POLYNOMIAL INEQUALITIES
FOR HILBERT SPACE OPERATORS

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ABSTRACT

A study is made of operator-valued representations of uniform algebras with the object of obtaining extensions of the von Neumann inequality for polynomials of contractions on Hilbert spaces. In the two-variable case sufficient conditions are obtained on the spectral sets of operators in order that they induce contractive operator-valued representations of certain uniform algebras. Subsequently a general dilation theorem is proven for normal operators and $H^\infty$-functional calculi are developed for some classes of operators.

Finally it is shown that the class of $Q$-algebras does not contain all the singly generated operator algebras that are semisimple.
PREFACE

This thesis is submitted for the degree of Doctor of Philosophy at the University of Edinburgh. The work is my own, except where specific mention to the contrary is made in the text, and the thesis has been composed by myself.

I would like to thank my research supervisor Dr A.M. Davie for many helpful suggestions and advice during the past three years of study, as well as Dr T.A. Gillespie for his being co-supervisor for this work during the past year. I would also like to thank the South African Council for Scientific and Industrial Research for their financial support during the period of research. Finally, thanks to Colin and Anne for their longanimity.
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The following notation will be used for subsets of the complex plane:

\[ \Delta = \{ z : |z| \leq 1 \}, \]
\[ D = \{ z : |z| < 1 \}, \]
\[ \Gamma = \{ z : |z| = 1 \}, \]

for any \( X \subset \mathbb{C} \), \( \partial X \) will denote the topological boundary of \( X \) and the Cartesian product of \( n \) copies of \( X \) will be written \( X^n \).

Let \( B(\mathcal{H}) \) denote the Banach algebra of all bounded linear transformations (or "operators") on a complex Hilbert space \( \mathcal{H} \). Elements of \( B(\mathcal{H}) \) of norm not greater than 1 are referred to as contractions.

In 1951 von Neumann [18] showed that for any contraction \( T \in B(\mathcal{H}) \),

\[ \|p(T)\| \leq \sup \{|p(z)| : z \in \Delta \} \]  \hspace{1cm} (1)

for all polynomials \( p \) with complex coefficients. A converse of this is also true; Foias [6] proved that if for any polynomial \( p \) with complex coefficients and any contraction \( T \) on a complex Banach space \( \mathcal{H} \) the inequality (1) holds, then \( \mathcal{H} \) is necessarily a Hilbert space.
In general, \( n \) commuting contractions \( T_1, T_2, \ldots, T_n \in \mathcal{B}(H) \) are said to satisfy the von Neumann inequality if for any polynomial \( p \) in \( n \) variables,
\[
\|p(T_1, T_2, \ldots, T_n)\| \leq \|p\|_\infty = \sup\{|p(z_1, z_2, \ldots, z_n)| : z_k \in \Delta \ (k = 1, 2, \ldots, n)\}.
\]

A simple proof of (1) follows from a well-known dilation theorem due to Sz.-Nagy (see [19]):

if \( T \in \mathcal{B}(H) \) is a contraction, then there exists a unitary operator \( U \in \mathcal{B}(K) \), for some Hilbert space \( K \) containing \( H \), such that
\[
T^n = P_H U^n |H \quad n = 0, 1, 2, \ldots,
\]
where \( P_H \) is the orthogonal projection on \( H \).

If \( p \) is any polynomial, then by the spectral theorem for unitary operators,
\[
p(T) = P_H \left( \int p(z) \, dE_z \right) |H,
\]
where \( E \) is the spectral measure of \( U \). For \( h, k \in H \) \((\|h\|, \|k\| \leq 1)\),
\[
|\langle p(T)h, k \rangle| = \left| \int p(z) \langle dE_z h, k \rangle \right| \\
\leq \sup\{|p(z)| : z \in \Gamma \} \|h\| \|k\|,
\]
and (1) follows. Furthermore, \( U \) is called the minimal unitary dilation of \( T \) if \( K \) is the closed linear span of \( \{U^n H : n = 0, \pm 1, \pm 2, \ldots\} \). For any contraction \( T \) its minimal unitary dilation is unique up to isomorphism.

Now let \( T_1, T_2 \in \mathcal{B}(H) \) be two commuting contractions.
A theorem of Ando (see [19]) shows that there exist commuting unitary operators $U_1, U_2 \in B(K)$, for some Hilbert space $K$ containing $H$, such that

$$T_1^n T_2^m = P_H U_1^n U_2^m |H|, \quad n, m = 0, 1, 2, \ldots,$$

where $P_H$ is the orthogonal projection on $H$. As for the single-operator case it follows that the contractions $T_1, T_2$ satisfy the two-variable von Neumann inequality. The pair $U_1, U_2$ is called a minimal unitary dilation of the pair $T_1, T_2$ if $K$ is the closed linear span of $\{U_1^n U_2^m H : n, m = 0, \pm 1, \pm 2, \ldots\}$. In contrast to the single-operator case, the minimal unitary dilation is not necessarily unique up to isomorphism.

In general, however, it was shown by Varopoulos [22] that for sets of more than two commuting contractions the von Neumann inequality fails to be true. For $n = 3$ specific counterexamples have independently been given by Kaijser and Varopoulos (see [23]) and Crabb and Davie (see [3]).

Let $X$ be a compact Hausdorff space and let $C(X)$ denote the Banach algebra of all continuous complex-valued functions on $X$ with the supremum norm. A closed subalgebra of $C(X)$ is called a uniform algebra. For a compact subset $X$ of $C^n$ let $R(X)$ be the uniform algebra of all functions in $C(X)$ that can be approximated uniformly on $X$ by rational functions that are analytic on a neighbourhood of $X$. Let $C_R(X)$ denote the real-valued functions
in $C(X)$ and let $H^\infty(X^\circ)$ be the collection of all bounded analytic functions on the interior $X^\circ$ of $X$ with the following norm
\[ \|f\|_\infty = \sup \{|f(z_1, z_2, \ldots, z_n)| : (z_1, z_2, \ldots, z_n) \in X^\circ\}. \]

1.1 DEFINITION A compact subset $X$ of $\mathbb{C}^n$ is said to be a joint spectral set for the $n$-tuple $(T_1, T_2, \ldots, T_n)$ of commuting operators in $B(H)$ if
(1) $\sigma(T_1, T_2, \ldots, T_n) \subset X$, where $\sigma(T_1, T_2, \ldots, T_n)$ denotes the joint spectrum of $(T_1, T_2, \ldots, T_n)$; and
(2) $\|f(T_1, T_2, \ldots, T_n)\| \leq \|f\| = \sup \{|f(z_1, z_2, \ldots, z_n)| : (z_1, z_2, \ldots, z_n) \in X\},$

for all rational functions $f$ that are analytic on a neighbourhood of $X$.

For the case $n = 1$ we refer to $X \subset \mathbb{C}$ as a spectral set of $T_1$.

For $n = 1, 2$ the $n$-variable von Neumann inequality shows that the closed unit disc in $\mathbb{C}$ (respectively, the closed bidisc in $\mathbb{C}^2$) is a spectral set (joint spectral set) for any contraction (pair of commuting contractions).

In Chapter 2 we consider conditions under which the Cartesian product of two compact sets of complex numbers is a joint spectral set for a pair of commuting operators, thus extending the abovementioned results for the bidisc. In connection with this a dilation theorem for normal operators
is proven.

Chapter 3 is devoted to obtaining extensions of \( H^\infty \)-functional calculi for commuting contractions. In particular we develop a single-variable \( H^\infty \)-functional calculus for some class of operators with a mild restriction on their spectral sets, and, in the two-variable case, for some pairs of operators with more severe restrictions on their spectral sets.

In Chapter 4 we generalize results by Varopoulos concerning cases for which the von Neumann inequality does not hold. This leads to the construction of a semisimple singly generated operator algebra that fails to be a Q-algebra.

First we introduce some well-known concepts that will be used. Let \( A \) be a uniform algebra on \( X \) and let \( \Phi \) be a homomorphism of \( A \) into \( \mathbb{C} \). A representing measure for \( \Phi \) is a probability measure \( m \) on \( X \), such that
\[
\Phi(u) = \int u \: dm, \quad u \in A.
\]
The set of all such representing measures for \( \Phi \) will be denoted by \( M_\Phi \). A set \( B \subset X \) is called \( \Phi \)-null if \( m(B) = 0 \) for all \( m \in M_\Phi \), and a measure \( \mu \) is \( M_\Phi \)-absolutely continuous if it vanishes on every \( \Phi \)-null set. The measure \( \mu \) is \( M_\Phi \)-singular if it is concentrated on a \( \Phi \)-null set. Every finite measure \( \mu \) on \( X \) has a unique \( \Phi \)-decomposition
\[ \mu = \mu_a + \mu_s \]

(2)

where \( \mu_a \) is \( M_\varphi \)-absolutely continuous and \( \mu_s \) is \( M_\varphi \)-singular. Two elements \( \vartheta \) and \( \varphi \) of the maximal ideal space \( M(A) \) of \( A \) are said to be equivalent if there exists a constant \( c > 0 \) such that

\[ c^{-1} < \frac{u(\vartheta)}{u(\varphi)} < c \]

whenever \( u \in \text{Re}(A), \ u > 0 \). This defines an equivalence relation on \( M(A) \) and the resulting equivalence classes are called the Gleason parts of \( M(A) \). It was proven by Glicksberg that if \( \vartheta \) and \( \varphi \) are in the same Gleason part of \( M(A) \), then the \( \vartheta \)-decomposition of \( \mu \) coincides with its \( \varphi \)-decomposition. If \( \vartheta \) and \( \varphi \) are in different Gleason parts, then the component \( \mu_a \) in (2) is \( M_\varphi \)-singular.

This justifies saying that (2) is the decomposition of \( \mu \) with respect to the Gleason part \( G (\varphi \in C) \), and that \( \mu_a \) is \( G \)-absolutely continuous and \( \mu_s \) is \( G \)-singular.

Let \( X \) be a compact Hausdorff space and suppose \( A \) is a uniform algebra on \( X \). Then \( A \) is called Dirichlet on \( X \) if \( \text{Re}(A) \) is uniformly dense in \( C_R(X) \), and \( A \) is called a Dirichlet algebra if \( A \) is Dirichlet on its Shilov boundary. If \( X \) is a compact subset of the complex plane the Shilov boundary of \( R(X) \) coincides with the topological boundary \( \partial X \) of \( X \) and hence we always implicitly refer to \( R(X) |_{\partial X} \) whenever \( R(X) \) is a Dirichlet algebra. The following result by Gamelin and Garnett (see [8]) characterizes
the Dirichlet algebra $R(X)$:

1.2 THEOREM  Let $X$ be a compact subset of the complex plane. Then $R(X)$ is a Dirichlet algebra if and only if

1. $R(\partial X) = C(\partial X)$,
2. each component of $X^0$ is simply connected, and
3. $R(X)$ is pointwise boundedly dense in $H^\infty(X^0)$ (i.e. each $f \in H^\infty(X^0)$ can be approximated pointwise on $X^0$ by a sequence in $R(X)$ that is bounded by $\|f\|_\infty$).

In particular, if $X$ is a compact subset of the complex plane that is finitely connected (i.e. the complement of $X$ consists of a finite number of components), then $R(X)$ is pointwise boundedly dense in $H^\infty(X^0)$ (see [7], VI 5.3).

As an example, if $X = \{z \in \Delta : \frac{|z - 1/2|}{1/2} \geq 1/2\}$ then $X$ is not simply connected, yet $R(X)$ is a Dirichlet algebra (see [7], VI 4.7).

If $A$ is a uniform algebra and $H$ is a Hilbert space the mapping $V : A \to B(H)$ is called a representation of $A$ if $V$ is an algebra homomorphism such that for some fixed positive number $K$,

$$\|V(u)\| \leq K\|u\|, \quad u \in A.$$  

Without loss of generality we assume $V(1) = I$. We call the representation contractive if $K = 1$. It follows that if
\( X \subset C^n \) is a joint spectral set for the \( n \)-tuple \((T_1, T_2, \ldots, T_n)\) of operators in \( B(H) \), there exists a unique contractive representation \( V : \mathbb{R}(X) \to B(H) \) that extends the homomorphism \( f \mapsto f(T_1, T_2, \ldots, T_n) \) on all rational functions which are analytic on a neighborhood of \( X \). For example, the closed unit disc is a spectral set for any contraction \( T \), hence \( T \) induces a unique contractive representation of the disc algebra.

If there exists a linear mapping \( V : A \to B(H) \), such that for some \( K > 0 \),
\[
\|V(u)\| \leq K \|u\|, \quad u \in A,
\]
then the Hahn-Banach extension theorem and Riesz representation theorem imply that there exist measures \( p_{xy} (x, y \in H) \) such that
\[
\|p_{xy}\| \leq K \|x\| \|y\| \quad \text{for } x, y \in H, \quad \text{and}
\]
\[
\langle V(u) x, y \rangle = \int u \, dp_{xy}, \quad u \in A, \quad x, y \in H.
\]
The measures \( p_{xy} \) are called elementary measures of \( V \).

For a survey of results on representations of uniform algebras induced by contractions, see [17].
CHAPTER 2

JOINT SPECTRAL SETS

In this chapter we are concerned with the following problem: assuming $S, T \in B(H)$ are commuting operators on some Hilbert space $H$ with spectral sets $X$ and $Y$ respectively, does it follow that $X \times Y$ is a joint spectral set for the pair $(S, T)$?

If $X = Y = \Delta$, then the answer is affirmative by the remarks in Chapter 1. Assume throughout this chapter that $S, T \in B(H)$ are commuting operators with spectral sets $X$ and $Y$ respectively. Two general results were obtained by Dash [4], viz.

2.1 THEOREM If $S$ and $T$ doubly commute (i.e. $ST = TS$ and $ST^* = T^*S$) and the complement of $Y$ is connected, then $X \times Y$ is a joint spectral set for $(S, T)$.

2.2 THEOREM If the complement of each of $X$ and $Y$ is connected and if the boundary of each of the components of their interiors is a simple closed curve, then $X \times Y$ is a joint spectral set for $(S, T)$.

We need the following
2.3 Lemma (Janas [11]) Let $A$ and $B$ be uniform algebras on the compact Hausdorff spaces $X$ and $Y$ respectively. Suppose $B$ is a Dirichlet algebra. Let $V : A \to B(H)$ be a representation of $A$ such that for some positive $M$, $\|V(v)\| \leq M\|v\|$ for $v \in A$. Suppose that $W : B \to B(H)$ is a contractive representation of $B$. If the representations $V$ and $W$ doubly commute (i.e. for $v \in A$ and $w \in B$,

$$V(v)W(w) = W(w)V(v) \quad \text{and} \quad V(v)W(w)^* = W(w)^*V(v)$$

then there is a unique representation $U : A \hat{\otimes} B \to B(H)$ (where $A \hat{\otimes} B$ denotes the completion of $A \otimes B$ in $C(X \times Y)$), such that

$$\|U(u)\| \leq M\|u\| \quad \text{for} \quad u \in A \hat{\otimes} B, \quad \text{and}$$

$$U(\sum_k v_k \otimes w_k) = \sum_k V(v_k)W(w_k) \quad \text{for} \quad v_k \in A, \; w_k \in B.$$

An application of this lemma will yield a generalization of 2.1. Let $R(Y)$ be a Dirichlet algebra. Let $V$ be the representation of $R(X)$ induced by $S$, and suppose $W$ is the representation of $R(Y)$ that is induced by $T$. Furthermore, $R(X) \hat{\otimes} R(Y) \subset R(X \times Y)$, and it is known that $R(X) \otimes R(Y) = Q$, where $Q = \{ h \in C(X \times Y) : \text{the mappings} \; x \mapsto h(x, y) \; (\text{for each} \; y \in Y) \; \text{and} \; y \mapsto h(x, y) \; (\text{for each} \; x \in X) \; \text{are in} \; R(X) \; \text{and} \; R(Y) \; \text{respectively} \}$, (see [7], VIII). Hence

$$R(X \times Y) \subset Q = R(X) \hat{\otimes} R(Y) \subset R(X \times Y),$$

so $R(X) \hat{\otimes} R(Y) = R(X \times Y)$. Thus by the remark that follows
1.2 we obtain an extension of 2.1:

2.4 Theorem If $S$ and $T$ doubly commute and $R(Y)$ is a Dirichlet algebra, then $X \times Y$ is a joint spectral set for $(S,T)$.

In what follows the objective will be to prove 2.10, thus generalizing 2.2. Suppose $U$ is a bounded region in the complex plane and $f \in C_{\mathfrak{R}}(\partial U)$, then for each $z \in U$ there exists a probability measure $\mu_z$ on $\partial U$ (the harmonic measure for $z$) such that if we define

$$\tilde{f}(z) = \int_{\partial U} f \, d\mu_z, \quad z \in U,$$

then $\tilde{f}$ is a bounded harmonic function on $U$ and

$$\lim_{z \to z_0} \tilde{f}(z) = f(z_0) \quad (z \in U)$$

for all $z_0 \in \partial U \setminus E$, where $E$ is a subset of $\partial U$ of capacity zero. Furthermore, $\mu_z$ is unique in the sense that if $\mu'_z$ and $\mu''_z$ are two harmonic measures on $\partial U$ for $z \in U$, then $\mu'_z = \mu''_z$ (see [20], III 44).

Let $z_1, z_2 \in U$, then Harnack's inequalities imply that there exists some $c > 0$, such that

$$c^{-1} \mu_{z_1} \leq \mu_{z_2} \leq c \mu_{z_1}.$$ 

Thus $\mu_{z_1}$ and $\mu_{z_2}$ define isomorphic $L^p$-spaces. If $f \in L^1(\mu_x)$ where $x \in U$, put $\tilde{f}(z) = \int f \, d\mu_z$, $z \in U$.

If there exists a sequence $(f_n)$ in $C(\partial U)$ such that $f_n \to f$ (n $\to \infty$) in $L^1(\mu_x)$, then $\tilde{f}_n \to \tilde{f}$ (n $\to \infty$) locally uniformly.
on $U$, where $\tilde{f}$ is harmonic on $U$.

In particular, for the case when $\partial U$ is connected, it follows from [20], I 11 that the abovementioned set $E$ is empty.

Suppose $\mathcal{B}$ is a $\sigma$-algebra of subsets of the space $Y$. The mapping $F : \mathcal{B} \to B(H)$ is called a semi-spectral measure if for each $h \in H$, $\sigma \mapsto \langle F(\sigma)h, h \rangle$ defines a finite positive measure on $\mathcal{B}$. The operator-valued function $A : Y \to B(H)$ is called simple if

$$A(y) = \sum_{j=1}^{m} R_j \chi_{\sigma_j}(y),$$

where $R_j \in B(H)$ and $\sigma_1, \sigma_2, \ldots, \sigma_m \in \mathcal{B}$ is a partition of $Y$. The bounded function $A : Y \to B(H)$ is called $\mathcal{B}$-measurable if there exists a sequence $(A_n)$ of simple functions such that

$$\sup\{\|A(y) - A_n(y)\| : y \in Y\} \to 0 \quad (n \to \infty).$$

The following theorem was proven by Mlak [15].

2.5 THEOREM Let $A : Y \to B(H)$ be a $\mathcal{B}$-measurable function such that $A(y)$ commutes with $F(\sigma)$ for all $y \in Y$ and $\sigma \in \mathcal{B}$. Then

$$\|\int_Y A(y) \, dF_Y\| \leq \|F(Y)\| \sup\{\|A(y)\| : y \in Y\}.$$

2.6 THEOREM If $T$ is a normal operator, then $X \times Y$ is a joint spectral set for $(S, T)$. 
Proof. Let $V: R(X) \rightarrow B(H)$ be the representation induced by $S$, and let $W: R(Y) \rightarrow B(H)$ be the representation induced by $T$. Let $E$ be the spectral measure of $T$. Suppose $v_1, v_2, \ldots, v_n \in R(X)$ and $w_1, w_2, \ldots, w_n \in R(Y)$, then using Fuglede's theorem,

$$\left\| \sum_k V(v_k) W(w_k) \right\| = \left\| \sum_k V(v_k) \int_Y w_k(y) \, dE_y \right\|$$

$$= \left\| \sum_k V(v_k) w_k(y) \, dE_y \right\|$$

$$\leq \sup \left\{ \left\| \sum_k V(v_k) w_k(y) \right\| : y \in Y \right\} \text{ by 2.5}$$

$$= \sup \left\{ \left\| W \left( \sum_k w_k(y) v_k \right) \right\| : y \in Y \right\}$$

$$\leq \sup \left\{ \left\| \sum_k v_k(x) w_k(y) \right\| : (x, y) \in X \times Y \right\}.$$ 

Hence the mapping

$$U: \sum_k v_k w_k \mapsto \sum_k V(v_k) W(w_k)$$

is well-defined and $U$ can be extended to a representation of $R(X) \hat{\otimes} R(Y)$, such that

$$\left\| U(u) \right\| \leq \left\| u \right\| \text{ for } u \in R(X) \hat{\otimes} R(Y).$$

Since $R(X) \hat{\otimes} R(Y) = R(X \times Y)$, the result follows.

2.7 LEMMA (Mlak [16]) Suppose $S \in B(H)$ has a spectral set $X$ and $(G_k)$ is the (possibly infinite) sequence of components of $X^0$. Then

$$S = \bigoplus_{k=0}^q S_k$$

$q \leq \infty,$

where

- (1) $S_0$ is a normal operator, $\sigma(S_0) \subset \partial X$;
- (2) $S_k$ has spectral set $\overline{G_k}$ for $k \geq 1$;
- (3) the representation of $R(\overline{G_k})$ induced by $S_k$ ($k \geq 1$) has associated elementary measures that are
Gₖ-Absolutely continuous.

Suppose A is a uniform algebra on X and Φ ∈ M(A).

The following two lemmas are found in [7], II 7.

2.8 Lemma Suppose E is an Fₙ-set and m(E) = 0 for all m ∈ Mₚ. Then there exists a sequence \( (f_n) \) in A, such that \( \|f_n\| ≤ 1 \) for all n, \( f_n → 0 \ (n → ∞) \) pointwise on E and \( f_n → 1 \) a.e. (m) for all \( m ∈ Mₚ \).

2.9 Lemma Suppose K is a weak-star compact convex set of positive measures on X. If the measure \( μ \) is singular to all elements of K, then there exists an Fₙ-set E, such that \( |μ|(X \setminus E) = 0 \) and \( k(E) = 0 \) for all \( k ∈ K \).

2.10 Theorem Let \( R(X) \) and \( R(Y) \) be Dirichlet algebras. Then \( XX Y \) is a joint spectral set for (S,T).

Proof. Assume that \( Y = Δ \) and \( G, G_2, ... \) are the components of \( X^0 \). The first step is to show that T reduces each of the subspaces of H which correspond to the decomposition of S in 2.7. For any fixed \( Gₖ \), the representation \( V: R(X) → B(H) \) induced by S decomposes into the direct sum of two representations of \( R(X) \),

\[ V = V_S \oplus V_a \]

in which the representation \( V_S \) is \( Gₖ \)-singular (i.e. \( V_S \))
has $G_k$-singular elementary measures $\nu_{xy}^{(s)} (x, y \in H)$, and
the representation $\nu_{a}$ is $G_k$-absolutely continuous (i.e. $\nu_{a}$
has $G_k$-absolutely continuous elementary measures $\nu_{xy}^{(a)}$
$(x, y \in H)$); see [14]. The restriction of $S$ to the
ranges of the projections $\nu_{a}(1)$ and $\nu_{a}(1)$ gives the
orthogonal decomposition

$$S = S_{s} + S_{a}.$$  

Since $R(X)$ is a Dirichlet algebra and $M_z$ therefore
consists of a unique element $\mu_z (z \in G_k)$, 2.8 and 2.9
imply that there exists a norm-bounded sequence $(u_n)$ in
$R(X)$, such that

$$u_n \to 1 \quad \text{a.e. } (\mu_z)$$
and

$$u_n \to 0 \quad \text{pointwise on some set } E \text{ for which } \mu_z (E) = 0.$$  

Let $g, h \in H$, then

$$\langle (u_n (S) - \nu_{a}(1))g, h \rangle = \langle (\nu_{a}(u_n) + \nu_{s}(u_n) - \nu_{a}(1))g, h \rangle$$
$$\leq \langle \nu_{a}(u_n - 1)g, h \rangle + \langle \nu_{s}(u_n)g, h \rangle$$
$$= \int (u_n - 1) d\nu_{gh} + \int u_n d\nu_{gh}.$$  

The first term vanishes as $n \to \infty$. Since $\nu_{gh}^{(s)} \perp \mu_z$,  

$$\nu_{gh} = \nu_{gh}^{(s)} \text{ in } 2.9; \text{ then } u_n \to 0 \quad \text{a.e. } |\nu_{gh}^{(s)}|,$$
so

$$\int u_n d\nu_{gh}^{(s)} \to 0 \quad \text{a.e. } (|\nu_{gh}^{(s)}|) \text{ as } n \to \infty.$$  

Hence the second term also vanishes as $n \to \infty$, i.e.

$$u_n(S) \to \nu_{a}(1) \quad \text{in the weak operator topology.}$$  

This means that $T$ commutes with the projections $\nu_{a}(1)$ and $\nu_{s}(1)$.

By repeated application of these arguments for each $G_k$ to
the singular representation of each previous decomposition
it follows that $T$ reduces the subspaces of $H$ that
correspond to the decomposition of \( S \) in 2.7. Hence
\[
T = \bigoplus_{k=0}^{q} T_k
\]
with \( T_k S_k = S_k T_k \) for \( k = 0, 1, \ldots, q \).

Let \( f \in \mathbb{R}(\mathbb{X} \times \Delta) \), then \( f(S, T) = \bigoplus_{k=0}^{q} f(S_k, T_k) \) and
\[
\|f(S, T)\| = \sup_{k} \|f(S_k, T_k)\|.
\]

Since \( S_0 \) is a normal operator it follows from 2.6 that
\[
\|f(S_0, T_0)\| \leq \sup \{\|f(z_1, z_2)\| : (z_1, z_2) \in \partial \mathbb{X} \times \Delta\}.
\]
Since \( \mathbb{R}(\mathbb{X}) \) is a Dirichlet algebra and each component of \( \mathbb{X}^0 \) is simply connected (see 1.2), let \( \Phi_k : G_k \rightarrow \mathbb{D} \) be a conformal mapping \( (k = 1, 2, \ldots, q) \). Fix an arbitrary \( k \).

There exists a sequence \( (u_n) \) in \( \mathbb{R}(\overline{G}_k) \) that converges pointwise to \( \Phi_k \) on \( G_k \), such that \( \|u_n\| \leq \|\Phi_k\|_{\infty} \) for all \( n \) (see 1.2).

For \( z \in G_k \) let \( \mu_z \) be the harmonic measure on \( \partial G_k \) and define \( H^\infty(\mu_z) \) to be the weak-star closure of \( \mathbb{R}(\overline{G}_k) \) in \( L^\infty(\mu_z) \). For \( g \in H^\infty(\mu_z) \) the function \( \tilde{g} \),
\[
\tilde{g}(z) = \int g \, d\mu_z \quad z \in G_k,
\]
denotes the harmonic extension of \( g \) on \( G_k \). Since \( \Phi_k = \tilde{h} \) for any weak-star adherent point of the sequence \( (u_n) \) in \( H^\infty(\mu_z) \), there exists a subsequence \( (u_{n_r}) \) which converges to \( h \) in the weak-star topology of \( L^\infty(\mu_z) \).

Let \( x, y \in H \), then for some \( G_k \)-absolutely continuous measure \( p_{xy} \),
\[
\langle u_{n_r}(S_k)x, y \rangle = \int u_{n_r} \, dp_{xy}.
\]
Since \( u_{n_r} \in L(\mu_z) \) and \( \mu_z \in M_z \).
since the mapping $g \mapsto \tilde{g}$ defines an isometric isomorphism from $H^\infty(\mu_z)$ onto $H^\infty(\sigma_k)$ and $h_1 = h_2$ (as elements of $H^\infty(\mu_z)$) whenever $\tilde{h}_1 = \tilde{h}_2 = q_k$.

If $f \in R(X \times \Delta)$ there exist open neighbourhoods $U$ and $W$ of $X$ and $\Delta$ respectively such that $f$ is analytic on some neighbourhood of $\overline{U} \times \overline{W}$. Also, since $R(\overline{U}) \otimes R(\overline{W})$ is uniformly dense in $R(\overline{U} \times \overline{W})$ (see discussion preceding 2.4) and by the continuity of the two-variable functional calculus for functions that are analytic on a neighbourhood of the joint spectrum (see [7], III 4) we restrict ourselves to $f \in R(X \times \Delta)$ of the form

$$f = \sum_j g_j h_j \quad \text{(finite summation)}$$

with $g_j \in R(\overline{U})$ and $h_j \in R(\overline{W})$ where $U \times W$ is some open neighbourhood of $X \times \Delta$. 
\[
\lim_{n \to \infty} \int (u_n - h) \, dp_{xy} = 0.
\]

Hence
\[
\lim_{n \to \infty} \langle u_n (S_k) x, y \rangle = \lim_{n \to \infty} \int u_n \, dp_{xy} = \int h \, dp_{xy},
\]
and
\[
|\int h \, dp_{xy}| \leq \lim_{n \to \infty} \sup \|u_n\| \|x\| \|y\| \leq \|h\| \|x\| \|y\|.
\]

Also \( \int h \, dp_{xy} \) is independent of the choice of sequence \((u_n)\).

Define therefore \( \Phi_k(S_k) = \lim_{n \to \infty} u_n(S_k) \), the limit taken in the weak operator topology. Then
\[
\|\Phi_k(S_k)\| = \sup \{ |\int h \, dp_{xy}| : \|x\| \|y\| \leq 1 \}
\]
\[
\leq \mu_{\text{ess}} \sup |h| = \sup \{|\Phi_k(z)| : z \in G_k\}
\]
\[
= 1.
\]

By the continuity of the two-variable functional calculus for functions which are analytic on a neighbourhood of the joint spectrum (see [7], III 4), and since \( R(X) \otimes R(\Delta) \) is uniformly dense in \( R(X \times \Delta) \) (see the discussion preceding 2.4), we restrict ourselves to \( f \in R(X \times \Delta) \) of the form
\[
f = \sum_j g_j h_j \quad \text{(finite summation)},
\]
where \( g_j \in R(X) \) and \( h_j \in R(\Delta) \).

Then
\[
f(S_k, T_k) = \sum_j g_j(S_k) h_j(T_k).
\]

Let \( q_j(z) = g_j(\Phi_k^{-1}(z)) \), then \( q_j \) is a bounded analytic function on \( D \). Let \((b^{(j)}_n)_n\) be a sequence in \( R(\Delta) \), where
\[
b^{(j)}_n(z) = q_j((1 - n^{-1})z).
\]
Then, as before,

\[ b_n^{(j)} \rightarrow b_j \quad (n \rightarrow \infty) \]

in the weak-star topology of \( L^\infty(m_z) \), where \( m_z \) is the harmonic measure for \( z \in D \) on \( \Gamma \), and \( \tilde{b}_j = q_j \).

If \( w = \Phi_k^{-1}(z) \), then by [7], VI 4.3,

\[
\int b_n^{(j)} \Phi_k \, dm_w = \int b_n^{(j)} \, dm_z \rightarrow \int b_j \Phi_k \, dm_w = \int b_j \cdot \Phi_k \, dm_w ,
\]

\( n \rightarrow \infty \). Since

\[
(b_j \cdot \Phi_k) = \tilde{b}_j \cdot \Phi_k = a_j \cdot \Phi_k = g_j
\]

and

\[
g_j(w) = \int g_j \, dm_w
\]

and since \( p_{xy} \) is \( G_k \)-absolutely continuous it follows that

\[
\langle b_n^{(j)}(\Phi_k(S_k))x,y \rangle = \int b_n^{(j)} \Phi_k \, dp_{xy} \rightarrow \int g_j \, dp_{xy} = \langle g_j(S_k)x,y \rangle
\]

\( (n \rightarrow \infty) \).

Hence

\[
\|f(S_k,T_k)\| = \sup \{ \| \sum g_j(S_k) h_j(T_k) x,y \| : \|x\|,\|y\| \leq 1 \}
\]

\[
= \sup \{ \lim \sup \langle b_n^{(j)}(\Phi_k(S_k))h_j(T_k)x,y \rangle : \|x\|,\|y\| \leq 1 \}
\]

\[
\leq \sup \{ \lim \sup \| \sum b_n^{(j)}(\Phi_k(S_k))h_j(T_k)x,y \| : \|x\|,\|y\| \leq 1 \}
\]

\[
\leq \sup \{ \lim \sup \| \sum b_n^{(j)}(\Phi_k(S_k))h_j(T_k) \| : \|x\|,\|y\| \leq 1 \}
\]

\[
= \lim \sup \| \sum b_n^{(j)}(\Phi_k(S_k))h_j(T_k) \|
\]

\[
\leq \lim \sup \sup \{ \| \sum g_j(\Phi_k^{-1}((1-n^{-1})z_1))h_j(z_2) \| : z_1,z_2 \in D \}, \text{by the von Neumann inequality}
\]

\[
\leq \sup \{ \sum g_j(z_1) h_j(z_2) \| : (z_1,z_2) \in G_k \times D \}, \text{by definition of } \Phi_k
\]

\[
\leq \sup \{ \sum g_j(z_1) h_j(z_2) \| : (z_1,z_2) \in X \times \Delta \}
\]
Thus $X \times \Delta$ is a joint spectral set for $(S,T)$. For arbitrary sets $X$ and $Y$ satisfying the requirements of the theorem we first decompose $S$ and $T$ into direct sums of operators that correspond to the partitioning of the set $X$ (as above). Then make a similar decomposition of all the component operators in a similar way, corresponding to the partitioning of $Y$. The general result follows by applying the above reasoning to this double decomposition.

2.11 REMARK Using the notation in 2.10, the conformal mapping $\varphi^{-1}_k : D \rightarrow G_k$ yields a bounded sequence $(h_n)$ in $R(\Delta)$ that converges to $g$ in the weak-star topology of $L^\infty(m_z)$, $z \in D$, where $\tilde{g} = \varphi^{-1}_k$. Hence for $x,y \in H$,

$$<h_n(\varphi_k(S_k))x,y> = \int h_n \cdot \varphi_k \, dp_{xy} \rightarrow \int g \cdot \varphi_k \, dp_{xy} \quad (n \rightarrow \infty)$$

$$= \int z \, dp_{xy}(z) = <S_k x, y> .$$

If we define $\varphi_k^{-1}(\varphi_k(S_k)) = \lim_n h_n(\varphi_k(S_k))$, the limit taken in the weak operator topology, then $S_k = \varphi_k^{-1}(\varphi_k(S_k))$.

It seems reasonable to ask whether the joint spectral set problem for two commuting operators can be solved for spectral sets $X$ and $Y$ whose interiors are multiply connected (a possibility not covered by 2.10), as 2.6
might suggest.

The proof of 2.10 consists essentially of decomposing the operators S and T into direct sums of restrictions to suitable reducing subspaces, in each of which case the two-variable von Neumann inequality can be applied. Furthermore, the von Neumann inequality is proven by using the fact that every contraction $V \in \mathcal{B}(H)$ has a (strong) unitary dilation $U \in \mathcal{B}(K)$, for some Hilbert space $K$ containing $H$, i.e.

$$p(V) = P_H p(U) | H$$

for all polynomials $p$, where $P_H$ is the orthogonal projection on $H$.

It is not known whether in general an operator $T \in \mathcal{B}(H)$ with spectral set $Y$ dilates in the following way,

$$f(T) = P_H f(N) | H \quad \text{for } f \in \mathcal{R}(Y),$$

where $N$ is a normal operator acting on a Hilbert space $K$ containing $H$, such that $\sigma(N) \subset \mathcal{B}Y$. This being generally true might then open the way to an extension of 2.10 to a wider class of sets for $X$ and $Y$. Lébow [13] showed that such dilations are possible if $\mathcal{R}(Y)$ is a Dirichlet algebra, and then deduced that for arbitrary $T$ (i.e. arbitrary spectral set $Y$) such a dilation is possible provided only polynomials $f$ are considered.
(I.G. Craw recently pointed out to the author that this theorem is contained in papers by W.B. Arveson, Acta Math., 123 (1969) and 128 (1972))
The following shows that in a special case the above dilation is possible for normal operators.

2.12 Theorem. Let $H$ be a separable Hilbert space and let $T \in B(H)$ be a normal operator with spectral set $\mathcal{Y}$. There exists a normal operator $N \in B(K)$ for some Hilbert space $K$ containing $H$, such that $\sigma(N) \subseteq \partial \mathcal{Y}$ and

$$f(T) = P_H f(N) \mid H \quad \text{for all } f \in R(Y).$$

Proof. Let $e_0, e_1, e_2, \ldots$ be an orthonormal basis for $H$. For $y \in Y^0$ let $\mu_y$ denote the harmonic measure on $\partial Y$, and for $y \in \partial Y$ let $\mu_y$ be the unit mass concentrated at $y$. Hence if $f \in C(\partial Y)$,

$$\int_{\partial Y} f \, d\mu_y = \begin{cases} \tilde{f}(y) & \text{for } y \in Y^0 \\ f(y) & \text{for } y \in \partial Y. \end{cases}$$

Let $E$ be the spectral measure of $T$. Let $K_0$ be the set of all continuous functions $f : \partial Y \to H$ where

$$f(y) = \sum_j \varphi_j(y) e_j \quad y \in \partial Y,$$

and

$$0 \leq \sum_{j,k} \int_Y \left( \int_{\partial Y} \varphi_j(y) \overline{\varphi_k(y)} \, d\mu_z(y) \right) \langle dE_z e_j, e_k \rangle < \infty.$$

For $f, g \in K_0$, $f(y) = \sum_j \varphi_j(y) e_j$, $g(y) = \sum_k \psi_k(y) e_k$, define an inner product $\langle \cdot, \cdot \rangle_{K_0}$ on $K_0$ by

$$\langle f, g \rangle_{K_0} = \sum_{j,k} \int_Y \left( \int_{\partial Y} \varphi_j(y) \overline{\psi_k(y)} \, d\mu_z(y) \right) \langle dE_z e_j, e_k \rangle.$$

This inner product is well-defined since $\varphi_j \overline{\psi_k} \in C(\partial Y)$ for all $j,k$. Let $K$ be the Hilbert space obtained from $K_0$. 

by factoring out the subspace \( \{ f \in K_0 : \langle f, f \rangle_{K_0} = 0 \} \) and by taking the completion. Denote the inner product on \( K \) by \( \langle \cdot, \cdot \rangle_K \).

Define \( N \in B(K) \) by
\[
(Nf)(y) = yf(y) \quad \text{for all } y \in \partial Y, \quad f \in K.
\]
By definition of the inner product on \( K \) it follows that for \( f \in K, \)
\[
(N^* f)(y) = \overline{y} f(y) \quad \text{for all } y \in \partial Y.
\]
Hence \( N \) is a normal operator. Let \( I \) denote the identity operator on \( K \) and suppose \( y \notin \partial Y \). Then for \( f \in K, \)
\[
((N - yI) f)(z) = (z - y) f(z) \quad z \in \partial Y.
\]
Define \( M \in B(K) \) by
\[
(Mf)(z) = (z - y)^{-1} f(z) \quad \text{for all } z \in \partial Y, \quad f \in K.
\]
Then \( M(N - yI) = I \), i.e. \( y \notin \sigma(N) \).
Hence \( \sigma(N) \subset \partial Y \).

Define \( \Phi : H \to K \) by \( h = \sum_j c_j e_j \rightarrow \sum_j \Phi_j(h)(\cdot) e_j \) where \( \Phi_j(h)(y) = c_j \) for all \( y \in \partial Y \). It follows that \( \Phi \) is linear, \( (1-1) \) and norm-preserving. Indeed,
\[
\| \Phi(h) \|^2 = \sum_{j,k} \int_{\partial Y} \int_{\partial Y} c_j \overline{c_k} \rho_{j,k}(z) \langle \rho_{j,k} \rangle_{j,k} \langle \rho_{j,k} \rangle_{j,k} \langle e_j, e_k \rangle
\]
\[
= \sum_{j,k} c_j \overline{c_k} \langle e_j, e_k \rangle
\]
\[
= \| h \|^2.
\]
Hence \( H \) can be identified with the set of all elements \( g \in K \) such that \( g(y) \) is a fixed element of \( H \), for all \( y \in \partial Y \). Let \( f \in R(Y) \). Then \( f(N) e_j = f(.) e_j \in K \)
and
\[ \langle f(N)e_j, e_k \rangle = \int_Y (\int_{\partial Y} f d\mu_z) \langle dE_z e_j, e_k \rangle = \int_Y f(z) \langle dE_z e_j, e_k \rangle = \langle f(T)e_j, e_k \rangle. \]

Hence \( P_H f(N) |H = f(T) \) for all \( f \in \mathbb{R}(Y) \).

2.13 COROLLARY With the same notation as in 2.12, if \( h \in H \) and \( x \in \partial Y \), then \( Th = xh \) if and only if \( Nh = xh \).

Proof. Suppose \( Nh = xh \). Then
\[ \| (T - xI)h \| = \| P_H (N - xI)h \| \leq \| (N - xI)h \| = 0. \]

Suppose \( Th = xh \). Write \( Nh = h_1 \oplus h_2 \) with \( h_1 \in H \) and \( h_2 \in K \oplus H \). Hence \( h_1 = P_H Nh = Th = xh \) and
\[ \| Nh \|^2 = |x|^2 \| h_2 \|^2. \]
Assume that \( h = \sum_j \beta_j e_j \).

Then
\[ \| Nh \|^2 = \langle Nh, Nh \rangle_K = \sum_{j,k} \beta_j \overline{\beta_k} \int_Y (\int_{\partial Y} y \overline{y} d\mu_z(y)) \langle dE_z e_j, e_k \rangle = \sum_{j,k} \beta_j \overline{\beta_k} \int_Y |z|^2 \langle dE_z e_j, e_k \rangle = \sum_{j,k} \beta_j \overline{\beta_k} \langle Te_j, Te_k \rangle = \| Th \|^2 = |x|^2 \| h \|^2. \]

Hence \( h_2 = 0 \) and so \( Nh = xh \).
2.14 REMARK In general one might hope that it is possible to drop the commutativity condition on the operators $T_1, T_2, \ldots, T_n$ in the von Neumann inequality by making an appropriate interpretation of $p(T_1, T_2, \ldots, T_n)$ for polynomials $p$ in $n$ variables. This interpretation would be the following: for each monomial $z_1^{r_1}z_2^{r_2} \cdots z_n^{r_n}$, the corresponding term in $p(T_1, T_2, \ldots, T_n)$ is given by

$$(n!)^{-1} \sum_{s} T^{s(1)}_{s(1)} T^{s(2)}_{s(2)} \cdots T^{s(n)}_{s(n)},$$

the summation extending over all permutations $s$ of $\{1, 2, \ldots, n\}$. However, Holbrook [10] has constructed a counterexample showing that for the case $n = 2$ the von Neumann inequality does not hold for non-commuting contractions.
For an operator $T \in B(H)$ with spectral set $X$ there exists a unique representation $R(X) \rightarrow B(H)$ that is induced by $T$. In this chapter we consider conditions under which there exist extensions to $H^\infty(X^0)$ of such representations.

In 1960 Sz.-Nagy and Foias (see [19]) initiated the development of a general $H^\infty$-functional calculus for operators on a Hilbert space by obtaining a one-variable $H^\infty$-functional calculus for contractions which are completely non-unitary (i.e. contractions for which there exist no non-zero reducing subspaces restricted to which they are unitary operators). More precisely, if the contraction $T \in B(H)$ is completely non-unitary, then for $f \in H^\infty(D)$ an operator $f(T)$ can be defined in such a way that there exists a norm-decreasing homomorphism $H^\infty(D) \rightarrow B(H)$ which extends the usual definition for polynomials. In addition, if $(f_n)$ is a bounded sequence in $H^\infty(D)$ which converges pointwise to $f$ on $D$, then $f_n(T) \rightarrow f(T)$ ($n \rightarrow \infty$) in the weak operator topology. This functional calculus is an extension to $H^\infty(D)$ of the representation of the disc algebra induced by $T$.

Similar functional calculi for certain classes of contractions have subsequently been developed by Briem, Davie
and Øksendal, Khahn and others (see [17]). However, since all the abovementioned functional calculi involve the von Neumann inequality and any contraction has spectral set $\Delta$ (which, of course, is a simply connected set) it seems worthwhile to investigate the existence of such functional calculi for operators whose spectral sets are multiply connected. First we consider the one-variable case.

3.1 THEOREM Let $T \in B(H)$ be an operator with spectral set $X$ such that $R(X)$ is pointwise boundedly dense in $H^\infty(X^0)$ and suppose that $T$ has no non-zero reducing subspace $H'$ such that $T|H'$ is a normal operator with spectrum contained in $\partial X$. Then there exists an algebra homomorphism $H^\infty(X^0) \to B(H)$ ($f \mapsto f(T)$) such that

1. $p(T)$ has the usual meaning if $p$ is a polynomial;
2. $\|f(T)\| \leq \|f\|_\infty$ for $f \in H^\infty(X^0)$;
3. if $\sup_n \|f_n\| < \infty$ and the sequence $(f_n)$ converges pointwise to $f$ on $X^0$, then $f_n(T) \to f(T)$ ($n \to \infty$) in the weak operator topology.

Proof. Let $(G_k)_{k=1}^q$ be the (possibly infinite) sequence of components of $X^0$. Then by 2.7 and the hypothesis on $T$, there exists the decomposition

$$T = \bigoplus_{k=1}^q T_k \quad (q \leq \infty),$$

where each $T_k$ has spectral set $\overline{G}_k$. Let $f \in H^\infty(X^0)$. 
There exists a bounded sequence \((u_n)\) in \(R(X)\) that converges pointwise to \(f\) on \(X^0\), such that \(\|u_n\| \leq \|f\|_{\infty}\) for all \(n\) (see [7], VI-5.3). For \(z \in G_k\), let \(\mu_z^{(k)}\) be the harmonic measure on \(\partial G_k\). As in the proof of 2.10 we obtain a subsequence \((u_{n_r})\) which converges to \(g_k\) in the weak-star topology of \(L^\infty(\mu_z^{(k)})\), where \(g_k = f|G_k\), for each \(k\). Define

\[
f(T) = \sum_{k=1}^{\infty} f(T_k),
\]

where \(f(T_k) = \lim_{n \to \infty} u_n(T_k)\), the limit taken in the weak operator topology.

(1) Suppose \(f(x) = \sum_j c_j x^j\) is a polynomial. Then

\[
f(x) = g_k(x) = \int g_k \, d\mu_x^{(k)}, \quad x \in G_k.
\]

But

\[
\int \sum_j c_j z^j \, d\mu_x^{(k)}(z) = \sum_j c_j \int z^j \, d\mu_x^{(k)}(z) = \sum_j c_j x^j.
\]

So \(g_k = f\) on \(\partial G_k\), for all \(k\). Let \(x = \bigoplus_k x_k\), \(y = \bigoplus_k y_k \in H\), then for some \(G_k\)-absolutely continuous elementary measures \(p_x^{(k)}\),

\[
\langle f(T)x, y \rangle = \sum_k \langle f(T_k)x_k, y_k \rangle
\]

\[
= \sum_k \int g_k \, dp_x^{(k)} \quad \text{as in the proof of 2.10}
\]

\[
= \sum_{k,j} c_j \langle T_k^j x_k, y_k \rangle
\]

\[
= \langle \sum_j c_j T^j \rangle x, y \rangle.
\]

If \(f \in H^\infty(X^0)\),

\[
\|f(T)\| = \sup_k \|f(T_k)\|
\]
which proves (2).

(3) As before, since \( f_n \to f \quad (n \to \infty) \) pointwise on \( X^0 \) it follows that \( f_n \to g_k \quad (n \to \infty) \) in the weak operator topology of \( L^\infty(\mu_z) \), where \( z \in C_k \) and \( g_k = f|_{C_k} \).

By repeating the arguments in the proof of 2.10 we have that \( f_n(T) \to f(T) \quad (n \to \infty) \) in the weak operator topology.

It remains to prove the multiplicativity of the mapping \( H^\infty(X^0) \to B(H) \). The proof of this is found in [2].

The theorem is complete.

3.2 REMARKS If the set \( X \) in 3.1 is symmetric about the real axis an additional conclusion holds:

(4) \( f(T)^* = \tilde{f}(T^*) \), where \( \tilde{f}(z) = \overline{f(z)} \).

This follows from (3) and the fact that (4) is true for functions in \( R(X) \).

By an earlier observation (following 1.2), 3.1 applies in particular when \( X \) is finitely connected.

In the two-variable case Khahn [12] gave the following generalization of a \( H^\infty \)-functional calculus for completely non-unitary contractions, due to Briem, Davie and Øksendal [2]:

\[
= \sup_k \sup \{ \left| \int g_k \, dp^{(k)} \right| : \|x\|, \|y\| \leq 1 \} \\
\leq \sup_k \|\tilde{g}_k\| \\
\leq \|f\|, 
\]
3.3 THEOREM Let $T_1, T_2 \in B(H)$ be commuting contractions and assume

(a) for $x \in H$, there exists a positive measure $\nu^{(k)}_x$ on $\Gamma$ ($\nu^{(k)}_x \ll m$, the Lebesgue measure on $\Gamma$), such that
\[
\langle T_k^r x, x \rangle = \int z^r \, d\nu^{(k)}_x(z) \quad \text{for} \quad r = 0, 1, 2, \ldots \quad (k = 1, 2);
\]

(b) there exists no non-zero subspace of $H$ that simultaneously reduces $T_1$ and $T_2$ and restricted to which both $T_1$ and $T_2$ are unitary.

Then there exists an algebra homomorphism $H^\infty(D^2) \to B(H)$ ($f \mapsto f(T_1, T_2)$) such that (1), (2), (3) of 3.1 hold for the two-variable case, as well as

(4) $f(T_1^*, T_2^*) = \bar{f}(T_1^*, T_2^*)$ where $\bar{f}(z_1, z_2) = f(\overline{z_1}, \overline{z_2})$;

(5) $f(T_1, T_2) = g(T_1)$ if $f(z_1, z_2) = g(z_1)$ for $g \in H^\infty(D)$.

3.4 REMARK Sz.-Nagy and Foias (see [19], II 6.4) proved that the spectral measure of the minimal unitary dilation of a completely non-unitary contraction is absolutely continuous with respect to Lebesgue measure on $\Gamma$. Hence any two commuting completely non-unitary contractions satisfy the hypotheses of 3.3.

The above theorem extends as follows:

3.5 THEOREM Let $S, T \in B(H)$ be commuting operators with spectral sets $X$ and $Y$ respectively, such that $R(X)$ and $R(Y)$ are Dirichlet algebras. Assume
(a) for $x \in H$, there exists a positive measure $\rho_x$ on $\partial X$ ($\rho_x$ is absolutely continuous with respect to the measure $\sum_{k=1}^{\infty} 2^{-k} \mu_{z_k}$ where $G_1, G_2, \ldots$ are the components of $X^0$ and $\mu_{z_k}$ is the harmonic measure on $\partial X$ for $z_k \in G_k$), such that $\langle u(S)x, x \rangle = \int u \, d\rho_x$ for $u \in \mathcal{R}(X)$. The same condition also applies to $T$ and $Y$.

(b) there exists no non-zero subspace $H'$ of $H$ that simultaneously reduces $S$ and $T$ and such that both $S \mid H'$ and $T \mid H'$ are normal operators with spectra contained in $\partial X$ and $\partial Y$, respectively.

Then there exists an algebra homomorphism $H^\omega(X^0 \times Y^0) \to B(H)$ ($f \mapsto f(S,T)$), such that (1), (2), (3) of 3.1 hold in the two-variable case.

Proof. Assume $X^0$ and $Y^0$ each consists of one component, and let $\phi: X^0 \to D$ and $\psi: Y^0 \to D$ be conformal mappings.

By the arguments in the proof of 2.10 there exists a bounded sequence $(u_n)$ in $\mathcal{R}(X)$ that converges pointwise to $\phi$ on $X^0$, such that $\phi(S) = \lim_n u_n(S)$ in the weak operator topology. Since $H^\omega(X^0) \to B(H)$ is a homomorphism (see 3.1), we have that for $x \in H$ and $r = 0, 1, 2, \ldots$,

$$\langle \phi(S)^r x, x \rangle = \lim_n \langle u_n(S)^r x, x \rangle$$

$$= \lim_n \int u_n^r \, d\rho_x, \quad \text{by (a)}$$

$$= \int g^r \, d\rho_x,$$

where $g = \phi$, since $\rho_x \ll \mu_z$ ($z \in X^0$) and $(u_n)$
converges in the weak-star topology to \( g \) in \( L^\infty(\mu_z) \).

But

\[
\int g^x d\rho_x = \int (g \cdot \varphi)_x^z d(\rho_x \varphi^{-1}) = \int z^x d(\rho_x \varphi^{-1})(z),
\]

where \( \rho_x \varphi^{-1} \ll \mu_z \varphi^{-1} \), the Lebesgue measure on \( \Gamma \), for some \( z \in X^0 \). Hence \( \varphi(S) \), and similarly \( \psi(T) \) satisfy the hypothesis 3.3(a).

Suppose now there exists a non-zero subspace \( H' \) of \( H \) such that \( \varphi(S) | H' \) and \( \psi(T) | H' \) are unitary operators. Let \( (v_n) \) be a bounded sequence in \( R(\Delta) \), such that

\[
S = \lim_n v_n(\varphi(S)) = \varphi^{-1}(\varphi(S)),
\]

the limit taken in the weak operator topology. Then \( H' \) reduces \( S \), for let \( h \in H' \), then for arbitrary \( y \in H' \),

\[
<Sh, y> = \lim_n <v_n(\varphi(S))h, y> = <h', y>
\]

for some \( h' \in H' \), since \( \varphi(S) \) reduces \( H' \). Similarly \( S^* \) reduces \( H' \). For \( x, y \in H' \),

\[
<S^*Sx, y> = <Sx, Sy>
\]

\[
= \lim_n <v_n(\varphi(S))x, y>
\]

\[
= \lim_n \lim_m <v_n(\varphi(S))x, v_m(\varphi(S))y>
\]

\[
= \lim_m \lim_n <v_n(\varphi(S))v_m(\varphi(S))^*x, y>,
\]

since \( \varphi(S) | H' \) is unitary. It follows that

\[
<S^*Sx, y> = <SS^*x, y>.
\]

So \( S \), and similarly \( T \) are normal operators on \( H' \).

Next we show that \( \varphi(S) \) and \( \psi(T) \) satisfy the hypothesis 3.3(b). Regard \( S \) as an operator on \( H' \) and let \( y \in \sigma(S) \cap X^0 \). For \( z \in X \), define \( g \in H^\sigma(x^0) \) by

\[
g(z) = (y - z)^{-1} (\varphi(y) - \varphi(z)).
\]

Then
\[(y - z)g(z) = \Phi(y) - \Phi(z) = \lim_n (\Phi(y) - u_n(z)).\]

Hence \[(yI_H, - S)g(S) = \lim_n (\Phi(y)I_H, - u_n(S)),\]

the limit taken in the weak operator topology for some subsequence \((u_n)_n\). Thus

\[(yI_H, - S)g(S) = \Phi(y)I_H, - \Phi(S).\]

If \(\Phi(y)I_H, - \Phi(S)\) has an inverse \(V\), then \(g(S)V\) is an inverse for \(yI_H, - S\), i.e. \(\Phi(y) \in \sigma(\Phi(S)) \subset \Gamma\), which is a contradiction. Then \(\sigma(S) \subset \partial X\) (and similarly for \(T\)), which contradicts the assumption (b). Therefore the contractions \(\Phi(S)\) and \(\Psi(T)\) satisfy 3.3(b) and now 3.3 may be applied to \(\Phi(S)\) and \(\Psi(T)\). For \(f \in H^\omega(X^0 \times Y^0)\) define \(g \in H^\omega(D^2)\) by

\[g(z_1, z_2) = f(\Phi^{-1}(z_1), \Psi^{-1}(z_2)).\]

Define \(f(S, T) = g(\Phi(S), \Psi(T))\). The multiplicativity of the mapping \(H^\omega(X^0 \times Y^0) \to B(H)\) follows from 3.3:

let \(f, g \in H^\omega(X^0 \times Y^0)\), then \((fg)(S, T) = h(\Phi(S), \Psi(T))\)
where \(h \in H^\omega(D^2),\)

\[h(z_1, z_2) = (fg)(\Phi^{-1}(z_1), \Psi^{-1}(z_2)) = (h_1 h_2)(z_1, z_2)\]

with \(h_1, h_2 \in H^\omega(D^2),\)

\[h_1(z_1, z_2) = f(\Phi^{-1}(z_1), \Psi^{-1}(z_2))\]

and \(h_2(z_1, z_2) = g(\Phi^{-1}(z_1), \Psi^{-1}(z_2)).\)

Apply 3.3, then \((fg)(S, T) = f(S, T)g(S, T).\)

(1) Since \(H^\omega(X^0 \times Y^0) \to B(H)\) is a homomorphism we need only show that if \(p(z_1, z_2) = z_1\), say, then \(p(S, T) = S.\)

By definition, \(p(S, T) = g(\Phi(S), \Psi(T))\) where
\[ g(z_1, z_2) = p(\varphi^{-1}(z_1), \psi^{-1}(z_2)) = \varphi^{-1}(z_1). \]

By 3.3(5) and 2.11,
\[ p(S, T) = \varphi^{-1}(\varphi(S)) = S. \]

(2) Let \( f \in H^\infty(X^0 \times Y^0) \), then if
\[ g(z_1, z_2) = f(\varphi^{-1}(z_1), \psi^{-1}(z_2)) \]
\[ \|f(S, T)\| = \|g(\varphi(S), \psi(T))\| \]
\[ \leq \sup \{|g(z_1, z_2)| : (z_1, z_2) \in D^2\}, \text{ by 3.3} \]
\[ \leq \sup \{|f(w_1, w_2)| : (w_1, w_2) \in X \times Y\}. \]

The property (3) follows from the corresponding result in 3.3.

If the interiors of \( X \) and \( Y \) each have an arbitrary number of components the theorem applies by decomposing \( S \) and \( T \) according to 2.7, then applying the above to the individual components of \( S \) and \( T \), and finally putting everything together again as in the proof of 2.10.

At this stage one might hope for a theorem similar to 3.1 in the two-variable case for commuting operators whose spectral sets have multiply connected interiors. The simplest case (where each spectral set is an annulus) will be examined.

Let \( X = \{z \in \mathbb{C} : a \leq |z| \leq 1\} \) (\( a > 0 \)), and \( f \in H^\infty(X^0) \). Since \( f \) is analytic on \( X^0 \) there exists a unique decomposition \( f = g + h \), where \( g \) is analytic on \( D \), \( h \) is analytic on \( D(a, \infty) = \{z \in \mathbb{C} : |z| > a\} \) and
\( h(\infty) = 0. \) For \( r \) satisfying \( a < r < 1, \)
\[
\sup\{|g(z)| : |z| < 1\} \leq \sup\{|g(z)| : |z| \leq r\} + \sup\{|h(z)| : r \leq |z| < 1\} + \sup\{|f(z)| : r \leq |z| < 1\}.
\]
Since \( f \in H^\infty(X^0), \) the right hand side of the inequality is finite and thus \( g \in H^\infty(D). \) Similarly, \( h \in H^\infty(D(a,\infty)). \)

Then
\[
g(z) = \begin{cases} 
(2\pi i)^{-1} \int_{|\xi| = 1} (\xi - z)^{-1} f(\xi) \, d\xi, & |z| < 1 \\
f(z) + (2\pi i)^{-1} \int_{|\xi| = a} (\xi - z)^{-1} f(\xi) \, d\xi, & |z| > a.
\end{cases}
\]
For \( a < |z| < 1, \)
\[
|g(z)| \leq \|f\|_\infty + a \sup \{|f(\xi)(\xi - z)^{-1}| : |\xi| = a, \ |z| = 1\} = (1 + a(1 - a)^{-1}) \|f\|_\infty = (1 - a)^{-1} \|f\|_\infty.
\]
Hence \( \|g\|_\infty \leq (1 - a)^{-1} \|f\|_\infty. \) Also
\[
\|h\|_\infty = \sup\{|h(z)| : z \in D(a,\infty)\} \leq \|f\|_\infty + \sup\{|g(z)| : a < |z| < 1\} \leq (1 + (1 - a)^{-1}) \|f\|_\infty = (2 - a)(1 - a)^{-1} \|f\|_\infty.
\]

3.6 **Theorem**. Let \( S, T \in B(H) \) be commuting contractions with spectral sets \( X \) and \( Y \) respectively,
\[
X = \{z \in \mathbb{C} : a \leq |z| \leq 1\} \quad (a > 0),
Y = \{z \in \mathbb{C} : b \leq |z| \leq 1\} \quad (b > 0).
\]
Suppose that there exists no non-zero reducing subspace of \( H \) restricted to which \( S \) is a normal operator whose spectrum
is contained in $\mathfrak{D} X$, and suppose $T$ satisfies a similar condition. Then there exists an algebra homomorphism $H^\infty(X^0 \times Y^0) \to B(H)$ ($f \mapsto f(S,T)$) such that (1) and (3) of 3.1 hold for the two-variable case, as well as

(2) $\|f(S,T)\| \leq c(a,b) \|f\|_\infty$, $f \in H^\infty(X^0 \times Y^0)$

where $c(a,b)$ is a constant depending on $a$ and $b$;

(4) $f(S,T)^* = \tilde{f}(S^*,T^*)$, where $\tilde{f}(z_1, z_2) = f(\overline{z_1}, \overline{z_2})$.

Proof. Let $\Phi: D(a,\infty) \to D, (z \mapsto a z^{-1})$,

$\Psi: D(b,\infty) \to D, (z \mapsto b z^{-1})$

be conformal mappings. The operators $S, T, \Phi(S)$ and $\Psi(T)$ are completely non-unitary commuting contractions; $\Phi(S)$ and $\Psi(T)$ are completely non-unitary by an argument in the proof of 3.5. Let $f \in H^\infty(X^0 \times Y^0)$, then $f$ decomposes uniquely $f = g_1 + g_2$, where $g_1 \in H^\infty(D \times Y^0)$,

$g_2 \in H^\infty(D(a,\infty) \times Y^0)$ and $g_2(y, y) = 0$ for all $y \in Y^0$. Moreover,

$$\|g_1\|_\infty \leq (1 - a)^{-1} \|f\|_\infty$$

and

$$\|g_2\|_\infty \leq (2 - a)(1 - a)^{-1} \|f\|_\infty.$$ 

Similarly, $g_1 = f_1 + f_2$, $g_2 = f_3 + f_4$ uniquely, where $f_1 \in H^\infty(D^2)$, $f_2 \in H^\infty(D \times D(b,\infty))$, $f_3 \in H^\infty(D(a,\infty) \times D)$ and $f_4 \in H^\infty(D(a,\infty) \times D(b,\infty))$, and $f_2(z, \infty) = 0$ for $z \in D$, $f_4(z, \infty) = 0$ for $z \in D(a,\infty)$. Hence

$$\|f_1\|_\infty \leq (1 - b)^{-1}\|g_1\|_\infty \leq (1 - a)^{-1}(1 - b)^{-1} \|f\|_\infty$$

and similarly,

$$\|f_2\|_\infty \leq (2 - b)(1 - a)^{-1}(1 - b)^{-1} \|f\|_\infty,$$

$$\|f_3\|_\infty \leq (2 - a)(1 - a)^{-1}(1 - b)^{-1} \|f\|_\infty.$$
By applying the Briem, Davie, Øksendal—functional calculus (see [2]), the operators \( f_k(S,T) \in B(H), \ k = 1, 2, 3, 4 \), are defined by (e.g.)
\[
f_2(S,T) = f_2(S,\psi^{-1}(\psi(T))),
\]
since \( f_2(z_1,\psi^{-1}(z_2)) \) is a function in \( H^\infty(D^2) \). Let
\[
f(S,T) = \sum_{k=1}^{4} f_k(S,T).
\]
There exists a sequence \( (h_n) \) in \( R(X \times Y) \) which converges pointwise to \( f \) on \( X^0 \times Y^0 \) such that \( \|h_n\| \leq \|f\|_\infty \) for all \( n \) (see [1]). For all \( n \),
\[
h_n = \sum_{k=1}^{4} g_k^{(n)}
\]
uniquely, where \( g_1^{(n)} \in R(\Delta^2), \ g_2^{(n)} \in R(\Delta \times \overline{D(b,\infty)}), \ g_3^{(n)} \in R(\overline{D(a,\infty)} \times \Delta), \ g_4^{(n)} \in R(\overline{D(a,\infty)} \times \overline{D(b,\infty)}) \) and for each \( k \), \( g_k^{(n)} \to f_k \) \( (n \to \infty) \) pointwise on the interior of the particular set.

Since \( \sup_n \|g_k^{(n)}\| < \infty \) and \( f_k(S,T) = \lim_{n} g_k^{(n)}(S,T) \) in the weak operator topology \( (k = 1, 2, 3, 4) \), property (1) trivially follows by decomposing a polynomial \( p \) in two variables,
\[
p = p_1 + 0 + 0 + 0,
\]
and applying the functional calculus in [2] to \( p_1 \).

(2) Let \( f \in H^\infty(X^0 \times Y^0) \), then
\[
\|f(S,T)\| = \sup\{|\sum_{k=1}^{4} f_k(S,T)x, y| : \|x\|, \|y\| \leq 1\}
\leq \lim_{n} \sum_{k=1}^{4} \|g_k^{(n)}(S,T)\|
\leq \lim_{n} \sum_{k=1}^{4} \|g_k^{(n)}\|
\[
\begin{align*}
&\leq (1-b)^{-1}(1-a)^{-1} + (2-b)(1-a)^{-1}(1-b)^{-1} \\
&\quad + (2-a)(1-a)^{-1}(1-b)^{-1} \\
&\quad + (2-a)(2-b)(1-a)^{-1}(1-b)^{-1} \lim_{n\to\infty} \|h_n\| \\
&\leq (3-a)(3-b)(1-a)^{-1}(1-b)^{-1} \|f\|_* \\
&= c(a,b) \|f\|_*
\end{align*}
\]

where the constant \(c(a,b) > 9\).

Property (3) follows by decomposing each element of a bounded pointwise converging sequence in \(H^\infty(X^0 \times Y^0)\) into four parts to which the functional calculus of \([2]\) is applied separately.

(4) Letting \(f = \sum_{k=1}^{4} f_k\) and \(f(z_1, z_2) = f(\overline{z_1, z_2})\),
\[f(S,T)^* = \sum_{k} f_k(S,T)^* = \sum_{k} f_k(S^*, T^*) = \tilde{f}(S^*, T^*)\]
by what was proven in \([2]\). As in 3.5, the mapping \(f \mapsto f(S,T)\) is multiplicative. The proof is complete.

3.7 REMARK It is not known whether it is possible to develop an \(H^\infty\)-functional calculus in which the constant \(c(a,b)\) is identically 1, or, at least independent of the nature of the multiply connected spectral sets \(X\) and \(Y\). A solution of the joint spectral set problem in Chapter 2 for such spectral sets \(X\) and \(Y\) would imply that a functional calculus exists.

We conclude this chapter by mentioning a result (i.e. 3.10) which is easy to prove once the methods used in the proofs of 2.10 and 3.5 are known.
Suppose $X$ and $Y$ are compact subsets of the complex plane, with $F_1, F_2, \ldots$ the components of $X^0$ and $G_1, G_2, \ldots$ the components of $Y^0$. Let $\mu_{x_k}$ be the harmonic measure on $\partial X$ for $x_k \in F_k$; let $\mu_{y_k}$ be the harmonic measure on $\partial Y$ for $y_k \in G_k$.

Define

$$\nu = \sum_{k=1}^{\infty} 2^{-k} \mu_{x_k}$$

and

$$\lambda = \sum_{k=1}^{\infty} 2^{-k} \mu_{y_k}.$$ 

Then $\nu$ and $\lambda$ are positive measures on $\partial X$ and $\partial Y$ respectively. Let $\mu$ denote the product measure of $\nu$ and $\lambda$ on $\partial X \times \partial Y$. Then $H^\infty(X^0 \times Y^0)$ can be identified with $H^\infty(\mu)$, the weak-star closure of $R(X \times Y)$ in $L^\infty(\mu)$ (see [7], VIII 11), where $R(X)$ and $R(Y)$ are Dirichlet.

3.8 DEFINITION A function $f \in H^\infty(X^0 \times Y^0)$ is outer if $\{fh : h \in R(X \times Y)\}$ is dense in $H^\infty(X^0 \times Y^0)$ in the weak-star topology of $L^\infty(\mu)$.

Briegel, Davie and Øksendal [2] proved

3.9 THEOREM (1) Let $f \in H^\infty(D^2)$ be outer and suppose $S, T \in B(H)$ are commuting completely non-unitary contractions. Then $f(S, T)$ is $(1 - 1)$ and has dense range.

(2) Suppose $f \in H^\infty(D^2)$ is not outer, then there exist commuting contractions $S$ and $T$ that are completely non-
unitary with \( f(S,T) = 0 \).

3.10 THEOREM (1) Let \( S, T \in \mathcal{B}(H) \) be commuting operators with spectral sets \( X \) and \( Y \) such that \( R(X) \) and \( R(Y) \) are Dirichlet algebras.

(*) Suppose \( S \) has no non-zero reducing subspace restricted to which it is a normal operator whose spectrum is contained in \( \partial X \); and suppose \( T \) satisfies a similar condition.
Then if \( f \in H^\infty(X^0 \times Y^0) \) is outer, \( f(S,T) \) is \((1-1)\) and has dense range.

(2) Suppose \( X \) and \( Y \) are compact subsets of the complex plane such that \( R(X) \) and \( R(Y) \) are Dirichlet algebras.
If \( f \in H^\infty(X^0 \times Y^0) \) is not outer, there exist commuting operators \( S', T' \in \mathcal{B}(H) \) with spectral sets \( X \) and \( Y \) satisfying (*) such that \( f(S',T') = 0 \).

Sketch of the proof. Without loss of generality we may restrict ourselves to the case where \( X^0 \) and \( Y^0 \) consist of one component each, as in the proofs of 2.10 and 3.5.
Let \( \Phi : X^0 \to D \) and \( \Psi : Y^0 \to D \) be conformal mappings.
Let \( f \in H^\infty(X^0 \times Y^0) \) and \( g(z_1, z_2) = f(\Phi^{-1}(z_1), \Psi^{-1}(z_2)) \).
Then \( g \in H^\infty(D^2) \). Furthermore, \( f \) is outer if and only if \( g \) is outer. If \( g \) is not outer, 3.9(2) implies that there exist commuting completely non-unitary operators \( U, V \) with \( g(U,V) = 0 \). Define commuting operators \( S' = \Phi^{-1}(U), \ T' = \Psi^{-1}(V) \) as in the proof of 2.10. From
methods used in the proof of 3.5 it follows that $S, T$ satisfy (*). Now apply 3.9(1) to $\Psi(S), \Psi(T)$ and $g \in H^\infty(D^2)$. 
A Banach algebra is called an operator algebra if it can be identified topologically with a closed subalgebra of $B(H)$, for some Hilbert space $H$. A commutative Banach algebra $A$ is said to be a $Q$-algebra if there exist a uniform algebra $B$ and a closed ideal $I$ of $B$ such that $A$ is isomorphic to $B/I$. By a result due to Cole (see [5]), such a quotient algebra $B/I$ is isometrically isomorphic to a subalgebra of $B(H)$, for some Hilbert space $H$.

In particular, the semisimple $Q$-algebras are characterized as those Banach algebras that are isomorphic to restriction algebras of uniform algebras to closed subsets of the maximal ideal space. A general characterization of $Q$-algebras in terms of polynomial inequalities is due to Craw (see [5]):

4.1 LEMMA A is a $Q$-algebra if and only if there exist positive numbers $M$ and $d$ such that whenever $a_1, a_2, \ldots, a_n \in A$ with $\|a_k\| \leq d$ for all $k$, and $P$ is a complex polynomial in $n$ variables without constant term satisfying $|P(z_1, z_2, \ldots, z_n)| \leq 1$ whenever $|z_k| \leq 1$ for all $k$, then $\|P(a_1, a_2, \ldots, a_n)\| \leq M$. 
Examples of Q-algebras are $L^p$ for $1 \leq p \leq \infty$ under coordinate-wise multiplication and $C^n[0,1]$ for every positive integer $n$. For all this and further information on Q-algebras, see [5] and [21].

Recently Varopoulos constructed a semisimple operator algebra that fails to be a Q-algebra. In this chapter we prove the existence of a semisimple singly generated operator algebra which is not a Q-algebra. Our methods are different from those of Varopoulos.

The following theorem is due to Varopoulos [23]:

4.2 THEOREM For every $M > 0$ there exist finitely many commuting contractions $U_1, U_2, \ldots, U_r \in B(H_M)$ on some complex Hilbert space $H_M$, satisfying $U_{j_1} U_{j_2} U_{j_3} U_{j_4} = 0$ for $j_1, j_2, j_3, j_4 = 1, 2, \ldots, r$, and there exists a complex homogeneous polynomial $P_M$ of degree 3 in $r = r(M)$ variables such that

$$\|P_M(U_1, U_2, \ldots, U_r)\| > M \|P_M\|_\omega.$$

From this theorem and Craw's criterion (4.1) it follows that there exist commutative operator algebras that are not Q-algebras. Another consequence of 4.2 is the fact that in general the von Neumann inequality fails for some commuting contractions ($n > 2$). From 4.5 it will
follow that in general the von Neumann inequality fails even if the \( n \) commuting contractions are contained in some singly generated operator subalgebra.

In an addendum to [23], the smallest value \( n_0 \) was determined for which there exist commuting contractions \( U_1, U_2, \ldots, U_{n_0} \) and a polynomial \( p \) in \( n_0 \) variables such that

\[
\|p(U_1, U_2, \ldots, U_{n_0})\| > \|p\|_\infty.
\]

It was shown that \( n_0 = 3 \); in particular,

4.3 THEOREM Let \( H \) be a five-dimensional Hilbert space with orthonormal basis \( \{e, f_1, f_2, f_3, g\} \). Let \( U_1, U_2, U_3 \in \mathcal{B}(H) \) be commuting contractions defined by

\[
U_j e = f_j, \quad U_j f_j = 3^{-1/2} g, \quad U_j f_k = -3^{-1/2} g \quad (j \neq k), \quad U_j g = 0,
\]

\( j, k = 1, 2, 3 \). Let \( p \) be the polynomial

\[
p(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2 - 2(z_1 z_2 + z_2 z_3 + z_1 z_3).
\]

Then

\[
\|p(U_1, U_2, U_3)\| > 5 = \|p\|_\infty.
\]

A similar result, involving an eight-dimensional Hilbert space was proven independently by Crabb and Davie [3].

4.4 REMARKS (1) Fix an arbitrary \( \epsilon > 0 \). If \( M > 0 \), then by 4.2 there exist commuting contractions \( U_1', U_2', \ldots, U_r' \) and a complex homogeneous polynomial \( P_M \) in \( r = r(M(1+\epsilon)^2) \) variables, such that
IIP(u_i, u_j, ..., u_r) > N(1 + \epsilon) \frac{1}{2} \|P_M\|_\infty.

Let \( U_m = (1 + \epsilon)^{-1/2} u_m \), \( m = 1, 2, ..., r \). By the homogeneity of \( P_M \),

\[ \|P_M(u_1, u_2, ..., u_r)\| > (1 + \epsilon)^{1/2} \|P_M\|_\infty, \]

where \( u_1, u_2, ..., u_r \) are commuting contractions of norm not greater than \((1 + \epsilon)^{-1/2}\).

(2) Suppose we have a finite number of simple functions on the \( r \)-dimensional torus \( \mathbb{T}^r \). Partition the torus into \( K \) components such that these simple functions are now defined with respect to this (finer) partitioning. Let \( s \) be a simple function with respect to this partitioning, such that the range of \( s \) consists of \( K \) distinct values and does not contain 0. Then the algebra generated by the given simple functions coincides with the algebra generated by \( s \).

The following theorem generalizes 4.2.

4.5 THEOREM For every \( M > 0 \) there exist finitely many contractions \( T_1, T_2, ..., T_r \) in a closed subalgebra of \( B(K_M) \) that is generated by a single element of \( B(K_M) \), for some complex Hilbert space \( K_M \), and there exists a complex homogeneous polynomial \( P_M \) of degree 3 in \( r = r(M) \) variables, such that

\[ \|P_M(T_1, T_2, ..., T_r)\| > M \|P_M\|_\infty. \]

Proof. Let \( M > 0 \), then by 4.4(1), 4.2 may be rephrased
Equivalence follows from Parseval's theorem for functions in $L^2(\mathbb{R}^n)$. 
as follows:

(*) for fixed $\varepsilon > 0$, there exist commuting operators $U_1, U_2, \ldots, U_r \in B(H_M)$ on some complex Hilbert space $H_M$, such that $\|U_m\| \leq (1 + \varepsilon)^{-1/2}$ for $m = 1, 2, \ldots, r$ and $U_{j_1} U_{j_2} U_{j_3} U_{j_4} = 0$ for $j_1, j_2, j_3, j_4 = 1, 2, \ldots, r$ and there exists a complex homogeneous polynomial $P_M$ of degree $3$ in $r = r(M)$ variables such that for some $h \in H_M (\|h\| \leq 1)$,

$$\|P_M(U_1, U_2, \ldots, U_r)h\| > M(1 + \varepsilon)^{1/2} \|P_M\|_\infty$$

Let $K_M = L^2(r^r)$. For $f \in K_M$, let $f_+$ denote the projection of $f$ on the span of $\{z_1^j z_2^{j_2} \cdots z_r^{j_r} : j_1, \ldots, j_r \geq 0\}$. Let $\|\cdot\|_2$ denote the $L^2$-norm on $K_M$. Define an equivalent norm to the $L^2$-norm on $K_M$ by

$$\|f\|^2 = \|f_+(U_1, U_2, \ldots, U_r)h\|^2 + \lambda\|f - f_+\|_2^2 + \varepsilon\|f\|_2^2,$$

where $\lambda$ is a sufficiently large positive number.

For each $m = 1, 2, \ldots, r$, let the operator $V_m \in B(K_M)$ be multiplication of elements of $K_M$ by the $m$-th coordinate. Then

$$\|V_1 f\|^2 = \|(V_1 f_+ + (V_1 f)_+ - V_1 f_+)(U_1, U_2, \ldots, U_r)h\|^2 + \lambda\|V_1 f - (V_1 f)_+\|_2^2 + \varepsilon\|f\|_2^2 \leq (1 + \varepsilon) \|(V_1 f_+)(U_1, U_2, \ldots, U_r)h\|^2$$

$$+ (1 + \varepsilon^{-1})\|(V_1 f_+) - V_1 f_+\|(U_1, U_2, \ldots, U_r)h\|^2 + \lambda\|V_1 f - (V_1 f)_+\|_2^2 + \varepsilon\|f\|_2^2.$$
\[ \leq (1 + \epsilon) \|f_+(u_1, u_2, \ldots, u_r) \| h^2 + c(1 + \epsilon^{-1}) \| (v_1 f)_+ - v_1 f_+ \|_2^2 + \lambda \|v_1 f - (v_1 f)_+\|_2^2 + \epsilon \|f\|_2^2, \]

where the positive constant \( c \) depends only on \( r \).

by choosing \( \lambda \geq c(1 + \epsilon^{-1}) \) and since

\[ (v_1 f)_+ - v_1 f_+ \perp v_1 f - (v_1 f)_+. \]

Hence \( V_1 \) and similarly also \( V_2, V_3, \ldots, V_r \) are contractions.

Let \( P \) be the polynomial obtained in (*).

If \( f \in K_M \) is the constant function with value \((\epsilon + \|h\|^2)^{-1/2}\) on the torus, then \( \|f\| = 1 \) and

\[ \|P(M(V_1, V_2, \ldots, V_r)) \| \geq \|P(M(V_1, V_2, \ldots, V_r) f) \| \]

\[ > (1 + \epsilon)^{-1/2} \|P(M(u_1, u_2, \ldots, u_r)) h\| \]

\[ > M \|P(M)\|_\infty. \]

Since the norm of \( K_M \) is equivalent to the \( L^2 \)-norm, each of the contractions \( V_m \) can be approximated by a multiplication operator \( T_m \in B(K_M) \) whose multiplier is a simple function on \( \Gamma^T \) that approximates the multiplier of \( V_m \) uniformly.

Hence \( T_1, T_2, \ldots, T_r \) are commuting contractions such that

\[ \|P(M(T_1, T_2, \ldots, T_r)) \| > M \|P(M)\|_\infty. \]

By 4.4(2), \( T_1, T_2, \ldots, T_r \) are contained in a closed subalgebra of \( B(K_M) \) that is generated by a multiplication operator \( S_M \in B(K_M) \) whose multiplier is a simple function on \( \Gamma^T \). The proof is complete.
We are now in a position to construct a semisimple singly generated operator algebra that is not a Q-algebra.

More precisely, we mean the following: assume $D_1$ is chosen such that the ranges of the multipliers of $S_M$ and $D_1^{-1}S_{M'} + 1$ are disjoint. Construct a polynomial having zeros the range of the multiplier of $S_M$, and think of this, acting on $D_1^{-1}S_{M'} + 1$ as the generator of the algebra containing $T_{1}^{(M' + 1)}$, $\ldots$, $T_{r(M' + 1)}$. Then construct polynomials in this new generator and take composition of the polynomials as the required $g_{m}^{(M' + 1)}$. 
For the purposes of the construction it is convenient to make certain modifications in the definitions which were introduced in the proof of 4.5. Suppose that the multipliers of the operators \( S_M \in B(K_M) \) \((M = 1, 2, \ldots)\) have disjoint ranges. Fix a positive integer \( M' \), then by 4.5 there exist a polynomial \( P_{M'} \) and contractions \( T_{(M')} = g_{(M')}^{(M')} (S_{M'}) \in B(K_{M'}) \) 
\((m = 1, 2, \ldots, r(M'))\) for some polynomials \( g_{1}^{(M')}, \ldots, g_{r(M')}^{(M')} \) without constant term, such that
\[
\|P_{M'} (T_{1}^{(M')}, T_{2}^{(M')}, \ldots, T_{r(M')}^{(M')})\| > M' \|P_{M'}\|_{\infty}.
\]
Let \( D_1 \) be a constant \((D_1 > 1)\) such that for each \( m \),
\[
\|g_{m}^{(M')} (D_1^{-1} S_{M'} + 1)\| \leq 1.
\]
For the case \( M = M' + 1 \), we obtain from 4.5 the polynomials \( g_{1}^{(M' + 1)}, g_{2}^{(M' + 1)}, \ldots, g_{r(M' + 1)}^{(M' + 1)} \) such that for each \( m \),
\[
\|g_{m}^{(M' + 1)} (D_1^{-1} S_{M'} + 1)\| \leq 1.
\]
We then modify these polynomials so that they vanish on the range of the multiplier of \( S_{M'} \). Hence:
\[
g_{m}^{(M' + 1)} (S_{M'}) = 0 \quad \text{for each } m.
\]
Next let \( D_2 \) be a constant \((D_2 > D_1)\) such that
\[
\|g_{m}^{(M') \left( D_2^{-1} S_{M'} + 2 \right)}\| \leq 1, \quad m = 1, 2, \ldots, r(M'),
\]
and
\[
\|g_{m}^{(M' + 1)} \left( D_2^{-1} S_{M'} + 2 \right)\| \leq 1, \quad m = 1, 2, \ldots, r(M' + 1).
\]
As before there exist polynomials \( g_{1}^{(M' + 2)}, \ldots, g_{r(M' + 2)}^{(M' + 2)} \) such that for each \( m \),
choose the constants $D_1, D_2, \ldots$ such that

$$\sup_j \| D_j^{-1} S_{M+j} \| < \infty,$$
By modifying these polynomials in the manner described above it follows that
\[ \|g_{m}(M'+2)(D^{-1}_1 S_{M'} + 2)\| \leq 1 \]
and
\[ \|g_{m}(M'+2)(D^{-1}_1 S_{M'} + 1)\| \leq 1, \quad m = 1, 2, \ldots, r(M'+2). \]

Continue this process, and let \( A \) be the operator algebra generated by 
\[ S = S_{M'} \oplus D^{-1}_1 S_{M'} + 1 \oplus D^{-1}_2 S_{M'} + 2 \oplus \ldots. \]
Then 4.1 implies that \( A \) is not a Q-algebra.

It remains to show that \( A \) is semisimple. Let \( A_{M} \) be the operator subalgebra of \( B(K_{M}) \) that is generated by \( S_{M} \).
Since \( A \) is an operator subalgebra of \( \bigoplus_{M=M'}^{\infty} A_{M} \), it is sufficient to prove that each \( A_{M} \) is semisimple. For if each \( A_{M} \) is semisimple and \( x \) is a non-zero element of 
\[ \bigoplus_{M=M'}^{\infty} A_{M}, \quad (x = \bigoplus_{M=M'}^{\infty} x_{M}), \text{ then} \]
\[ \lim_{n \to \infty} \|x^n\|^{1/n} = \lim_{n \to \infty} (\sup_{M} \|x_{M}^n\|^{1/n}) > 0. \]
Thus \( \bigoplus_{M=M'}^{\infty} A_{M} \) is semisimple, and similarly \( A \) is semisimple.

Let the range of the multiplier of \( S_{M} \) be positive, then \( S_{M} \) is a positive operator on \( L^2(\Gamma^{r}) \). Let \( A_{M}' \) be the operator subalgebra of \( B(L^2(\Gamma^{r})) \) generated by \( S_{M} \), then \( A_{M}' \) is semisimple. Since by 4.5(*) we may choose \( \epsilon \geq 1 \) in the definition of the norm of \( K_{M} \), and therefore have that
\[ \|f\| \geq \epsilon^{1/2} \|f\|_{2} \geq \|f\|_{2}, \]

it follows that for any polynomial \( p \) for which \( p(S_{M}) \neq 0 \),
\[ \lim_{n \to \infty} \|p(S_M)^n\|^{1/n} \geq \lim_{n \to \infty} \|p(S_M)^n\|_2^{1/n} > 0, \]

by the semisimplicity of \( A_M \). Hence \( A_M \) is semisimple, and the construction is complete.

4.6 THEOREM  There exist commuting contractions \( T_1, T_2, T_3 \) in a closed subalgebra of \( B(K) \) that is generated by a single element of \( B(K) \), for some complex Hilbert space \( K \), and there exists a polynomial \( p \) in three variables, such that

\[ \|p(T_1, T_2, T_3)\| > \|p\|_\infty. \]

The proof of 4.6 follows from 4.3 in the same way as the proof of 4.5 follows from 4.2.

From 4.6 it can also be seen that the remark preceding 4.3 concerning the smallest number of commuting contractions for which the von Neumann inequality does not hold, can be extended to commuting contractions which are contained in a singly generated operator subalgebra.
REFERENCES


8. T.W. Gamelin and J. Garnett, "Pointwise bounded


