Declaration

I declare that this thesis has been composed by myself and that the work contained is entirely my own, unless clearly and explicitly stated otherwise.

James Sutherland.
Acknowledgements

Clive Greated and Bill Easson at Edinburgh University gave me the opportunity to do this PhD almost four years ago. I hope that they have not regretted it since, as I have not regretted taking up their offer. I wish to express my thanks to them, both for the supervision given and the leeway allowed during this period. Without Wim Morris and the experience gained from working with him at B.P. I would have missed out on a great deal. I wish to thank him for this glimpse of a different world and for his tolerance throughout and B.P. for their sponsorship of this project through the C.A.S.E. award.

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Abstract

A number of accurate measurements of the kinematics under the crests of regular waves and two-component uni-directional wave groups have been made in a laboratory wave flume using Particle Image Velocimetry. The waves were in intermediate to deep water, with relative depths in the range $d/gT^2 = 0.05$ to $0.085$ and were of moderate to high relative steepnesses, in the range $H/gT^2 = 0.005$ to $0.018$. (Here $d$ is water depth, $T$ wave period, $H$ wave height and $g$ gravitational acceleration.) The main conclusions are:

1. Regular waves were accurately modelled using an implementation of high order fourier theory by Rienecker and Fenton, providing Stokes second (zero mass transport) definition of wave celerity was used.
2. Steep, near-breaking two-component waves were modelled accurately using superposition stretching, a derivative of linear theory. The input for this is the measured wave spectrum, including first and second harmonics. The second harmonic contribution was found to be significant.
3. The kinematics in the crests of different waves of a given height and period can vary considerably. Here, differences of over 20% were noticed at the crest.
4. Wave group length affects the internal wave kinematics.
5. Measurements must be made above the level of the wave troughs and should be made above the mean water level also, if experimental results are to have much credence.
6. Particle image velocimetry proved to be an excellent measurement technique to use for measuring velocities as it was capable of measuring close to the free surface of high waves, with a high degree of accuracy.
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Chapter 1

Introduction.

1.1 The importance of fluid loading

The study of water wave kinematics plays a pivotal role in ocean and coastal engineering. The design of offshore structures is significantly affected by the perceived value of extreme storm environmental loading. For a typical steel spaceframe offshore structure the extreme environmental loading is dominated by wave forces. Prasthoffer [86] gives a ratio of 60% loading due to waves, 30% to currents and 10% to wind for a slimline jacket under the 100 year design load. The wave contribution is somewhat reduced for semisubmersibles and is down to 30% for a tension leg platform which remains significant. Moreover, for such a spaceframe structure the calculation of the extreme wave loading is likely to be dominated by a drag term proportional to the square of the calculated local wave velocity according to the universally used Morison's equation [76].

Therefore, knowledge of wave kinematics is necessary to ensure the integrity of offshore structures, the safety of the personnel manning them and the safe and economic extraction of the hydrocarbon fuel resources found on the world's continental shelves. To this end the measurement of internal wave kinematics plays an
important role in calibrating and guiding the development of the wave kinematics and wave loading models used in the design of offshore structures. In particular, measurements of the internal kinematics of moderate to extreme waves and wave groups (as presented here) are of relevance to the extreme loading case. This is especially so as the measurements include results from the trough to crest region where both velocities and free surface errors are greatest, and where few measurements have been made. Such measurements are still needed despite the technological advances of the past four decades as no comprehensive wave kinematics package has been agreed by the researchers in the field.

1.2 Wave Loading and Design

1.2.1 Initial Formulation of the Design Process

The design process for wave and current loading on offshore structures has been refined considerably since the first offshore rigs were constructed in the late 1940s for the Gulf of Mexico. In the first generation of wave force projects [114] the internal kinematics were inferred from wave height records as there were no instruments capable of making reliable, calibrated wave velocity measurements in field conditions. Indeed the first kinematics measurements made under storm conditions offshore were not conducted until 1973 [34].

The first quantitative formulation for wave loading was derived by MacCamy and Fuchs [67] who formulated a diffraction theory for large diameter vertical cylinders in 1945. In 1950 Morison et. al. [76] presented an equation for 'the force exerted by unbroken surface waves on a cylindrical object, such as a pile, which extends from the bottom upward above the wave crest.' This equation became universally known as the Morison equation and is applicable for cases where there is no significant diffraction of the wave by the pile (generally taken as when the
dominant wavelength is greater than about five times the pile diameter). As this regime covers many offshore structures, Morison's Equation has become the most commonly used equation in the calculation of forces on offshore structures. The Morison Equation calculates force per unit length and contains two terms, namely:

\[ F_D = A \text{drag force proportional to the square of the water particle velocity } (u) \text{ with the proportionality term dependent on the cylinder diameter } (D) \text{ and a drag coefficient } (C_D) \text{ determined experimentally.} \]

\[ F_M = \text{An inertia force proportional to the horizontal component of the accelerative force exerted on the mass of water (of density } \rho \text{) displaced by the pile. Again the proportionality term is the product of a geometrical term and an inertia coefficient } (C_M) \text{ determined experimentally.} \]

Morison's equation may, therefore be written as:

\[ F = F_D + F_M = \frac{1}{2} \rho C_D D u |u| + \frac{1}{4} \rho \pi D^2 C_M \frac{du}{dt} \]  \quad (1.1)

Much work has been done to try and determine accurate values for drag and inertia coefficients \((C_M \text{ and } C_D)\) in different flow regimes and even to determine whether this simple linear-quadratic equation has any validity in a given flow regime [97, 93, 98, 82].

The application of Morison's Equation requires knowledge of wave kinematics and accelerations and these are acquired through the concept of the design wave as part of the design environmental conditions. The design process is split into three steps, assumed independent. These are

**Derivation of extreme environmental conditions.** Statistical processes are applied to sources of environmental data collected from near the proposed site or hindcast from the most appropriate and longest running records available. Design criteria covering winds, waves and currents for a N-year return
period are produced. For waves this amounts to calculating the most probable wave height that will be exceeded once every N years. A corresponding wave period will also be calculated for the same position (and hence water depth). These define the N-year design wave which it is assumed can be modeled by a regular, monochromatic wave extending to plus and minus infinity.

**Calculation of wave kinematics.** The second stage involves modeling the design wave to produce the local wave kinematics. A standard wave theory is used and the presence of the structure is ignored.

**Calculation of local force.** The calculated wave kinematics are used in Morison's Equation to calculate the local forces which can, for example, subsequently be integrated to give the total moment on the structure about any fixed point, such as the sea bed.

### 1.2.2 Development of the Design Practice

The first generation of Gulf of Mexico structures proved unreliable as the maximum wave loading forces were underpredicted. A return period of 25 years had been chosen for the design wave and a drag coefficient ($C_D$) of 0.5 was used in Morison's Equation. A regular wave theory (Stokes Vth) was used to calculate the local wave kinematics. This assumes that the kinematics of an extreme wave can be represented by a higher order regular unidirectional wave theory. No account was made of the extra drag caused by marine fouling or of the effect of currents on the kinematics. The unreliability of these structures was such that some 21 Gulf of Mexico platforms designed before 1970 have collapsed or suffered a major structural integrity failure due to extreme wave loading.

Much research has subsequently been done into factors affecting the design process and by the beginning of the 1970s there had been an increase in design wave
heights and storm currents had been added to wave kinematics. Moreover, regulatory bodies were demanding the use of 50 or 100 year return periods for design environmental conditions, and drag coefficient values of 0.6–0.8 to recognise the effects of marine fouling [4, 82]. Current practice is to define a 100 year return period wave, current, storm surge and wind; to calculate them separately and to assume they occur simultaneously and in the same direction in determining the overall design loads. The result is a design condition with an effective return period of greater than the 100 years of each of the components.

These steps mean that for a typical spaceframe structure in a given location the total design wave forces have increased by a factor of 2.5 to 3 [7]. No platform designed since 1970 has suffered major structural damage due to wave loading. The period from 1970–1989 inclusive encompassed about 70,000 platform years of exposure to marine conditions [25] so it can be seen that the reliability of such platforms is high, though unknown. This does not mean that there is no need for further research, however. What can be said is that the science of calculating offshore loading is reaching a state of some maturity and that it is now possible to review and even consider replacing the simplistic 3-stage design process.

1.2.3 The Current Design Practice

The current design process is a very simple idealisation of a highly complex situation: but it is an idealisation that has served a very useful purpose over a number of years. As such it is not one that can be thrown away lightly. Certainly the attitude 'if it ain't broke don't fix it' may be said to apply here and changes will only be made if they can be shown to produce both conservative designs as at present [121] and a more physically realistic model of wave loading. Moreover, the logic of the American Petroleum Institute RP2A work (on recommended Practice for planning, designing and constructing fixed offshore structures [4] ) is to go to a load resistance factor design rather than a working stress design. Snell [84, page
states that that 'will inherently require us to put lower partial factors on the loads that are well understood, essentially top side weight, and higher partial factors on the loads that are not particularly well understood — predominately the environmental loads.' Therefore the importance of understanding wave velocities and forces will play a role of increasing importance in the next few years. There is evidence that the process could be improved and there are two different ways of approaching the problem.

- The first is to maintain essentially the same approach but adjusting one stage at a time to reflect a better data set or more accurate theory. This process may involve simply replacing a coefficient or an equation with a more accurate one while retaining the three stage design process.

- The second approach is to replace the three stage time-independent deterministic design process with an alternative approach. Vugts [119] suggests the introduction of a model with probabilistic and time dependent 3-D features to reflect the random wave environment in a realistic manner. Any such alternatives may well be left using the Morison's Equation anyway due to the lack of a suitable replacement. Three possible approaches suggested [119] are:

1. Time domain simulations.
2. Frequency domain simulations.
3. Shell's NEW WAVE model [25].

In Shell's NEW WAVE model a variation of linear theory for wave velocities called $\delta$-stretching [31, 92, 115] (and Chapter 2) is used with a surface profile derived from the autocorrelation function of the surface elevation as derived from a random wave spectrum and an amplitude calculated from the significant wave height and duration of a storm. Thus the benefits of a higher order theory are swapped for those of a 3-D linear-based model of the sea. The idea is to produce best-estimate
velocities (rather than conservative ones) and from there to apply these in a wave loading formula incorporating best-estimate coefficients (rather than hybrid ones, see below). Wave kinematics research can aid in the calibration necessary for the verification of such theories by looking at specific regular and non-regular waves (as presented here) and comparing the two approaches.

1.2.4 The role of Wave Kinematics Research in the Design Process

The different stages of the design process will now be reviewed in turn, with reference to the role of wave kinematics research. It is not the role of this research to look at the statistical means of hindcasting design wave parameters or of obtaining them from wave height records. It is however the role of this wave kinematics research to look at the outcome of those processes — the design wave parameters $H$, $T$ and $D$ — and to question whether the velocities calculated by regular wave theory from these parameters do represent the most severe case obtainable from the highest sea state. It is currently assumed that the calculated velocities are conservative.

Once statistical information has been obtained on the most probable design sea state then wave kinematics have to be derived. This can be done by a variety of theoretical methods (see Chapter Two) all of which require validation and common acceptance before being used in design. In recent years the appropriateness of the combination of the regular wave assumption and higher order theories in modeling storm conditions has been called into question. In particular the use of Directional Gaussian Linear Wave Theory (DGLWT) incorporating a stretching factor has become a common alternative. In DGLWT the sea is assumed to be composed of the sum of a large number of individual small waves, each with its own amplitude, phase, frequency and direction and each conforming to the
infinitesimal amplitude limitation of linear theory. The advantages of DGLWT are that it can model (if only crudely) many of the three-dimensional aspects of the ocean surface and is easy to compute. The main disadvantage in applying the theory lies in the limit of applicability of linear theory being waves of infinitesimal amplitude. Any sea state even approaching storm conditions will be of sufficient amplitude to ensure the presence of non-linearities. As a result the theory produces large free surface boundary errors. Wheeler [124], Chakrabarti [13], Rodenbusch and Forristal [92] (δ-stretching) and others have all proposed stretching modifications to linear theory. The modifications were introduced because of observed discrepancies between linear theory and measurements.

The calculation of velocities is somewhat complicated by the presence of currents. The original loading calculations ignored current. A typical modern British practice is simply to add the 100-year current and wave velocities, although there are a number of approaches used as part of different overall packages. The American Gulf of Mexico practice is to add the appropriate current, which will be a fraction of the anticipated maximum. Research is now appearing [122, 32, 79, 31] on the joint probability of waves, currents and wind. The aim is to make a more accurate judgement of the most severe combined wave and current loading. For example, Harris [45] shows that, for storm data, the tidal component in the total averaged co-linear current is negligible. Moreover, Lambrakos [58] and Taylor [112] have reported the effect of current blockage by a platform, for which Taylor has developed a model based on actuator disc theory. In current blockage a platform diverts part of the current around it, thereby reducing the flow velocity around the platform. This factor has a bearing on the value of any current measured and to be used in design. Blockage also has a bearing on the applicability of comparisons between full scale velocity measurements and lab measurements usually made without the presence of a structure. Wave/current probabilities and blockage are outwith the scope of this research, though with the above comment on the applicability of comparisons noted. What is significant is that a greater level of understanding
is being achieved about a factor affecting wave loading that was previously not understood.

The calculation of forces produced by velocities are of relevance in understanding the motivations behind the results to be presented. This leads to an appraisal of Morison's Equation, equation 1.1. There have been a number of attempts to replace or reform Morison's Equation, see Rainey [88], Lighthill [60] or Sarpkaya and Isaacson [98, pages 122–126] for example, since its inception. None has achieved widespread acceptance and Morison's Equation has provided acceptable answers in a range of situations, particularly at low or high Keulegan Carpenter numbers (see below). Therefore it seems likely that any revised wave loading formulation will have to rely on it, for the foreseeable future at least. It has two terms and two experimentally determined coefficients. The best-estimate values of the coefficients vary according to such factors as:

**Reynolds Number** $R_E$ relates the mass times acceleration forces to the viscous forces from shearing of the fluid. The value of Reynold's Number has a large effect on drag coefficient, but for the loading regime of interest (flow round offshore structures) the flow is post-critical ($R_E \geq 2 \times 10^5$) and drag coefficient is independent of Reynold's Number, based on leg diameter.

**Keulegan Carpenter Number** $K_C$ gives the relative size of oscillation amplitude related to the diameter of a member and gives a measure of the relative importance of the drag and inertia components of the total flow. At low $K_C$ numbers ($K_C \leq 10$) the inertia term ($F_M$) dominates Morison's equation and at high $K_C$ ($K_C \geq 25$) the drag dominates. The appropriate $K_C$ range for extreme loading will depend on the structure and its location.

**Roughness Ratio** $\kappa/D$ is an important factor in determining the forces exerted by waves. The roughness is caused by marine fouling, corrosion etc and small levels of roughness can increase forces substantially. Here $\kappa$ is a measure of the mean roughness height and $D$ is the member diameter. Typical ratios
for platforms lie in the range $10^{-4} < \kappa/D < 10^{-1}$. The ratio is hard to measure, however, and different distributions of marine growth can give the same $\kappa/D$ value but different $C_D$ values.

Method of calculating Coefficients. There are several methods of obtaining values for the drag and inertia coefficients. Experiments can be done in a laboratory or at sea in a number of ways and the coefficients can be calculated by a variety of methods. It is important to note that experimental calibrations of the coefficients rely on the ability to know both the forces exerted on an element of cylinder by a moving fluid and the exact kinematics of that fluid. This is commonly done by inferring the kinematics using a wave theory or by measuring the kinematics a finite distance away. Both methods introduce uncertainties.

Flow Coherence. In a lab (particularly in a U-tube or towing experiment) flow velocity is essentially uniform across the span of a section used to measure forces. The force measuring section is usually not more than one or two cylinder diameters in length. However, on a real structure it is the net force on a whole span of a tubular member which is of interest and it is unlikely that the flow will be properly coherent along the length of a single member. This may lead to a high $C_D$ value for one section but a spatially averaged value which is lower.

The Department of Energy Background report [82] has investigated the effect of flow conditions, roughness and current (vol. 1, pp185-91, sections 3.2.11 and 3.2.12) and produced best estimates of mean values of $C_D$ and $C_M$. These best estimates were dependent on $K_c$, $\kappa/D$ and ratio of wave particle velocity to current velocity. The report concluded that the use of these best estimates is unjustified because of the high scatter in measured values. Therefore a graph of $C_D$ and $C_M$ values against roughness is presented to give the same total loading as the best estimate values (for post-critical flows at all $K_c$ values). It shows
linear relationships between the coefficients and \( \log \kappa/D \). For drag coefficient the values increase from a smooth cylinder value of 0.5 to values in the range 1.0 to 1.3 for typical levels of near-surface roughness.

The Department of Energy: Guidance on design, fourth draft edition [83] then gives the following values for \( C_D \) and \( C_M \):

\[
\begin{align*}
C_D &= 0.7 \text{ (marine growth)} \\
C_D &= 0.6 \text{ (no marine growth)} \\
C_M &= 1.7 \text{ (extreme conditions)} \\
C_M &= 2.0 \text{ (fatigue conditions)}
\end{align*}
\]

The report justifies using values significantly lower than those predicted from research on the basis of conservatism in the following assumptions:

1. the waves are long-crested
2. water particle motions are calculated by regular wave theories
3. no shielding effects on the structure are included
4. independent extreme values of wave and current are combined.

Thus \( C_D \) has evolved into a hybrid coefficient reflecting both the available knowledge of drag on cylinders and an experimental adjustment. It does not represent the best estimate of that parameter for the anticipated flow regime.

(Therefore tinkering with the coefficients in the present approach can only lead to small improvements in specific cases and will never solve the problem. Moreover, existing design formulations can give a coefficient of variation of the load side of the equation of order of 60% [86].)

The aim of research into fluid loading and design is to produce a more physically realistic and cost effective design process, reflecting best estimate values as far as possible and with knowledge of the degree of accuracy attained. At the present
time realistic efforts are being made to assess the effects of joint probabilities and current blockage, and realistic $C_D$ values are being derived for the extreme loading case. The industry has not reached a consensus on how to make a best estimate of the wave kinematics involved. There is a consensus that high order regular wave theory is conservative, but there is no consensus on by how much.

If realistic estimates of drag coefficient are to be used in future (and that is to be recommended as the basis of a more realistic view of wave loading) then what is required is a better understanding of the magnitude of, and error in, the four assumptions above. The latter two are the subject of ongoing research as briefly reported above and the fourth is certainly conservative in the determination of an N-year design sea state from independent N-year design wave and current conditions. The first two assumptions are believed to be conservative as the sea is 3-D in nature so that

- for a given resultant wave height the component wave heights add algebraically, whereas the velocities and accelerations add vectorially — at least according to linear theory.

- the loading on a structure becomes less well correlated as the wave crest length decreases due to directional spreading.

In particular the magnitudes and errors in the kinematics of extreme wave situations must be better understood. To that end, this thesis reports on the results of wave kinematics experiments made on a range of waves, including many results from near the free surface.
1.3 Wave Kinematics Measurements

Wave theories always represent an effort at approaching a model of an idealised wave state. Therefore there must always be a question mark over their applicability in real situations. As a consequence, wave theories have always been checked against wave measurements. These are notoriously difficult to make particularly near the free surface, where the velocities are greatest. This is especially important for calculating the drag term in Morison’s Equation as the peak forces are generated by the highest velocities. These are generated in the wave crests where theoretical errors are greatest and it is most difficult to measure. Nevertheless much time and effort has been spent on measuring waves both at sea and in lab conditions. There are advantages in each approach.

1.3.1 Laboratory experiments

Laboratory experiments have controlled conditions and the freedom to vary one parameter at a time or to repeat experiments as often as necessary. The wave generation and measurement should be accurate and calibrated and results can be concentrated in the area of flow of greatest interest. Research techniques such as LDA and PIV (see below) are non-intrusive so do not disturb the flow. Moreover these techniques can be used to measure above the trough level. The effects of tidal current and winds can be removed (though some wave flumes include current and/or wind generation facilities) leaving wave generated flow.

The disadvantages include that the results must, of necessity, be small scale so may not be directly applicable to a full scale structure. Moreover, flumes produce 2-D waves (velocity components in two dimensions only) so no 3-D effects can be accounted for. LDA and PIV measurements are more difficult to make in 3-D wave basins. The results will be affected by what are known as tank effects. These
include reflected waves, mass transport (induced currents), tank resonances, wave harmonics and problems caused when an exponential decay in wave oscillation amplitude with depth is approximated by a linear decay of paddle oscillation amplitude with depth. These factors do not invalidate flume measurements but care must be taken to specify what precautions have been taken to minimise and quantify these effects.

Laboratory Techniques

The main lab techniques used to make velocity experiments will now be described briefly. For a fuller description of the techniques used in this thesis, see chapter 3 and for a general review of lab techniques see Wessels et al [123].

Laser Doppler Anemometry (LDA) [23, 22] Two coherent light beams (from a split laser beam) intersect at one point in the flow field to form a small measuring volume. In this volume a pattern of Young’s fringes is formed by optical interference of the two beams. As small particles in the flow pass through this volume they scatter light at a frequency dependent on their velocity and this light is detected. From this a continuous time series of velocities is recorded for a small volume in the flow.

Particle Image Velocimetry (PIV) is an essentially non-intrusive measurement technique derived from the application of speckle photography to a fluid flow [39, 37, 3]. The technique involves taking a multiple exposure photograph of an illuminated plane of the wave, parallel to the wall of the flume. The flume had first been seeded with suitable reflecting particles (conifer pollen). Optical and digital analysis of the photographic negative produces a grid of velocity vectors at regular intervals throughout the flow. Thus a large number of points are measured in a plane in the same short period of time.
Wave gauges are used to provide a record of wave height with time at a point in the horizontal plane. They generally work by measuring the change in capacitance or resistance between two conducting rods as the surface profile rises and falls. Their records can be used to produce a wave spectrum (amplitude or energy against frequency) by the use of a fourier transform.

Laboratory Experiments; Regular Waves

A number of regular wave experiments have been made in wave flumes. These include:

Nath and Kobune[77], 1978, who made measurements of random and periodic waves at Oregon State University. The flume used was 104 × 3.7 × 4.6m (length, width, water depth). Wave height with time measurements were made with a sonic wave profiler and velocities were measured at one height below the lowest trough depth using a propeller current meter and a TSI hot film anemometer. The circulation in the flume was considered to be negligible. Measured maximum velocities were compared favourably with periodic theory especially under large waves. The kinematics were estimated by fitting a periodic wave to the measured wave profile, even for random wave states. The results show that the crest velocities were overpredicted by regular wave theory and trough velocities were more negative than predicted by regular wave theory (linear and Dean’s stream function — see chapter two). Nath and Kobune concluded that it was reasonable to use periodic wave theory to predict conditions in large waves at sea. Sobey [102] points out that the circulation would not be zero and repeated calculations for the same data using a zero mass-transport condition which introduced a uniform shift with depth to the velocities, reducing the calculated velocities. This ensured that the fit to the measured results improved and gave little cause to doubt the predictive capacity of appropriate steady wave theory.
Bullock and Short[10], 1985 measured regular waves in the range 0.29 to 0.71Hz in a flume of 39.6 x 1.26 x 1m. A capacitance wave gauge measured wave elevations and a DISA LDA measured velocities. A minimum of 240 wave periods were allowed to let a steady flow pattern form. The wave spectrum was limited to the first 5 harmonics and a drift pattern matching that of the conduction solution of M.S. Longuet-Higgins was observed. The drift velocity is commonly of the same size as the second harmonic contribution to the flow. The conclusions are that horizontal velocity is not always well predicted by Stokes 1st, 2nd or 5th order wave theories. Errors of the order of 10% are possible. Discrepancies between theory and measurement are a function of the experimental set up and are caused by the wave generation, beach design, channel geometry and location of measurement.

Gudmestad et al[44], 1988 compared measurements of regular waves from Delft Hydraulics and California Institute of Technology to seven different theories. The waves were generated in flumes of 50 x 1 x 0.85m and 40 x 1.1 x 0.6m and in each case surface elevation and velocity records (using LDA) were made near the free surface. They noted a set down effect (a lowering of the mean surface elevation) and the setting up of a negative mean velocity (opposite to the direction of wave propagation). Both effects started as the wave field became established. The authors concluded that they may be calculated using the concept of radiation stress. They also concluded that without the current then their own variation of linear theory fitted the data best, but with the return current then Stokes Vth and Dean's Stream Function theories fitted best. Skjelbreia and Torum [99] however report that the linear theories were applied by using a single wave height and period (a fact not brought out in the paper). They contest that if a wave spectrum had been used then the higher harmonics would have caused the results to fit better and provide a greater rate of increase of velocity near the free surface.

Zhang, Randall and Spell[126], 1991 measured a regular and a two-component wave field and compared the results to Wheeler stretching, linear extrapolation and a nonlinear numerical scheme. The linear and Wheeler theories were cal-
culated from the measured wave spectrum. In the regular wave case the cut-off frequency was chosen to exclude the second and higher harmonics. This was justified by saying that 'it is well known in the Stokes expansion that [higher] harmonics make no contribution to the wave kinematics in deep water,' and cite Yuen and Lake [125] in support. They found that Wheeler stretching underpredicted crest and trough velocities for the regular case while linear extrapolation and numerical theory were very close to the measured results.

Laboratory experiments; Irregular Waves

A number of the more relevant irregular wave measurement programmes are outlined below:

Vis[118], 1980, made measurements in a flume at Delft Hydraulics. Irregular waves of moderate steepness and intermediate depths were measured by LDA and by a wave height gauge. Velocities under troughs were found to be higher (in magnitude) than those measured under the crest at the same height. Linear theory was found to give a reasonable description of the velocity field especially for velocities near the free surface under the steepest wave trough. All frequency components with a spectral density less than 10% of the peak value were ignored in the theoretical calculations.

Bosma and Vugts[9], 1981, conducted an extensive series of tests in a 50 × 1 × 0.85m flume, also in Delft. Nine spectra were measured, including extreme conditions. Velocity results were obtained using LDA for waves up to the maximum wave steepness without breaking. Results were obtained at 6 elevations in the flume, including still water level (SWL) and one standard deviation of wave surface elevation above and below SWL. The results were compared to linear theory calculations based on measured wave spectra in the range 0.05 to 2.5Hz. The conclusions were that linear wave theory (LWT) provided a valid and adequate description of 2-D irregular waves, but needs modification in order to describe
the most extreme events. The maximum crest velocities were smaller than theory while the maximum trough velocities were greater (more negative) than theoretical values (a similar conclusion to Nath and Kobune). Calculations from higher order wave theories were worse than those from the linear random wave model.

Melville and Rapp[75], 1988, conducted a set of experiments in a flume at the Scripps Institute of Oceanography. The flume is $28 \times 0.5 \times 0.6$m and an original set-up was used to measure the horizontal component of velocity at the surface. A LDA rig was set up with an unusually long measuring volume positioned vertically so that it always crossed the free surface. The water was dyed and the surface was seeded so that a signal would result from the surface only. Simultaneous measurements were made of the free surface elevation a few millimeters away. The comparison of velocity and displacement time series shows that there are large random velocity excursions in breaking waves.

Klinting and Jakobsen[56], 1989, reported on a set of results from the Danish Hydraulics Institute deep water test facility. This is $20 \times 30 \times 3$m and spectra with a peak period of 1 to 2.3 seconds were generated with significant wave heights of up to 0.25m. The waves were unidirectional and generated from JONSWAP spectra. An ultrasonic current meter was mounted on a moving frame controlled by a wave gauge so that the current meter remained a fixed distance below the measured height of the surface. This distance varied between 6 and 18cm, depending on the test. The conclusions were:

- for a real sea, linear theory predicts higher velocities than second order irregular theory.
- second order irregular theory works well below the troughs.
- crest velocities are overestimated when extrapolation is used.
- on replacing extrapolation with coordinate stretching (similar to Wheeler stretching) the crest velocities are underpredicted.
Wheeler stretching works well under crests especially for lower recording levels.

Tørum and Skjelbreia[113] and Skjelbreia et al [99], both 1989, wrote an initial and a more detailed report of a set of experiments at the Norwegian Hydrotechnical Laboratory in Trondheim. Velocity and elevation records of regular and irregular waves were made in a $33 \times 1.02 \times 1.8$ m flume. Velocity measurements were made at different elevations including some above the trough, using LDA. They found that LWT gave substantial overprediction of the velocity beneath a steep crest (measured at mean water level) while LWT's Wheeler stretching derivative (see chapter 2) was only out by a few percent. They also noted a rapid increase in horizontal velocity as the surface is approached and emphasised the importance of measurements made near the surface.

Kim, Randall Kraft and Boo[55], 1990 generated an extreme transient wave similar in form to one measured in hurricane Camille and a regular Stokes wave with the same overall characteristics. They found that 'due to particular asymmetries not present in the Stokes wave, the transient wave kinematics under the crest are shown to be much more severe above still water level and somewhat less severe below.' The experiments were carried out at Texas A& M University in a $37 \times 0.91 \times 1.22$ m flume.

Zhang, Randall and Spell[126], 1991, as stated above, measured a regular and a two-component wave field and compared the results to Wheeler stretching, linear extrapolation and a nonlinear numerical scheme. The two-component waves were generated by the separate generation of two regular wave trains, letting the lower frequency train overtake the higher frequency one. The periods and amplitudes were in the ratio 1:3 with the higher frequency having the higher amplitude. The waves were measured using wave gauges and LDA. The linear and Wheeler theories were calculated from the measured wave spectrum. In the regular wave case the cut-off frequency was chosen to exclude the second and higher harmonics. For
the two-component waves the cut-off frequency was just above double the higher frequency. In contrast to their findings in the regular wave case, they found that Wheeler stretching and linear theory proved to be more accurate for the two-component case, while linear extrapolation proved overestimated crest and trough velocities significantly. The authors therefore concluded that the most appropriate choices of spectrum depended on the bandwidth of the sea being measured but this author considers that their choice of filter cut-off frequency was probably just as important.

There have, of course, been many other experimental laboratory programs to measure different aspects of water wave kinematics. The above are just some of the recent papers with priority given to those measuring kinematics in or near the trough to crest region in deep or intermediate water depths (ie well away from the surf zone). Many other experimental programs have sought to measure different aspects of wave kinematics — for example wave induced drift currents have been measured by Russell and Orsorio [94], 1957 to the present day, Swan [109, 110]. Some of these will be discussed later.

The above papers demonstrate that different experimental apparatus, procedures and programs lead to different conclusions and interpretations. As a result there is no universally agreed method of calculating wave kinematics in every situation. Dean [20], in a review of the (1989) state of the art says that ‘existing capabilities to predict kinematics in the laboratory under regular wave conditions appear to be within the accuracy (≈ 5%) of inherent errors in the measurements.’ He goes on to say that the ‘predictive capability of irregular waves within the lab environment appears to be adequate except in the case where strong nonlinearities exist, such as near the free surface.’ As this area is of great interest he calls for ‘more accurate measurements and comparisons to theory,’ and the building of accurate data sets of near surface kinematics in extreme waves. The NATO Advanced Research Workshop (ARW) on Water Wave Kinematics [78] at which Dean was presenting had a working group on water wave kinematics. Its summary report stated that
'the quantity and variety of lab measurements available is significant but are still lacking in the important near surface and bottom zones.' The summary report recommended laboratory measurements be made on '2D regular waves and groups of waves for a larger range of parameters to verify analytical and numerical procedures,' as part of a larger program of experiments still needed in wave kinematics research.

1.3.2 Field Experiments

Field experiments are made on the full-scale 3-D flow and measurements are made in all the conditions a structure experiences in the measuring period. Conversely the probabilistic nature of the sea surface means that the conditions of interest may not occur during the time of recording. The instruments must be robust (yet still sometimes fail) so lack accuracy. Flow measurements are often difficult to calibrate and may drift from the calibrated value. Moreover they are often restricted to events below the lowest trough (so results must be extrapolated to the crest) and instruments are subject to biological fouling. The interaction of the structure with the flow is unknown. Often one component of the measurement is missing so the measurements may not be fully 3-D. It is reported that the Waverider buoy is unable to follow the surface of the most severe waves. Field experiments are also costly and time consuming.

Field Experiments and Results

Forristall, Ward, Borgman and Cardone [34] and Forristall, Ward, Cardone and Borgman [33] both 1978, reported on the results of wave kinematics measurements in tropical storm Delia. This occurred in the Gulf of Mexico in 1973 and was the first time wave kinematics had been measured in storm conditions. Three electromagnetic current meters were used, with the highest at 80%
of water depth, as well as a wave staff to record surface elevation. They found a considerable scatter between the measured velocities and those calculated from unidirectional wave theories. Stokes fifth theory was found to overpredict the measured crest velocities and underpredict trough velocities. Dean [20] comments that this is 'entirely consistent with the presence and phasing of a forced second order wave,' forced by radiation stress as proposed by Longuet-Higgins and Stewart [65].

Forristall et al found that the use of irregular unidirectional theories did not substantially improve comparisons but that a good fit could be obtained from a directional wave spectrum based on linear theory. Therefore an assumption of linearity is less damaging than one of uni-directionality.

Vartdal, Krogstad and Barstow [117], 1989 reported on the results from the 'Wave Direction Calibration Project' (WADIC) in the central North Sea. In this experiment velocity measurements were made in and below the splash zone by two types of current meter and wave height records were obtained from a laser array and from buoys. Wadic extends previously reported wave kinematics experiments to higher sea states, including one with very little directional spreading. They found that wave profiles from buoys are more symmetric than those derived from fixed platforms. Moreover the steepest waves do not always occur in the highest sea states. They also found that the sub-surface current meter results matched DGLWT reasonably well and those in the splash zone compared favourably with linear theory using conditional simulation.

Forristall, Gutierrez, Ward and Marshall [31] analyzed results taken from the Cognac Platform during Hurricane Frederic. Cognac is in the Gulf of Mexico in 1024 feet of water. Measurements were made of wind, waves, surface current and strain in two of the structural members. They calculated strain from Morison's Equation using variations of linear theory. The wave records were reconstructed from the electromagnetic current meter as earlier records had shown a linear rela-
relationship between wave height and electro-magnetic current meter readings. The Morison’s coefficients were taken from Rodenbusch and Kallstrom [93]. Results were used from two sets of measurements, one when there was a very small current and one when there was a significant current. In both cases kinematics were calculated using δ-stretching [93] with a 3-D spectrum, Wheeler stretching [124] with a 3-D and a 2-D spectrum and Wheeler stretching with a 2-D spectrum, then reduced by a factor of 0.8.

They found that for the low current case, wave forces from 3-D δ-stretching model compared reasonably well to the measured strain. Utilising 3-D Wheeler stretching for the kinematics resulted more in a shift in the mean value than in the range while using 2-D wheeler stretching only made a small change. Reducing the 2-D Wheeler kinematics by a factor of 0.8 clearly led to smaller results than measured. In the high current case, the δ-stretching wave force model fits the oscillations in force with time well but with a substantial DC bias. Utilising Wheeler stretching reduces the thrust under the crests so reducing the range of forces below that measured. Using 2-D waves with Wheeler stretching increased the range of forces from the 2-D case and changed the DC bias; the overall result was similar to the δ results. The current was then uncoupled from the waves by squaring the wave (3-D Wheeler) and current velocities before (instead of after) adding them. This produced forces which were clearly too low in this case.

The overall conclusion was that the 3-D δ-stretching with coupled currents produced the best directional wave force model. The use of 2-D Wheeler stretching was not as good but the disagreement was not enough to rule out the use of this model. Uncoupling waves and currents produced results that were too low. The mean thrust was underpredicted in all cases for the significant current case; increasing the current by an arbitrary 30% produced a better fit.

The working group on the measurement of wave kinematics from the NATO Advanced Research Workshop (ARW) on Water Wave Kinematics [78] states that
the 'quantity of field measurements is inadequate and the quality ... is low. Instrument reliability is very uncertain as instruments are often improperly calibrated.' Dean [20], from the same conference states that field kinematics can be predicted to within about 30%.

1.4 Justification, Aim and Summary of Work Performed.

The justification for the research presented here lies in the desire to determine the correct kinematics for any given situation offshore. These are needed in the determination of a more physically realistic model for the calculation of wave loads on offshore structures. At the moment there is no universally accepted method of calculation of wave kinematics in a given situation. As a consequence, all theories have to be calibrated against measurements.

Accurate experimental data has been obtained only rarely near the free surface, and it is in this region that velocities and free surface errors are greatest. Therefore this is the most important region in the calculation of wave-induced forces. Moreover it is the region where the differences between kinematics predictions from high order regular wave theories and from variations on linear theory should be most apparent. (The higher order theories seek to minimise the free surface boundary condition errors, while the latter aim to reduce errors in the free surface profile at the expense of increased errors in the free surface boundary condition.)

Therefore, the aim of this research became to test current design practice and to improve the current database by measuring accurately close to the free surface. The chosen method was to make a set of accurate measurements of wave kinematics on a number of different waves using the Particle Image Velocimetry (PIV) technique developed at Edinburgh [39, 37, 38, 41]. These measurements
concentrate on the crest-to-trough region and cover a number of different wave conditions, namely:

**Monochromatic, regular waves.** These are the simplest and most accurately defined waves that can be generated and examples have been measured at a number of different steepnesses.

**Simple two-component groups in a flume.** This is the easiest non-regular case and is the simplest way to compare the internal kinematics of different waves of the same height and period and to see how wave group behaviour varies from theory as wave steepness increases.
Chapter 2

Wave Theories

This chapter describes the main wave theories used in this thesis for the prediction of water wave kinematics, serves as a background to comparisons made with experimental data later in the thesis. It contains a section on the theoretical regions of applicability of the theories used.

A method of applying each of the theories to the measured wave information has been devised. Computer programs were obtained for implementing the higher order solutions from Professor J. Chaplin and Dr. G. Klopman, whose kindness is much appreciated. The author wrote the programs for calculating the velocity profiles from the other methods. The input for the linear order programs was the measured wave spect\textsuperscript{lin}, while the higher order solutions were determined from the measured wave height and period.

2.1 Governing Equations

The Navier-Stokes equations were derived over a period of 15 years (1827-1845) as the equations of motion of an incompressible, Newtonian fluid with constant, though not necessarily zero, viscosity. Their derivation can be found in standard
textbooks on fluid mechanics. They reduce to the following form when gravity is the only body force exerted:

\[
\begin{align*}
\frac{Du}{Dt} &= -\frac{1}{\rho} \frac{\partial (p + \rho gh)}{\partial x} + \nu \nabla^2 u \\
\frac{Dv}{Dt} &= -\frac{1}{\rho} \frac{\partial (p + \rho gh)}{\partial y} + \nu \nabla^2 v \\
\frac{Dw}{Dt} &= -\frac{1}{\rho} \frac{\partial (p + \rho gh)}{\partial z} + \nu \nabla^2 w
\end{align*}
\]  

(2.2)

where \(u, v, w\) represent velocity components in \(x, y\) and \(z\) directions with \(z\) vertical, \(t\) as time, \(\rho\) as density, \(p\) as pressure, \(\nu\) as kinematic viscosity and \(h\) as height above a horizontal datum line.

It was stated above that the fluid was assumed to be incompressible. The principle of continuity requires the conservation of mass expressed in the fact that the divergence of the velocity vector be zero, i.e.

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]  

(2.3)

The fluid can also be assumed to be irrotational, that is to say that the fluid elements undergo only translation and straining and no rotation. Mathematically this can be expressed as

\[
\omega_x = \omega_y = \omega_z = 0
\]  

(2.4)

where \(\omega_x, \omega_y\) and \(\omega_z\) are components of the rotation vector (and give a measure of the vorticity):

\[
\begin{align*}
\omega_x &= \frac{1}{2} \left\{ \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right\} \\
\omega_y &= \frac{1}{2} \left\{ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right\} \\
\omega_z &= \frac{1}{2} \left\{ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right\}
\end{align*}
\]

Thus, no generation or loss of vorticity is allowed — unlike in a real fluid where vorticity is generated, diffused, carried by convection and ultimately diffused by viscosity. If the flow is assumed to be irrotational then it is possible to define
a continuous, differentiable scalar function $\phi$, a function of position and time, such that the gradients of $\phi$ satisfy the irrotationality condition (equation 2.4) automatically. The gradients of the velocity potential, as $\phi$ is known, give the velocity at that time and place i.e.

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}$$

(2.5)

Moreover, substituting $\phi$ into the continuity equation 2.3 results in a second order linear differential equation which is known as the Laplace equation:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

(2.6)

The Laplace Equation is the governing field equation for linear theory. To find a solution for it requires knowledge of the appropriate boundary conditions. These are the bed condition and the free surface kinematic and dynamic boundary conditions. The bed condition assumes that you are dealing with a flat horizontal bottom boundary across which there is no flow. Then, to solve the Laplace equation for velocity potential, at the bed (at $z = -d$);

$$\frac{\partial \phi}{\partial z} = 0$$

(2.7)

At the free surface (air/water interface) the kinematic boundary condition states that any particle at the free surface ($z = \eta(x, y, t)$) will not leave it. Mathematically this can be represented by:

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y}$$

(2.8)

The dynamic boundary condition states that the pressure difference across the free surface results in a force normal to the boundary, due to surface tension:

$$p = p_a + \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

(2.9)

where $\sigma =$ surface tension and $R_1$ and $R_2$ are radii of curvature of the surface in any two orthonormal directions. Taking $\sigma = 0$ for the moment and using
the condition of irrotationality enables the pressure to be determined from the Bernoulli equation:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \left\{ (\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 + (\frac{\partial \phi}{\partial z})^2 \right\} + g\eta = F(t), \text{ for } z = \eta \quad (2.10)$$

This equation may be derived from the unsteady Navier-Stokes equation, assuming irrotational flow and zero kinematic viscosity, expressing velocities in terms of the velocity potential and integrating with respect to the spatial coordinate.

Therefore the boundary conditions, equations 2.7, 2.8 and 2.9, are non-linear, containing products of derivatives of $\phi$ and $\eta$. The Laplace equation 2.6 is linear but is difficult to solve because of its boundary conditions. The most common method of solving it is to use some form of approximation.

### 2.2 Linear Wave Theory Solution

The equations of motion and governing field equations outlined in section 2.1 are most commonly solved using the assumption that the waves are 'small' (in height compared to water depth and wavelength). Solutions of this form are known as Linear Wave Theory (LWT) or Small Amplitude Wave Theory or Airy Wave Theory. A brief derivation of LWT is given below for the simple case of a regular wave train. This can be expanded upon to include many components from many directions using the principle of linear superposition. This can be applied as the non-linear terms, being of the order of the square of a small quantity, are assumed to be negligible.

Define a coordinate system $(x, y, z)$ with $x$ in the direction of wave propagation, $z$ vertically upward from still water level and $y$ orthogonal to them both. The waves are assumed to be in the $x$-$z$ plane, progressing in the positive $x$ direction over a smooth horizontal bed in water of depth $d$. Assume that there is no vorticity and there are no currents (though a uniform current can be added). Moreover,
assume that the waves maintain a constant form and there is no surface tension. Figure 2.1 is a definition diagram of such a wave, where $H$ is wave height, $\lambda$ is wave length, $k$ the associated wavenumber ($k = 2\pi/\lambda$) $\omega$ the circular frequency and $c$ the wave speed or celerity ($c = f\lambda = \omega k$).

The solution for $\phi$ is assumed to be a power series in terms of the non-dimensional perturbation parameter $\epsilon$, which is small and is defined in terms of the wave slope $\epsilon = \frac{1}{2}kH$. Then:

\[
\eta = \sum_{n=1}^{\infty} \epsilon^n \eta_n \tag{2.11}
\]

\[
\phi = \sum_{n=1}^{\infty} \epsilon^n \phi_n \tag{2.12}
\]

Laplace's equation then becomes:

\[
\epsilon \nabla^2 \phi_1 + \epsilon^2 \nabla^2 \phi_2 + \cdots + \epsilon^n \nabla^2 \phi_n + \cdots = 0 \tag{2.13}
\]

As the linear wave theory is a first-order solution, only the first term of the series for $\phi$ and $\eta$ are substituted in the free surface boundary conditions. The free surface boundary conditions may be solved for $z = 0$, as wave heights are assumed
to be so small. In this case they reduce to

\[ \frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial t} = 0 \text{ at } \eta = 0 \] (2.14)

\[ \frac{\partial \phi}{\partial t} + g\eta = 0 \text{ at } \eta = 0 \] (2.15)

These may then be combined to produce

\[ \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0 \] (2.16)

\[ \eta = -\frac{1}{g} \left( \frac{\partial \phi}{\partial t} \right)_{z=0} \] (2.17)

As the waves are periodic, with period \( T \), it follows that \( \phi(x, z, t) = \phi(x - cT, z, t) \)
so the solution can be obtained by a separation of variables technique with

\[ \phi = Z(z)\Phi(x - cT) \] (2.18)

Substituting this into the Laplace equation and determining the resultant constants from the boundary conditions then gives the velocity potential as

\[ \phi = \frac{gH}{2\omega} \frac{\cosh(k(z + d))}{\cosh kd} \sin \theta \] (2.19)

where \( \theta = (kx - \omega t) \) and \( \omega = 2\pi/T \). This shows that the velocity potential is periodic in \( x \) with a wavelength \( \lambda = 2\pi/k \), \( k \) being the wavenumber. Moreover, substituting \( \phi \) from equation 2.19 into the equation for the boundary condition 2.16 relates wavenumber to frequency through the linear dispersion relation:

\[ \omega^2 = gk \tanh kd \] (2.20)

This in turn gives the wave speed or celerity:

\[ c = \omega/k \Rightarrow c^2 = \frac{\omega^2}{k^2} = \frac{g}{k} \tanh kd \] (2.21)

Another concept of interest is that of group velocity \( c_G \) which is the speed of a group of waves (and the speed at which energy is transmitted) rather than the speed of the individual waves. Its value can be determined by considering a simple
group of two waves of equal amplitude and slightly different frequencies. These are added together, and the speed of the wave envelope is calculated giving a group velocity of:

\[
C_G = \frac{\partial \omega}{\partial k} = \frac{1}{2} \left[ 1 + \frac{2kd}{\sinh 2kd} \right]
\]  \hspace{1cm} (2.22)

In deep water this simplifies to \( C_G = \frac{1}{2} c \), for small \( A \).

### 2.2.1 Linear Theory Results

The velocity potential as determined in equation 2.19 can then be used to determine horizontal and vertical velocities (from the appropriate derivatives of \( \phi \)) accelerations (from the temporal derivatives of the velocities) and displacements (from integrating over the velocities). The resulting expressions are written using wave amplitude \( a \) rather than wave height \( H \), \( a = \frac{1}{2} H \), using a phase angle \( \theta = kx - \omega t + \chi \) measured from \( \theta = \chi \) at \( x = 0 \) at \( t = 0 \). The resulting equations are:

**Velocity Potential** \( \phi(x, z, t) = \frac{a \cosh k(z + d)}{\omega \cosh kd} \sin \theta \) \hspace{1cm} (2.23)

**Dispersion Relation** \( \omega^2 = gk \tanh kd \) \hspace{1cm} (2.24)

**Surface Elevation** \( \eta(x, t) = a \cos \theta \) \hspace{1cm} (2.25)

**Horizontal Velocity** \( u(x, z, t) = a\omega \frac{\cosh k(z + d)}{\sinh kd} \cos \theta \) \hspace{1cm} (2.26)

**Vertical Velocity** \( w(x, z, t) = a\omega \frac{\sinh k(z + d)}{\sinh kd} \sin \theta \) \hspace{1cm} (2.27)

**Horizontal Displacement** \( \xi(x, z, t) = -a \frac{\cosh k(z + d)}{\sinh kd} \sin \theta \) \hspace{1cm} (2.28)

**Vertical Displacement** \( \zeta(x, z, t) = a \frac{\sinh k(z + d)}{\sinh kd} \cos \theta \) \hspace{1cm} (2.29)

The ratio of hyperbolic functions in velocity potential, horizontal velocity, etc. is known as the depth decay term and determines how quickly the values decrease with depth. The symbols in table 2.1 are used in the equations for linear theory.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>$f$</td>
</tr>
<tr>
<td>Period</td>
<td>$T = 1/f$</td>
</tr>
<tr>
<td>Angular Frequency</td>
<td>$\omega = 2\pi f$</td>
</tr>
<tr>
<td>Amplitude</td>
<td>$a$</td>
</tr>
<tr>
<td>Water Depth</td>
<td>$d$</td>
</tr>
<tr>
<td>Elevation above still water level</td>
<td>$z$</td>
</tr>
<tr>
<td>Elevation above bed</td>
<td>$s = d + z$</td>
</tr>
<tr>
<td>Wavelength</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>Wavenumber</td>
<td>$k = 2\pi/\lambda$</td>
</tr>
<tr>
<td>Celerity (wave speed)</td>
<td>$c = \omega/k$</td>
</tr>
<tr>
<td>Group Velocity</td>
<td>$c_g$</td>
</tr>
<tr>
<td>Gravitational Acceleration</td>
<td>$g = 9.81$</td>
</tr>
<tr>
<td>Wave Phase Angle</td>
<td>$\theta = kx - \omega t$</td>
</tr>
<tr>
<td>Wave Slope</td>
<td>$\epsilon = ak$</td>
</tr>
<tr>
<td>Surface Tension</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>Water Density</td>
<td>$\rho$</td>
</tr>
</tbody>
</table>

Table 2.1: Water Wave Parameters

**2.2.2 Further Approximations to Linear Theory**

Further approximations can be made depending on the circumstances. These include deep and shallow water approximations, including surface tension and the use of many components linearly superimposed to model a non-regular sea-state.

Deep and Shallow Water Approximations can be derived for the appropriate ranges of water depth to wavelength ratio. This can be expressed by the value $kd(= 2\pi d/\lambda)$ which is a non-dimensional measure of wave steepness. For deep
water:

\[
kd \geq \pi \Rightarrow \begin{cases} 
\tanh kd \approx 1 \\
\sinh kd \approx \cosh kd \approx \frac{1}{2} e^{kd}
\end{cases}
\]

and substituting these into the equations for linear theory (2.23–2.29) above gives the deep water approximations in table 2.2.

<table>
<thead>
<tr>
<th>Deep Water Approximations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dispersion Relation</td>
</tr>
<tr>
<td>Surface Elevation</td>
</tr>
<tr>
<td>Horizontal Velocity</td>
</tr>
<tr>
<td>Vertical Velocity</td>
</tr>
</tbody>
</table>

Table 2.2: Deep water linear wave approximations.

Similar approximations can be made for the shallow water case where

\[
kd \leq \pi/10 \Rightarrow \begin{cases} 
\cosh kd \approx 1 \\
\sinh kd \approx \tanh kd \approx kd
\end{cases}
\]

but these will not be given as they are not generally of relevance to the waves measured in this thesis.

**Surface Tension** was assumed to be zero in the derivation of the dynamic boundary condition. If the surface tension term had been retained then the dispersion relation would have become:

\[
\omega^2 = gk(1 + \epsilon_\sigma) \tanh kd
\]  

(2.30)

where \( \epsilon_\sigma = \frac{\sigma k^2}{\rho g} \) gives the relative importance of surface tension and gravity. Typical values are \( \sigma \approx 0.074 \text{Nm}^{-1} \), \( \rho \approx 10^3 \text{kgm}^{-3} \) and \( g = 9.81 \text{ms}^{-2} \), giving \( \sigma/(\rho g) \approx 8 \times 10^{-6} \). (Note that \( \sigma \) varies with temperature and surface contamination.) Therefore,

\[
1 + \epsilon_\sigma \approx 1, \text{ unless } k > 10^2
\]
Surface tension should only be ignored if the latter condition is met. For a water depth \( d > 0.54 \text{m} \) (as used in the experiments presented here) all waves with \( k \gg 10^2 \) are deep water waves so \( \omega^2 = gk(1 + \epsilon) \). Linear theory should only be applied to a measured spectrum up to a cut-off frequency determined from the degree of accuracy required. Table 2.3 shows the percentage change in wavenumber between the case with surface tension and the case without for three frequencies.

<table>
<thead>
<tr>
<th>( f/\text{Hz} )</th>
<th>( k_t )</th>
<th>( k )</th>
<th>( \Delta% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.0</td>
<td>129</td>
<td>145</td>
<td>12</td>
</tr>
<tr>
<td>7.0</td>
<td>165</td>
<td>199</td>
<td>20</td>
</tr>
<tr>
<td>8.0</td>
<td>199</td>
<td>258</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 2.3: Wavenumber percentage change with surface tension.

These changes in \( k \) would have an effect on the values calculated for the water velocities. Therefore as the waves measured in this research have virtually no energy at all at 6Hz (determined from the measured spectra) a cut-off frequency of 6Hz is applied for the calculation of velocities and no account is taken of surface tension.

### 2.2.3 Linear Superposition: the Irregular Wave Case

One further important principle is used in determining velocities in non-regular cases; that of linear superposition. As this is a linear theory the linear sum of any combination of individual solutions to Laplace’s Equation is also a solution. Therefore an irregular 2 or 3-D sea can be modeled by adding together components until the surface profile matches the required one. This is a very useful attribute as the sea is 3-D and probabilistic and the most common record of actual sea
conditions is a wave height with time record. Such a record can be represented as an infinite sum of individual regular waves at closely spaced frequency intervals i.e.

$$
\eta(x, t) = \sum_{i=1}^{\infty} \eta_i(x, t) = \sum_{i=1}^{\infty} a_i \cos \theta_i \quad \text{and} \quad \phi_i = a_i x_i - \omega_i t \quad (2.31)
$$

where the symbols are as before but subscript \( i \) refers to the \( i \)th component wave. Performing a fourier transform on a wave height record produces output of this form. These results for wave amplitudes and phases can be translated into velocities and accelerations using the relationships derived earlier for the regular wave case. The values are calculated for the individual wave components, treating them as individual free waves, and then added to give the overall value. The resulting equations are given below, but now stating explicitly that \( \phi, u, w, \eta, \xi \) and \( \zeta \) are functions of position and time:

$$
\phi(x, z, t) = \sum_{i=1}^{\infty} \phi_i(x, z, t) = \sum_{i=1}^{\infty} g a_i \frac{\cosh k_i (z + d)}{\cosh k_i d} \sin \theta_i \quad (2.32)
$$

$$
\omega_i^2 = g k_i \tanh k_i d \quad (2.33)
$$

$$
\eta(x, t) = \sum_{i=1}^{\infty} \eta_i(x, t) = \sum_{i=1}^{\infty} a_i \cos \theta_i \quad (2.34)
$$

$$
u(x, z, t) = \sum_{i=1}^{\infty} u_i(x, z, t) = \sum_{i=1}^{\infty} a_i \omega_i \frac{\cosh k_i (z + d)}{\sinh k_i d} \cos \theta_i \quad (2.35)
$$

$$
w(x, z, t) = \sum_{i=1}^{\infty} w_i(x, z, t) = \sum_{i=1}^{\infty} a_i \omega_i \frac{\sinh k_i (z + d)}{\sinh k_i d} \sin \theta_i \quad (2.36)
$$

$$
\xi(x, z, t) = \sum_{i=1}^{\infty} \xi_i(x, z, t) = \sum_{i=1}^{\infty} -a_i \frac{\cosh k_i (z + d)}{\sinh k_i d} \sin \theta_i \quad (2.37)
$$

$$
\zeta(x, z, t) = \sum_{i=1}^{\infty} \zeta_i(x, z, t) = \sum_{i=1}^{\infty} a_i \frac{\sinh k_i (z + d)}{\sinh k_i d} \cos \theta_i \quad (2.38)
$$

The principle is illustrated in figure 2.2 For the case of 3-D seas (waves in any direction in the \( x-y \) plane) \( x \) is replaced by \( x \cos \alpha_i + y \sin \alpha_i \) when the \( i \)th wave makes an angle \( \alpha_i \) with the \( x \)-axis.
Figure 2.2: Linear superposition of surface elevations ($\eta_i$) and horizontal velocities ($u_i$) of two waves, 'a' and 'b'.

2.3 Stretching approximations to Linear Theory

Theoretically Linear Wave Theory is applicable only up to still water level. Performing a Taylor series expansion about still water level allows values to be calculated up to an elevation $z = \eta_i$ for the $i$th wave. This produces much the same result as merely using LWT with $0 < z < \eta_i$ which is what is done here above still water level. This procedure leads to a phenomenon known as *high frequency contamination* when there is more than one wave component present. Consider the case under the crest of a two-component sea state where the surface elevation is $\eta = \eta_1 + \eta_2$. If wave two is a small wave of high frequency then it may be said to be riding on top of wave one, possibly at an elevation greater than its own wavelength ($= e^{kz} > e^{2z}$). For high frequency, deep water waves the velocity components are proportional to $e^{kz}$ so at the surface $u_2 \propto e^{k\eta} \gg e^{k\eta_2}$. This leads to very high contributions from high frequency components in the wave crests where the waves are well outside their theoretical limits.

Apparent discrepancies between measured and predicted linear theory values lead
Wheeler [124] to suggest the first of several stretching approximations to linear theory. In general these theories violate the Laplace Equation 2.6 beyond first order but do not suffer from high frequency contamination. The most common of these approximations are detailed in the following sections.

### 2.3.1 Wheeler stretching

The stretching approximation of Wheeler [124] is a linear filtering technique introduced as a consequence of the results from wave tank studies. It avoids the problems of high frequency contamination by stretching the vertical coordinate $z$ such that the velocity value calculated for still water level previously (from equation 2.35) is now calculated for the surface. All other vertical coordinates are stretched or compressed from the bed accordingly, using the same transformation in $z$, namely:

$$z_w + d = (z + d) \frac{1}{1 + \eta/d} \equiv \beta(z + d)$$

(2.39)

$z_w + d = \beta(z + d)$ is an effective height which is always less than $d$ and has the same ratio to still water height as the actual height bears to the free surface. This has the effect of reducing velocities calculated in the crest and enhancing those found under the troughs of waves. Forristall [29] notes that using Wheeler’s method results in a lower kinematic boundary condition error than using linear wave theory.

Therefore, for example, the equation for horizontal velocity 2.35 becomes:

$$u(x, z, t) = \sum_{i=1}^{\infty} u_i(x, z_w, t) = \sum_{i=1}^{\infty} a_i \omega_i \frac{\cosh k_i(z + d)}{\sinh k_i d} \frac{\left(\frac{d}{d + \eta}\right)}{\cos \theta_i}$$

(2.40)

The subscript $w$ denotes calculation by Wheeler stretching. The corresponding expressions for vertical velocity etc. are derived from the standard equations (2.32 to 2.38) using the same substitution. The relationship between linear theory and both Wheeler and Chakrabarti stretching (see below) is illustrated in figure 2.3.
Figure 2.3: Comparison of variation in horizontal velocity with depth between linear \((u)\) and Wheeler \((u_w)\) theories (left) and Chakrabarti \((u_c)\) and Wheeler theories (right). All velocity profiles are for the same wave.

2.3.2 Chakrabarti stretching

A feature of linear wave theory is that the surface boundary conditions are satisfied at still water level. Therefore the expression for pressure is not valid for positive \(z\). Chakrabarti [13] reported that the pressure term can be changed so that the dynamic pressure satisfies the boundary condition completely. This is also the approach to kinematics taken by Lo and Dean [61]. It results in a velocity potential of:

\[
\phi_c(x, z, t) = \sum_{i=1}^{\infty} \frac{a_i}{\omega_i} \frac{\cosh k_i(z + d)}{\cosh k_i(\eta + d)} \sin \theta_i \tag{2.41}
\]

The transformation in this case is in the denominator of the depth decay term and is

\[
\cosh k_i d_c = \cosh k_i(\eta + d) \tag{2.42}
\]

This leads to an expression for the horizontal component of velocity \(u_C\) given by:

\[
u_c(x, z, t) = \sum_{i=1}^{\infty} a_i \omega_i \frac{\cosh k_i(z + d)}{\sinh k_i(\eta + d)} \cos \theta_i \tag{2.43}\]
2.3.3 Superposition Stretching

Pawsey and Dello Stritto [85] noted that while Wheeler stretching suppresses the extrapolation of high frequency components well above their region of theoretical applicability it also suppresses the extrapolation of the dominant wave in an irregular sea state above still water level. Such waves can be extended up to their own wave amplitude with little loss of accuracy, so it was proposed that instead of waves being stretched to still water level, each component could instead be stretched up to its instantaneous wave elevation. In other words instead of the elevations being altered in the ratio \( d/(d + \eta) \) in the calculation of the velocity potential, they could be altered in the ratio \((d + \eta_i)/(d + \eta)\). The transformation is illustrated in figure 2.4. They called the technique superposition stretching. The
transformation in the vertical coordinate is

\[ z_{ss} + d = (z + d) \frac{d + \eta_i}{d + \eta} \]  

(2.44)

where the subscript \( ss \) denotes superposition stretching. The velocity potential \( \phi_{ss} \) and horizontal component of velocity \( u_{ss} \) become:

\[
\phi_{ss}(x, z, t) = \sum_{i=1}^{\infty} \phi(x, z_{ss}, t) = \sum_{i=1}^{\infty} a_i \frac{\cosh k_i(z + d) \frac{d + \eta_i}{d + \eta}}{\cosh k_i d} \sin \theta_i
\]

(2.45)

\[
u_{ss}(x, z, t) = \sum_{i=1}^{\infty} u(x, z_{ss}, t) = \sum_{i=1}^{\infty} a_i \omega_i \frac{\cosh k_i(z + d) \frac{d + \eta_i}{d + \eta}}{\sinh k_i z} \cos \theta_i
\]

(2.46)

The corresponding expressions for vertical velocity etc. are derived from the standard equations (2.32–2.38) using the same transformation.

### 2.3.4 Extrapolation

Extrapolation is a modification of linear theory which assumes that the vertical partial derivative of a kinematic variable is a constant above the still water level and equal to the value given by linear theory at still water level. The horizontal component of velocity is then:

\[
u_{ex}(x, z, t) = u(x, 0, t) + z \frac{\partial u(x, 0, t)}{\partial z}, \text{ for } 0 < z < \eta
\]

(2.47)

A similar equation exists for the vertical component of velocity.

### 2.3.5 Delta stretching

Rodenbusch and Forristall [92] contest that “Stretching” and “extrapolation” provide a lower and an upper bound respectively for water velocities. They propose the adoption of an interpolation between the two, which they called delta-stretching (\( \delta \)-stretching).
This incorporates the following transformation for the vertical coordinate:

$$z_\Delta = \begin{cases} 
(z + d_\Delta) \frac{d_\Delta + \Delta \eta}{d_\Delta + \eta} - d_\Delta & \text{for } z > -d_\Delta \text{ and } \eta > 0 \\
\Delta & \text{otherwise}
\end{cases} \quad (2.48)$$

where $\Delta$ is the delta stretch parameter and $d_\Delta$ is the depth above which the kinematics are to be stretched. The velocities are then calculated by substituting the vertical coordinate $z_\Delta$ for $z$ according to the following criteria:

if $z_\Delta < 0$ then $z_\Delta$ is used in the equation for linear theory 2.35.

if $z_\Delta > 0$ then the kinematics are calculated by extrapolation.

It follows that, for example, if $\Delta = 0$ and $d_\Delta = d$ then the equations become those of Wheeler stretching. Moreover, if $\Delta = 1$ and $d_\Delta = d$ then pure extrapolation arises (as the equation for the coordinate transformation 2.48 collapses to $z_\Delta = z$ so linear theory is used for $z < 0$ and extrapolation is used above). Any other combination of $\Delta$ and $d_\Delta$ results in a velocity profile in three horizontal bands. In the lowest band, $z \leq -d_\Delta$ and the velocities are calculated from linear theory:

$$u_\Delta(z) = u_L(z) \quad (2.49)$$

where $u_\Delta(z)$ and $u_L(z)$ are the velocities calculated by delta stretching and linear theory at an elevation $z$ respectively.

In the middle band, $-d_\Delta \leq z \leq z_\Delta$, where $z_\Delta$ is the value of $z$ at which $z_\Delta = 0$ and the velocities are calculated by a stretching method, using the transformation in the $z$ coordinate given by 2.48:

$$u_\Delta(z) = u_L(z_\Delta) \quad (2.50)$$

It can easily be shown that $z_\Delta$ is given by

$$z_\Delta = \frac{d_\Delta \eta (1 - \Delta)}{d_\Delta + \Delta \eta} \quad (2.51)$$
In the upper band, \( z \geq z_{\Delta 0} \) and the velocities are calculated by extrapolation:

\[
        u_\delta(z) = u_L(0) + z_\Delta \frac{\partial u_L(0)}{\partial z_\Delta} \quad (2.52)
\]

\( \Delta \) and \( d_\Delta \) are empirical coefficients determined by comparison with data. Rodenbusch and Forristall used comparisons with field and laboratory tests [92] to give the following values:

\[
\Delta = 0.3 \quad \text{and} \quad d_\Delta = 2\sigma_n
\]

where \( d_\Delta = 2\sigma_n = \) significant wave displacement = 2\( \times \) standard deviation of free surface elevation below mean water level. Note also that stretching is applied only to crests (\( \eta > 0 \)) as velocities at a given height below still water level are reported to be higher under troughs than crests [30]. The authors point out that with the above definition the depth \( d_\Delta \) is small compared to many random wave heights and that the equation 2.48 becomes ill-behaved if \( \eta < d_\Delta \).

### 2.4 Dean’s Stream Function

In 1965 Dean [18] presented a stream function representation of a nonlinear gravity wave. The stream function \( \psi \) is a scalar quantity, describing both the geometry and the components of velocity at a point, as well as the flow rate between any two streamlines in a 2-D flow. The velocity components are given by:

\[
        u = \frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{\partial \psi}{\partial x} \quad (2.53)
\]

and the definition does not require that the flow is irrotational. If, however, it is irrotational then the Laplace Equation for the stream function is formed:

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \nabla^2 \psi = 0 \quad (2.54)
\]

and the stream function is orthogonal to the potential function.
2.4.1 Dean’s Solution

Deans solution was defined so that \( \psi \) solved the Laplace equation 2.54 and the bottom boundary condition. The flow was assumed to be incompressible and the water depth was considered uniform. The reference frame used moved with the wave speed or celerity \( c \) so the problem becomes one of steady flow. The stream function \( \psi(x, z) \) then gives:

\[
\frac{\partial \psi}{\partial z} = u - c \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -w
\]  

(2.55)

The five conditions to be met as stated by Chaplin are:

1. For irrotationality: \( \nabla^2 \psi = 0 \) everywhere.

2. At the sea bed: \( \frac{\partial \psi}{\partial z} = 0 \) at \( z = -d \) (no flow through bed).

3. Kinematic free surface boundary condition: \( z = \eta(x) \) is a flow boundary so the local velocity vector is tangential to it \( \frac{\partial n}{\partial x} = w/(u - c) \).

4. Dynamic free surface boundary condition: the pressure is zero at the free surface so Bernoulli’s equation 2.10 becomes \([ (u-c)^2 + w^2]/2g + \eta = R \) on \( z = \eta(x) \) where \( R \) is the total head and is a constant.

5. Wave is periodic in \( x \) with interval \( \lambda \) and is symmetrical about the plane through the crest or trough.

All the conditions except 3 and 4 are satisfied by:

\[
\psi = -\frac{\lambda}{T} z + \sum_{n=1}^{N} a_n \sinh \left( \frac{2\pi n (d + z)}{\lambda} \right) \cos \left( \frac{2\pi n x}{\lambda} \right)
\]  

(2.56)

Here \( N \) is the order of the solution. The third condition is met by identifying the free surface as a streamline. \( \psi_n \) is unknown and differs according to the total mass transport. Therefore the problem is to determine the unknowns such that the dynamic free surface boundary condition is satisfied. In Dean’s formulation
of the problem the parameters of $\psi, a_1$ and $\lambda$ were chosen by a numerical perturbation procedure to provide the best fit to the kinematic and dynamic boundary conditions. The disadvantage was that the wave height was not an independent parameter, but was obtained from an iterative procedure involving complete solutions for different values of the total head. In 1980 Chaplin [14] reformulated the stream function solution. He non-dimensionalised by taking $\lambda$ and $\psi_\eta$ as the length and stream function scale respectively. The non-dimensionalised parameters are:

$$
\Psi = \frac{\psi}{\psi_\eta} : X = \frac{x}{\lambda} : D = \frac{d}{\lambda} : S = \frac{z + d}{\lambda} : A = \frac{a}{\psi_\eta}
$$

(2.57)

The independent parameters are depth, wave height and period ($d, H$ and $T$) and the unknowns are $D$ and $S_1 \ldots S_J$, the surface elevations at a number of points distributed evenly in the horizontal direction between crest and trough. Since the wave height is specified it follows that $S_J = S_1 - HD/d$. Moreover as depth is a constant the remaining values of $S_j$ can be found by numerical integration involving a weighting term at each point. The solution is found by iteratively reducing the sum of the squares of the errors in the free surface boundary condition. The formulation is more complicated than Dean’s but results are obtained directly and as the unknowns are surface elevations the starting point for the iterations can be (and throughout this thesis is) linear theory. The input parameters for the first program are $d/\lambda_0$ and $H/\lambda_0$ (where $\lambda_0$ is the deep water linear theory wavelength $\lambda_0 = gT^2/(2\pi)$). This gives non scaled results of the correct profile which are then scaled in the second program by inputing the wave period $T$. Chaplin’s programs are now commercially available as ‘MSTR56.’

In aiming to assess his program Chaplin found that his solution converged up to 99% of limiting wave height at a relative depth of $d/\lambda_0 = 0.2$. Moreover it also gives maxima for total head and wavelength before the highest waves are reached. This agrees with the exact solution of Cokelet [15].
2.5 Rienecker and Fenton Fourier Approximation

The stream function solutions of Dean and Chaplin may be categorised as fourier approximation methods (being based on the use of truncated fourier expansions for field quantities). They assume an expansion of the stream function $\psi$ so as to satisfy the Laplace equation and the bed boundary equation and solve the resulting nonlinear equations for each of the fourier coefficients. Both methods generate hyperbolic functions in deep water and neither allows mass transport to be specified in the determination of wave speed. In effect the Eulerian mean velocity is set to zero and a non-zero mean mass transport condition will be set up. In a flume or tank with fixed ends the time mean mass transport will be zero.

In an effort to formulate a theory applicable at all depths Rienecker and Fenton [91] developed another numerical method based on fourier approximation techniques where the only approximation is in the truncation of the fourier series.

2.5.1 Rienecker and Fenton’s Solution

The solution is obtained for a 2-D periodic wave, symmetrical about the crest, propagating without change of form over a horizontal bed. The flow is considered irrotational and incompressible and the frame of reference moves with the same speed as the waves so the steady state solution is sought. The velocities are defined as for the stream function solution 2.55 as is the Laplace equation 2.54. All variables are non-dimensionalised with respect to mean water depth and gravity. Again the pressure is assumed constant at the free surface so the Bernoulli equation takes a similar form to that in Chaplin’s stream function solution. The boundary conditions to be satisfied at the bed and free surface are expressed as:

$$\psi(x, -d) = 0$$  \hspace{1cm} (2.58)
Here \( Q \) is a constant giving the volume rate of flow per unit length perpendicular to the \( x, z \) plane. Using symmetry about the crest allows the stream function \( \psi(x, z) \) to be written:

\[
\psi(x, z) = B_0(z + d) + \sum_{j=1}^{N} B_j \frac{\sinh jk(z + d)}{\cosh jk\gamma} \cos jkx \tag{2.60}
\]

The values of \( B_0 \) to \( B_N \) are constant for a given wave. The assumption that \( N \) is finite is the only approximation made in this method of solution (for irrotational, incompressible flow). An arbitrarily defined reference level \( \gamma \) has been introduced because of the exponential behaviour of the hyperbolic terms at high \( j \) values. Making \( \gamma \approx \eta + d \) removes the mathematical problems associated with the hyperbolic functions at high values and allows deep water waves to be studied.

Equation 2.60 can be substituted into the free surface boundary condition 2.59 and Bernoulli’s equation. The problem is then solved numerically by satisfying those two equations at \( N + 1 \) equally spaced points between crest and trough. The resulting \( 2N + 2 \) equations have \( 2N + 5 \) variables so other equations are necessary for a solution. These are expressions for the unit mean depth (unit as the nondimensionalisation was done with respect to mean water depth) wave height and celerity. The mean depth is found from the average value of surface elevation above the bed. The wave height is defined by \( H = \eta_0 - \eta_N \) as for Chaplin’s solution and the wave speed or celerity \( c \) is given by \( \lambda/T \) or \( 2\pi/kT \). Therefore \( kcT - 2\pi = 0 \). In this formulation it is possible to specify the assumption used in determining the wave celerity. For example, it is possible to specify the value of mean particle drift velocity, or mass transport velocity \( c_s \) which is appropriate \((c_s = 0 \text{ for flume tests})\). That allows \( c \) to be specified by \( c - c_s - Q = 0 \).

The \( 2N + 6 \) equations derived from the above form a closed system for the \( 2N + 5 \) unknown variables. This can be solved using Newton’s technique for the solutions of a set of nonlinear equations. The details of the solution are given in Rienecker and Fenton’s 1981 paper [91]. It was found that the number of iterations required
to approach the solution to a required degree of accuracy was independent of wave height. In some instances, for example above 95\% of the maximum deep water wave height, the method does not converge at all unless more fourier components are used. In all cases the fourier theory results are calculated from the measured wave height, period and water depth and from the mean mass transport (zero).

2.6 Regions of Applicability

The different wave theories outlined above are derived for different circumstances and sets of conditions. The question of which is the most accurate solution in a given set of circumstances is both important and difficult to answer. This decision can be made on the basis of comparison to experiments (as performed in the following chapters) or using theoretical considerations. Dean [19] has compared several wave theories theoretically by seeing which has the lowest free surface dynamic boundary condition errors for each of a series of regular waves. He chose to use the dynamic boundary condition, as the Laplace equation and the bed boundary conditions are linear and the kinematic boundary condition is always matched by the stream function solution. The results are shown in figure 2.5.

The process of determining which theory produces the lowest errors in the dynamic free-surface boundary condition errors does not necessarily determine which theory produces the lowest errors in wave kinematics predictions, however. Moreover, all theories are based on specific idealisations, often centring on Laplace’s equation thereby excluding viscosity, compressibility and vorticity. Hence even a highly accurate solution is only a good solution to a highly idealised wave condition.
Figure 2.5: Best-fit wave theory to dynamic free surface boundary condition (after Dean).
The Dean's stream function and the Rienecker and Fenton fourier approximation methods were the only higher order theories used out of a number of possible choices. The most common alternatives are the Stokes Fifth and Cnoidal theories. Cnoidal theory is derived for shallow water situations, outwith the range of relative depths used here, and is therefore ignored. The reasons for not using Stokes Theory are given below.

The Stokes Fifth solution assumes that the dependent variables can be represented as a fourier cosine series. The series is truncated at a finite order and the fourier components are truncated power series in wave steepness \((kH/2)\). The coefficients are found successively and analytically at each order from the free surface boundary conditions. Unfortunately the solution does not converge for very steep waves where the wave speed and energy are not monotonic functions of wave height.

The Rienecker and Fenton fourier approximation method starts from the same assumption as the Stokes Fifth and differs only in the final stage; the fourier coefficients being computed numerically to satisfy both the free surface boundary conditions and the dispersion relationship. This allows the solution to be obtained right up to the theoretical breaking limit of the waves and allows calculations to be made (in principle) to any order. It is for these reasons that the fourier method was chosen in preference to the Stokes Fifth. The theories are demonstrably correct within their ranges of validity and they are based on similar fundamentals, but the greater range of the fourier approximation and the fact that high waves were being measured lead to the adoption of the fourier technique.
Chapter 3

Experimental Apparatus for Fluid Kinematics Measurements

Two main measurement techniques were used in this thesis; Particle Image Velocimetry (PIV) for the measurement of wave velocities and wave gauges for the measurement of wave surface elevations. The techniques and apparatus used in these measurements are described in this chapter as is the wave flume in which the measurements were made.

3.1 Particle Image Velocimetry

In chapter 1 the role of wave kinematics research in the development of offshore design practice was discussed. In fact the general development of fluid dynamic theory has relied consistently on measurements for verification and to stimulate areas of research. Velocity is often the most useful measurable quantity and many devices, such as Pitot tubes or hot-wire anemometers have been used to provide point velocity measurements (see Wessels [123] for example for a review of modern laboratory techniques). Such measurements are comparative, however and must be calibrated which can lead to considerable uncertainty in the results. They
also intrude physically into the flow which disturbs the flow to a greater or lesser degree.

The development of Laser Doppler Anemometry (LDA) resulted in the first accurate, non-intrusive point flow measurement technique. Durst et al. [23] and Durrani and Greated [22] describe the technique in detail. In LDA coherent laser beams cross forming a small measurement volume (typically of order 1mm$^3$ for flume wave measurements). The optical interference there sets up a diffraction pattern and the light scattered off microscopic particles suspended in the flow is measured as the particles cross the diffraction pattern. The frequency of the signal gives the velocity, depending on the geometry of the beams.

The advantages of LDA are that it is non-obtrusive and accurate. The disadvantages are that it measures at one point only so that a flow must be accurately repeatable if the spatial structure of the flow pattern is to be determined. Other techniques such as streak photography were developed to help perform flow visualisations to determine the structure of the flow.

In all flow visualisations a second material (solid, liquid or gas) is added to the fluid as a tracer. The flow is photographed and the structure can be determined in a number of ways. A line drawn through all particles that have passed through a point in space is called a streak line and is most easily obtained by continuously injecting a tracer at one point and photographing using a short exposure. A path line can be determined by photographing a particle for a long time and is a Lagrangian measurement. Streamlines are lines tangential to the velocity vector at every point at an instant. They are more difficult to measure but this can be done by having many particles in the flow and photographing them for a short time so each particle image is a short streak. The direction of the streak gives the tangent to the velocity and these tangents may be joined to form a smooth curve. This form of streak photograph can be used to provide rough velocity information at a point and in a short time interval. The disadvantages of such
measurements are that they commonly require sophisticated analysis techniques (see Hesselink [46] for a review) and lack accuracy. Moreover the flow should be predominantly two-dimensional in character as information about motion towards the camera is not obtained (unless a stereo-photography technique is used). The illumination is commonly provided by an intense, thin sheet of light.

The desire to achieve a greater degree of accuracy from full field measurements was the driving force behind the development of what became known as Particle Image Velocimetry or PIV.

3.1.1 The Development of the PIV technique

The breakthrough in combining the accuracy of LDA with the multi-point nature of flow visualisation came in 1977 when Scattered Light Speckle Photography [26, 5] was first applied to the measurement of flow velocities in papers by Barker and Fourney [6], Dudderar and Simpkins [21] and Grousson and Mallick [42].

Scattered light speckle photography was derived for rough solid surfaces. These are illuminated by laser light which scatters off many points. The random phase relationship of the light creates a speckle pattern when photographed. If the object is moved slightly and photographed again then the speckle pattern will have moved on the film and a 'specklegram' is formed. This can be analyzed quantitatively in one of two ways; either by the repeated analysis of small areas of the film using a small laser (the Young's fringe technique outlined by Burch and Tokarski [11]) or by a full-field spatial filtering technique described by Celaya [12].

In the initial applications of this method to a fluid, the flow was heavily seeded with reflecting particles, and illuminated by a double pulse of laser light (which was usually expanded into a sheet by the use of cylindrical lenses). This light sheet is in the plane of motion of the flow, which should have low velocity components
Figure 3.1: PIV Illumination system as seen from camera (top left) and plan view (bottom right)

out of this plane. The light sheet is photographed so that both images of the seeding particles are imaged on the photographic negative. The resulting full field, double exposure negative can then be analyzed by the methods mentioned above. (Further details of the above can be obtained in the following sections.) Figure 3.1 shows a PIV illumination system of this sort.

In the original papers [6, 21, 42] the seeding concentration was so high that a speckle pattern was generated. The range of velocities that could be measured was constrained by the speckle size and the need to avoid decorrelation between successive speckle patterns. Moreover any out of plane motion leads to decorrelation of the speckle pattern. This problem is reduced by lowering the seeding density so that individual particle images are formed on the film. The new seeding density regime lead to a new name — Particle Image Velocimetry — for the technique.
The two main analysis methods still apply to the lower seeding density regime. In the analysis of small areas method (the Young's fringe method) a laser beam is shone through a small area of the photographic negative. The pattern of seeding particles is the superposition of a random distribution of seeding particles plus their corresponding images displaced by the flow in the time $\tau$ between the two illuminating pulses. If the flow is sufficiently regular each particle displacement will be approximately equal and the laser beam generates a dominating set of Young's fringes (determined by the most common separation) in the back focal plane of a converging lens. The spacing and orientation of the fringes is proportional to the spacing and orientation of the particle displacements from which the velocity can be calculated using the photographic magnification $M$ and the time $\tau$. A velocity map can be generated by repeating the above over a grid of points covering the negative. A schematic diagram of the Young's fringe method is shown in figure 3.2. This is the method used for all PIV analysis here.

Iwata et al. [50] used multiple exposures (> 2) and resolved particle images to improve the signal to noise ratio. The use of multiple exposures is now widespread, with a balance being struck between time between exposures, number of exposures and the total length of time the camera shutter must be open. For a steady flow, the exposure time can be long, containing many exposures but for a non-steady
flow the length of exposure should be small, relative to the timescale of the flow structure, or detail will be lost.

The principles of PIV remain the same to this day, although the methods used have grown in sophistication. The following subsections outline some of the advances made and the equipment used in this thesis.

Directional Ambiguity

It is impossible to determine the direction of the flow from a PIV negative, as all seeding images are similar. In many flows this is not a problem as a priori information can be used to determine the direction of the flow. This is the case for the water waves measured. The problem can be solved without a priori information if necessary [2, 17, 27, 1, 69].

Other Analysis Methods

The Young's fringe method of analysis was used for all the measurements. Other means of analysing PIV negatives exist and include using two optical fourier transforms [16]. Some methods work directly from the image plane and these are reviewed by Moriatis [1]. The full-field method [12] produces velocity contours through the use of a spatial filter in the fourier plane of the negative.

Good reviews of the PIV technique can be found in Adrian [3, 53, 54], Gray [36] or Lourenco [66] where the development of techniques is discussed in greater detail.
3.1.2 A Description of the Photographic Apparatus

PIV was first applied to the study of water waves by Gray [39, 37]. Studies using the technique have been carried out on breaking waves by Skyner [100, 101], on groups of waves by Sutherland [108, 106] and on waves on beaches by Quinn [87]. The PIV apparatus is shown in figures 3.3 and 3.4. Figure 3.3 shows a general view of the laboratory during a PIV experiment. The flume can be seen on the left, with the measurement area (actually a volume due to the finite width of the laser beam) illuminated by the blue/green light of the laser. The parabolic mirror of the scanning system can be seen below it and the camera facing it.

The Acorn Archimedes micro-computer which controls the wave paddle and triggers the camera can also be seen and the beam path can be traced back under
Figure 3.4: Scanning beam PIV apparatus
the flume to the laser on the table on the right hand side. The photograph in figure 3.4 shows the scanning beam illumination system.

**Camera, Lens and Film**

The PIV photographs were taken with a Hasselblad 500EL/M camera with a Zeiss Planar CF 80mm f/2.8 flat focus lens. This was chosen to minimise distortion. The film used was T MAX 100 which is of 120 format (i.e. negative size = 56.5 x 56.5mm²). This film is rated 100 ASA and has a high resolution of 200 lines/mm. As such it produces an image with a resolution of \( \approx 11,300 \times 11,300 \) points (compared to a video or ccd camera with a resolution of up to \( 2048 \times 2048 \) points). The shutter was triggered (via a relay switch) by the Acorn Archimedes computer which controlled the wave generation and sampled the wave gauge signal.

The above factors make the combination of lens, camera and film ideally suited to the full-field measurement of an area as it can then be analysed accurately, using a relatively small interrogation spot area, over a large grid of points.

The optical axis of the camera is lined up on a small reference marker on the side of the flume at the mean water level. This is done as the crest velocities are of most interest here, image distortions are smallest near the optical axis and velocities near the crest are difficult to measure due to reflections off the underside of the wave free surface. The effects are reduced by having the camera close to the surface.

The shutter speed was chosen so that multiple exposures (> 2) were taken. This has the effect of sharpening the Young's fringes due to the increased particle image pair density and the increased correlation between pairs separated by the flow. The choice of shutter speed balances the desire for multiple images with the preferred scan period (see following section) and the need to 'freeze' the wave.
motion (to limit velocity gradients in the analysis region). A shutter speed of 1/60s was chosen with a typical scan period of 4 – 5ms to ensure 3 or 4 images of each particle.

Laser and Scanning Beam System

The laser used is a Spectra Physics 15W CW Argon-Ion laser, model 171. The wavelengths produced range from 454–515nm and CW stands for ‘continuous wave’, i.e. the laser is not pulsed, but produces a continuous beam. This is reflected off special high-power laser mirrors onto the scanning beam system.

The purpose of the illumination is to produce multiple well-exposed, sharp images of the seeding particles. Therefore the illumination must occur for a short period of time (relative to the period between exposures) so that the motion of the particles is frozen and the displacement between particles is large compared to the size of the particles. This necessitates the use of a high powered light source, such as the laser used here. The coherence of the light is not used, only its intensity and collimation.

The original PIV results were made using a beam expanded into a sheet so that an area was covered and the beam often chopped mechanically to produce the pulses. The scanning beam system of Gray [36, 38, 40], as seen in figure 3.3 and shown below in figure 3.5, was used to optimise the use of the light. In it the laser beam is not expanded but is reflected off an octagonal rotating mirror onto a parabolic mirror and up into the flow. As the octagonal mirror rotates the beam is swept along the parabolic mirror. Also, as the octagonal mirror is at the focus of the parabola, the beam reflected off the parabola at one point is parallel to the beam reflected off the parabola, at any other point. Here the direction of the beam off the parabolic mirror is vertically up through the glass base of the flume. This provides a non-divergent scan through the measurement region.
In this system, the rotating mirror is turned by its own motor, controlled by an adjustable control circuit. The scan rate was measured by placing a photodiode at the end of the parabolic mirror and detecting its signal using a Thurlby Digital Storage Analyser and a 20MHz CRO. This allowed the signal to be stored and measured to an accuracy of around 0.2%.

The choice of scanning rate is vital in the production of high quality PIV negatives, as the dynamic range that can be resolved is limited by the analysis system. The ratio of maximum over minimum measurable velocities is just under 10. Therefore the scan rate must be adjusted to try to make the maximum velocities produce the maximum separation measurable on the negative (about 0.25mm). This is done initially by using a higher order theory to estimate the maximum wave velocities then adjusting the scan rate according to the photographic magnification. The scan rate may be altered depending on the results of the first run of the experiment.

One of the advantages of the scanning-beam system over the expanded sheet is
an increase in the percentage of available light used. Moreover, the exposure of
the seeding particles is increased on average by a factor equivalent to the ratio
of scanning period over pulse duration (length of time it takes for the beam to
pass a point). The average pulse duration is given by the length of scan (width of
measurement area) divided by the beam width and is approximately 200 for the
system used. This means that much larger areas and/or higher velocities can be
measured with a scanning beam system than with an expanded beam using the
same laser.

The main disadvantages of the scanning beam technique are that a small system-
atic error is introduced due to the movement of the seeding and there is a variation
of pulse duration with position, affecting exposures and the size of the measured
velocities. In PIV it is the positions of particles at successive illuminations that
is recorded. The known time \(\tau\), however, is that between successive illuminations
of the same point. The small movements of the particles between illuminations
thus causes a small change in the time between illuminations. The variation in
pulse duration is due to the fact that the constant angular rotation of the octag-
onal mirror does not translate into a constant horizontal scanning velocity. This
affects the systematic error and the exposure of the particles.

In Gray et al. [40] the error caused by the particle displacement is calculated using
the average scan velocity (horizontal velocity of the beam through the measure-
ment region). The modified expression for measured horizontal particle velocity
\(v_x\) is [40]:

\[
v_x = \frac{\Delta x}{\tau} \left( \frac{L}{L - \Delta x} \right) = \frac{\Delta x}{\tau} \left( \frac{\tau \overline{v}_x}{\tau \overline{v}_x - \Delta x} \right)
\]  

(3.61)

where \(\Delta x\) is the measured horizontal particle displacement, \(L\) is the scan length
and \(\overline{v}_x\) the average scan velocity. The same paper also shows that the scan velocity
is not constant and derives the following expression which is proportional to the
velocity change in terms of the angle the beam is deflected off the rotating mirror:

\[
\frac{\partial x}{\partial \theta} = \frac{L}{1 + \cos \theta}
\]  

(3.62)
If the scan velocity $\frac{\partial x}{\partial t}$ at a position $x$ is derived explicitly in terms of the scan period $\tau$ and geometrical factors of the system (as in appendix A) it is shown to be:

$$\frac{\partial x}{\partial t} = \frac{2\pi(L^2 + x^2)}{N\tau L}$$

(3.63)

where $N$ is the number of faces in the rotating mirror and $x$ the horizontal position along the scan length (measured from $x = 0$ below the rotating mirror, hence $0 \leq x \leq L$).

When equation 3.63 for the scan velocity is substituted for the expression for the average scan velocity in equation 3.61, then the systematic error in the calculation of the horizontal velocity can be calculated and the expression for the measured velocity is shown (in appendix A) to be:

$$v_x = \frac{\Delta x}{\tau} \frac{1}{1 - \xi}$$

(3.64)

Here $\xi$ is given by:

$$\xi = \frac{\Delta x LN}{2\pi(L^2 + x^2)}$$

(3.65)

It can be seen that the measured velocity $v_x$ should be multiplied by the correction factor $(1 - \xi)^{-1}$. For any PIV system covering a reasonable area $\Delta x \ll L$ so $\xi$ is negligible. Here, for example, $L = 630\, mm$, $N = 8$ and $\Delta x < 3\, mm$ so $\xi \leq 0.006$. Therefore the maximum correction factor is 1.006 and the value in the centre of the PIV negatives is 1.005. This systematic error was considered to be too small to be worth calculating explicitly for each analysis point and so was treated as a relative error of $0.006/0.994$ in the calculation of the total analysis error (equation 3.67).

The variation in pulse duration with position affects the exposure of the seeding. There must be sufficient intensity in the laser beam to ensure adequate exposure
at the shortest pulse durations or the signal-to-noise ratio will decrease and points may not be measurable.

Seeding and the Measurement Volume

The flume is seeded with conifer pollen. These particles are approximately spherical, with a diameter of $\approx 70\mu m$. They are close to neutrally buoyant when wet, and take several minutes to drift to the surface when in the flume. This suggests that they will follow the fluid motions accurately and so are suitable for use as PIV seeding.

Conifer pollen is found to scatter the laser light reasonably well and provides clear images on the film. Moreover, the size is approximately the optimal value for the apparatus as set up. The photographic magnification is $\approx 0.08$ so the pollen images are $\approx 5\mu m$ in diameter which is the same as the minimum resolvable size of image on the film.

The illuminated measurement volume is situated in the center of the flume, parallel to the glass walls. It is approximately $630mm \times 3mm$ in horizontal cross section and extends up from the flume bed to the free surface. It is important that the plane of the measurement volume be the plane of motion of the flume, or there may be a non-negligible out-of-plane motion (where plane refers to the plane of the measurement volume). This leads to de-correlation of the output signal as seeding particles will enter and leave the measurement volume during exposure, sometimes forming one image only.

The photographic magnification is determined by placing a sheet of perspex marked with a rectangular grid in the flume in the location of the illuminated region. This is then photographed and the magnification determined from measuring the photographed grid. The camera must not be moved between taking
this photograph and the PIV. This takes into account the magnification effect caused by refraction at the water/glass and glass/air boundaries. The magnification increases slightly with increasing distance off the optical axis but this effect can be shown to be small [36, pages 62–65] and is ignored. The results of greatest interest are in the wave crests and the camera is positioned so that they are near the optical axis of the system.

3.1.3 A Description of the Analysis System

The analysis system used is based on the Young’s fringe technique. The previous description of this merely pointed out how to produce Young’s fringes (see diagram in previous section) and stated that the flow velocities could be determined from their separation and orientation. Much work has gone into producing accurate, efficient and quick methods of obtaining that information over a grid of points covering a PIV negative. A schematic diagram of the analysis system used here is shown in figure 3.6.

The PIV negative is mounted on two micro-translational stages, allowing horizontal and vertical movement which is controled by the analysis computer, an Acorn Archimedes 440. The PIV negative is analysed in small regions (defined by the size of the laser beam interrogation spot). After each spot is analysed the stages move the negative so that a new area is interrogated by the beam. The process is repeated over a grid of points until a velocity map is built up, covering the whole negative.

The interrogation is provided by a He-Ne laser with maximum power 0.5mW and wavelength $\lambda = 633\text{nm}$. This is shone through a pair of polarisers to control the beam power, and a spatial filter to limit the spatial noise in the beam. The resulting beam creates the interrogation spot on hitting the PIV negative. This is one focal length from the converging lens which is centred on the beam path.
and a further focal length from the CCD video camera, also on the beam path. There is a small optical stop in the center of the video lens to remove the DC (un-deviated) component of the beam.

The Young’s fringe diagram shows that the lens is used to perform a two-dimensional fourier transform on the light from the interrogation spot of the PIV negative. Each seeding particle image acts as a point source and hence each pair of images acts as a pair of coherent light sources and produces a set of Young’s fringes in the fourier plane (the back focal plane of the lens where the video camera is situated).

The random distribution of the seeding particles in the flow produces a random set of fringes which form speckle-type background noise. Each pair of successive images of the same point also produces a set of fringes related to the flow characteristics. If there are no great velocity gradients across the analysis region then the flow will have superimposed a similar separation between successive images of all the particles and each set of images will form similar fringes. This leads to the
reinforcement of that fringe pattern and the formation of a dominant set of fringes caused by the flow (as shown in figure 3.7). These fringes are perpendicular to the flow direction and their separation is inversely proportional to the magnitude of the seeding displacement.

The processing of this fringe information can be done automatically by a number of methods. The first methods to be developed [49, 52] were semi-automatic and involved the compression of the data into a 1-dimensional array using integration or a cylindrical lens. The autocorrelation of the array gives the fringe periodicity.

The system used here is a two-dimensional fourier transform technique first developed by Huntley (1986) [47]. Huntley's method involves a fully automated fringe analysis based on the two-dimensional fourier transformation of the fringe intensity distribution.

Here, the fringe pattern is digitised by a video camera and then averaged to a
Figure 3.8: PIV autocorrelation plane from a Young's fringe pattern.

64 × 64 array. An average diffraction halo, the average of 40 sets of fringe patterns from random positions and random angles throughout the flow, is then subtracted from it. This models the diffraction halo caused by single particle images. A numerical fourier transform is carried out on the resulting array [36]. The fourier transform of the fringe intensity equals the convolution of the amplitude of the interrogation-spot image field [53]. This results in an autocorrelation plane with a self-correlation peak, symmetrical and opposite displacement peaks and noise peaks. The effect of subtracting the diffraction halo is to approximately zero the self-correlation peak. This aids the resolution and detection of small particle separations [47]. The self-correlation peak is not usually completely removed so the central area of the self-correlation plane is set to zero. An example of an autocorrelation plane is shown in figure 3.8.

The dominant separation is then found by locating the displacement peaks and measuring their distance from the center (the function is symmetrical about its
centre). This is done by locating the position of the highest correlation value then calculating the centroid of the surrounding displacement peak.

Other forms of Young's fringe analysis occur but Huntley [48] has shown that the two-dimensional fourier transform method gives more accurate results than other methods used.

The results, in arbitrary displacement units, are converted into velocities by the analysis program using the photographic magnification, the scan period and a scale factor of the analysis system.

### 3.1.4 PIV Measurement Accuracy

The formula for calculating PIV velocities is

\[ v_{x,y} = \frac{CM}{(1 - \xi)\tau} s_{x,y} \]  

(3.66)

where \( v_{x,y} \) is the \( x \) (horizontal) or \( y \) (vertical) velocity component in the illumination plane, \( C \) is a scale factor of the system, \( M \) the photographic magnification, \( \xi \) a correction term derived earlier, \( \tau \) the scan period and \( s_{x,y} \) is the measured displacement peak location in the autocorrelation plane. An estimate of the error introduced by each term is given in the following paragraphs.

Each factor \( A \) has an error \( \sigma_A \) and a corresponding relative error \( \sigma_A/A \). The analysis rig scale factor \( C \) is estimated from analysing known measurements. It has a relative error of 0.01/11.16. The magnification \( M \) is obtained from measurements of a photograph of a known grid. Its value varies systematically with position and randomly due to measurement errors. A typical relative error is 1/200. The scan period \( \tau \) is determined using a digital storage analyser and so the measurement is accurate to around 1 part in 500.
The errors in $\xi$ depend on geometrical factors and $s_z$ and are taken to be small compared to $\xi$. As the term $(1 - \xi)$ is itself treated as one, $\xi$ can be treated as an uncertainty, with relative error $\xi/(1 - \xi) \leq 0.006/0.994$.

The uncertainty in the location of the displacement peak is the most difficult to judge. It involves seeding density, velocity gradients, optics (both recording and analysing), photographic grain noise, out-of-plane motion, sampling and random correlation noise among other factors. Gray [36, pages 93–113] and Quinn et al. [87] have investigated these effects for the measurement of waves in the flume used here and for the analysis rig used here. Systematic errors have been minimised by the careful choice of components and random ones investigated using a Monte Carlo simulation. An analysis of the problem by Keane and Adrian [53, 54] produces similar results.

There is a random error and a systematic error in determining $s_z$ and $s_y$, and they both depend on the velocity gradient (assuming that, as here, the other main parameters are set to values close to their optimal ones). The papers of Keane and Adrian [53] and Quinn et al. [87] are used to obtain values for the systematic error ($\sigma_{s_z}$) and the random error ($\sigma_{s_y}$) as percentages of the maximum measurable velocity, based on the highest measured velocity gradients. As the errors increase with velocity gradient the errors determined will be the maximum for any of the waves measured. The calculated values are 0.3% and 1.1% for random and systematic errors respectively.

The relative uncertainty in $v_x$ can be given by

$$
\frac{(\sigma_{v_x})^2}{v_x} = \left( \frac{\sigma_C}{C} \right)^2 + \left( \frac{\sigma_M}{M} \right)^2 + \left( \frac{\sigma_T}{T} \right)^2 + \left( \frac{\sigma_{s_x}}{s_x} \right)^2 + \left( \frac{\sigma_{s_y}}{s_y} \right)^2 + \left( \frac{\xi}{1 - \xi} \right)^2
$$

There is a similar equation for $v_y$. The error is taken to comprise a relative component of approximately 1% and an absolute error of about 1.1% of the maximum measurable velocity. The scan period is adjusted (section 3.1.2) so that the anticipated crest velocity equals the maximum measurable and the experiment is
repeated if the measured velocity is too high, so the maximum measured velocity is close to the upper limit. Therefore the absolute error may increase slightly but should be lower than about 1.3% of the highest measured velocity.

The total relative error increases as the velocities decrease, due to the increasing relative importance of the absolute error. The total error, however, decreases as velocities decrease and should always be less than 2% of the maximum measured velocity.

3.1.5 PIV output.

The output of a PIV experiment is a multi-exposure negative of a large flow area (here \(500\text{mm} \times 500\text{mm}\)). The raw output of the analysis of the negative is a set of results in a grid of points covering the negative. Each result comprises a pair of coordinates, horizontal and vertical velocity components and a measure of the displacement peak visibility at that point (a measure of the signal-to-noise ratio). The output then undergoes post-processing to remove spurious results. The processed output can then be used in comparisons with theories [106, 108, 101] or to look at flow structures [72, 73] and turbulence etc.

An example of a print of a PIV negative is given in figure 3.9. The surface elevation can be easily traced where the scanning laser hits the free surface. The menisci at either side of the flume can also be seen above and below this and the trace from the lower meniscus obscures some particle images.

The corresponding velocity map is shown in figure 3.10. The map is shown after post-processing of the PIV analysis output. The analysis system provides a result for every point on the analysis grid. Obviously this extends above the surface elevation at certain points on the grid and a spurious result is produced there when noise is measured. These results are removed by post-processing of the
data. The velocity map is displayed on the screen of the analysis computer. Data points from above the surface are removed by selecting them with the mouse and deleting. Other spurious points may be deleted in this way also. These may be caused by a lack of seeding or illumination or lack of correlation due to out-of-plane motion. They tend to have a very low signal-to-noise ratio and show no continuity in magnitude or direction with their immediate neighbours.

The second major element of the post-processing is the scaling of positions and velocities back to full scale. This is done using the measured photographic magnification, measured scan period and a scaling factor of the analysis rig. Errors in these values are discussed in section 3.1.4.

The horizontal and vertical coordinate magnification of the velocity map in figure 3.10 is not exactly the same as in the photograph but is similar. Some points are missing, after being deleted in the post-processing. The occurrence of some points like these is inevitable in a run of negatives due to the drift and dispersion of seeding and the random nature of seeding distribution.

In the analysis presented in the following chapters the majority of this data is, in fact, only used as a qualitative check on the results. All the quantitative comparisons are done on velocities beneath the wave crests. A subsection of the velocity results only are needed for this and a copy of the full field data is edited down to a few columns of data under the crest. The choice of which columns to retain was made by selecting the block at the centre of the negative where the horizontal crest velocities were highest and the vertical velocities lowest. The wave crests were at the centre of the negative due to the wave gauges iteration procedure (see section 3.2). The measured wave phases were close to zero, but were not exactly zero. Therefore a block of results is used so that the results at the gauge measurement point and the highest results could be presented together. The block is typically 2 to 4 columns wide, which corresponds to about 2 to 4% of a wavelength.
Figure 3.9: Print of a PIV negative
Figure 3.10: PIV velocity map
The results from a block of data from beneath the wave crest can be presented as a velocity profile beneath the wave crest. An example of this is shown in figure 3.11. There are a few points at each elevation as results from 2 to 4% of a wavelength are shown to cover any error in fitting PIV measurements to theoretical results. No attempt was made to fit error bars to the PIV results as the magnitude of the errors was so small at the crest. The spread in velocities due to using a block of results provided a suitable measure of the uncertainty in fitting theoretical profiles to the measurements, as the wave phases measured were not always exactly zero.

3.1.6 Application of PIV to the Surface of Water Waves.

One of the main advantages of PIV is to be able to measure velocities above the level of the wave troughs. There is a problem measuring in the immediate vicinity of the free surface, however, due to reflections off the free surface. As a check on
the validity of using PIV to estimate the surface velocities, some PIV experiments were done to measure the surface velocities of waves and comparisons were made with scanning-beam PIV results for the same wave.

The surface PIV was performed [107] using two pairs of camera flashguns to illuminate the seeded wave surface (see figure 3.12). The camera was mounted on a beam above the flume, with the optical axis of the camera pointing vertically down towards the center of the flume. The camera is focussed on the elevation of the wave crest to be measured. Seeding is provided by sprinkling dry conifer pollen onto the wave surface.
The micro-computer which controls the wave generation triggers the camera, which in turn triggers the delay box. The delay box sets off the first pair of flashguns (1A and 1B) simultaneously and then sets off the second pair (2A and 2B) a known time later, see figure 3.13. The time delay is measured using a photodiode connected to a CRO. This also checks that both flashguns in a pair go off together. The resulting double image PIV negative is analysed in the usual way. The results are less accurate than the results presented in the later chapters as the CRO was not as accurate, so there is a greater error in the velocities.

Moreover, as the photographed surface is not flat, results away from the wave crest (at the center of the negative) have a different magnification and a higher out-of-plane component than results at the crest. Nevertheless the results beneath a point are compared to the results beneath the same point using scanning-beam PIV in figure 3.14.

The scanning-beam results tend towards the value of the surface result to within the margin of error. Similar results are found for other points under the same
wave. Therefore it was decided that the results of the scanning-beam PIV would suffice to define the wave kinematics and no more surface PIV was performed.

3.2 Wave Gauges

Surface elevations were measured in the wave flume using resistance-type wave gauges. They determine elevation by sampling the resistance between two parallel metal conductors and depend on the difference in resistivity between water and air. As the length of conductor immersed in water changes, so the voltage across the conductors changes and this is measured by the microcomputer which also controls the wave generation. The change in voltage is taken to vary linearly with surface elevation so wave surface elevation is determined by subtracting the voltage given at the mean water line and multiplying by a calibration constant. A diagram of a wave gauge is shown in figure 3.15.
The wave surface must be between the two insulating blocks for a reading to be made. The secondary conductor is used to determine the conductivity of the water. The calibration factor and the voltage at mean water level are determined experimentally each time the gauges are used. The linearity of the gauges was tested experimentally and the results are shown in figure 3.16. The results show a linear relationship between voltage and elevation, to within experimental error.

The wave gauges (2 or 4 at a time) are used in an iterative scheme to achieve the desired sea state. A wave-group was generated, sampled and analysed (see following sections). The input parameters were then altered to ensure that the amplitudes of the two wave components were in the desired ratio and the phases were close to zero. The above process was repeated until the desired accuracy was achieved.
3.2.1 Wave Samples, Sampling Rate and Frequency Number $j$.

The output from the gauges is an elevation versus time record, samples at 40Hz. An example of such a record is given in figure 3.17.

![Figure 3.17: Wave surface elevation record.](image)

The sampling rate of 40Hz was chosen to give approximately 40 samples per wave, thereby allowing reasonably accurate determination of zero-crossing times, wave heights, etc. Moreover, the wave generation frequencies were limited to $f = j/25.6$Hz, where $j$ is an integer. This ensures that all possible wave states will repeat after 25.6s. At a sampling rate of 40Hz, the 1024 samples in a wave record will be taken in exactly 25.6s. Thus there will be an integer number of complete repetitions of the sea state in the sampling period.

The integer $j$ used in generating the wave ($f = j/25.6$Hz) is called the frequency number and is a convenient way to represent the frequency. It is easier to recall an integer than a real number and easier to calculate the relationship between two integers than two reals. For example, $j = 30$ corresponds to 1.172Hz and comparing $j = 45$ to $j = 30$ is easier than comparing 1.758Hz to 1.172Hz. Therefore the frequencies and frequency ranges used will often be referred to by their frequency numbers only. To get a frequency in Hertz divide by 25.6.
3.2.2 Wave Spectra

The 1024 sample wave gauge records are fourier transformed to produce wave amplitude spectra. The output of this process is an amplitude and phase at each of a series of discrete frequencies separated by 1/25.6Hz. Each one represents the energy contained within the frequency range of 1/25.6 centred on $f = j/25.6$. The phases are calculated for the start of the record and are measured from zero at a wave crest. An example of a wave amplitude spectrum is given in figure 3.18. The phases of the input waves and their harmonics are close to zero, but the rest of the phases are spread randomly so the phases are not shown, only amplitudes.

The spectra are calculated using two gauges to average the effect of small ripples and take the form of a series of high spikes at, or at a frequency associated with one of the input waves. The spectrum on the left shows a regular, one component wave at a frequency of 30/25.6Hz =1.17Hz. The lower peaks at double and triple that frequency represent the wave's bound harmonics. These are bound to the lower frequency wave and travel at its celerity. In the calculation of the linear theories each component is treated as a separate free wave, travelling at its own celerity, however. The spectrum on the right is of a two-component wave group and contains a greater number of significant components. This occurs as there are a greater number of frequencies related to the input (at twice the frequencies and
at the sum or difference of the two, for example, see section 5.2).

The wave amplitude spectra are used as the inputs for the linear and stretching theories described in chapter 2.

3.3 The Wave Flume

The experiments were all carried out in the short wave flume in the Physics Department of the University of Edinburgh. This is 6m long by 0.3m wide and has a mean water depth of 0.54m. The bed is flat and the walls and bed in the center 2m section are made out of 19mm thick glass to allow optical axis. The waves are generated at one end using a single hinged wavemaker and travel up the flume where they are absorbed by an expanded aluminium beach. A cross section of the flume is shown in figure 3.5 and it can be seen in figure 3.3.

The waves produced are two-dimensional in nature. The wavemaker used is of the single flap type produced by Edinburgh Designs [24]. The flap, or paddle pivots on a fabric hinge fixed to the bottom edge of the wavemaker box. The paddle is dry-backed, and has a sealed, waterproof, fabric gusset between it and the water.

The paddle is driven by a servo-motor via a drive belt and springs are used to offset the hydro-static force. A piezo-electric transducer is located between the drive belt and the paddle and measures the force on the paddle. Also the position is calculated from an encoder on the motor. These are used to optimise the paddle absorption. The paddle therefore acts as an absorber of reflected waves as well as a wave generator [96].

The absorbing beach is made from expanded aluminium mesh, held in a wedge-shaped metal framework. The horizontal cross-section of the wedge is almost triangular and is constant with elevation. This beach has a reflection coefficient
which decreases as wave amplitudes increase. The wave gauges can be used to split the spectra into incident and reflected components to determine reflection coefficients. This procedure (which relies on the assumption of wave linearity) breaks down in the most extreme cases and at high frequencies. Therefore it was used only to calculate a reflection coefficient at the wave generation frequencies; the spectra used were simply the average from two gauges.

The calculated wave reflection coefficient was found to decrease with increasing wave amplitude (keeping to non-breaking waves in the range 0.75–1.5Hz). The average values were just over 5.5% for amplitudes less than 20mm, 2.4% for amplitudes greater than 20mm and 1.6% for amplitudes greater than 30mm.

Any wave flume has certain resonance frequencies associated with waves which have an integer multiple of wavelengths equal to the length of the flume. These frequencies were calculated using linear theory and a number of spectra were checked to see if significant components were present at those frequencies. There were no significant tank resonances found on the wave spectra. Here 'significant' is defined as having an amplitude greater than 0.3mm.
Chapter 4

Monochromatic Waves

Chapter one included a brief review of some of the recent experiments conducted on the wave-induced kinematics of regular (i.e. single frequency) waves. The continued interest in this, the simplest case, reflects both the historical importance of the regular wave case as a form of bench-mark test and the interest in the continued use of regular wave kinematics in the design process. As a result of this continued interest and due to the fact that regular waves can be specified and reproduced relatively accurately, a limited selection of regular waves will be compared to theory in some detail in this chapter.

These waves are of the same relative height and depth as the groups of waves to be considered in chapter 5 and will, in fact, be used in comparison with waves of the same gross characteristics (of height and depth) in chapter 6. The concept of relative steepness \((H/gT^2)\) gives a measure of the height to wavelength ratio of a wave. This gives an approximate measure of how non-linear a wave can be expected to be. The theoretical breaking limit for a regular wave (the maximum steepness a wave can attain before becoming unstable) calculated by Stokes is \(H/gT^2 = 0.027\), at which point the surface profile forms two smooth curves, meeting at a point at the crest. The Stokes limit is still the most commonly applied breaking criterion in deep-water. Experiments in wave flumes give a
breaking criterion of approximately $\frac{H}{gT^2} = 0.021$ [89, 116] for a variety of regular and irregular waves. Breaking may be caused by higher order instabilities such as the Benjamin-Feir instability [8] or the presence of sub or super-harmonics. The relative depth of a wave $(d/gT^2)$ gives a measure of the depth to wavelength ratio. This helps to determine the relative validities of the different wave theories. Waves become deep when their wavelength becomes less than approximately half the water depth. At that point the decay of amplitude of the wave is so great that the wave induced motion is almost zero at the bed. The amplitude of the oscillations are approximately zero and deep water approximations may be used. This occurs at a relative depth given by $d/gT^2 \approx 0.08$ or $d/\lambda \approx \frac{1}{2}$. Waves become shallow when their depth becomes less than about 5% of the wavelength (i.e. $d/\lambda \approx \frac{1}{20}$). This occurs at a relative depth of approximately $d/gT^2 = 0.0025$. The waves between these two limits are described as being intermediate in depth. All the waves looked at here are intermediate to deep in character.

4.1 Wave Data and Statistics

The waves used in the comparisons in this chapter have the following attributes listed in figure 4.1.

<table>
<thead>
<tr>
<th>$H$ (m)</th>
<th>$T$ (s)</th>
<th>$\sigma_T$ (s)</th>
<th>$j$</th>
<th>$H/gT^2$</th>
<th>$d/gT^2$</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0714</td>
<td>0.84</td>
<td>0.01</td>
<td>30</td>
<td>0.0103</td>
<td>0.0782</td>
<td>e9n9</td>
</tr>
<tr>
<td>0.0874</td>
<td>0.854</td>
<td>0.003</td>
<td>30</td>
<td>0.0123</td>
<td>0.0760</td>
<td>e8n7</td>
</tr>
<tr>
<td>0.0957</td>
<td>0.855</td>
<td>0.005</td>
<td>30</td>
<td>0.0134</td>
<td>0.0753</td>
<td>e9n8</td>
</tr>
<tr>
<td>0.1121</td>
<td>0.85</td>
<td>0.02</td>
<td>30</td>
<td>0.0159</td>
<td>0.0764</td>
<td>e8n9</td>
</tr>
</tbody>
</table>

Table 4.1: Regular wave attributes

Here $\sigma_T$ is the standard deviation in the wave period, calculated from approxi-
approximately 25 waves. ‘Ref’ is the experiment reference number, and \( j \) is the frequency number for generating the wave. All waves are generated at frequencies given by \( f = j/25.6 \) Hz. The surface profiles of the waves are shown in figure 4.1.

The regularity of the waves can be gauged from the small values of the standard deviation in the values of the periods. In each case (and henceforth, unless otherwise stated) the heights and periods are ‘zero down crossing’ values. In other words the start of each wavelength in an elevation versus time record is the point at which the surface elevation decreases through zero.

The surface elevation records are used to produce the wave amplitude spectra shown in figure 4.2. The amplitudes are rounded to the nearest 0.1 mm and the frequency range shown is 0-4 Hz. There is no significant energy present above this frequency so the upper frequency cut-off was put at this level. The wave gauges were designed to operate best at around 1 Hz, however, so the performance at 1 Hz is not optimal. Therefore in most cases the frequency cut-off was put at
2.9Hz (corresponding to a frequency of 75/25.6Hz or 2.5× the wave generation frequency). The effect of the choice of upper and lower frequency cut-offs will be investigated in section 4.3. The choice was made on the basis of comparisons with wave group results as well as regular wave results.

4.2 Frequency and Amplitude effects

The amplitude spectra displayed in figure 4.2 show the expected form; namely a dominant amplitude component at the wave generation frequency (known as the fundamental component or first order harmonic) and smaller components at the higher harmonics. These harmonics are bound to the primary component, and hence travel at its celerity. In the calculations of linear theory and its derivatives the components are not considered to be bound, however.
In the use of linear theory and other first-order theories, a choice has to be made concerning the minimum size of amplitude component to be utilised in the calculations. There is an amplitude cut-off, below which all amplitudes are ignored in the calculations of surface elevations and velocities. This value has been set to 0.3mm in each case, to remove the effects of low-amplitude noise from the calculations. The effect of altering the low-amplitude cut-off value can be seen in figure 4.3 where linear and Wheeler stretching theories are calculated in the range \( j = 1 - 75 \) \((f = j/25.6)\) using minimum cut-offs of 0.1, 0.3 and 0.5mm.

The linear and Wheeler stretching velocity profiles are identical for the cases of the amplitude limits of 0.3 and 0.5mm. This occurs as there is no wave component with an amplitude between the two limits in the example given and the level of the signal noise is lower than 0.3mm. In each case the 0.1mm limit gave higher results for a given elevation due to the presence of noise at the lower level. The 0.3mm filtering level was used throughout the thesis.

In comparing the contribution of the harmonics to their frequency ranges the velocity profile resulting from the harmonic only was calculated and compared to the velocity profile of the frequency range about it. The primary component \((j=30)\) velocity profile was compared to that of the range \( j=15 \) to 45 and the second harmonic \((j = 60)\) profile to that of the range \( j = 46-75 \). In all cases \( f = j/25.6 \text{Hz}, \) as usual. In every case the harmonic dominated the velocity profile to such an extent that the lines for the harmonic and the range overlapped. In each case the primary component and its range form the higher value curves to the right and the harmonic contributes the two lower value curves. This result (repeated for all the regular waves) highlights the dominance of the harmonic components in determining the velocity profile of a regular wave.

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Figure 4.3: The effect of varying the minimum amplitude used in calculating velocity profiles under the crest for linear (top) and Wheeler stretching theories. Also (right) the relative contribution of the harmonics to their frequency ranges for the same theories (amplitude limit = 0.3mm).
4.3 Comparison with Linear Theory

Linear theory is derived for waves of infinitesimal amplitude and, as such, is not strictly applicable to the waves being measured. It is still used in many circumstances due to the ease of use (particularly the relative simplicity of the computing required) and to its ability to model irregular sea states. In particular, directional gaussian linear wave theory (DGLWT) is the favoured technique of many offshore engineers modelling free-dimensional sea-states. Moreover it is the basis for the many stretching theories outlined in chapter 2, so the results it produces will be examined in some detail in this section. In all cases an upper velocity cut-off has been set at a value of 0.7m/s so nothing calculated beyond the limit of the axes is shown. The theoretical curves for each frequency range of a given wave all extend up to the common surface elevation.

The results for the four waves detailed in section 4.1 are given in figure 4.4. In each of the first four cases 'a' to 'd' the measurements of horizontal velocity taken under the crest of the wave are compared to the velocity profiles calculated from the measured spectra over different frequency ranges, using linear theory. The wave generation frequency \( f_{30} \) was given by \( j = 30 \) or \( f_{30} = 30/25.6 \text{Hz} \) as before so the range \( j=1-15 \) corresponds to \( f < 15/25.6 = f_{30}/2 \) and is designed to determine the extent of the low-frequency signal detected. In each case the low frequency contribution was negligible (no more than 2mm/s) and was obscured by the vertical axis.

The frequency range \( j=1-45 \) corresponds to \( f \leq 45/25.6 \text{Hz} \) or \( f \leq \frac{3}{2}f_{30} \) which incorporates the fundamental component but not the second harmonic. The results match the form of the experimental results quite well, a result which shadows those of Gudmestad [44] and Zhang, Randall and Spell [126]. In each case only the energy around the peak frequency was considered and the form of the results matched the measurements to within a few percent. Here the theoretical veloc-
Figure 4.4: Linear theory velocity profiles compared to measurements under the crests of regular waves.
It should be pointed out that only the results from above the elevation -0.2m are displayed on the graphs whereas the mean water depth of the tank is 0.54m. This approach is justified on the basis that it is the highest velocities that are of greatest interest for the calculation of drag-dominated forces and these occur near the crest. Moreover, the PIV system has a limited dynamic range and is set up to work best at the velocities that occur near the crest. As a result the crest is emphasised. In each case the variation between experimental results and theories shows no significant dependence on elevation, indicating the presence of a low frequency or d.c. effect.

The situation in a closed wave flume corresponds to Stokes second definition of wave celerity where the wave travels at such a speed that the average horizontal momentum is zero (there is zero mass transport). It is common, however, to calculate wave theories on the basis of Stokes' first definition of celerity. This states that the wave travels at a speed such that the average horizontal particle velocity is zero (there is a zero mean eulerian velocity). Traditionally the first definition was considered to be valid in the open sea and the second in closed flumes. The return current necessary to ensure that there was no mass transport was considered to be a tank effect.

Recent experiments [44, 51] have shown that the return current evolves immediately under the waves, as the waves travel into calm water and before there can be any reflections from the far (beach) end of the flume. This shows that the effect is not due to the finite length of the tank but is caused by the wave itself. Gudmestad et al. [44] suggest that it can be explained by radiation stress.

Therefore, if a wave measurement made in a closed flume is compared to a wave theory calculated using Stokes first definition of celerity then a discrepancy is introduced. In the case of linear theory, the two definitions of celerity produce the same result. Therefore there is no theoretical alteration that can be made to
the results to take account of the apparent presence of a return current without applying a higher order correction.

The frequency range \( j=1-75 \) corresponds to \( f \leq 75/25.611 \text{Hz} \) or \( f \leq \frac{5}{4} f_{30} \). This incorporates both the fundamental component and the second harmonic and all components up to \( 2.93 \text{Hz} \). The results match those of the range \( j=1-45 \) up to approximately -0.15m. At this point the higher frequency velocity component, caused almost entirely by the second harmonic component (see section 4.2) becomes significant. This leads to substantial overprediction of the measured velocities at mean water level (by 30% for the lowest wave to 55% for the highest) and considerably greater overprediction near the free surface. Such levels of over-prediction are clearly not acceptable (particularly as the drag term in Morison’s equation is proportional to velocity squared). This triggered the development of the various stretching techniques which reduce the extent of the overprediction whilst allowing the use of a technique which is both relatively simple computationally and allows for the modelling of irregular sea states.

The results for the frequency range \( j=1-105 \) are also presented. These results merely show an exaggerated version of the results for the \( j=1-75 \) range, above. The differences are caused by the third harmonic, a low amplitude but high frequency component the equation for which is applied well outside its region of validity. This phenomenon is known as high frequency contamination.

### 4.4 Comparison with Wheeler stretching

The results for linear theory in the previous section (4.3) demonstrated the need either for a low frequency cut-off point excluding the second harmonic or for some form of stretching technique. Wheeler stretching [124] is in many ways the most established and respected of the many stretching methods now available. In it
the vertical coordinate is stretched from the mean water line to the free surface. The effect of this is to make the velocity calculated for MWL in linear theory the highest velocity at the free surface in Wheeler stretching (see section 2.3.1).

The graphs for Wheeler stretching applied to the waves detailed in the previous two sections are shown in figure 4.5. The graphs are the same as before but using Wheeler stretching calculations instead of linear theory ones. The conclusions are rather different, as might be expected.
In this case Wheeler stretching in the range $j=1-45$, incorporating the fundamental component only, seriously underestimates the measured velocities above mean water line. The differences at the level of the trough are relatively small (10% for the highest wave and 12% for the lowest) however. Therefore the use of a measuring system which can only measure below trough height would not bring out the errors in the Wheeler method so well and nor would the differences between the frequency ranges be so great.

The results for Wheeler stretching in the frequency range $j=1-75$ incorporating the primary component and the second harmonic show a much better match to the form of measured results. In particular the rate of increase in velocity with elevation in the crest to trough region is far superior to that in the range $j=1-45$. At the level of the trough the differences between experiment and theory were 4% for the highest wave and 3% for the lowest. In each case (except, just, the lowest; graph 'a') the velocity profile for the range $j=1-75$ matched the highest measured velocity at the highest measured point under the crest, within experimental error. The only apparent fault is that perhaps the rate of increase of velocity with elevation is too high right at the crest. If so it is a relatively small error compared to the others seen so far.

The results for the range $j=1-105$ show small increases in velocity over the $j=1-75$ results near the crest. The maximum difference is 12% at the crest of the second highest wave (e9n8). The results in the lower range are to be preferred due to the lack of accuracy of the gauges at the higher frequencies.

4.5 **Comparison with Chakrabarti Stretching**

Chakrabarti stretching [13] relies not on the stretching of the vertical coordinate but on a redefinition of the water depth. The water depth in the depth decay
function is replaced by the bed to free surface distance (see section 2.3.2) thereby reducing the velocity profile under the crest. In deep water the solution tends to the Wheeler stretching value — the value given by linear theory at MWL — at the free surface.

The results produced by Chakrabarti stretching for the velocity profiles under the crests of the previously specified waves are shown in figure 4.6. The form of the results is similar to that of Wheeler stretching. The range j=1-15 again produced only negligible results which are obscured by the elevation axis. The frequency

Figure 4.6: Comparison of Chakrabarti stretching velocity profiles with experimental results for regular waves.
range \( j=1-45 \) again underpredicts in the crest to trough region but provides the basis for the accurate prediction below around \(-0.1\,\text{m}\). The frequency range \( j=1-75 \) introduces the rapid increase in velocity with elevation, as the free surface is approached, that is necessary if the experimental results are to be matched. A small increase in velocity near the free surface is introduced if the range \( j=76-105 \) is included.

### 4.6 Comparison with Superposition Stretching

Superposition stretching \([85]\) is similar to Wheeler stretching in that it relies on the stretching of the vertical coordinate. In this case it is the distance from the bed to the elevation of that component (calculated from its amplitude and phase) that is stretched to the free surface (see section 2.3.3). In the case of regular waves the primary component is dominant so the results produced are similar to those of linear theory only not quite so extreme. The same graphs as used previously are shown in figure 4.7 for superposition stretching.

In this case there is a consistent overprediction of the velocity profiles for the \( j=1-45 \) range (although slightly less than for the linear comparisons). As with linear theory this is a low frequency or a d.c. effect. The results from the \( j=1-45 \) range around the primary component are better than from the higher ranges incorporating higher harmonics. In fact it is only in the case of the highest wave measured that the error in the \( j=1-45 \) range comparison is significantly above the experimental error near the crest. In this case there is a consistent overprediction of the velocities ranging from 16% at \(-0.18\,\text{m}\) to 17% at the level of the wave troughs to 12% at the level of the highest PIV measurement. All percentages have been rounded up to the nearest integer value. All comparisons have been made against the highest measured velocity at that elevation.
Figure 4.7: Comparison of superposition stretching velocity profiles with experimental results for regular waves.
The corresponding percentage errors for the frequency range \( j=1-75 \) are 16% at \(-0.18\text{m}\), 23% at the trough level and 32% at the highest measured point. In all cases the larger frequency range performed less well.

4.7 Comparison with Extrapolation

Extrapolation is an attempt to reduce the high rate of increase in velocity with elevation above MWL in linear theory by applying the velocity gradient calculated at MWL to all levels above it. The result of this is a straight line velocity profile above MWL. The accuracy of the results (shown in figure 4.8) is critically dependent on the linear theory results at MWL.

The conclusions are much as those for linear theory were. The fit of the results is adequate only for the range \( j=1-45 \). The higher frequency ranges produce inordinate errors (>25%) in the crest to trough region. This is partly because the second harmonic generates a velocity term which is significant compared to that from the fundamental harmonic at mean water level in all four cases. If any form of extrapolation is to be used over the higher frequency range, it can only be accurate at the crest if it stretches from a point lower than MWL. That in turn is not very satisfactory as a straight line graph over a longer range cannot hope to match the experimental results throughout.
Figure 4.8: Comparison of linear extrapolation velocity profiles with experimental results for regular waves.
4.8 Comparison with Delta Stretching

Delta stretching involves both the use of stretching the vertical coordinate and extrapolation above the new zero elevation point. It also involves the use of two parameters (delta stretch $\Delta$, and a depth $d_{\Delta}$) the values for which have to be determined empirically. The method used by the originators in their paper [92] involved the use of a least squares method to determine the best-fit values of the parameters. Thus the velocity profiles were fitted to the available data in the expectation that the values chosen would continue to be close to the optimal values for subsequent experiments. It is hard therefore to make comparisons on the basis of a limited number of experiments — either the parameters are conditioned to fit the data in which case the comparisons should be good or, if a previously determined set of values are used, then it can be argued that a different set might have performed better. The values used here were those of the original authors. The justification for this is that the parameters should be valid in a range of situations or it becomes impossible to use the theory without measuring in the anticipated conditions for a long period.

The results for the four waves are shown in figure 4.9. In this case the performance was best for the range $j=1$-45. The results for $j=1$-75 were too high in the crest to trough region. This arises as there is no stretching below $z = -d_{\Delta}$ where linear theory is used and there is a non-negligible velocity component from the second harmonic at that elevation. This would seem to argue for an increase in the value of $d_{\Delta}$. Currently $d_{\Delta} = 2 \times \sigma_{\eta}$ (the standard deviation in free surface elevation) which works out at just over two-thirds of the wave elevation for the waves measured.

In the next chapter the highest wave in a group will be considered. In this case the standard deviation of free surface elevation is going to be smaller for a given height of measured wave so the delta stretching will be performed from even closer to
Figure 4.9: Comparison of delta stretching velocity profiles with experimental results for regular waves.
the surface. Therefore it seems unlikely that the present definition of $d_\Delta$ in terms of $\sigma_n$ will be of much use here.

### 4.9 Comparison with higher order theories

The two higher order, regular wave theories used in comparison with the experimental data were Chaplin’s reformulation of Dean’s stream function solution and the Rienecker and Fenton fourier approximation method. The stream function method was formulated assuming Stokes first (zero mean eulerian velocity) definition of wave celerity (see section 4.3) whereas the fourier solution was formulated assuming the second (zero mass transport) definition of wave celerity. The latter assumes that there is no average mass transport and is the definition valid for a closed wave flume situation, as here. The Rienecker and Fenton fourier solution also gave a value for the mean Eulerian velocity. The velocity values produced by the theory may be interpreted as being made up of the linear sum of a wave component and a drift, or current, component. If that drift component is assumed to be uniform with depth then the two can be separated simply by subtracting the mean Eulerian drift from the sum to give the wave component only. The values for the regular waves used are given in table 4.2. $u_e$ is the mean Eulerian velocity in m/s and $u_{pivm}$ is the maximum velocity measured using PIV.

<table>
<thead>
<tr>
<th>$H/gT^2$</th>
<th>$u_e$</th>
<th>$u_{pivm}/u_e$</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0103</td>
<td>-0.009</td>
<td>291</td>
<td>e9n9</td>
</tr>
<tr>
<td>0.0123</td>
<td>-0.012</td>
<td>292</td>
<td>e8n7</td>
</tr>
<tr>
<td>0.0134</td>
<td>-0.015</td>
<td>258</td>
<td>e9n8</td>
</tr>
<tr>
<td>0.0159</td>
<td>-0.020</td>
<td>220</td>
<td>e9n8</td>
</tr>
</tbody>
</table>

Table 4.2: Relative importance of the mean eulerian velocity.
Figure 4.10: Comparison of higher order velocity profiles from Dean's stream function and Rienecker and Fenton fourier methods with experimental results.

As may be expected, the magnitude increases with steepness and also increases as a fraction of the highest PIV measurement made. The results of comparing these theories to the measurements are shown in figure 4.10.

The Rienecker and Fenton fourier method performed better than the stream function largely because of the use of the different definition of celerity. In all cases the theories matched the form of the measured results well. In the majority of cases the theoretical prediction for the fourier method was just higher than the value of the highest measurement at that point. The only exception, when the difference
increased above a few percent, was in the trough to crest region of the highest wave where the difference remained less than 10%. In all cases the higher order theories could be used as a conservative means of obtaining a velocity profile for the measured waves.

4.10 Comparison Between Techniques

The previous sections have shown at some length how the various linear and higher order theories perform in comparison to the measured data. In this section the methods are compared with each other. The first such comparison is shown in figure 4.11. In this the six methods — linear, Wheeler, Chakrabarti, superposition, extrapolation and delta — based on first order, linear theory have their velocity profiles plotted for the frequency range \( j=1-75 \ (f \leq 2.93\text{Hz}) \).

This figure shows that the methods may be split into three groups depending on their results. The main advantage of this is that only one method of a group need be plotted on any one graph thereby reducing the considerable clutter that comes from trying to compare results from 8 methods and a set of experiments. The Wheeler and Chakrabarti methods of stretching form the first group. They tend to the same value at the crest and as both theories underpredict the results except at the crest then the Wheeler stretching results can be shown to provide the better fit to the results.

The second group consists of the superposition and delta stretching results which tend to similar values at the crest. The third and final group is that of linear and extrapolation methods which are identical up to MWL and which both massively overestimate the velocity profile above MWL.

When the above process is repeated for the range \( j=1-45 \ (f \leq 1.76\text{Hz}) \) which excludes the second harmonic the results (figure 4.12) show a similar grouping.
Figure 4.11: Comparison between the velocity profiles of the six first order methods — linear, Wheeler, Chakrabarti, superposition, extrapolation and delta — for the frequency range $j=1-75$
Figure 4.12: Comparison between the velocity profiles of the six first order methods — linear, Wheeler, Chakrabarti, superposition, extrapolation and delta — for the frequency range $j=1-45$
of results. The delta stretching profiles sit alone between groups one and three, however, and the superposition stretching profiles are very similar to the linear and extrapolation profiles in the third group with the highest values.

The fact that the results form groups allows a subset of all the results to be plotted with the experimental results for comparison. The following results are plotted on figures 4.13 and 4.14 for the frequency ranges j=1-75 and j=1-45 respectively; experimental, Wheeler, superposition, delta and fourier.

It can be seen that Wheeler stretching including the second harmonic performs better than Wheeler stretching without. The same applies to Chakrabarti stretching but in all cases the Wheeler stretching performs better than Chakrabarti. The only apparent problem is that the results appear to overpredict the velocity gradient above mean water level — in the highest waves the results tend to underpredict the velocities measured above MWL, except at the crest. Nevertheless the values tend towards those produced by the higher order Rienecker and Fenton fourier approximation at the crest, so the two theories produce equal drag forces (per unit length) at the crest using the Morison equation.

The fourier approximation method is to be preferred to the stream function because it incorporates the second definition of wave celerity and produces results consistently closer to the measurements. The theory matches the experimental results well, erring on the side of slight overprediction (thereby being slightly conservative while remaining relatively accurate at the same time). The highest errors occurred with the highest wave where the theory overpredicted the measured results by less than 10%.

Delta stretching can be transformed into Wheeler stretching or extrapolation by the appropriate choice of coefficients. In both those cases the stretching is performed over the full water depth ($d_\Delta = d$). The recommended distance below MWL at which stretching should begin is given by Rodenbusch and Forristall [92] as $d_\Delta =$
Figure 4.13: Comparison between selected theories and experiments for the frequency range \( j = 1-75 \).
Figure 4.14: Comparison between selected theories and experiments for the frequency range $j=1-45$. 

Comparison between selected methods

$H/gT^2 = 0.0159$
$j=1-45$

EBN9
(a)

Comparison between selected methods

$H/gT^2 = 0.0134$
$j=1-45$

E9n8
(b)

Comparison between selected methods

$H/gT^2 = 0.0123$
$j=1-45$

e9n7
(c)

Comparison between selected methods

$H/gT^2 = 0.0103$
$j=1-45$

e9n9
(d)
2 x standard deviation of free surface elevation, a distance which is substantially less than the wave height even in regular waves. This makes the vertical range over which stretching occurs a lot less than the water depth. In the cases measured here this distance is inadequate to cope with the non-negligible contribution of the second harmonic at the elevation $-d_\Delta$.

The superposition, linear and extrapolation methods all overpredicted the velocity profiles by substantial amounts above the MWL for the higher frequency range. The superposition stretching produced the lowest results of the three in each case. However, its results in the lower frequency range, incorporating the primary component only, match those of the fourier method well. The results at the crest differ by a couple of percent only but the superposition method becomes slowly greater than the fourier method away from the crest. In the case of the highest wave, the superposition velocity is about 1% greater at the crest and 7% greater at -0.1m.

### 4.11 Conclusions

The previous section outlined the four theories which were the best representatives of their type at modelling regular wave crest velocity profiles. They are:

**Fourier method.** Rienecker and Fenton’s method was both accurate yet slightly conservative in its predictions — ideal for use in a Morison equation.

**Wheeler stretching.** This modelled the form of the results very well when both the primary component and second harmonic were used. If anything it slightly underpredicted velocities just below the crest but its peak velocities matched those of the fourier method well.
**Delta stretching.** This did not perform as well as the other theories listed here. It could have been made identical or very similar to the Wheeler stretching through the calculation of a new set of stretching parameters though.

**Superposition stretching.** This matched the experimental and fourier values very well if the frequency range was limited to around the primary component only. It overpredicted the results at lower depths.

The velocity profiles for the above theories are shown in figure 4.15. The preferred version would certainly be the fourier approximation method with the Wheeler a close second. The effect of using superposition with the smaller frequency range produced good results but only by ignoring the measured second harmonic component. As this was a significant fraction of the primary component in amplitude (from 11% to 17%) this would appear unwise.
Figure 4.15: Comparison between measured crest velocity profiles and the best theories (including the frequency range for the linear based theories)
Chapter 5

Wave Groups

The concept of the wave group, perhaps most simply put as a series of wave crests all above a threshold height, is neither new nor is it confined to fluid dynamicists. The idea that every nth wave (7th, 50th, 200th or whatever) will be the highest is one that has entered popular folklore and encapsulates the idea that the elevation of successive crests, defining a wave envelope, will slowly rise and fall in a regular pattern. Nature is not so accommodating as to arrange such a degree of regularity however. Nevertheless, there is widespread acceptance of evidence from field records that wave grouping exists [28, 80, 71, 70, 105, for example] but it has taken much work to produce a more acceptable mathematical model than the one above.

Most of the theoretical work on wave groups, outlined below, has concentrated on the derivation of statistical expressions for such terms as the length of time between and the number of waves in a run of waves above a positive threshold elevation. The details of such expressions are not of interest here, as this thesis is involved with measuring wave kinematics, making comparisons with the common theories and determining their relative accuracies. However, the approach now taken is to move from measuring regular waves to one of measuring simple wave groups; the same combinations of frequencies at different wave heights.
The justification for this is that the actual sea surface is irregular in nature and the use of simple two-component wave groups is the simplest way to obtain a non-regular wave pattern while retaining an accurately defined wave. In this way waves of the same gross characteristics can be placed in different environments and comparisons made. Therefore the rest of this chapter comprises sections on the following topics:

5.1 The development of wave group theory.

5.2 Beats and superharmonics in simple groups.

Two sections on comparisons between waves then follow:

5.3 Two waves of equal magnitude but different frequency combine to form a group. The central wave (phases ≈ 0) is measured using PIV for each of 4 combinations of wave amplitude, and the results compared to the methods used in chapter 4. This case is analogous to the comparisons made in chapter 4 but using the central (highest) wave in a group instead of a regular wave.

5.4 Two waves of equal steepness (amplitude times wavenumber, or \(ak\)) but different frequency combine to form a group. The central wave is measured (as above) for 4 combinations of wave amplitude. This is also analogous to the experiments in chapter 4.

5.5 Conclusions.

5.1 The Development of Wave Group Theory.

The starting point for the mathematical description of water wave grouping was the adaptation of Rice's work [90] on noise in electrical circuits to water waves by
Longuet-Higgins [62, 63]. The work was extended or modified by Rye [95], Nolte and Hsu [80], the two original authors and others before being reviewed by Goda [35], Longuet-Higgins [64] and Medina and Hudspeth [74]. In these papers the sea state is considered as being made up of a number of uncorrelated waves and the surface displacements will be Gaussian if the linear superposition principle remains valid.

The wave spectrum is usually truncated, for example Longuet-Higgins uses components in the range 0.5 to 1.5 times the peak frequency. This is justified by noting that observers interested in groups are not interested in small waves riding on dominant waves and by noting small increases in error between measured and theoretical numbers of zero crossings as the frequency range increases. Nolte and Hsu [80] truncate using a sine filter, $\sin(\pi f/2f_p)$ for $f \leq 2 \times f_p$ and 0 for $f > 2 \times f_p$ (with $f_p$ being the spectral peak frequency).

In this approach a wave envelope function which joins the successive wave peaks together is characterised by a Rayleigh probability density function. A threshold level is established above mean water level and statistics concerning the length of time the envelope remains above the threshold and the time separation between zero up-crossings follow. These theoretical results match measured ones closely, after the record is filtered as above [80, 64, 35].

Kriebel and Dawson [57] noted that errors they observed were due to high waves having higher crests and shallower troughs than a simple sine wave about zero. They re-derived the same statistics without using the Rayleigh assumption but using an amplitude-modulated Stokes wave to give the free-surface elevation, as shown by Tayfun [111]. For example the probability function for wave crest amplitude is shown to be the Rayleigh term modified by a term dependent on wave steepness. The envelope theory is then re-worked to include non-linearities to second order.
Results from PIV studies could be used to build up probability density functions for velocities to complement those from surface elevation studies. The maximum wave crest velocity could be measured under every crest of an irregular sea state using a succession of PIV photographs. (The crests and/or troughs could be detected using wave gauges which would trigger the camera when the rate of change of surface elevation with time passed through zero in the appropriate way.) If enough crests were considered then probability density curves could be plotted for both wave crest amplitudes and wave crest velocities, or the two could be plotted as a joint crest amplitude-velocity probability distribution. Thus a direct velocity equivalent of the crest amplitude probability distributions used by Kriebel and Dawson [57] could be obtained using PIV and the statistical methods applied to amplitudes could be applied to velocities and tested against the crest velocity distribution.

An alternative approach would be to use the wave height record (from the wave gauge that triggered the camera) with the PIV results to calculate the relative steepness \((H/gT^2)\) of each wave. The predicted crest velocity (as a function of celerity) for a given wave steepness and theory could be used with the measured crest velocity to produce probability density curves of the ratio of measured to predicted velocity for a given wave steepness. This would give a measure of the probability of a wave deviating from the theoretical value and would show the size of the most common deviations.

Statistical work with velocities and velocity distributions, as suggested above, is now possible using PIV and may be used with existing data on amplitude distributions to gain a better indication of the most probable velocities within a given wave.
Sobey and Liang [103] present the complex envelope function as an analysis method for wave records. This retains phase information previously not used and produces a link between envelope amplitude and phase traces that can act as a 'signature' for wave groups, via the use of cross-correlograms and coherence spectra.

5.1.1 Results of Interest from Wave Group Research

Although this thesis does not deal with the statistical analysis of wave records, some useful information can be obtained from wave grouping papers, in particular the results of a paper by Su [105].

Rye [95], for example, notes that grouping increases during wave growth and explains this through the link between grouping and spectral peakedness derived first by Goda. The wave spectrum tends to be more peaked or more narrow-banded in character during wave growth than during saturation or decay and because grouping increases with spectral peakedness, more grouping will be seen during growth than after.

The papers based on sea wave records which note the existence of wave grouping [28, 80, 71, 70, 105, for example] provide an indication that the approach of using simple wave groups can give a realistic representation of the surface profile of an extreme sea state. Real storm records do show high waves in relatively simple wave groups with smoothly varying wave envelope elevations around the highest crest, though not in all cases of course.

Kriebel and Dawson [57] note that 'runs of high waves occur much more frequently than linear theory would predict.' Nolte and Hsu [80] emphasise the need for large data sets as the deviation from linear theory only becomes pronounced at very low probabilities i.e. for the highest waves.
Rye [95], Longuet-Higgins [64] and Ochi [81] all note that the average number of waves in a group may be under two for high thresholds (such as an elevation above half the significant wave height — the average height of the highest third of all waves). The definition of a wave group given at the start of the chapter (a series of wave crests all above a threshold height) requires at least two crests above the threshold so the definition of a group must be modified to something like 'where the wave envelope is above a threshold height'.

The statistics for the number of waves above the threshold height are not as important as the relative heights of the waves around the highest for this thesis. For this the statistics of the extreme wave group (EWG) — the relative amplitudes and periods of the waves around the highest wave in a record — are more important. A paper by Su [105] has investigated just these statistics for data gathered in the Gulf of Mexico by the Ocean Data Gathering Program [120].

Goda [35] had previously noted that the extreme wave group had a much greater length of run than the average wave group (for a threshold at the significant wave height). This implies that the highest wave does not exist alone and that the highest wave is more likely than any other wave to appear in a group.

Su [105] investigated the height, period and steepness for each of 7 waves per wave record, $W_{-3}, W_{-2}, W_{-1}, W_0, W_1, W_2, W_3$, namely the highest ($W_0$) the three preceding waves ($W_{-3}, W_{-2}, W_{-1}$) and the three following ones ($W_1, W_2, W_3$). Each wave has a height $H_j$, period $T_j$ and steepness $S_j$. The heights were normalised by the square root of the zeroth spectral moment, $\sqrt{\bar{m}_0}$, and the periods by the mean wave period of the record. Statistics were then derived for the average values and for different bands of values of the spectral peakedness parameter.

Here the $i$th spectral moment $m_i$ is defined as:

$$m_i = \int_0^\infty f^i p(f) \, df$$

(5.68)

$f$ is the frequency and $p(x)$ is the probability density of $x$. Goda defined spectral...
peakedness $Q_p$ as;

$$ Q_p = m_0^2 \int_0^\infty 2f p^2(f) \, df $$

(5.69)

The value of $Q_p$ is always greater than unity and increases as the spectrum becomes narrower.

The average envelope of the EWG was found to be rather symmetrical in height; $H_0 = 6.3, \bar{H}_{+1} = 3.9, \bar{H}_{+2} = 2.7$ and $\bar{H}_{+3} \approx 2.5$, where the overline denotes an average value. The mean wave height for a linear, narrow band Gaussian sea is $H = 2.51\sqrt{m_0}$. The average wave period calculated for the same sea would be 1.0 but the measured values were $\bar{T}_0 \approx \bar{T}_{+1} \approx 1.3$ and $\bar{T}_{+2} \approx \bar{T}_{+3} \approx 1.1$. This meant that the average wave steepness for the highest waves was $S_0 = 0.2$, while those of the other waves ranged from 0.13 to 0.15.

Rebaudengo Lando et al. [59] performed numerical simulations in the time domain using a random phase method and analysed the statistics of characteristic groups. Their conclusions matched those of Su to within about 5% with regard to the relative heights of the waves in a group and the symmetry of the wave heights about the central wave in the group.

These figures [105, 59] indicate that the extreme wave in a record, on average, appears in a symmetrical group containing three waves significantly higher than the average value. The mean wave group length for the EWG was just under 2.5 for a threshold level given by $4\sqrt{m_0}$. This agrees well with Goda's value [35] of 2.36 and is higher than the length of run of the average wave group ($\approx 1.5$) at the same threshold level.

The effect of increasing the spectral peakedness is to increase the average wave heights, to reduce the average central period slightly and thus to increase the average central wave steepness slightly. In other words the EWG becomes longer and the highest waves steeper as the wave spectrum narrows.
5.2 Beats and Superharmonics in Simple Wave Groups

Consider 2 waves of equal amplitude $A$ and frequencies $f_1$ and $f_2$ (where $f_i = j_i/25.6$) and the frequencies are similar but not the same. Then the surface elevations $\eta_1(t)$ and $\eta_2(t)$ are given by:

$$\eta_1(t) = A_1 \cos(2\pi f_1 t + \phi_1)$$ (5.70)

and

$$\eta_2(t) = A_2 \cos(2\pi f_2 t + \phi_2)$$ (5.71)

where $\phi_i$ represents the phase of the $i$th wave and all measurements are made at the same point. Using the principle of superposition gives the surface elevation when the two components are generated together as

$$\eta(t) = \eta_1(t) + \eta_2(t) = 2A \cos \left\{ 2\pi \frac{f_1 + f_2}{2} t + \frac{\phi_1 + \phi_2}{2} \right\} \cos \left\{ 2\pi \frac{f_1 - f_2}{2} t + \frac{\phi_1 - \phi_2}{2} \right\}$$ (5.72)

This can be considered as a wave of frequency $(f_1 + f_2)/2$ and amplitude $2A$ modulated by an envelope varying at frequency $(f_1 - f_2)/2$. Here an iterative process is used to set the phases to zero at the measurement time ($\phi_i \approx 0$ at $t = x = 0$) so the surface elevation may be expressed as:

$$\eta(t) = \eta_1(t) + \eta_2(t) = 2A \cos \left\{ 2\pi \frac{f_1 + f_2}{2} t \right\} \times \cos \left\{ 2\pi \frac{f_1 - f_2}{2} t \right\}$$ (5.73)

A beat (a maximum in amplitude) occurs when $\cos \pi(f_1 - f_2)t = \pm 1$. Therefore beats occur at a frequency $f_1 - f_2$ Hz and the wave pattern is repeated (the envelope repeats) at a frequency $\frac{f_1 - f_2}{2}$ Hz. All the frequencies used for wave generation here
can be expressed as \( f_i = j/25.6\text{Hz} \) where \( j \) is an integer known as the frequency number. It follows that:

- Frequency of envelope = \( \frac{j_1 - j_2}{2 	imes 25.6} \text{Hz} \)
- Frequency of waves = \( \frac{j_1 + j_2}{2 	imes 25.6} \text{Hz} \)
- Number of waves per envelope = \( \frac{j_1 + j_2}{j_1 - j_2} \)

This chapter takes a similar form to the previous one, only instead of looking at the variation in behaviour of regular waves with decreasing height, the central waves in simple groups are used. The groups are chosen to be as short as possible as they are thus as least like the regular waves as they can be, while still being simple two-component groups with a simple relationship between the two amplitudes in a group. Therefore, all of the waves measured in this chapter have components with frequency numbers \( j_1 = 35, j_2 = 25 \) so the simple linear theory of beats would give a wave frequency of \( \frac{60}{51.2} \text{Hz} = 1.172\text{Hz} \) with a period of 0.853s, the same as the regular waves measured in chapter 4. The waves would be modulated by an envelope that repeats completely at a frequency of \( \frac{10}{51.2} \text{Hz} = 0.195\text{Hz} \) with a period of 5.12s. The number of waves per envelope is \( \frac{60}{10} = 6 \) and the number per beat = 3. Table 5.1 shows results for the frequency combinations used in this chapter and the following one. These numbers are all for the simple linear addition of two primary components of the same amplitude.

However, the above ignores the bound long waves and harmonics that are found if the Laplace equation 2.6 is solved to second order. Long waves of frequency \( f_1 - f_2 \) are associated with wave set-down, whereby the average surface elevation under the centre of a wave group is lower than MWL. This decrease in potential energy compensates for the excess of kinetic energy there and helps maintain an energy balance. The long waves travel at the group velocity. The second harmonics are a double frequency phenomenon and result in a sharpening of the wave crests and a
<table>
<thead>
<tr>
<th>$j_1$</th>
<th>$j_2$</th>
<th>Wave frequency (Hz)</th>
<th>Envelope frequency (Hz)</th>
<th>Waves per envelope</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>25</td>
<td>60/51.2=1.17</td>
<td>10/51.2=0.195</td>
<td>6</td>
</tr>
<tr>
<td>33</td>
<td>27</td>
<td>60/51.2=1.17</td>
<td>6/51.2 =0.117</td>
<td>10</td>
</tr>
<tr>
<td>31</td>
<td>29</td>
<td>60/51.2=1.17</td>
<td>2/51.2 =0.039</td>
<td>30</td>
</tr>
<tr>
<td>30</td>
<td>-</td>
<td>30/25.6=1.17</td>
<td>regular</td>
<td>$\infty$</td>
</tr>
<tr>
<td>28</td>
<td>20</td>
<td>48/51.2=0.937</td>
<td>8/51.2=0.156</td>
<td>6</td>
</tr>
<tr>
<td>26</td>
<td>22</td>
<td>48/51.2=0.937</td>
<td>4/51.2=0.078</td>
<td>12</td>
</tr>
<tr>
<td>25</td>
<td>23</td>
<td>48/51.2=0.937</td>
<td>2/51.2=0.039</td>
<td>24</td>
</tr>
<tr>
<td>24</td>
<td>-</td>
<td>24/25.6=0.937</td>
<td>regular</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Table 5.1: Wave Group Combinations

flattening of the wave troughs. The second harmonics include two self-interaction terms at $2j_1$ and $2j_2$ (as described by Stokes [104]) and a mutual term at $j_1 + j_2$.

To investigate whether these harmonics were present to any extent the harmonic components from two waves, e8n2 and e8n12, used in the following section were plotted in figure 5.1. These wave groups both have equal amplitude components with frequency numbers of $j = 25, 35$. The measured surface profiles and amplitude spectra can be seen in figure 5.2. The components in wave E8N2 are about twice those in wave E8N12.

In the harmonics diagram 5.1 the top two diagrams on each side represent the calculated surface elevation against time for the primary components at $j = 25$ and 35. All elevation records were calculated from the measured wave spectrum component at that frequency, which provided amplitude and phase at a time of 12 seconds. The third graph gives the sum of the previous two, the total first order time series. It can be seen that the first order contributes the bulk of the total time series. Moreover the periods of the central waves (peaking at about 12s) match the period calculated from the beating theory, see table above, to within
Figure 5.1: Non-dimensionalised elevation ($z/H$) versus time (s) records for harmonic components of two measured waves, E8N2 on the left and E8N12 on the right. The vertical scale of the graphs of 2nd order components (4th to 8th from top) are exaggerated by a factor of 5.
experimental error.

The fourth graph in each column ($j = 10$) is the first of the second order graphs and like all the second order graphs has its elevation axis exaggerated by a factor of 5 compared to the first order ones. The following three represent the two self-interaction terms ($j = 50$ and $j = 70$) and the mutual term ($j = 60$). The last but one shows the sum of the three second order (not long) harmonic terms. The contribution of the second order terms to the surface elevation is clearly not insignificant, amounting to just under 18% of the magnitude of the first order total at the crest of wave E8N2. The second order contribution (as a percentage of the wave height) is much smaller for the less steep wave (E8N12) as can be seen by comparing the second order total harmonic graphs. The increase in second order effects with increasing wave steepness is to be expected.

The bottom graph shows the sum of the first and second order totals. The differences between it and the first order total graph can most clearly be seen at the most extreme peaks and troughs.

A more detailed look at the contributions of the different harmonics to the surface profiles of natural waves is provided in a paper by Mansard, Sand and Klinting [68]. In comparisons with sea data they found that the contribution of the bound harmonics to the crest level varied from around 30%, at $d/gT^2 \approx 0.01$ or $d/L_0 \approx 0.08$, to 10%, at $d/gT^2 \approx 0.1$ or $d/L_0 \approx 0.8$ ($L_0$ is the deep water linear wavelength). This is in keeping with the results here and show that there can be a considerable harmonic contribution to wave crest elevations in a typical North Sea situation. Mansard et al. noted a corresponding harmonic velocity component of up to 10% in intermediate depths. The following sections will investigate the contributions of the harmonics to the velocities of several groups of waves.
5.3 Comparisons with Two-component, Equal Amplitude Groups.

The waves investigated in this section are from four two-component groups with frequencies of 25/25.6Hz and 35/25.6Hz, i.e. \( j=25 \) and 35 where \( f = j/25.6 \). The central wave in each group is measured using PIV and wave gauges (after an iteration scheme was used to ensure the component phases were \( \approx 0 \)). The measured amplitudes of the two components were approximately equal (again through the use of an iterative process) but each group had a different component amplitude. In this way the velocities under the crest of four waves of different height, but each the centre of a similar group (in terms of frequencies) are measured and compared. Some of the details of the measured waves are given in table 5.2.

<table>
<thead>
<tr>
<th>wave</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experiment Reference</td>
<td>e8n2</td>
<td>e8n3</td>
<td>e8n11</td>
<td>e8n12</td>
</tr>
<tr>
<td>Amplitude (mm) for ( j = 25 )</td>
<td>20.0</td>
<td>18.2</td>
<td>15.4</td>
<td>9.7</td>
</tr>
<tr>
<td>Phase (rad) for ( j = 25 )</td>
<td>-.02</td>
<td>-0.01</td>
<td>-.01</td>
<td>0.0</td>
</tr>
<tr>
<td>Amplitude (mm) for ( j = 35 )</td>
<td>20.8</td>
<td>19.2</td>
<td>15.0</td>
<td>9.6</td>
</tr>
<tr>
<td>Phase (rad) for ( j = 35 )</td>
<td>0.01</td>
<td>0.02</td>
<td>-.01</td>
<td>0.02</td>
</tr>
<tr>
<td>Height (mm)</td>
<td>91.6</td>
<td>77.7</td>
<td>62.4</td>
<td>36.1</td>
</tr>
<tr>
<td>Period (s)</td>
<td>0.79</td>
<td>0.81</td>
<td>0.83</td>
<td>0.86</td>
</tr>
<tr>
<td>Relative steepness ( (H/gT^2) )</td>
<td>.0151</td>
<td>.0121</td>
<td>.0091</td>
<td>.0050</td>
</tr>
<tr>
<td>Relative depth ( (D/gT^2) )</td>
<td>.0826</td>
<td>.0842</td>
<td>.0791</td>
<td>.0746</td>
</tr>
<tr>
<td>Mean eulerian velocity (mm/s)</td>
<td>-14.3</td>
<td>-10.3</td>
<td>-6.6</td>
<td>-2.2</td>
</tr>
<tr>
<td>Standard deviation of ( \eta ) (mm)</td>
<td>20.8</td>
<td>18.5</td>
<td>14.9</td>
<td>9.5</td>
</tr>
<tr>
<td>Steepness ( ak ) for ( j = 25 )</td>
<td>79</td>
<td>72</td>
<td>61</td>
<td>38</td>
</tr>
<tr>
<td>Steepness ( ak ) for ( j = 35 )</td>
<td>156</td>
<td>144</td>
<td>113</td>
<td>72</td>
</tr>
</tbody>
</table>

Table 5.2: Wave details from equal amplitude groups.
In the table the height and period are those of the central wave measured from the wave gauges using the zero down-crossing method. $\eta$ represents surface elevation. The mean eulerian velocity is that calculated by the Rienecker/Fenton fourier approximation method. In the calculation of the steepness $ak$ the wavenumber $k$ is calculated from the amplitude spectrum using a third order theory.

Points to note from the table are given in the next two paragraphs. The amplitudes of the components are all within 5.5% of each other and all phases are in the range -0.02 to 0.02 radians. There is a steady increase in period on going from the steepest wave to the least steep where the value tends towards the average regular wave value of 0.853s. This is not what would result from the simple linear superposition of the different component waves. The differences in period also manifest themselves in the relative depths. The mean eulerian velocity increases with increasing relative steepness.

The standard deviation ($\sigma$) of $\eta$ (surface elevation) is equivalent to $\sqrt{m_0}$. For a narrow-banded sea state the wave amplitudes are characterised by a Rayleigh distribution. In this case the significant wave height $H_\frac{1}{3}$ (originally defined as the average height of the highest third of all waves) becomes $H_\frac{1}{3} = 4\sqrt{m_0}$. In sea states where this condition is not met the two methods of calculating $H_\frac{1}{3}$ will give different answers.

Cases ‘a’ to ‘c’ here have significant wave heights defined from $\sigma$ of over 90% of the highest wave height and case ‘d’ has one of over 100%, thereby showing that it is not narrow-banded. The significant wave heights for case ‘a’, experiment e8n2, are 87mm (average of highest $\frac{1}{3}$ of waves) and 83mm ($4\sqrt{m_0}$). The sample size for calculating these was only 18 waves however. The corresponding figures for e8n11 are 61 and 60mm respectively (sample of 21 waves).

The surface profiles and amplitude spectra for the waves are shown in figure 5.2. The surface profile diagrams have a vertical scale of ±0.1m and the time series go
Figure 5.2: Amplitude Spectra and Surface Elevation Records for Equal Component Amplitude Wave Groups from 6.9 to 17.1s. All measurements were made at a time of 12s.
5.3.1 Comparisons with Stretching Methods.

The comparisons between the PIV results and the stretching methods in the frequency range $j = 1 - 75$ can be seen in figure 5.3. The frequency $f$ is given by $f = j/25.6$Hz. This range includes both the first or primary harmonics (at $j = 25$ and $j = 35$) and second harmonics (at $j = 50, 60$ and $70$). As before, the measured results from 2 or 3 close horizontal positions are shown at the same elevation. This means that a missing data point does not leave a gap and covers any error in matching a PIV position to the wave gauge position.

The methods can be split into two main groups for comparison. The first group comprises Wheeler, Chakrabarti and Superposition results. This is in contrast to the behaviour of the regular waves in chapter 4 where Superposition stretching was grouped (with Delta stretching) away from Wheeler and Chakrabarti — see section 4.10. The conclusions from Wheeler and Chakrabarti stretching are similar to those of chapter 4. In each case the results tended to the same value at the crest, with the Chakrabarti values lower than the Wheeler at lower levels. The theoretical results of both methods matched the measured velocity profiles of the lower two waves ('c' and 'd') to within the margin of experimental error at all elevations.

In the steepest wave cases (‘a’ and ‘b’) Wheeler and Chakrabarti were both found to underpredict the measured results within the crest to trough region. The results for the second highest wave differed by less than about 5% and were worst just below MWL, but returned to the predictions at the highest measured point. Recall that the mean eulerian velocity was around 2% of the maximum velocity for this wave, calculated by the fourier approximation method. If a positive shift of this amount had been applied uniformly to the experimental results then the conclusion would have been that the methods underpredicted measured velocities slightly in all cases.
Figure 5.3: Velocity profiles under the crest of the central waves in a group of 2 equal amplitude components, compared to stretching methods in the range \( j = 1 - 75 \). The method labels refer to all graphs.
It is only in the case of the highest wave \((H/gT^2 = 0.0151)\) that the results differ significantly from Wheeler and Chakrabarti methods. The differences are largest around the crest to trough region. At the trough level (-35mm) the difference between Wheeler results and the highest measurements was \(\approx 16\%\). The corresponding figures at MWL and the highest measurement (at 30mm) were \(\approx 17\%\) and \(\approx 21\%\) respectively. The crest elevation was 57mm, and all the measured results were higher than theory. This represents a significant under-prediction of velocities in the trough to crest region.

The conclusions from superposition stretching are that the method tends to overpredict the experimental results slightly (typically by around 5\%). This can largely be accounted for by the mean eulerian velocity and experimental error. The only instance where the results underpredict the measurements is around the crest to trough region of the highest wave. Here the highest experimental results are greater than the ones from the theory, by \(\approx 9\%\) from trough through to crest level. Note that, as with Wheeler and Chakrabarti this effect is already established at the few percent level at the trough elevation.

The second set of results in figure 5.3 incorporates linear, extrapolation and delta stretching methods. The linear and extrapolation methods both have the same velocity profiles below the MWL. Both methods seriously overestimate the velocities at MWL and neither is adequate for the calculation of peak forces using the Morison equation. The delta stretching is again very ill-conditioned for this comparison and is also an inappropriate choice for engineering purposes without re-conditioning. This occurs as the choice of delta-stretching parameters made by Rodenbusch et al. on the basis of comparisons with their experiments was not the optimum choice for these tests. This certainly shows that the choice of best parameter values is not universal and that the accurate use of the method requires more data-collection to determine the best choice in a given set of circumstances.

The results are repeated with comparisons in the frequency range \(j = 1 - 45\),
incorporating the primary component \( (j = 30) \) but not the second harmonic \( (j = 60) \). The results are shown in figure 5.4.

The results are again grouped as before, except in the case of the lowest wave \( (H/gT^2 = 0.005) \) where all results match the experimental velocity profile closely and the groups merge. The conclusions about the methods are somewhat different, however. The Wheeler and Chakrabarti profiles get successively worse as the wave steepness increases, with the Wheeler stretching underpredicting the measured velocity at the highest point of the steepest wave by just over 30%.

The superposition stretching performs adequately for the lower waves but significantly underpredicts the velocities under the highest wave. The method performs accurately below an elevation of \(-0.1m\) (not allowing for any correction due to the mean eulerian drift) but the underprediction has risen to \( \approx 15\% \) by the (preceding) trough elevation and \( \approx 24\% \) at the highest measured point.

Linear theory performed well for the steepest wave and the lowest but was less successful in between. In comparison with the steepest wave measurements it was found that the method overpredicted the measurements by \( \approx 20 mms^{-1} \) at an elevation of \(-20mm\) and at \(-10mm\). This compares to the mean eulerian velocity of \(-14 mms^{-1}\). The divergence from the measurements decreased with increasing elevation until the results matched to within the margins of experimental error in the crest to trough region.

The fit to the second highest wave was not nearly so good. The results were constantly overpredicted with the overprediction increasing from \( \approx 8\% \) at \(-0.2m\) to \( \approx 12\% \) at the trough elevation to \( \approx 15\% \) at the highest measurement.

The extrapolation method is the same as linear method up to MWL then is a straight line extrapolation of it, whereas the linear method above MWL is an extension of the use of the same equations. The number of points available is low
Figure 5.4: Velocity profiles under the crest of the central waves in a group of 2 equal amplitude components, compared to stretching methods in the range $j = 1 - 45$. Two parallel sets of measured velocities are shown.
for making comparison between extrapolation and extension above MWL however, so the methods should be treated as equally valid.

It was shown in the previous section (5.2) that the second harmonic contributed significantly to the maximum crest elevation measured in the group — about 18% for case ‘a’. The contributions of the second harmonics to the velocities under the crests of the same two waves are shown in table 5.3 for 3 stretching methods at 3 elevations. The 3 elevations are (1) the crest, (2) at 90% of the distance from the bed to the crest and (3) at 70% of the distance from the bed to the crest. The second harmonic contribution was zero at the bed.

<table>
<thead>
<tr>
<th>Elevation →</th>
<th>crest</th>
<th>90% height</th>
<th>70% height</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theory/Wave</td>
<td>j45</td>
<td>j75</td>
<td>Δ%</td>
</tr>
<tr>
<td>Linear (a)</td>
<td>0.540</td>
<td>1.020</td>
<td>89</td>
</tr>
<tr>
<td>Wheeler (a)</td>
<td>0.364</td>
<td>0.486</td>
<td>33</td>
</tr>
<tr>
<td>Superp (a)</td>
<td>0.408</td>
<td>0.539</td>
<td>32</td>
</tr>
<tr>
<td>Linear (c)</td>
<td>0.323</td>
<td>0.500</td>
<td>55</td>
</tr>
<tr>
<td>Wheeler (c)</td>
<td>0.253</td>
<td>0.319</td>
<td>26</td>
</tr>
<tr>
<td>Superp (c)</td>
<td>0.275</td>
<td>0.344</td>
<td>26</td>
</tr>
</tbody>
</table>

Table 5.3: Comparison between first and second order contributions from linear based theories.

The left hand column gives the theory used and the wave (case ‘a’ or ‘c’) it was applied to. The columns \( j45 \) and \( j75 \) give the velocities (in \( ms^{-1} \)) for those frequency ranges, starting from 1. The column \( Δ\% \) gives the second order contribution as a percentage of the first. It can be seen that the highest 2nd order contributions come from the linear method (as expected) and that the other results were very similar. The second order contributions were higher for case ‘a’ than ‘c’ which matched the higher second order contribution to the maximum crest elevation from case ‘a’ (18% compared to 14%). The elevations were also higher for case
It is interesting to note that the various stretching methods were introduced to model discrepancies between measurements and linear theory, yet the difference between the theories is minimal at 70% of the bed-crest elevation. This corresponds to an elevation of \(-124\, mm\) and \(-136\, mm\) for 'a' and 'c' respectively. Therefore measurements must be made above this elevation for a reasonable choice to be made between theories. At 90% of bed-crest height (\(-5\, mm\) and \(-20\, mm\) for 'a' and 'c') the difference between linear and the rest is apparent enough but a choice between Wheeler and superposition cannot be made from considering the relative size of first and second order contributions. These change equally on ascending a given wave.

5.3.2 Comparisons with Fourier and Stream Function Theories

The measurements were then compared to the results from the stream function and fourier theories in figure 5.5. These are both higher order, regular wave theories (see sections 2.4 and 2.5 respectively). The stream function program was formulated according to Stokes’ first definition of wave celerity and the second definition was chosen for the fourier method. The second definition should prove the more accurate, for a flume situation.

The results of the comparison show that both theories perform adequately well in most of the cases. It is only in the case of the highest wave (case 'a') that the errors rise above about \(25\, mms^{-1}\), and only then above MWL. The underprediction at the highest measurement was 11%. The difference between the two theories at \(-0.2\, m\) was \(17\, mms^{-1}\), comparable to the mean eulerian velocity calculated by the fourier theory \((14\, mms^{-1})\). The fourier theory matched the measurements and the stream function results were higher, as expected. These were from fifth order
Figure 5.5: Velocity Profiles for Equal Amplitude Groups, Compared to Higher Order Theories.
calculations as these were found to give exactly the same results as ninth order.

5.3.3 Summary of Results.

The previous two sections showed the comparisons between each of the methods and the experimental results. This section draws together some of the best of those comparisons and makes some conclusions. The methods chosen are linear theory in the range covering the fundamental harmonics \((j = 1 - 45)\), Wheeler, superposition and delta stretching in the range covering the fundamental and second harmonics \((j = 1 - 75)\) and the fourier approximation method. The linear theories were calculated using the amplitude spectra as inputs, while the fourier theory used the measured wave height and period with the depth to determine its values. The choice was not made to determine the very best — Chakrabarti \(j = 1 - 75\) performs better than the delta used — but to reflect the best from each group of results. The results are somewhat inconclusive as no theory managed to match the experimental velocity profiles best for each wave. They are shown in figure 5.6.

The conclusions drawn from these comparisons are:

**Fourier method.** This performs well for all but the trough to crest region of the highest wave 'a' where the theory underpredicts the results by just over 10% at the highest measurement.

**Linear theory.** This performs very well in the frequency range covering the fundamental harmonics \((j = 1 - 45)\) for the highest and lowest waves. Unfortunately it overpredicts the measurements from the other two waves. The error is as high as 15% in the trough to crest of wave 'b.' Linear theory significantly overpredicts the measured velocities for all 4 waves when used with the larger frequency range \((j = 1 - 75)\).
Figure 5.6: Comparison Between Velocity Profiles under Equal-Amplitude Groups and Selected Theories.
**Wheeler stretching.** This theory models the results of the lower three waves adequately using the larger frequency range (including second order terms). It does, however, underpredict the results from the highest wave by about 20% at the highest point. It underpredicts the measurements for all 4 waves when used with the smaller frequency range (comprising first order terms).

**Superposition stretching.** This theory produced results which were just on the upper limit of the measurements for the lower 3 waves, using the larger frequency range. Unfortunately it underpredicted the velocities in the trough to crest region of the highest wave by around 10%. It is interesting to note that the percentage difference hardly varied over the crest to trough region; only the magnitude increased with elevation. The results from using the lower frequency range were worse in all cases.

**Delta stretching.** The delta stretching results were not well conditioned for these circumstances. This is the same as for the regular waves in chapter 4 and in both cases the stretching must be performed from a much lower height for the results to fit well with the measurements.

The conclusions that can be drawn for the other main wave kinematics methods are that Chakrabarti stretching produces results which are consistently lower than the Wheeler results and so Wheeler stretching is to be preferred. The differences between extrapolation and linear theory are not large enough to enable a reasonable choice to be made between the two.

The main differences from the conclusions of the regular wave studies of chapter 4 are that:

- Linear theory ($j = 1 - 45$) performed a lot better for the highest wave.
- Wheeler stretching ($j = 1 - 75$) performed worse for the highest wave.
- Fourier method has not remained conservative in all cases.
• Superposition stretching in the range $j = 1 - 75$ performs dramatically better than for regular waves, while in the range $j = 1 - 45$ the performance is a lot worse.

5.4 Comparisons with Two-component, Equal Component Steepness Groups.

The waves investigated in this section are again the central waves in four two-component groups, as in the previous section. Here however, the two component waves had the same wave steepness $a_1k_1$ rather than the same amplitude. An iterative scheme was used again to ensure that the two steepnesses were equal and their phases zero at the measuring point in space and time. The steepnesses were calculated from the amplitude spectra with the wavenumber $k$ being calculated from frequency by theory. Some of the details of the measured waves are given in table 5.4.

As before the frequencies used were $j = 25$ and $j = 35$ and the symbols have the same meaning. The steepnesses of the primary components are within 5% for each group. The variation in the central wave period is less than for the equal-amplitude groups and does not follow the pattern of decreasing period with increasing steepness shown in those waves. The standard deviation of surface elevation increases relative to the central wave height from case ‘a’ to ‘d’ as the wave height and steepnesses decreases. This is the same trend as for the equal amplitude waves.

The amplitude spectra and surface profiles for the waves are shown in figure 5.7. All measurements were made at a time of 12s.
Figure 5.7: Amplitude Spectra and Surface Elevation Records for Equal Steepness Wave Groups
<table>
<thead>
<tr>
<th>wave</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experiment Reference</td>
<td>e9n3</td>
<td>e9n5</td>
<td>e9n2</td>
<td>e9n6</td>
</tr>
<tr>
<td>Amplitude (mm) for ( j = 25 )</td>
<td>31.9</td>
<td>30.1</td>
<td>26.6</td>
<td>23.7</td>
</tr>
<tr>
<td>Phase (rad) for ( j = 25 )</td>
<td>0.07</td>
<td>-0.01</td>
<td>-0.03</td>
<td>-.01</td>
</tr>
<tr>
<td>Amplitude (mm) for ( j = 35 )</td>
<td>16.7</td>
<td>16.1</td>
<td>14.3</td>
<td>12.9</td>
</tr>
<tr>
<td>Phase (rad) for ( j = 35 )</td>
<td>0.10</td>
<td>-0.01</td>
<td>-0.00</td>
<td>-.02</td>
</tr>
<tr>
<td>Height (mm)</td>
<td>109.0</td>
<td>101.5</td>
<td>86.4</td>
<td>74.4</td>
</tr>
<tr>
<td>Period (s)</td>
<td>0.87</td>
<td>0.91</td>
<td>0.89</td>
<td>0.88</td>
</tr>
<tr>
<td>Relative steepness ( (H/gT^2) )</td>
<td>.0148</td>
<td>.0125</td>
<td>.0110</td>
<td>.0099</td>
</tr>
<tr>
<td>Relative depth ( (D/gT^2) )</td>
<td>.073</td>
<td>.066</td>
<td>.069</td>
<td>.0718</td>
</tr>
<tr>
<td>Mean eulerian velocity (mm/s)</td>
<td>-18.6</td>
<td>-15.7</td>
<td>-11.7</td>
<td>-9.0</td>
</tr>
<tr>
<td>Standard deviation of ( \eta ) (mm)</td>
<td>26.2</td>
<td>24.7</td>
<td>21.7</td>
<td>19.5</td>
</tr>
<tr>
<td>Steepness ( ak ) for ( j = 25 )</td>
<td>126</td>
<td>119</td>
<td>103</td>
<td>94</td>
</tr>
<tr>
<td>Steepness ( ak ) for ( j = 35 )</td>
<td>126</td>
<td>121</td>
<td>108</td>
<td>97</td>
</tr>
</tbody>
</table>

Table 5.4: Wave details from equal component steepness groups.

5.4.1 Comparisons with Stretching Methods

The comparisons between the PIV results and the stretching methods in the frequency range \( j = 1 - 75 \) (including fundamental and second harmonics) can be seen in figure 5.8. In this case the delta stretching results are not shown as they remained ill-conditioned. The methods split into the same groups as for the previous section and the results are similar.

Chakrabarti stretching produced results lower than Wheeler stretching below the crest and both underpredicted the measured velocity profiles for the higher waves. The superposition stretching performed better than both the other methods and matched the results from all 4 waves accurately. The linear and extrapolation results massively overestimated the velocities in all cases. The best method for
Figure 5.8: Velocity Profiles Under the Crest of the Central Waves in an Equal Steepness Group Compared to Stretching methods ($j = 1 - 75$).
Figure 5.9: Velocity Profiles Under the Crest of the Central Waves in an Equal Steepness Group Compared to Stretching Methods ($j = 1 - 45$).

The range $j = 1 - 75$ is the superposition stretching.
The comparisons with the same methods in the range $j = 1 - 45$ are shown in figure 5.9. The conclusions for Wheeler, Chakrabarti and superposition are not surprising. Each underpredicts the results in all cases with superposition being the best of the three. In each case the previous result for $j = 1 - 75$ is better. Conversely the results for linear and extrapolation are better in the range $j = 1 - 45$ as they are reduced. The results for the lower three waves are overpredicted as are the results for the steepest wave under the crest. The linear results tend to the highest measurements at the crest though.

5.4.2 Comparisons with Higher Order Theories

The measured velocity profiles are compared with the higher order regular wave theories in figure 5.10. The results of the comparisons show that the fourier method using Stokes' second definition of mass transport performs better than the stream function results using the first. Both theories underpredict the results above MWL however. The fourier theory underpredicts the velocity at the highest measurement of the largest wave by around 9% and of the second biggest by around 13%. The differences only become apparent above MWL, suggesting that this is a high frequency effect. This fits with the situation of the smaller amplitude, higher frequency wave riding on top of the lower frequency dominant wave as is the case here.

5.4.3 Summary of Results

The previous two sections show comparisons between measured velocity profiles and theories. This section states some conclusions that can be drawn from these results and presents a comparison between the measurements and selected methods. These are Wheeler and superposition ($j = 1 - 75$), linear ($j = 1 - 45$) and the fourier method and the results are in figure 5.11.
Figure 5.10: Velocity Profiles Under the Crest of the Central Waves in an Equal Steepness Group Compared to Higher Order Theories.
Figure 5.11: Comparison Between Selected Methods and Equal-steepness Group Measurements
The conclusions that can be drawn from the comparisons with two-component, equal-steepness wave groups measured under the central wave are:

**Superposition stretching** \((j = 1 - 75, \text{ including the 2nd harmonics})\). This theory performs as the best of the methods, when the return current is not taken into account. If it is then the theory marginally underpredicts the results, which lie between the predictions for the superposition and linear methods. Results are all underpredicted using the lower frequency range.

**Wheeler stretching** \((j = 1 - 75)\). This underpredicts all the results above about twice the trough depth below MWL. The degree of underprediction is even greater for the lower frequency range.

**Chakrabarti stretching** . This performs similarly to Wheeler but with even lower results under the crest. The Wheeler stretching is more accurate in all cases.

**Linear theory** \((j = 1 - 45, \text{ containing the fundamental harmonics})\). This overpredicts all the results except the highest results from the highest wave. If however the mean eulerian velocity (from the fourier method) was added to the linear results then the measured results would be quite close to the theoretical ones. This would leave linear theory as a slightly conservative theory in all cases bar a negligible underprediction at a few points. Using the higher frequency range produced a massive overestimation of all velocities. The conclusions for extrapolation are the same as the above for linear theory.

**Fourier theory**. This performed excellently below MWL but was unable to model the rapid rises in velocity above there. Stream function theory performed slightly worse as it assumed a mean eulerian velocity of zero.
5.5 Conclusions

A summary of the chapter's results is given below:

- Fourier theory performs accurately below MWL for both sections. It is unable to model the rapid increase in velocities above MWL in either case however.

- Linear theory (over the smaller frequency range of just first order waves) overpredicts the results in all cases. In both sets of comparisons the results do tend to the measured ones at the crest of the highest wave. Linear theory may be regarded as a conservative theory that becomes more accurate at the highest velocities, on the evidence of these two sets of comparisons. The results from linear theory often tended towards the same values as those from superposition theory (including 2nd harmonics) at the crest.

- Wheeler stretching (including 2nd harmonics) tended to underpredict the measurements in the highest waves in both sets of experiments.

- Superposition stretching matches the measured velocities most often. The only time that it does not fit to within the limits of experimental accuracy is in the crest region of the highest wave where the underprediction is within \( \approx 10\% \).

This chapter serves to illustrate the limitations of higher order theory in modelling wave groups and the problems lie in the most important area — in the crests of the highest waves. Chapters 4 and 5 also show the difference between the ability of the linear-based methods to model regular waves and groups of waves. Wheeler stretching is the best theory in comparison with the regular waves but performs worse than superposition stretching in the group cases. Superposition stretching overpredicts all the velocities for the regular waves in chapter 4.
Chapter 6

Waves of the Same Characteristics

The previous two chapters have considered the behaviour of single waves or two-component groups of waves as the amplitude of the component waves has increased. Measurements made have been compared to theories and it has been shown that some theories that model the experimental results well at low wave amplitudes fail to model the wave behaviour near the free surface in higher waves.

This chapter contains more of these comparisons but also contains comparisons between waves of the same gross characteristic, such as height and period. The reason for this is that the common design practice (see section 1.2.2) is to approximate the highest wave in a sea by a regular wave of the same height and period and to hope that the kinematics calculated for the regular wave match those of the highest wave. This assumption is tested by the experimental comparisons presented here.
6.1 Comparisons Between Waves of the Same Characteristics in Different Group Environments, Part 1; \( H/gT^2 \approx 0.015. \)

In this section the waves used are of essentially the same steepness but exist in groups made up of different frequencies. Therefore the gross characteristics of height and period remain the same but the waves exist in a different group pattern. All 4 wave groups are based on a frequency of 24/25.6Hz and therefore have an average frequency number 'j' of 24. The wave group length decreases on going from case 'a' to 'd' as the separation of the frequency numbers increases. Some details of the measured waves are given in table 6.1.

The amplitude spectra and surface elevation records for the measured waves are shown in figure 6.1.

Initially an iterative scheme was used to get the input (or primary) wave components to have the same total amplitude. The amplitudes of the final two groups 'c' and 'd' had to be reduced to prevent the onset of wave breaking in the flume. Wave breaking is here defined as the generation of a broken (non-smooth) wave surface (usually at the front edge of the wave crest). The wave periods were very similar except for case 'd' where the wave period was less. This results in case 'd' having a higher relative wave steepness \( H/gT^2 \) rather than component steepness \( a_k \). The ratio of central wave height to standard deviation of surface elevation \( H/\sigma_n \) here increased on going from the regular wave case 'a' to the shortest group 'd.' This is to be expected due to the periods of destructive interference between the waves in the group.

The surface elevation records for this section were recorded from the measurement point. To show a wave group with an equivalent of the measured wave at the centre, it was necessary to use a later section of the wave record. The section
<table>
<thead>
<tr>
<th>wave</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
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<td>e1n101</td>
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<td>e1n98</td>
</tr>
<tr>
<td>1st Frequency No. $j_1$</td>
<td>24</td>
<td>23</td>
<td>22</td>
<td>20</td>
</tr>
<tr>
<td>2nd Frequency No. $j_2$</td>
<td>—</td>
<td>25</td>
<td>26</td>
<td>28</td>
</tr>
<tr>
<td>Amplitude (mm) for $j_1$</td>
<td>77.7</td>
<td>39.3</td>
<td>37.1</td>
<td>34.1</td>
</tr>
<tr>
<td>Phase (rad) for $j_1$</td>
<td>-.24</td>
<td>-.21</td>
<td>-.23</td>
<td>-.20</td>
</tr>
<tr>
<td>Amplitude (mm) for $j_2$</td>
<td>—</td>
<td>39.2</td>
<td>37.8</td>
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</tr>
<tr>
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<td>-.40</td>
</tr>
<tr>
<td>Height (mm)</td>
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<td>154.0</td>
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<td>Period (s)</td>
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<td>1.06</td>
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</tr>
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<td>Relative steepness ($H/gT^2$)</td>
<td>0.0149</td>
<td>0.0146</td>
<td>0.0147</td>
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<td>Relative depth ($D/gT^2$)</td>
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<td>0.048</td>
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<td>Mean eulerian velocity (mm/s)</td>
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<td>Standard deviation of $\eta$ (mm)</td>
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<td>39.2</td>
<td>37.4</td>
</tr>
<tr>
<td>Steepness $ak$ for $j_1$</td>
<td>0.285</td>
<td>0.134</td>
<td>0.118</td>
<td>0.093</td>
</tr>
<tr>
<td>Steepness $ak$ for $j_2$</td>
<td>—</td>
<td>0.155</td>
<td>0.160</td>
<td>0.165</td>
</tr>
</tbody>
</table>

Table 6.1: Details of $H/gT^2 \approx 0.015$ Waves.
Figure 6.1: Amplitude Spectra and Surface Elevation Records for Equal Steepness Waves in Different Groups
chosen is centred on a time 12.8s after the measurement time, as this is half the maximum repeat time. Therefore the wave records 'a', 'c' and 'd' show the equivalent of a wave record centred on the measurement point, only recorded at a later time. In case 'b' the envelope repeat time is 25.6s so what is seen is the wave beat following that with the measured wave, hence the trough at the centre.

The vertical asymmetry between the maximum crest elevations and minimum trough ones is obvious in this diagram (with a vertical axis from -0.1 to +0.1m).

### 6.1.1 Comparisons with Stretching Theories

The waves here are based on an average frequency number of 24 rather than 30, as previously. This means that the boundary between first and second harmonic changes from $j = 45$ to $j = 36$. The boundary between second and third harmonic is now at $j = 59$ and that between third and fourth is at $j = 84$. The upper frequency limit was previously put at $j = 75$ but is here increased to $j = 84$ ($f = 2.89\text{Hz}$) to allow an integer number of the harmonic ranges to be considered.

The comparisons between the measured velocity profiles and the stretching theories in the ranges $j = 1 - 84$ and $j = 1 - 59$ are shown in figures 6.2 and 6.3 respectively. The former covers the first to the third harmonic inclusive and the latter covers the first and second.

The conclusions that can be drawn are that both Wheeler and Chakrabarti theories underpredict measured velocities in all 4 cases. The fit is best at the top and bottom and worst around MWL. Including the third harmonic had only a marginal effect on the regular wave, case 'a', adding only 1.5% to the crest velocity for example. The effect of the third harmonic increased as the groups became shorter and the number of components above the minimum amplitude threshold of 0.3mm increased. Its contribution in cases 'c' and 'd' were 7% and 13% respec-
Figure 6.2: Comparison Between Stretching Theories ($j = 1 - 84$) and Measured Velocity Profiles Under the Crests of 4 Waves of the same Steepness in Different Environments.
Figure 6.3: Comparison Between Stretching Theories (j = 1 - 59) and Measured Velocity Profiles Under the Crests of 4 Waves of the same Steepness in Different Environments.
tively. The theories underestimated the velocities by a significant amount when only the first harmonic was used.

Superposition stretching overestimated the velocities for the regular wave (as in chapter 4) for both frequency ranges shown. The theory modelled the results for the two longer groups, waves ‘b’ and ‘c’, well, overestimating the results above MWL only, and then by just a few percent. If however the results were shifted to take into account the mean eulerian flow (typically $-33\text{mms}^{-1}$) then the overestimation at the crest would be less and there would be underestimation below MWL approximately equal to the mean eulerian flow.

The effect of including the third harmonic in the calculations for the superposition was small for the regular wave but had increased to about 7% at the surface for case ‘c’. The corresponding figure for case ‘d’ was almost 13%. This had decreased to under 3% at MWL. This third harmonic frequency range helped the technique to increase its velocities near the crest so that the degree of underprediction (not overprediction as in all the other cases) was reduced to about 5% at the highest measurement point.

Linear theory and extrapolation significantly overpredict the measurements in all four cases for both frequency ranges shown. The fit for the range $j = 1 - 36$ including the first harmonic only is a lot better, as can be seen in figure 6.6. The effect of including the third harmonic becomes noticeable only for the shortest two groups, ‘c’ and ‘d’ and is largest ($\approx 60\%$ of the crest total) for ‘d’, as before.

Delta stretching performs better at predicting the crest velocity of the shortest group’s central wave than any of the other theories in the range $j = 1 - 59$. Nevertheless superposition stretching performs better for the higher range and the delta theory is still ill-conditioned overall.
6.1.2 Comparisons with Higher Order Theories

The velocities under the crests and the preceding troughs of the waves in this section were measured using PIV. The PIV measurements were done at the same place along the wave flume but at half a (regular) wave period apart. The measured velocity profiles under the crest and trough are compared to the profiles generated by Fourier theory in figure 6.4. These velocity profiles are generated for a regular wave from the measured zero downcrossing wave height and period. The accuracy of the Fourier theory (where accuracy is defined by the difference between the Fourier solution and the exact mathematical solution) is determined by the number of Fourier components used. Here eight are used [91]. The profiles are calculated for both Stokes first (zero mean Eulerian velocity) and second (zero mass transport) definitions of wave celerity.

The second definition of wave celerity is found to perform better than the first in all four cases. The first definition of celerity is still probably the most commonly applied one and theoretical waves calculated using it produce a non-zero mass transport. This was traditionally assumed to be the case in the sea where the horizontal boundaries can be considered to be at infinity. The second definition of celerity was considered to be valid in a finite-length wave flume or tank, where the finite length would ensure the creation of a ‘mean Eulerian velocity,’ opposite to the direction of propagation of the waves, which would balance the forward momentum created by the wave motion. This logic is now being questioned due to the measurement of a mean Eulerian velocity under water waves in a flume as soon as the waves start to be generated [44]. This implies that the mean Eulerian velocity is a feature of the wave and is not generated by reflection off the end walls of the flume. If so then Stokes second definition of wave celerity is the correct one in all cases and not just in flumes. The appropriateness of using the second definition in flumes is demonstrated in figure 6.4.

The Fourier theory models the measurements very well in both crest and trough...
Figure 6.4: Velocity profiles for equal steepness waves compared to fourier theory. Both Stokes' first and second definition of wave celerity are used in calculating velocities from the measured height and period.
regions of the wave in the longest 3 groups ('a' to 'c'). There is no evidence to doubt the predictive capacity of regular wave theories on the basis of comparison with those waves. In particular there is no reason to doubt the capacity of a regular wave theory to model a regular wave. In light of this it is somewhat ironic to recall that many of the stretching approximations to linear theory were introduced to take into account the differences between measurements and regular wave theories set up using Stokes first definition of wave celerity [43, 80, for example]. As many such differences are due to the choice of definition of wave celerity these modifications were introduced partly for the wrong reasons, so their success at modeling many of the results may be regarded as somewhat fortuitous.

The central wave in the shortest group, case 'd,' does diverge from the regular wave predictions. At the elevation of the highest measurement the fourier theory underpredicts the measurement by about 17%, an error that is reduced to zero by MWL. The match is not perfect under the trough either, except near the free surface. In both cases (trough and crest) this velocity profile has more the appearance of a smaller wave with a higher frequency component riding on top.

6.1.3 Comparisons Between Experiments and Between Theories.

The waves used in this comparison are all of approximately the same height and period. As a consequence the results from one wave can be plotted against the results from another to see how waves of similar characteristics compare. To reduce the effects caused by the differences in the waves the results are non-dimensionalised.

This is done by dividing the velocities by $gT$ and the elevations by $H$. The factors $g$, $H$ and $T$ are used for the following reasons; $g$ as the waves are gravity waves and $T$ and $H$ as the measured wave period and height are the most common parameters.
used in defining waves. Therefore the vertical axes show results from elevations of 1.5 (or 2) times the wave height below MWL up to 1 wave height above MWL. The wave crest are at about +0.5 on this scale, although the actual value will increase from almost exactly +0.5 for a low, linear wave to higher values for a more extreme, non-linear wave. The horizontal axes show velocities divided by a scaling factor of $gT = 2\pi C_0$ where $C_0$ is the deep water linear theory wave celerity. It follows that the non-dimensional velocity 0.08 corresponds to $0.50 \times C_0 = C_{90}$ where $C_{90}$ is the deep water linear theory wave group velocity.

The comparisons between the measured results and between the velocity profiles are shown in figure 6.5. This figure shows four graphs (on the same scale). The top graph shows the different sets of measured horizontal (and non-dimensionalised) velocity profiles under the crests of the central waves (component phases $\approx 0$) in each group. This shows that the results for the waves ‘a’ to ‘c’ (the regular wave and the two longest groups) follow substantially the same pattern. Deep water linear wave theory gives a maximum crest velocity of $u/gT = 0.063$ for waves of steepness $H/gT^2 = 0.015$, if $H$ and $T$ are used as the inputs. The experimental results of cases ‘a’ to ‘c’ tend to a value close to this at the crest.

The corresponding velocity for a wave of steepness $H/gT^2 = 0.016$ (case ‘d’, wave e1n98) is $u/gT = 0.069$. The measured value is just over 0.08 which corresponds to a value just over the deep water linear wave theory group celerity. Note, however, that the wave is not entirely deep water and if non-dimensionalisation is done with respect to (non-deep) linear wave theory then the values tend to the linear theory wave group celerity.

The shortest wave group (d) produced the highest velocities above MWL and the most negative beneath the trough. The (non-dimensionalised) velocity of the wave in the shortest wave group was about 30% higher than the highest of the other measured velocities (at half the wave height).
Figure 6.5: Comparisons between the Velocity Profiles from Different Waves.
The second graph in figure 6.5 shows the values for the velocity profiles calculated from Fourier theory. The results for cases ‘a,’ ‘b’ and ‘c’ follow a similar spread to those from the experiments, with the steepest wave producing the highest velocities in and around the wave crest. The results from the shortest group (case ‘d’) produce the highest non-dimensional velocities. This is the second highest, but steepest, wave. The results all reduce to the same value at about one wave height below MWL.

The third graph on figure 6.5 shows the velocity profiles from Wheeler stretching including the first to third harmonic ranges. Below MWL the results are very similar. Above MWL the wave in the shortest group provides the highest velocities while the others are very similar. This is due to the relative sizes of the higher frequency contributions as shown by the amplitude spectra in figure 6.1. The number and amplitude of the higher frequency components increases on going from the regular wave to the shortest group. If the results are plotted for the frequency range encompassing only the first two harmonics of each component then all the Wheeler results match each other to within 3% at the crests.

The bottom graph of figure 6.5 shows the four velocity profiles generated by superposition stretching. Here the regular wave has velocities greater than the groups of waves — the difference being of the order of 10% below MWL. The result for the shortest group increases towards the regular wave value above MWL. The two intermediate groups produce very similar results.

The measured velocity profiles are compared to a selected group of methods in figure 6.6. The selected methods are linear theory using the frequency range \( j = 1 - 36 \) (including fundamental harmonics only), Wheeler and superposition stretching (using the first to third harmonics inclusive) and Fourier theory.

The results from linear and Fourier theories followed the same pattern. The Fourier results are typically around 10% less than the linear at an elevation of \(-1.5H\) with
Figure 6.6: Comparison Between the Measured Velocity Profiles and those from Selected Theories. Waves have $H/gT^2 \approx 0.015$. 

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the difference generally reducing as elevation increases towards the crest. This difference is approximately the same as the mean eulerian current calculated by the fourier theory. In both cases the theoretical values match the form of the results for cases ‘a’ to ‘c’ but underpredict the results for the crest of case ‘d’ significantly.

Wheeler stretching fares worse than either linear or fourier theory at predicting the velocities in the crest to trough region. It does however come close to their values at the crests of the waves.

Superposition stretching performs the worst of all the theories at predicting the velocities of cases ‘a’ to ‘c’ above half the crest elevation over MWL. It is however best at predicting the velocities in case ‘d’ and the fit to the non-regular waves is always adequate below MWL.

6.1.4 Summary of Results

The waves measured in this section all had approximately the same gross characteristics. They were all based on the same central frequency of 24/25.6Hz (i.e. frequency number \( j = 24 \)) and were a regular wave (case ‘a’) and the central wave in three two component groups. The groups had component waves of the same amplitude, with the frequency separation increasing from case ‘b’ to case ‘d’. The waves did not have the same measured zero-downcross period, however. The period decreased as the envelope period decreased. This resulted in the wave from the shortest group having the highest relative steepness. Nevertheless the waves were considered to be close enough to be compared both with theories and with the other waves — providing a suitable non-dimensionalisation was used in the comparisons between waves.

The graphs showing the main comparisons have already been shown and discussed
Measurements. The measurements of the similar waves gave similar results except for the wave in the shortest group. This was close to breaking and produced velocities around 30% higher than in any of the other waves. The non-dimensionalised velocity of this wave tends to a value about 20% greater than maximum non-dimensionalised velocity predicted by linear theory for a wave of that steepness. The waves in the three longest groups (including the regular wave) tend at the crest to the maximum non-dimensionalised velocity predicted by linear theory for a wave of that steepness.

Fourier theory. The fourier theory modelled the results for the regular wave very accurately using Stokes' second (zero mass transport) definition of wave celerity (see figure 6.4). This theory also matched the results for the two non-extreme wave groups very accurately. It only failed to model the results for the near-breaking wave in the central group.

Linear theory. This theory matched the form of the fourier theory very well when the frequency range was limited to that around the fundamental frequencies. This is somewhat surprising due to the difference in complexity of the theories and the difference in the inputs used in the calculations (the amplitude spectra for linear theory and the measured $H$ and $T$ for fourier). The results were all greater than the fourier results by an amount approximately equal to the mean eulerian current calculated by fourier theory. The closeness to the fourier theory meant that linear theory also failed to model the measured high velocities in the crest of the extreme wave in the short group.

Wheeler stretching. This underpredicts the velocities in the crest to trough region for all four waves. The frequency range used included the first to third harmonics. It matches the results well enough at an elevation of $-1.5H$ and
also tends to the measured results in the crests of the waves in cases 'a' to 'c'. It underpredicted the velocities in the crest of the extreme case 'd'.

Superposition stretching. This performed best at modelling the velocity profile of the shortest group 'd' but was worst at modelling the regular wave 'a'. The results for the other two groups show an accurate match with the experimental results below about half the crest elevation then an increasing overprediction of the results above.

General observation. None of the theories could match the rapid increases in velocity with elevation in the crest to trough region shown by the wave in the shortest group. The theory that came closest, superposition stretching, overestimated the crest velocity of the other waves above MWL.

6.2 Comparisons Between Waves of the same Characteristics in Different Group Environments, Part2; \( H/gT^2 \approx 0.012. \)

This section contains four further comparisons between waves with the same relative steepness. In this case the waves are of a lower steepness than before and the waves are based on a higher central frequency of \( 30/25.6 \)Hz. They include the central wave in a group made up of two waves with the same component steepness, \( ak \), two waves from groups with the same component amplitude and a regular wave. Some details of the measured waves are given in table 6.2.

The waves are compared in detail with the various theories in other sections. The wave group length decreases on going from case 'a' to case 'c' which has the same length as case 'd'. Cases 'c' and 'd' are from groups of the same length composed from components of the same frequencies and producing waves of the
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<th>c</th>
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<td>.0121</td>
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<td>.075</td>
<td>.084</td>
<td>.066</td>
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<td>0.130</td>
<td>0.144</td>
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Table 6.2: Measured Details of $H/gT^2 \approx 0.0125$ waves
same steepness. Only the ratio of amplitudes is different.

One point of interest from the table is that the waves vary in height and period — it is only the ratio \( \frac{H}{gT^2} \) that remains about equal. This varied over a very small range of 0.0121 to 0.0125 only, close enough for the steepnesses to be taken as equivalent. The amplitude spectra and surface elevation records are shown in figure 6.7.

6.2.1 Comparisons between Experiments and Between Theories.

The differences in height and period between cases ‘c’ and ‘d’ can be seen in both the table and the surface elevation diagrams in figure 6.7, so the results are non-dimensionalised as before in figure 6.8 where the results from different waves are compared. The same four graphs are plotted here as in the previous section (figure 6.5).

The top graph shows non-dimensionalised experimental velocities against non-dimensionalised elevation. The results show a spread of ±10 to 15% around a central value for the velocity at a given elevation. The results from case ‘d’ (equal \( ak \) components, shortest group) are the highest at a given elevation above MWL with the results from case ‘c’ (equal amplitude components, shortest group) the highest below.

The second graph shows the non-dimensional velocity profiles produced by fourier theory. The results are virtually identical. The third graph in figure 6.8 is of the Wheeler stretching results for the frequency range \( j = 1 - 75 \) up to and including the second harmonics. Here the results for case ‘d’ (equal \( ak \), shortest group) produce the highest velocities above MWL. The other three cases provide similar velocity profiles.
Figure 6.7: Amplitude Spectra and Surface Elevation Records for Equal Steepness Waves in Different Groups, Part 2; $H/gT^2 \approx 0.012$. 
Figure 6.8: Comparison Between Velocity Profiles from Different Waves. All Waves have $H/gT^2 \approx 0.0125$. 

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The bottom graph in figure 6.8 shows the non-dimensionalised velocity profiles calculated using superposition stretching also in the frequency range up to and including the second harmonics. The results have similar profiles, but with cases 'a' (regular wave) and 'd' (equal $ak$, shortest group) producing higher crest velocities than the equal component amplitude groups (cases 'b' and 'c').

Overall the theoretical velocity profiles produced a lower spread in results than the experimental profiles. Case 'd' provided the highest values both for the experimental velocity profiles and for Wheeler and superposition profiles. The regular wave and the other groups (cases 'a' to 'c') produced very similar results for all graphs in figure 6.8 except for the superposition profiles where the regular wave produced higher velocities.

The measured velocity profiles are compared to a selected group of theories in figure 6.9. The selected theories are linear theory (using the frequency range $j = 1 - 45$ including fundamental harmonics only), Wheeler and superposition stretching (using the fundamental and second harmonics) and fourier theory.

Fourier theory performs very well for cases 'a' and 'b' which are the most regular in nature but fails to model the results in the trough to crest region in case 'd' (equal $ak$, shortest group). Linear theory overestimates the velocities for all four cases and is outperformed by Wheeler stretching. This theory performs very well for cases 'a' to 'c' but produces slight underestimations of the crest velocities in case 'd'. Superposition stretching, on the other hand, produces results which match the velocity profile of wave 'd' accurately but which overestimate the crest velocities of the other three cases. The degree of overprediction increases as the wave groups become longer.
Figure 6.9: Comparison Between the Measured Velocity Profiles and those from Selected Theories.
6.2.2 Summary of Results

The second set of comparisons for waves of the same steepness \((H/gT^2)\) differs from the first in two main ways. The steepness of the waves is lower \((\approx 0.012\) against \(\approx 0.015\)) and the waves measured have different characteristics. The central frequency is higher \((30/25.6\text{Hz})\) and the waves again are a regular wave, and the central wave in three two-component groups. Cases 'b' and 'c' are equal component amplitude groups — the first long, the second as short as the shortest group in the first set of comparisons. Case 'd' is a group with equal component steepnesses \((ak)\) and the same frequencies used in case 'c'. The conclusions for the second set of comparisons between waves of the same relative steepness are as follows:

Experimental results. The non-dimensionalised velocity profiles follow the same pattern as the theoretical ones but with a greater spread in values at a given elevation. The profiles from the shorter wave groups show the greater velocities, with the equal component steepness \((ak)\) case being higher than the equal amplitude case above MWL.

Linear theory. This overestimates the results in all four cases and is outperformed by Wheeler stretching.

Wheeler stretching. This performed as the most accurate of the theories for the most regular of the cases and for the shortest group with equal amplitude components. It did, however, underpredict the velocities for the most extreme case (defining most extreme as being the one with the highest velocities for a given value of steepness) \(i.e.\) case 'd'.

Fourier theory. This performs well as a conservative theory below MWL. It is not able to model the rapid increases in velocity near the crest of the shortest wave groups.
Superposition stretching. This models the rapid increase in velocities of the waves from the shortest groups accurately, unlike any of the other theories. It overpredicts the crest velocities of the other cases, however, with the overprediction worsening as the waves become more regular.

6.3 Comparisons Between Waves of the Same Characteristics in Different Group Environments, Part 3; equal $ak$.

The waves in this section all have the same total component steepness for their first order waves. In other words

$$\sum_n a_n k_n = \gamma$$

(6.74)

Here $a_n$ and $k_n$ are the amplitude and wavenumber of the $n$th (fundamental harmonic) component and $\gamma$ is the total of the component steepnesses. In this section $\gamma \approx 0.25$.

In the previous chapter one set of comparisons is between waves where the components have equal steepnesses ($ak$). The last section has comparisons between waves with the same wave steepness ($H/gT^2$). In this section the same frequency combinations are used as in the previous section but the amplitudes are altered so that the total component steepness of the input waves was the same in all cases. The waves are compared to one another and to theories to see if there is any dependency on total component wave steepness or if any conclusions can be drawn. Some details of the measured waves are given in table 6.3.

If the overall wave steepness $a_t k_t$ is calculated from the measured height and period, assuming $a_t = H/2$ and $k_t = (\frac{2\pi}{g})^{21}$, then $a_t k_t$ is proportional to $H/gT^2$. The table shows that the waves do not have the same overall steepness. They do
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Table 6.3: Measured details of equal $\sum \alpha k$ waves.
however have the same steepnesses (±2%) calculated from the sum of the steepnesses of the input component waves. Thus the four waves have the same total component steepnesses, but from different combinations of component steepnesses, resulting in different overall wave steepnesses.

The wave spectra and surface elevation records are shown in figure 6.10.

6.3.1 Comparisons between Experiments and Between Theories.

The wave velocities are again non-dimensionalised by dividing by $gT$ and the elevations by dividing by $H$ for the comparisons between waves shown in figure 6.11. This shows comparisons between the measured velocity profiles from each wave in the top graph and comparisons between the theoretical velocity profiles from a different theory in each of the other three.

The top graph shows clear differences between the results. The wave from the short group with equal component steepnesses (case ‘d’) has velocities which are over 25% higher than those from any other wave at the elevation of the highest measurements. The wave from the long group with equal amplitude components (case ‘c’) provided the lowest velocities at any given elevation. The crest velocities vary according to wave steepness ($H/gT^2$) rather than component steepness ($ak$). The crest velocity of the highest steepness wave (case ‘d’, $H/gT^2 = 0.0148$) matches that expected from deep water linear wave theory using $T$ and $H$, rather than the amplitude spectrum, as input. The results from the other waves are all below the results expected from deep water linear wave theory.

The magnitude of the velocities at a given depth vary with wave height or relative steepness. Case ‘d’ has the largest steepness and height and case ‘c’ has the lowest. The other two sets of velocity profiles are similar and come from waves of similar
Figure 6.10: Amplitude Spectra (top) and Surface Elevation Records for the Equal $ak$ Waves.
Figure 6.11: Comparisons Between Waves With the Same $\sum ak$ for Fundamental Components and Between Theoretical Predictions for these Waves. Velocities are non-dimensionalised by dividing by $gT$ and elevations by dividing by $H$. 
steepnesses and heights.

The second and third graphs in figure 6.11 show the velocity profiles from fourier theory and Wheeler stretching. The profiles in the graphs show a similar form to the experimental results. In both cases the order of the profiles at the crest (lowest crest velocity to highest) is 'b', 'a', 'c' and 'd'. Case 'c' has a lower wave height than 'a' but a greater steepness due to its slightly lower period.

The bottom graph is similar to those above except that the regular wave, case 'a', has a higher velocity profile, relative to the other waves. This is the same as for previous cases.

The measured velocity profiles are compared to a selected group of theories in figure 6.12. The selected velocities are linear theory (using the frequency range \( j = 1 - 45 \) including fundamental harmonics only), Wheeler and superposition stretching (using the fundamental and second harmonics) and fourier theory.

Fourier theory performs very well for cases 'a' to 'c' (although overpredicting the results for case 'b' slightly) but fails to model the results above MWL in case 'd' (equal \( a \kappa \), shortest group). Linear theory overestimates the velocities for all four cases, with the degree of difference increasing from 'a' to 'c'. The results for case 'd' are better above MWL as the rapid increase in measured velocities with elevation brings the measured results back up to the theoretical predictions. Linear theory is, on average, outperformed by Wheeler stretching. This theory performs very well for cases 'a' to 'c' but produces slight underestimations of the velocities in case 'd' above about one wave height below MWL. Superposition stretching, on the other hand, produces results which match the velocity profile of wave 'd' accurately but which overestimate the crest velocities of the other three cases. The degree of overprediction increases as the wave groups become longer.
Figure 6.12: Comparison Between the Measured Velocity Profiles and those from Selected Theories.
6.3.2 Summary of Results

This set of comparisons between waves of the same total fundamental component steepness, \( \sum a_i k_i \approx 0.25 \), comprised comparisons between a regular wave (case 'a') and the central waves from three two-component groups. The groups comprised a long ('b') and a short ('c') equal amplitude group and a short equal component steepness group ('d').

**Experimental results.** The non-dimensionalised velocity profiles follow the same pattern as the theoretical ones but with a greater spread in values at a given elevation. The profile from the shortest wave group (with equal steepness components) shows the greatest velocities, up to around 25% higher than any of the others. The crest velocities vary with wave steepness \( (H/gT^2) \) however, with the highest steepnesses producing the highest velocities. All the crest velocities are lower than would be expected from deep water linear theory by up to 10%, except for the highest steepness wave where the two values are the same.

**Linear theory.** This overestimates the results in all four cases. The size of overestimation is not much greater than the size of the mean eulerian velocity at an elevation of about twice the wave height below MWL (at \(-2H\)). Moreover it varies only slowly with elevation.

**Wheeler stretching.** This performed as the most accurate of the theories for the most regular of the cases and for the shortest group with equal amplitude components. It tends to underpredict answers slightly, even without the mean eulerian velocity being taken into account. It clearly underpredicts the velocities for the most extreme case (defining most extreme as being the one with the highest velocities for a given value of steepness) i.e. case 'd'.

**Fourier theory.** This performs well as a conservative theory below MWL. It is not able to model the rapid increases in velocity near the crest of the short
wave group with equal steepness components.

**Superposition stretching.** This models the rapid increase in velocities of the wave from the shortest group (with equal steepness components) accurately, unlike any of the other theories. It overpredicts the crest velocities of the other cases, however, with the overprediction worsening as the waves become more regular.

### 6.4 Comparisons Between the Steepest Waves and Less Steep Waves of the Same Component Frequencies.

The first section of this chapter showed how a high two-component wave at the centre of a short wave group could produce velocities 30% greater than those of similar waves in longer groups. The highest velocities measured were almost exactly half the deep water linear theory wave celerity (calculated from the measured zero-downcross period) or almost exactly the linear theory group velocity.

The wave which produced these velocities was both the steepest (in terms of $H/gT^2$) and from the shortest group of those measured. This section makes four sets of comparisons using this wave, a similar but less steep wave, a steeper wave from a longer group and a wave similar to that one but less steep. The highest velocities wave is compared to one with the same components but lower amplitudes to see if the same behaviour occurs at lower steepnesses. It is also compared to a steeper wave from a much longer group to see if similar behaviour can be seen there. The third comparison is between the steep wave from the long group and a similar wave with the same components but lower amplitudes to test the variation in behaviour with steepness. The final comparison is between the two lower waves which have the same steepness but are from different combinations of frequencies.
and amplitudes.

Some details of the measured waves are given in table 6.4 and their amplitude spectra and surface elevation records are shown in figure 6.13.

The waves with the same component frequencies have very similar periods, independent of steepness.
Figure 6.13: Amplitude spectra (top) and surface elevation records for the two high waves and their lower equivalents.
6.4.1 Comparisons between experiments and theories.

The velocity profiles under the crests of the central waves of the groups are non-dimensionalised as before. The elevations about MWL are divided by the wave height and the velocities are divided by $gT$, which amounts to dividing by $2\pi C_0$ where $C_0$ is the deep water linear theory celerity. This means that the maximum non-dimensionalised velocity plotted, 0.08, corresponds to $C_0/2 = C_{\phi}$ where $C_{\phi}$ is the deep water linear theory wave group velocity. The results of this are shown in figure 6.14.

The topmost graph in figure 6.14 shows the experimental results. The two steep, near-breaking waves show very similar results, although they are of different wave steepnesses. Both results tend at the crest to a non-dimensional velocity equivalent to about the deep water linear theory group velocity. The longer period wave from the shortest group (case 'c,' wave e1n98 which was used in section 6.1) had a result higher than this, but it is not entirely deep water ($\tanh kd \neq 1.00$) so the linear theory group velocity is greater than the linear theory deep water group velocity (see section 2.2). Non-dimensionalising by the linear theory group velocity would have given a result just under 1, showing both results tending to linear theory group velocity at the crest.

Non-dimensionalising the velocity profiles by the fourier theory wave celerity, $c_f$, gives a similar result, see figure 6.15. This time the results are all under 0.5, with the results from wave e9n8 (longer group) tending to a lower value of about 0.45 at the crest. This compares to the fourier theory ratio of crest velocity ($u_{fc}$) over wave celerity ($c_f$) of $\frac{u_{fc}}{c_f} = 0.46$ for e8n8 and a ratio of $\frac{u_{fc}}{c_f} = 0.43$ for e1n98.

The two lower waves of steepness $H/gT^2 = 0.0125$ also have very similar non-dimensionalised velocity profiles. This is not surprising, bearing in mind the results of section 6.2, where two-component waves with equal amplitude components produce similar velocity profiles under the crest of the central wave. The maxi-
Figure 6.14: Comparison between velocity profiles for two steep waves and their lower equivalents. Velocities non-dimensionalised by $gt$ (proportional to the deep water linear theory wave celerity).
Figure 6.15: Comparison between non-dimensionalised, measured velocity profiles. Velocities non-dimensionalised with respect to the fourier theory wave celerity $c_f$.

The lower wave from the short group does not behave like the higher wave from the short group (i.e. it does not produce velocities much greater than other waves of the same steepness). Rather, the lower wave behaves like another equal component amplitude wave of the same steepness. This shows that the higher wave behaves as it does because of its severity (high steepness, short group length) rather than just because of its group length.

The two waves from the longer group (waves e8n8 and e8n4) both tend towards the maximum crest velocities expected for a wave of that steepness calculated using linear theory. For the higher wave (e8n8) the maximum expected wave velocity is $0.080/gT$ and for the lower wave (e8n4) it is $0.050/gT$. 

The second graph in figure 6.14 shows the fourier theory velocity profiles, calcu-
lated from wave height and period. The two lower waves of equal steepness again give very similar wave profiles. The two higher waves give profiles which diverge above one wave height below MWL and are over 10% different at the crest.

The bottom two graphs show the results of Wheeler and superposition stretching. At the lower elevations, the velocities split into two pairs of profiles. The waves with the lower relative depth show the higher velocities, as is expected. At the upper elevations the results from the two steeper waves separate. The results from the longer group wave (e8n8) are greater than from the short group wave (e1n98). The superposition stretching is particularly high, as can also be seen in figure 6.16 where experiments are plotted against theories.

These graphs show that although superposition stretching is the most accurate theory at predicting the velocities in the high wave from the short group (wave e1n98) it overpredicts the velocities in the crest of the high wave in the long group (wave e8n8, case 'a'), as it does the crest velocities in regular waves.

6.4.2 Summary of Results

The waves measured in this section had markedly different characteristics. The main conclusions concern the relative sizes of the measured velocities in the crest regions and their relationship to the expected value for a wave of that steepness.

The two lower waves \( H/gT^2 = 0.0125 \) tend to the expected non-dimensionalised velocities at their crest, as does the high wave from the long group. The only wave which does not match the predicted non-dimensionalised velocity for a wave of that steepness is the high wave in the short group. This tends to a non-dimensionalised velocity 20% higher than the maximum from linear theory. This wave is close to breaking and tends to a crest velocity equal to the linear theory wave group celerity.
Figure 6.16: Comparison between experiments and theories.
The results from the two highest waves show that the velocities are not solely dependent on wave steepness, but that higher velocities can be obtained on approaching breaking.

The superposition stretching overpredicted the results in the crest of the high, long group, wave just as it does for regular (infinite group length) waves. Wheeler stretching performs better for this case, but the opposite situation applies to the high wave in the short group, where superposition stretching outperforms Wheeler stretching. Fourier theory, with $T$ and $H$ as the inputs gives a very accurate match to the measurements from the high wave in the long group.

6.5 Conclusions

This chapter deals with three sets of comparisons between sets of waves which have the same characteristics, but each embedded in a different environment. (A fourth set of comparisons looks at steep wave groups and their lower equivalents.) The first three sets all make different comparisons — the first two with waves of the same $H/gT^2$ steepness and the third with waves of the same component $ak$ steepness — but all use a regular wave and a short group with 3 waves per beat (6 waves per repetition). Despite the fact that the groups were all different, some observations can be made that emerge from more than one group.

The main aim of the thesis was to compare regular waves with irregular ones of the same characteristics. This and other comparisons with theories are performed in the following sections.

Comparisons between waves of the same characteristics. The figures of the velocity profiles of different waves of similar gross characteristics (figures 6.5, 6.8, 6.11 and 6.14) clearly show differences in velocities between
the waves. In particular, figure 6.5 in section 6.1 shows the velocities in the near-breaking wave of the shortest equal-amplitude group to be some 30% higher than those of the regular wave in the same comparison. The non-dimensionalised velocity of the shortest-group wave tends to a value about 20% greater than maximum non-dimensionalised velocity predicted by linear theory for a wave of that steepness. The waves in the three longest groups (including the regular wave) tend at the crest to the maximum non-dimensionalised velocity predicted by linear theory for a wave of that steepness (calculated from $H$ and $T$ rather than the amplitude spectra).

When a similar comparison is made between the equivalent waves at a lower steepness ($H/gT^2 = 0.0125$ rather than 0.015) the difference is reduced to a few percent and the highest result comes from a wave from a group of the same length but with components of equal component steepnesses $a_k$. The decrease in difference between regular and short-group wave profiles as the wave steepness is reduced is indicative of the decreases in wave non-linearities when this happens. The relative steepness of the waves measured is therefore an important factor to note in any comparison between experiments and theories and great care must be taken not to use conclusions from low steepness experiments in designing for the high steepness case.

The final set of comparisons (section 6.4) demonstrates that the difference between the velocity profiles in section 6.1 is not due to steepness alone. This is shown by comparing the shortest-group wave from section 6.1 to a wave from a steeper, longer group in figure 6.14. The two velocity profiles are very similar, showing that the same profile can be got from different waves just as different profiles can be measured in similar waves.

**Comparisons between results from equal $H/gT^2$ sets.** The first two sets of results (sections 6.1 and 6.2) contain three directly comparable cases; 'a', 'b' and 'd' from section 6.1 can be compared with 'a', 'b' and 'c' from section 6.2 respectively. The cases 'a' are all regular waves and the 'b's are
all long, equal amplitude groups. The cases 'c' and 'd' mentioned are short (3 waves per beat) equal amplitude groups. In all cases the steepnesses from section 6.1 are greater than those from section 6.2.

The differences in conclusions between the two sets are considerable. Wheeler stretching is the best theory in the lower steepness comparisons (see figure 6.9), then fourier. Comparing a higher order theory using Stokes first definition of celerity to the Wheeler results would give an even stronger preference in favour of the stretching technique over the higher order theory. In contrast, Wheeler stretching underpredicts the results in the trough to crest region of all the cases with the higher steepnesses (although the comparisons are better right at the crests). Moreover fourier stretching performs better and only fails in the crest region of the near-breaking wave.

In other words the different theories perform with different degrees of accuracy at different steepnesses. It is only at the higher steepness, that differences from Wheeler stretching become apparent. Moreover results taken below the trough level cannot always indicate the behaviour above (see figure 6.6, case ‘d’) and the theory which performs best in the extreme cases (superposition stretching) does not always perform best in the less extreme. Also the use of a regular wave in experiments to determine the choice of theory to be used is to be discouraged.

Comparison between equal amplitude and component steepness groups.

In sections 6.2 and 6.3 cases ‘c’ and ‘d’ were from short groups with equal component amplitudes and equal component steepnesses \((a_k)\) respectively. In both cases the equal component steepness cases give higher velocities above MWL in the non-dimensionalised velocity profiles (top graphs in figures 6.8 and 6.11). Moreover, in both cases the higher velocities favoured the comparison with superposition stretching rather than Wheeler stretching and were further from the fourier results. This agrees with the conclusions of chapter 5, that superposition stretching works best in the most extreme
cases, but modifies it. The change is to note that the equal component steepness groups begin to favour superposition stretching at a lower steepness than the equal amplitude groups.

Observation of the waves in the flume leads to the conclusion that the equal component steepness groups break at a lower \((H/gT^2)\) steepness than the equal amplitude groups. Therefore the superposition stretching may be said to favour those waves close to breaking.

Some comments based on comparisons with each of the theories are given below as the second aim of the thesis was to prepare a qualitative assessment of the various commonly used wave theories.

**Fourier Theory.** This proves to be a very accurate theory both for the regular waves and for the long groups (15 to 6 waves per beat). The comparisons with the stream function theory (using Stokes first definition of wave celerity) and in particular figure 6.4, which shows results under troughs as well as crests, demonstrate the improvement gained from using Stokes second definition of celerity rather than the first.

Fourier theory was, however, unable to model the rapid increases in velocities with elevation above MWL in the waves from the least regular of the groups measured. In all cases these groups were the shortest (3 waves per beat) and provided the highest measured velocities in the group.

**Linear Theory.** The basic first order theory generally performed very well when the frequency range was limited to first order components only \((f \leq \frac{3}{2}f_{av}\) where \(f_{av}\) is the average frequency of the two components). The exception to this is the very steep wave from the long group in section 6.4. The velocity profiles for the regular waves and long groups had the same form as those from Fourier theory, but were always larger. The difference was commonly about the size of the mean eulerian current calculated using Fourier theory.
The differences between the linear and fourier theories were greater for the waves from the equal component steepness \((ak)\) groups than for the waves from the equal component amplitude groups.

The differences between the results from the extension of linear theory above MWL (here just called linear theory) and the extrapolation of linear theory above MWL (here called extrapolation) are not sufficient to establish conclusively which is better. The results from the first set of comparisons would suggest a preference for extension over extrapolation.

**Wheeler Stretching.** This theory performs as the best of the theories when the steepness is low and the waves are from equal amplitude component groups (i.e. cases 'a' to 'c' of the last three sets of comparisons). It underpredicts the velocities in the crest to trough regions of the steepest waves and the short equal component steepness groups. The measurements were underpredicted by over 10% above MWL in the shortest group from the highest steepness cases and by similar amounts in the shortest group, equal component steepness cases.

In all cases Chakrabarti stretching produced lower velocities than Wheeler stretching in the same situation so Wheeler stretching was preferred.

**Superposition stretching.** This theory provides the best match to the experimental results in all the most extreme cases measured here. (Extreme is defined as providing the highest velocities for a given steepness.) The theory overestimates the velocities above MWL in all other cases though. Below MWL the results tend to match those of fourier theory very closely. The exception is the regular wave cases where the results are always larger than fourier theory and experiment and tend towards the linear theory value below trough depth.
Chapter 7

Summary and Conclusions

The first section of this chapter is a summary of the aims and the results of this thesis. As such it contains details of the comparisons between similar waves, the comparisons between wave models and measurements and the implications of these comparisons for fluid loading. The second section is on recommended research to be undertaken and there is a third section of conclusions.

7.1 Summary of Research.

The research presented in this thesis had three main aims. These were:

1. To compare the kinematics within regular waves to those within irregular waves with the same characteristics.

2. To perform a relative assessment of the accuracy of various methods of predicting wave kinematics.

3. From the above, to come to some conclusions that can be drawn regarding the theories to be used for fluid loading and to compare them to those of others in the same field.
The method chosen for achieving those aims was to measure the velocities under the crests of regular (i.e. single input frequency) waves and the central waves of uni-directional two-component wave groups in a wave flume, using Particle Image Velocimetry (PIV). This is a full-field photographic technique which records the velocities within a plane covering a significant section of a wave in a short interval relative to the timescale of the flow. Here a square photograph with sides about a quarter to a third of a wavelength long is taken in less than 2% of the wave period. Automated optical and digital analysis of a grid of points covering the resulting PIV negative allows a velocity map to be determined for each wave. The velocity profiles under the crest are then obtained by selecting the appropriate section of the grid.

Regular and two-component groups were chosen for the following reasons. Regular waves are the most accurately defined waves available and are those that have been most often studied in the past. The study of regular waves and the computer modelling of them is well advanced, to the point where the theoretical free surface boundary condition errors are minimal in some higher order solutions, such as those used here. These theories tend to be for regular waves, however, so comparisons with accurately measured regular waves are included. They are also included to compare them with irregular waves of the same average characteristics.

Two-component groups were used as a model for an irregular sea-state because the simplicity allows every case to be analysed in detail. It was thus relatively easy to produce a series of waves from similar groups but with different wave heights, or a series of waves with the same steepness but different group lengths. Moreover, the relative simplicity allowed the crest velocities and elevations to be approximated, thereby allowing the PIV system to be adjusted for each wave.
7.1.1 Comparisons Between Waves of the Same Characteristics.

The comparisons between waves of the same characteristics were made in chapter six. This was achieved by manipulating regular waves and simple two-component wave groups using an iterative scheme until the desired characteristic was obtained. The kinematics under the crests of the chosen waves were then measured using PIV, non-dimensionalised and compared. The main conclusions from these comparisons are:

- The velocity profiles under the crests of the waves of steepnesses \( H/gT^2 \approx 0.015 \) show that the velocities in the near-breaking shortest group wave are some 30% larger than those of the regular wave (or the waves from the longer groups) in the same comparison (figure 6.5). This is a significant difference even when the fact that the shortest-group wave is slightly steeper than the others is taken into account. It becomes apparent only above mean water level (MWL).

- The velocity profiles under the crests of the waves of the same group lengths but steepnesses \( H/gT^2 \approx 0.0125 \) show no large differences in crest velocities (figure 6.8). The above wave groups have equal amplitude components.

- The highest crest velocities from the \( H/gT^2 \approx 0.0125 \) waves came from the equal component steepness short group (where the higher frequency has the lower amplitude). This has a crest velocity almost 10% higher than that from the equal component amplitude short group (figure 6.8 and see the section on the fit to the theories below).

- The waves with equal total component steepnesses all had different velocity profiles, the magnitude of which varied with wave steepness \( H/gT^2 \) (see figure 6.11).
It can be seen from the above that waves of similar steepness (and height) can have substantially different velocity profiles. This is particularly true of the steepest waves where the velocity profiles are most important for the calculation of extreme environmental loads. Moreover these loads tend to be dominated by the drag component of the Morison equation which is proportional to the square of the velocity. This implies that, for example, a 20% increase in measured velocity over predicted results in a 44% increase in measured force over predicted.

7.1.2 Relative Assessment of Wave Theories.

The second of the aims of this thesis is to perform a relative assessment of the accuracy of various methods of predicting wave kinematics. This was performed in chapters 4, 5 and 6. In chapters 4 and 5, the measured velocity profiles under the crests of waves of different steepnesses, but made from the same combinations of frequencies, were compared to different methods of calculating their kinematics to see which was the most accurate. In chapter 6 the waves of the same characteristics were modelled as well. The linear and stretching methods of calculating velocities all used the wave amplitude spectra as their inputs whilst the wave steepness and the velocities for the higher order theories were calculated using the zero down crossing period and height.

Comparisons with Regular Waves.

A summary of the findings for the comparisons with regular waves made in chapters 4 and 6 are given below:

Fourier theory of Rienecker and Fenton, using the correct (second) definition of wave celerity was found to be the best method of determining the velocities under the crests of regular waves. The results were generally very accurate and in the
only instance where they were out by more than about 3% they were conservative
and the error was less than 10%. The benefits of using the second (correct) over the
first definition of wave celerity are apparent in all the comparisons. The second
(zero mass transfer) definition performs better in all cases. In particular when
the crests and preceding troughs are measured (figure 6.4) it is found that the
overprediction of velocities under the crest and underprediction of the magnitude
of trough velocities produced using Stokes first definition of celerity disappear
when the second is applied.

Wheeler stretching modelled the form of the results very well when the fre-
quency range included the first and second order harmonics. If anything the
results tended to be underpredicted in the crest-to-trough zone, but were a more
accurate fit to the results at the crest and below the wave troughs. Chakrabarti
stretching produces results which are very similar to Wheeler stretching. The
results were always lower under the crest and tended to the Wheeler result at
the free surface. The Wheeler theory gave a better fit to the measurements in all
cases.

The implementation of linear theory (which used a simple extension of lin-
ear theory above MWL) matched the results surprisingly well when limited to
a frequency range which excluded second and higher harmonics. The results all
overpredicted the measurements, with the degree of overprediction increasing as
the wave steepness increases and the overprediction being greatest near the free
surface. The wave results were massively overpredicted if the larger frequency
range was used.

Superposition stretching performed very similarly to linear theory using the
same frequency range, as most of the wave energy was concentrated in the input
component.
Comparison with Wave Groups

Chapters 5 and 6 involved wave groups of different lengths and steepness. In chapter 5 short groups (which repeated after 6 waves) of different steepnesses were generated and modelled in much the same way as in chapter 4. The short group length was used so that the waves were in an environment as different from the regular wave case as possible. In chapter 6 the wave groups were of different length but the wave steepnesses were all similar within a comparison.

A summary of how the various theories perform with increasing wave steepness is given in the following paragraphs:

**Fourier Theory** proved to be an accurate theory below MWL, providing the second (zero mass transport) definition of wave celerity was used. It was, however, unable to model the rapid increases in velocities with elevation above MWL in the most extreme cases. Extreme is defined here as having the highest steepness, or the highest velocities for a given steepness. In all cases these groups were the shortest and they had the least regular surface profile. Therefore the above conclusion is expected as the fourier theory is for regular waves.

**Wheeler Stretching** tended to underpredict the velocities in the crests of the most extreme groups (i.e. the steepest waves and the short equal component steepness groups). It performed very well at modelling the low steepness waves, particularly the groups with equal component amplitudes, which had lower velocity increases above MWL than waves of the same steepness from equal component steepness groups. In all cases Wheeler performs better than Chakrabarti stretching.

**The linear theory** results performed surprisingly well when the input from the frequency spectra were limited to the first order frequency range. The results were always greater than the experimental measurements (often by an amount similar
to the mean eulerian velocity calculated by fourier theory) and followed the form of the experimental results, producing higher velocities for short groups than for long groups of the same steepness. The results were all greatly overpredicted if the input frequency range was increased to include the second harmonics.

**Superposition stretching** provided the best match to the experimental results in all the most extreme cases measured here, and was the only method to model their rapid increases in velocity with elevation above MWL. The theory overestimates velocities in the trough-to-crest region of the regular waves and longer wave groups, though.

**Modelling of Extreme Waves**

This section concentrates on the results from the breaking and near-breaking cases featured in this thesis. Generally the intention was to measure steep non-breaking waves, and in one case (e1t98) the amplitudes had to be reduced to prevent breaking and in another (e9n3) the wave surface was actually broken, producing a spilling breaker. These waves both fit the description of extreme waves given earlier and the definition can now be taken to include waves close to breaking.

In chapter 6 the waves were non-dimensionalised by dividing by $gT$ which is proportional to dividing by $C_0$, the deep water linear wave celerity. Therefore the crest velocity can be determined as a fraction of the deep water linear theory wave celerity from this diagram. Linear theory can also be used to determine the crest velocity at the free surface, $u_c$, as a fraction of water celerity depending on the wave steepness, $H/gT^2$.

The extreme wave e1n98 has a steepness $H/gT^2 = 0.016$, calculated from its measured wave height and period. Its measured value of crest velocity is close to
the linear wave theory group celerity, almost 20% higher than the crest velocity predicted by linear wave theory. Its crest velocity is also close to the group celerity calculated from fourier theory, some 13% higher than the fourier crest velocity. The other waves it was compared to (in section 6.1) had lower steepnesses of about 0.015 and their crest celerities were close to those predicted by linear wave theory or fourier theory.

The spilling breaker (wave e9n3) also had a steepness of just under 0.015. Its crest velocity was close to the linear theory one for that steepness, and less than the group celerity $c_g$. The steepest wave measured (from a long group and having steepness $H/gT^2 = 0.018$) has a crest velocity $u_c$ close to the linear theory prediction for a wave of that steepness i.e. $u_c = c_g$. Conversely the linear theory velocity profile calculated from the spectrum for that wave gives a large overprediction of the measured velocities. Other, lower waves also have velocity profiles which tended to the linear theory value at the free surface for waves of their steepness.

**General Assessment of Wave Modelling**

- No one theory performs better than all others, in all circumstances. The fourier theory performs best for regular waves and the superposition for the least regular.

- The use of the correct (second) definition of wave celerity is to be preferred wherever possible.

- The choice of filtering (both in minimum amplitude and frequency range used) is important. The choice of amplitude below which all measured amplitudes are set to zero should be just above the noise level. The choice of frequency range, in particular whether to include the second harmonics, is of great importance. Different conclusions are reached when different
frequency ranges are used.

- Measurements made below trough level provide little guidance as to the accuracy of models. Measurements must, therefore, be made above trough level, and should be made above MWL also if results are to have much credence. The differences between measurements and theoretical velocity profiles are greatest near the free surface.

7.1.3 The Implications for Fluid Loading

The implications of the results for the fluid loading process can now be discussed. The consequences for the calculation of extreme loading in particular will be considered. This implies the need to select the theory which is best in the most extreme cases, not in the majority of cases. On this basis the most suitable method of modelling extreme waves is superposition stretching, based on the measurements made here. However, this comes with the proviso that no theory is the best in all cases and the performance of superposition stretching is best for the shortest wave groups.

The measurements presented show that the kinematics of the wave of given period and height in the shortest wave group can be considerably greater than those of a wave of the same characteristics in a regular sea. This becomes obvious only at a reasonable steepness, when the short group wave approaches breaking, yet reaches a significant level (over 20%) in the wave crest. This is the area of greatest importance in extreme wave loading, yet is the region where fewest accurate measurements have been obtained.

The conclusion that superposition stretching gives the best fit to the results depends on the waves being of a sufficiently great steepness and the group length being sufficiently short. If those conditions are not met, then another choice could justifiably be made. This conclusion can only be made because Particle Image
Velocimetry can make accurate kinematic measurements above the Mean Water Line.

Measurements should, therefore, be made on breaking/near-breaking waves and on wave groups rather than regular waves, if they are to be used in the determination of an extreme wave design procedure. Comparisons with lower steepness waves cannot always be taken to apply to higher cases. This should always be borne in mind when considering any set of results, particularly when it is considered that many sets of results are made at lower steepnesses than here.

The previous paragraph should act as a warning and discourage anyone from applying the above conclusions in a situation far different from the conditions the results were measured under. However, solutions to engineering problems are needed in the short term as well as the long and for this the best information must be applied. Therefore the applicability of these results for the extreme case should be considered.

Su [105] and Rebaudengo Lando et al. [59] have shown that the extreme wave tends to occur in a group, called the extreme wave group (EWG) [35]. Su finds that the extreme wave group has 3 large waves, with the highest, central wave having a height approximately 1.6 times that of the wave before and after it. This rules out regular waves and the two-component long groups used here as a realistic representation of the design wave. The short groups fit the criteria partially but were neither symmetrical enough in wave height, nor did they tend to have the correct ratio of wave heights to fit the EWG exactly. Nevertheless, although not models of the EWG, the results are from high to breaking waves in short groups.

Therefore the importance of these results to the development of the extreme wave loading design practice is in providing accurate measurements near the free surface in a number of extreme waves and indicating that superposition stretching should be used in their modelling. These are just the sort of measurements that were
called for to verify analytical and numerical procedures in the summary report of the NATO Advanced Research Workshop [78] on 'Water Wave Kinematics', for example.

7.2 Suggestions for future course of action

There are two main approaches that can be taken when deciding on the best future course of action. The first is to recognise that as neither the non-linear interaction between individual waves nor the criteria for wave breaking are fully understood, more research on simple wave groups with a limited number of components is both necessary and justified. Moreover, as until recently the number of accurate measurement made above the wave trough level has been very small, the use of PIV for these measurements would be ideal.

The second main approach is based on the fact that the waves in the sea are three-dimensional in nature and may be represented by a continuous spectrum and that therefore the use of two, uni-directional components cannot fully represent the sea. Therefore, realistic wave spectra should be used, and three-dimensional measurements made wherever possible.

If the first approach is taken then more measurements of the transitions of wave groups up to breaking should be done. This is where the greatest differences from regular waves appear. In particular experiments should be done to see if the waves start to break when their crest surface velocities become larger than the group velocity. This is suggested by the results, but there are too few of them here to justify this claim.

Another approach to this problem is already underway in Edinburgh, where PIV measurements have been made of a crossing wave pattern in the three-dimensional wave basin. The crossing waves are generated by sending a plane wave at an angle
to the (plane) glass end wall of the basin. The waves reflect off the end wall and form a crossing pattern with the incident waves. PIV is used to measure the kinematics in a plane parallel to the end wall, several centimetres from it.

A further set of experiments which would be of interest would be a modelling of an extreme wave group [105] in a two-dimensional tank. This could be done simply, using a finite number of components to model a group of the correct form, perhaps as a scaled down version of a north sea design wave. An alternative approach would be to model an equivalent group within a sea-state produced by a spectrum. This approach has not been tried at Edinburgh yet.

If the second approach is taken then the same apparatus as is used in this thesis could be used to generate different uni-directional wave spectra. PIV could be used to measure under the highest waves, and the resulting velocity measurements compared to different theories, as here, to see if the different theories were affected by the spectral widths, wave steepness or length of wave group within the spectrum, for example. This would require either that the amplitudes and phases of the spectrum components were known, or that there be a continuous sampling of wave elevation in the measurement area, with a triggering process for the PIV camera based on a wave elevation criterion.

A further alternative would be to adopt the statistical approach of section 5.1 and to measure the crest velocity under every crest in an irregular sea using PIV. The results could be used to form a crest velocity probability density curve analogous to those now done for crest amplitudes [57, for example].
7.3 Conclusions

A number of accurate measurements of the kinematics under the crest of regular waves and two-component uni-directional wave groups have been made using Particle Image Velocimetry. The waves were in intermediate to deep water, with relative depths in the range $d/gT^2 = 0.05$ to 0.085 and were of moderate to high steepnesses, in the range $H/gT^2 = 0.005$ to 0.018. (Here $d$ is water depth, $T$ wave period, $H$ wave height and $g$ gravitational acceleration.) The main conclusions are:

- Regular waves were accurately modelled using an implementation of high order fourier theory by Rienecker and Fenton, providing Stokes second (zero mass transport) definition of wave celerity was used.

- Steep, near-breaking two-component waves were modelled accurately using superposition stretching, a derivative of linear theory. The input for this is the measured wave spectrum, including first and second harmonics. The second harmonic contribution was found to be significant, even though the waves were in intermediate to deep water.

- The kinematics in the crests of different waves of a given height and period can vary considerably. Here, differences of over 20% were noticed at the crest, depending on the component build-up of the waves.

- Wave group length affects the internal wave kinematics.

- Measurements must be made above the level of the wave troughs and should be made above the mean water level also, if experimental results are to have much credence.

- Particle image velocimetry proved to be an excellent measurement technique to use for measuring velocities as it was capable of measuring close to the free surface of high waves, with a high degree of accuracy.
It should be noted that none of the stretching alterations to linear theory have any physical basis and they were introduced because of observed discrepancies between higher order theories and existing measurements. The amount of data available was limited, however, and an incorrect definition of wave celerity was commonly used in calculating the higher order theories. However, if a stretching theory has to be used then it should be superposition stretching as that is conservative in the most extreme cases measured here.
Appendix A

Scanning-Beam Analysis

The purpose of this appendix is to derive an equation for the systematic error introduced in the scanning beam system by the movement of the seeding and the non-uniform scan velocity through the measurement area. In the paper of Gray et al. [40] the error caused by the movement of the particles is calculated, assuming an average scanning velocity of the beam through the measurement region. Moreover the same paper derives an expression for the scanning velocity which is dependent on horizontal position within the measuring volume and geometrical factors of the system. This appendix combines the two for the first time.

A.1 Systematic error in time between pulses

The time $\tau$ is the time between successive illuminations of the same point. In P.I.V. however it is the positions of particles at successive illuminations that is recorded. The small movements of the particles between illuminations thus causes a small change in the time between illuminations. Let

$\Delta t$ be the time between successive illuminations of the same particle,
Δx be the recorded horizontal distance travelled by a particle between exposures, 
v_x be the horizontal component of velocity of the particles between exposures.

This value is assumed to be constant throughout the measurement area throughout the measurement time.

⇒ Δx = v_x Δt \hspace{1cm} (A.75)

The time between exposures is given by:

Δt = τ - \frac{Δx}{\frac{∂x}{∂t}} \hspace{1cm} (A.76)

where τ is the scan period and \( \frac{∂x}{∂t} \) is the horizontal velocity of the beam through the measurement volume, or scan velocity. This assumes that Δx is so small that \( \frac{∂x}{∂t} \) can be treated as a constant within the length Δx. Therefore:

\[ v_x = \frac{Δx}{Δt} = \frac{Δx}{τ - \frac{Δx}{\frac{∂x}{∂t}}} \] \hspace{1cm} (A.77)

To calculate this, the scan velocity must be known. An average value may be assumed [40], but unfortunately the constant angular velocity of the rotating mirror does not translate into a constant horizontal scan velocity of the laser beam through the measurement volume. The scan velocity is dependent on the horizontal position and is given by:

\[ \frac{∂x}{∂t} = \frac{∂x}{∂θ} \frac{∂θ}{∂t} \] \hspace{1cm} (A.78)

where θ is the angle between incident and reflected directions of the scanning beam off the rotating mirror (and hence \( \frac{∂θ}{∂t} \) is the angular velocity \( ω_b \) at which the beam scans). Also x = 0 when θ = 0 so the origin of the horizontal axis is on the axis of symmetry of the parabola.

Now each of the N faces of the rotating mirror scans the beam over 4π/N radians. The mirror rotates at a constant angular velocity, \( ω_m \) which scans the beam at a
constant angular velocity $\omega_b$ twice that of the mirror.

$$\Rightarrow \omega_b = 2\omega_m = 4\pi F = \frac{\partial \theta}{\partial t}$$  \hspace{1cm} (A.79)

where $F$ is the frequency of rotation of the mirror (Hz) so $\tau = 1/FN$. Therefore:

$$\frac{\partial \theta}{\partial t} = \frac{4\pi}{N\tau}$$  \hspace{1cm} (A.80)

All that is required is the expression for the rate of change of horizontal beam position on the parabolic mirror, with angle. This is derived from the equation of the parabola using the cosine rule (bearing in mind the fact that the beam contact with the rotating mirror is at the focus of the parabola). The equation of the parabola is given by:

$$y^2 = x^2/2L$$  \hspace{1cm} (A.81)

The vertical coordinate is $y$ and the horizontal $x$ and the origin is on the horizontal and vertical axes of symmetry of the parabola, which is of length $L$ and height $L/2$. The laser beam is reflected from the rotating mirror at point $(0,L/2)$ and the point of contact between beam and parabolic is at $(x,y)$. Therefore, as in Gray et al. [40]:

$$x = L\sqrt{1 - \frac{2\cos \theta}{1 + \cos \theta}}$$  \hspace{1cm} (A.82)

$$\Rightarrow \frac{\partial x}{\partial \theta} = \frac{L}{1 + \cos \theta}$$  \hspace{1cm} (A.83)

Gray et al. presented a graph of effectively the rate of change in scan velocity against angle ($\theta$) to show that a variation exists. The expression for $\theta$ can, however, also be given in terms of $L$ and $x$ which allows the scan velocity to be explicitly expressed as a function of position, etc. as follows:

$$\cos \theta = \frac{L^2 - x^2}{L^2 + x^2}$$  \hspace{1cm} (A.84)
so:

\[
\frac{\partial x}{\partial \theta} = \frac{L}{1 + \frac{L^2 - x^2}{L^2 + x^2}} = \frac{L^2 + x^2}{2L}
\]  

(A.85)

The above results (equations A.85 and A.80) can be combined to give

\[
\frac{\partial x}{\partial t} = \frac{\partial x}{\partial \theta} \frac{\partial \theta}{\partial t} = \frac{(L^2 + x^2)}{2L} \cdot \frac{4\pi}{N\tau} = \frac{2\pi(L^2 + x^2)}{N\tau L}
\]  

(A.86)

The systematic error in the calculation of the horizontal velocities can now be computed

\[
v_x = \frac{\Delta x}{\tau - \frac{\Delta x N \xi}{\tau N}} = \frac{\Delta x}{\tau - \frac{\Delta x N \xi}{2\pi(L^2 + x^2)}}
\]  

(A.87)

This simplifies to:

\[
v_x = \frac{\Delta x}{\tau} \cdot \frac{1}{1 - \xi}
\]  

(A.88)

where \( \xi \) is given by:

\[
\xi = \frac{\Delta x LN}{2\pi(L^2 + x^2)}
\]  

(A.89)

The solution is recognisable as the solution without correction multiplied by a calculable correction factor, dependent only on position, the measured image separation, and known factors of the PIV system.
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Papers Published

The following three papers based on the contents of this thesis have been published. Copies of the papers are given on the following pages.


THE DEVELOPMENT OF A PARTICLE IMAGE VELOCIMETRY TECHNIQUE FOR THE MEASUREMENT OF SURFACE VELOCITIES OF WATER WAVES

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SUMMARY

A method of measuring velocity distributions on the surface of waves is described, based on the principles of Particle Image Velocimetry (P.I.V.). Results obtained from this method are compared to results from three established methods and close agreement is found both in the form and magnitude of the results obtained. The advantages of this method include the ability to look at single waves and three-dimensional seas, all without the need for a high powered laser.
NOMENCLATURE

P.I.V. = Particle Image Velocimetry
L.D.A. = Laser Doppler Anemometry
M = magnification of recording system
ΔT = time between pulses
T = length of pulse
a = wave amplitude
H = 2a = wave height
T = wave period
ω = circular wave frequency
k = wavenumber
s = wave steepness
D = water depth
x = horizontal distance from wavemaker along tank
y = horizontal distance across tank
z = height above mean water level
Ux = horizontal velocity along tank
Uy = horizontal velocity across tank
g = gravitational acceleration.

1. INTRODUCTION

The study of velocity distributions in water waves is of great interest to engineers working, for example, on ocean structures. With the highest velocities occurring at (or near) the surface of the water, and with forces proportional to the square of velocities, the measurement of surface velocities is of considerable importance in calculating extreme wave loading. In the past few years several theoretical models (often based on a Dean's stream function (ref. 1), Stokes theory (ref.2) or a time stepping process (ref.3)) have been developed to calculate the internal velocity fields of waves.

The traditional probes (hot wire anemometers, propellers, etc.) have also been used to obtain experimental results and in the last few years two optical methods, Laser Doppler Anemometry (L.D.A.) and Particle Image Velocimetry (P.I.V.) have been developed for measurements within a wave. This paper gives a description of a new application of P.I.V. to the measurement of velocities at the surface of waves and compares results with three established methods - namely scanning beam P.I.V., linear theory and a variation of a Dean's stream function developed by Professor J.R. Chaplin (ref.1). The comparison can then be used to show whether results are unduly affected by surface layer effects caused by, for example, surface tension or friction with air at the boundary or whether the surface measurements match those of established methods. If the latter is the case then this fundamentally simple technique can be used to ascertain maximum velocities without the need for specially constructed wave flumes or high powered lasers for illumination.

One reason for developing this new method is that there are problems in applying all the established methods, mentioned above. The analytical and computational methods are difficult to apply to practical situations and often fail when a wave approaches breaking. Moreover, the common non-linear programs are all two-dimensional and linear programs break down in extreme cases. Traditional probes are limited by their response accuracy, suffer calibration problems and may even be sufficiently large to prevent their taking results right up to the surface. Recent developments in Laser Doppler Anemometry (ref.4) have allowed accurate measurements of the velocity or acceleration at a point in a wave to be made from records as short as 0.5ms. However, to produce a vector map of velocities under a wave requires a highly repeatable wave. This leads to
obvious problems when it comes to investigating the kinematics of pseudo-random waves generated from ocean spectra, for example. These problems require almost instantaneous recording of the whole flow field over a large section through a wave profile. Many of these problems were resolved by the development of scanning beam Particle Image Velocimetry (P.I.V.) (refs. 5,6).

P.I.V. is an essentially non-intrusive velocity measuring technique derived from the application of laser speckle photography to a fluid flow. In P.I.V., two or more images of a plane of a seeded flow are recorded photographically. Provided that the Image/Object magnification and the time separating the formation of successive images are known, then the velocity field can be calculated by optical and digital analysis of the separations between particle images. This system, though limited in its dynamic range, provides a velocity map of a large area, obtaining its data almost instantaneously. Unfortunately, both L.D.A. and the scanning beam P.I.V. previously developed rely on the use of specially designed two-dimensional wave flumes with transparent walls and base (at least until the new generation of physically small, unobtrusive fibre optic L.D.A. systems are improved). These are necessary to allow the illumination and recording of the area of interest. Therefore P.I.V. and L.D.A. rely on the measurement of two-dimensional waves (that is, waves with motion in two dimensions) being sufficient to model sea states which are, by nature three-dimensional.

Surface P.I.V. does not suffer from these constraints. Although the system described here has been developed on a two-dimensional tank for simplicity and to allow comparison with the established methods it should prove equally applicable to the three-dimensional case.

2. P.I.V. METHODS

2.1. The Wave Flume

The results for both P.I.V. methods were made on the same two-dimensional wave flumes at Edinburgh. This is 6m long, 0.3m wide with a mean water depth of 0.54m. The waves are generated by a single absorbing paddle driven by a microcomputer, via a force transducer. An expanded aluminium beach absorbs the waves at the far end of the tank and measurements are made in a 2m section of the middle. The presence of the beach and force transducer ensure that there is minimal reflection of waves from the ends of the tank. The narrowness of the tank and the close fit of the wavemaker ensure that the waves are two-dimensional in nature.

The use of the microcomputer allowed a wide choice of waves to be available and ensured a high degree of repeatability.

2.2. Photographic Recording for Surface P.I.V.

A Nikon camera with flat focus lens was mounted in a stable configuration with its film plane horizontal, a known distance from the undisturbed surface of the flume [Figure 1]. The surface of the water was seeded with neutrally buoyant conifer pollen particles (average diameter 0.07mm). The surface was illuminated by two sets of flashguns, with two flashguns per set to prevent shadows behind steep waves. The required wave was generated by a micro which also triggered the camera shutter release. This in turn activated a delay box which set off the two sets of flashguns a known time apart [Figure 2]. Provided that the shutter was open during both flashguns' operations the film should contain double exposures of each seeding particle. The surface separations and orientations of those particles give the surface horizontal speed and direction at that point. Care must be taken to ensure that the ratio of pulse length to pulse separation of the illumination was sufficiently small to allow a reasonable result to be obtained (ref. 7). (See section on errors).
2.3 Photographic Recording for Scanning Beam P.I.V.

In this well established technique, particle velocities are again obtained using a multiple exposure photograph. In this case the bulk of the fluid is seeded using neutrally buoyant particles. A multiple exposure photograph is taken of the seeding particles' motion within a vertical plane through the wave, parallel to the side of the flume. It is assumed that all motion of the two-dimensional wave is in the plane photographed. This plane is illuminated by a 15W C.W. Argon Ion laser, using a scanning beam technique, implemented with a dynamic reflection system. In this system the laser beam is narrowed, collimated and reflected from an octagonal rotating mirror onto a parabolic recollimating mirror. This is positioned beneath the middle of the flume and directs the beam vertically upwards through the glass base of the wave flume. As the octagonal mirror rotates, the laser beam scans over the parabolic mirror and the vertical beam sweeps over the area of interest at a constant speed [Figure 3].

An analysable photograph is produced provided that sufficient laser light is scattered off the seeding particles to form clear images and provided that the shutter is open for long enough to capture two or more images of each particle.

2.4. Analysis of P.I.V. Photographs.

At Edinburgh an automatic system is used to provide information on the spatially-averaged value of separations between particle images within a small area of film (about 1mm. across). This information, coupled with Image/Object magnification (M) and the time separation between illuminating pulses (ΔT) is sufficient to allow the average velocity within that small area to be ascertained. A number of measurements are made over a grid of points to build up a picture of the overall particle, and thus flow, velocity distribution.

The information on separations is obtained by coherently illuminating a small area of film. The diffracted light passing through the photograph is collected and observed as Young's fringes (refs. 8,9). The separation and orientation of the fringes give a measure of average particle separation and direction of motion of the flow within the area illuminated. This information is obtained by digitising the fringes. A Fast Hartley transform (refs. 5,6,9) is then performed on the rows then the columns of the digitised data. Subtraction of the fringe pedestal before transforming reduces the size of the d.c. component (ref.10) so that a good approximation to the fringe frequency can be found by locating the spatial frequency with maximum signal height. An interpolation procedure is used to locate the maximum from the discrete spectrum. The position of the maximum, in the frequency plane, gives the spatial frequency which is converted into a velocity by multiplication by a scale factor. This works as the autocorrelation of the photographic density distribution over the analysis region and the Young's fringes form a Fourier transform pair. Performing the Hartley transform on the fringes produces the autocorrelation, minus the d.c. component (the correlation of each point with itself). The scale factor comprises a calibration term, an Image/Object ratio term and a pulse separation term. The calibration term relates the values produced by the analysis rig in arbitrary units of distance to an actual average separation between successive images on the negative. Typically it has an associated relative error of three percent due to the spread in results caused by finite size of the irregularities in shape and distribution of images on the calibration slides. Conversion from distance on negative to velocity is achieved by multiplying by the Image/Object ratio and dividing by pulse separation. These values have typical relative errors of two and three percent respectively.

3. WAVE THEORIES

3.1. Linear Theory

Starting with wave amplitude (a), period (T) and depth (D), the deep water
approximation for the dispersion formula \((\omega^2 - gk)\) gives wavenumber, \(k\). An assumed cosinusoidal surface profile is calculated for time \(t=0\), taking \(x=0\) at the crest, defining \(z\) as vertical displacement above still water level. The horizontal component of velocity, \(U_x\), can then be calculated using the formula

\[
U_x = \omega a e^{kz} \cos(kx).
\]

3.2. Dean's Stream Function

A computer program developed by J.R. Chaplin (Ref. 1) was used to provide horizontal components of velocity and surface profile at a number of positions along the given wave. This program solved the case of the two-dimensional irrotational periodic wave based on a Dean's stream function representation (15th order) taking water depth \((D)\), wave height \((H=2a)\) and period \((T)\) to uniquely define the wave. The formulation of the wave theory used by Chaplin is rather more complicated than Dean's but results are obtained directly. Chaplin's program also has the advantage that since the unknowns define the surface profile, a good starting point for the iterative solution can be obtained from linear or cnoidal theory.

4. RESULTS

The results were all obtained for a monochromatic wave of frequency 1.44 Hz. and amplitude, \(a=40\)mm generated in a depth of 54cm of water. Its steepness was calculated because, although the question of breaking wave parameters is a contentious one, a simple parameter is required to ensure that the wave used was relatively steep. This is the area of greatest interest as it is here that numerical models start to lose their validity. Defining wave steepness \('s'\) as \(s=H/(T^2 g/2\pi)\) gives a value for steepness of 0.11. Longuet-Higgins (refs. 11,12) suggests a value of \(s=0.14\) for breaking.

The results for linear theory and the Dean's stream function theory are obtained from the relevant formulae and are shown in figure 4.

A scanning beam P.I.V. photograph of the wave was taken [Figure 5] and analysed to produce a velocity map [Figure 6]. This shows a set of arrows with the length of each arrow proportional to the speed at that point and the direction of each arrow giving the direction of the flow. The 180 degree directional ambiguity associated with each measurement was resolved using a priori information about the direction of the flow. The base of the arrow gives the location of the centre of the region of the negative that produced the result. All points are in the vertical plane through the centre of the tank illuminated by the scanning laser beam. The surface component of velocity was obtained simply by taking the value at the highest point in each column. This means that the results produced are not for the surface but are an averaged result for a small area below the surface. The extent to which the measured values can be made to approach the surface depends on a number of factors including the magnification factor, the time separation between pulses, the seeding, and the size of grid used for analysis. These factors were not optimised for the photograph taken so results are only from within 1 or 2cm below the surface.

For the purposes of comparison, the distances along the tank had to be converted into distances from the crest. The crest was located simply by identifying maximum and minimum velocities and their locations and fitting them to the graph accordingly. The location of points in the two P.I.V. experiments was fixed in relation to one another by use of reference points. The results for surface P.I.V. were presented in the form of \(x\) and \(y\) coordinates (distance along the flume and across it respectively) \(x\) and \(y\) components of velocity and visibility. The latter gives a measure of the validity of the results (being somewhat akin to signal-to-noise ratio). In theory there should not be any
component of velocity across the flume and the results give an average velocity \( U_{av} = -1.7 \text{cm}^2\text{s}^{-1} \) with a standard deviation of 2.5 cm\(^2\text{s}^{-1}\). A consistent set of non-zero \( y \) velocities would best be explained by a misalignment within the system. Those that occurred are probably due to the small ripples that are formed on the surface of the flume.

All results contain an error of approximately 5\% and most results for a given distance from the crest lie within a spread given by this error. In two cases there were instances of especially low velocities compared to the average. These both occurred when successful velocity measurements were made close to the wall of the flume. In these cases the discrepancy was attributed to edge effects, caused by drag between wall and water. The results for horizontal component of velocity along the tank versus distance from the wavecrest are shown in figure 7. Figure 8 shows how a vertical line of scanning beam results tend towards the surface result measured directly above.

The results produced show close agreement in their form and in their magnitudes. As might have been suspected the scanning beam values were, on average, lower than the surface values. The theoretical lines (represented here by the Dean's Stream function line as the two are so similar) fall between the two sets of experimental data, giving close agreement between all four methods.

5. ERRORS

The main limitation of surface P.I.V. measurements include the facts that

- the object plane is not flat
- only 2 flashes were used in the system
- there is a limit to the pulse width to separation ratio

The first point arises because the object plane is the surface of the water. The effect of this is to create differences in measured velocities where there were none in the wave. For example, 2 particles a set distance apart at a crest subtend a larger angle at the camera than two similar particles the same distance apart at a trough, thus giving images of different separation. Corrections can only be applied if the surface profile is known. The off-axis distortion caused to a flat plane by the lens is minimal. The effect of wave height can be minimised by moving the camera as far away from the wave as possible but this results in a loss of resolution. The problem also links in with the limited dynamic range of the analysis system. The maximum separation between successive images of a particle must be less than a fraction of the analysis beam diameter (ref. 13) with that fraction commonly taken to be one half. The minimum separation is one particle diameter so that two discrete particle images are formed. Therefore for a given time between the formation of images there is only a certain range of velocities that can be measured. Water waves have a wide range of horizontal velocities; both positive and negative. It is therefore currently impossible to accurately measure over a whole wavelength and moreover the current system cannot distinguish between positive and negative velocities - though the problem could be removed by the use of an image shifter (refs. 11, 15).

One of the main features of the surface P.I.V. results obtained so far has been the poor signal-to-noise ratio allied with low fringe visibility. A primary reason for this is the fact that only 2 pulses of light are used to form two images of each seeding particle in each photograph. It can be shown that increasing the number of exposures improves the sharpness of the interference fringes (ref. 13). This occurs as increasing the number increases the correlation between the particle pairs of equal separation and orientation within the analysis region and also increases artificially the seeding density.

Another limitation lies in the ratio of pulse width to pulse separation. Ideally the ratio is as small as possible: \( \tau \ll \Delta \tau \). This also helps to prevent the images of the particles becoming streaked, so narrowing the diffraction halo.

48.6
within which the fringes appear. In practice this places a limit on the minimum pulse separation that can be used. A ratio of 1:3 represents about the highest ratio possible. Increasing the ratio increases the rate at which the intensity of the fringes decreases on moving away from the centre of the diffraction pattern (ref. 16).

6. CONCLUSIONS

It has been demonstrated that surface P.I.V. can be used to obtain velocity distributions on the surface of waves generated in a laboratory flume. The results obtained (for horizontal components of velocity only) are in close agreement both in form and magnitude with those obtained using three established methods, namely linear theory, Dean's stream function theory and scanning beam P.I.V. Further work must be done on the effect of representing the curved surface of the wave by a flat image plane, as this leads to spurious results. Work will also be done on theoretical and experimental analysis of other factors determining the accuracy of the system and it is worth noting that a substantial reduction in the scale factor error can be made without much extra effort. The advantages of the method include the ability to look at single waves and three-dimensional seas, all without the need for a high-powered laser.

6.1 Acknowledgements

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The authors would like to thank D.J. Skyner for his help in programming the microcomputer.

7. REFERENCES


Fig. 1. Experimental Set-up for Surface P.I.V.

Fig. 2. Recording Apparatus for Surface P.I.V.

Fig. 3. Experimental Set-up for Scanning Beam P.I.V.

Fig. 4. Comparison of Wave Theory Results.
Fig. 5. Scanning Beam Photograph.

Fig. 6. Scanning Beam Velocity Map.

Fig. 7. Comparison of Results.

Fig. 8. Elevation vs. Velocity (below a point).
THE EFFECT OF FREQUENCY SPREADING ON A
TWO-COMPONENT SEA STATE

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ABSTRACT
The kinematics under the crest of the highest waves in five two-
dimensional two component wave groups are presented. The highest wave
in each group had the same gross characteristics of wave height and
period but each group has a different separation between the two
fundamental components. The measurement technique used, Particle
Image Velocimetry, allowed velocity measurements to be made up to
about 90% of the wave height.

The measured kinematics were compared with Linear Theory, its Wheeler
Stretching derivative and Dean's Stream function (taken to 21st
order). It was found that Linear & Dean's theory performed well at
all depths for monochromatic waves and long groups but that all the
theories underpredict crest kinematics in the shortest wave groups.
Wheeler stretching did not perform well near the surface of any of the
waves.
INTRODUCTION

Over the years various approaches have been taken to the calculation of environmental forces. In the case of wave loading the prediction of the kinematics within the crest of extreme waves is vital. In determining the design criteria a statistical process is used to determine a design wave of given height and period. That wave is then modelled computationally using a high order wave theory - the choice of which depends on a number of factors (such as combinations of water depth, wave height and period) recently reviewed (Barltrop 1989). The design wave is assumed to be monochromatic and extending to infinity and the forces on a structure are predicted from the local velocities and accelerations calculated for the wave.

Much work has gone into attempts to verify or improve upon the methods used in these calculations. For example Fenton (1985) published a correction to the Stokes Vth theory of Skjelbreia and Hendrickson (1962). Cokelet (1977) derived an exact solution for waves of finite amplitude in any depth of water, although this is seldom applied in engineering design. Dean (1965) formulated his theory in terms of the stream function and published tables of coefficients to facilitate its use. Chaplin (1980) reformulated this solution by taking water depth, wave height and period as the independent parameters and using the surface elevations as the unknown. This theory can now be taken to any order without recourse to tables.

Linear or Airy theory is still used for many purposes because of its simplicity, its low computation time, its easy capacity for absorbing many components and its ability to model three dimensional sea-states.
Wheeler (1970), Chakrabarti (1971), Gudmestad and Connor (1986) and Lo and Dean (1986) have all proposed stretching modifications to linear theory which reduce the errors in the free surface boundary conditions but which, in doing so, violate the Laplace equation. Wheeler's modification was proposed because of observed discrepancies between measurements and linear theory, in particular the high frequency contamination observed in the crests of waves. This apparent error arises from straightforward extrapolation of linear theory outside its strict bounds (0 < Z < d). This results in the magnitude of the velocity term (governed by the product of wavenumber k and elevation Z) increasing swiftly above its theoretical limit for elevations above mean water level. The effect is particularly severe for high frequency and therefore high wavenumber and low amplitude waves. Wheeler's solution was to transform the vertical coordinate so that the velocity at the surface was given by the velocity previously calculated for mean water level with all other coordinates being stretched or compressed between the surface and the bed. This has the effect of reducing crest velocities and increasing the magnitude of trough velocities. The other stretching modifications are just variations or developments of Wheeler's.

Measured Kinematics

The experimental study of wave kinematics has had a rather haphazard past with traditional flowmeters unable to record well (if at all) above the depth of the lowest troughs. It is only since the advent of Laser measuring systems - in particular Laser Doppler Anemometry - that accurate velocities have been measured by, for example, Bosma and Vugts (1981), Bullock and Short (1985), Easson and Created (1984),

Even with the benefit of accurate measurements no systematic or uniform method has appeared for modelling wave kinematics, particularly in the crest to trough region. This is due partly to the difficulties inherent in most Laser Doppler measurements in this region caused by interruptions to the signal occurring when the measurement volume crosses the water/air boundary.

Melville and Rapp (1988) attempted to get round this problem by measuring velocities at the surface using a L.D.A. system mounted above their wave flume with beams deflected onto a vertical axis with the elongated measuring volume covering the range of surface elevations used. This in conjunction with a wave gauge, produced a series of horizontal surface velocity components and wave height records with time used to investigate wave breaking.

Griffiths, Easson and Greated (1987) used L.D.A. to measure velocities above and below still water level in monochromatic waves breaking on beaches in a 2-D flume. This was achieved using a phase locked trigger pulse to link the wave phase to time of measurement. A digital transient recorder with microprocessor was used for analysis. They obtained measurements to a level of 95% of the wave height for waves breaking on a 1 in 30 slope. This is considerably higher up the wave than the results achieved by continuous recording and the results obtained reflected this. They found that the kinematics computed from Linear, Stokes Vth and Dean's 3rd order theories all "underestimated the true velocities over a region of the top third of the crest above still water level". Moreover they found that this was a recurring
phenomenon over a range of conditions in breaking waves and that the size of the under-prediction was not trivial (up to 50%). As all the forms of stretching proposed reduce crest kinematics, this result suggests that stretching theories should not be applied to the extreme case of breaking wave crest kinematics and casts into doubt their validity in the high crest region in all circumstances.

The phase locked approach is not applicable to a random sea. Skjelbreia and Torum (1989) obtained L.D.A. results above mean water level for regular and irregular waves over a flat bed. The results in the crest region (in effect any results above mean water level) did suffer from the effects of signal interruption and so coverage of particle velocities in the crest to trough region remains incomplete using L.D.A. The authors here concluded that linear theory significantly overpredicted the wave kinematics in the crest - but this was based on results taken at mean water level. They also concluded that velocities predicted using Wheeler stretching give a much better fit than ones from Linear theory.

The Sea-States Investigated.
Traditional design criteria result in the choice of a monochromatic design wave. The extreme case kinematics are then calculated for the crest of this wave, regular in form and preceded and followed by identical waves. Real sea states are vastly different from this, commonly being represented by a continuous spectrum and there have been no conclusive studies to show that the kinematics of the design wave match those of a wave of the design criteria - period and height in a given depth - in a real sea state.
It would, of course, be impossible to compare directly the kinematics of two waves of given height and period - one in a monochromatic sea, the other in storm conditions - and draw a definite conclusion as to the causes of the differences. There are too many influences (such as the spread of frequencies, random phases, currents, three-dimensionality and the disruption to the energy balance caused by wave breaking) for this to happen. The effects can, however, be isolated and investigated individually. Here a monochromatic wave is compared to the simplest of two-dimensional groups, consisting of two components of equal amplitude measured when in phase and at the trough immediately before. This isolates the effect of frequency spreading on the wave kinematics and should allow useful comparisons to be made.

The measurement technique used here is Particle Image Velocimetry (P.I.V.). This system generates a whole field velocity map within a vertical plane in a 2-D wave flume almost instantaneously (8 milliseconds) and allows measurements to be made up to 10 to 20 mm below the crest height. The waves measured were a single component wave of steepness

\[
\frac{H}{gT^2} = 0.015, \quad \frac{d}{gT^2} = 0.048
\]

and four two-component groups each with approximately the same maximum wave height as the single component case and with the average component frequency equal to that of the single component case. The separations of the frequencies was varied, altering the number of waves in each wave packet and the repeat time of the wave packets created, (Fig. 1).
Fig. 1. Wave Height versus Time for case A (0.937 Hz), case B (0.898 & 0.977 Hz) and case E (0.781 & 1.094 Hz).

Velocity measurements were made over a grid of points (separated by approximately 10 mm) covering the crest or trough region. The horizontal components of velocity below crest and trough were then compared to the Dean's Stream function formulation of Chaplin (1980), Linear theory and its Wheeler stretching derivative.

The main purpose of the paper is to compare measured kinematics with linear theory, Wheeler Stretching and Dean's Stream function theory predictions and to investigate the effect of the spreading of a simple group away from the monochromatic case.
PARTICLE IMAGE VELOCIMETRY

PIV is an essentially non-intrusive measurement technique derived from the application of speckle photography to a fluid flow (see Dudderar, Meynart and Simpkins (1988) for a review). Gray and Greated (1988a) applied the technique to the measurement of water waves in a two-dimensional flume. The technique involves taking a multiple exposure photograph of an illuminated plane of the wave, parallel to the wall of the tank. The flume had first been seeded with suitable reflecting particles (in this case conifer pollen). These are small enough and close enough to neutral buoyancy to follow the flow precisely, so each moves according to the local flow velocity conditions (Fig. 2).

![Fig.2. P.I.V. photograph.](image)

The illumination of the vertical plane was achieved in our case using a 15W C.W. Argon Ion laser. The "scanning beam" method of Gray and Greated (1988b) was adopted to prevent the streaking of images and to maximise the percentage of available light power being used.
A camera is focused on the illuminated plane and holding the shutter open for long enough to record two or more scans allows the velocity field over the area of the photograph to be obtained by using an automated system of optical & digital analysis of small areas of the developed negative (Gray and Greased 1988a). This involves shining a low-powered laser through a small area of the negative (here about .75mm in diameter), capturing the fringes with a C.C.D. camera at the focal point of a bi-convex lens and performing a fast Hartley transform (analogous to an FFT) on the data. The velocity is found by locating the centre of mass of the spatial frequency peak of maximum height in the transform plane (having first removed the D.C. component). This is then multiplied by a scale factor giving a velocity. Thus, sufficient information to produce a velocity map over a significant area (here approximately 60 x 60cm) is recorded in a short time compared to the timescale of the flow (here \( \frac{1}{125} \) seconds).

The accuracy of the system depends on three sources of error. The magnification was calculated by taking a photograph of an accurate grid placed in the flume. Forty measurements of the magnification were made at different places and the results were averaged, producing an error of 1%. The time between light pulses was very accurately known with an error of about 0.25% by using a digital storage analyser. The remaining error is that associated with locating the signal peak in the transform plane. The procedure of Gray (1989) was followed to obtain an overall estimate of the value of this error. For a typical fringe visibility (which gives a measure of the quality of the result) the relative error decreases with velocity from approximately 1.5% at lower speeds near the flume bed to about 0.5%
at the higher speeds in the crest region. For lower fringe visibilities the error increases and a spurious result may even be given. The overall error may then be said to lie within the range 1 to 2.5% except for the occasional region where the seeding density is too low due to the random distribution of the seeding.

EXPERIMENTAL APPARATUS

The measurements were made in the 6m wave flume at Edinburgh University. This has a still water level of 0.54m and is 0.3m wide. Optical access to the P.I.V. measurement region is obtained by having the walls and base of the central 2m section made out of 20mm thick glass.

The waves are generated by a single hinged paddle which is of the absorbing type (Salter 1982). This incorporates a force feedback mechanism which equates the force on the water to the sum of the drive signal and a filtered velocity signal, thereby simultaneously generating waves and absorbing reflections. The waves travel through the measurement region and are absorbed by an expanded aluminium beach which has a reflection coefficient of between 2 and 5% over the frequency range used. The drive signal to the paddle is provided by a microcomputer which also triggers the P.I.V. camera and samples from four wavegauges.

The gauges are of the resistance type and are arranged in 2 sets at different positions along the tank. One set was initially positioned in the centre of the measurement region and was used to obtain wave
height records with time. These were used to test tank repeatability and as the tank transfer function assumes that the waves are linear, the fourier transform of the wave height record was separated into amplitude and phase spectra. Each was used in an iterative scheme to achieve the desired wave conditions. Once these were achieved the wavegauges were moved out of the measurement region.

One photograph was taken for each test run at either 15 or 14.47 seconds to record the highest crest or preceding trough. A delay of 15 seconds was sufficient to allow the waves to become established and for reflections (of order 2 to 5% for the central frequency range considered) to have reached the measurement region. A delay of 2 to 3 minutes between runs allowed the tank to settle between obtaining results.

**The Test Waves**

The easiest way to compare the kinematics of a monochromnatic wave with ones of the same gross characteristics but comprised of more than one component was to utilise the phenomenon of beating. Consider two waves of equal amplitude $A$ and frequencies $f_1$ and $f_2$. Then the elevations are given by

\[ \eta_1 = A \cos(2\pi f_1 t + \phi_1) \]
\[ \eta_2 = A \cos(2\pi f_2 t + \phi_2). \]

The surface elevation of the sum of the two is given by

\[ \eta = \eta_1 + \eta_2 = 2A \cos \left[ 2\pi \left( \frac{f_1 + f_2}{2} \right) t + \frac{\phi_1 + \phi_2}{2} \right] \cos \left[ 2\pi \left( \frac{f_1 - f_2}{2} \right) t + \frac{\phi_1 - \phi_2}{2} \right]. \]
This takes the form of waves of amplitude 2A (and height 4A) and frequency \((f_1+f_2)/2\) Hz, modulated by an envelope of frequency \((f_1-f_2)/2\) Hz. The iterative scheme was used to ensure that \(\phi_1 = \phi_2 = 0\).

A beat (maximum amplitude) occurs when the envelope term equals \(\pm 1\) so beats occur at a frequency \(f_1-f_2\) Hz. Note that the whole wave packet repeats at a frequency \((f_1-f_2)/2\) Hz. The wavemaker is set up to generate waves of frequency \(n/25.6\) Hz, \(n\) an integer. This ensures that sampling takes place over an integral number of wavelengths. Therefore a central frequency of 0.9375 Hz, corresponding to \(n=24\) was chosen. Four other combinations as detailed below, were chosen to give an integer number of waves per beat.

<table>
<thead>
<tr>
<th>Case</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f_1) (Hz)</td>
<td>.937</td>
<td>0.898</td>
<td>0.859</td>
<td>0.820</td>
<td>0.781</td>
</tr>
<tr>
<td>(f_2) (Hz)</td>
<td>-</td>
<td>0.977</td>
<td>1.016</td>
<td>1.055</td>
<td>1.094</td>
</tr>
<tr>
<td>(f_1-f_2) (Hz)</td>
<td>-</td>
<td>0.078</td>
<td>0.156</td>
<td>0.234</td>
<td>0.312</td>
</tr>
<tr>
<td>(N)</td>
<td>-</td>
<td>12</td>
<td>6</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>(A_1) (mm)</td>
<td>80.0</td>
<td>40.0</td>
<td>38.2</td>
<td>39.0</td>
<td>36.4</td>
</tr>
<tr>
<td>(A_2) (mm)</td>
<td>-</td>
<td>39.2</td>
<td>38.0</td>
<td>38.7</td>
<td>35.7</td>
</tr>
<tr>
<td>(\phi_1) (rad)</td>
<td>.029</td>
<td>-0.04</td>
<td>-0.01</td>
<td>-0.02</td>
<td>-0.01</td>
</tr>
<tr>
<td>(\phi_2) (rad)</td>
<td>-</td>
<td>0.02</td>
<td>0.00</td>
<td>-0.03</td>
<td>-0.03</td>
</tr>
<tr>
<td>(H_R) (mm)</td>
<td>167</td>
<td>164</td>
<td>156</td>
<td>164</td>
<td>161</td>
</tr>
</tbody>
</table>
\( f_1 \) and \( f_2 \) are the frequencies of the primary harmonics. \( A_1 \) and \( A_2 \) are their amplitudes, taken from the amplitude spectrum.

\( \phi_1 \) and \( \phi_2 \) are their phases at measurement time \((t=15s)\), taken from the phase spectrum.

\( N \) is the number of waves per beat

\( H_R \) is the total height of the wave measured, taken from the wave height record with time.

**Results**

Amplitude spectra for the five wave cases studied are shown in figure 3. The presence of multiple bound harmonics (higher harmonics of the fundamental frequencies which travel with them) is to be expected. The quality of the wave generation is reflected in the fact that the majority of the components are in the range 0.1 to 1 mm compared to fundamentals of order 35 to 80 mm. It should be noted that the amplitudes of the fundamentals for case E (with the greatest separation of the two fundamental frequencies) had to be reduced to prevent the occurrence of spilling breakers. This is despite the fact that the wave steepness calculated using the measured height and average component frequency \((H/L = 0.09)\) was only 65% of the theoretical limit for breaking. The other waves of the same average characteristics did not start spilling at this height.

**Wave Kinematics**

The velocities under the crests and troughs of the waves measured are shown in figure 4. The measurements were made over a grid of points
Case A.
\(F_1 = 0.9375\text{Hz.}\)
\(F_2 = 0.9766\text{Hz.}\)

Case B.
\(F_1 = 0.8984\text{Hz.}\)
\(F_2 = 0.9766\text{Hz.}\)

Case C.
\(F_1 = 0.8594\text{Hz.}\)
\(F_2 = 1.0156\text{Hz.}\)

Case D.
\(F_1 = 0.8203\text{Hz.}\)
\(F_2 = 1.0547\text{Hz.}\)

Case E.
\(F_1 = 0.781\text{Hz.}\)
\(F_2 = 1.094\text{Hz.}\)

Figure 3. Amplitude Spectra
separated by 1 mm on the film (corresponding to 9.9 mm in the wave). The grid size was ten by thirty points covering the crest or trough area. The horizontal component of velocity from the three adjacent columns with the highest velocities nearest the surface were plotted against the elevation of the measurement area. These results show a very low degree of scatter, confirming the reliability and accuracy of the system. The values predicted for Chaplin's (1980) version of Dean's Stream function theory (taken to 21st order), linear theory and its Wheeler stretching derivative are also plotted. The Chaplin's stream function values are calculated using the wave height measured from the time record and the average frequency and are plotted up to the surface elevations calculated by the theory. The linear and Wheeler stretching values are calculated using the amplitudes of the components taken from the amplitude spectra with their individual frequencies and with phases set to zero at the crest. None of the bound harmonics are included and the surface elevations at the crest and trough are given by the sum of the two amplitudes.

The trends of the measured results match the form of the theoretical curves (at least for the near-monochromatic cases) but are all offset to the side of lower velocities. This phenomenon has been noted previously by Sobey (1989) and can probably be attributed to the choice of definition of phase velocity used. The theoretical values were calculated assuming the validity of Stokes first definition of phase velocity (zero mean Eulerian current) and assuming that there were no currents. The finite boundaries of a wave flume mean that Stokes 2nd definition of phase velocity (giving zero mass transport) is valid. This may well account for the observed discrepancy but this effect was not calculated or allowed for in the presentation of results.
Figure 4. Horizontal Velocities vs. Elevation

Results from:
- Experiments and the theories below:
  - Stream Fun.
  - Linear
  - Wheeler stretching

Figure 4. Horizontal Velocities vs. Elevation
CONCLUSIONS

The authors reached the following conclusion for the waves measured (with average properties $H/gT^2 = 0.015$, $d/gT^2 = 0.048$).

The kinematics of the highest wave in a short, two-component, two-dimensional group are not the same as those of a wave of the same characteristics in a regular sea. This is most noticeable at the crest and for the shortest groups - the velocities for case E (the shortest group with only 3 waves per beat) are about 15% higher than for the regular wave at the highest measurement point. Lower down the wave the theories overpredict the velocities measured, especially for the shortest groups. The measured results are, however, slightly offset from the predictions of 21st order Dean's theory. This may be attributed to the definition of phase velocity employed with the calculations and this will affect the magnitude of the differences observed.

The high order Dean's stream function velocities match the velocities from the monochromatic waves very well (allowing for the point above). The theory also performs well for all but the widest spreads of frequencies (the shortest groups). Linear theory also performs well for the single component sea (case A). It also underpredicts the measured velocities at the crest for the shortest group (case E), despite the presence of the highest wavenumber component. This might have been expected to lead to the high frequency contamination (and hence an overprediction of the velocities) mentioned earlier. There was no trace of this at the crests but there was some overprediction in the range -50 to -250 mm. The presence of currents and the definitions of phase velocity would however have to be investigated.
thoroughly before this could be said to be anything more than a possible tank effect. Wheeler stretching does not perform well for any of the waves. This becomes obvious only when measurements approach the surface emphasising the need for measurements to be made as close to the surface as possible.

Therefore we conclude that Chaplin's formulation of Dean's Stream function (taken here to 21st order) and linear theory both perform well in the calculations of extreme velocities over the full range of elevations for all but the shortest groups.

All the theories underpredict the velocities at the crest (top 25% of wave height) of the near-breaking combination of waves with the shortest group (3 waves per beat). Wheeler stretching does not provide a good estimate of maximum velocities for any of the wave cases studied.

The last two points become obvious only when measuring well above mean water level into the crest. Therefore for extreme wave loadings a technique with the ability to measure accurately well above mean water level (such as Particle Image Velocimetry) must be used before firm conclusions can be drawn.

ACKNOWLEDGEMENTS

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Crest Kinematics in Wave Groups

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1 Abstract

The measured kinematics under the crest of the highest wave in some simple two-dimensional wave groups are presented. The highest wave in each group has the same gross characteristics of height and period but each is embedded in a different wave environment. The measurement technique used, Particle Image Velocimetry, allows accurate velocity measurements to be made up to 95% of the wave height.

The measured kinematics show significant differences between wave groups for very high crests, although this is not reflected well in comparisons with standard theories (Wheeler stretching and Dean's Stream function).

2 Key Words

Measured Kinematics, Wave Groups, Particle Image Velocimetry.

3 Introduction

3.1 Loading and Design

Extreme wave loading has long been recognised as producing the single most significant contribution towards the total environmental load upon a typical steel spaceframe offshore structure. Therefore the ability to model accurately the extreme storm conditions, and from them the peak loads and overturning moments, is of considerable importance in the design of such structures.

Current practice in the design of these structures starts by determining certain design criteria. Statistical processes are applied to sources of environmental data collected from near the site or hindcast from the most appropriate and longest running records available. Design criteria covering winds, waves and currents for a (normally) 100 year return period are produced. In the case of waves, this amounts to calculating a most probable maximum wave height and its corresponding period for the given depth.

The second stage (assumed to be independent from both the first and third) involves modelling the design wave to produce the local wave kinematics. A higher order wave theory is used (normally Stokes' Vth, but the choice depends on the ratios of depth, wave height and period). The design wave is therefore assumed to be monochromatic, unidirectional, two-dimensional and regular (to allow the higher order theory to be applied). The presence of the structure is ignored in calculating the kinematics. The third stage combines the current and wave kinematics and calculates the local forces by the application of a Morison's-type equation (assuming no interaction between structure and flow).

3.2 Higher Order Theories and Irregular Seas

In recent years the appropriateness of the combination of the regular wave assumption and higher order theories in modelling storm conditions has been called into question. In particular the use of Directional Gaussian Linear Wave Theory (D.G.L.W.T.) incorporating a stretching factor has become a common alternative. In DGLWT the sea is assumed to be composed of the sum of a large number of individual small waves, each with its own amplitude, phase, frequency and direction and each conforming to the infinitesimal amplitude limitation of linear theory. The advantages of DGLWT are that it can model (even if only crudely) many of the three-dimensional aspects of the ocean surface and is easy to compute. The main disadvantage in applying the theory lies in the limit of applicability of linear theory being waves of infinitesimal amplitude. Any sea state even approaching storm conditions will be of sufficient amplitude to ensure the presence of non-linearities. As a result the theory produces large free surface boundary condition errors.

Wheeler (1970), Chakrabarti (1971) and others have all pro-
posed stretching modifications to linear theory. These reduce the errors in the free surface boundary conditions but, in doing so, violate the Laplace equation (the governing field equation of linear theory). The modifications were introduced because of observed discrepancies between linear theory and measurements, however, and not because of boundary conditions, so this violation attracts less interest from an engineering viewpoint than it might.

3.3 Sea States Investigated

The debate between the two approaches — applying a higher order, more accurate theory to a limited idealisation of the sea-state or a first order theory applied to a more realistic representation of the sea-state — centres on the applicability of using a regular wave to represent the most probable highest wave. There have been no conclusive studies to show whether or not the crest kinematics of the design wave match those of the design criteria — period and height in a given depth — in a real sea state. This paper takes a first step in that direction by addressing the effects of wave 'groupiness'. The crest kinematics of the highest wave in five simple wave groups are presented. In each case the gross characteristics of the highest wave (period and height) were kept the same but each was embedded in a different wave environment. This was achieved by limiting the groups to two (or one) component. As the frequencies of the two components were spread out from the central frequency (while maintaining the same average frequency) the wave became modulated with the length of the wave group inversely proportional to the frequency separation (see Figure 1).

Whole field velocity maps from underneath the crests of the highest waves were produced using the non-intrusive Particle Image Velocimetry (PIV) technique.

This paper follows on from one presented at the Society for Underwater Technology conference 'Environmental Forces on Offshore Structures and Their Prediction' by the authors (Sutherland et al 1990). In it the same measurements are compared to a limited application of linear wave theory and its Wheeler stretching derivative. In those applications linear theory and Wheeler stretching were applied to the two primary (input) components only and no account was taken of reflected waves. Here a full implementation of Wheeler Stretching is included, as are a number of other engineering approximations to linear theory.

The experiments were carried out in the short (6m) wave flume at Edinburgh University. The advantages of this approach are that the conditions are well controlled and monitored and that the use of this special facility allows accurate measurements to be taken over a large area in a short time. The disadvantages lie in the limitation of measuring two-dimensional (unidirectional) waves only and the difficulties of drawing conclusions valid in the sea from experiments conducted, no matter how accurately, in a wave flume.

4 Experimental Apparatus

4.1 The Wave Flume

The measurements were made in the 6m wave flume which has a still water depth of 0.54m and a width of 0.3m, see figure 2. Optical access to the PIV measurement region is obtained by having the walls and base of the central 2m section made out of 20mm thick glass.

The waves are generated by a single hinged paddle which is of the absorbing type (Salter, 1982). This incorporates a force feedback mechanism which equates the force on the water to the sum of the drive signal and a filtered velocity signal, thereby simultaneously generating waves and absorbing reflections. The drive signal to the paddle is provided by a microcomputer which also triggers the PIV camera and samples from four wavegauges.

The gauges are of the resistance type and are arranged in two sets at different positions along the flume. They were used in an iterative scheme to achieve the desired wave conditions. The wave height records were fourier transformed to produce amplitude and phase spectra. The input parameters were then altered to ensure that the amplitudes of the two pri-
mary wave components became equal, that the same maximum wave height was achieved in each case and that the two components were in phase at the measurement accuracy point. The whole process was repeated until the desired accuracy was achieved.

4.2 Particle Image Velocimetry (PIV)

PIV is an essentially non-intrusive measurement technique derived from the application of speckle photography to a fluid flow (see Keane and Adrian (1990) for a review). Gray and Greated (1989a) applied the technique to the measurement of water waves in a two-dimensional wave flume. The technique involves taking a multiple exposure photograph of an illuminated plane of the wave, parallel to the wall of the flume. The flume had first been seeded with suitable reflecting particles (conifer pollen). These are small enough and close enough to neutral buoyancy to follow the flow precisely, so each moves according to the local flow conditions.

The illumination is pulsed using the scanning-beam system derived by Gray and Greated (1989b). In this the beam from a 15W CW laser is collimated, reflected off an octagonal rotating mirror onto a parabolic recollimating mirror and scanned through the area of interest. If the camera shutter is opened for longer than two or more (typically four) pulse separations then multiple images of the seeding particles are produced. The scanning beam method was adopted to prevent the streaking of particle images (by maximising the pulse separation to pulse length ratio) and to maximise the percentage of available light power used in illuminating each particle. As a consequence, sufficient information to produce a velocity map over a significant area is recorded in a short time compared to the timescale of the flow (1/125 seconds compared to an average wave period of over a second).

4.3 PIV Analysis

A velocity map can be produced from the resulting multiple-exposure photograph of the wave by using an automated system of optical and digital analysis of small areas of the developed negative (Gray and Greated, 1989a). This involves shining a low-powered laser through a small area of the negative (here about 1mm in diameter), capturing the fringes with a C.C.D. camera at the focal point of a bi-convex lens and performing a Fourier transform on the resulting fringe data. The velocity is found by locating the centre of mass of the spatial frequency peak of maximum height in the correlation plane. The location of this peak gives a direct measure of the most common particle separation and direction of translation within the small illuminated area. Repeating this process automatically over a grid of points covering the area of interest allows a detailed velocity map to be produced.

The accuracy of the system depends on three main sources of error: in the magnification of the photograph, the pulse separation and the error in locating the signal peak in the transform plane. The last error decreases as velocity increases for a typical fringe visibility. The resulting error is estimated to be approximately 2% for the crest velocities (for which the errors are minimised) increasing as velocities decrease down the wave.

5 Wave Theories

Linear theory assumes that the sea can be represented by a sum of individual small waves. It depends on the wave height $H$ being much less than the wavelength $\lambda$ or water depth $D$, i.e. $H \ll \lambda, D$. The dispersion relation becomes $\omega^2 = gk \tanh(kD)$ and the horizontal component of velocity at a height $z$ measured upwards from mean water level, $U_z$, is given by the sum of terms from each component wave:

$$U_z = \sum_{i=1}^{\infty} A_i \omega_i \frac{\cosh(k_i(z + D))}{\sinh(k_iD)} \sin(\theta_i)$$

where $\omega$ is circular frequency, $k$ is the wavenumber ($k = 2\pi/\lambda$) and $g$ is gravitational acceleration. In deep water the amplitude term tends towards an exponential $e^{kz}$ which increases very rapidly above mean water level, particularly at high frequencies. This leads to the phenomenon of high frequency contamination when small, high frequency waves are lifted up above mean water level by much longer waves, thus producing conditions where $k > 1$ and a correspondingly large exponential term ensues.

The stretching approximation of Wheeler (1970) attempts to avoid this problem by stretching the vertical coordinate such that the velocity value calculated for mean water level previously is now calculated for the surface. All other vertical coordinates are stretched or compressed from the bed accordingly, using the same transformation: in the calculation of $U_z$,

$$(z + D) \rightarrow (z + D) \frac{D}{D + \eta}$$

where $\eta$ represents the instantaneous surface elevation. This has the effect of reducing velocities calculated at the crest and enhancing those found under the troughs of waves.

A variation of this approach stretches the vertical coordinate of each component so that the velocity calculated for the individual component height $\eta_i$ (the component amplitude and phase about mean water level) is that given to the surface. In the calculation of $U_z$, $$(z + D) = \alpha(\eta + D)$$

where $0 \leq \alpha \leq 1$ and $\alpha$ represents proportionally how far up the wave the elevation is. The stretching performed is

$$(z + D) = \alpha(\eta + D) - \alpha(\eta + D).$$

Wheeler stretching implies a discontinuous pressure at the air-water boundary so Chakrabarti (1971) proposed a modification (to equation 1) so that the stretching satisfies the dynamic boundary condition. In the calculation of $U_z$, $$(\sinh(k \eta) - \sinh(k(D + \eta))).$$

A further approach is to use linear theory up to mean water level then to extrapolate from there to the surface by assuming that the vertical partial derivative of velocity remains constant above MWL. In the calculation of $U_z$, for $z \geq 0$

$$U_z = U_0 + z \frac{\partial U_0}{\partial z}$$

Dean (1965) formulated his solution to the monochromatic wave problem in terms of the stream function. Chaplin (1980)
reformulated this solution by taking water depth, wave height and period as the independent parameters and using the surface elevations as the unknown and it is Chaplin's programme for regular waves that is used here.

6 The Test Waves
6.1 Wave Generation
A wave of the same period and height is most easily generated in a range of environments by utilising the phenomenon of beating. Consider two waves of equal amplitude $A$ and frequencies $f_1$ and $f_2$. Then the wave surface elevations are given by:

$$\eta_1 = A \cos(2\pi f_1 t + \theta_1)$$  \hfill (6)

$$\eta_2 = A \cos(2\pi f_2 t + \theta_2)$$  \hfill (7)

The surface elevation of the sum of the two is given by:

$$\eta = 2A \cos(2\pi \left(\frac{f_1 + f_2}{2} + \frac{\theta_1 + \theta_2}{2}\right)) \cos(2\pi \left(\frac{f_1 - f_2}{2} + \frac{\theta_1 - \theta_2}{2}\right))$$  \hfill (8)

This takes the form of waves of amplitude $2A$ (and height $4A$) and frequency $(f_1 + f_2)/2$, modulated by an envelope of frequency $(f_1 - f_2)/2$. (The iterative scheme ensured that $\theta_1 \approx \theta_2 \approx 0$). The wavemaker is set up to generate waves of frequency $n/25.6$ Hz, $n$ an integer. A frequency of 0.9375 Hz, corresponding to $n = 24$ was chosen as the central frequency, that of the regular wave. Four two-component combinations, as detailed in the table below, were chosen to give an integer number of waves per group, the number (N) decreasing as the frequency separation increased. The iterative use of the wavegauges ensured that the highest wave in each group corresponded closely to the regular wave (case A).

<table>
<thead>
<tr>
<th>Case</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$ (Hz)</td>
<td>0.938</td>
<td>0.898</td>
<td>0.859</td>
<td>0.820</td>
<td>0.781</td>
</tr>
<tr>
<td>$f_2$ (Hz)</td>
<td>-</td>
<td>0.977</td>
<td>1.016</td>
<td>1.055</td>
<td>1.094</td>
</tr>
<tr>
<td>$f_2 - f_1$</td>
<td>-</td>
<td>0.078</td>
<td>0.156</td>
<td>0.234</td>
<td>0.312</td>
</tr>
<tr>
<td>N</td>
<td>$\infty$</td>
<td>12</td>
<td>6</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$A_1$ (mm)</td>
<td>76.2</td>
<td>38.2</td>
<td>35.5</td>
<td>34.8</td>
<td>33.0</td>
</tr>
<tr>
<td>$A_2$ (mm)</td>
<td>-</td>
<td>38.4</td>
<td>38.0</td>
<td>37.8</td>
<td>33.9</td>
</tr>
<tr>
<td>$H_R$ (mm)</td>
<td>167</td>
<td>164</td>
<td>156</td>
<td>164</td>
<td>162</td>
</tr>
</tbody>
</table>

Case A has relative steepness $H/gT^2 = 0.015$ and relative depth $D/gT^2 = 0.048$. $H_R$ is the wave height taken from the wave height record. Wave height records from two positions along the flume were used to separate the incoming and reflected components. Phase and amplitude spectra were produced and the amplitude spectra for cases A, B and E are shown in figure 3. The reflection coefficients from the primary (input) components show a range from under 1% to 8% in amplitude as the percentage of reflection present. The spectra were limited to 8Hz as surface tension is already having a considerable effect at that frequency so the waves no longer obey the linear dispersion equation. The amplitudes below 0.1 mm were not plotted in figure 3 as this is below the resolving power.
of the wavegauge. The presence of bound harmonics is to be expected (as is a degree of set-down at a frequency equal to the difference in frequency between the primary components). The harmonics travel 'bound' to the wave, at the speed of the primary component but any variation of linear theory used to calculate the velocities based on the measured spectrum will treat the bound harmonics as free waves. It should be noted that the amplitudes of case E (with the shortest wave group) had to be reduced to prevent the occurrence of spilling breakers. This is despite the fact that the wave steepness calculated using the measured wave height and average component frequency \( H/L = 0.09 \) was only 65% of the theoretical limit for breaking. The other waves did not start to break at this height.

7 Wave Kinematics

The velocities measured under the crests of the chosen waves are shown in figure 4. The measured results from case A (monochromatic case) through to case D (the second widest separation) correspond very closely to each other. A significant increase in the highest velocities can be seen in case E (the shortest wave group) for a similar wave. This difference only becomes manifest above mean water level but the measured velocity at 95% of wave height is over 25% larger for the wave in the shortest wave group (case E) compared to the regular wave (case A). The difference is almost zero at mean water level, however, a fact which emphasizes the significance of measurements obtained above mean water level.

This behaviour is not unexpected though, particularly as the wave components in the shortest group (case E) had to be reduced to prevent the occurrence of spilling breakers. Case E is therefore tending to the case of a spilling wave on a flat bed. Griffiths (1989) has shown that for monochromatic spilling waves on a flat bed, the velocity increases rapidly with height in the crest, tending to the wave celerity at the surface. This rapid increase is limited to the region nearest the crest and the rate of increase of horizontal velocity with height increases with the development of the spilling. In pre-breaking waves the velocities tend to a lower value than the celerity and the rate of increase of velocity with height is correspondingly not so great. In the short wave group used here the crest grows more rapidly and the wave energy is focussed more quickly than in the longer wave groups, but the wave is not yet spilling. Therefore the presence of a region in the crest where velocities increase more rapidly with height than in the monochromatic wave case is to be expected, as is the fact that the velocities tend to a value lower than the wave celerity.

The measured kinematics were also plotted against the theoretical predictions of Dean's stream function (taken to 15th order) and the variations of linear theory from section 6. Plots of elevation against velocity are shown in figure 5. The values calculated using Chaplin's stream function were obtained using the wave height taken from time records. The values for the variations on linear wave theory came from the measured spectra. The precise values of the high frequency and low amplitude limits used in the calculation were only found to have a small effect: there was a change of approximately 3% in velocity on going from an amplitude cut-off of 0.1mm to one of 0.4mm for case A (Wheeler stretching). The higher limit was used to reflect the resolving power of the wavegauges. Similarly the only difference between going from applying a frequency cut-off at 4Hz to one at 8Hz was a small change, normally a reduction, in the velocities in the uppermost 5% of the wave due to very high frequency components of limited size. As a result only components in the range 0 to 4Hz and with amplitudes greater than 0.4mm were used in the calculations.

The effect of set-down becomes apparent at the lower frequency end of the spectrum, however. In each spectrum there is a peak at the frequency corresponding to the difference between the two input frequencies and this contributes to the velocities calculated by Wheeler stretching. The effect is negligible for low frequency separations (and so is not plotted) but becomes noticeable at the higher frequency separations. The plot for cases E and D in figure 5 therefore shows the results of using the components in the range 0—4Hz as well as the range 0.5—4Hz in the calculation of Wheeler stretching velocities. Even allowing for this, the Wheeler stretching underpredicts the measured velocities in all cases above mean water line. The match is much better well below mean water level, where most measurements are made. The other engineering approximations are calculated for the range 0 to 4Hz only.

The Chakrabarti stretching values are all lower than the Wheeler stretching ones, although the two sets of results converge at the surface. The stretching to component height performs better than any of the other stretching modifications, as the velocities calculated are higher. The fit is better for cases A and B (with long wave groups) than for cases C and D (shorter groups) but even then the calculated values tended to the measured at the surface.

None of the stretching modifications to linear theory coped with the rapid increases in the crest velocities present in the near-breaking wave (case E). The only theory that matched case E both well below MWL and near the surface was the extrapolation. This technique significantly overestimated the velocities at the crest of all the other wave cases and its success in the crest is completely dependent on the rate of change.

![Wave Kinematics](image-url)
8 Conclusions

The measurements presented show that the kinematics of the wave of given period and height in the shortest wave group are greater than those of a wave of the same characteristics in a regular sea. This becomes apparent only as the wave approaches breaking (only one wave out of five showed significant changes from the others; yet reaches a significant level (over 25%) in the wave crest. This is the area of greatest importance in extreme wave loading, yet is the region where fewest accurate measurements have been obtained. This result suggests that wave groupiness plays an important part in the internal kinematics of the individual crests and makes questionable the use of a regular train of waves for the design process.

The experimental results were bounded, above and below, by the extrapolation technique and the stretching approximations to linear theory. Of these the stretching to component height technique performed best for all cases, though it did not perform as well as the higher (15th) order Dean's stream function theory. It is only in the near-breaking case that this fails to provide a close match to the experimental results in the crest. It is only in the near-breaking case that extrapolation provides a close match to the experimental results in the crest.

The differences between the highest wave in the shortest group and the other waves of the same gross characteristics of period and height only became significant above MWL and near breaking. Moreover, the differences between measurement and theory are also greatest there. These results demonstrate the problems with using either a regular wave theory or an approximation to linear wave theory (particularly those calibrated from velocity measurements made below MWL) in modelling a sea-state. They also indicate the need for future work to be done on wave groupiness, extending the current research into random seas and then looking at three-dimensional effects, to justify current design practice.

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10 References


