectively. In this section an extension of the second method of § 2 is used to analyse the orbits passing close to triple collision/expansion in the two configurations. As in the previous
section the analysis is used to estimate the distribution of the final binary's binding energy in the case of zero total energy $h$, and also to predict a relation between $h$ and the probability of ionisation, for small positive $h$. The results for the two configurations are compared to see which types of encounter are more likely to produce tight and loose binaries, and which is more likely to result in total ionisation. Initially, we consider the case of equal masses.

3.1 Eigenvalues: The Siegel Exponents

As in one dimension, in two and three dimensions the exact solutions for triple collision/expansion have coordinates where the time-dependence is of the form:

$$x \propto |t|^{2/3}.$$  

As before, close triple encounters may be expressed as small perturbations from these solutions of the form:

$$x = C|t|^{2/3}(1 + dx_1|t|^{\lambda_1} + dx_2|t|^{\lambda_2} + dx_3|t|^{\lambda_3} + dx_4|t|^{\lambda_4} + ..) . \ (47)$$

In two dimensions there are four degrees of freedom (in the rest frame of the centre of mass), and so there are eight eigenvalues in each of the two configurations. In the equilateral configuration these are:

- $-1$, $\frac{2}{3}$, $0$, $-\frac{1}{3}$, $-\frac{1}{6}(1 + \sqrt{13})$ repeated, $-\frac{1}{6}(1 - \sqrt{13})$ repeated;

these results are obtainable from equation (32) of Siegel & Moser(1971) and the formulae on page 88, by observing that their eigenvalues differ from ours simply by a change of sign. Similarly, in the collinear configuration the eigenvalues are:

- $-1$, $\frac{2}{3}$, $-\frac{1}{6}(1 + \sqrt{(237/5)})$, $-\frac{1}{6}(1 - \sqrt{(237/5)})$,

[as in one dimension], and
In three dimensions the extra two degrees of freedom add four new eigenvalues in each case. In the equilateral configuration these are:

$0$, $-\frac{1}{3}$, $-\frac{1}{6}(1 + i\sqrt{51/5})$, $-\frac{1}{6}(1 - i\sqrt{51/5})$.

In the collinear configuration they are:

$0$, $-\frac{1}{3}$, $-\frac{1}{6}(1 + i\sqrt{51/5})$, $-\frac{1}{6}(1 - i\sqrt{51/5})$.

As previously, the eigenvalue $-1$ corresponds to a small translation in $t$, and $\frac{2}{3}$ to systems having a small non-zero total energy $h$. Similarly it is quite easily shown that the eigenvalues $0$, $-\frac{1}{3}$ correspond to a small rotation of the system, and to having a small non-zero angular momentum, respectively.

### 3.2 Final Binary Binding Energy For $h=0$

As in one dimension the coordinates are of the form (47). We shall omit the eigenfunction corresponding to non-zero energy, and choose time such that exact collision/expansion would occur at $t=0$. The solution ceases to be valid when one of the perturbations reaches order unity, and, as in the one-dimensional problem (§ 2.3), this sets the scale, $r$, of the triple system at the time at which a binary begins to form.

Again, as in one dimension, the binary binding energy, $E$, is given to order of magnitude by an expression of the form:

$$E \propto |dx_i|^2/(3\text{Re}(\lambda_i))$$

where $i$ labels that term in the perturbation which first reaches order unity. (The only distinction from the one-dimensional case is that some of the eigenvalues are complex, and it is only the real part which determines the size of the perturbation.) As before, the relevant eigenfunctions have eigenvalues with negative real part for triple
collision, and positive for triple expansion. The probability
distribution for the binding energy is found by assuming that the
coefficients $dx_1$ of these eigenfunctions are independently and uniformly
distributed in the vicinity of $dx_1 = 0$. (This will be approximately true
if the probability distribution of the initial conditions in phase space
is locally uniform in the vicinity of those initial conditions which
lead to exact collision/expansion.)

At this point it is worth adding that the above argument (for
determining the energy of the resulting binary) can be made more
rigorous by the method of matched asymptotic expansions (Waldvogel
1976). Waldvogel shows, for example, that the escape speed of the final
single body, after an equilateral near-collisional encounter, can be
written as

$$v = \left[ (a_3^*)^{1/(3a_3^*)} \right] \tilde{v},$$

(Waldvogel's equation (53)) where $a_3^* = \frac{1}{6}(-1-\sqrt{13})$, $a_3^*$ is proportional to
the coefficient, $dx_1$, of the term in our equation (47) in which $\lambda_1 = a_3^*$,
and $\tilde{v}$ is independent of $a_3^*$. Thus the energy of escape, and so the
energy of the binary also, is given by an equation of the form (48).
Waldvogel's analysis assumes that $a_3^*$ is not so small that one of the
other terms of the expansion (47) becomes important sooner. However,
within the scope of this assumption, it provides a more rigorous
foundation for (48) than our scaling argument.

As an alternative illustration consider the two-body problem. In a
rectilinear collision/expansion of zero total energy, the relative
coordinates scale as $|t|^{2/3}$ (as for triple collision/expansion). If we take the exact collision/expansion to be along the x-axis, then the relative coordinates are

$$x = Ct^{2/3}, \quad y = z = 0,$$

where $C = g^{1/3}$ in the units of § 2.1.1.

We may again consider small perturbations of the form

$$x = Ct^{2/3}(1 + dx|t|^\lambda), \quad y = Ct^{2/3}dy|t|^\lambda, \quad z = Ct^{2/3}dz|t|^\lambda.$$

The resulting eigenvalues are $-1, \frac{2}{3}, 0$ twice, $-\frac{1}{3}$ twice, corresponding to time translation, non-zero energy, a small rotation and a small non-zero angular momentum. Consider the perturbation with $\lambda = -\frac{1}{3}$, $dx = dz = 0$. It is easy to show that the magnitude of the total angular momentum $|l| \propto |dy|$. Hence the perturbation is (relatively) of order one when $|t|^{1/3} \sim |l|$, i.e. $|x| \sim C|l|^2$. Now, from the exact theory of two-body motion the minimum distance between the two bodies is approximately $r_{\text{min}} = |l|^2$, which is of the same order as the value of $|x|$ arrived at above by considering the time when the relative perturbation reaches order unity.

3.3 A Comparison Of The Probabilities Of Forming Very Tight Binaries Near Triple Collision For The Two Configurations

Binaries formed by systems passing very close to triple collision are tightly bound (i.e. they have a high binding energy). In the equilateral case in two dimensions, the eigenfunctions determining final binary binding energy are the three with eigenvalues

$$\lambda = -\frac{1}{6}(1 + 13^{1/2}) \text{ twice, } -1/3$$

(i.e. those with negative real part). In accordance with the above
discussion, the probability distribution for the binding energy is found
by assuming that the coefficients of these eigenfunctions are locally
distributed independently and uniformly on a small three-dimensional
volume enclosing the point $dx_i = 0$.

Denoting the three relevant eigenvalues by $\lambda_1$, $\lambda_2$, $\lambda_3$, the binding
energy is given by (48), where $\lambda_i$ is the eigenvalue corresponding to the
term $dx_i |t|^\lambda_i$ which first becomes of order unity. The time $t_i$ at which
this occurs is given by $|t_i| \approx |dx_i|^{-1/\text{Re}(\lambda_i)}$. Since $t < 0$ as we
approach triple collision, the first term to reach order unity is the
one for which $|t_i|$ is largest, i.e. $|dx_i|^{1/\text{Re}(\lambda_i)}$ is smallest. Hence
(48) now becomes

$$E \propto \min_{1 \leq i \leq 3} |dx_i|^{2/(3\text{Re}(\lambda_i))}$$

It follows that the probability of obtaining a binary with binding
energy exceeding $E_0$ is given by

$$P(E > E_0) = P(\min |dx_i|^{2/(3\text{Re}(\lambda_i))} > \frac{E_0}{\text{const.}})$$

$$= P\left[|dx_1|^{2/(3\text{Re}(\lambda_1))} > \frac{E_0}{\text{const.}} \text{ and } |dx_2|^{2/(3\text{Re}(\lambda_2))} > \frac{E_0}{\text{const.}} \right]$$

$$\text{and } |dx_3|^{2/(3\text{Re}(\lambda_3))} > \frac{E_0}{\text{const.}}$$

$$= P\left[|dx_1| < \text{const.} \times E_0^{3\text{Re}(\lambda_1)/2} \text{ and } |dx_2| < \text{const.} \times E_0^{3\text{Re}(\lambda_2)/2} \text{ and } |dx_3| < \text{const.} \times E_0^{3\text{Re}(\lambda_3)/2} \right]$$

$$= \prod_{i=1}^{3} P\left[|dx_i| < \text{const.} \times E_0^{3\text{Re}(\lambda_i)/2} \right]$$

$$= \prod_{i=1}^{3} E_0^{3\text{Re}(\lambda_i)/2}$$

$$= \alpha E_0^{-\alpha} \quad (49)$$

where
\[ \alpha = \frac{1}{4}(1 + \sqrt{13}) + \frac{1}{4}(1 + \sqrt{13}) + \frac{1}{2} \]
\[ = \frac{1}{2}(2 + \sqrt{13}) \]
\[ \simeq 2.80 . \quad (50) \]

In the collinear case (in two dimensions) there are four relevant eigenfunctions, those with eigenvalues
\[ -\frac{1}{6}(1 + \sqrt{(237/5)}), -\frac{1}{6}(1 + i\sqrt{(51/5)}), -\frac{1}{6}(1 - i\sqrt{(51/5)}), -\frac{1}{3} . \]

Therefore in this case the probability distribution is:

\[ P(E > E_0) \propto E_0^{-\alpha} \]

where

\[ \alpha = \frac{1}{4}(1 + \sqrt{(237/5)}) + \frac{1}{4} + \frac{1}{4} + \frac{1}{2} \]
\[ = \frac{1}{4}(5 + \sqrt{(237/5)}) \]
\[ \simeq 2.97 . \]

This probability decreases (as \( E_0 \) increases) more quickly than the result for the equilateral configuration. Therefore we may conclude that very tight binaries are more likely to be formed by passing close to triple collision in the equilateral configuration than in the collinear configuration (in two dimensions). Also, for large \( E_0 \), the probability distribution for \( E \) should be of the form (49); i.e. approximately proportional to \( E_0^{-2.80} \).

Progressing to three dimensions, the new eigenfunctions added have eigenvalues which were listed at the end of § 3.1. The new binding energy probability distribution for the equilateral configuration is therefore

\[ P(E > E_0) \propto E_0^{-\alpha} \]

where

\[ \alpha = \frac{1}{2}(2 + \sqrt{13}) + \frac{1}{2} + \frac{1}{2} \]
Similarly, the addition of new eigenfunctions in the collinear case changes the general form of the binding energy probability distribution to

\[ P(E > E_0) \propto E_0^{-\alpha} \]

where

\[ \alpha = \frac{1}{4}(9 + \sqrt{(237/5)}) \]
\[ \approx 3.97 \]

Hence the ratio of the probability distributions for large \( E_0 \) has the same behaviour in three dimensions as in two dimensions: the equilateral configuration is more likely to produce tight binaries.

3.4 A Comparison Of The Probabilities Of Forming Very Loose Binaries Near Triple Expansion For The Two Configurations

Binaries formed by systems passing very close to triple expansion with \( h=0 \) are "loose" (of low binding energy). The scale of the final binary may be estimated along lines similar to our discussion of § 3.3. Here, however, \(|t| \) increases as the triple system expands, and in (47) we are concerned with terms in which \( \text{Re}(\lambda_1) \) is positive. Therefore, in the equilateral case in two dimensions, the eigenfunctions determining the final binary binding energy are those with eigenvalue

\[ \lambda_2 = \frac{1}{6}(\sqrt{13} - 1) \]

(There are two of these.) In other respects the argument is unchanged, and so the probability distribution for the binding energy is:

\[ P(E < E_0) \propto E_0^\beta \]

where
In the collinear case there is just one relevant eigenvalue, viz. 
\[ \lambda = \frac{1}{5}(\sqrt{237/5} - 1) \]
and so here the probability distribution is
\[ P( E < E_0 ) \propto E_0^{\beta} \]
where
\[ \beta = \frac{1}{4}(\sqrt{237/5} - 1) \]
\[ \simeq 1.47 \]
When \( E_0 \) is sufficiently small this result is negligible compared to that for the equilateral configuration. Therefore, in two dimensions a very wide binary is more likely to result from a near-equilateral configuration than from a near-collinear one, and the probability distribution for small \( E_0 \) will be of the form (52). These results are equally true for three dimensions, as all the new eigenvalues added by the extra dimension have a negative or zero real part, and therefore are not relevant.

Incidentally, the dominance of the equilateral configuration should not be understood as meaning that passage close to this configuration is more common but rather that passage close to it causes greater expansion than passage equally close to the collinear configuration.

3.5 Relative Probabilities Of Ionisation For Small Positive h

In two and three dimensions, exactly as in one dimension, ionisation can occur in a system with small positive total energy, \( h \), provided that it passes close to triple expansion. As before (§ 2.4), we can use two
different methods to find a relation between $h$ and the probability of ionisation. Here, however, we describe only the second method. The first method, which depends upon an assumption that we have not attempted to justify, leads to the same results.

In the second approach, the occurrence of ionisation depends on whether the perturbation term due to non-zero energy (with eigenvalue $\lambda = \frac{2}{3}$, cf. § 2.2) reaches order unity before any of the other perturbation terms. The relevant perturbation terms, apart from that due to the non-zero total energy, are the same as those which determine the formation of loose binaries when $h=0$ (§ 3.4): as in that case, the two- and three-dimensional problems are identical because the third dimension only adds irrelevant eigenvalues with negative real part. The similarity of the two cases is perhaps to be expected, as ionisation is just the limit of loose binary formation.

As an example, let us consider the equilateral configuration in two dimensions. Including only the relevant eigenfunctions (i.e. those with $\text{Re}(\lambda) > 0$), the perturbation expansion for any coordinate is given by

$$x = C |t|^{2/3} \left( 1 + dx_1 |t|^\lambda_1 + dx_2 |t|^\lambda_2 + dx_3 |t|^\lambda_3 \right),$$

where $\lambda_1 = \frac{2}{3}$, $dx_1$ is proportional to $h$, and $\lambda_2 = \lambda_3 = \frac{1}{6}(\sqrt{13} - 1)$. The first perturbation term becomes of order unity when $|t| \sim |dx_1|^{-1/\lambda_1}$, and at this time the magnitudes of the other two terms are $|dx_i| |dx_1|^{-\lambda_i/\lambda_1}$, $i = 2, 3$. By the arguments of § 2.4.2, the encounter results in ionisation if both terms are still small at this time, i.e. if $|dx_i| < |dx_1|^\lambda_i/\lambda_1$, $i = 2, 3$. Assuming as before that $dx_2$ and $dx_3$ are approximately uniformly and independently distributed in a small
neighbourhood of $dx_2 = dx_3 = 0$, and recalling that $dx_1 \propto h$, it is easily seen that the probability of ionisation varies with $h$ as

$$P_{\text{ion}} \propto h^\beta$$

where

$$\beta = \frac{\lambda_2 + \lambda_3}{\lambda_1} = \frac{1}{2}(\sqrt[3]{13} - 1) \approx 1.30.$$

As already mentioned, this result is also valid for scattering in three dimensions. As in § 3.4, the corresponding result for the collinear configuration, in either two or three dimensions, is that $P_{\text{ion}} \propto h^\beta$ where $\beta \approx 1.47$. Again this is negligible for sufficiently small positive $h$.

Incidentally, it is on this issue that we differ from Grujić and Simonović (1988), who appear to have had a very similar method to determine the dependence of $P_{\text{ion}}$ on $h$. However, they assert that the only configuration for the three bodies is a collinear one, and give results for this case.

3.6 Numerical Integrations In Two Dimensions

3.6.1 Initial Conditions

In order to test the foregoing predictions, the one-dimensional program described in § 2.5.1 was modified to integrate two-dimensional systems. The initial conditions were selected at random according to appropriate probability distributions, in contrast to the one-dimensional grid search which was carried out in the one-dimensional case.
Taking the centre of mass as being fixed, a three-body system has four degrees of freedom, and so a given scattering event is determined by eight initial conditions. These can be further reduced by rotation of the coordinate system, time translation, and scaling of the coordinates. If the total energy $h = 0$, this leaves us with a system in which we have four initial conditions, which we chose to be as follows:

(a) The apsidal angle of the binary measured from the direction of motion of the third star when it was at infinity ($u$);

(b) The impact parameter ($p$) of the third star at infinity;

(c) The eccentricity of the binary ($e$); and

(d) The initial true anomaly of the binary ($v$).

The integrations commenced with the third star twenty units distant from the binary, and the binary's semi-major axis ($a$) equal to one. All masses were unity, and $G = 1$.

We now describe the probability distributions used in the generation of the initial conditions. First, the range of $p$ was restricted to $(-\sqrt{18}, \sqrt{18})$, so that all encounters are included in which the third star passes within a distance approximately 2.69 of the centre of the binary, according to the Keplerian approximation. More importantly, this range includes all encounters with small total angular momentum. (We are concerned with encounters close to triple collision or expansion, and these solutions have zero total angular momentum.) Indeed in our computations the total angular momentum can vanish only if $|p| \leq \frac{\sqrt{3}}{2}$. Note also that $p$ may be positive or negative, so that both prograde and retrograde encounters are included.
To describe the probability distribution of $u$, $e$ and $v$, let $r$ be the relative position vector of the binary components, and $\dot{r}$ their relative velocity. The microcanonical distribution of $r$ and $\dot{r}$, corresponding to a given value of the binding energy of the binary, readily leads to the distribution

\[ f(e) = \frac{e}{\sqrt{1-e^2}} \]

for the eccentricity; (cf. Jeans(1929) for the analogous calculation in three dimensions). Assuming next that the mean anomaly of the relative motion is uniformly distributed on $(0,2\pi)$, the conditional distribution of the true anomaly is

\[ f(v|e) \propto \frac{1}{(1 + e \cos v)^2} . \]

The apsidal angle $u$ is uniformly distributed on $(0,2\pi)$.

In all 50,000 integrations from different initial conditions were performed. Of these 10,525 were found to result in a binary of less than half the initial binding energy, and 1,650 were found to increase the initial binding energy by a factor exceeding 2.

### 3.6.2 Comparison Of Numerical And Analytic Results

The analyses in §§ 3.3, 3.4 predict that the distribution of the final binding energy $E$ of a binary, after an encounter passing very close to either triple collision or triple expansion, is given approximately by a simple power law ((49) and (52), respectively). We now show how these predictions may be tested by comparison with the results of numerical scattering experiments discussed in § 3.6.1. There are two main tests: one is a direct test of (49) and (52), and the
other, slightly more indirect, makes use of the principle of detailed balance.

Figures 5 and 6 show cumulative frequency plots for the low- and high-energy tails respectively. While the trend of the experimental results follows the predictions quite satisfactorily, a detailed comparison of the asymptotic slope with the predicted value is rendered uncertain by the small numbers of encounters leading to very high and very low values of $E$.

A quantitative comparison may be attempted along the following lines, where we concentrate on the high energies by way of illustration; clearly a similar test can be applied to the low energies. The prediction given in (49) is meant to be correct at sufficiently high $E$. Let us suppose then that a random variable $E$ takes values above some lower cutoff $E_0$, with probability distribution \( F(E) = \left( \frac{E}{E_0} \right)^{-\alpha}, \quad E > E_0 \). (54)

If $E_1, \ldots, E_N$ are $N$ independent observations of $E$, the maximum likelihood estimator of $\alpha$ is easily shown to be given by

\[
\frac{1}{\hat{\alpha}} = < \ln \left( \frac{E_1}{E_0} \right) >.
\]

It is easy to show that this is an unbiased estimator of $\frac{1}{\alpha}$, and that the variance of $\ln \left( \frac{E}{E_0} \right)$ is $\frac{1}{\alpha^2}$. Therefore 95% confidence limits for $\frac{1}{\alpha}$ have been obtained by computing $< \ln \left( \frac{E_1}{E_0} \right) > \pm \frac{1.96}{\sqrt{N}}$. In applying these estimates, the cutoff must be chosen sufficiently high so that only results in the asymptotic regime of the theoretical prediction are included, but not so high as to cause large statistical fluctuations.
Figure 5: Results of 3-body scattering calculations in the plane at zero total energy. $E$ is the binding energy of the final binary, and $n$ is the number of cases in which $E$ is less than the abscissa value. The straight line (on this log-log plot) shows the theoretical result (52), with arbitrary normalisation.
Figure 6: Results of 3-body scattering calculations in the plane at zero total energy. $E$ is the binding energy of the final binary, and $n$ is the number of cases in which $E$ greater than the abscissa value. The steeper line represents the theoretical result (49) whose logarithmic slope is nearly $-2.80$, while the shallower curve shows for comparison a line with slope $-2.5$. 
through the small number of scattering events in the sample. The results show that the theoretical value lies within the estimated confidence limit provided that $E_0 > 2$. (Recall that the initial energy of the binary was $E = \frac{1}{2}$.) For this value of $E_0$ the estimated confidence interval for $\alpha$ is $(2.30, 2.91)$. (The theoretical value is 2.80.)

A similar analysis was carried out for the relatively more numerous scatterings which led to small final values of $E$. Here one includes only those scatterings which led to values below an appropriate cutoff $E_0$. The theoretical prediction (52) is that $E$ has the probability distribution

$$F(E) = \left[ \frac{E}{E_0} \right]^{\beta} \quad \text{for } E < E_0,$$

where $E_0$ is an energy sufficiently low for the asymptotic theory to be valid, and $\beta \approx 1.30$. Again the theoretical exponent lies within the estimated 95% confidence interval, provided that $E_0 < 0.03$. At this point the confidence interval is $(1.15, 1.31)$. It is relatively more precise than the corresponding result for the high-energy encounters, because of the much larger number of encounters involved (see below). Perhaps this explains why the asymptotic theory is found to hold only for encounters which reduce the energy of the binary by a factor of at least about 16, whereas it was mentioned above that the theoretical result is also consistent with the data when the energy is increased by a factor of only about 4.

Though the result for scatterings leading to small energies is more sensitive than the corresponding result for large energies, it does not
allow the asymptotic slope to be determined to better than about one decimal place. An independent result can be obtained by exploiting the principle of detailed balance. For two-dimensional scatterings this can be expressed by the statement (Hut & Heggie 1991)

\[ v_3 E^{-2} \frac{d\sigma}{dE},(E'|E) = v_3'E'{-2} \frac{d\sigma}{dE} (E|E') , \]

in which the symbols have the following meanings. The "differential cross-section" for scatterings which lead to a change in the binding energy of the binary from \( E \) to \( E' \) is denoted by \( \frac{d\sigma}{dE},(E'|E) \). To understand this let us consider a binary of energy \( E \) lying in a spatially uniform field of single stars whose speed relative to the binary is \( v_3 \) and whose (two-dimensional) space density is \( n \). Then the rate of encounters leading to a binding energy in the small range \((E', E'+dE')\) is \( n v_3 \frac{d\sigma}{dE} , dE' \). By conservation of energy in each such encounter, the resultant relative speed of the binary and the single star is \( v_3' \), given by

\[ \frac{1}{3} mv_3^2 - E = \frac{1}{3} mv_3'^2 - E' = h \]

(in the case of equal masses). When the total energy in the rest frame of the three masses, \( h \), is zero, as in the scatterings we have considered, detailed balance leads to the still simpler result

\[ \frac{d\sigma}{dE},(E'|E) = \left[ \frac{E}{E'} \right]^{3/2} \frac{d\sigma}{dE} (E|E') . \] (55)

Now let us suppose that \( \frac{d\sigma}{dE} (E'|E') = C E^{-\alpha-1} \) when \( E >> E' \) (cf. (54)). Then the total cross-section for encounters leading to energies \( E >> kE' \), where \( k \) is some large number, varies as \( C(kE')^{-\alpha} \). Now for fixed \( k \) it is clear that this total cross-section varies as the scale of the target binary, i.e. as \( E'^{-1} \), and so \( C \) varies as \( E'^{(\alpha-1)} \). Hence

\[ \frac{d\sigma}{dE} = C'E'^{\alpha-1} E'^{-\alpha-1} , \] (56)
where $C'$ is another constant. Substituting this result, which is valid for $E >> E'$, in (55), we find that

$$\frac{d\sigma}{dE'}(E'|E) = C'E'\alpha^{-5/2}E^{-\alpha+1/2} \quad \text{(for } E' << E). \quad (57)$$

Thus we have proved that the probability of very low resulting energies also satisfies a power law. Furthermore, we have established a relationship between the exponents in the two cases, and it is easily seen to be satisfied by our theoretical results (49) and (52).

Now let $\sigma_1$ be the total cross-section for encounters with a binary of energy $E$ such that $E' > kE$, where $k > 1$. Provided that $k$ is sufficiently large for (56) to be valid (with $E$ and $E'$ interchanged) we find that

$$\sigma_1 = \frac{C'}{\alpha} k^{-\alpha} E^{-\alpha}. \quad \text{Likewise, if } \sigma_2 \text{ is the total cross-section for events such that } E' < \frac{1}{k} E,$$

it follows from (57) that

$$\sigma_2 = \frac{C'}{\alpha-3/2} k^{\alpha+3/2} E^{-\alpha}. \quad \text{Therefore,}

\frac{\sigma_1}{\sigma_2} = \left[ 1 - \frac{3}{2\alpha} \right]^{-3/2},$$

and this relation may be used to estimate $\alpha$ from the results of a set of scattering experiments. If $n_1$ ($n_2$) experiments result in a binary with final binding energy $E'$ in the range $E' > kE$ ($E' < \frac{1}{k} E$), then we may estimate approximately that $\frac{\sigma_1}{\sigma_2}$ has a 95% confidence interval bounded by $\frac{n_1}{n_2} \left( 1 \pm 1.96\sqrt{\frac{n_1+n_2}{n_1 n_2}} \right)$. In this way an approximate confidence interval for $\alpha$ can be found. From our experiments we determine that the theoretical value ($\alpha \approx 2.80$) lies within this confidence interval provided $k > 3$. At this value of $k$ the confidence interval is $(2.57, 3.00)$. This is a less precise determination of $\alpha$ than was
obtained from the direct investigation of low-energy scattering events alone, because of the relative paucity of events in which the final energy of the binary is large. Nevertheless, it serves to illustrate that a single theory can explain not only the slopes of the results illustrated in Figures 5 and 6 (at small and large energies) but also the relative normalisation.

3.7 Summary

We shall now summarize these results for the equal-mass case. The extension of the theory to the unequal masses case is straightforward and is detailed in Heggie & Sweatman(1991).

The probabilities are all expressed in terms of Siegel exponents, $\lambda$, which are given in Table 1. Some of these exponents correspond to perturbations which have a particularly simple interpretation, given in the footnote to the table.

From the exponents, three probabilities may be computed. If $E$ is the final binding energy of a binary, and $h$ is the total energy of the three-body system in the rest frame of its barycentre, then

\[(i) \quad P(E \geq E_0) \propto E_0^{-\alpha} \quad (E_0 \text{ large, } h=0) \quad (58)\]

where

$$\alpha = - \frac{3}{2} \sum \text{Re} \lambda_i \quad (\text{cf. § 3.3});$$

$$\text{Re} \lambda_i < 0$$

$$\lambda_i \neq -1$$
(ii) \( P(E < E_0) \propto E_0^\beta \quad (E_0 \text{ small, } h=0) \) (59)

where

\[
\beta = \frac{3}{2} \sum \text{Re} \lambda_i \quad (\text{cf. § 3.4});
\]

\[
\lambda_i > 0 \quad \lambda_i \neq 2/3
\]

(iii) \( P(E < 0) \propto h^\beta \quad (h \text{ small, positive}) \) (60)

(cf. § 3.5). This last result is the probability of ionisation.
<table>
<thead>
<tr>
<th>One dimension (all eigenvalues)</th>
<th>Collinear Configuration</th>
<th>Equilateral Configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2}{3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\frac{1}{6} \left( 1 \pm \sqrt{\frac{237}{5}} \right)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Two dimensions (all eigenvalues)</th>
<th>Collinear Configuration</th>
<th>Equilateral Configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2}{3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\frac{1}{3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\frac{1}{6} \left( 1 \pm \sqrt{\frac{237}{5}} \right)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\frac{1}{6} \left( 1 \pm i\sqrt{\frac{51}{5}} \right)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Third dimension (extra eigenvalues)</th>
<th>Collinear Configuration</th>
<th>Equilateral Configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\frac{1}{6} \left( 1 \pm i\sqrt{\frac{51}{5}} \right)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Table 1: Siegel exponents for equal masses</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) non-zero energy</td>
<td>e) rotation about 2nd axis</td>
</tr>
<tr>
<td>b) time-shift</td>
<td>f) angular momentum about 2nd axis</td>
</tr>
<tr>
<td>c) rotation about 3rd axis</td>
<td>g) rotation about 1st axis</td>
</tr>
<tr>
<td>d) angular momentum about 3rd axis</td>
<td>h) angular momentum about 1st axis</td>
</tr>
</tbody>
</table>
4. The Connection Problem

At this point it is necessary to raise an issue which further complicates the derivation of results for special choices of the masses. Our theory was based implicitly on the assumption that triple systems passing close to triple collision do not also pass close to triple expansion. If this were to happen, then the reduction in scale in the first configuration might be more-or-less reversed in the second configuration.

The problem is most easily understood in the context of the one-dimensional problem discussed in § 2, because we may imagine that the positions of the two "spikes" in Figure 3 will depend on the masses (if we were to vary the stars' relative masses), and there may be choices of the masses where the spikes coalesce and disappear. From another point of view, in Figure 1 we are concerned with the possibility that orbits emanating from the lower rest point flow into the upper one.

In general the question of the existence of such an orbit may be termed a "connection problem", and it has been studied by several authors (McGehee 1974, Simó 1981, Moeckel 1983, Susín 1988) because of its rôle in the classification of possible orbital motions in the three-body problem. For example it is known (Simó & Susín 1990) that, for certain masses, there is such a connecting orbit between the equilateral collision and expansion singularities. Since the equilateral configuration played the dominant rôle in our discussions in §§ 3.3-3.5, it is desirable to consider this matter further. In order to analyse the problem in as simple a setting as possible we consider the
two-dimensional problem, and specialise to the case in which the total energy (in the barycentre frame) and total angular momentum both vanish. As in the rest of this chapter we shall take the stars to have equal masses; however, a similar argument applies for unequal masses (Heggie & Sweatman (1991)). It is known (Waldvogel 1982) that in this case the analogue of McGehee's manifold, which was sketched in Figure 1, is a four-dimensional space, which is called the "non-rotating collision manifold", $N$. It has equilibria, corresponding to the usual possible configurations for exact collision and expansion. In the vicinity of the equilateral configurations the linearised flow has four independent solutions proportional to $|t|^\lambda$, where the values of $\lambda$ are the four values $-\frac{1}{6}(1 \pm \sqrt{13})$, repeated. Two are positive, and we denote these by $\lambda_1, \lambda_2$, while the other two, denoted by $\lambda_3, \lambda_4$, are negative.

First of all we show how to obtain results analogous to those of § 3.3 in this picture. Let us take the origin of local coordinates at the equilibrium corresponding to collision in the equilateral configuration, so that the linearised flow is given by

$$(x_1, x_2, x_3, x_4) = (\alpha_1 |t|^{\lambda_1}, \alpha_2 |t|^{\lambda_2}, \alpha_3 |t|^{\lambda_3}, \alpha_4 |t|^{\lambda_4})$$

where $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$ are constants. Then contraction of the triple system stops as soon as either the third or fourth component becomes of order unity, (Figure 7), i.e. when $|t| = \max\{|\alpha_3|^{-1/\lambda_3}, |\alpha_4|^{-1/\lambda_4}\}$. Again the length scale of the triple system is of order $|t|^{2/3}$, and so the energy of a resulting binary is of order

$$E \approx \min\{|\alpha_3|^{2/3\lambda_3}, |\alpha_4|^{2/3\lambda_4}\}.$$  
Assuming that values of $\alpha_3$ and $\alpha_4$ are
Figure 7: Flow near equilateral triple collision, in the non-rotating collision manifold. Only the part of the flow which diverges away from the equilibrium point at the origin is sketched. Contraction of the triple system effectively stops as soon as a point flows out of the unit square. OC is part of an orbit which connects the origin with another equilibrium point representing equilateral triple expansion; OC' is a neighbouring orbit.
nearly uniformly distributed close to $\alpha_3 = \alpha_4 = 0$ we find that

$$P(E > E_0) \approx P(|\alpha_3| < E_0^{3\lambda_3/2} \text{ and } |\alpha_4| < E_0^{3\lambda_4/2})$$

$$\approx E_0^{3(\lambda_3 + \lambda_4)/2}$$

(61)

$$\approx E_0^{-(1 + \sqrt{13})/2}$$

This result differs from (50), which is the value of $\alpha$ to be used in (58), only because we are considering a restricted set of initial conditions — those with vanishing angular momentum.

Now we consider how the result should be modified if there is an orbit connecting the equilibrium point under discussion with the other point representing expansion in the equilateral configuration. This orbit must lie (asymptotically) in the $x_3, x_4$ plane (it is shown as OC in Figure 7), and any system which moves close to this orbit (e.g. OC') will eventually spend a long (scaled) time close to a configuration of triple expansion. However, in order for the length scale of the system to expand by a very large factor during this phase of the motion, C' must be very close to C. The position of C' is determined by the ratio $\alpha_3/\alpha_4$, and so there is a very narrow range of values of this ratio for which our previous estimates of the scale of the final binary are wrong. Furthermore, the greater the contraction in the vicinity of 0, the smaller is the range of values of $\alpha_3/\alpha_4$ for which our estimates are incorrect. Thus our previous estimate, given by (61), for the measure of the set of initial conditions leading to a binary energy $E > E_0$, is asymptotically correct for sufficiently large $E_0$.

The foregoing argument shows that the existence of a connection in
this case does not vitiate the results of §§ 3.3-3.5. This fact depends on the dimensionality of the space in which the dynamics occurs; in the McGehee case of Figure 1, the orbits emerging from the lower equilibrium form a curve, and then if there was a connection between the two equilibria, any orbit passing close to one would have to pass close to the other. However, in the spatially two-dimensional case that we have considered with the aid of Figure 7, the set of orbits emanating from the equilibrium point at 0 is two-dimensional, but only a single orbit connects the two equilibria.

5. Discussion And Conclusions

5.1 A Comparison With Results In The Literature.

The work of this chapter is concerned with the three-body scattering problem for encounters in which the three bodies pass close to either a collinear or an equilateral central configuration, either expanding or contracting. In the latter case the three bodies pass close to a triple collision, and there are many papers discussing such encounters from an analytic or numerical point of view. Very few, however, are concerned with the statistics of the outcome.

First we consider a number of papers discussing one-parameter families of close triple approaches, where the results can be interpreted in terms of Siegel exponents. If the family is parameterized by \( d \), \( d = 0 \) corresponding to triple collision, the solutions can be written approximately in a form like (47), where we suppose the coefficients \( dx_j \) are proportional to \( d \). Arguing as in § 3.2, we see that the contraction of the triple system stops at a time when the largest
perturbation term becomes of order unity. When $|d|$ is small enough, the relevant term is the one with the most negative value of $ReA_1$, and then the value of $t$ is given by $|t| \alpha d^{-1/\lambda_1}$. Thus the length scale, and the semi-major axis, $a$, of the final binary, vary as $d^{-2/3\lambda_1}$.

For equal masses in the equilateral configuration, the results of §3.1 show that $\min(\lambda_1) = -\frac{1}{6}(1 + \sqrt{13})$, and so $a \alpha d^4/(1 + \sqrt{13})$, i.e. $a \alpha d^{0.869}$ approximately. Szebehely(1974) numerically computed a sequence of encounters close to equilateral triple collision and found $a \alpha v_0^{0.87}$, where $v_0$ is Szebehely's notation for our parameter $d$. Alexander(1986, Table II) gives two examples of encounters close to triple collision in the same configuration, and his results imply $a \alpha 6x^{0.868}$ approximately, where $6x$ is his parameter.

Such studies of one-parameter families of encounters illustrate some aspects of the theory, but our emphasis in this chapter has been on statistical results. There appear to be relatively few published accounts of scattering experiments in which (a) the initial conditions have been correctly randomized, and (b) the sample size is sufficiently large. The main source for our purposes is Hut & Bahcall(1983). Among the many series of scattering experiments they conducted, they discussed the cross-section $\sigma$ for ionisation of an initially circular binary, their result being

$$\sigma \alpha (v - 1)^{1.3}.$$  

Here $v$ is the initial relative velocity of the incoming third star and the barycentre of the binary, measured in units of the critical velocity required for the energy, $h$, of the three-body system to vanish in the
rest frame of their centre of mass. Thus \((v - 1) \propto h\) if \(|h|\) is small, and so \(\sigma \propto h^{1.3}\). By comparison, in § 3.5 we found that the equilateral configuration dominates the ionisation cross-section in the case of equal masses, and that the probability of ionisation varies as
\[
P(E < 0) \propto h^\beta
\]
where
\[
\beta = -\frac{1}{2} (1 - \sqrt{13})
\]
\(\approx 1.30\).

This closely agrees with Hut and Bahcall's numerical answer, based though it was on a sample of only 52 encounters with \(1 < v < 1.1\). However, using the method of § 3.6.2, we estimate the 95% confidence bounds for their exponent to be \([1.0, 1.8]\), and so the close agreement is rather fortuitous.

Now we turn to previous theoretical discussions relevant to the theory of this paper. We concentrate here on three-dimensional encounters leading to binaries with very large binding energy. On the basis of two or three kinds of approximate theory, Heggie(1975, equations (4.11), (5.4), (5.19) and (5.25)) obtained results which can be expressed asymptotically in the form
\[
P(E > E_0) \propto E_0^{-7/2}.
\]
It was later realized (Hut & Heggie 1991) that this exponent essentially reflects the small volume of phase space available at high binary binding energies. It can be seen from the results of the present chapter that this result is incorrect. Thus in the case of equal masses, the correct exponent is about 3.80, by (51). (In fact the old theoretical result, (62), is correct only for unequal masses, in the limit in which
one mass much exceeds the other two [Heggie & Sweatman(1991)]. A phase-space volume argument has been used by Mikkola & Hietarinta(1989) to suggest a theoretical form to be fitted to the results of a number of one-dimensional scattering experiments. They noted differences in detail between this prediction and experiment, and indeed, the results of the present chapter show that the correct exponent differs from that expected on grounds of phase-space volume.

5.2 Conclusions

In this chapter we have enlarged the theoretical understanding of three-body gravitational scattering. The method is unlike most others used for the approximate analytical study of three-body scattering in that it does not attempt to break down an encounter into one or more two-body encounters; all three bodies enter essentially in the analysis. Though the necessary analytic tools have been used in atomic scattering theory for many years (Wannier 1953), their application to the gravitational problem so far has been rather modest. With regard to the statistics of three-body scattering, the only other results of which we are aware are given in the independent work of Grujić & Simonović(1988). Their results have been extended by including the equilateral configuration which for equal masses is the dominant one.

The results are easiest to consider in the case when, $h$, the total energy of the three stars, in the rest frame of their barycentre, is zero. We have given the form of cross-section, $\sigma$, for encounters which lead to a single star and a binary of a given binding energy $E$, in the two extreme cases where $E$ is either very large or very small. In the
first case
\[ \sigma(E > E_0) \propto E_0^{-\alpha} \]
and in the second
\[ \sigma(E < E_0) \propto E_0^\beta \]
for certain exponents \( \alpha, \beta \). (The theory does not allow the constants of proportionality to be determined, but an application of the theory of detailed balance allows the two constants (and the two exponents) to be related. In the case of two dimensions this is discussed in § 3.6.2.)

The main results of the theory are values for \( \alpha \) and \( \beta \). For one-dimensional scattering of equal masses the results are given by (29) and (31); for two-dimensional scattering of equal masses the results are given in (50) and (53); and for three-dimensional scattering of equal masses we have (51) and (53).

The results are also approximately applicable in situations where \( h \) is non-zero. If \( h \) is small and positive, the cross-section for destruction (ionisation) of a binary is given approximately by \( \sigma \propto h^\beta \), with the same values of \( \beta \) as quoted in the previous paragraph, and it is clear from the discussion of § 2.3 and on physical grounds that (63), which is probably this chapter's most important result, is approximately correct when \( |h| \) is small compared with the final binding energy of the binary. It follows that the same power law applies (for sufficiently large binding energies) not only for cross-sections at fixed \( h \), but also for "thermalised" cross-sections, in which the result is averaged over a Maxwellian (or other) distribution for the initial relative velocity of the binary and the third star.
The Development Of An N-Body Code For The Edinburgh Concurrent Supercomputer

"Many hands make light work." Proverb

1. Introduction

We now turn our attention from N=3 to large N. We shall study systems which are roughly spherically symmetric, and may show behaviour present in an entire globular star cluster. To perform this work, an N-body code was developed to run upon the Edinburgh Concurrent Supercomputer (E.C.S.). This development is the subject of the present chapter. The ensuing chapters analyse aspects of simulations performed using the program.

N-Body computer codes integrate through time the equations of motion for a collection of particles moving under forces between them. In the program developed during this study, the force involved is that due to gravity acting within a group of stars. For any individual star it is computed by directly summing the forces from each of the other N-1 stars in the system, treating the stars as point masses. The calculated value can then be used to predict the position and velocity of that specific star in the immediate future, using the known motion of the star at the time of summing the forces.

The aim of the work described in this chapter was to write an N-body code to run on the E.C.S.. This computer differs from standard ones in that it is a parallel rather than serial machine; that is it operates by
performing many processes simultaneously in different physical locations rather than one process at a time at a single location. If we consider an Ancient Greek armed with an abacus to be a simple standard computer, then another Greek with a larger abacus may be a more powerful computer. However, there is a limit beyond which it is hard to further increase computer power (the abacus becomes unmanageable). At this stage we could construct a parallel computer by bringing together a collection of several Greeks with abaci and persuading them to work together. The metaphor gives some idea of where the difficulties lie for the programmer of a parallel machine: the problem to be tackled needs to be divided evenly amongst the processors (Greeks) to prevent inefficiency; also one must consider the communication system between the processors.

The eventual program went through a succession of stages during development, as modifications were introduced to an initial version. The parallel framework of the code and the master/slave structure present in all the stages were suggested by an earlier N-body parallel code written by Duncan Roweth to run on the E.C.S.. (This used the "leapfrog algorithm" which is explained later [§ 3.3.2].) In the final program, the algorithm used to evaluate the position of a particle was based upon those used on serial machines by Wielen(1967) and Aarseth(1985 and references therein). The code developed has been used for systems in which the stars all have the same mass; however, with small modifications it would be ready for use in a multimass simulation.

The chapter begins with a description of the computer environment. We go on to look at the program and its development, starting with the
overall structure, and then proceeding to study the development of particular parts of the program: the communications, computational algorithm, structure, and starting and finishing sequences. Having described the program its performance is analysed. For later studies we require a program to integrate stars in a fixed potential; this second program is described in the penultimate section. The final section describes the simulations performed with the programs. The data from these simulations are to be used in the later chapters.

Throughout this chapter and in the following ones we shall use the units of Heggie & Mathieu(1986) unless otherwise stated. That is we shall take $G=1$, $M=1$, and $E = -\frac{1}{4}$ where $G$ is the gravitational constant, $M$ the total mass and $E$ the total energy.

2. Computing Environment

The E.C.S. is an array of several hundred transputers. Each transputer contains a processor (rated at about 1 Mflop), memory and communications. Each transputer has four "hard links": that is connections that can be joined to other transputers to send messages between them. By making many such connections, transputers can be linked together to form large "concurrent" systems. The program whose development is described in this chapter runs on such a system.

To program effectively in this environment, account must be taken of the time required to communicate between transputers; for computing efficiency it must be minimized. In the chain of communication it takes much longer to send messages between different transputers, than
internally between an individual transputer's processes (along "soft links"). In addition, one must avoid deadlock: the halting of one process to await input from another, when this second process in turn is awaiting further output from the first.

To use the full power of the machine, the dominant calculational part of the problem to be tackled must be divided into a number of nearly equal parts that can be solved independently on separate transputers. If the problem is not divided evenly, then some of the transputers will be left idle whilst others are still finishing off another part of the work.

3. The Program

3.1 Overall Structure

Right from the first version, the program could be divided into two sections: the master (1 transputer) controls the system, initialises the calculation, collects the results, and is connected with input/output devices; the slaves (s transputers) perform the bulk of the calculations. Information about the stars needed for a particular transputer's calculations are stored in that transputer's memory in an array (called "world") with the successive data for different bodies arranged sequentially.

The program's operation is started by the master. It either reads in or generates a set of initial data for each star from which calculations can begin. It then communicates this data to the slaves and initiates the main sequence of operation. During this process the slaves
numerically integrate the equations of motion of the stars, monitored and guided by the master. Once the stars have had their motion integrated to a preset time, the master terminates the main sequence of operation and assisted by the slaves, finds any information required about the stars. This it lists to a file. A long integration is composed of several short runs during which final data is listed at the end of each run and read into the next as initial data.

In the more detailed study of the program and its development which follows, we shall in succession consider its different constituent parts. We begin with the communication network between processors which remained more or less the same throughout program development (§ 3.2). Then we look at the algorithm used for the slaves' calculations, which underwent several changes (§ 3.3). Alongside these changes there were some adaptations made to parts of the larger program and these are detailed in §§ 3.4, 3.5. The procedures used at the beginning and end of the run are commented upon in § 3.6 and § 3.7, respectively.

3.2 Communications

The master and slave transputers are joined together in a simple loop

```
master slave slave slave

input/output

slave slave slave slave
```

Diagram 1: Transputer communications
(Diagram 1). Data is sent around this loop in one direction passing successively through the transputers. If the message is intended for the master, it is stopped there. Otherwise it is passed on by the transputers until terminated at the original sender, by which time it has passed through them all. The segments of program that run on each transputer are further divided into a main process that does the calculations and two buffer processes (Diagram 2).

![Diagram 2: A transputer's internal processes and communications](image)

The buffers facilitate communication between transputers. One buffer (Inbuf) collects messages arriving from the incoming hard link and passes these on to the main process. The other (Outbuf) collects messages from the main process and sends them out on the outgoing hard link. Together, by temporarily storing messages, they ensure that the hard links between processors are used efficiently, and without "deadlocking". As soon as the Outbuf of a transputer receives a message (from the transputer's main process), it sends it across the hard link to the Inbuf of the next transputer. This occurs independently of the main processes which may continue to work. The message stays at the Inbuf until the main process on this transputer is ready to receive it. When the main process is ready, the message is sent to it across the
soft link from the *Inbuf*. This communication is much more rapid than the one across the hard link that would have now occurred had not the message already been passed to the *Inbuf*. The *master* has an additional hard link running from the main process to the input/output devices.

The buffers' sole function is to relay messages. To improve computational speed, an attempt was made to use the *Inbuf* process to sort the arriving messages into ones to send on to the main process of that transputer and ones to send directly to the *Outbuf*. The change was not implemented in the final program as it actually caused a slight reduction in computation speed. In retrospect it seems likely that this slow down was caused by overworking the *Inbuf*: it was too busy sorting messages to be ready to receive messages from the neighbouring transputer's *Outbuf*. This problem would be avoided by introducing a further process on each transputer to do the sorting, leaving the *Inbuf* to do its original job.

Communication time could also be improved by refining the way in which the transputers are linked together. One could arrange the hard links into a branching structure to minimize the distance that messages have to travel between transputers. However, more significant gains are to be made by developing the calculation algorithms which tend to dominate the communications, and it is into this area that we proceed in the next section.

### 3.3 The Integration Algorithm In The Main Sequence Of Operation

During the main sequence of operation the *slaves* numerically
integrate through time the stars' equations of motion, under the master's control. It is done by the repetition of a collection of operations. Consider a particular cycle. To begin with all the stars have their positions estimated at a time advanced from that of the previous cycle. Then, for a number of the stars, the force on them due to the others is calculated. This new data shall be used in later cycles' estimations of position. These stars are said to have been "updated"; they have had their equations of motion integrated to the time used in the cycle.

To parallelise this section of operation the star updates are divided amongst the transputers: different slaves update different stars at the same time. As an illustration of such a cycle we shall outline the one contained within the final program, which had individual timesteps (explained in § 3.3.3). The cycle begins with the master finding the next s stars to be updated (where s is the number of slaves). Then each slave updates one of these stars. Each slave repeats a loop: receive message from the master, stating which body to update; predict the positions of all the bodies and also the velocity of the body to be updated; calculate the new force on the body to be updated and compute its derivatives using divided differences (a method given in Wielen(1967) and described in more detail in § 3.3.5); add corrector terms to the updated body's predicted position and velocity and calculate the next time by which it must be updated; send the new data for the updated body to the other transputers and receive from them their new data on updated bodies.
Before we look at the initial algorithm used we should think about the statistical validity and accuracy of such algorithms.

3.3.1 Statistical Validity And Accuracy

We need to have some idea of the algorithm's statistical validity: does our solution resemble the exact solution starting from the same initial conditions? During an N-Body simulation, numerical errors accumulate and the system's coordinates deviate exponentially from the exact solution to the equations of motion and initial conditions. For typical codes, the error of the integrated model outstrips the accuracy of the computer in a few crossing times (Heggie(1991)). However, with a sufficiently high numerical accuracy it is hoped that the global properties of the simulation will resemble those of the exact solution (cf. Heggie(1988, 1991)). The deviation over the run of the total energy, an integral of the motion, is reckoned to be a good measure of the validity of these statistics.

The number of bodies and to a lesser extent number of transputers influence accuracy; however, a more easily varied third influence is the size of the timestep (the difference in time between updates). For N-body codes, the timestep generally takes the form of a product $A\tau$, where $\tau$ is a time determined by local conditions and $A$ is a constant. By varying $A$ we alter the size of the timesteps and hence indirectly can adjust the error in the energy.

3.3.2 Algorithm Of The First Program: The Lockstep Leapfrog Algorithm

The first working program used the "leapfrog" algorithm to integrate
the motions of the stars through time. It is a lockstep algorithm, i.e. in each cycle all the particles have their positions and velocities updated to the same time. In fact in the program all the particles were updated together at constant intervals \( \Delta t \). For each particle the cycle begins with the calculation of the force due to the other particles. Then the velocity is updated by the formula

\[
v_{\text{new}} = v_{\text{old}} + f \Delta t . \tag{64}
\]

Finally, the position is given by

\[
r_{\text{new}} = r_{\text{old}} + v_{\text{new}} \Delta t . \tag{65}
\]

(See Diagram 3.) \( f \) is the force per unit mass on the individual particle before the update, \( v \) is its velocity and \( r \) its position. The routine is repeated at every update.

Find the forces on all of the bodies

Find new velocities of all bodies using (64)

Find new positions of all bodies using (65)

Diagram 3: Leapfrog lockstep algorithm

The algorithm is called the leapfrog algorithm because in its most effective form the sequence of velocities for the bodies are taken to be
at times differing by half a timestep from those of its positions (rather than concurrent with them). Then, the absolute truncation error in the energy per unit time is $O(\Delta t^2)$. To arrive at this result consider the updates required to integrate the system through a fixed period of time. Consider a velocity update. We are setting

$$v_{n+\frac{1}{2}} = v_{n-\frac{1}{2}} + f_n \Delta t$$

but (by Taylor's expansion)

$$f_n = f_{n-\frac{1}{2}} + \frac{f_{n-\frac{1}{2}} \Delta t}{2} + O(\Delta t^2)$$

and also

$$v_{n+\frac{1}{2}} = v_{n-\frac{1}{2}} + f_{n-\frac{1}{2}} \Delta t + \frac{1}{2} \frac{f_{n-\frac{1}{2}} \Delta t^2}{2} + O(\Delta t^3),$$

and so the truncation error for this update is $O(\Delta t^3)$ per step. Similarly a position update has truncation error $O(\Delta t^3)$ per step. The potential and kinetic energies will have error of the same order as the position and velocity respectively and so the total energy has error $O(\Delta t^3)$ per step. Over a unit interval of time the number of steps taken is $1/\Delta t$ and hence over the run the total energy is preserved to an accuracy of $O(\Delta t^2)$.

With concurrent velocities and positions, as in my implementation of the algorithm, the error in the total energy is $O(\Delta t)$. Table 2 shows the changes in energy (to three significant figures) measured at the end of several runs. The runs have the same fixed length and the same initial conditions but have differing timesteps. The table clearly shows that the change in energy depends linearly on $\Delta t$. (There were 25 bodies.)
<table>
<thead>
<tr>
<th>Δt</th>
<th>change in energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0530</td>
</tr>
<tr>
<td>0.01</td>
<td>0.00133</td>
</tr>
<tr>
<td>0.001</td>
<td>0.000134</td>
</tr>
<tr>
<td>0.0005</td>
<td>0.0000669</td>
</tr>
<tr>
<td>0.0002</td>
<td>0.0000267</td>
</tr>
</tbody>
</table>

Table 2: Linear error dependence for first program

The algorithm in the actual code could be described as being an Euler method and that in which the position and velocity are updated at alternate half-intervals as a midpoint method (see e.g. Press et al. (1986) pp. 550-551).

The algorithm is parallelised by dividing the star updates evenly amongst the slaves. We also only store the data for N/s bodies on each slave (although this will change during the program's subsequent development). (The number of stars in the system was chosen so as to be a multiple of the number of slaves.) After updating, the stars' velocities and positions are sent around the ring of transputers. They shall be used by each slave to find the new forces on its own N/s stars for the next update. A message contains data for all the stars stored on the transputer that sent it and hence its size is proportional to N/s. To go around the loop past all the slaves a message must travel across s+1 hard links. Communication time will be the product of these two factors; that is approximately of order N. In comparison the
calculations made on each slave in between these communications are approximately \( O(N^2/s) \), as on each slave there are \( N/s \) body updates each involving adding up forces from \( N-1 \) other bodies. Comparing the order of the calculation and communication terms per body update, we see that the calculations are dominant for sufficiently many stars on each processor.

### 3.3.3 Ideas For Algorithm Development From The Lockstep Leapfrog Program

From the lockstep leapfrog algorithm a number of possible improvements may be made. In the final program we implement individual timesteps, variable timesteps, and a higher order in the prediction of orbits. These types of adaptations are explained by Aarseth (1985) and customarily programs of this kind are called "Aarseth-type" after his famous "NBODY" series of codes.

By individual timesteps we mean that the individual stars are to be updated at different time intervals. Some stars (e.g. those in the core: a denser region) have many closer interactions with other stars which change their orbits rapidly; they require frequent update. Other stars (e.g. those in the halo: a less dense region) have less frequent, more distant interactions and change orbit more slowly; they require less frequent update. We can save upon computation by only updating a star when it is strictly necessary. In the lockstep program the stars are all updated together at the smallest interval required by any star and so all the other stars are updated prematurely.

As mentioned in the previous paragraph stars may change their orbits. Also, stars on different orbits require to be updated at different
intervals. So to allow for the stars which change orbit and hence require to be updated more or less frequently we introduce variable timesteps. The interval between updates is not fixed but is varied so as to be of optimal length for efficiency at a given accuracy: not too long because this would create too large errors and not too short as this would make extra work.

By using higher order formulae to estimate a star's position and velocity it is not generally necessary to update stars as frequently as with a lower order formula. So the introduction of a higher order algorithm lets the program make less updates; however, it requires more calculations per update. These gains and losses are to be balanced against each other: the addition of a few orders beyond the lockstep leapfrog algorithm resulted in improvements in computing time; however, the loss from extra calculations would eventually overtake the gains from fewer updates if too many higher order terms were introduced. (The subject is discussed further in § 4.2.)

3.3.4 Introducing Individual Variable Timesteps

The first stage in developing the algorithm was the removal of lockstep and the introduction of individual, variable timesteps; that is the first two improvements mentioned in the last section were written into the program together.

Here a difference emerges between our program and standard sequential codes: the first modification (individual timesteps) is only partially implemented. To fit in with the overall structure (§ 3.1) bodies are not
updated strictly individually but rather a fraction of them are updated at once, one star per slave. This enables us to obtain most of the gains in speed of strictly individual timesteps, whilst, at the same time, spreading the calculational load across the array and minimizing communication. The extra computation forced by premature update of bodies is studied in § 4.1.

The first individual variable timestep used was a multiple of \( r_{\text{min}} / v_{r_{\text{min}}} \), where \( r_{\text{min}} \) is the distance from the star in question to its nearest neighbour and \( v_{r_{\text{min}}} \) is the relative velocity of this nearest star. This timestep is comparatively easy to find - one can record \( r_{\text{min}} \) whilst performing the force calculations, (during which the distances between the body being updated and all other stars are found), and then find \( v_{r_{\text{min}}} \), (using the predictor to calculate that nearest star's velocity at the present time). (This form of timestep might be improved in a mathematical sense, if it were instead the smallest ratio of relative distance apart divided by relative velocity, but the advantage gained has to be balanced with the extra calculation time required to find the relative velocity for all of the bodies.)

With individual timesteps, there has to be a section of program that determines which bodies require their orbit data updated next. For each body, the time by which it must be updated along with times of previous update are recorded with the rest of its data in world. Now, as for the lockstep program, individual slave transputers store and perform the calculations for a subset of the bodies (N/s of them). As previously, there are partially overlapping calculation and communication phases;
however, during the calculation phase only one body is updated on each slave, rather than all \( N/s \). In order to update this body we require all the stars' positions at this time. They are in general calculated from truncated Taylor series ("force-polynomials") which are based on the estimates for position and its derivatives found at the time of the last update \( t_{\text{old}} \) (see § 3.3.5). For this initial stage of program development, with just positions and velocities used, we are taking

\[
\mathbf{r}(t) = \mathbf{r}_{\text{old}} + \mathbf{v}(t - t_{\text{old}}) \quad .
\]

This process of predicting the positions of the bodies we shall call the extrapolations. At the end of the run all the bodies' positions and velocities are found by an extrapolation from their previous values.

With individual timesteps, there has to be a section of program that determines which bodies require their orbit data updated next. After the body update, each slave uses a heapsort-like routine (see Appendix 1) to find the star stored on it with the smallest time before next update. This body shall be updated next upon the slave. The smallest times (one per slave) are sent to the master. A straight comparison of the \( s \) messages arriving at the master finds the smallest time of all the \( N \) bodies. This time is now sent around the loop of slaves so that they can use it to perform their next update.

3.3.5 Changing To A Higher Order Algorithm

After implementing individual variable timesteps the next progression made was to increase the order of the algorithm. Accompanying this change a new form of individual variable timestep was also introduced, which is described in § 3.3.6. The new algorithm is shown as a flow
On all slaves find the positions of all the bodies using the predictors.

On each slave find the velocity of the body to be updated using its predictor.

On each slave find the force on the body to be updated due to the other bodies.

On each slave find a new predictor for the updated body.

On each slave correct the position and velocity of the updated body.

On each slave find a new timestep for the updated body.

---

Diagram 4: The higher order algorithm

The algorithm used is a multistep integrator similar to that of
NBODY1 (Aarseth 1985), (but of one order lower in the final program). It is based upon the formulae given by Wielen (1967): a predictor-corrector scheme. The structure and properties of such algorithms are discussed and summarized by Makino (1991). Essentially the predictor part of the method estimates the position of a star using a Newtonian extrapolating polynomial (a "force polynomial"). This is a polynomial fit to a number of previous values of the star's acceleration, and its immediately previous position and velocity. The predictor is used to find the position and velocity of a star whenever required in between the star's own updates, (for other star's updates or at the end of a run). When it is time for a star itself to be updated, the predictor is used to find provisional values for its position and velocity. The predictors for other stars are also used to find their positions at this time. The force on the star being updated is calculated. The star's new acceleration together with the previous acceleration values involved in its predictor are used to obtain a new predictor and corrections to its provisional position and velocity. The new values of position and velocity are those that would be given by a polynomial of one order higher than the predictor, fitted to the same previous accelerations but also taking into account the acceleration calculated for the present time. Finally the next time of update for the star is found.

In practice we are only required to store the coefficients of the predictor and the update times for each star. (Each of the latter corresponds to a previous acceleration that is used in the predictor [there will be two less of these than the coefficients of the predictor as the order of the force polynomial is two less than that of the
position].) When the previous accelerations of a star are required for use in its corrector and its next predictor, Wielen's formulae are rearranged to recover them from the coefficients of the current predictor. The program in its final form stored twenty items of data per star (in array world): the mass (taken to be 1/N throughout), five coefficients in each coordinate for the position's predictor, three previous update times and the time of next update.

The correction terms improve the accuracy of the position and velocity at update; however, their primary use is as an estimate of the error before correction. We shall be using this in the construction of our new choice of timestep, which is described next.

3.3.6 New Timestep

To go with the higher order algorithm a more sophisticated formula is used for the timesteps. The individual timestep that is computed for each star is chosen to keep the estimated error in energy per update within a bound (ε) preset by the user.

The energy of a single star (per unit mass) is

\[ E = \frac{1}{2} v^2 + \varphi(r) , \]

where \( \varphi \) is the potential at its position. So the error in this quantity is

\[ \delta E = v \cdot \delta v + \delta r \cdot \nabla \varphi(r) \]  \hspace{1cm} (67)

and

\[ |\delta E| \leq |v| |\delta v| + |\delta r| |f| , \]  \hspace{1cm} (68)

where \( f \) is acceleration. The errors in the position and velocity, (δr
and \( \delta v \), respectively, are taken to be the first terms that are not included in their respective Taylor Series. For the timestep actually used in the update, these are precisely the correction terms mentioned in the preceding section.

It is reasonable to equate the error of a position or velocity series with the first truncated term. Press & Spergel (1988) have shown that these terms will dominate higher order ones with a small enough timestep. In my program the actual truncation errors in position and velocity are of one order higher than their added on corrections. However, we cannot find a timestep based on these error terms. Although we know their order we do not have any way of estimating their coefficients, and hence sizes, and it is these that we require for the timestep calculation. (There is an explanation in Press et al. (1986), pp. 555-556.) Therefore the correction terms themselves are taken as the truncation errors.

From (68) we obtain two conditions:

\[
|v| |\delta v|, |f| |\delta r| \leq \epsilon
\]

(69)

where \( \epsilon \) is a preset constant: the energy error bound. We shall now show how to find a provisional timestep that is the largest satisfying both these conditions. Upon completing an update we find two timesteps, \( \Delta t_1 \) and \( \Delta t_2 \). The first is the one that would have made the term \( |v| |\delta v| \) equal to \( \epsilon \), and the second is that which would have made \( |\delta r| |f| \) equal to \( \epsilon \). For these calculations we take \( |v| \) and \( |f| \) to be constant, assigning them their values at the actual update. \( |\delta v| \) and \( |\delta r| \) are treated as being the first truncated terms from the Taylor series for
velocity and position, respectively, (constant multiples of powers of
the respective timesteps $\Delta t_1$ and $\Delta t_2$). That is

\[ |\delta v| = A \Delta t_1^n \quad \text{and} \quad |\delta r| = B \Delta t_2^{n+1} , \]

where $n$ is the order of the predictor for estimating position between
updates. $A$ and $B$ may be found from the known values of $|\delta v|$, $|\delta r|$ and $\Delta t$
at the performed update (the correction terms and actual timestep). The
relationships between these quantities is

\[ |\delta v| = A \Delta t^n \quad \text{and} \quad |\delta r| = B \Delta t^{n+1} . \]

Using (67), $|\delta E|$ would have been of the order of $\varepsilon$ if we had taken $\Delta t$
to be the minimum of $\Delta t_1$ and $\Delta t_2$, and so this value is provisionally taken
to be the timestep to the next update.

In order to prevent the timestep from growing too fast, a further
restriction is enforced. The provisional timestep is replaced by 1.4
times the previous timestep if the latter value is smaller (this
stability factor is recommended by Aarseth(1985)). So the new timestep
finally takes the value

\[ \Delta t = \min \left( \left[ \varepsilon(A|v|)^{-1} \right]^\frac{1}{n} , \left[ \varepsilon(B|f|)^{-1} \right]^\frac{1}{n+1} , 1.4 \Delta t_{\text{last}} \right) . \]

(70)

For the eventual program $n = 4$ and this becomes

\[ \Delta t = \min \left( \left[ \varepsilon(A|v|)^{-1} \right]^{\frac{1}{4}} , \left[ \varepsilon(B|f|)^{-1} \right]^{\frac{1}{5}} , 1.4 \Delta t_{\text{last}} \right) . \]

(During the simulations it was found that the timestep was hardly ever
that associated with the term $|\delta r| |f| ; |v| |\delta v|$ was generally the
dominant term in the error of the energy.)
3.4 The Development Of Structure From The Lockstep Leapfrog Program

In the lockstep leapfrog program the slave processors were only required to store data on a fraction of the stars, data on the rest being sent in from other processors as necessary. When body updates took place each slave predicted the position of N/s bodies (a process of order N/s per cycle) and then communicated this information to all the other slaves (a process of order N per cycle). As mentioned in § 3.3.2 these processes were dominated by the force calculations for the body updates (which were a process of order N^2/s per cycle). However, after the introduction of individual, variable timesteps, only 1 body is updated in a cycle on each slave rather than N/s. This makes the force calculations per cycle of order N (N-1 gravitational forces are found between the body to be updated and the other stars). This is now of the same order as the communication (s updates are performed per cycle), and so communication time becomes important in the overall duration of the program operation. At this stage in the development, the program's speed was improved by a change in structure. Instead of having the slaves each only store a fraction of the stars' data, the new program had them each store it all. The slaves are made to duplicate each other's work by all simultaneously predicting the positions of all the bodies. This removes the necessity for the communication of the bodies' positions around the ring of transputers at the beginning of each cycle. The change does introduce extra work of order N per cycle (predicting N bodies' positions on each slave); however, the time taken to do this proves to be smaller than the aforementioned communication process (that is of the same order) which it is replacing.
3.5 Timestep Sorting On Master

Following the previous change another simple improvement is to have the timestep sorting routine entirely on the master. This sorting is a comparatively small part of the computing time, being of order $(\log_2 N)$ per body update (see Appendix 1 for more details). Timesteps for the updated bodies are sent to the master and there the smallest $s$ of them are found using a heapsort-like routine similar to those which were formerly used on the slaves (again see Appendix 1). By having the data for all of the bodies on all of the transputers we are able to update the coordinates of any body on any transputer. Hence we may update the $s$ bodies with the smallest time to next integration, rather than the $s$ bodies each of which have the smallest time on one of the slaves (the previous situation).

3.6 Initialisation

The overall operation of the master and slave transputers during the main part of a run have been described. Before these routines can start the system must be initialised: a set of data for the bodies must be generated or read from a file and sent to the appropriate transputers. To begin with, the initial conditions were generated by a subprogram that ran on the master transputer prior to the main code. Later, this subprogram was made into a separate program. At the start the initial positions and velocities of the stars in the model are set up using computer generated random numbers to fit a distribution function. For the simulations described in this study all stars have the same mass $(1/N)$ and the distribution taken was that of a Plummer model:
\[ f(E) = \kappa_1 (-E)^{7/2} \quad \text{for } E < 0 \]
\[ = 0 \quad \text{for } E \geq 0 , \]

where
\[ E = \frac{1}{2} v^2 + \phi \quad \text{(see e.g. Spitzer(1987))} \]

\( \kappa_1 \) is a constant, \( v \) is the magnitude of the velocity of the star, \( \phi \) its gravitational potential (taken to be a spherically symmetric function of position), and so \( E \) is energy per unit mass of the star. The distribution is a function of purely the magnitudes of the position and velocity vectors and not their directions, and so these vectors are isotropically distributed. (The generation of these initial data is given in more detail in Appendix 2.) Any further data required for the bodies are calculated from these initial positions and velocities, (e.g. times for update, coefficients for a Taylor expansion for the position of the body). (The reader may again wish to refer to Appendix 2.)

Contained within the more advanced codes for the master is a section to read in random initial conditions, or equivalently initial conditions that are the final data for a previous run. The master stores these data in array \textit{world} before sending them in sections to the slaves, around the loop. There are \( s \) sections (one for each slave); each section contains the initial data for \( N/s \) bodies. In the earlier programs each slave only recorded the information from one section but in later programs they recorded the data for all of the bodies.

To end the initialisation, the program finds the bodies to be updated and an initial timestep for them. This is done in the same way as in the main part of the code. Once this information has been broadcast to the
transputers, the main part of the code's operation can begin.

3.7 The Duration Of The Integration

Initially, when the timesteps were of a fixed length, a prescribed number of integrations were performed. With the introduction of variable timesteps the run was more directly limited by a preset time to which the equations were to be integrated. During the run the master monitors the time of the next update; when this exceeds the limit the master broadcasts a message that directs the slaves to cease their main operations. The current state of world is recorded for future runs, then the slaves update the positions and velocities to their values at the time limit and find any information required (e.g. the total kinetic and potential energies, the position of the centre of the core or the Lagrangian Radii).

4. The Performance Of The Program

During the development of the program, timed runs were performed at successive stages. A timer incorporated in the program recorded how long it took to update the system through a period of time. This excluded the time to set the program going and that to record the results at the end: the objective was to relate the parameters of the integration with the time spent doing the main calculations. The system also recorded the number of body updates that were required to perform the integration. In the next section, formulae are found for the time to update the system using the completed program and varying the numbers of bodies(N), and transputers. The program is compared with Aarseth's NBODY1 (Aarseth 1985) running on the Edinburgh mainframe computer.
4.1 Tests Performed On The Final Program

The results presented here were obtained from a set of $N$ identical masses whose initial positions and velocities were generated from a Plummer model (as given in § 3.6). $N$ was varied over a range of multiples of 2 between 16 and 2048, and the number of slaves between 1 and 128. The systems were all integrated through one of our time units which is $\frac{1}{2\sqrt{2}}$ crossing times. The energy error bound ($\epsilon$) is taken to be $2^{-7} \times 10^{-2}$ for all these numerical experiments.

Empirically, for most of the results, the processing time per update of a star (in seconds) is given to within 10% by the formula

$$\frac{1}{8000} N + \frac{7}{4000} s^{1/4}$$

The two contributions are mainly due to computation and communication, respectively.

The first term, which is due to computation, is proportional to $N/s$. This is because for every body update the calculations are dominated by the prediction of the positions of all the bodies and the computation of their contributions to the force upon the body being updated. Both of these are of order $N$; however, $s$ bodies are updated at once, one per slave, giving the $N/s$ proportionality.

The power of $s^{1/4}$ on the second (communication) term is used purely because it is a good fit to the results. If the time to communicate between any two transputers in a system of any size was constant then this term too would be constant: there are $s$ updates at once and the information from them is simultaneously passed around a loop of $s$
transputers (if we ignore the master). In fact as the number of transputers in the system grows the communication time between them also grows and so the communication term is not independent of $s$.

Figure 8 shows the linear relationship between number of bodies ($N$) and processing time per body update, for various numbers of slaves ($s$). Also shown is Aarseth's NBODY1 running on a 2 Mflop scalar machine. The different slopes of the graphs indicate the effect on computational power of increasing the number of slaves, while the different $y$-intercepts indicate the presence of the second term in (71).

The main part of the calculational work done is $21N$ double precision multiplications per update (these are elaborated upon during the next section). So the computer is working at the rate $\frac{168 s}{(1 + 14N^{-1/4}s^{5/4})}$ thousand double precision multiplications per second, giving for large $N$ (when computation dominates communication) approximately 168 thousand double precision multiplications per second per processor.

Using larger $s$ the program takes less time per body update, but it also forces some updates sooner than are required by the timestep criterion. (For example suppose that we have 2 stars which require update at times 0.010 and 0.015, respectively. If we update them simultaneously it must be to the time 0.010 and in this case the second star is updated 0.005 time units earlier than required. That star will be eventually updated more times than if it had been initially updated at 0.015 and only updated thereafter when strictly necessary.) Figure 9
Figure 8: The linear relationship between number of bodies in a stellar system (N) and computing time per body update, for various numbers of slaves (s). (The different numbers of slaves are represented by different lines as indicated [the line labelled 16s shows the results for an array with sixteen slave transputers].) Also shown is Aarseth's NBODY1 running on a 2 Mflop scalar machine.
Figure 9: The total number of body updates required to integrate a stellar system through one time unit plotted against number of bodies (\( N \)), for various numbers of slaves (\( s \)) (labelled as in Figure 8).
Figure 10: The computing time to update a stellar system through one time unit plotted against number of bodies (N), for various numbers of slaves (s) (labelled as in Figure 8). (a) linear scale; (b) log-log plot.
shows how the number of body updates taken to integrate a system through a time unit relates to N, for various s. As N increases the disadvantage of simultaneously updating s bodies gradually disappears. This graph indicates how much extra computation is done as we increase the number of slaves tackling a given N-body system. So for instance with 1000 bodies, 4 slaves do nearly the same amount of computation between them as 1 slave would have done, however, 64 slaves do over twice as much as either of these arrays.

Figure 10 shows graphs of the computing time to update the system through a time unit against N, again for various s. For a given value of N the lowest line on these graphs will give the optimal (fastest) number of slaves to integrate the equations of motion, allowing for both extra body updates and longer interprocessor communication times.

4.2 Optimal Order

In programs of this type there is an optimal order for computational efficiency: the higher order schemes require an increasing number of calculations per body update so that there is a point beyond which the computer time involved calculating the higher order terms balances out the saving in calculations due to having longer timesteps. The program developed here seemed to be reaching the point where only small gains were to be made by increasing the algorithm order any higher. In Press & Spergel (1988) the authors study the choice of order in Aarseth-type N-body codes. We shall temporarily adopt their notation whilst studying the results of their paper. They define the extrapolatable interval (τ_E) from a time t_0 to be the maximum timestep such that an extrapolated
value (\( \dot{f} \)) of the acceleration (found using the predictor part of the algorithm) has a fractional accuracy bounded by a limit (\( \varepsilon \)) for all smaller timesteps, i.e. \( \tau_E \) satisfies the implicit equation

\[
\frac{|\delta f|}{|f|} = \frac{|f(t_0 + \tau) - \dot{f}(t_0 + \tau, t_0, M, \tau)|}{|f(t_0 + \tau)|} < \varepsilon, \quad 0 \leq \tau \leq \tau_E.
\]

\( M \) is the number of values of acceleration used in the predictor (one less than its own order). (We see that this is a similar criterion to that implied in § 3.3.6. Total energy \( E \) is an integral of the system and so we may take that criterion to be \( |\delta E|/|E| \leq \kappa \), where \( \kappa \) is a constant which is the energy error bound divided by the total energy.) Press & Spergel found that the ratio of the extrapolatable interval (\( \tau_E \)) to a local timescale (\( \tau_A \)) is typically constant to within a factor of two or three. They approximate the mean value of this ratio (\( <s> \)) by a function of \( \varepsilon \) and \( M \),

\[
<s> \approx 0.3 \left( \frac{\varepsilon}{0.07} \right)^{1/M} \quad \text{where} \quad s = \frac{\tau_E}{\tau_A}, \quad \text{and} \quad \tau_A = \left| \frac{f}{\dot{f}} \right|^{1/2}.
\]

They also give a formula for the optimal value of \( M \) for a given value of \( \varepsilon \). It is arrived at by taking the force calculation as being dominated by evaluating the extrapolating polynomials, the time for which scales with \( M \). Next the computation time is taken to be proportional to the force calculation time divided by \( <s> \), and this function is minimized to find the optimal \( M \). In fact the calculation time scales as (\( M + R \)) where \( R \) is a constant. This is because the force calculations for the updated particle are of the same order of magnitude as the extrapolations, but do not depend on their order \( M \).

In my program \( R \) is 4, deduced as follows. There are \( M + 1 \) double
precision multiplications involved in the estimation of a position coordinate for each of the $N$ bodies (from the predictor). Therefore in each update there are $3N(M+1)$ double precision multiplications. Also, the force upon the body to be updated must be found and this involves adding contributions from all of the other bodies. Each contribution involves finding the distance between two stars and then finding contributions to the force upon the updated star in each direction; these involved nine double precision multiplications (four calculations to find the distance, two to find mass over distance cubed and three to multiply these by the individual coordinates of the radial vector). In total the calculations associated with finding the force contribute $9N$ calculations per update and therefore $R = 4$.

Again differing from Press & Spergel, we assert that the optimal $M$ should actually be that with a fixed error per unit of time. Therefore we take $\frac{\xi}{<s>}$, rather than $\epsilon$ (as used by Press & Spergel), as being constant. Call this constant $\xi$; then use of (72) gives

$$<s> = \left( \frac{0.3^M \xi}{0.07} \right)^{1/(M-1)} \quad (73)$$

The computer time to update the system through a fixed interval of its own time ($T_M$) is proportional to $\frac{(M + R)}{<s>}$. If we regard this as being a function of $M$ we may use it to determine the optimal $M$ for a given $\xi$. We take the natural logarithm of the quantity, and then set its derivative with respect to $M$ to be zero. This gives the following formula to be satisfied by the optimal $M$:

$$\frac{(M-1)^2}{(M+R)} + (M+R) \ln \left( \frac{0.3^M \xi}{0.07} \right) = 0 \quad .$$
Solving as a quadratic in \((M-1)\), we find that
\[
M = 1 - \frac{1}{2} \ln \left[ \frac{0.07}{0.3} \right] \pm \frac{1}{2} \left[ \left( \ln \left[ \frac{0.07}{0.3} \right] \right)^2 - 4(1+R) \ln \left[ \frac{0.07}{0.3} \right] \right]^{1/2}.
\] (74)

Taking \(R=4\), we must now estimate \(\xi\). In NBODY1 when \(M=4\), a typical choice of timestep would be
\[
\eta = 0.03
\]
(Aarseth\(1985\)). Press & Spergel comment that the local timescale
\[
\left( \frac{\ln |f|}{|\vec{f}|| \vec{f}'| + |\vec{f}|} \right)^{1/2}
\]
behaves very similarly to \(\tau_A\) as given in (72) (an earlier timestep of Aarseth) and so we shall set \(<\sigma>\) to be \(\eta^{1/2}\). Now we can invert (73) to derive
\[
\xi = \left( \frac{0.07}{0.3^4} \right)^{3/2} \eta. \quad (75)
\]
We shall fix the value of \(\xi\) to get a similar error per time unit to that of NBODY1 with \(\eta = 0.03\). Substitute (75) into the formula for optimal \(M\) (74) to obtain
\[
M = 1 - \frac{1}{2} \ln \left[ \frac{\eta^{3/2}}{0.3^3} \right] \pm \frac{1}{2} \left[ \left( \ln \left[ \frac{\eta^{3/2}}{0.3^3} \right] \right)^2 - 4(1+R) \ln \left[ \frac{\eta^{3/2}}{0.3^3} \right] \right]^{1/2}.
\] (76)

Now \(\ln(\frac{\eta^{1/2}}{0.3})\) is negative, so we shall choose the positive root in order to make \(M\) positive. If we put in our value \(R = 4\) we obtain
\[
M \approx 4.810
\]
Therefore we conclude that the optimal order for \(M\) with this error per time unit is 5 (or 4). Comparing the time to update the system through a fixed interval of time with \(M = 5\) with that of another value of \(M, P\),
gives
\[
\frac{T_P}{T_5} = (P+4) \left( \frac{P \xi}{0.07} \right)^{-1/(P-1)} \frac{1}{9} \left( \frac{0.3 \xi}{0.07} \right)^{1/4}
\]
\[
= \frac{(P+4)}{9} \left[ \frac{0.3 \xi}{0.07} \right]^{1/4 - 1/(P-1)}
\]
\[
= \frac{(P+4)}{9} \left[ \frac{\sqrt{3}}{3} \right]^{3/4 - 3/(P-1)}
\]

(Recall \( T_M \propto \left[ \frac{M + R}{<s>^2} \right] \).) The values of this ratio for \( P \) equal to 4 and 3 are approximately 1.020 and 1.174, respectively. The first result shows that the optimum \( M \) is indeed 5 rather than 4; however, it also indicates that NBODY1 would not be greatly improved by an increase in its order (at this accuracy). My final program has \( M = 3 \); as I was aiming for errors similar to those for NBODY1 as above, my algorithm should take less than \( \frac{6}{5} \) the time taken with the optimum order algorithm.

Makino(1991) has also studied the problem of optimal order. He differs with the results of Press & Spergel in that he asserts that the optimal order is dependent upon \( N \). This may be true; however, his results are not appropriate for integrations at the accuracy of the standard N-body codes in use, and hence are not applicable here. (His predictor of optimal order becomes negative unless the relative error in energy is very small for large numbers of bodies.)

5. Fixed-potential Models

The program that has been described earlier in this chapter integrates through time a star cluster model with a self-consistent potential. It will be of interest to know what would happen in a model of stars with the same initial positions and velocities but moving in a fixed potential (that corresponding to the distribution function used to
generate the stars' initial data [cf. § 3.6]). In such a model the stars move independently from each other: there are no star-star interactions. The stars move unperturbed on their original orbits for all time. The motivation for constructing these models is for them to act as a kind of "control" for comparison with the self-consistent simulations; they give an insight into what observed behaviour is intrinsic to having a finite collection of stars with a variety of periods and orbits, and what is due to self-consistent effects. The predictability of a star's motion in the fixed potential also proves useful. (Motion in the self-consistent potential is not so predictable.) The strategy of later chapters is to simulate systems in self-consistent and fixed potentials and collect data at regular intervals of simulation time. The data are then analysed. The results for the fixed-potential simulations may also be compared with values calculated by another method using the predictability of the fixed-potential orbits (as is done in Chapter IV § 3, where experimental results are compared to an exact calculation).

The integrations are performed using essentially the same program as before but with the ideal force replacing the self-consistent one. As has been previously mentioned the models to be used in the latter part of this dissertation have the Plummer distribution. This leads to a central force; a force towards the centre with a constant magnitude at any given radius. The individual star orbits the geometric centre of the model in a fixed plane, with constant radial and angular periods dependent only upon its (constant) energy and angular momentum. The stars are started with a small step size from the initial conditions as in the self-consistent problem, so that the true Taylor series may be
established before any significant errors occur. During the integrations, the deviation in total energy and the number of integration steps (body updates) are monitored to check upon accuracy per step as in the self-consistent model.

6. The Simulations

The computer code described in the main part of this chapter was used to simulate two self-consistent single-mass star clusters which had 10048 and 1024 stars, respectively. Two corresponding fixed-potential models were also created, so that there were four models in total. Information from the integrations is recorded at 282 successive 0.1 time intervals. This interval was chosen to facilitate a study of Lagrangian radii oscillation as explained in the next chapter (IV § 2). The main items of data recorded are the potential and density centres, corresponding Lagrangian radii, and for the self-consistent models the energies of a randomly chosen subgroup of the stars. (These will be constant for the stars in the fixed-potential models.) The performance of the integration process itself is also monitored, with records kept of such things as the deviation of total energy and the number of integration steps. The energy error bound (see § 3.3.5) was taken to be $2^{-7} \times 10^{-1}$ for the 1024-body self-consistent model and both the fixed-potential models. For the 10048-body self-consistent model it had to be decreased to $2^{-8} \times 10^{-1}$ in order to maintain the total energy's value to within $1.7 \times 10^{-4}$ per 0.1 time interval. Generally, (as observed by Makino(1991)), the energy errors in such an integration tend to cancel each other out, and so for instance in the 1024-body self-consistent model, though there were changes in some of the 0.1
intervals of nearly $2 \times 10^{-5}$, the deviation over the entire run (28.1 time units) was less than $1 \times 10^{-4}$. 
IV

Lagrangian Radii Oscillations

"And certain stars shot madly from their spheres"
William Shakespeare: "A Midsummer Night's Dream"

1. Introduction

This chapter studies Lagrangian radii oscillations. A Lagrangian radius is the radius of an imaginary sphere about the centre of a stellar system, and containing a fixed proportion of its mass. Such radii are often used to monitor and analyse spatial evolution. Reasonably persistent fairly regular oscillations of these radii have been previously observed by Heggie(1989) in a nearly isothermal model within a bounding sphere. The periods of these oscillations were comparable with and nearly proportional to the local circular orbital periods (the times for a star to complete a circular orbit about the centre with the local radius). Their persistence was unexpected - one would expect oscillations of this kind to die away rapidly by phase-mixing or Landau damping (cf. Binney & Tremaine(1987)). Such oscillations are of interest because the collective movement of a group of stars will affect the (self-consistent) gravitational potential. This in turn may affect the orbits of individual stars and cause relaxation (changes in energy and other orbital parameters of stars mainly due to two-body encounters). In addition to Heggie's numerical results, there have also been a number of theoretical studies of radial oscillations in collisionless models (e.g. Mathur(1989), Sridhar(1989), Sridhar & Nityananda(1989), Palmer & Papaloizou(1987), Palmer & Papaloizou(1988), Palmer, Papaloizou & Allen(1989)); these shall also be referred to later.
We shall begin this chapter (§ 2) by outlining the measurements made during the simulations. (Recall that these were described at the close of the previous chapter (III § 6).) In § 3, we present the results, which were similar to those of Heggie's simulation even though his model was very different. The fixed-potential models help to give us an insight into these oscillations, and help in their analysis. The chapter finishes with a comparison with previous work by other authors (§ 4) and the conclusions (§ 5).

2. Measurement From The Simulations

In the four simulations, sixteen Lagrangian radii (each one, one sixteenth of the total mass out from the last) were measured from both the potential and density centres. (The definition of these centres is given in Chapter V § 2.) We do not use the geometric centre or centre of mass because the dense group of stars forming the gravitational heart of the self-consistent cluster does not stay at either of these points but rather moves about in an oscillatory fashion (the subject of a later discussion [Chapter V]). By taking a more precise estimate of the position the stars are orbiting about we shall isolate the radial oscillations from this effect. In practice, the measurements from the two centres are pretty similar; in the following discussions those from the density centre are used as its motion is smoother. We shall comment further upon the choice of these centres in the context of fixed-potential models in § 3.2. As mentioned in Chapter III § 6 the Lagrangian radii are sampled at 0.1 time intervals; the reasoning behind this choice of interval is that the Lagrangian radii should be determined at sufficiently frequent intervals to resolve the sought
after oscillations. The period of these will now be derived.

As mentioned in the introduction to this chapter Heggie's results had oscillations with a period comparable to that of a star on the local circular orbit. This is one local timescale; another is the epicyclic period: the period of small radial oscillations about a circular orbit. We shall consider the latter timescale in addition to the former as it is the period of a local radial oscillation and we are looking for radial oscillations. Binney & Tremaine(1987) discuss epicyclic motions. They observe that epicyclic and circular orbital frequencies ($\kappa(r)$ and $\Omega(r)$ respectively) are related by

$$\kappa^2 = r \frac{\partial}{\partial r} (\Omega^2) + 4\Omega^2 .$$

(77)

In general, for reasonable models of globular star clusters, the two local frequencies are comparable and

$$\Omega \lesssim \kappa \lesssim 2\Omega .$$

(78)

$\kappa \sim 2\Omega$ in the centre of a distribution if it is sufficiently homogeneous; $\frac{\kappa}{\Omega}$ then progressively decreases with radius and may tend to $\Omega$ if stars' orbits become near-Keplerian. The two timescales are smallest in the centre of the distribution. (There is a further theoretical discussion of epicyclic orbits in § 3.5.1.)

In a Plummer model the potential is given by

$$\varphi = - \frac{GM}{(R_p^2 + r^2)^{1/2}} ,$$

(79)

(cf. Spitzer(1987), p.13), where $R_p$ is the Plummer scale radius ($\frac{3\pi}{16}$ in our units). Therefore
The interval between measurements was chosen to be 0.1 so as to be a magnitude of less than $\frac{1}{10}$ that of the epicyclic period, and adequate for resolving oscillations with a comparable period.

3. Analysis Of The Results

3.1 Observed Oscillations

The resulting time series for the Lagrangian radii are to be analysed. First, we may directly look at the numerical values of the radii themselves as a function of time. The radii containing a quarter, a half and three quarters of the total mass are plotted for each of the four simulations in Figure 11. (Note that the numbers of stars in the simulations (1024 or 10048) are divisible by 16; hence these fractions of the total mass are a whole number of stars.) Overall, the graphs do seem to show some evidence of periodic oscillations. There is a similarity between the motions exhibited by the fixed-potential models and those exhibited by the corresponding self-consistent models, which suggests that these motions are due more to the sum of contributions
Figure 11: The Lagrangian radii plotted against time:
(a) self-consistent 10048-body model, (b) fixed-potential 10048-body model, (c) self-consistent 1024-body model, (d) fixed-potential 1024-body model. (i) 4th Lagrangian radius, (ii) 8th Lagrangian radius, (iii) 12th Lagrangian radius, of 16 Lagrangian radii, i.e. radii containing \( \frac{4}{4}, \frac{2}{2} \) and \( \frac{1}{2} \) of the cluster's mass. The abscissa and ordinate are labelled on (a)(i) and are the same for all the graphs.
from individual star orbits than star-star interactions (which do not take place in the fixed-potential models).

We can get a crude initial impression of the variances of the Lagrangian radii and their frequencies of oscillation; from here we may deduce the approximate relationship between these values for the different radii and different kinds and sizes of model. The frequencies in models with different numbers of stars appear similar but the variances appear smaller in the 10048-body models. The variances appear to be smaller and the frequencies larger in the fixed-potential models than in the self-consistent ones. Smaller radii appear to have smaller variances and larger frequencies than larger radii.

As well as the effects that we are interested in the self-consistent 1024-body results show overall (upward or downward) trends in the radii. The reason for this is that the simulations are sufficiently long to be significantly affected by core collapse (see e.g. Spitzer (1987)). Figure 12 shows the Lagrangian radii measured from the density centre for the 1024-body self-consistent model. Overlaid on the graph are results for a 1000-body Plummer distribution gas model (cf. Heggie & Ramamani (1989)) kindly provided by Dr. Douglas C. Heggie. The self-consistent model Lagrangian radii shows the trends predicted by the gas model: the contraction of the inner radii and the expansion of the outer radii. These are the features of core collapse.

To proceed from these first impressions, we shall compute the variances of the Lagrangian radii, and use the autocorrelations of these
Figure 12: Core collapse. All 15 of the Lagrangian radii measured in the 1024-body self-consistent model are plotted against time (solid lines). Superimposed upon this graph are the corresponding Lagrangian radii as predicted by a gas model (dashed lines).
radii to give a better impression of the frequencies. A theoretical
connection will be found relating the values of the variances and
frequencies for the different situations.

3.2 A Related Quantity: The Mass Within A Fixed Radius

It proves useful to compare the radial oscillations with those of a
closely related variable: the mass $M(t)$ contained within a fixed radius.
(The fixed radii taken are the means of the observed Lagrangian radii.)
This quantity differs from the associated Lagrangian radius in that it
is a discrete quantity rather than a continuous one. However, in the
continuum approximation, the mass can be expressed as a function of the
corresponding Lagrangian radius. Consider a Lagrangian radius, $R(t)$,
with mean $R_0$. Let the fixed mass in $R$ be $M_0$, and let the variable mass
contained within $R_0$ be $M(t)$. Suppose in a small interval of time $\Delta M$
changes from $M_0$ to $M_0 + \delta M$ and $R$ changes from $R_0$ to $R_0 + \delta R$. Compare the
masses inside $R$ and $R_0$ at the end of the interval. The difference will
be mass contained between two radii

$$M_0 - M = 4\pi \int_{R_0}^{R} \rho(r)r^2 \, dr , \quad (82)$$

where $\rho$ is the density at time $t$. The left-hand side is equal to $(-\delta M)$,
and so for small oscillations ($\delta R \ll R_0$) we can linearise about $R_0$ to
get

$$\delta M \approx -4\pi \rho_o(R_0)R_0^2 \delta R , \quad (83)$$

where $\rho_o$ is the density when $R=R_0$. For the unperturbed Plummer model,
the density

$$\rho(r) = \frac{3M_*}{4\pi R_p^3} \left( \frac{1}{1 + \frac{r^2}{R_p^2}} \right)^{5/2}$$
and the mass within a radius \( r \)

\[
M(r) = M_t \frac{r^3/R_0^3}{(1 + r^2/R_0^2)^{3/2}}
\]

(84)

(cf. Spitzer(1987), p.13), where \( R_0 \) is the Plummer scale radius and \( M_t \) the total mass. Our expression becomes

\[
\delta M \simeq - \frac{3 M_0}{R_0[1 + R_0^2/R^2]} \delta R.
\]

(85)

Implicit in the above derivation is the assumption that the model used is spherically symmetric about the geometric centre. In our use of the quantity "mass within a fixed radius" we shall centre the radii on the geometric centre. This differs from the approach in the simulations where Lagrangian radii were measured from the density or potential centres. Using these latter centres has the advantage that these are the centres used in the self-consistent models; however, the gravitational force in the fixed-potential model is towards the geometric centre which is an advantage for this choice of centre. In fact we assert that there would only be a small difference between measurements for the Lagrangian radii from two of these different centres. Consider the following argument. Take an imaginary sphere of fixed radius superimposed upon an instantaneous set of positions of the stars of our fixed-potential system. If the sphere's centre is moved about within the same range as the various different choices for centre, the number of stars contained within the sphere will not change by very much: as stars pass out of one side of the sphere, others will pass in through the other side.

For the reasons given above, we argue that the quantity "mass within
a fixed radius" measured from the geometric centre should have a virtually identical autocorrelation to that of the corresponding Lagrangian radius measured from the density centre. The results from the simulations which are given later supported this belief (§ 3.4.3). Our motivation for considering mass within a fixed radius is that this quantity is easier to analyse exactly (cf. § 3.4). We shall now proceed to investigate various autocorrelations.

3.3 The Autocorrelations of the Lagrangian radii time series

The autocorrelations for the radii in all four of the simulations were estimated to give an indication of the underlying periodicity and the coherence of their oscillations. By autocorrelation we mean autocovariance normalised by dividing by variance (autocovariance with zero lag). In turn autocovariance is defined as

\[ < R(t)R(t+\lambda) > - < R >^2 = < (R(t) - < R >)(R(t+\lambda) - < R >) >, \]

where \(< >\) denotes time average; for example

\[ R(t)R(t+\lambda) = \lim_{t \to \infty} \frac{1}{t-0} \int_0^t R(t)R(t+\lambda)dt, \]

and

\[ R(t) = \lim_{t \to \infty} \frac{1}{t-0} \int_0^t R(t)dt. \]

R(t) is the value of the Lagrangian radius (or in general the function whose autocovariance is being determined) at time t, and \(\lambda\) is the "lag" (the time difference between two parts of the time series of R that are being compared). In the case of a finite discrete time series, the computation of autocorrelation is discussed in Press et al (1986). For our set of results at time intervals of 0.1 in the range \([0, 28.1]\) we will use the estimate for autocovariance...
\[
< \delta R(t) \delta R(t + \lambda) > = \frac{1}{282} \sum_{i=1}^{282} \delta R(t_i) \delta R(t_i + \lambda) \quad (86)
\]

with

\[
\delta R(t) = R(t) - < R > \quad 0 \leq t \leq 28.2
\]

\[
= 0 \quad \text{otherwise}.
\]

We use the estimate \( \frac{1}{282} \sum_{i=1}^{282} R(t_i) \) for \( < R > \). The lags (\( \lambda \)) here are restricted to integer multiples of 0.1.

The estimate above corresponds to taking twice the autocovariance of an infinite discrete time series for \( R(t_i) - < R > \) after it has been multiplied by a characteristic function \( \chi(t) \)

\[
\chi = 1 \quad 0 \leq t \leq 28.1
\]

\[
= 0 \quad \text{otherwise} ,
\]

and then replicated at intervals of 56.4 time units (i.e. padded with an equal number of zeros as there are data points and then periodically repeated [there are 282 data points separated by intervals of 0.1]). Alternatively, the autocovariance of the time series as taken here may be regarded as a weighted mean of the values of \( \delta R(t_i) \delta R(t_i + \lambda) \) that are known. This estimate is better than taking the autocovariance of the data replicated at intervals of 28.2, without the padding by zeros. The latter approach, which was also tried, pollutes the autocovariance with itself, as the result includes contributions that have both lags of \( \lambda \) and \( (28.2 - \lambda) \).

Naturally there are difficulties which arise trying to estimate the
autocovariance of a time series from a discrete finite sample. One is the problem of aliasing due to sampling the time series at discrete intervals of time ($\Delta T$). Any underlying periodic behaviour with period less than $2\Delta T$ will be mapped onto those with a period which is greater than $2\Delta T$. This inaccuracy is not a major problem providing that the dominant contributors to the autocovariance have periods greater than $2\Delta T$. This is certainly true in our fixed-potential situation as the contributions there are related to the stars' radial periods. In the following section we shall give a further illustration that this possible problem is not important by using the estimates of autocovariance for the masses in fixed radii. As has already been observed the behaviour of the Lagrangian radii in the self-consistent model appears similar to that in the fixed potential, and one would expect similar underlying frequencies. Therefore aliasing should not be a major problem for the study of these oscillations.

Another source of difficulties is the fact that the sample is over a finite time interval. This is more of a problem. The most accurate estimates will be those for small lags where the zero-padding bias is having little effect and there are many results of the simulation contributing to the estimate. Fortunately, again, a comparison of the fixed-potential autocovariances for Lagrangian radii with those for the mass contained within the mean Lagrangian radii can be used to clarify the size of this inaccuracy as we shall proceed to do in the following section.
3.4 Autocorrelations Of The Masses In Fixed Radii

3.4.1 Exact Autocovariance

Consider a fixed radius in the fixed-potential model (measured from the geometric centre [cf. § 3.2]). The mass in that radius $M(t)$ may be found at any time from the initial conditions: the radial equation of motion for each star is individually integrated and the masses of all the stars that lie within the fixed radius added up. We can estimate the autocovariance from a number of values of $M(t)$ as done for the Lagrangian radii in § 3.3 (this approach is used in § 3.4.2); however, it is also possible to find a precise expression for the autocovariance of the mass $M(t)$ contained within a fixed radius of a fixed-potential model: i.e. for the infinite time series. In fact, the autocovariance of the infinite time series only depends upon the radial periods of the stars and the proportion of time they spend within the fixed radius. The result is derived in the ensuing calculations.

Define $M_1(t)$ to be the $i$th star's contribution to the total mass within the fixed radius ($R_1$), i.e.

$$M_1(t) = \begin{cases} m & |r_1| < R_1 \\ 0 & |r_1| > R_1 \end{cases}.$$

(In our units the total mass $M = 1$, and so an individual star's mass $m = \frac{1}{N}$, where $N$ is the number of stars). We are dealing with a fixed potential in which the motion of the different stars are independent. Therefore we may assume that the $N$ different functions $M_i(t)$ are uncorrelated, and so
\[
< M(t)M(t+\lambda) > - < M(t) >^2 \\
= \sum_{i=1}^{N} < M_i(t)M_i(t+\lambda) > - \sum_{i=1}^{N} < M_i(t) >^2 
\] (87)

Now let \( T_1 \) be the period of the \( i \)th star, and \( \alpha_i \) the time during that period the star spends within the fixed radius (\( R_1 \)). Then

\[
< M_1(t) > = m \frac{\alpha_i}{T_1} 
\] (88)

and

\[
< M_1(t)M_1(t+\lambda) > = \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} M_1(s)M_1(s+\lambda) \, ds \\
= \frac{1}{T_1} \int_{0}^{T_1} M_1(s)M_1(s+\lambda) \, ds \\
= \frac{m}{T_1} \frac{(T_1 + \alpha_1)}{2} \\
= \frac{m}{T_1} \frac{(T_1 - \alpha_1)}{2} 
\] (if the origin of time is chosen to coincide with apocentre). So for \( \alpha_i \leq \frac{T_1}{2} \) we have

\[
< M_1(t)M_1(t+\lambda) > = \frac{m^2}{T_1} (\alpha_i - \{\lambda\}_1) \\
\text{if } \{\lambda\}_1 \leq \alpha_i \\
= 0 \\
\text{otherwise} 
\] (89)

and for \( \alpha_i \geq \frac{T_1}{2} \) we have

\[
< M_1(t)M_1(t+\lambda) > = \frac{m^2}{T_1} (\alpha_i - \{\lambda\}_1) \\
\text{if } \{\lambda\}_1 \leq T_1 - \alpha_i \\
= \frac{m^2}{T_1} (2\alpha_i - T_1) \\
\text{otherwise} 
\] (90)

\( \{\lambda\}_1 \) denotes \(|\lambda_{p_1}|\), where \( \lambda_{p_1} \) is defined implicitly by the relations

\[
\lambda_{p_1} = \lambda + kT_1, \text{ for some } k \in \mathbb{Z}, \quad (91) \\
-\frac{1}{2} T_1 < \lambda_{p_1} \leq \frac{1}{2} T_1 . 
\]

Collecting these cases together

\[
< M_1(t)M_1(t+\lambda) > = \frac{m^2}{T_1} (\alpha_i - \min[\{\lambda\}_1, \alpha_i, T_1 - \alpha_i]) , 
\] (92)
and so the autocovariance of the mass can be expressed as

\[
\langle M(t)M(t+\lambda) \rangle - \langle M(t) \rangle^2 = \sum_{i=1}^{N} \langle M_1(t)M_1(t+\lambda) \rangle - \langle M_1(t) \rangle^2 = m^2 \sum_{i=1}^{N} \left[ \frac{1}{T_1} \alpha_1 - \left( \frac{1}{T_1} \alpha_1 - \frac{1}{T_1} \min[\{\lambda\}, \alpha_1, T_1 - \alpha_1] \right) \right],
\]

(93)

The radial periods of the stars \((T_1)\), and amount of that period spent within the fixed radii \((\alpha_1)\) can be numerically computed from the initial conditions. The time taken for a star to move from radius \(r_1\) to \(r_2\) \((r_1 < r_2)\) is given by the integral

\[
\int_{r_1}^{r_2} \frac{1}{r} dr = \int_{r_1}^{r_2} \frac{1}{2(2E - \phi(r)) - \frac{h^2}{r^2}} dr,
\]

(94)

where \(E\), \(\phi\) and \(h\) are respectively the energy, potential and angular momentum for the star. At the minimum and maximum values of \(r\), \(r_{\min}\) and \(r_{\max}\), the star has no radial velocity and so the integrand has a singularity. We can regularise by the substitution for \(r\) by \(\theta\) where

\[
r = \frac{1}{2}[(r_{\max} + r_{\min}) + (r_{\max} - r_{\min})\sin\theta], \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.
\]

(95)

Now the time is given by

\[
\int_{\theta_1}^{\theta_2} \frac{1}{2f(r(\theta))} (r_{\max} - r_{\min})\cos\theta d\theta.
\]

(96)

\(r_{\max}\) and \(r_{\min}\) satisfy

\[
\frac{h^2}{2r^2} + \phi(r) = E,
\]

and so substituting for \(\phi\) the Plummer potential (79), we obtain a cubic for \(r^2\):
\[ r^6 + \left( \frac{R_p^2 - \frac{h^2}{E} - \frac{1}{E^2}}{E} \right) r^4 + \left( \frac{\frac{h^4}{4E^2} - \frac{R_p^2 h^2}{E}}{2} \right) r^2 + \frac{R_p^2 h^4}{4E^2} = 0 \]. (97)

We can find \( E \) and \( h \) from the initial conditions, and then the minimum and maximum radii are given by the square roots of the two positive roots for the cubic in \( r^2 \) (the third is negative as the product of the roots is negative). With these we can evaluate the time taken for the star to travel between any two radii, using (95) to find the corresponding \( \theta_1 \) and \( \theta_2 \) and then numerically integrating (96). With \( r_{\text{min}} \) and \( r_{\text{max}} \) as our two radii we find half the radial period \( T_1 \), and with \( r_{\text{min}} \) and \( R_1 \) (the fixed radius) we find half the time the star spends within \( R_1 \) in a radial period \( \frac{T_1}{2} \).

An attempt was made to measure the radial periods directly from the fixed potential simulations. During a run, the positions and times of the stars' apocentres and pericentres were crudely estimated at update times using a low-order extrapolation from the values of radial velocity and position. Unfortunately, this information was rather patchy, as it missed some turning points in radius and spotted others twice. However, those values that were found agreed with, and provided a useful check upon, the more accurate values computed from the initial conditions as described earlier above.

3.4.2 Autocovariance Estimated From A Discrete Finite Time Series

As well as computing the exact autocovariance for the mass (93), for comparison we shall also estimate its value from a discrete finite time series using the same method as that used with the Lagrangian radii (cf. § 3.3). To do this we must first generate the finite time series for \( M \);
this requires a further item of information for each star: the initial phase of its orbit. The phase is the proportion of the radial period elapsed from pericentre at time 0, multiplied by $2\pi$. We do not calculate it explicitly, but rather obtain data that is equivalent and use that directly to find the time series. We can find the size of the time difference between time 0 and the nearest time of pericentre (either positive or negative) by the same method as used to find the period in § 3.4.1 above. By finding whether a star is moving in or out we may then determine how much longer it will spend inside/outside the fixed radius before crossing it. After this crossing, the star alternately spends periods $\alpha_1$ within the radius, and periods $(T_i - \alpha_1)$ without. Hence we can easily find whether the body is inside or outside the radius at any of the discrete time intervals. By summing the contributions from all the stars we can then find the mass in $R_1$.

To illustrate the fact that the data is being sampled frequently enough at 0.1 time intervals, the mass autocovariance was also estimated from data taken at 0.01 intervals. Figure 13 parts (a) and (b) both show estimates for the autocorrelation of the mass within the same two fixed radii over the finite time interval $[0, 28.1]$; however, the samples used in the estimates were taken at different time intervals of 0.01 and 0.1 time units. The graphs are virtually identical.

3.4.3 A Comparison Of The Mass Autocorrelation and Corresponding Lagrangian Radius Autocorrelation At Given Radii In The Fixed Potential

With the close relation between the mass within the mean of a Lagrangian radius and the Lagrangian radius itself (cf. § 3.2), we would
Figure 13: Radius and Mass autocorrelations in the 10048 body fixed-potential model: (a) estimated mass autocorrelation with ten times as many points as usual (2820 points); (b) usual estimated mass autocorrelation (282 points); (c) corresponding estimated Lagrangian radius autocorrelation (282 points). (i) Fifth Lagrangian radius; (ii) eleventh Lagrangian radius. The ordinates and abscissae are as labelled on (a)(i).
expect their autocorrelations to be similar. Figure 13 part (c) shows the estimated autocorrelations of the Lagrangian radii that correspond to the mass autocorrelations plotted on that figure. It is readily seen that these pairs of graphs are almost identical at the radii shown. Thus it is reasonable to use the autocorrelations for mass interchangeably with those of Lagrangian radius for the fixed-potential models.

3.5 Comparing The Different Models And Different Radii

Figures 14 and 15 show graphs of the estimated autocorrelations (from normalising (86)) for the self-consistent models' Lagrangian radii along with those estimated in the same way for the corresponding masses in the fixed-potential model and also the exact value for the latter (from normalising (93)). As in Figure 11 the self-consistent models' oscillations appear comparable with those in a fixed potential; however, they appear to have a slightly larger period. (The distance between maxima in an autocorrelation gives an estimate of the period for the corresponding oscillation.) The autocorrelation for the 1024-body self-consistent model (Figure 15(a)) is affected by the core collapse as referred to in § 3.1, and so the inner and outer radii autocorrelations have a general drift downwards superimposed upon their oscillations.

Figure 16 shows the times corresponding to the maxima and minima of the estimated Lagrangian radii autocorrelations and mass autocorrelations plotted against the proportions of mass contained within the associated Lagrangian radii. Additionally, it gives the corresponding values for the maxima and minima of the exact autocorrelations for mass. Finally, for further comparisons, the figure
Figure 14: Autocorrelations in the 10048-body models: (a) self-consistent model estimated autocorrelation of Lagrangian radii; (b) corresponding estimated autocorrelation of mass for a fixed-potential model; (c) exact autocorrelation for the mass in the fixed-potential model. (i) Fourth Lagrangian radius; (ii) eighth Lagrangian radius; (iii) twelfth Lagrangian radius. The axes are as labelled in (a)(i).
Figure 15: Autocorrelations as in Figure 14 but for the 1024-body models: (a) self-consistent model estimated Lagrangian radii autocorrelation; (b) corresponding fixed-potential model mass autocorrelation estimate; (c) exact autocorrelation for the mass in the fixed-potential model. (i) Fourth Lagrangian radius; (ii) eighth Lagrangian radius; (iii) twelfth Lagrangian radius. Axes as labelled in (a)(i).
Figure 16: The minima (circles) and maxima (crosses) of the autocorrelations plotted against their corresponding Lagrangian radii. The radii are numbered according to the proportion of the cluster's mass within them (in sixteenths). The axes of all eight graphs are identical and are labelled in just (a)(i). (a) 10048-body model; (b) 1024-body model. (i) Self-consistent estimate; (ii) fixed-potential estimate; (iii) fixed-potential exact. (c) Corresponding theoretical local timescales for the Plummer model, the circles are odd multiples of the half periods and the crosses whole multiples of the period: (i) epicyclic period; (ii) circular period.
also includes graphs showing multiples of the half periods for epicyclic oscillations and circular orbits at the corresponding mean radii (as these are local timescales [cf. § 2]). By superimposing the graphs from this figure we can see that the periods involved in the oscillations of the 10048-body models are comparable to those of the corresponding 1024-body models. It appears that the frequency of these oscillations is a function of mass density, independently of the number of bodies involved. A more detailed comparison shows that the dependence of the frequency of the Lagrangian radii oscillations upon radius (for these Plummer models) appears to be similar to that of the circular frequency but multiplied by a constant. In the fixed-potential simulations the period of the Lagrangian radii oscillations appear about 0.6 times the local circular period. For the self-consistent oscillations the period appears closer to 0.7 times the local circular period. The oscillations have a period slightly longer than the epicyclic ones towards the centre but a period slightly shorter towards the outside of the model. A possible explanation for this is the following argument. At a small radius the local stars would on average have a radial period similar to or longer than the epicyclic one, as they would either be on orbits remaining close to that radius or near the pericentre of orbits going out to larger radii. Likewise, at large radii the stars on average have a shorter radial period than the local epicyclic one, as they tend to be near the apocentre of their orbits.

3.5.1 Differences In Frequency And Variance For The Various Models

As already mentioned, there are higher frequencies of oscillation in the fixed-potential models, and a greater amplitude of oscillation in
the self-consistent models. To understand this, consider a Lagrangian radius' oscillation in the fixed-potential model. The radius decreasing corresponds to the majority of stars within the local radial neighbourhood moving inwards; similarly an increase corresponds to the majority of local stars moving outwards. In the corresponding self-consistent model the density inside a given local radius increases (or decreases), as the majority of the stars move inwards (or outwards). By Newton's Theorems for spherically symmetric potentials (see e.g. Binney & Tremaine(1987), pp.34-36) the force on a star is the same as that due to an object concentrated at the origin with the total mass of all the matter at smaller radii than the star. So for any star, as the density at smaller radii increases (or decreases) from the values of the unperturbed model, so does the force on the star towards the centre. While the majority of the stars move inwards, the force towards the centre at a given radius increases, and so the stars move in further than in the fixed potential and take longer to reach a minimum radius. Similarly, while the majority of the stars move outwards, the force decreases, and so they move further out and take longer to reach maximum radius than in the fixed potential. So the motion in the self-consistent potential produces a larger variation in Lagrangian radius but a lower frequency of oscillation.

As an example of this effect we shall consider an (infinite) uniform spherical distribution. For the unperturbed model the spatial density is a constant value ($\rho_0$) everywhere. We assume that the stars at a given radius have the directions of their velocities distributed uniformly on the tangential plane and that they have the exact angular velocity ($\Omega$)
required for a circular orbit ($\Omega^2 = \frac{4\pi G \rho_0}{3}$, where $G$ is the gravitational constant). We now perturb this model by letting the stars perform epicyclic oscillations (small oscillations on either side of the circular orbit). The radial equation of motion of a star in a spherical potential is

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{GM(r)}{r^2}$$

where $h$ is angular momentum (per unit mass) and $M(r)$ is the mass contained within $r$ in the distribution that produces the potential. In this particular model

$$h^2 = \frac{4\pi G \rho_0 a^4}{3},$$

where $a$ is the radius of the circular orbit with this angular momentum. Therefore, for a small radial perturbation from the circular orbit ($\delta r$) we have

$$\delta \ddot{r} \approx \left. \frac{\partial}{\partial r} \left[ \frac{h^2}{r^3} - \frac{GM(r)}{r^2} \right] \right|_{r=a} \delta r$$

$$= -\kappa^2 \delta r . \quad (98)$$

$\kappa$ is the epicyclic frequency of the star. If we suppose that the stars in the locality of a Lagrangian radius are moving in and out in phase, then the Lagrangian radius' oscillations have the epicyclic frequency.

For the fixed-potential model

$$M(r) = \frac{4\pi \rho_0 r^3}{3},$$

and so in that case

$$\kappa^2 = \frac{16\pi}{3} \frac{G \rho_0}{3} .$$

In the self-consistent case, the bodies' combined motion inwards alters the potential; $M(r)$ is now a constant

$$M(r) = \frac{4\pi \rho_0 a^3}{3} ,$$
and so

\[ \kappa^2 = \frac{4}{3} \pi G \rho_0 \]

Though different from our models in that the radial oscillations are epicyclic and ours are not (cf. § 3.5 earlier), this model gives an illustration of the "slowing down" of radial oscillations in the self-consistent case: here, in this self-consistent model, the Lagrangian radii oscillations occur at half the frequency of those in the corresponding fixed-potential situation.

<table>
<thead>
<tr>
<th>radius</th>
<th>10048-body models</th>
<th>1024-body models</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>s-c</td>
<td>f-p</td>
</tr>
<tr>
<td>1</td>
<td>2.1475E-05</td>
<td>1.4695E-05</td>
</tr>
<tr>
<td>2</td>
<td>1.6803E-05</td>
<td>1.1025E-05</td>
</tr>
<tr>
<td>3</td>
<td>1.4847E-05</td>
<td>1.1900E-05</td>
</tr>
<tr>
<td>4</td>
<td>1.6470E-05</td>
<td>1.2290E-05</td>
</tr>
<tr>
<td>5</td>
<td>1.8349E-05</td>
<td>1.4389E-05</td>
</tr>
<tr>
<td>6</td>
<td>2.2831E-05</td>
<td>1.7249E-05</td>
</tr>
<tr>
<td>7</td>
<td>2.2903E-05</td>
<td>1.7025E-05</td>
</tr>
<tr>
<td>8</td>
<td>2.7860E-05</td>
<td>1.9907E-05</td>
</tr>
<tr>
<td>9</td>
<td>3.3729E-05</td>
<td>2.2871E-05</td>
</tr>
<tr>
<td>10</td>
<td>5.1644E-05</td>
<td>2.5767E-05</td>
</tr>
<tr>
<td>12</td>
<td>1.0928E-04</td>
<td>8.7892E-05</td>
</tr>
<tr>
<td>13</td>
<td>1.5498E-04</td>
<td>1.3119E-04</td>
</tr>
<tr>
<td>14</td>
<td>4.3453E-04</td>
<td>4.1282E-04</td>
</tr>
<tr>
<td>15</td>
<td>9.3973E-04</td>
<td>4.3350E-04</td>
</tr>
</tbody>
</table>

Table 3: The variances of the Lagrangian radii
The values shown are for the usual fifteen Lagrangian radii in the 10048-body and 1024-body, self-consistent (s-c) and fixed potential (f-p) models (see § 3.5.2).
3.5.2 Relating The Frequency And Variance In The Various Models

The variances of the Lagrangian radii oscillations are shown in Table 3. There is a further connection between these variances and the corresponding frequencies that relates the different models. For each Lagrangian radius \( R \) one can assign an energy to its oscillation

\[
\frac{1}{2} M(R) \omega^2 \text{Var}(R),
\]

(99)

where \( \omega \) is the frequency of oscillation, \( \text{Var}(R) \) is the variance of the Lagrangian radius and \( M(R) \) is an as yet unknown quantity which represents the mass involved in the oscillation. One would suppose \( M(R) \) to depend solely upon the local distribution function, and for it therefore to be the same in all four models. It will be of order unity as it will be some fraction of the total mass \( M \), and this has been set equal to 1 unit. Figure 17 shows the quantity

\[
\frac{\langle v^2 \rangle}{N \omega^2 \text{Var}(R)}
\]

(100)

plotted against Lagrangian radius for the various models (where \( N \) is the number of particles and \( \langle v^2 \rangle \) the local variance of the velocity of a particle). This quantity's relative constancy across the 4 different models implies that, for a particular Lagrangian radius, the energy assigned above to an oscillation (99) is proportional to, and comparable with, the mean kinetic energy of a local star \( \frac{1}{2} m \langle v^2 \rangle \), independently of the number of bodies and the kind of model (fixed-potential or self-consistent), as long as the total mass is the same.

We can shed some light upon this topic by using the relationship between the Lagrangian radii oscillations and those of the mass within a fixed radius which was described in § 3.2. Returning to the notation of
Figure 17: The quantity (100), \[ \frac{<v^2>}{Na^2\text{Var}(R)} \], for the fifteen different Lagrangian radii in the four models. \( \omega \) is the frequency of oscillation, \( \text{Var}(R) \) the variance of the Lagrangian radius, \( N \) the number of particles and \( <v^2> \) the local variance of the velocity of a particle. The lines represent different models: (j) self-consistent 10048-body model; (k) fixed-potential 10048-body model; (l) self-consistent 1024-body model; (m) fixed-potential 1024-body model. The Lagrangian radii are numbered according to the proportions of the cluster's mass within them (in sixteenths), and this appears upon the abscissae. The ordinate in graph (a) is on a linear scale and that in graph (b) on a logarithmic scale.
that section we have

$$\text{Var}(R) = \langle \delta R^2 \rangle$$

$$\approx \frac{\langle \delta M^2 \rangle}{16\pi^2 \rho_0 (R_0) R_0^4},$$

(101)

for small oscillations of a spherically symmetric system (from equation (83)). The denominator of this expression is a function of radius but not of the number of bodies. In § 3.4.1, we obtained an exact value for the numerator in a fixed-potential discrete model:

$$\langle \delta M^2 \rangle = \langle M(t) M(t+\lambda) \rangle - \langle M(t) \rangle^2$$

$$= \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{T_1} \alpha_i - \left( \frac{1}{T_1} \alpha_i \right)^2 - \frac{1}{T_1} \min[\{\lambda_i\}, \alpha_i, T_1 - \alpha_i] \right],$$

(cf. equation (93)). Recall that the $T_1$ are the radial periods of the stars, the $\alpha_i$ are the amount of that period spent within the fixed radii, and the $\{\lambda_i\}$ are a function of the lag and individual stars' radial periods. These three quantities are distributed independently to the number of stars and so the mean of the expression being summed is a function of just radius (we shall call this $g(R_0)$). Therefore, if we estimate $\langle \delta M^2 \rangle$ at a given radius in a model with $N$ stars by taking its mean across several models, we obtain

$$\langle \delta M^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{T_1} \alpha_i - \left( \frac{1}{T_1} \alpha_i \right)^2 - \frac{1}{T_1} \min[\{\lambda_i\}, \alpha_i, T_1 - \alpha_i] \right]$$

$$= \frac{1}{N} \sum_{i=1}^{N} \frac{1}{T_1} \min[\{\lambda_i\}, \alpha_i, T_1 - \alpha_i]$$

$$= \frac{1}{N} g(R_0)$$

(102)

This result together with (101) show that the variance of a Lagrangian radius is inversely proportional to the number of stars in a system, and so if $\omega$ is independent of $N$ (as is indeed the case [cf. § 3.5]), the
quantity (100) will be independent of \( N \) also.

This relationship between the energy of the radial oscillations and the energy of a star is analogous to one that can be derived between the global energy of sound waves in a gas and the local energy of molecules in the gas. As an illustration, consider a gas of mean density \( \rho_0 \) in a cubical box with side \( L \) (\( V = [0,L]^3 \)), with sound waves propagating only in the \( x \)-direction. Then there are stationary waves in which the displacement is

\[
\xi = A \sin \frac{n \pi x}{L} \cos \left[ \frac{n \pi c t}{L} + \epsilon \right]
\]

where \( A \) is a constant (cf. Coulson & Jeffrey (1977), pp.104-116). For simplicity we take \( \epsilon = 0 \), and so the density perturbation is

\[
\delta \rho(t) = - \rho_0 \frac{\partial \xi}{\partial x} = - \rho_0 A \frac{n \pi}{L} \cos \frac{n \pi x}{L} \cos \frac{n \pi c t}{L} .
\] (103)

(Note \( \delta \rho \) is a function of position as well as time.) We can determine \( A \) after setting \( t = 0 \):

\[
\int \int \int_V \delta \rho(0) \cos \frac{n \pi x}{L} \, dx \, dy \, dz = - \rho_0 A \frac{n \pi}{L} \frac{L^2}{2}
\]

\[
\Rightarrow \quad A = - \frac{2}{n \pi \rho_0 L^2} \int \int \int_V \delta \rho(0) \cos \frac{n \pi x}{L} \, dx \, dy \, dz
\]

\[
= - \frac{2}{n \pi \rho_0 L^2} \int \int \int_V \left( \rho_0 + \delta \rho(0) \right) \cos \frac{n \pi x}{L} \, dx \, dy \, dz .
\] (104)

Now let us think of the gas as consisting of \( N \) particles of mass \( m \) at positions \( r_i = (x_i, y_i, z_i) \), and so \( \rho = \sum_{i=1}^{N} m \delta(r-r_i) \), \( \delta(x) \) here is the Dirac delta function. By (104)

\[
A = - \frac{2m}{n \pi \rho_0 L^2} \sum_{i=1}^{N} \cos \frac{n \pi x_i}{L} .
\]
For random positions of the particles, \(< A > = 0\), but
\[
< A^2 > = \frac{4m^2}{n^2 \pi^2 \rho_0 L^2} \frac{N}{2} = \frac{2m^2 N}{n^2 \pi^2 \rho_0 L^2} .
\] (105)

The kinetic energy of the wave is
\[
\iiint \frac{1}{2} \rho_o \left[ \frac{\partial \xi}{\partial t} \right]^2 dx dy dz = \frac{1}{2} \rho_o \left[ \frac{n \pi c}{L} \right]^2 \iiint A^2 \sin^2 \left[ \frac{n \pi x}{L} \right] \sin^2 \left[ \frac{n \pi c t}{L} \right] dx dy dz,
\]
and the total energy is equal to the maximum value of the kinetic energy
\[
E = \frac{1}{2} \rho_o \left[ \frac{n \pi c}{L} \right]^2 \frac{A^2}{2} L^3
= \frac{1}{4} \rho_o A^2 \left( \frac{n \pi c}{L} \right)^2 L .
\] (106)

Taking the mean value of this expression and substituting for \(< A^2 >\) from (105) we have
\[
< E > = \frac{1}{4} \rho_o \left( \frac{n \pi c}{L} \right)^2 L \frac{2m^2 N}{n^2 \pi^2 \rho_0 L^2}
= \frac{1}{2} \rho_o c^2 \frac{m \pi}{2} \frac{\gamma}{\rho_0 L}
= \frac{1}{2} m \gamma c^2 .
\] (107)

Now speed of sound in an ideal gas at temperature \(T\) is
\[
c = \sqrt{\frac{\gamma k T}{m}}
= \sqrt{(\gamma < v_x^2>) .}
\] (108)

where \(\gamma\) is the ratio of specific heat capacity at constant pressure to specific heat capacity at constant volume (\(\gamma\) is a constant), \(k\) is Boltzmann's constant, and \(< v_x^2 >\) is the mean square velocity of a gas particle in one direction (see e.g. Binney & Tremaine(1987) pp.674).

Substituting this value (108) into our expression for \(E\) (equation (107)) we obtain
\[
< E > = \frac{1}{2} \gamma k T
= \frac{1}{2} \gamma m < v_x^2 > .
\] (109)

Hence we have shown that the mean kinetic energy of a particle in one
direction \( \frac{1}{2} m < v_x^2 > \) is comparable to the energy of the stationary sound wave in the gas.

The relationship between the energy of oscillation and the kinetic energy of a local star may be further illustrated by returning to the uniform spherical model used earlier. Again consider the epicyclic oscillations about a particular radius. Suppose that the stars in the two kinds of model (self-consistent and fixed-potential) have the same radial velocity when they pass through the radius of the corresponding circular orbit; i.e. for all stars \( \delta r \) (the radial velocity) is the same in the self-consistent model as in the fixed-potential model when \( \delta r = 0 \) (zero radial displacement from the circular orbit). (This follows if we take the velocity distribution of the stars to be the same in the two kinds of model at this unperturbed position.) By integrating the epicyclic equation (98) we have

\[
\frac{\delta \dot{r}^2}{2} + \kappa^2 \frac{\delta r^2}{2} = E ,
\]

where \( E \) is a constant with the units of energy per unit mass. When the star crosses its unperturbed circular orbit (\( \delta r = 0 \))

\[
E = \frac{\delta \dot{r}^2}{2} .
\]

As mentioned above we supposed that \( \delta \dot{r} \) is the same in both kinds of model at this point and therefore so is \( E \). Now at apocentre or pericentre \( \delta \dot{r} \) is zero, and so

\[
\kappa^2 \frac{\delta r^2}{2} = E .
\]

So the frequency and amplitude of oscillation in the self-consistent and fixed potentials here are indeed related in the same way as in the simulations, (i.e. smaller frequency corresponds to larger amplitude).
We shall summarize the results of this and the previous subsections. For a system of a given total mass, energy and size, the frequency of the oscillation at a given radius is independent of the number of stars in the model. So the period of an oscillation is a constant multiple of the crossing time. However, the amplitude of the oscillation varies so as to keep the "energy" of the oscillation $\frac{1}{2} M(R)\omega^2 \text{Var}(R)$ proportional to the average kinetic energy of a star in the system. This is inversely proportional to $N$ and would disappear in the continuum limit ($N \to \infty$).

4. A Comparison With Previous Work By Other Authors

Several authors have investigated oscillation and stability in collisionless models. Mathur(1989) shows the possibility of radial "oscillation modes" in some spherically symmetric self-consistent models where a radial perturbation in the corresponding fixed-potential model would disappear because of phase-mixing. Papers by Sridhar(1989), Sridhar & Nityananda(1989), Palmer & Papaloizou(1987), Palmer & Papaloizou(1988) and Palmer, Papaloizou & Allen(1989) give other examples of such modes and models. Essentially in one of these oscillation modes the time-dependent part of the gravitational force that is due to the perturbation is just sufficient to maintain the oscillation, its amplitude neither increasing nor decreasing. However, all these models are anisotropic, whereas the simulations described in this thesis are based on a distribution which is isotropic ($f=f(E)$). Also, in my model $\frac{df}{dE} < 0$ and in these conditions a model is stable to radial perturbations and all radial oscillations die away by Landau damping (see e.g. Palmer, Papaloizou and Allen(1989)).
In Heggie's work (Heggie 1989), he simulated the middle of a nearly isothermal model: the innermost 3000 stars of a total distribution of 20,000. He commented that the Lagrangian radii oscillations increased in amplitude and in period with radius, and that the periods were nearly proportional to the period of circular orbits of the same radius. These results, though from a different model, are similar to mine. Heggie also states that the persistence of these oscillations is surprising in view of the effects of phase-mixing. Having results for the fixed-potential model, in which phase-mixing also occurs, helps to clarify the situation. There appears to be an inherent periodicity and amplitude of oscillation associated with the Lagrangian radii of a discrete distribution of stars. However, the autocorrelations show that the oscillations of these Lagrangian radii do die away as expected. It would seem that the oscillations look persistent because they are constantly re-excited by fluctuations in the discrete distribution of mass. It would appear that in self-consistent models the individual oscillations die away in much the same way as in the fixed-potential models, but are similarly re-excited by fluctuations.

Heggie also states that the oscillations would lead to changes in the gravitational potential which enhance the relaxation of the stellar energy distribution. The earlier work of this chapter (in the subsections of § 3.5) shows that the motion of the stars does indeed have a significant effect on the gravitational potential in the self-consistent model, as shown by the lengthening of the period and increase in amplitude of the Lagrangian radii oscillations; however, effects on relaxation still require to be determined.
5. Conclusions

Star clusters have inherent periods and amplitudes of oscillation associated with their Lagrangian radii. These are comparable in fixed-potential and self-consistent models, suggesting that the origin of these oscillations is largely kinematic: they are caused mainly by the motions of the stars, and not by any subtle interaction between their motions and the forces they exert. The oscillations appear periodic locally but will not do so over a long time scale. A local oscillation does not persist - though it may seem to from the appearance of the estimates of its autocorrelation over a fixed time interval - phase-mixing does occur and this will damp out local oscillations. However, this phase-mixing merely generates another discrete collection of stars with a similar distribution and so there will be another similar local oscillation. So, although not persistent in the sense of a local oscillation lasting forever, the oscillations are persistent in the sense that they will always exist with the same kind of frequency and amplitude. Such oscillations in self-consistent models do affect the gravitational potential and hence may affect the orbits and energies of individual stars. The effects on gravitational potential are indicated by a lower frequency and larger amplitude in the oscillations of self-consistent models than in the corresponding oscillations in a fixed potential.

For any N-body model generated to fit a specific distribution function with either self-consistency or a fixed-potential, the frequency of oscillations at a particular radius is a constant proportion of the crossing time independently of the number of bodies.
(If we fix the scale radius and total mass the frequencies stay the same.) However, as the number of stars increase the amplitudes of radial oscillations change in such a way as to preserve the ratio of the kinetic energy of a local star to the energy of the oscillation (quantity (99), which is of order 1 [cf. § 3.5.2]). There is another way of explaining the diminishing size of oscillations as we increase the number of bodies in a system of fixed size and fixed total mass. The larger number of bodies of smaller individual mass have a "smoothing out" effect upon the density, reducing the size of the inherent fluctuations present in the discrete system. The fluctuations and hence our oscillations would vanish in the continuum limit.
Core Wandering

"I wandered lonely as a cloud"
William Wordsworth

1. Introduction

The core of a globular star cluster is a denser subgroup of the stars that forms towards the centre of the whole distribution. In the self-consistent model it acts as the gravitational heart of the system — it is about the core that the outer stars orbit. (Recalling Newton's 1st Theorem in a spherically symmetric system, it is only the mass within a given radius that contributes to the gravitational force on a star at that radius (see e.g. Binney & Tremaine(1987), pp.34-36).) As mentioned in Chapter IV § 2 the densest part (the core) of our N-body star cluster models does not remain fixed at one point or have a constant velocity (as the centre of mass does) but rather moves about in an oscillatory fashion. This phenomenon, "core wandering", has been previously observed in the simulations by Makino & Sugimoto(1987) and Heggie(1989)).

Movements of the core are important because they will affect the orbits of individual stars. In many simplified simulations of globular star clusters, the models are forced to be symmetric and so it will be important in the future to determine what are the effects of a moving core which is not fixed at the geometric centre. During the simulations described in Chapter III § 6 the position of the centre of the core was monitored at intervals of 0.1 time units. We took account of the effects of the core's motion in the study of Lagrangian radii (Chapter IV). In the present chapter the motions themselves are studied in more detail.
We begin by looking at possible definitions of the centre of a cluster core (§ 2). Then we move on to study the results of the simulations (§ 3). In § 4 we compare these results with those reported previously by other authors and draw some conclusions.

2. Defining The Centre Of The Core Of A Globular Star Cluster

The centre of an entire distribution of stars is usually taken to be at either the geometric centre or at the centre of mass. For the fixed-potential situation the stars move about the geometric centre, and in a self-consistent potential the centre of mass moves uniformly and gives some idea of the overall motion of the cluster. However, the core does not always coincide with either of these points and so we need a method of estimating where it is centred. We shall describe two approaches which define respectively the "potential" and the "density" centres.

The potential centre is the measure that was used by Heggie (1989). We define a softened potential for the ith star

$$ N \sum_{j=1, j \neq i}^{N} - G \frac{m^2}{(r_{ij}^2 + r_c^2)^{1/2}} $$

where $r_{ij}$ is the distance between the ith and jth stars, and $r_c$ is comparable with the core radius (0.4 for my simulations). (As usual, $G$ is the gravitational constant and $m$ is the mass of an individual star.) Then the potential centre is defined to be at the position of the star with the minimum (most negative) softened potential. The idea behind this estimate of the centre is that the densest part of the cluster will
be roughly where the potential is minimal; however, this is only really true in a smoothed out system. Therefore, to reduce the effects of closely interacting pairs of stars or even hierarchical groups, the softening parameter $r_C$ is introduced. Being comparable with the core radius, it is sufficiently small to resolve the core adequately but also large enough to remove the effects of compact groups within the core.

The density centre was used by Makino & Sugimoto (1987). The definition used here is described in detail in Casertano & Hut (1985) who generalised an idea introduced by von Hörner (1963). In the version we use here, we find the distance to the 6th closest star for each body, denoted by $r_{6i}$ for the $i$th star. Then the position of the density centre is defined to be at the point

$$\sum_{i=1}^{N} \left[ \frac{r_i}{r_{6i}} \right] \left( \sum_{j=1}^{N} \left[ \frac{1}{r_{6j}} \right] \right)^{-1},$$

where $r_i$ is the position vector of the $i$th star. This is a density-weighted average of the positions of the stars. The local density about a star used in this definition is the density in the sphere between a star and its sixth-nearest neighbour, excluding both these stars (the sphere is centred on the local star); i.e. the local density is five times the individual star mass divided by the volume of the sphere. The choice of sixth-nearest neighbour is recommended in Casertano & Hut as being a good compromise between the need to avoid bias due to three- or four-particle subsystems whilst being sufficiently close to maintain the locality of the measure.
3. Results From The Simulations

3.1 Direct Measurements

The Figures 18 and 19 show the position of the potential and density centres in self-consistent and fixed-potential simulations for 10048 and 1024 bodies, respectively. In the self-consistent models, the constant motion of the centre of mass of the system has been subtracted, to make clearer the other superimposed movements of the centres. These can now be compared with those of the fixed-potential models.

Both fixed-potential models show some regularity in the motion of their cores. The oscillations appear similar in period between the 10048- and 1024-body models but the latter have larger amplitudes. Perhaps, these oscillations are an inherent part of those shown for the self-consistent models, in a similar way to the relationship between the oscillations of the fixed-potential models' Lagrangian radii and the corresponding oscillations in the self-consistent models. However, there appear to be other motions of the centres in the self-consistent models that are not observed in the fixed potential model, and these require further investigation.

It is observed that the graphs of the potential centre are much "noisier" than the corresponding ones for the density centre. The potential centre is at a specific body and this changes during most of the 0.1 time intervals, so it is possible that this change between bodies is the major contributor to the "noise". To investigate the effect one could measure the position of the potential centre at smaller time intervals sufficient to resolve its motion between changes in the
Figure 18: The coordinates of the centre of the core plotted against time for the 10048-body models. (a) self-consistent model's potential centre; (b) self-consistent model's density centre; (c) fixed-potential model's potential centre; (d) fixed-potential model's density centre. (i), (ii) and (iii) are the x-, y- and z-coordinates, respectively. The coordinate axes are as labelled in (a)(i).
Figure 19: The coordinates of the centre of the core plotted against time as in Figure 18, but for the 1024-body models. (a) self-consistent model's potential centre; (b) self-consistent model's density centre; (c) fixed-potential model's potential centre; (d) fixed-potential model's density centre. (i), (ii) and (iii) are the x-, y- and z-coordinates, respectively. The coordinate axes are as labelled in (a)(i).
body at which it is placed. One would then compare the contributions to
the motion of the centre that are due to the motion of the body at
potential centre with those due to the body at potential centre
changing. The large "kinks" in some of the coordinates of the density
centre (e.g. that in the y-coordinate of the self-consistent 1024-body
simulation after about 24 time units) are still to be explained.

3.2 Autocorrelation And Variance

We shall now use the techniques of Chapter IV § 3.3 to try to further
expand our understanding of the results detailed in the last section. We
estimate the autocovariances of the coordinates of the core (after
centre of mass motion has been removed) using (86), and then from these
extract the autocorrelations and variances.

The autocorrelations for the 10048- and 1024-body models are shown in
Figures 20 and 21, respectively. It is immediately noticeable that the
graphs for the fixed-potential simulations appear different from those
with self-consistency suggesting that there is something more to the
core-wandering present in the latter (realistic) systems than the
underlying motions that are there in the former. Comparison of the
graphs of the 10048- and 1024-body (self-consistent and fixed-potential)
models does not reveal any obvious differences and so it would appear
that the timescales associated with core-wandering are largely
independent of the number of stars in the model. The autocorrelations
taken of the coordinates of the potential centres are generally damped
out very rapidly due to the "noisiness" of their motions mentioned
earlier (§ 3.1). Some of the autocorrelations do individually appear to
Figure 20: The autocorrelations of the coordinates of the centre of the core for the 10048-body models. (a) self-consistent model's potential centre; (b) self-consistent model's density centre; (c) fixed-potential model's potential centre; (d) fixed-potential model's density centre. (i), (ii) and (iii) are the x-, y- and z-coordinates, respectively. The coordinate axes are as labelled in (a)(i).
Figure 21: The autocorrelations of the coordinates of the centre of the core as in Figure 20, but for the 1024-body models. (a) self-consistent model's potential centre; (b) self-consistent model's density centre; (c) fixed-potential model's potential centre; (d) fixed-potential model's density centre. (i), (ii) and (iii) are the x-, y- and z-coordinates, respectively. The coordinate axes are as labelled in (a)(i).
show some periodicity but there does not appear to be any consistent frequency present across all the coordinates of any of the models. Also the "periods" are rather long for us to be sure of them in simulations of this limited time duration. The periods present appear longer than a crossing time ($2\sqrt{2}$ in our units); however, with the above qualifications upon the results, we can deduce little more with confidence about any typical periodicity associated with core wandering.

The variances of the coordinates for the cores are shown in Table 4 together with their sums (the overall variances of the cores), and the roots of these sums (the standard deviations of the cores). The latter

<table>
<thead>
<tr>
<th></th>
<th>x-coord.</th>
<th>y-coord.</th>
<th>z-coord.</th>
<th>total</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 S p</td>
<td>3.3524E-04</td>
<td>2.9601E-04</td>
<td>2.7602E-04</td>
<td>9.0728E-04</td>
<td>3.0121E-02</td>
</tr>
<tr>
<td>d</td>
<td>1.2650E-04</td>
<td>9.6894E-05</td>
<td>6.4455E-05</td>
<td>2.8785E-04</td>
<td>1.6966E-02</td>
</tr>
<tr>
<td>F p</td>
<td>3.2186E-04</td>
<td>2.6815E-04</td>
<td>2.4031E-04</td>
<td>8.3033E-04</td>
<td>2.8815E-02</td>
</tr>
<tr>
<td>d</td>
<td>2.6919E-05</td>
<td>2.9863E-05</td>
<td>2.1473E-05</td>
<td>7.8256E-05</td>
<td>8.8462E-03</td>
</tr>
<tr>
<td>2 S p</td>
<td>9.4441E-04</td>
<td>1.6362E-03</td>
<td>7.8942E-04</td>
<td>3.3700E-03</td>
<td>5.8052E-02</td>
</tr>
<tr>
<td>d</td>
<td>3.0519E-04</td>
<td>1.1860E-03</td>
<td>3.3638E-04</td>
<td>1.8276E-03</td>
<td>4.2750E-02</td>
</tr>
<tr>
<td>F p</td>
<td>1.3569E-03</td>
<td>9.6124E-04</td>
<td>1.3101E-03</td>
<td>3.6283E-03</td>
<td>6.0235E-02</td>
</tr>
<tr>
<td>d</td>
<td>6.3492E-04</td>
<td>2.6157E-04</td>
<td>3.0415E-04</td>
<td>1.2006E-03</td>
<td>3.4650E-02</td>
</tr>
</tbody>
</table>

Table 4: Variances and standard deviations of the cores
1. 10048-body models; 2. 1024-body models; S. self-consistent; F. fixed-potential; p. potential centre; d. density centre.
values are a measure of the amplitude of the core wandering. Comparing the fixed-potential and self-consistent models, they seem to have a similar amplitude of movement of potential centres, but the latter has a slightly larger amplitude of movement of density centre. Comparing the results for the 10048- and 1024-body models the former have smaller amplitudes of movement. The factor involved varies between about two and six and so it appears likely that the actual relationship between amplitude and number of bodies is an inverse power law (amplitude inversely proportional to the root of the number of bodies) as was the case for the Lagrangian radii oscillations in the last chapter.

4. Discussion And Conclusions

4.1 A Comparison With Other Author's Results

Both Heggie's and Makino & Sugimoto's results are for more centrally concentrated distributions than that of the simulations presented here. In Heggie's simulation he comments that the amplitude of the core oscillations exceeds the radius of the core. This is not true in this simulation though the motions are noticeable in comparison with the size of the core. Perhaps this is because the core is much larger in these rather unconcentrated Plummer models.

Makino & Sugimoto observe that their core oscillation is a superposition of a rapid oscillation upon a slower movement of a larger timescale than the crossing time. Perhaps, the results of the self-consistent simulations presented here could be subdivided in the same way. Makino & Sugimoto attribute the rapid oscillations of the core to the reaction when stars are ejected from the core. This may indeed be
an important contribution to the motion of the core; however, it is clearly not the only factor contributing to the rapid oscillations as these are also present in the fixed-potential models where there is no interaction between stars. As Makino & Sugimoto state, the fact that the slower motion of the core has a period longer than a crossing time is explained by observing that the restoring force upon the core returning it towards the centre of mass is much smaller than the force upon a star oscillating about the core: the core moves under a force due to its interaction with the other spread out stars (the "halo"), whereas the individual stars move under a force usually dominated by the concentrated core.

4.2 Conclusions

The cores of both fixed-potential and self-consistent systems make noticeable movements. The amplitude of this motion is larger for systems with fewer stars, and it may possibly be inversely proportional to the square root of the number of stars. Some of the movements of self-consistent model cores may be attributed to the inherent motions present in fixed-potential systems; however, other phenomena seem to have a rôle (one possible such phenomenon is the escape of stars from the core). There appears to be some motion on a larger timescale than the crossing time. As observed by Heggie(1989), the motion of the core is of importance as it could influence the effects of relaxation (changes in energy and other orbital parameters of stars due mainly to two-body encounters). More detailed work needs to be done to investigate the amplitudes, frequencies and causes of core wandering.
Heapsorts have been mentioned as the method used for finding the body/bodies with the smallest next update time. These procedures are based on those given in Press et al. (1986) pp. 229-234. Essentially, their algorithm takes a selection of numbers (times of next update) and orders it by size. Here, for this program, separate subsections of the algorithm are used rather than the whole, but we shall first look at it in its entirety. For convenience an algorithm is described that puts the smallest numbers on the top of the heap, rather than the largest as in Press et al..

The algorithm has two main parts, "heaping" and "promotion", both of which have order $N \log_2 N$, where $N$ is the number of objects being sorted. The heaping part involves the arrangement of the numbers into a binary tree (which is as near symmetric as possible). We shall call the node at the centre of the tree's branches "the top of the heap", and the nodes at the ends of the branches "the base of the heap". For the ordered heap, the number at the top is the smallest. For any position in the middle of the heap the two numbers below are larger, and the number above is smaller. However, there may be numbers higher in the heap on other branches that are larger, or numbers lower down on other branches that are smaller. An example of an ordered tree is shown in Diagram 5.

To arrange the numbers in this form the base of the heap and the level immediately above are filled with numbers. (The base will not be a complete row unless the total number of numbers to be stored in the heap
is of the form $2^n - 1$ with $n$ a positive integer, and so the base is filled from the left as far as possible.) Now all the numbers on the upper row are individually compared with the numbers below them. If a number is larger than one or both of the pair immediately below, then its position is swapped with that of the smaller number below. Following this stage, the next row up is assigned numbers and they are swapped with those in the rows below them as necessary. The process is repeated until the whole heap has been filled with numbers.

Diagram 5: An ordered heap of six numbers.
The numbers are at the nodes and the "branches" are represented by line segments. There is a gap at the right of the base as six is not one less than an integer power of two.

Now to the "promotion". During this stage the numbers are taken in order of size from the heap. The smallest number is on the top of the heap, and it is removed. It is replaced by the last number on the right of the base of the heap (the number of numbers in the heap is being reduced by one). The next smallest number is one of the two numbers in the row below the top, it is moved up ("promoted") to swap positions with the number originally from the base. The latter number is in turn compared with the numbers now immediately below it and if necessary
swapped with one of them. The process is continued until an ordered heap has been created. Now we remove the second smallest number (overall) from the top of the heap and repeat the whole process. The process is repeated \( N \) times: once for each of the bodies. At the end the numbers removed form an ordered array.

In the earlier programs in which the extrapolated bodies are passed around, there are subsections of the heapsort procedure on each of the slaves. Each slave stores the data for \( N/s \) bodies, and only ever updates one of these particular bodies. At the start of the run of the program, their times of next update are arranged in the heap using the "heaping" part of heapsort. We only require the smallest time of next update on each slave, for sending to the master and for determining the body to be updated next on that slave, and so this is removed from the top of the heap. After update, the new time of next update is placed on the top of the heap, and we repeat the last part of the heaping procedure i.e. swapping this member of the top row with the rows below as necessary to preserve the heap structure. The heaps are kept in existence during the whole of the run without ever needing to rebuild them.

For the later programs, in which all the slaves store the positions of all the bodies, the ordering of the times of next update, and determination of which bodies to update next, is done entirely in one large heap of \( N \) update times upon the master. As has been mentioned elsewhere, this was an improvement on the previous arrangement: any \( s \) of the stars can now be updated at the same time. The master initially arranges the timesteps into the heap using the heaping part of heapsort.
Before the round of updates the master must find the smallest time (just the top of the heap), and the bodies corresponding to the s smallest times of update. Those s smallest times, and hence associated bodies, are found by "promoting" s bodies off the top of the heap using a subsection of the "promotion" part of heapsort. At the end of the updates the s new update times are returned to the master. They are successively placed in the spaces at the base of the heap and are then swapped with times above as is required to recreate the heap structure. Now the smallest s times may be promoted off the top of the heap again and the process repeated.

A2. Generating Initial Conditions To Fit A Plummer Model

For the simulations carried out, all the bodies are chosen to have the same mass, \( \frac{1}{N} \), where N is the number of bodies. As mentioned in § 3.6, their positions and velocities are randomly generated to satisfy the Plummer model distribution function

\[
f(E) = \begin{cases} 
\kappa_1 (-E)^{7/2} & \text{for } E < 0 \\
0 & \text{for } E \geq 0 
\end{cases}
\]

\( \kappa_1 \) is a constant and E is energy per unit mass of the body. In order to generate a model on the computer from this probability generating function, we must express the individual stars' positions and velocities in terms of independent random variables uniformly distributed on the unit interval. (The random number generator simulates data of this kind.) Due to the spherical symmetry of the distribution of position and velocity we shall use spherical polars in our description of both. Note that the main program works in Cartesian coordinates, so the data is converted into this form at the end of the initialisation program. The
eventual program uses at least seven independent computer-generated random numbers, uniform on \([0, 1]\), for each body: one each for the spherical angles \((\theta_r, \psi_r, \theta_v, \psi_v)\) describing the direction of the radial coordinate from the geometric centre \((r)\) and the direction of the velocity \((v)\); one for the magnitude of the radial coordinate \((r)\); and several pairs for the magnitude of the velocity \((v)\). A useful strategy is to make use of the fact that the total probability distribution function of a random variable \(x\), \(F(x) = \int_{-\infty}^{x} g(x) \, dx\) (where \(g(x)\) is the probability density function for \(x\)), is itself a random variable with uniform distribution between 0 and 1:

\[
\begin{align*}
    \text{for } y < 0 & \quad P(F < y) = 0; \\
    \text{for } 0 \leq y \leq 1 & \quad P(F < y) = F(F^{-1}(y)) = y; \\
    \text{for } y > 1 & \quad P(F < y) = 1.
\end{align*}
\]

The easiest coordinates to generate are the four angular coordinates, as \(r\) and \(v\) are isotropic. For any isotropic vector with spherical polar angle coordinates \((\theta, \psi)\), the probability that \((\theta, \psi)\) lies in a range \(A = [\theta_1, \theta_2] \times [\psi_1, \psi_2]\) \((0 \leq \theta_1 < \theta_2 \leq \pi, 0 \leq \psi_1 < \psi_2 \leq 2\pi)\) is

\[
\int_A \sin\theta \, d\theta d\psi,
\]

independently of \(r\). Also, \(\theta\) and \(\psi\) are independent and so we have probability density functions for them \(g_\theta(\theta) = \frac{1}{2} \sin\theta\) and...
As $\psi$ is uniform on $[0, 2\pi]$ we take $\psi = 2\pi X$, where $X$ is a computer generated random number. We use the aforementioned strategy of using the total distribution function for $\theta$ and set $F_\theta(\theta) = \frac{1}{2} (1 - \cos \theta)$ equal to another computer generated number. Now we can find $\cos \theta$ and $\sin \theta$. We do not need to know the actual value of $\theta$ for either $r$ or $v$ as we are going to convert into Cartesian coordinates.

The next coordinate to be found is $r$. The units used here (Heggie & Mathieu (1986)) set $M = 1$, $G = 1$, $E = -\frac{1}{4}$ and so the density is

$$p(r) = \frac{3}{4\pi R_p^2 \left[ 1 + \frac{r^2}{R_p^2} \right]^{5/2}}, \quad R_p = \frac{3\pi}{16},$$

(cf. Spitzer (1987), p.13), and the mass within $r$

$$M(r) = \frac{r^3}{R_p^3 \left[ 1 + \frac{r^2}{R_p^2} \right]^{3/2}}.$$

These quantities give the probability density for $r$ and the total distribution function for $r$, respectively. To generate $r$ we set $M(r)$ equal to a computer generated number $X$ and find $r$ as a function of $X$:

$$r = \frac{R_p X^{1/3}}{(1 - X^{2/3})^{1/2}}.$$

Now finally for $v$ another strategy is required. $r$ and $v$ are not independent, so having generated a value for $r$ from its probability distribution we must now look at the distribution of $v$ conditional on that $r$:

$$f(v|r) = \frac{f(r,v)}{f(r)} = \frac{\kappa_1 (-E)^{7/2}}{\rho(r)}$$
and so
\[ f(v|\mathbf{r}) = \frac{k_1\left(-\frac{1}{2}v^2+\varphi\right)^{7/2}4\pi v^2}{\rho(\mathbf{r})} \quad 0 \leq v \leq \sqrt{-2\varphi} \]
\[ = 0 \quad \text{otherwise.} \]

Note \( f(v|\mathbf{r}) \) is zero at \( v = 0 \), or \( \sqrt{-2\varphi} \), and is positive in between, and so for the given \( \mathbf{r} \) there exists a maximum value of \( f(v|\mathbf{r}) \), \( f_{\text{max}}(\mathbf{r}) \) (actually the value at \( v = \sqrt{-4\varphi/9} \)). Now define
\[ g_r(v) = \frac{f(v|\mathbf{r})}{f_{\text{max}}(\mathbf{r})}. \]
\[ g_r(v) = \frac{v^2\left(-\frac{1}{2}v^2+\varphi\right)^{7/2}}{-\frac{4}{9}v^2\left(\frac{2}{9}v^2+\varphi\right)^{7/2}} = \frac{19683\sqrt{7}}{9604} \frac{v^2\left(-\frac{1}{2}v^2+\varphi\right)^{7/2}}{(-\varphi)^{9/2}}. \]

\( g_r \) takes a value between 0 and 1. We use one random variable to chose a provisional value for \( v \) by setting \( v = X\sqrt{-2\varphi} \), and then use another, \( X_2 \), to ensure that \( v \) has an appropriate distribution, by discarding our provisional value for \( v \) and starting again if \( X_2 > g_r(v) \).

An easy check on the initial conditions is to find the kinetic energy of the whole system (\( T \)) and to check that this is consistent with the statistical bounds for the model. We have
\[ T = \sum_{i=0}^{N} \frac{1}{2} m_i v_i^2 = \frac{1}{2N} \sum_{i=0}^{N} v_i^2 \]
where \( v_i \) is the magnitude of the velocity of the \( i \)th star. So, by the independence of the stars, for any \( i \)
\[ <T> = \frac{1}{2} <v_i^2> \quad \text{and} \quad \text{var}(T) = \frac{1}{4N^2} \sum_{j=0}^{N} \text{var}(v_j^2) = \frac{1}{4N} \text{var}(v_i^2) \]
(where \( <x> \) denotes the expectation of \( x \) and \( \text{var}(x) \) the variance).

Also, the initial distribution is chosen to satisfy the virial theorem (see e.g. Binney & Tremaine(1987), pp.211-219) and so
\[ <T> = -E = \frac{1}{4}, \] so \[ <v_1^2> = \frac{1}{2}. \]

Now
\[ <v_1^4> = \frac{I_4}{I_0}, \] where \[ I_n = \int \int v^n f(r,v) \, dv \, dr \]

and so
\[ I_n \propto \int \int v^{n+2} \left( -\frac{1}{2} v^2 - \varphi(r) \right)^{7/2} \, dv \, r^2 \, dr. \]

Put \( v = (\sqrt{2} \varphi) \sin \theta. \) Now
\[ I_n \propto \int_0^{\infty} \int_0^{\pi/2} (-\varphi)^{n/2+1} \sin^{n+2} \theta (-\varphi \cos^2 \theta)^{7/2} (\sqrt{2} \varphi) \cos \theta \, d\theta \, r^2 \, dr \]
\[ = 2^{n/2} \left( -\varphi \right)^{n/2+5} \int_0^{\pi/2} \sin^{n+2} \theta \cos^2 \theta \, d\theta. \]

\( \varphi \) is the potential (given in Spitzer (1987)) and in our units
\[ \varphi = -\frac{1}{(R_p^2 + r^2)^{1/2}}, \quad R_p = \frac{3\pi}{16}. \]

Now let \( r = R_p \tan \psi, \) whence
\[ I_n \propto 2^{n/2} \int_0^{\pi/2} \left( \frac{\cos \psi}{R_p} \right)^{n/2+5} R_p^2 \tan^2 \psi \sec^2 \psi \, d\psi \int_0^{\pi/2} \sin^{n+2} \theta \cos \theta \, d\theta \]
\[ = \left[ \frac{2}{R_p} \right]^{n/2} \int_0^{\pi/2} \left( \frac{\cos \psi}{R_p} \right)^{n/2+1} \tan^2 \psi \, d\psi \int_0^{\pi/2} \sin^{n+2} \theta \cos \theta \, d\theta. \]

Now, we may evaluate \( I_4/I_0 \) from standard integrals (see e.g. Gradshteyn & Ryzhik (1965)),
\[ \int_0^{\pi/2} \cos^{\mu-1} \psi \sin^{\nu-1} \psi \, d\psi = \frac{1}{2} B\left( \frac{\mu}{2}, \frac{\nu}{2} \right) = \frac{\Gamma\left( \frac{\mu}{2} \right) \Gamma\left( \frac{\nu}{2} \right)}{2\Gamma\left( \frac{\mu + \nu}{2} \right)} \quad (\mu, \nu > 0), \]
where \( B \) and \( \Gamma \) are the beta and gamma functions, respectively, and
\[ \Gamma(x+1) = x\Gamma(x). \]

So
\[
<v_i^4> = \frac{1}{4} \frac{1}{10} \left( \frac{2}{R_p} \right)^{4/2} \frac{\Gamma(2) \Gamma^{3/2} \Gamma^{7/2} \Gamma^{9/2}}{\Gamma^{7/2} \Gamma^{7/2} \Gamma^{7/2} \Gamma^{9/2}} \Gamma(6) \Gamma(8) \Gamma(1) \Gamma^{3/2} \Gamma^{3/2} \Gamma^{9/2} \\
= \left( \frac{32}{3\pi} \right)^2 \frac{1}{28}
\]

and

\[
\text{var}(T) = \frac{1}{4N} \text{var}(v_i^2) = \frac{1}{4N} (<v_i^4> - <v_i^2>^2) = \frac{1}{N} \left( \frac{64\pi^2}{63} - \frac{1}{16} \right).
\]

A 95% confidence interval for T is

\[
<T> \pm 1.96\sqrt{\text{var}(T)} = \frac{1}{4} \pm \frac{1.96}{\sqrt{N}} \left( \frac{64\pi^2}{63} - \frac{1}{16} \right)^{1/2}.
\]

In addition to initial positions and velocities, the more advanced programs require initial conditions with higher order derivatives of these and "times of previous update". The policy adopted here is similar to that in Aarseth(1985): for each body coefficients of its predictor are calculated exactly from the initial conditions, its timestep is found, and then the times of previous update are assigned multiples of this timestep.
References


