CATEGORIES

AND

DISTRIBUTIVELY GENERATED NEAR-RINGS

BY

SURAIYA JABEEN MAHMOOD

Presented for the degree of
Doctor of Philosophy in Mathematics
at the
University of Edinburgh
May 1979
Dedicated

to

the memory of my late father

Mohammad Bashir Qureshi
The following record of research work is submitted as a thesis for the degree of Doctor of Philosophy, in the University of Edinburgh, having been submitted for no other degree. Except where acknowledgement is made, the work is original.

Mrs. Suraiya Jabeen Mahmood
Acknowledgement

I am deeply grateful to my supervisor Dr. J.D.P. Keldrum, for suggesting the problem and his invaluable help, encouragement and generous guidance throughout the course of this work. My sincere gratitude is due to Dr. L. O'Carroll, whose constant help and valuable suggestions made it possible to work on and improve a large part of this thesis. I am thankful to all the members of the teaching and secretarial staff of the department of Mathematics, for their cooperation and friendliness. Thanks are due to the Government of Pakistan, in helping me with finance.

I must emphasise that my husband Mahmood and daughters Saima, Saadia and Uzma, who suffered a lot due to my negligence during the last three years, deserve my heartiest gratitude. Without the cooperation and encouragement of my husband, it would have been impossible for me to complete this work.

Thanks are due to my parents, by whose efforts I have reached this stage, and to my mother in particular for creating my interest in this subject.
Abstract

The notions of upper and lower faithful d.g. near-rings, first given by J.D.P. Meldrum [11], are generalized and used to prove that the category $\mathcal{B}$ of all faithful d.g. near-rings is a reflective as well as a coreflective subcategory of the category $\mathcal{A}$ of all d.g. near-rings. $\mathcal{A}$ and $\mathcal{B}$ are also shown to be complete and cocomplete.

Let $\mathcal{C}$ be an algebraic category satisfying certain conditions and let $\mathcal{D}$ be a reflective subcategory of $\mathcal{C}$ such that each reflection homomorphism is onto. In chapter 3, it is proved that if this is the case, then the existence of limits in $\mathcal{D}$ implies the existence of limits in $\mathcal{C}$. An abstract version of this result is also given in this chapter.

If $S$ is an inverse semigroup of endomorphisms of an additive group $G$, then $S$ is shown to be an inverse semigroup of partial isomorphisms of $G$ such that each $s \in S$ gives a splitting of $G$ and these splittings are shown to satisfy certain conditions. This result and its converse are given in chapter 5. Moreover, in this chapter, it is proved that for an inverse semigroup $S$, each d.g. near-ring $(R,S)$ is faithful, having a faithful representation on its additive group $(R,+)$.

Throughout this work, a few categories are involved. In chapter 4, the existence of some natural functors, among these categories, is shown.

In the last chapter, the concept of a group d.g. near-ring, given by J.D.P. Meldrum [12], for a faithful d.g. near-ring, is generalized.
Introduction

A d.g. near-ring is not necessarily faithful. But J.D.P. Meldrum, in [11], proved that with each d.g. near-ring we can associate two faithful d.g. near-rings, namely the upper and lower faithful d.g. near-rings. In chapter 2, we generalize the notions of upper and lower faithful d.g. near-rings and make a correction in the latter case. Then using these ideas, we show that the category \( \mathcal{B} \), of all faithful d.g. near-rings, is a reflective as well as a coreflective subcategory of the category \( \mathcal{A} \) of all d.g. near-rings. Moreover, we prove that the categories \( \mathcal{A} \) and \( \mathcal{B} \) are complete and cocomplete.

It may be interesting to mention that initially we proved the existence of products and general limits first in \( \mathcal{B} \) and then, with the help of this result, we demonstrated the existence of products and general limits in \( \mathcal{A} \), which is unusual. When we looked into the structure of the product object in \( \mathcal{A} \), we noticed that the existence of products in \( \mathcal{A} \) can be proved directly. But the original proof provided a general result of algebraic categories which satisfy certain conditions. We give this in chapter 3, where we also prove an abstract version of this result.

The generating semigroup plays an important role in the structure and behaviour of a d.g. near-ring; for example,

1) if \( S \) has a left identity \( e \), then \( e \) is a left identity of \((R,S)\) and so \((R,S)\) is faithful having a faithful representation
2) if the generating semigroup $S$ is a group, then $(R, S)$ is faithful.

Therefore it was natural to think about the behaviour of d.g. near-rings generated by other types of semigroups, for example, inverse semigroups. J.D.P. Meldrum thought about this and conjectured that "every d.g. near-ring $(R, S)$, over an inverse semigroup $S$, is faithful, having a faithful representation on $(R, +)$". We prove this conjecture in chapter 5. In that chapter, we also study the representations of an inverse semigroup and prove that, if an inverse semigroup $S$ is contained in the semigroup of endomorphisms of a group $G$, then $S$ is an inverse semigroup of partial isomorphisms of $G$ such that each $s \in S$ gives a splitting of $G$, satisfying certain conditions. We prove the converse of this result as well.

In considering the behaviour of d.g. near-rings, we relate a few categories, namely the categories of all d.g. near-rings, of all faithful d.g. near-rings, of all semigroups, of all groups, of all sets and of $(R, S)$-groups for a d.g. near-ring $(R, S)$. In chapter 4, we give a few natural functors among these categories, and their properties.

J.D.P. Meldrum, in [12], constructed a group d.g. near-ring $(R(G), SG)$ for a faithful d.g. near-ring $(R, S)$ over a multiplicative group $G$. In chapter 6, we generalize this idea and construct a
group d.g. near-ring for any d.g. near-ring \((R, S)\) on a multiplicative group \(G\), as a homomorphic image of \((\overline{R}(G), SG)\), where \((\overline{R}, S)\) is the upper faithful d.g. near-ring for \((R, S)\).

For the contents of §1.1, we refer to [13], except for where it is mentioned otherwise and the definition of an adjunction, for which the reference is made to [10]. Since we often work in the categories consisting of algebraic systems, we give some definitions and results in §1.2, from [3], which deals with algebraic categories only.

Upto now there is only one book published on near-rings [15], in which almost all the material available on this subject is gathered systematically. So we refer to this book in general for the contents of §§1.4, 1.5, and 1.6. But occasionally we mention the alternative sources, where the topics we need are not covered fully in this book. Note that a few of our terms are different from that of [15].

In [10], [13] and [15] the maps are written on the left. But, to be consistent with our work to follow, we have shifted them to the right.

For the contents of §5.1, we refer to [9], [1] and [2], and for that of §6.1 we refer to [12]. For the results of group theory, which we have used, reference is made to [16] and [14].
## Contents

Acknowledgement iv
Abstract v
Introduction vi

### Chapter 1. Preliminaries

- 1.1. Categories and functors 1
- 1.2. Algebraic categories 13
- 1.3. The category $G$ of all groups 15
- 1.4. Near rings 16
- 1.5. D.G. Near-rings 19
- 1.6. Representations 23

### Chapter 2. Categories and d.g. near-rings

- 2.1. The upper and lower faithful d.g. near-rings 27
- 2.2. Colimits in $A$ 34
- 2.3. Limits in $A$ 41

### Chapter 3. Surjective reflections

- Introduction 47
- 3.1. Surjective reflections 48

### Chapter 4. Some functors

- 4.1. Functor: $A \times S \rightarrow G$ 58
- 4.2. Functors: $A \leftrightarrow S$ 69

### Chapter 5. Inverse semigroups of endomorphisms

- 5.1. Preliminaries 73
Chapter 1

Preliminaries

1.1. Categories and Functors

Definition 1.1.1. A category \( \mathcal{A} \) is a class of objects together
with a class \( \bigcup (A,B) \in \mathcal{A} \) \( \text{Hom}(A,B) \) which is a disjoint union of sets
\( \text{Hom}_{\mathcal{A}}(A,B) \), possibly empty, of arrows, called morphisms with domain
\( A \) and codomain \( B \), for all pairs \( A, B \) of objects of \( \mathcal{A} \).
Furthermore for objects \( A, B, C \) of \( \mathcal{A} \) and \( \alpha \in \text{Hom}_{\mathcal{A}}(A,B) \),
\( \beta \in \text{Hom}_{\mathcal{A}}(B,C) \) a composition is defined as \( \alpha \beta \in \text{Hom}_{\mathcal{A}}(A,C) \) which
satisfies:

1) whenever the compositions make sense we have
\[ (\alpha \beta) \gamma = \alpha (\beta \gamma) \]

2) for each object \( A \in \mathcal{A} \), there exists \( I_A \in \text{Hom}_{\mathcal{A}}(A,A) \) such
that \( \alpha I_A = \alpha \), \( I_A \beta = \beta \) whenever the compositions make sense.

A category \( \mathcal{A} \) is called a small category if the class of
objects of \( \mathcal{A} \) is a set. In this case the class of morphisms
\( \bigcup (A,B) \in \mathcal{A} \) \( \text{Hom}_{\mathcal{A}}(A,B) \), being a union of sets indexed over a set, is
also a set.

We normally write \( A \in \mathcal{A} \) to mean that \( A \) is an object in \( \mathcal{A} \)
and \( \alpha : A \rightarrow B \) to mean that \( \alpha \in \text{Hom}_{\mathcal{A}}(A,B) \).

Examples 1.1.2.
The collection \( \mathcal{G} \) of all groups with group homomorphisms.

2) The collection \( \mathcal{A} \) of all abelian groups with group homomorphisms.

3) The collection of all sets and set mappings.

4) The collection of all near-rings and near-ring homomorphisms.

5) The collection of all d.g. near-rings and d.g. near-ring homomorphisms.

Definition 1.1.3. A category \( \mathcal{B} \) is called a subcategory of a category \( \mathcal{A} \) if

1) \( \mathcal{B} \subseteq \mathcal{A} \), i.e., the objects of \( \mathcal{B} \) are also objects of \( \mathcal{A} \).

2) \( \text{Hom}_\mathcal{B}(A,B) \subseteq \text{Hom}_\mathcal{A}(A,B) \), for all objects \( A, B \) in \( \mathcal{B} \).

3) The composition of two morphisms in \( \mathcal{B} \) is the same as their composition in \( \mathcal{A} \).

4) \( I_A \in \text{Hom}_\mathcal{B}(A,A) \) is the same in \( \mathcal{B} \) as in \( \mathcal{A} \) for all \( A \in \mathcal{B} \).

If furthermore \( \text{Hom}_\mathcal{B}(A,B) = \text{Hom}_\mathcal{A}(A,B) \) for all \( A, B \in \mathcal{B} \),
then \( \mathcal{B} \) is called a full subcategory of \( \mathcal{A} \).

Example. 1.1.2.(2) is a full subcategory of 1.1.2.(1) whereas 1.1.2.(3) is a subcategory of 1.1.2.(4) which is not full.

Definition 1.1.4. The dual category \( \mathcal{A}^* \) of a category \( \mathcal{A} \) has the same class of objects as \( \mathcal{A} \) and is such that

\[ \text{Hom}_\mathcal{A}(A,B) = \text{Hom}_{\mathcal{A}^*}(B,A) \]

for all \( A, B \) in \( \mathcal{A} \). The composition \( \beta \alpha \) in \( \mathcal{A}^* \) is defined as the composition \( \alpha \beta \) in \( \mathcal{A} \). Clearly \( (\mathcal{A}^*)^* = \mathcal{A} \) and conse-
quently every result about categories actually embodies two results.
If a statement \( p \) is true for a category \( \mathcal{A} \), then there is a dual statement \( p^* \) which will be true for \( \mathcal{A}^* \). If the assumptions on \( \mathcal{A} \) used to prove \( p \) hold also in \( \mathcal{A}^* \), then \( p^* \) is true for \( (\mathcal{A}^*)^* = \mathcal{A} \).

Definition 1.1.5. A morphism \( \alpha \in \text{Hom}(A,B) \) is called a monomorphism if \( f \alpha = g \alpha \) implies that \( f = g \) for all morphisms \( f, g \) with codomain \( A \). \( \alpha \) is called an epimorphism if \( \alpha f = \alpha g \) implies that \( f = g \) for all morphisms \( f, g \) with domain \( B \). In other words, \( \alpha \) is a monomorphism (epimorphism) if \( \alpha \) is right(left) cancellable.

The notion of an epimorphism is dual to that of a monomorphism.

Theorem 1.1.6. a) If \( \alpha, \beta \) are epimorphisms (monomorphisms) and \( \alpha \beta \) is defined then \( \alpha \beta \) is an epimorphism (a monomorphism).

b) If \( \alpha \beta \) is an epimorphism (a monomorphism) then \( \beta(\alpha) \) is an epimorphism (a monomorphism).

Definition 1.1.7. A morphism \( \theta : A \to B \) is said to be an isomorphism if there exists a morphism \( \theta' : B \to A \) such that \( \theta \theta' = I_A \) and \( \theta' \theta = I_B \). A morphism (isomorphism) \( \alpha : A \to A \) is called an endomorphism (automorphism) of \( A \).

Definition 1.1.8. In a category, if \( \alpha : A' \to A \) is a monomorphism, \( A' \) is called a subobject of \( A \) and we refer to \( \alpha \) as the inclusion of \( A' \) in \( A \). Sometimes we write \( \alpha : A' \subseteq A \). If \( \alpha \) is
not an isomorphism we call $A'$, a proper subobject of $A$. The composition of a monomorphism $a : A' \rightarrow A$ with a morphism $f : A \rightarrow B$ is often denoted by $f / A'$ and is called the restriction of $f$ to $A'$.

Definition 1.1.9. Given two morphisms $\alpha, \beta : A \rightarrow B$ in a category $\mathcal{A}$, a morphism $\gamma : K \rightarrow A$ is called an equalizer of $\alpha$ and $\beta$ if $\gamma \alpha = \gamma \beta$ and if for every morphism $\gamma' : K' \rightarrow A$ with $\gamma' \alpha = \gamma' \beta$ there exists a unique morphism $\mu : K' \rightarrow K$ such that $\mu \gamma = \gamma'$. We write $K = \text{Equ}(\alpha, \beta)$.

Usually $K$ is called the equalizer and $\gamma$ is viewed as an inclusion map because of the following result:

Theorem 1.1.10. If $\gamma$ is an equalizer of $\alpha, \beta$ then $\gamma$ is a monomorphism. Any two equalizers of $\alpha, \beta$ are isomorphic subobjects of $A$.

Definition 1.1.11. Let $\{ u_\lambda : A_\lambda \rightarrow A \}_{\lambda \in \Lambda}$ be a family of subobjects of $A$. We call a morphism $u : A' \rightarrow A$ the intersection of the family if for each $\lambda \in \Lambda$ there exists a unique morphism $v_\lambda : A' \rightarrow A_\lambda$ such that $u = v_\lambda u_\lambda$ and furthermore if every morphism $v : B \rightarrow A$ factoring through each $u_\lambda$, factors uniquely through $u$.

Consider a diagram

\[
\begin{array}{ccc}
A' & \rightarrow & B' \\
\downarrow & & \downarrow \\
A & \rightarrow & B
\end{array}
\]
where \( f \) is any morphism and the vertical morphisms are monomorphisms.

Then we say that the subobject \( A' \) is carried into the subobject \( B' \) by \( f \) if there exists a morphism \( g : A' \longrightarrow B' \) (necessarily unique) making the above diagram commutative.

**Definition 1.1.12.** The union of a family \( \{ A_\lambda : \lambda \in \Lambda \} \) of subobjects of an object \( A \) is defined as a subobject \( A' \) of \( A \) which is preceded by each of the \( A_\lambda \), and which has the following property: If \( f : A \longrightarrow B \) is a morphism and each \( A_\lambda \) is carried into some subobject \( B' \) by \( f \), then \( A' \) is also carried into \( B' \) by \( f \). The object \( A' \) is denoted by \( \bigvee \{ A_\lambda : \lambda \in \Lambda \} \).

**Definition 1.1.13.** Let \( f : A \longrightarrow B \) be in \( \mathcal{C} \). Then the image of \( f \) is defined as the smallest subobject of \( B \) which \( f \) factors through; that is a monomorphism \( u : I \longrightarrow B \) is the image of \( f \) if \( f = f' \cdot u \) for some \( f' : A \longrightarrow I \) and if \( u \) precedes any other monomorphism into \( B \) with the same property. The object \( I \) is usually denoted by \( \text{Im} \, f \).

**Definition 1.1.14.** An object \( 0 \in \mathcal{C} \) is called a zero object if \( \text{Hom}(0,A) \) and \( \text{Hom}(A,0) \) have precisely one morphism each, for each \( A \in \mathcal{C} \). A morphism \( f : A \longrightarrow B \) in \( \mathcal{C} \) is a zero morphism if it factors through a zero object.

For the following two definitions we assume that the category has a zero object.
Definition 1.1.15. [10] The cokernel of \( f : A \rightarrow B \) is a morphism \( g : B \rightarrow D \) such that

1) \( f g = 0 : A \rightarrow D \),

2) if there is a morphism \( h : B \rightarrow C \) such that \( f h = 0 \), then \( h = g h' \) for a unique morphism \( h' : D \rightarrow C \).

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow f & & \downarrow g \\
B & \rightarrow & D
\end{array}
\]

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow f & & \downarrow g \\
B & \rightarrow & D \\
\downarrow h & & \downarrow h' \\
C & & C
\end{array}
\]

Cokernels are epimorphisms.

Definition 1.1.16. For \( f : A \rightarrow B \) in \( \mathcal{C} \), we call a morphism \( u : K \rightarrow A \), the kernel of \( f \), if \( u f = 0 \) and if for every morphism \( u' : K' \rightarrow A \) with \( u' f = 0 \), there exists a unique morphism \( y : K \rightarrow K' \) such that \( y u = u' \).

\[
\begin{array}{ccc}
K & \rightarrow & A \\
\downarrow u & & \downarrow f \\
A & \rightarrow & B
\end{array}
\]

\[
\begin{array}{ccc}
K & \rightarrow & A \\
\downarrow y & & \downarrow u' \\
K' & \rightarrow & K'
\end{array}
\]

Kernels are monomorphisms. We usually write \( K = \ker f \).

Definition 1.1.17. Let \( \{ A_\lambda : \lambda \in \Lambda \} \) be a family of objects in a category \( \mathcal{C} \). An object \( A \in \mathcal{C} \) together with a family \( \{ p_\lambda : A \rightarrow A_\lambda \} \) of morphisms is the product of \( \{ A_\lambda : \lambda \in \Lambda \} \) if whenever there is a family
\[ \{ \alpha_{\lambda} : B \rightarrow A_{\lambda} \}_{\lambda \in \Lambda} \] of morphisms in \( \mathcal{A} \), then there exists a unique morphism \( \phi : B \rightarrow A \) such that \( \phi \circ \alpha_{\lambda} = p_{\lambda} \) for each \( \lambda \in \Lambda \). We write \( A = \prod_{\lambda \in \Lambda} A_{\lambda} \) and call the \( p_{\lambda}'s \) projections.

Dually we define the coproduct \( \{ u_{\lambda} : A_{\lambda} \rightarrow A \}_{\lambda \in \Lambda} \) in \( \mathcal{A} \) of \( \{ A_{\lambda} : \lambda \in \Lambda \} \). We write \( A = \bigsqcup_{\lambda \in \Lambda} A_{\lambda} \) and call the \( u_{\lambda}'s \) injections.

Definition 1.1.18 A triple \((A,M,d)\), where \( A \) is a set whose elements are called vertices, \( M \) a set whose elements are called arrows and \( d \) a function \( : M \rightarrow \Lambda \times \Lambda \), is called a diagram scheme. If \( m \circ d = (\lambda, \mu) \) we say \( \lambda \) is the origin and \( \mu \) the extremity of \( m \).

Definition 1.1.19 Let \( \mathcal{A} \) be a category. A diagram \( D \) in \( \mathcal{A} \) over a scheme \((A,M,d)\) is a function which assigns to each vertex \( \lambda \in \Lambda \) an object \( A_{\lambda} \) of \( \mathcal{A} \) and to each arrow \( m \) with origin \( \lambda \) and extremity \( \mu \) a morphism \( m \circ D = m_{\lambda, \mu} \in \text{Hom}(A_{\lambda}, A_{\mu}) \).

Definition 1.1.20 A diagram, in which any two different finite compositions of morphisms with same origin and extremity are equal, is called a commutative diagram.

Definition 1.1.21 If \( D \) is a diagram in \( \mathcal{A} \) over a scheme \((A,M,d)\), we say that a family \( \{ \alpha_{\lambda} : A \rightarrow A_{\lambda} \}_{\lambda \in \Lambda} \) of
morphisms is compatible for \( D \) if for every arrow \( m \in M \) the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{a_\lambda} & A \lambda \\
\downarrow & & \downarrow \mu \lambda m \\
A & \xrightarrow{a_\mu} & A \mu
\end{array}
\]

is commutative.

**Definition 1.1.22** A family \( \{ a_\lambda : A \longrightarrow A_\lambda \}_\lambda \in \Lambda \) is a limit for \( D \) in \( \mathcal{A} \) if it is compatible for \( D \) and if for every compatible family \( \{ \beta_\lambda : B \longrightarrow A_\lambda \}_\lambda \in \Lambda \) for \( D \) in \( \mathcal{A} \), there exists a unique morphism \( \phi : B \longrightarrow A \) such that \( \phi a_\lambda = \beta_\lambda \) for each \( \lambda \in \Lambda \). If \( \{ \beta_\lambda : B \longrightarrow A_\lambda \}_\lambda \in \Lambda \) is also a limit for \( D \) then \( \phi \) is an isomorphism.

If a category \( \mathcal{A} \) has all limits for all diagram schemes, then it is called a complete category.

Dually we can define the notions of cocompatible family, colimit for a diagram and cocomplete category.

**Definition 1.1.23** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two categories. A covariant functor \( T : \mathcal{A} \longrightarrow \mathcal{B} \) is a function which assigns to each object \( A \in \mathcal{A} \) an object \( A T \in \mathcal{B} \), and to each morphism \( \alpha : A \longrightarrow B \) in \( \mathcal{A} \) a morphism \( \alpha T : A T \longrightarrow B T \) such that
1) if $a \beta$ is defined in $\mathcal{A}$, then $(a \beta) T = (a T) (\beta T)$,

2) $I_A T = I_{A T}$ for each $A \in \mathcal{A}$.

Definition 1.1.24 If $\mathcal{A}'$ is a subcategory of $\mathcal{A}$, then the covariant functor $I : \mathcal{A}' \to \mathcal{A}$ such that $A I = A$ for all $A \in \mathcal{A}'$, and $a I = a$ for all morphisms $a$ in $\mathcal{A}'$ is called the inclusion functor of $\mathcal{A}'$ in $\mathcal{A}$.

Definition 1.1.25 Let $S$, $T$ be two covariant functors from a category $\mathcal{A}$ to a category $\mathcal{B}$. Suppose that for each $A \in \mathcal{A}$, there is a morphism $\eta_A : A S \to A T$ such that for every morphism $a : A \to B$ in $\mathcal{A}$ the diagram

$$
\begin{array}{c}
A S \\
\downarrow a S \\
B S
\end{array} \\
\begin{array}{c}
\eta_A \\
\downarrow \\
\eta_B
\end{array} \\
\begin{array}{c}
A T \\
\downarrow a T \\
B T
\end{array}
$$

is commutative. Then $\eta : S \to T$ is called a natural transformation from $S$ to $T$. If each $\eta_A$ is an isomorphism then $\eta$ is called a natural equivalence.

Definition 1.1.26 Let $\mathcal{A}$ and $\mathcal{B}$ be two categories together with two covariant functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$. Suppose that there is a function $\phi$ which assigns to each pair $(A, B)$
of \mathcal{A} \times \mathcal{B} a bijection \phi_{A,B} which in turn assigns to each morphism \( f : A \rightarrow B \) a morphism \( f \phi = f \phi : AF \rightarrow B \) such that for all morphisms \( h : A' \rightarrow A \), \( k : B \rightarrow B' \),

\[(h f) \phi = h F \cdot f \phi \quad (f \cdot k G) \phi = f \phi \cdot k \cdot \]

Then we say that \((F,G,\phi)\) is an adjunction and \(F\) is the coadjoint of \(G\) and \(G\) is the adjoint of \(F\).

Remark 1.1.27 We are using the terms of adjoint and coadjoint in the sense of Mitchell [13].

Definition 1.1.28 Let \( \mathcal{A}' \) be a subcategory of a category \( \mathcal{A} \). A coreflection for \( A \in \mathcal{A} \) in \( \mathcal{A}' \) is an object \( AT \in \mathcal{A}' \) together with a morphism

\[ \theta_A : A \rightarrow AT \]

such that for every object \( A' \in \mathcal{A}' \) and every morphism \( \alpha : A \rightarrow A' \) there exists a unique morphism \( \beta : AT \rightarrow A' \) such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\theta_A} & AT \\
\downarrow{\alpha} & & \downarrow{\beta} \\
A' & &
\end{array}
\]

is commutative. Equivalently, denoting by \( I \) the inclusion functor: \( \mathcal{A}' \rightarrow \mathcal{A} \), \( \theta_A : A \rightarrow AT \) defines a coreflection.
of \( A \) in \( \mathcal{A} \) if the function

\[
\text{Hom}_{\mathcal{A}}(AT, A') \longrightarrow \text{Hom}_{\mathcal{A}}(A, A')
\]

induced by \( \theta_A \) is a one to one correspondence for all \( A' \in \mathcal{A}' \).

From the uniqueness of the morphism \( \beta : AT \to A' \) it follows that any two coreflections of \( A \) in \( \mathcal{A} \) are isomorphic. If \( \mathcal{A}' \) is a full subcategory of \( \mathcal{A} \), then every object of \( \mathcal{A} \) which is in \( \mathcal{A}' \) is its own coreflection via the identity morphism.

Dually, for \( A \in \mathcal{A} \), \( \theta_A : AT \to A \) is called the reflection of \( A \) in \( \mathcal{A} \) if \( AT \in \mathcal{A} \) and if for every morphism \( \alpha : A' \to A \) with \( A' \in \mathcal{A} \) there is a unique morphism \( \beta : A' \to AT \) making the diagram

\[
\begin{array}{ccc}
AT & \xrightarrow{\theta_A} & A \\
\downarrow{\beta} & & \downarrow{\alpha} \\
& A' & \\
\end{array}
\]

commutative.

If every object of \( \mathcal{A} \) has a coreflection (reflection) in \( \mathcal{A}' \), then \( \mathcal{A}' \) is called a coreflective (reflective) subcategory of \( \mathcal{A} \).

In this case \( T : \mathcal{A} \to \mathcal{A}' \) becomes a covariant functor, called the coreflector (reflector) of \( \mathcal{A} \) in \( \mathcal{A}' \), which assigns to each morphism \( \alpha : A \to B \) of \( \mathcal{A} \) the unique morphism
\[ a \circ T : AT \longrightarrow BT \quad \text{in} \quad \mathcal{A} \quad \text{such that the diagram} \]

\[
\begin{array}{cccc}
A & \xrightarrow{\theta_A} & AT & \quad \xleftarrow{\theta_A} \quad AT \\
\downarrow{\alpha} & & \downarrow{a \circ T} & \quad \downarrow{\alpha} \\
B & \xrightarrow{\theta_B} & BT & \quad \xleftarrow{\theta_B} \quad BT
\end{array}
\]

is commutative. It is easy to check the naturality of the \(1 - 1\) correspondence \(\phi : \text{Hom} \_\_ (AT, A') \longrightarrow \text{Hom} \_\_ (A, A' \circ \text{Hom})\)

\[
(\phi : \text{Hom} (A', AT) \longrightarrow \text{Hom} (A \circ \text{Hom}, A) ) \quad \text{in both A and A'}.
\]

Therefore we have:

Theorem 1.1.29 If \(T : \mathcal{A} \longrightarrow \mathcal{A}'\) is a coreflector (reflector) functor then \(T\) is a coadjoint (adjoint) of the inclusion functor \(I : \mathcal{A}' \longrightarrow \mathcal{A}\).

Conversely, if the inclusion functor \(I : \mathcal{A}' \longrightarrow \mathcal{A}\) has a coadjoint (adjoint) \(T\) then \(T\) is the coreflector (reflector) of \(\mathcal{A}\) in \(\mathcal{A}'\).

Theorem 1.1.30 Let \(\mathcal{A}'\) be a full reflective subcategory of a category \(\mathcal{A}\). If a diagram \(D\) in \(\mathcal{A}'\) over a scheme \((A, M, d)\), has a colimit \(\{ a_\lambda : A_\lambda \longrightarrow A \}_{\lambda \in \Delta} \) in \(\mathcal{A}\), then \(AT \cong A\) and \(\{ a_\lambda : A_\lambda \longrightarrow AT \}_{\lambda \in \Delta} \) is a colimit of \(D\) in \(\mathcal{A}'\), \(T\) being the reflector of \(\mathcal{A}\) in \(\mathcal{A}'\).

Theorem 1.1.31 Let \(\mathcal{A}'\) be a full reflective subcategory of a category \(\mathcal{A}\). If a diagram \(D\) in \(\mathcal{A}'\) over a scheme \((A, M, d)\),
has a limit \( \{ p_\lambda : A \rightarrow A_\lambda \}_{\lambda \in \Lambda} \) in \( \mathcal{A} \) then it has a limit \( \{ \theta p_\lambda : AT \rightarrow A_\lambda \}_{\lambda \in \Lambda} \) in \( \mathcal{A} \), where \( \theta : AT \rightarrow A \) is the reflection of \( A \) in \( \mathcal{A} \), with \( T \) the reflector of \( \mathcal{A} \) in \( \mathcal{A} \).

Definition 1.1.32. Let a category \( \mathcal{A} \) have a property \( p \), where \( p \) can be intersections (unions, images, kernels, cokernels, intersection of kernels). Then we say that \( \mathcal{A} \) is a category with \( p \).

1.2. Algebraic Categories

By an algebraic category we mean a category with objects having certain algebraic structure or structures, for example, rings, groups, modules, near-rings etc., and having as morphisms the structure preserving mappings. We call morphisms in this case, homomorphisms (of the structure of the objects). We say a homomorphism is an epimorphism (monomorphism, isomorphism) if it is a surjection (injection, bijection). We note that an epimorphism (monomorphism, isomorphism) is also an epimorphism (monomorphism, isomorphism) in the sense of 1.1.5.

Throughout this section \( \mathcal{A} \) denotes an algebraic category.
The notions of binary relation and an equivalence relation are defined as usual.

Definition 1.2.1. An equivalence relation $\rho$ on $A \in \mathcal{A}$ is called a congruence if it is a subobject of $A \times A$.

Definition 1.2.2. The composition of two binary relations $\phi$ and $\psi$ on $A \in \mathcal{A}$ is defined as follows:

$$\phi \circ \psi = \{ (x,y) \in A \times A : (x,z) \in \phi, (z,y) \in \psi \text{ for some } z \in A \}.$$

Let $f : A \to B$ be a homomorphism in $\mathcal{A}$. Then $\sigma = f \circ f^{-1}$ is an equivalence relation on $A$.

Definition 1.2.3. $\sigma$ in the above result is called the kernel of $f$ and is written as $\text{Ker } f$.

Theorem 1.2.4. The kernel of a homomorphism is a congruence.

For $A \in \mathcal{A}$ and for any congruence $\sigma$ on $A$, $A / \sigma$ is an object of $\mathcal{A}$ and $\sigma^\downarrow : A \to A / \sigma$ is an epimorphism with $\text{Ker } \sigma^\downarrow = \sigma$.

Theorem 1.2.5. Let $f : A \to B$ be an epimorphism and $g : A \to C$ be any homomorphism such that $\text{Ker } f \subseteq \text{Ker } g$.

Then there exists a unique homomorphism $h : B \to C$ such that $fh = g$. 
1.3. The Category $G$ of all Groups

Let $G \in G$ be any group. If $G$ is generated as a group by a subset $X$, we write $G = Gp\{X\}$. In this case each element $g$ of $G$ is written as $g = \epsilon x_1 + \epsilon x_2 + \cdots + \epsilon x_n$, $x_i \in X$, $\epsilon_i = \pm 1$, $i = 1, \ldots, n$, $n$ a finite integer.

Definition 1.3.1. An element $\epsilon x_1 + \cdots + \epsilon x_n$ of $G = Gp\{X\}$ is called a relation in $G$ relative to $X$ if it is 0 in $G$.

Definition 1.3.2. Let $G = Gp\{X\}$ and $H$ be any group. Then we call a mapping $\alpha : X \to H$ well-defined if whenever $\epsilon x_1 + \cdots + \epsilon x_n$ is a relation in $G$ relative to $X$,

$\epsilon x_1 \alpha + \cdots + \epsilon x_n \alpha$ is 0 in $H$.

Theorem 1.3.3. Let $G = Gp\{X\}$ and $H$ be any group. Then a well-defined mapping $\alpha : X \to H$ can be extended to a group homomorphism $\overline{\alpha} : G \to H$.

The category $G$ is complete and cocomplete. In this case the coproducts are called free products and we write $G = \ast H$ to mean that $G$ is the free product of the family $\{H_\lambda : \lambda \in \Lambda\}$ of groups.

Theorem 1.3.4. Let $G$ be the free product of a family $\{H_\lambda : \lambda \in \Lambda\}$ of groups. Then each non-zero element $g$ of $G$ is uniquely expressible as
\[ g = h_{\lambda_1} + \ldots + h_{\lambda_r}, \]

\[ 0 \neq h_{\lambda_i} \in H_{\lambda_i}, \quad \lambda_i \neq \lambda_{i+1}, \quad i = 1, \ldots, r - 1. \]

Theorem 1.5. Let \( G = H \ast K \), for \( G \), \( H \) and \( K \in \mathcal{G} \), and let \( N \) be a normal subgroup of \( K \). Then

\[ G/N \cong H \ast (K/N), \]

where \( N^G \) is the normal closure of \( N \) in \( G \).

Note. The results of § 1.2 and § 1.3 will be used frequently and since they are well known we will not mention them explicitly.

1.4. Near-Rings

Definition 1.4.1. A (left) near-ring is a set \( R \) together with two binary operations \( + \) and \( \cdot \) such that

1. \( (R,+) \) is a group, not necessarily commutative,

2. \( (R,\cdot) \) is a semigroup,

3. for all \( r_1, r_2 \) and \( r_3 \) in \( R \),

\[ r_1(r_2 + r_3) = r_1r_2 + r_1r_3, \]

i.e., the elements of \( R \) satisfy the left distributive law.

Examples 1.4.2.

1. The set \( \text{T}(G) \) of all mappings of a group \( G \) into itself with pointwise addition and multiplication as composition of mappings,
forms a near-ring.

2) The subset $T_0(G)$ of $T(G)$ consisting of all mappings which map 0 onto 0, is also a near-ring.

3) Let $R$ be a commutative ring with identity. Then the set $R[x]$ of polynomials with usual addition and with the operation of substitution as multiplication, forms a near-ring.

4) Let $G$ be a non-abelian group and $\text{End}(G)$ be the semigroup of endomorphisms of $G$. $\text{End}(G)$ is not closed under addition, but the set of all finite sums and differences of the elements of $\text{End}(G)$ is closed under addition and multiplication, and forms a near-ring. We denote this near-ring by $E(G)$.

Theorem 1.4.3. In a near-ring $R$,

$$r \cdot 0 = 0 \quad \text{and} \quad r(-r) = -rr,$$

for all $r, r \in R$.

Remark 1.4.4. In general $0 \cdot r \neq 0$ and $(-r) \cdot r' \neq -rr'$. For example in $T(G)$, $0 \cdot f \neq 0$ and $(-f) \cdot f' \neq -(f \cdot f')$ in general.

Definition 1.4.5. A near-ring $R$ is said to be zero symmetric if $0 \cdot r = 0$ for all $r \in R$.

$T_0(G)$ in 1.4.2.(2) is a zero symmetric near-ring.

Definition 1.4.6. A mapping $\alpha$ from a near-ring $R$ to a near-
ring. This called a near-ring homomorphism if

1) \( a \) is a group homomorphism from \((R,+)\) to \((T,+),\)

2) \( a \) is a semigroup homomorphism from \((R,\cdot)\) to \((T,\cdot)\).

We can define epimorphism, monomorphism and isomorphism in the usual way.

Definition 1.4.7. A subset \( S \) of \( R \) is said to be a sub-near-ring of the near-ring \( R \) if \( S \) itself is a near-ring with respect to the same addition and multiplication.

\( T_0(G) \) is a sub near-ring of \( T(G) \).

Definition 1.4.8. We call a normal subgroup \( I \) of a near-ring \( R \) an ideal if

1) \( R I \subseteq I \)

2) for all \( r, r' \in R \) and all \( a \in I \)

\((r + a)r' - rr' \in I\)

A normal subgroup of a near-ring \( R \) satisfying (1) is called a left ideal of \( R \), while a normal subgroup satisfying (2) is called a right ideal of \( R. \)

It is easy to see that \( I \) is an ideal of a near-ring \( R \) if and only if \( r \equiv r'(\text{mod } I) \) is a congruence relation on \( R. \) For every ideal \( I \), the quotient set \( R/I \) is a near-ring.

Theorem 1.4.9. (Homomorphism Theorem)

1) If \( I \) is an ideal of a near-ring \( R \) then the canonical map
\[ \pi : R \rightarrow R/I \] is a near-ring homomorphism and \( R/I \) is a homomorphic image of \( R \).

2) Conversely, if \( \alpha : R \rightarrow T \) is a near-ring epimorphism then \( R/\text{Ker} \, \alpha \cong T \).

Theorem 1.4.10 Let \( \alpha : R \rightarrow T \) be a near-ring epimorphism. Then there is a one to one, order-preserving, correspondence between the sub-near-rings (ideals) of \( R \) containing \( \text{Ker} \, \alpha \) and the sub-near-rings (ideals) of \( T \).

If \( \pi : R \rightarrow R/I \) is the canonical epimorphism then for all ideals \( J \) of \( R \), such that \( J \supseteq I \),

\[ \pi(R)/\pi(J) \cong R/J. \]

Theorem 1.4.11. If \( I \) and \( J \) are ideals of a near-ring \( R \), then \( I \cap J \) is an ideal of \( R \) and \( J + I/I \cong J/I \cap J \).

Definition 1.4.12. A group \( G \) called an \( R \)-group, \( R \) a near-ring, if there exists a mapping \( \mu : G \times R \rightarrow G \) defined by \((g, r)\mu = gr\) such that for all \( r, r' \in R \), \( g \in G \)

\[ g(r + r') = gr + gr' \]

\[ g(rr') = (gr)r' \]

Obviously \((R, +)\) is an \( R \)-group for every near-ring \( R \).

1.5. D.G. Near-Rings

Definition 1.5.1. An element \( d \) of a near-ring \( R \) is said
to be distributive if \((r + r')d = rd + r'd\) for all \(r, r' \in R\). A near-ring in which every element is distributive is called a distributive near-ring.

The endomorphisms of \(G\) are distributive in the near-ring \(T_0(G)\) (1.4.2.(2)).

Remark 1.5.2. In a near-ring the set of distributive elements forms a multiplicative semigroup.

Definition 1.5.3. A near-ring \(R\) is said to be a d.g. near-ring (distributively generated near-ring) if \((R,+)\) is generated by a semigroup \(S\) of distributive elements of \(R\). \(S\) need not be the whole set of distributive elements of \(R\). Because of the importance of the generating semigroup we write a d.g. near-ring as \((R,S)\).

For any non abelian group \(G\), \((E(G),\text{End}(G))\) is a d.g. near-ring.

Remark 1.5.4. Every element \(r\) of a d.g. near-ring \((R,S)\) is a finite sum of the form \(r = \sum_{i=1}^{n} \epsilon_i s_i\), \(\epsilon_i = \pm 1\), \(s_i \in S\), \(i = 1, \ldots, n\).

We will represent \(r \in (R,S)\) in this way without further reference.

Theorem 1.5.5. Let \((R,S)\) be a d.g. near-ring. Then

1) \(\forall r \in (R,S)\) and \(\forall s \in S\)

\[(-r)s = -(rs) = -S(e+s) = -(sr)\]
2) \((R,S)\) is zero symmetric,

3) \(\forall \ r, r' \in (R,S) \text{ and } \forall \ s \in S\)

\[
(r + r') (s) = -(r + r') s = -(rs + r's) = -r's - rs
\]

4) if \(r = \sum_{i=1}^{n} \epsilon_i s_i\), \(r' = \sum_{j=1}^{m} \mu_j t_j\), \(s_i, t_j \in S\)

\[
r r' = \sum_{j=1}^{m} \mu_j \left( \sum_{i=1}^{n} \epsilon_i s_i t_j \right)
\]

**Theorem 1.5.6.** Let \((R,S)\) be a d.g. near-ring. Then
1) \(R\) is distributive if and only if \(\forall \ a, b, c, d \in R\),

\[
ab + cd = cd + ab,
\]

i.e., if and only if \(R^2\) is abelian,

2) \((R,+)\) is abelian if and only if \((R,+,\cdot)\) is a ring.

**Theorem 1.5.7.** Let \((R,S)\) be a d.g. near-ring. Then
1) if \(S\) contains a left (right, two-sided) identity \(e\) then \(e\) is a left (right, two sided) identity for \(R\).

2) if \(R\) contains exactly one left (right) identity then it is the two-sided identity of \(R\).

**Theorem 1.5.8.** 1) If a near-ring \(R\) contains a d.g. near-ring \(T\) then \(R\) is not necessarily d.g.

2) A sub-near-ring of a d.g. near-ring need not be d.g.

3) Every homomorphic image of a d.g. near-ring is a d.g. near-ring.

**Theorem 1.5.9.** Let \((R,S)\) be a d.g. near-ring. A normal sub-
Theorem 1.5.10 [11] Let \((R,S)\) be a d.g. near-ring, \(X\) a subset of \((R,S)\). The ideal generated by \(X\) is the normal subgroup of \((R,+)\) generated by

\[ R \times S = \{ rxs, rx, xs, x : r \in R, x \in X, s \in S \}. \]

Definition 1.5.11. Let \((R,S)\) and \((T,U)\) be two d.g. near-rings. A mapping \(\alpha : (R,S) \rightarrow (T,U)\) is called a d.g. near-ring homomorphism if \(\alpha\) is a near-ring homomorphism from \(R\) to \(T\) which maps \(S\) into \(U\).

Remark 1.5.12. The kernel of a d.g. near-ring homomorphism is an ideal and one can state the homomorphism and isomorphism theorems for d.g. near-rings.

Theorem 1.5.13. [11] Let \((R,S)\) and \((T,U)\) be two d.g. near-rings. A group homomorphism \(\alpha : (R,+) \rightarrow (T,+)\) is a d.g. near-ring homomorphism if \(\alpha\) is a semigroup homomorphism from \(S\) to \(U\).

Let \(S\) be a multiplicative semigroup. Let \((\text{Fr}(S),+)\) be the free group on the set \(S\). This group consists of all finite sums

\[ \sum_{i} \epsilon_{i}s_{i}. \]

By defining
Theorem 1.5.14. 1) The multiplication $\cdot$ is well defined.

2) $(\text{Fr}(S), +, \cdot)$ is a d.g. near-ring generated by $S$.

3) For every d.g. near-ring $(T, U)$, every semigroup homomorphism $\theta : S \rightarrow U$ can be extended uniquely to a d.g. near-ring homomorphism $\phi : (\text{Fr}(S), S) \rightarrow (T, U)$.

4) Every d.g. near-ring $(R, S)$ is a homomorphic image of $(\text{Fr}(S), S)$.

5) $(\text{Fr}(S), S)$ is determined uniquely to within d.g. near-ring isomorphism.

Definition 1.5.15. $(\text{Fr}(S), S)$ is called the free d.g. near-ring on the multiplicative semigroup $S$.

1.6. Representations

Definition 1.6.1. Let $(R, S)$ be a d.g. near-ring. A group $G$ is called an $\text{(R, S)}$-group if there exists a d.g. near-ring homomorphism $\theta : (R, S) \rightarrow (E(G), \text{End}(G))$. The map $\theta$ is often omitted and we write $gr$ for $g(r\theta)$, $g \in G$, $r \in (R, S)$.

2) Let $S$ be a multiplicative semigroup. A group $G$ is said to be an $S$-group if there exists a semigroup homomorphism $\theta$ from $S$ to $\text{End}(G)$ and we write $gs$ for the element $g(s\theta)$ of $G$.

In (1) \{ (2) \} $\theta$ is said to be a representation of $(R, S)$.
Remark 1.6.2. If \((R,S)\) is a d.g. near-ring, then an \(R\)-group \(G\) is an \((R,S)\) group if and only if \(G\) is an \(S\)-group under the induced representation.

Theorem 1.6.3. [6] Let \((R,S)\) be a d.g. near-ring and \(G\) an \((R,S)\)-group. A subgroup \(H\) of \(G\) is an \((R,S)\)-group if and only if it is an \(S\)-group.

Theorem 1.6.4. [7] Every \(S\)-group \(G\), \(S\) a multiplicative semigroup, is an \((\text{Fr}(S),S)\) group.

2) Let \((R,S) \cong (\text{Fr}(S),S) / I\) be a d.g. near-ring. Then an \(S\)-group \(G\) is an \((R,S)\)-group if and only if \(G I = \{0\}\).

Definition 1.6.5. Let \(H\) and \(G\) be \((R,S)\)-groups \(\{ S\text{-groups}\}\).

A group homomorphism \(\alpha : H \longrightarrow G\) is called an \((R,S)\)-homomorphism \(\{ S\text{-homomorphism}\}\) if for all \(h \in H\), \((hr)\alpha = (ha)r\) for all \(r \in (R,S)\) \(\{ (hs)\alpha = (ha)s\), for all \(s \in S\}\).

Definition 1.6.6. A subset \(X\) of an \((R,S)\) group \(G\) is called a free \((R,S)\)-generating set or a basis of \(G\) if for every \((R,S)\)-group \(H\) and for every mapping \(\alpha : X \longrightarrow H\) there exists a unique \((R,S)\)-homomorphism \(\phi : G \longrightarrow H\) extending \(\alpha\) and such a group is called a free \((R,S)\)-group.

The next result shows that for any non-empty set \(X\) we can construct a free \((R,S)\)-group with \(X\) as its basis.
Theorem 1.6.7. Let $(R,S) \cong (\text{Fr}(S),S)/I$ be a d.g. near-ring and $X$ be any set. Let $\text{Fr}(X,S)$ be the free group generated by the set $\{x, s_x : x \in X, s \in S\}$ and

$$\text{Fr}(X,S) \cap I = \text{Grp}\{g a : g \in \text{Fr}(X,S), a \in I\}.$$ 

Then $\text{Fr}(X,R,S) = \text{Fr}(X,S)/\text{Fr}(X,S) \cap I$ is the free $(R,S)$-group on $X$, where $\text{Fr}(X,S) \cap I$ is the normal closure of $\text{Fr}(X,S) \cap I$ in $\text{Fr}(X,S)$.

Definition 1.6.8. Let $(R,S)$ be a d.g. near-ring and $G$ an $(R,S)$-group with representation $\theta$. Then

1) the representation $\theta$ is called faithful if $\text{Ker}\theta = \{0\}$,

2) if $(R,S)$ has an identity element $1$, the representation $\theta$ is said to be unitary and $G$ a unitary $(R,S)$-group if $\theta$ maps $1$ onto the identity endomorphism of $G$.

Theorem 1.6.9. Let $(R,S)$ be a d.g. near-ring with a left identity. Then $(R,S)$ has a faithful representation on $(R,+)$. Theorem 1.6.10 If a d.g. near-ring has a faithful representation, then it has a faithful representation on $\text{Fr}(x,R,S)$, the free $(R,S)$-group on one generator.

We call a d.g. near-ring faithful if it has a faithful representation. J.D.P. Meldrum constructed examples of d.g. near-rings which are not faithful.

Theorem 1.6.11. Let $G$ be an $(R,S)$-group. If for every non
zero $r \in (R,S)$ there is an element $g = g_r \in G$ such that $g r \neq 0$, then $(R,S)$ has a faithful representation on $G$.

Theorem 1.6.12. [11] Let $(R,S)$ be a faithful d.g. near-ring and $G = Fr(x,R,S)$, the free $(R,S)$-group on one generator. Then $G = G_1 \ast G_2$, the free product of $G_1$ and $G_2$, where $G_1$ is the free cyclic group and $G_2$ is a group isomorphic with $(R,+)$.

Definition 1.6.13 Let $G$ be an $(R,S)$-group, $(R,S)$ a d.g. near-ring. An $(R,S)$-subgroup $H$ of $G$ is called a submodule of $G$ if it is normal in $G$.

Definition 1.6.14. Let $(R,S)$ be a d.g. near-ring and $\mathcal{C}(R,S)$ be the category of all $(R,S)$-groups. Then the coproduct object in $\mathcal{C}(R,S)$ of a family of objects in $\mathcal{C}(R,S)$ is called the free $(R,S)$-product of that family.

Theorem 1.6.15. [12] Let $\{ H_{\lambda} : \lambda \in \Lambda \}$ be a family of $(R,S)$-groups with $\{ X_{\lambda} : \lambda \in \Lambda \}$ as their $(R,S)$-generating sets. Then $Fr(X,R,S)/K$ is the free $(R,S)$-product of $\{ H_{\lambda} : \lambda \in \Lambda \}$, where $X$ is the disjoint union of the family $\{ X_{\lambda} : \lambda \in \Lambda \}$ of sets and $K$ is the submodule generated by $\{ \text{Ker} \theta_{\lambda} : \lambda \in \Lambda \}$ for the natural homomorphisms $\theta_{\lambda} : Fr(X_{\lambda},R,S) \rightarrow H_{\lambda}$, $\lambda \in \Lambda$. 

26
Chapter 2

Categories and D.G. Near-Rings

2.1. The Upper and Lower Faithful D.G. Near-Rings

The lower faithful d.g. near-ring for a d.g. near-ring \((R,S)\) is a faithful d.g. near-ring \((R,S')\) with a d.g. near-ring epimorphism 
\[ \theta : (R,S) \twoheadrightarrow (R,S') \]
such that

1) \( S \theta = S' \)

2) if \( \psi : (R,S) \twoheadrightarrow (T,U) \) is a d.g. near-ring homomorphism, where \((T,U)\) is a faithful d.g. near-ring, then there exists a unique d.g. near-ring homomorphism \( \phi : (R,S) \twoheadrightarrow (T,U) \) such that \( \theta \phi = \psi \).

The upper faithful d.g. near-ring for a d.g. near-ring \((R,S)\) is a faithful d.g. near-ring \((\overline{R},S)\) with a d.g. near-ring epimorphism 
\[ \theta : (\overline{R},S) \twoheadrightarrow (R,S) \]
such that

1) \( \theta / S = \text{identity} \),

2) if \( \phi \) is any d.g. near-ring homomorphism from a faithful d.g. near-ring \((T,U)\) to \((R,S)\) then there exists a unique d.g. near-ring homomorphism \( \phi : (T,U) \twoheadrightarrow (\overline{R},S) \) such that \( \phi \theta = \phi \).

Now we will show that for any d.g. near-ring \((R,S)\) upper and lower faithful d.g. near-rings exist. First we prove the existence of lower faithful d.g. near-rings.

**Theorem 2.1.1.** Let \((R,S)\) be a d.g. near-ring and

\[ A = A(R,S) = \{ r \in (R,S) : Gr = 0 \} , \]

27
where \( G = \text{Fr}(x, R, S) \), the free \((R, S)\)-group on one generator \( x \).

Then the quotient d.g. near-ring \((R, S)/A\) is the lower faithful d.g. near-ring for \((R, S)\).

We prove this theorem in an exactly similar way to theorem 4.3 [11]. So we need the following two lemmas of the same paper.

**Lemma 2.1.2.** Let \( \theta : (R, S) \longrightarrow (T, U) \) be a d.g. near-ring homomorphism and \( G \) a \((T, U)\)-group with representation \( \phi \). Then \( G \) is an \((R, S)\)-group and the kernel of the representation \( \mu \) of \((R, S)\) on \( G \) is the inverse image under \( \theta \) of \( \ker \phi \).

**Proof.** \( \theta \phi \) is a d.g. near-ring homomorphism from \((R, S)\) to \((E(G), \text{End}(G))\). Therefore \( \mu = \theta \phi \) is a representation of \((R, S)\) on \( G \). Let \( r \in \ker \mu \) for some \( r \in (R, S) \). Then \( 0 = \mu r = r \theta \phi \), i.e., \( r \in [\ker \phi] \theta^{-1} \). Therefore \( \ker \mu \subseteq [\ker \phi] \theta^{-1} \). On the other hand if \( r \in [\ker \phi] \theta^{-1} \), then \( r \theta \in \ker \phi \) and \( \mu r = r \theta \phi = 0 \). This shows that \( [\ker \phi] \theta^{-1} \subseteq \ker \mu \). Hence we get \( \ker \mu = [\ker \phi] \theta^{-1} \).

**Lemma 2.1.3.** Let \((R, S)\) be a d.g. near-ring and \( G = \text{Fr}(x, R, S) \) be the free \((R, S)\)-group on one generator \( x \). Let
\[
A = A(R, S) = \{ r \in (R, S) : G r = 0 \}.
\]
If \( \phi \) is any representation of \((R, S)\) then \( \ker \phi \) contains \( A \).

**Proof.** Let \( H \) be an \((R, S)\)-group with representation \( \phi \).
Suppose that $\ker \phi \not\subseteq A$, and let $r \in A - \ker \phi$. Then there exists an $h \in H$ such that $h(r\phi) \neq 0$. Map $x$ to this $h \in H$. Since $G$ is the free $(R,S)$-group on one generator, this map extends to an $(R,S)$-homomorphism $\alpha : G \rightarrow H$. Therefore, as $xr = 0$ in $G$,

$$0 = (xr)\alpha = (x\alpha)r\phi = h(r\phi)$$

in $H$. This is a contradiction. Hence $\ker \phi$ contains $A$.

Proof of Theorem 2.1.1. Since $A$ is an ideal of $(R,S)$ we get a quotient d.g. near-ring $(R,S)/A$, which is faithful having a faithful representation on $G$. We denote $(R,S)/A$ by $(R,S)$. Let $\theta$ be the natural homomorphism from $(R,S)$ to $(R,S)$. Then $S = \{ s + A : s \in S \}$, i.e., $S\theta = S$. Note that $\theta/S$ need not be the identity mapping.

Now let $\psi$ be a d.g. near-ring homomorphism from $(R,S)$ to a faithful d.g. near-ring $(T,U)$ and $H = Fr(x,T,U)$ be the free $(T,U)$-group on one generator $x$. By lemma 2.1.2 $H$ is an $(R,S)$-group with representation $\psi \eta$, where $\eta$ is the representation of $(T,U)$ on $H$. Since $(T,U)$ is faithful, by 1.6.10 it has a faithful representation on $H$. Therefore $\ker \eta = \{0\}$, and we get $\ker \psi \eta = \ker \psi$. By lemma 2.1.3, $A \subseteq \ker \psi \eta = \ker \psi$. So there exists a unique d.g. near-ring homomorphism $\phi : (R,S) \rightarrow (T,U)$ such that $\theta \phi = \psi$, i.e., the following diagram is commutative.
This completes the proof.

Let \((R, S)\) be a d.g.near-ring and \(G = \mathbb{C}p[x] \ast R_x\) be the free product of the free group \(\mathbb{C}p[x]\) on one generator and the group \(R_x \cong (R, +)\). Define for \(s \in S\) a map \(\overline{s} : G \rightarrow G\) by

\[
x \mapsto S_x \quad \text{and} \quad r_x \mapsto (rs)_x.
\]

Then since \(S\) is a distributive semigroup of \((R, S)\), each \(\overline{s}\) is an endomorphism of \(R_x\). Therefore by the property of free products each \(\overline{s}\) extends to an endomorphism of \(G\), again denoted by \(\overline{s}\).

Now the map \(\beta : s \mapsto \overline{s}\) is a semigroup homomorphism from \(S\) to \(\text{End}(G)\), for

\[
x \overline{s} \overline{t} = (x \overline{s}) \overline{t} = (s_x) \overline{t} = (s \overline{t})_x = x \overline{s} \overline{t}
\]

\[
r_x \overline{s} \overline{t} = (r_x \overline{s}) \overline{t} = (r \overline{s})_x \overline{t} = (r \overline{s} \overline{t})_x = r_x \overline{s} \overline{t}
\]

\(\forall s, t \in S\). Moreover \(\beta\) is injective, for \(s \neq t\) in \(S\) implies that \(s_x \neq t_x\) in \(R_x\) and \(x \overline{s} = (s_x) \neq t_x = x \overline{t}\), i.e., \(\overline{s} \neq \overline{t}\) in \(\text{End}(G)\). So we identify \(\overline{S}\) with \(S\) and say that \(S\)
is a semigroup of automorphisms of $G$. This $S$ generates a d.g. near-ring $(\overline{R}, S)$ which has a faithful representation on $G$. Now we prove the following theorem.

**Theorem 2.1.4.** $(\overline{R}, S)$ is the upper faithful d.g. near-ring for $(R, S)$.

**Proof.** Let $I_S$ be the obvious semigroup isomorphism from $S$ contained in $\overline{R}$ to $S$ contained in $R$, and let

$$\epsilon_1 s_1 + \ldots + \epsilon_n s_n = 0$$

in $(\overline{R}, +)$. Then $0 = x(\epsilon_1 s_1 + \ldots + \epsilon_n s_n)$ in $G$, $x \in X$,

$$= \epsilon_1 s_1 x + \ldots + \epsilon_n s_n x,$$

$$= (\epsilon_1 s_1 + \ldots + \epsilon_n s_n) x,$$

$$= r x,$$

$r \in (R, +)$. But as $(R, +) \cong (R, +) x = 0$ in $(R, +)$ if and only if $r = 0$ in $(R, +)$. So

$$\epsilon_1 s_1 + \ldots + \epsilon_n s_n = 0$$

in $(R, +)$. So $I_S$ extends to a group homomorphism $\theta$ from $(\overline{R}, +)$ to $(R, +)$. By 1.5.13, $\theta$ is a d.g. near-ring homomorphism from $(\overline{R}, S)$ to $(R, S)$. Clearly $\theta$ is onto and $\theta/S$ = identity.

Now let $(T, U)$ be a faithful d.g. near-ring with a d.g. near-ring homomorphism $\psi : (T, U) \rightarrow (R, S)$. Then by the homomorphism theorem for d.g. near-rings we have
where $I = \text{Ker } \psi$ is an ideal of $(T, U)$, let $H = \text{Fr}(x, T, U)$ be free $(T, U)$-group on one generator $x$. Since $(T, U)$ is faithful, by 1.6.12, $H = \text{Gp}[x] \ast T_x$, the free product of the free group $\text{Gp}[x]$ and $T_x \cong (T, +)$. The homomorphism $\psi : (T, +) \longrightarrow (R, +)$ induces a homomorphism $\psi_x : t_x \longrightarrow (t \psi)_x$ from $T_x$ to $R_x$.

Clearly $\text{Ker } \psi_x = I_x$ and $T_x \psi_x = (T \psi)_x \cong (T/I, +)$.

We define a map $\phi' : H \longrightarrow G$ by

$$x \longmapsto x \text{ and } t_x \longmapsto (t \psi)_x.$$ 

By the property of free products this map extends to a homomorphism $\bar{\psi} : H \longrightarrow G$, such that

$$\bar{\psi}/T_x = \psi_x \text{ and } \bar{\psi}/\text{Gp}[x] = \text{identity}.$$ 

Now since $I_x U \subseteq (I U)_x$ and $I$ is an ideal of $(T, U)$, we get $I_x U \subseteq I_x$.

This implies that $I_x$ is a $U$-group. Therefore the normal closure $I_x^H$ of $I_x$ in $H$ is also a $U$-group and hence by 1.6.3, it is a $(T, U)$-group. Thus $H/I_x^H$ is a $(T, U)$-group and we get

$$H/I_x^H \cong \text{Gp}[x] \ast T_x/I_x \quad (1.3.5),$$

$$\cong \text{Gp}[x] \ast (T \psi)_x,$$

$$= G' \subseteq G = \text{Gp}[x] \ast R_x.$$ 

Thus we have proved that $G$ contains a subgroup $G' \cong H/I_x^H$.
which is a \((T,U)_\text{group}\). The action of \(U\) on \(H/\text{I}_X\) is the same as that of \(U\psi\) on \(G'\). Therefore the semigroup homomorphism
\[
\psi : U \subseteq T \longrightarrow S \subseteq \overline{R}
\]
extends to a group homomorphism \(\phi : (T,+) \longrightarrow (\overline{R},+)\). Again by 1.5.13, \(\phi\) is a d.g. near-ring homomorphism from \((T,U)\) to \((\overline{R},S)\) such that \(\phi / U = \psi\). Also for \(u \in U\),
\[
u \phi \theta = u \psi \theta = u \psi .
\]
This shows that \(\phi \theta\) and \(\psi\) agree on \(U\). So they do so on \((T,U)\).
Hence \(\phi \theta = \psi\) as d.g. near-ring homomorphisms. Clearly \(\phi\) is unique with this property.

From now on we will denote by \(\mathcal{A}\) the category of all d.g. near-rings and by \(\mathcal{B}\) the category of all faithful d.g. near-rings. Then \(\mathcal{B}\) is a full subcategory of \(\mathcal{A}\). From § 2.1 we see that the lower faithful d.g. near-ring \((\overline{R},S)\) for \((R,S)\) defines a coreflection in \(\mathcal{B}\) of \((R,S) \in \mathcal{A}\) and since every d.g. near-ring in \(\mathcal{A}\) has a lower faithful d.g. near-ring associated with it, \(\mathcal{B}\) is a coreflective subcategory of \(\mathcal{A}\). By denoting \(-(R,S) = (R,S) F\) we get a covariant functor \(F : \mathcal{A} \longrightarrow \mathcal{B}\) which is a coadjoint of the inclusion functor \(I : \mathcal{B} \longrightarrow \mathcal{A}\) (1.1.29).

Similarly the upper faithful d.g. near-ring \((\overline{R},S)\) defines a reflection of \((R,S) \in \mathcal{A}\) in \(\mathcal{B}\) and \(\mathcal{B}\) is a reflective subcat-
category of \( R \) since each \( (R,S) \in \mathcal{R} \) has a reflection \( (\overline{R},\overline{S}) \) in \( \mathcal{B} \).

Again by denoting \( (\overline{R},\overline{S}) = (R,S) G \) we get a covariant functor

\[
G : \mathcal{R} \longrightarrow \mathcal{B}
\]

which is an adjoint of the inclusion functor

\[
I : \mathcal{B} \longrightarrow \mathcal{R}
\]

(1.1.29).

We now go on to show that the categories \( \mathcal{R} \) and \( \mathcal{B} \) are complete and cocomplete. But since \( \mathcal{B} \) is a reflective subcategory of \( \mathcal{R} \), by theorems 1.1.30 and 1.1.31, it is sufficient to prove the existence of limits and colimits in \( \mathcal{R} \).

2.2. Colimits in \( \mathcal{R} \)

First we prove the existence of special colimits in \( \mathcal{R} \), namely the coproducts. Let \( \{(R_{\lambda},S_{\lambda}) : \lambda \in \Lambda\} \) be a family of d.g. near-rings in \( \mathcal{R} \). For each \( \lambda \in \Lambda \), we have the free d.g. near-ring \( F_{\lambda} = (Fr(S_{\lambda}),S_{\lambda}) \) such that \( F_{\lambda}/A_{\lambda} \cong (R_{\lambda},S_{\lambda}) \) for some ideal \( A_{\lambda} \) of \( F_{\lambda} \).

Let \( S^\circ = \bigoplus_{\lambda \in \Lambda} S_{\lambda} \) be the free product of the family \( \{S_{\lambda} : \lambda \in \Lambda\} \) of semigroups. Then we have another free d.g. near-ring \( (Fr(S^\circ),S^\circ) \). Let \( \{u_{\lambda} : S_{\lambda} \longrightarrow S^\circ\}_{\lambda \in \Lambda} \) be the family of semigroup monomorphisms. By the property of free groups each \( u_{\lambda} \) extends to a group monomorphism from \( (Fr(S_{\lambda}),+) \) to \( (Fr(S^\circ),+) \) again denoted by \( u_{\lambda} \). Hence each \( u_{\lambda} \) is a d.g. near-ring homomorphism from \( (Fr(S_{\lambda}),S_{\lambda}) \) to \( (Fr(S^\circ),S^\circ) \) (1.5.13). Since the free d.g. near-rings are unique up to d.g. near-ring isomorphism (1.5.14), for
each $\lambda \in \Lambda$ identifying $(\text{Fr}(S^*), S^*)$ with its isomorphic image in $(\text{Fr}(S^*), S^*)$, we can consider it as a sub d.g. near-ring of $(\text{Fr}(S^*), S^*)$.

Now let $A$ be the ideal of $(\text{Fr}(S^*), S^*)$ generated by $\{ A_{\lambda} : \lambda \in \Lambda \}$.

Then we prove the following theorem on similar lines to 1.1 [12].

**Theorem 2.2.1.** $\{ a_{\lambda} : (R_{\lambda}, S_{\lambda}) \rightarrow (\text{Fr}(S^*), S^*) / A \}_{\lambda \in \Lambda}$ is the coproduct of $\{ (R_{\lambda}, S_{\lambda}) : \lambda \in \Lambda \}$ in $B$. 

**Proof.** For each $\lambda \in \Lambda$, $A \cap F_{\lambda}$ is an ideal of $(\text{Fr}(S^*), S^*)$ containing $A_{\lambda}$. So there exists a unique d.g. near-ring homomorphism $\beta_{\lambda} : F_{\lambda} / A_{\lambda} \rightarrow F_{\lambda} / A \cap F_{\lambda}$ such that the diagram

$$
\begin{array}{ccc}
F_{\lambda} & \xrightarrow{\pi_{\lambda}} & F_{\lambda} / A_{\lambda} \\
\gamma_{\lambda} \downarrow & & \downarrow \beta_{\lambda} \\
F_{\lambda} / A \cap F_{\lambda} & & \\
\end{array}
$$

where $\pi_{\lambda}$ and $\gamma_{\lambda}$ are natural homomorphisms. Commutes for each $\lambda \in \Lambda$. By the isomorphism theorem for d.g. near-rings we get $F_{\lambda} / A \cap F_{\lambda} \cong F_{\lambda} + A / A$, for each $\lambda \in \Lambda$. Thus we get a d.g. near-ring homomorphism $a_{\lambda} : (R_{\lambda}, S_{\lambda}) \rightarrow (\text{Fr}(S^*), S^*) / A$ for each $\lambda \in \Lambda$, which is a composition of the following previously defined homomorphisms

$$(R_{\lambda}, S_{\lambda}) \rightarrow F_{\lambda} / A_{\lambda} \rightarrow F_{\lambda} / A \cap F_{\lambda} \rightarrow F_{\lambda} + A / A \rightarrow F / A,$$

where $F / A = (\text{Fr}(S^*), S^*) / A$. For each $\lambda \in \Lambda$, $a_{\lambda}$ maps
Let \((T,U)\) be any d.g. near-ring in \(\mathcal{H}\) together with a family \(\{ \phi_\lambda : (R_\lambda, S_\lambda) \rightarrow (T,U) \}_{\lambda \in \Lambda} \) of d.g. near-ring homomorphisms. This gives us a family \(\{ u_\lambda : S_\lambda \rightarrow U \}_{\lambda \in \Lambda} \) of semigroup homomorphisms. Since \(u_\lambda : S_\lambda \rightarrow S^*\) is the free product of \(\{ S_\lambda : \lambda \in \Lambda \}\), there exists a unique semigroup homomorphism \(\theta : S^* \rightarrow U\) such that \(u_\lambda \theta = \phi_\lambda\) for each \(\lambda \in \Lambda\). Again by the property of free groups \(\theta\) extends to a group homomorphism \((\text{Fr}(S^*), +) \rightarrow (T, +)\), again denoted by \(\theta\). Hence \(\theta\) is a d.g. near-ring homomorphism from \((\text{Fr}(S^*), S^*)\) to \((T, U)\). Now for each \(\lambda \in \Lambda\)

\[
(\text{Fr}(S_\lambda), +) \theta = \text{Gp}[S_\lambda \theta],
\]

\[
= \text{Gp}[S_\lambda \phi_\lambda].
\]

Therefore \((\text{Fr}(S_\lambda), +) \theta = (R_\lambda, S_\lambda) \theta = (S_\lambda, +) \phi_\lambda\) for each \(\lambda \in \Lambda\).

But \((R_\lambda, S_\lambda) \cong (\text{Fr}(S_\lambda), S_\lambda) / A_\lambda = F_\lambda / A_\lambda\). This shows that \(A_\lambda \subseteq \text{Ker} \theta\) for each \(\lambda \in \Lambda\), whence \(A \subseteq \text{Ker} \theta\). Thus we get a unique d.g. near-ring homomorphism \(\phi : (\text{Fr}(S^*), S^*) / A \rightarrow (T, U)\) such that \(\pi \phi = \theta\), where \(\pi\) is the natural homomorphism from \((\text{Fr}(S^*), S^*)\) to \((\text{Fr}(S^*), S^*) / A\). Now we consider the following diagram.
For each \( \lambda \in \Lambda \) and \( s_\lambda \in S_\lambda \), we have from above

\[
s_\lambda a_\lambda \phi = (s_\lambda + A) \phi = s_\lambda \pi \phi = s_\lambda \theta = s_\lambda \phi_\lambda.
\]

This shows that \( a_\lambda \phi \) and \( \phi_\lambda \) agree on the generators of \( (R_\lambda, S_\lambda) \), so they do so on \( (R_\lambda, S_\lambda) \). Hence \( a_\lambda \phi = \phi_\lambda \) for each \( \lambda \in \Lambda \), i.e.

the above diagram is commutative for each \( \lambda \in \Lambda \). Moreover we show that \( \phi \) is unique with this property. For that, let \( \psi \) be another d.g. near-ring homomorphism from \( (\text{Fr}(S^e), S^e) / \Lambda \) to \( (T, U) \) such that \( a_\lambda \psi = \phi_\lambda \) for each \( \lambda \in \Lambda \). Then

\[
s_\lambda a_\lambda \psi = s_\lambda \phi_\lambda = s_\lambda a_\lambda \phi
\]

or

\[
(s_\lambda + A) \psi = (s_\lambda + A) \phi.
\]

Since the set \( \{ s_\lambda + A : s_\lambda \in S_\lambda, \lambda \in \Lambda \} \) generates \( (\text{Fr}(S^e), S^e) / \Lambda \) and \( \psi \) and \( \phi \) agree on this set, therefore they agree on \( (\text{Fr}(S^e), S^e) / \Lambda \). Hence \( \psi = \phi \).

To complete the proof we need to show that \( a_\lambda \), for each \( \lambda \in \Lambda \), is a monomorphism. For that let \( (T, U) = (R_\mu, S_\mu) \) for \( \mu \in \Lambda \) and
\[ \phi_\lambda = \delta_\lambda \mu, \text{ where } \delta_\lambda \mu = 1 \text{ if } \lambda = \mu \\
= 0 \text{ if } \lambda \neq \mu. \]

Therefore we get \( a_\lambda \phi = \delta_\lambda \mu \), for each \( \lambda \in \Lambda \), and hence

\[ a_\mu \phi = I(R_\lambda, S_\lambda) \]

for each \( \lambda \in \Lambda \). Therefore by 1.1.6 \( a_\mu \) is a monomorphism for each \( \mu \in \Lambda \).

Now we prove the existence of general colimits in \( \mathcal{A} \). Let \( D \) be a diagram in \( \mathcal{A} \) over a scheme \( (M,A,d) \). By theorem 2.2.1 the coproduct \( * (R_\lambda, S_\lambda) \) of the family \( \{ (R_\lambda, S_\lambda) : \lambda \in \Lambda \} \) of d.g. near-rings involved in \( D \) exists in \( \mathcal{A} \) with the family

\[ \{ a_\lambda : (R_\lambda, S_\lambda) \rightarrow * (R_\lambda, S_\lambda) \} _{\lambda \in \Lambda} \]

of d.g. near-ring monomorphisms. Let \( K \) be the ideal of \( * (R_\lambda, S_\lambda) \) generated by

\[ \cup \{ \text{Image}(a_\lambda - m_\lambda \mu a_\mu) : m \in M \} \].

Then we get a quotient d.g. near-ring \( * (R_\lambda, S_\lambda)/K \) with natural homomorphism \( \pi \) from \( * (R_\lambda, S_\lambda) \) to \( * (R_\lambda, S_\lambda)/K \). We have

\[ \text{Image}(a_\lambda \pi - m_\lambda \mu a_\mu \pi) = [\text{Image}(a_\lambda - m_\lambda \mu a_\mu)] \pi, \]

\[ = 0. \]

Therefore the family \( \{ a_\lambda \pi : (R_\lambda, S_\lambda) \rightarrow * (R_\lambda, S_\lambda)/K \} _{\lambda \in \Lambda} \)

is cocompatible for \( D \), i.e. for each \( m \in M \) we have a commutative diagram
Then we prove the following theorem:

**Theorem 2.2.2.** \{ \alpha_\lambda \pi : (R_\lambda, S_\lambda) \to (R_\lambda, S_\lambda) \} \quad \lambda \in \Lambda

is a colimit of the diagram \mathcal{D} in \mathcal{A} over a scheme \((A, M, d)\).

**Proof.** We have seen that \{ \alpha_\lambda : \lambda \in \Lambda \} is a cocompatible family of d.g. near-ring homomorphisms for \mathcal{D} \}. Let \((T, U)\) be a d.g. near-ring in \mathcal{A} with a cocompatible family

\{ f_\lambda : (R_\lambda, S_\lambda) \to (T, U) \} \quad \lambda \in \Lambda

of d.g. near-ring homomorphisms for \mathcal{D}. By the property of coproducts there exists a unique d.g. near-ring homomorphism \(\phi : \ast (R_\lambda, S_\lambda) \to (T, U)\), \lambda \in \Lambda

such that for each \(\lambda \in \Lambda\) we have a commutative diagram
Now for each $m \in M$, we have

$$\{ \text{Image}(a_\lambda - m \lambda \mu a_\mu) \} \phi = \text{Image}(a_\lambda \phi - m \lambda \mu a_\mu \phi)$$

$$= \text{Image}(f_\lambda - m \lambda \mu f_\mu)$$

By the compatibility of $\{ f_\lambda : \lambda \in \Lambda \}$ for $D$,

$$\text{Image}(f_\lambda - m \lambda \mu f_\mu) = 0.$$  

Therefore $\text{Image}(a_\lambda - m \lambda \mu a_\mu) \subseteq \text{Ker } \phi$, for each $m \in M$, and hence $K \subseteq \text{Ker } \phi$. So there exists a unique d.g. near-ring homomorphism $\psi : \ast (R_\lambda S_\lambda)/K \longrightarrow (T, U)$ such that we have a commutative diagram

```
\[
\begin{array}{ccc}
\ast (R_\lambda S_\lambda) & \xrightarrow{\pi} & \ast (R_\lambda S_\lambda)/K \\
\downarrow{\phi} & & \downarrow{\psi} \\
(T, U) & & (T, U)
\end{array}
\]
```

Now for each $\lambda \in \Lambda$, we have from above diagrams,

$$(a_\lambda \pi) \psi = a_\lambda (\pi \psi),$$

$$= a_\lambda \phi,$$

$$= f_\lambda.$$  

Finally, we prove that $\psi$ is unique with this property. For that let $\psi' : \ast (R_\lambda S_\lambda) \longrightarrow (T, U)$ be another d.g. near-ring homomorphism such that $(a_\lambda \pi) \psi' = f_\lambda$ for each $\lambda \in \Lambda$. Then from
the uniqueness of \( \phi \) with \( a_{\lambda} \phi = f_{\lambda} \) for each \( \lambda \in \Delta \), we get
\[
\pi \phi' = \phi
\]
and then from the uniqueness of \( \phi \) such that \( \pi \psi = \phi \), we get \( \psi' = \psi \). This completes the proof.

2.3. Limits in \( R \)

First we look for the products in \( R \). Let \( \{(R_{\lambda}, S_{\lambda}) : \lambda \in \Delta \} \)
be a family of d.g. near-rings in \( R \), and let \( Q = \prod_{\lambda \in \Delta} R_{\lambda} \) be the
cartesian product of the family \( \{ R_{\lambda} : \lambda \in \Delta \} \) considered as near-rings. Then \( Q \) is a near-ring which is not necessarily d.g. Let
\( S = \prod_{\lambda \in \Delta} S_{\lambda} \) be the cartesian product of the family \( \{ S_{\lambda} : \lambda \in \Delta \} \) of
semigroups. Then \( S \) is a sub-semigroup of \( Q \). Now for all \( (r_{\lambda})_{\lambda \in \Delta} \),
\( (r'_{\lambda})_{\lambda \in \Delta} \) belonging to \( Q \), and for all \( (s_{\lambda})_{\lambda \in \Delta} \) belonging to \( S \) we have
\[
\{(r_{\lambda})_{\lambda \in \Delta} + (r'_{\lambda})_{\lambda \in \Delta}\} (s_{\lambda})_{\lambda \in \Delta} = (r_{\lambda} + r'_{\lambda})_{\lambda \in \Delta} (s_{\lambda})_{\lambda \in \Delta},
\]
\[
= ((r_{\lambda} + r'_{\lambda}) s_{\lambda})_{\lambda \in \Delta},
\]
\[
= (r_{\lambda} s_{\lambda} + r'_{\lambda} s_{\lambda})_{\lambda \in \Delta},
\]
\[
= (r_{\lambda} s_{\lambda})_{\lambda \in \Delta} + (r'_{\lambda} s_{\lambda})_{\lambda \in \Delta},
\]
\[
= (r_{\lambda})_{\lambda \in \Delta} (s_{\lambda})_{\lambda \in \Delta} + (r'_{\lambda})_{\lambda \in \Delta} (s_{\lambda})_{\lambda \in \Delta}.
\]
This shows that the semigroup \( S \) is distributive in \( Q \). This \( S \)
generates a sub d.g. near-ring \( (R, S) \) of \( Q \). Let
\[
\{ p_{\lambda} : Q \rightarrow R_{\lambda} \}_{\lambda \in \Delta}
\]
be the near-ring projection homomorphisms.
Then for each $\lambda \in \Lambda$, $p_\lambda$ maps $S$ onto $S_\lambda$ and hence $(R,+)$ onto $(R_\lambda,+)$. Therefore, for each $\lambda \in \Lambda$, $q_\lambda = p_\lambda/(R,S)$ is a d.g. near-ring epimorphism onto $(R_\lambda,S_\lambda)$. Now we prove the following theorem:

**Theorem 2.3.1.** \{ $q_\lambda : (R,S) \longrightarrow (R_\lambda,S_\lambda)$ \}_{\lambda \in \Lambda} is the product of \{ $(R_\lambda,S_\lambda)$ : $\lambda \in \Lambda$ \} in $\mathcal{R}$.

**Proof.** Let $(T,U)$ be a d.g. near-ring in $\mathcal{R}$ with a family \{ $\phi_\lambda : (T,U) \longrightarrow (R_\lambda,S_\lambda)$ \}_{\lambda \in \Lambda} of d.g. near-ring homomorphisms. Then we have a family \{ $\phi_\lambda : U \longrightarrow S_\lambda$ \}_{\lambda \in \Lambda} of semigroup homomorphisms and since \{ $q_\lambda : S \longrightarrow S_\lambda$ \}_{\lambda \in \Lambda} is the product of \{ $S_\lambda : \lambda \in \Lambda$ \}, we get, for each $\lambda \in \Lambda$, a commutative diagram

![Diagram](https://via.placeholder.com/150)

for a uniquely defined semigroup homomorphism $\phi$. Now let

$$\epsilon u_1 + \cdots + \epsilon u_n = 0 \text{ in } (T,U).$$

Then, for each $\lambda \in \Lambda$,

$$\left( \epsilon u_1 + \cdots + \epsilon u_n \right) \phi_\lambda = 0 \text{ in } (R_\lambda,S_\lambda).$$

Therefore, for each $\lambda \in \Lambda$, we have in $(R_\lambda,S_\lambda)$,

$$0 = \epsilon u \phi_\lambda + \cdots + \epsilon u \phi_\lambda.$$
\[
\begin{align*}
\epsilon u \phi q_\lambda + \cdots + \epsilon u \phi q_\lambda &= (\epsilon u \phi + \cdots + \epsilon u \phi) q_\lambda.
\end{align*}
\]

Hence \( \epsilon u \phi + \cdots + \epsilon u \phi = 0 \) in \( Q \), and since

\( \epsilon u \phi + \cdots + \epsilon u \phi \) belongs to \( (R,S) \), \( \phi \) extends to a group homomorphism from \( (T,+') \) to \( (R,+') \). We denote this homomorphism also by \( \phi \). Thus \( \phi \) is a d.g. near-ring homomorphism from \( (T,U) \) to \( (R,S) \). Also we have \( \phi q_\lambda = \phi_\lambda \), for each \( \lambda \in \Lambda \), as d.g. near-ring homomorphisms. Moreover, the uniqueness of \( \phi \) as a semigroup homomorphism with \( \phi q_\lambda = \phi_\lambda \), for each \( \lambda \in \Lambda \), gives us its uniqueness as a d.g. near-ring homomorphism with the same property.

**Corollary 2.3.2.** If \( \{ q_\lambda : (R,S) \to (R_\lambda,S_\lambda) : \lambda \in \Lambda \} \) is the product in \( \mathcal{R} \) of \( \{ (R_\lambda,S_\lambda) : \lambda \in \Lambda \} \), then

\( \{ q_\lambda : S \to S_\lambda : \lambda \in \Lambda \} \) is the product of \( \{ S_\lambda : \lambda \in \Lambda \} \) in the category of all semigroups.

**Proof.** If \( \{ q_\lambda : (R,S) \to (R_\lambda,S_\lambda) : \lambda \in \Lambda \} \) is the product in \( \mathcal{R} \) of \( \{ (R_\lambda,S_\lambda) : \lambda \in \Lambda \} \) then we have a family

\( \{ q_\lambda : S \to S_\lambda : \lambda \in \Lambda \} \) of semigroup epimorphisms. Let \( U \) be a semigroup with a family \( \{ f_\lambda : U \to S_\lambda : \lambda \in \Lambda \} \) of semigroup homomorphisms. Then we get a family \( \{ \overline{f}_\lambda : (Fr(U),U) \to (R_\lambda,S_\lambda) : \lambda \in \Lambda \} \) of d.g. near-ring homomorphisms, where \( (Fr(U),U) \) is the free d.g. near-ring on \( U \) and \( \overline{f}_\lambda/U = f_\lambda \), for each \( \lambda \in \Lambda \). From the hypot-
thesis, there exists a unique d.g. near-ring homomorphism \( \phi \) from \((\mathcal{F}(U), U)\) to \((R, S)\) such that \( \phi q_\lambda = \overline{f_\lambda} \) for each \( \lambda \in \Lambda \). Obviously \( \phi q_\lambda = f_\lambda \) for each \( \lambda \in \Lambda \) as semigroup homomorphisms, and \( \phi \) from \( U \) to \( S \) is unique with this property. Hence \( \{ q_\lambda : S \rightarrow S_\lambda \}_\lambda \in \Lambda \) is the product of \( \{ S_\lambda : \lambda \in \Lambda \} \) in the category of all semigroups.

Before proving the existence of general limits in \( \mathcal{R} \), we give a result, from theory of categories, which we are going to use.

Theorem 2.3.3. [13] Let \( \mathcal{C} \) be a category with products and finite intersections and let \( D \) be a diagram in \( \mathcal{C} \) over a scheme \((\Lambda, M, d)\). Then a limit for \( D \) is given by the compositions

\[ \bigcap \{ \text{Eq}(p_\lambda m_D, p_\mu) : m \in M \} \subseteq \prod_{\lambda \in \Lambda} A_\lambda \xrightarrow{p_\lambda} A_\lambda, \]

where \( m d = (\lambda, \mu) \), \( m D : A_\lambda \rightarrow A_\mu \) and \( p_\lambda \) represents the \( \lambda \)-th projection.

Now let \( \mathcal{S} \) be the category of all semigroups with zero element. Then in \( \mathcal{S} \) any homomorphism \( \alpha : U \rightarrow V \) maps \( 0_U \) onto \( 0_V \).

Also \( \mathcal{S} \) is a category with intersections and products, since

\[ \cap \{ U_\lambda : \lambda \in \Lambda \}, U_\lambda \in \mathcal{S} \] always contains the zero element and if

\[ U = \prod_{\lambda \in \Lambda} U_\lambda \] then \( (0_U)_\lambda \) is the zero element of \( U \).

Let \( D \) be a diagram in \( \mathcal{R} \) over a scheme \((\Lambda, M, d)\). By theorem 2.3.1 the product \( \{ q_\lambda : (R, S) \rightarrow (R_\lambda, S_\lambda) \}_\lambda \in \Lambda \) of the
family \{ (R, S) : \lambda \in \Lambda \} of d.g. near-rings involved in \( D \), exists in \( A \). Without loss of generality we can assume that \( 0 \in S, \) for each \( \lambda \in \Lambda \), i.e. we can consider each \( S, \lambda \) in \( B \).

By corollary 2.3.2, \( \{ q, \lambda : S \rightarrow S, \lambda \} \) is the product of the family \( \{ S, \lambda : \lambda \in \Lambda \} \) of semigroups involved in the corresponding diagram \( D \) in \( B \). Let \( S' = \cap \{ \text{Equ}(q, \lambda, m, \mu), q, \mu : m \in M \} \subseteq S \).

Then \( S' \) is not empty since \( 0 \in S' \). Let \( q, \lambda^l = q, \lambda / S' \) for each \( \lambda \in \Lambda \). Then by theorem 2.3.3, \( \{ q, \lambda^l : S' \rightarrow S, \lambda \} \) is a limit for \( D \) in \( B \). \( S' \) generates a sub d.g. near-ring \( (R, S') \) of \( (R, S) \) and \( q, \lambda^l \) is a d.g. near-ring homomorphism from \( (R, S') \) to \( (R, S, \lambda) \) for each \( \lambda \in \Lambda \). Now we prove the following theorem:

**Theorem 2.3.4.** \( \{ q, \lambda : (R, S') \rightarrow (R, S, \lambda) \} \) is a limit for \( D \) in \( A \).

**Proof.** First we prove that the family \( \{ q, \lambda : \lambda \in \Lambda \} \) is compatible for \( D \), i.e., the following diagram is commutative for each \( m \in M \).

```
\begin{tikzcd}
(R, S') \arrow{r}{q, \lambda} \arrow{d}{q, \mu} & (R, S, \lambda) \arrow{d}{m, \lambda, \mu} \\
(R', S') & (R, S, \mu) \arrow{u}{q, \mu}
\end{tikzcd}
```
For all \( r \in (R', s') \) and for all \( m \in M \), we get from above:

\[
\begin{align*}
qr^i_m \mu &= (\epsilon s^i + \cdots + \epsilon s^n) qr^i_m \mu, \\
&= \epsilon s^i qr^i_m \mu + \cdots + \epsilon s^n qr^i_m \mu, \\
&= \epsilon s^i q^i \mu + \cdots + \epsilon s^n q^i \mu, \\
&= (\epsilon s^i + \cdots + \epsilon s^n) q^i \mu, \\
&= qr^i \mu,
\end{align*}
\]

i.e. \( qr^i_m \mu = q^i \mu \) for each \( m \in M \). Hence \( \{ q^i : \lambda \in \Lambda \} \) is compatible for \( D \) in \( A \).

Let \((T, U)\) be a d.g. near-ring with a family, compatible for \( D \), \( \{ \phi^i : (T, U) \longrightarrow (R^i, S^i) \} \) of d.g. near-ring homomorphisms. Then by the property of products there exists a unique d.g. near-ring homomorphism \( \phi : (T, U) \longrightarrow (R, S) \) such that \( \phi q^i = \phi^i \) for each \( \lambda \in \Lambda \). Since \( \{ q^i : S^i \longrightarrow S^i \} \lambda \in \Lambda \) is a limit in \( \mathcal{S} \) for \( D \), \( \phi \) factors uniquely through \( S^i \). Therefore \( \phi \) maps \( U \) into \( S^i \). Hence \( \phi \) maps \( (T, +) \) into \( (R', +) \) so that \( \phi \) factors through \( (R', S') \) as a d.g. near-ring homomorphism. Clearly this factorisation is unique. Hence we have proved the theorem.
Chapter 3

Surjective Reflections

Introduction

In some cases, proving the existence of limits, in a subcategory, may be easier than proving this in the bigger category; for example, the existence of products in $\mathcal{B}$, the category of faithful d.g. near-rings, can be proved in the following elementary way:

Let $\{ (R_\lambda, S_\lambda) : \lambda \in \Lambda \}$ be a family of faithful d.g. near-rings. Let $G_\lambda$ be a faithful $(R_\lambda, S_\lambda)$-group, for each $\lambda \in \Lambda$. Then we can show that $S = \prod_{\lambda \in \Lambda} S_\lambda$ is a semigroup of endomorphisms of the group $G = \prod_{\lambda \in \Lambda} G_\lambda$. This semigroup $S$ generates a d.g. near-ring $(R, S)$ inside $E(G)$. Each semigroup projection homomorphism $p_\lambda : S \to S_\lambda$ extends to a group homomorphism $p_\lambda : (R, +) \to (R_\lambda, +)$. It is easy to see that

$\{ p_\lambda : (R, S) \to (R_\lambda, S_\lambda) : \lambda \in \Lambda \}$ is the product in $\mathcal{B}$ of

$\{ (R_\lambda, S_\lambda) : \lambda \in \Lambda \}$.

As mentioned before, with the help of products and general limits in $\mathcal{B}$, we were able to prove the existence of products and limits in $\mathcal{C}$. The reason for this is that each reflection homomorphism in this case is surjective. In this chapter we generalize this method for all algebraic categories satisfying a certain condition, which holds in the case of d.g. near-rings.
3.1. Surjective Reflections

Let \( \mathcal{C} \) be an algebraic category. Let \( \{ a_\lambda : C \to C_\lambda \}_{\lambda \in \Lambda} \) be a set of morphisms in \( \mathcal{C} \) and let

\[
\rho_\lambda = \{ (x, y) \in C \times C : x a_\lambda = y a_\lambda \}
\]

denote the kernel of \( a_\lambda, \lambda \in \Lambda \). Suppose further that \( p = \cap \{ \rho_\lambda : \lambda \in \Lambda \} \).

Note that \( p \neq \emptyset \), since \( (c, c) \in p \) for each \( c \in C \) and each \( \lambda \in \Lambda \).

Let \( \mathcal{D} \) be a reflective subcategory of \( \mathcal{C} \) such that each reflection homomorphism \( \theta_\mathcal{D} : C \to C \) is surjective, \( T \) being the reflector functor. Suppose that for a diagram \( D \) in \( \mathcal{D} \) over a scheme \( (\Lambda, \lambda, d) \) a limit exists in \( \mathcal{D} \) and consider the same diagram in \( \mathcal{C} \). Let \( \{ C_\lambda : \lambda \in \Lambda \} \) be the family of objects of \( \mathcal{C} \) involved in \( D \). Then we have the corresponding diagram \( D \) in \( \mathcal{D} \), involving the family \( \{ C_\lambda T : \lambda \in \Lambda \} \) of objects of \( \mathcal{D} \), and for each \( m \in M \), the following diagram is commutative,
where $\theta_\lambda$'s are reflection homomorphisms.

Let $\{ p_\lambda : C \longrightarrow C_\lambda \} \lambda \in \Lambda$ be a limit of the diagram $D$ in $\mathcal{C}$. Put $q_\lambda = p_\lambda \theta_\lambda$ for each $\lambda \in \Lambda$. Then

$$\sigma_\lambda = \{(x, y) \in C \times C : x q_\lambda = y q_\lambda \}$$

is the kernel of $q_\lambda$ for each $\lambda \in \Lambda$. Let $\sigma = \cap \{ \sigma_\lambda : \lambda \in \Lambda \}$. From above $\sigma^\cap : C \longrightarrow C/\sigma$ and each induced homomorphism $\phi_\lambda : C/\sigma \longrightarrow C_\lambda$ are in $\mathcal{C}$ and $\sigma^\cap \phi_\lambda = q_\lambda, \lambda \in \Lambda$. We claim that $\{ \phi_\lambda : C/\sigma \longrightarrow C_\lambda \} \lambda \in \Lambda$ is a limit of $D$ in $\mathcal{C}$.

First we show that this family is compatible for $D$. For each $m \in M$ we have from above:

$$\sigma^\cap \phi_\lambda m_\mu = q_\lambda m_\mu = p_\lambda \theta_\lambda m_\mu = p_\lambda (m_\mu T) \theta_\mu = p_\mu \theta_\mu = q_\mu$$

Since $\sigma^\cap$ is a surjection, we get $\phi_\lambda m_\mu = \phi_\mu$, for all $m \in M$.

Thus $\{ \phi_\lambda : C/\sigma \longrightarrow C_\lambda \} \lambda \in \Lambda$ is compatible for $D$.

Now let $\{ f_\lambda : A \longrightarrow C_\lambda \} \lambda \in \Lambda$ be any compatible family for $D$, in $\mathcal{C}$. Then we get a family $\{ f_\lambda T : A T \longrightarrow C_\lambda T \} \lambda \in \Lambda$ in $\mathcal{D}$, which is compatible for $D$, because for all $m \in M$ we have:

$$(f_\lambda T)(m_\mu T) = (f_\lambda m_\mu T) = f_\mu T.$$

By the property of limits there exists a unique homomorphism

$\phi : A T \longrightarrow C$ such that $\phi p_\lambda = f_\lambda T$, for each $\lambda \in \Lambda$. Thus we get the following commutative diagram.
where \( \theta : AT \to A \) is the reflection homomorphism for \( A \in \mathcal{C} \).

Now let \( \rho = \{(a, b) \in AT \times AT : a\theta = b\theta \} \). Then \( \rho \) is a congruence on \( AT \) which is the kernel of \( \theta \). Therefore, for each \( (a, b) \in \rho \) and for each \( \lambda \in \Lambda \), we have from the above diagram:

\[
\begin{align*}
    a \theta f_\lambda &= b \theta f_\lambda , \\
    a (f_\lambda T) \theta_\lambda &= b (f_\lambda T) \theta_\lambda , \\
    a \phi p_\lambda \theta_\lambda &= b \phi p_\lambda \theta_\lambda , \\
    a \phi q_\lambda &= b \phi q_\lambda .
\end{align*}
\]

This shows that \( (a, b) \in \rho \). So there exists a unique homomorphism \( \psi : A \to C/\sigma \) such that \( \theta \psi = \phi \sigma^H \), since \( \theta \) is a surjection. Now, for each \( \lambda \in \Lambda \), again from the above diagram we get:

\[
\theta \psi \phi_\lambda = \phi \sigma^H \phi_\lambda = \phi p_\lambda \theta_\lambda = (f_\lambda T) \theta_\lambda = \theta f_\lambda .
\]

Again as \( \theta \) is a surjection, we get from here \( \psi \phi_\lambda = f_\lambda \), for each \( \lambda \in \Lambda \).
Next we show that $\psi$ is unique with this property. We first show that for $c, d \in C/\sigma$, $c \phi_\lambda = d \phi_\lambda$ for each $\lambda \in \Lambda$, implies that $c = d$. There exist $c', d' \in C$ such that $c' \sigma^\lambda = c$ and $d' \sigma^\lambda = d$. This implies that

$$(c' \sigma^\lambda = d' \sigma^\lambda, \text{ for each } \lambda \in \Lambda),$$

$$(c' q_\lambda = d' q_\lambda, \text{ for each } \lambda \in \Lambda),$$

$$((c', d') \in \sigma),$$

$$c = c' \sigma = d'. \sigma = d. \quad (I)$$

Now let $\psi' : A \rightarrow C/\sigma$ be a homomorphism in $\mathcal{C}$ such that $\psi' \phi_\lambda = f_\lambda$ for each $\lambda \in \Lambda$. Then for all $a \in A$, we have

$$a \psi' \phi_\lambda = a f_\lambda = \psi \phi_\lambda,$$

for each $\lambda \in \Lambda$. Therefore from (I) we get a $\psi' = \psi$ for all $a \in A$. Hence $\psi' = \psi$. Thus we have proved the following theorem:

Theorem 3.1.1. Let $\mathcal{C}$ be an algebraic category with the following property: "For every family $\{ \alpha_\lambda : A \rightarrow A_\lambda \mid \lambda \in \Lambda \}$ of homomorphisms in $\mathcal{C}$, the homomorphism $\rho : A \rightarrow A/\rho$ and the induced homomorphisms $\overline{\alpha}_\lambda : A/\rho \rightarrow A \alpha_\lambda, \lambda \in \Lambda$, are in $\mathcal{C}$, where $\rho = \cap \{ \rho_\lambda : \lambda \in \Lambda \}$ and $\rho_\lambda$ is the kernel of $\alpha_\lambda$ for each $\lambda \in \Lambda"$. Let $\mathcal{D}$ be a reflective subcategory of $\mathcal{C}$ such that each reflection homomorphism $\theta : D \rightarrow \mathcal{C}$ is surjective, where $T : \mathcal{C} \rightarrow \mathcal{D}$ is the reflector functor. If for a diagram $D$ in $\mathcal{D}$ over a scheme $(\Lambda, M, d)$ a limit exists
in \( \mathcal{D} \), then for same diagram in \( \mathcal{C} \) a limit exists in \( \mathcal{C} \).

We prove the dual of the above result in the following form, using definitions 1.1.12 and 1.1.13.

**Theorem 3.1.2.** Let \( \mathcal{C} \) be an algebraic category with unions and surjective images. Let \( \mathcal{D} \) be a coreflective subcategory of \( \mathcal{C} \) such that each coreflection \( \theta_C : C \rightarrow CT \) is a monomorphism onto \( \text{Im} \theta_C \) and \( \theta_C^{-1} : C \theta_C \rightarrow C \) is in \( \mathcal{C} \), where \( \theta_C : \mathcal{C} \rightarrow \mathcal{D} \) is the coreflector functor and \( \theta_C \theta_C^{-1} = I_C \). If for a diagram \( D \) in \( \mathcal{D} \) a colimit exists in \( \mathcal{D} \), then for the same diagram in \( \mathcal{C} \) a colimit exists in \( \mathcal{C} \).

**Proof.** Let \( \{ C_\lambda : \lambda \in \Lambda \} \) be the family of objects involved in the diagram \( D \) in \( \mathcal{C} \). Then we have the corresponding diagram in \( \mathcal{D} \) involving the family \( \{ C_\lambda T : \lambda \in \Lambda \} \) of objects such that for each \( m \in M \) the following diagram is commutative,

\[
\begin{array}{ccc}
C_\lambda & \xrightarrow{\theta_\lambda} & C_\lambda T \\
m_\lambda \mu \downarrow & & \downarrow m_\lambda \mu T \\
C_\mu & \xrightarrow{\theta_\mu} & C_\mu T
\end{array}
\]

where \( \theta_\lambda : C_\lambda \rightarrow C_\lambda T \), for each \( \lambda \in \Lambda \), is the coreflection in \( \mathcal{D} \) of \( C_\lambda \in \mathcal{C} \).

Let \( \{ \alpha_\lambda : C_\lambda T \rightarrow C \}_\lambda \in \Lambda \) be a colimit of \( D \) in \( \mathcal{D} \).

Then \( C' = \bigcup \{ \text{Im} \theta_\lambda \alpha_\lambda : \lambda \in \Lambda \} \) is a subobject of \( C \). Now we have
Clearly \( \phi_{\lambda} \cdot i_{\lambda} = \theta_{\lambda} \alpha_{\lambda} \), where \( i_{\lambda} : C' \rightarrow C \) is the inclusion morphism. We claim that \( \{ \phi_{\lambda} : C_{\lambda} \rightarrow C' \}_{\lambda \in \Lambda} \) is a colimit of \( D \) in \( C \). For each \( m \in M \) we have from above:

\[
m_{\lambda}(\phi_{\mu} \cdot i_{\mu}) = m_{\lambda} \theta_{\lambda} \alpha_{\mu} = \theta_{\lambda}(m_{\lambda} \alpha_{T}) \alpha_{\mu} = \theta_{\lambda} \alpha_{\lambda} = \phi_{\lambda} \cdot i_{\lambda}.
\]

Since \( i_{\lambda} \) is a monomorphism we get

\[
m_{\lambda}(\phi_{\mu} \cdot i_{\mu}) = \phi_{\lambda} \cdot i_{\lambda}
\]

for each \( m \in M \). So \( \{ \phi_{\lambda} : C_{\lambda} \rightarrow C' \}_{\lambda \in \Lambda} \) is cocompatible for \( D \). Now let \( \{ f_{\lambda} : C_{\lambda} \rightarrow B \}_{\lambda \in \Lambda} \) be a cocompatible family for \( D \). Then, since \( \{ f_{\lambda T} : C_{\lambda T} \rightarrow BT \}_{\lambda \in \Lambda} \) is cocompatible for \( D \) in \( \mathcal{D} \), we get a commutative diagram:

\[
\begin{array}{cccccc}
C_{\lambda} & \xrightarrow{f_{\lambda}} & B \\
\downarrow{f_{\lambda}} & & \downarrow{f_{\lambda T}} \\
C_{\lambda T} & \xrightarrow{\phi} & BT
\end{array}
\]

for a uniquely defined morphism \( \phi : C_{\lambda} \rightarrow BT \).

Let \( \psi = \phi / C' \), i.e. \( \psi = i_{C'} \phi \). Then for each \( \lambda \in \Lambda \) we have:

\[
\phi_{\lambda} \cdot i_{\lambda} \psi = \phi_{\lambda} \cdot i_{\lambda} \cdot i_{C'} \phi = f_{\lambda T} \theta.
\]

So \( (\text{Im} \theta_{\lambda} \alpha_{\lambda}) \cdot i_{\lambda} \phi \subseteq B \theta \), for each \( \lambda \in \Lambda \), since images are surjective. By definition of \( C' \), \( (C') \cdot i_{C'} \phi \subseteq B \theta \). Then
is defined and we get
\[
(\psi_\lambda \cdot \lambda)^{-1}(\psi \cdot \lambda^{-1}) = \psi_\lambda \cdot \lambda (\iota_C \cdot \phi) = \theta_\lambda \cdot \lambda \cdot \phi \cdot \phi^{-1} = \theta_\lambda (f_\lambda \cdot \theta)^{-1} \\
= f_\lambda \cdot \theta \cdot \phi^{-1} = f_\lambda,
\]
for each \( \lambda \in \Lambda \). Moreover it can be easily seen that \( \psi \cdot \lambda^{-1} \) is unique with this property. This completes the proof.

Now we give an abstract version of theorem 3.1.1.

Let \( \mathcal{C} \) be a category with kernels, cokernels and intersection of kernels. Let \( \mathcal{D} \) be a reflective subcategory of \( \mathcal{C} \) such that each reflection morphism \( \theta_C : CT \rightarrow C \) is the cokernel of its kernel. Suppose that for a diagram \( D \) in \( \mathcal{D} \) a limit exists in \( \mathcal{D} \) and consider the same diagram in \( \mathcal{C} \). Let \( \{ C_\lambda : \lambda \in \Lambda \} \) be the family of objects of \( \mathcal{C} \) involved in \( D \). Then we get the corresponding diagram \( D \) in \( \mathcal{D} \), involving the family \( \{ C_\lambda : \lambda \in \Lambda \} \) of objects of \( \mathcal{C} \), such that for each \( m \in M \) we have the following commutative diagram:

\[
\begin{array}{ccc}
C_\lambda T & \xrightarrow{\theta_\lambda} & C_\lambda \\
\downarrow m_{\lambda \mu} T & & \downarrow m_{\lambda \mu} \\
C_\mu T & \xrightarrow{\theta_\mu} & C_\mu
\end{array}
\]

Let \( \{ p_\lambda : C \rightarrow C_\lambda T \}_\lambda \in \Lambda \) be a limit of \( D \) in \( \mathcal{D} \). Let \( u_\lambda : X_\lambda \rightarrow C \) be the kernel of \( q_\lambda = p_\lambda \cdot \theta_\lambda : C \rightarrow C_\lambda \), for
each $\lambda \in \Lambda$. Then $u : X = \cap \{ X_\lambda : \lambda \in \Lambda \} \to C$ exists.

Let $\pi : C \to L$ be the cokernel of $u : X \to C$. Now we have $u q_\lambda = 0$, for each $\lambda \in \Lambda$. Therefore by the definition of cokernels there exists a unique morphism $\phi_\lambda : L \to C_\lambda$, for each $\lambda \in \Lambda$, such that $\pi \phi_\lambda = q_\lambda$ and we get a commutative diagram:

$$
\begin{array}{ccc}
X_\lambda & \xrightarrow{u_\lambda} & L \\
\uparrow{\gamma_\lambda} & & \uparrow{\pi} \\
C & \xrightarrow{q_\lambda} & C_\lambda \\
\downarrow{u} & & \downarrow{\phi_\lambda} \\
X & \xrightarrow{\gamma} & C_\lambda
\end{array}
$$

where $\gamma_\lambda : X \to X_\lambda$ is a unique morphism such that $\gamma_\lambda u_\lambda = u$.

So we get a family $\{ \phi_\lambda : L \to C_\lambda \}_{\lambda \in \Lambda}$ of morphisms in $\mathcal{C}$, which we now show to be compatible for $D$ as follows:

$$
\pi \phi_\lambda m_\lambda \mu = q_\lambda m_\lambda \mu = p_\lambda \theta_\lambda m_\lambda \mu = p_\lambda (m_\lambda \mu T) \theta_\mu = p_\mu \theta_\mu = q_\mu = \pi \phi_\mu.
$$

But $\pi : C \to L$ is an epimorphism, by 1.1.15. So we get:

$$
\phi_\lambda m_\lambda \mu = \phi_\mu, \text{ for each } m \in M.
$$

Now let $\{ f_\lambda : B \to C_\lambda \}_{\lambda \in \Lambda}$ be a compatible family for $D$ in $\mathcal{C}$. Then we get a compatible family $\{ f_\lambda T : BT \to C_\lambda T \}_{\lambda \in \Lambda}$ for $D$ in $\mathcal{D}$. Let $\nu : K \to BT$ be the kernel of $\theta : BT \to B$. Then, for each $\lambda \in \Lambda$, we get the following commutative diagram:
Therefore for each \( \lambda \in \Lambda \), we get:

\[
\nu \phi q_\lambda = \nu \phi p_\lambda \theta_\lambda = \nu (f_\lambda T) \theta_\lambda = \nu \theta f_\lambda = 0 f_\lambda = 0.
\]

Hence there exists a unique morphism \( a_\lambda : K \rightarrow X_\lambda \), for each \( \lambda \in \Lambda \), such that \( a_\lambda u_\lambda = \nu \phi \). Now by the definition of intersections, there exists a unique morphism \( a : K \rightarrow X \) such that \( a y_\lambda = a_\lambda \), for each \( \lambda \in \Lambda \). Therefore we get:

\[
\nu \phi \pi = a_\lambda u_\lambda \pi = a y_\lambda u_\lambda \pi = a u \pi = a 0 = 0.
\]

Since \( \theta : BT \rightarrow B \) is the cokernel of its kernel, there exists a unique morphism \( \psi : B \rightarrow L \) such that \( \theta \psi = \phi \pi \). Hence, for each \( \lambda \in \Lambda \), we get

\[
\theta \phi \phi_\lambda = \phi \pi \phi_\lambda = \phi q_\lambda = \theta f_\lambda.
\]

By 1.1.15, \( \theta \) is an epimorphism. So we get \( \psi \phi_\lambda = f_\lambda \), for each \( \lambda \in \Lambda \). Clearly \( \psi \) is unique with this property.

Example. Let \( \mathcal{C} \) be the category of all finitely generated abelian groups and \( \mathcal{D} \) be the subcategory of \( \mathcal{C} \), of all finitely
generated torsion free abelian groups. Then \( \mathcal{C} \) and \( \mathcal{D} \) satisfy the hypotheses of the above abstract result.

We leave the statement of the dual of this abstract result, as one can state it in analogy with theorem 3.1.2.
Chapter 4

Some Functors

Throughout this chapter we will use the following notation.

\( \mathcal{A} \) = the category of all d.g. near-rings.
\( \mathcal{B} \) = the category of all faithful d.g. near-rings.
\( \mathcal{S} \) = the category of all sets.
\( \mathcal{G} \) = the category of all semigroups.
\( \mathcal{G} \) = the category of all groups.
\( \mathcal{G}_{(R,S)} \) = the category of all \((R,S)\)-groups for a d.g. near-ring \((R,S)\).

### 4.1. Functor: \( \mathcal{A} \times \mathcal{S} \to \mathcal{G} \)

First of all we give a result which we are going to use in this section.

**Theorem 4.1.1.** [12] Let \( \text{Fr}(X,R,S) \) be the free \((R,S)\)-group on the set \( X \). Let \( X \) be the disjoint union of the subsets \( \{ X_\lambda : \lambda \in \Lambda \} \). Then \( \text{Fr}(X,R,S) \cong \bigast_{\lambda \in \Lambda} (R,S) \text{Fr}(X_\lambda,R,S) \), the free \((R,S)\)-product of the family \( \{ \text{Fr}(X_\lambda,R,S) : \lambda \in \Lambda \} \) of subgroups of \( \text{Fr}(X,R,S) \).

**Remark 4.1.2.** If \((R,S) \in \mathcal{B}\), then by 4.1.1. we have
\[
\text{Fr}(X,R,S) \cong \bigast_{x \in X} (R,S) \text{Fr}(x,R,S),
\]

\[
\cong \bigast_{x \in X} (R,S) \{ G_p[x] \ast R_x \}, \text{ by 1.6.12.}
\]

Therefore \( \text{Fr}(X,R,S) \) is a group generated by the set.
[x, r_x : x ∈ X, r ∈ (R,S)].

Now we prove a few technical lemmas.

Lemma 4.1.3. Let \( h : (R,S) \rightarrow (T,U) \) be a homomorphism in \( \mathcal{B} \). Then there exists an \((R,S)\)-homomorphism

\[
\bar{h} : Fr(X,R,S) \rightarrow Fr(X,T,U),
\]

which maps \( r_x \) to \( (rh)_x \) and \( x \) to \( x \) for all \( x \in X, r \in (R,S) \), \( X \) being any set.

Proof. By lemma 2.1.2 \( Fr(X,T,U) \) is an \((R,S)\)-group. Since \( Fr(X,R,S) \) is the free \((R,S)\)-group on \( X \), there exists a unique \((R,S)\)-homomorphism \( \bar{h} : Fr(X,R,S) \rightarrow Fr(X,T,U) \) extending the identity map on \( X \) (1.6.6). Clearly \( x \bar{h} = x \), for all \( x \in X \).

Now \( r_x \bar{h} = (xr) \bar{h} = (x \bar{h})(rh) = x(rh) = (rh)_x \).

Remark. The identity \( d.g. \) near-ring homomorphism \( I_{(R,S)} \) gives us the identity \((R,S)\)-homomorphism

\[
I_{Fr(X,R,S)} : Fr(X,R,S) \rightarrow Fr(X,R,S),
\]

for every set \( X \).

Lemma 4.1.4. Let \( \alpha : X \rightarrow Y \) be a mapping in \( \mathcal{B} \), and let \( (R,S) \in \mathcal{B} \). Then there exists an \((R,S)\)-homomorphism

\[
\alpha : Fr(X,R,S) \rightarrow Fr(Y,R,S)
\]
such that

\[
\alpha / X = \alpha \quad \text{and} \quad r_x \alpha = r_x \alpha .
\]
Proof. Let \( i_Y \) be the inclusion map

\[ Y \longrightarrow \text{Fr}(Y,R,S). \]

Then \( a i_Y \) is a mapping from \( X \) to \( \text{Fr}(Y,R,S) \). So there exists a unique \((R,S)\)-homomorphism

\[ a : \text{Fr}(X,R,S) \longrightarrow \text{Fr}(Y,R,S) \]

extending the map \( a i_Y \). Therefore \( a / X = a i_Y = a \).

Now \((r_x)a = (xr)a = (xa)r = (x a)r = r_x a\).

Remark. \( i_X: X \longrightarrow X \) gives rise to the identity \((R,S)\)-homomorphism \( i_{\text{Fr}(X,R,S)}: \text{Fr}(X,R,S) \longrightarrow \text{Fr}(X,R,S) \).

Lemma 4.1.5. For \((R,S) \in \mathcal{C} \) and \( X \in \mathcal{S} \), the groups \( \text{Fr}(X,R,S) \) and \( \text{Fr}(X,R,S) \) are isomorphic.

Proof. Let \( \phi, \mu \) be the representations of \( \text{Fr}(X,R,S) \) and \( \text{Fr}(X,R,S) \) as \((R,S)\)-and \((R,S)\)-groups respectively. Let \( \theta \) be the natural homomorphism from \((R,S)\) to \((R,S) = (R,S)/A\), where \( A \) is as defined in lemma 2.1.3. Then by the same lemma, \( \text{Ker} \phi \supseteq A \).

By lemma 2.1.2, \( \text{Fr}(X,R,S) \) is an \((R,S)\)-group with representation \( \theta \mu \). Since \( \text{Fr}(X,R,S) \) is the free \((R,S)\)-group on \( X \), there exists a unique \((R,S)\)-homomorphism

\[ \psi : \text{Fr}(X,R,S) \longrightarrow \text{Fr}(X,R,S) \]

extending the identity map

\[ i_X: X \subseteq \text{Fr}(X,R,S) \longrightarrow X \subseteq \text{Fr}(X,R,S). \]
Obviously $\psi$ is an epimorphism.

Now $r \in \text{Ker } \phi$ implies that $g(r \phi) = 0$ in $\text{Fr}(X, R, S)$ for each $g \in \text{Fr}(X, R, S)$, and therefore $g(r \phi) \psi = 0$ in $\text{Fr}(X, R, S)$.

So $0 = g(r \phi) \psi = (g \psi) \epsilon \theta \mu = h(\varphi \theta \mu)$ for all $h$ in $\text{Fr}(X, R, S)$ since $\psi$ is an epimorphism. Therefore $r \theta \mu = 0$ in $(E(H), \text{End}(H))$, where $H = \text{Fr}(X, R, S)$. Hence $r \theta = 0$ in $(R, S)$ as $\mu$ is a faithful representation. Therefore $r \in A$ and we get $\text{Ker } \phi \subseteq A$. Hence we have $\text{Ker } \phi = A$. So there exists a unique d.g. near-ring homomorphism $\eta: (R, S) \rightarrow (E(G), \text{End}(G))$, where $G = \text{Fr}(X, R, S)$, such that $\theta \eta = \phi$. Thus $\text{Fr}(X, R, S)$ is an $(R, S)$-group with representation $\eta$. Again as $\text{Fr}(X, R, S)$ is the free $(R, S)$-group on $X$, there exists a unique $(R, S)$-homomorphism $\alpha : \text{Fr}(X, R, S) \rightarrow \text{Fr}(X, R, S)$ extending the identity map of $X$.

Now for $x(r \phi) \in \text{Fr}(X, R, S)$ we have

$$\{ x(r \phi) \} \psi \alpha = \{ (x \psi) r \theta \mu \} \alpha$$

$$= \{ x(r \theta \mu) \} \alpha$$

$$= (x \alpha) r \theta \eta$$

$$= x(r \theta \eta)$$

$$= x(r \phi) .$$

Similarly we can show that $\{ x(r \theta \mu) \} a \psi = x(r \theta \mu)$.
Now the set $Y = \{ x, x(r\psi) : x \in X, r \in (R,S) \}$ generates the group $\text{Fr}(X,R,S)$ while the set 

$$Z = \{ x, x(r \theta \mu) : x \in X, r \in (R,S) \}$$

generates the group $\text{Fr}(X,R,S)$. Also $\psi \alpha$ maps $Y$ identically onto itself and $\alpha \psi$ maps $Z$ identically onto itself. Therefore $\psi \alpha = I_{\text{Fr}(X,R,S)}$ and $\alpha \psi = I_{\text{Fr}(X,R,S)}$ and hence $\text{Fr}(X,R,S)$ is isomorphic to $\text{Fr}(X,R,S)$.

Now we are in a position to prove the following theorem:

Theorem 4.1.6 A function $P : \mathcal{B} \times \mathcal{S} \rightarrow \mathcal{G}$, defined by $((R,S), X) \rightarrow \text{Fr}(X,R,S)$ is a covariant functor such that $P/\mathcal{B} = P_X$ and $P/\mathcal{S} = P_{(R,S)}$ are covariant functors for each $X \in \mathcal{S}$ and each $(R,S) \in \mathcal{B}$ respectively. Also $P$ is natural in both the components.

We divide this theorem into three parts and prove them as separate theorems.

Theorem 4.1.7 For each $X \in \mathcal{S}$, $P_X : \mathcal{B} \rightarrow \mathcal{G}$, defined by $(R,S) \rightarrow \text{Fr}(X,R,S)$ is a covariant functor.

Theorem 4.1.8 For each $(R,S) \in \mathcal{B}$, $P_{(R,S)} : \mathcal{S} \rightarrow \mathcal{G}$, defined by $X \rightarrow \text{Fr}(X,R,S)$ is a covariant functor.

Theorem 4.1.9 For each $\alpha : X \rightarrow Y$ in $\mathcal{S}$ and each
h : (R,S) \rightarrow (T,U) in \mathcal{B}, there are natural transformations

\begin{align*}
a (R,S) : P_X & \longrightarrow P_Y \\
h (x) : P_{(R,S)} & \longrightarrow P_{(T,U)}
\end{align*}

respectively.

Remark 4.1.10 Let H be an (R,S)-group for (R,S) \in \mathcal{B}, with two (R,S)-homomorphisms \(a\) and \(\beta\) from \(\text{Fr}(X,R,S)\) to \(H\). In order to prove that \(a = \beta\), by 4.1.2, it is sufficient to show that they agree on \(\{ x, r_x : x \in X, r \in (R,S) \}\). This will be a common argument in the proofs of these theorems. We will use this without further mention.

Proof of theorem 4.1.7. We write \(\text{Fr}(X,R,S) = (R,S) P_X\).

Let \(h : (R,S) \rightarrow (T,U)\) be a homomorphism in \(\mathcal{B}\). By 4.1.3 we get an (R,S)-homomorphism \(\overline{h} : \text{Fr}(X,R,S) \rightarrow \text{Fr}(X,T,U)\) and we write \(\overline{h} = h P_X\). Since \(I_{(R,S)}\) in \(\mathcal{B}\) gives rise to \(I_{\text{Fr}(X,R,S)}\) in \(\mathcal{C}\), we get \(I_{(R,S)} P_X = I_{\text{Fr}(X,R,S)}\).

Now let \(h : (R,S) \rightarrow (T,U)\) and \(f : (T,U) \rightarrow (Q,V)\) be homomorphisms in \(\mathcal{B}\). Then we have homomorphisms

\begin{align*}
h P_X & : \text{Fr}(X,R,S) \rightarrow \text{Fr}(X,T,U) \\
f P_X & : \text{Fr}(X,T,U) \rightarrow \text{Fr}(X,Q,V) \\
(h \circ f) P_X & : \text{Fr}(X,R,S) \rightarrow \text{Fr}(X,Q,V)
\end{align*}
in $G$. To complete the proof we show that $(hP_X)(fP_X) = (hf)P_X$.

For $x \in X$, $r \in (R,S)$, by 4.1.3, we have:

$$x(hP_X)(fP_X) = x(fP_X) = x = x(hf)P_X$$

and

$$r_x(hP_X)(fP_X) = (r h)_x(fP_X) = (r h f)_x = r_x(hf)P_X.$$ 

Hence we get $(hP_X)(fP_X) = (hf)P_X$. This proves that $P_X$ is a covariant functor.

Proof of theorem 4.1.8. We write $Fr(X,R,S) = X P(R,S)$ and let $a : X \rightarrow Y$ be a map in $G$. By 4.1.4 there exists an $(R,S)$-homomorphism $a : Fr(X,R,S) \rightarrow Fr(Y,R,S)$. We denote $a$ by $a P(R,S)$. Since $I_X$ gives rise to the identity $(R,S)$-homomorphism $I_{Fr(X,R,S)}$ in $G$ we write $I_X P(R,S) = I_X P(R,S)$.

Now let $a : X \rightarrow Y$ and $\beta : Y \rightarrow Z$ be maps in $G$.

By 4.1.4 we get $(R,S)$-homomorphisms:

$$\alpha P(R,S) : Fr(X,R,S) \rightarrow Fr(Y,R,S)$$
$$\beta P(R,S) : Fr(Y,R,S) \rightarrow Fr(Z,R,S)$$

For $x \in X$ and $r \in (R,S)$, by 4.1.4, we have:

$$x \{a P(R,S)\} \{\beta P(R,S)\} = (x a)\{\beta P(R,S)\} = x(\alpha \beta) = x[(\alpha \beta) P(R,S)],$$
$$r_x \{a P(R,S)\} \{\beta P(R,S)\} = (r_x a')\{\beta P(R,S)\} = r_x(\alpha \beta) = r_x[(\alpha \beta) P(R,S)].$$
Hence \( \alpha P_{(R,S)} \{ h P_{(R,S)} \} = \{ (\alpha \beta) P_{(R,S)} \} \), which completes the proof.

Proof of theorem 4.1.9. Let \( h : (R,S) \to (T,U) \) be a homomorphism in \( \mathcal{B} \) and \( \alpha : X \to Y \) be a map in \( S \). By theorems 4.1.3, 4.1.4, 4.1.7, 4.1.8 we get the following diagram in \( G \):

\[
\begin{array}{ccc}
Fr(X,R,S) & \xrightarrow{\alpha P_{(R,S)}} & Fr(Y,R,S) \\
\downarrow h P_X & & \downarrow h P_Y \\
Fr(X,T,U) & \xrightarrow{\alpha P_{(T,U)}} & Fr(Y,T,U)
\end{array}
\]

where

\[
Fr(X,R,S) = X P_{(R,S)} = (R,S) P_X ,
\]

\[
Fr(X,T,U) = X P_{(T,U)} = (T,U) P_X ,
\]

\[
Fr(Y,R,S) = Y P_{(R,S)} = (R,S) P_Y ,
\]

\[
Fr(Y,T,U) = Y P_{(T,U)} = (T,U) P_Y .
\]

Now by the same theorems, we have, for each \( x \in X \) and \( r \in (R,S) \):

\[
x \{ h P_X \} \{ \alpha P_{(T,U)} \} = \{ x( h P_X)\{ \alpha P_{(T,U)} \} \}
\]

\[
= x \{ \alpha P_{(T,U)} \}
\]

\[
= x \alpha ,
\]

and \( x \{ \alpha P_{(R,S)} \}\{ h P_Y \} = \{ x(\alpha P_{(R,S)})\}\{ h P_Y \} \)
Thus \( x \{ h \ P_x \} \{ a \ P_T \} = x \{ a \ P_{(R,S)} \} \{ h \ P_Y \} \),

and \( r_x \{ h \ P_x \} \{ a \ P_T \} = r_x \{ a \ P_{(R,S)} \} \{ h \ P_Y \} \), for

\[
\begin{align*}
\{ h \ P_x \} \{ a \ P_T \} &= \{ r_x \{ h \ P_x \} \{ a \ P_T \} \} \\
&= \{ (r \ h) \{ h \ P_x \} \{ a \ P_T \} \} \\
&= \{ (r \ h \ x) \{ a \ P_T \} \} \\
&= \{ (r \ x \ h) \{ a \ P_T \} \} \\
&= \{ r_x \{ a \ P_T \} \} \{ h \ P_Y \} \\
&= r_x \{ a \ P_{(R,S)} \} \{ h \ P_Y \} .
\end{align*}
\]

Hence we get

\[
\{ h \ P_x \} \{ a \ P_T \} = \{ a \ P_{(R,S)} \} \{ h \ P_Y \} ,
\]

i.e. the above diagram is commutative. Hence \( h \) and \( a \) define

the natural transformations:

\[
\begin{align*}
h(X) : P_{(R,S)} &\longrightarrow P_{(T,U)} , \\
a(R,S) : P_X &\longrightarrow P_Y
\end{align*}
\]

respectively. This completes the proof of theorem 4.1.9.

Thus we have proved theorem 4.1.6.

Remark 4.1.11. \( F \times I \mathcal{S} \) and \( G \times I \mathcal{S} : \mathcal{A} \times \mathcal{S} \longrightarrow \mathcal{B} \times \mathcal{S} \),

where \( F \) and \( G \) are the functors defined in chapter 2, are covari-
ant functors, which are coadjoint and adjoint respectively of the inclusion functor $I_{B} \times I_{S} : B \times S \rightarrow A \times S$.

The following main theorem of this section becomes an obvious corollary of theorem 4.1.6, by lemma 4.1.5.

Theorem 4.1.12. $C : A \times S \rightarrow C$ defined by $((R,S),X) \mapsto \text{Fr}(X,R,S)$ is a covariant functor such that $C / A = C_X$ and $C / S = C(R,S)$ are covariant functors for each $X \in S$ and each $(R,S) \in A$ respectively. Also $C$ is natural in both components.

Note that $C_X = F \mathcal{F}_X$ and $C(R,S) = P(R,S) F$. Also $P(R,S)$ for each $(R,S) \in B$ is a functor from the category $S$ to the category $\mathcal{C}(R,S)$, which is a subcategory of $\mathcal{C}$. Clearly $\mathcal{C}(R,S)$ is not a full subcategory of $\mathcal{C}$.

The forgetful functor $Q(R,S) : \mathcal{C}(R,S) \rightarrow S$ is the functor which forgets the $(R,S)$-group structure on the objects of $\mathcal{C}(R,S)$, i.e. if $G$ is an $(R,S)$-group then $G Q(R,S)$ is the underlying set $G$ of $G$. If $\alpha : G \rightarrow H$ is an $(R,S)$-homomorphism in $\mathcal{C}(R,S)$ then $\alpha Q(R,S) = \alpha$.

Theorem 4.1.13. The forgetful functor $Q(R,S) : \mathcal{C}(R,S) \rightarrow S$ is an adjoint of the functor $P(R,S) : S \rightarrow \mathcal{C}(R,S)$.

Proof. Let $\alpha : X \rightarrow Y$ be a mapping in $S$ and let $\beta : \text{Fr}(X,R,S) \rightarrow G$, $\theta : G \rightarrow H$ be homomorphisms in $\mathcal{C}(R,S)$. Then we have a diagram:
where \( \phi : \text{Hom}_{\mathcal{C}(R,S)}(Y, P(R,S)) \rightarrow \text{Hom}_{\mathcal{S}}(Y, G Q(R,S)) \), for each pair \( (Y, G) \in \mathcal{S} \times \mathcal{C}(R,S) \), is defined by \( \beta \phi = \beta Y \). In this case it is rather trivial to show that \( \phi \) is well defined and a bijection, therefore we are not going to prove this. But in the next section we will give the proof of a similar fact, since it is not so obvious in that case.

It is easy to see that the above diagram is commutative. Therefore

\[
\{ (\alpha P(R,S)) \beta \} \phi = \alpha (\beta \phi) \quad \text{and} \quad (\beta \theta) \phi = (\beta \phi)(\theta Q(R,S)).
\]

Hence \( Q(R,S) \) is an adjoint of \( P(R,S) \).

Corollary 4.1.14. For each \( (R,S) \in \mathcal{R} \), there exists a covariant functor \( Q(R,S)_F : \mathcal{C}(R,S)_F \rightarrow \mathcal{S} \), which is an adjoint.
Proof. Obvious.

4.2. Functors. $\mathcal{H} \xrightarrow{\mathcal{G}}$

We define $M : \mathcal{H} \to \mathcal{J}$ by $S \mapsto (\text{Fr}(S), S)$ for each $S \in \mathcal{J}$ and for each $\alpha : S \to U$ in $\mathcal{J}$, $\alpha M$ to be the induced d.g. near-ring homomorphism $(\text{Fr}(S), S) \to (\text{Fr}(U), U)$ in $\mathcal{J}$. Then $\alpha M / S = \alpha$. Now we prove the following theorem:

**Theorem 4.2.1.** $M : \mathcal{J} \to \mathcal{H}$ is a covariant functor.

Proof. Clearly $I_S : S \to S$ in $\mathcal{J}$ gives rise to $I_{(\text{Fr}(S), S)} : (\text{Fr}(S), S) \to (\text{Fr}(S), S)$ in $\mathcal{H}$. Therefore $I_S M = I_{S M}$. Now let $f : S \to U$ and $g : U \to T$ be homomorphisms in $\mathcal{J}$. Then we get homomorphisms $f M$, $g M$ and $(f \circ g) M$ in $\mathcal{H}$. To complete the proof we have to show that $(f M)(g M) = (f \circ g) M$. For each $s \in S$ we have:

$$s [(f M)(g M)] = s(f M)(g M) = (s f)(g M) = (s f) g = s(f g)$$

and $s [(f \circ g) M] = s(f \circ g)$, i.e. $(f \circ g) M / S = (f M)(g M) / S$.

Hence by the property of free d.g. near-rings $(f g) M = (f M)(g M)$.

Now we define another functor $N : \mathcal{H} \to \mathcal{J}$ by $(R, S) N = S$ for each $(R, S) \in \mathcal{H}$, and $\theta N = \theta / S$ for each homomorphism $\theta : (R, S) \to (T, U)$ in $\mathcal{J}$. Then we prove the
following theorem:

Theorem 4.2.2. \( N : \mathcal{H} \rightarrow \mathcal{S} \) is a covariant functor.

Proof. By definition of \( N \), \( I_{(R,S)} N = I_{(R,S)} N \).

Now let \( f : (R,S) \rightarrow (T,U) \) and \( \varepsilon : (T,U) \rightarrow (Q,V) \) be homomorphisms in \( \mathcal{H} \). Then for each \( s \in S \), by definition of homomorphisms in \( \mathcal{H} \), we get

\[
s \{ (f N)(\varepsilon N) \} = (s(\varepsilon N))(s N) = (s f)(\varepsilon N) = s \{ (f \varepsilon) N \}.
\]

Hence \( (f N)(\varepsilon N) = (f \varepsilon) N \), which completes the proof.

Theorem 4.2.3. \( N \) is an adjoint of \( M \).

Proof. To prove this theorem we show the existence of a bijection

\[
\phi = \phi(U, (R,S)) : \text{Hom}_\mathcal{H}(U, (R,S)) \rightarrow \text{Hom}_\mathcal{S}(U, (R,S) N)
\]

for each pair \( (U, (R,S)) \in \mathcal{H} \times \mathcal{H} \), which is natural in both components. For that let \( \alpha \in \text{Hom}_\mathcal{H}(U, (R,S)) \) and define

\[
\alpha \phi = \alpha / U \in \mathcal{H}.
\]

It follows from the definition of homomorphisms in \( \mathcal{H} \), that \( \phi \) is well defined. Now let \( \alpha, \beta \in \text{Hom}_\mathcal{H}(U, (R,S)) \) such that \( \alpha \phi = \beta \phi \), i.e. \( \alpha / U = \beta / U \).

Therefore by the property of free d.g. near-rings \( \alpha = \beta \). So \( \phi \) is an injection. Also for \( \gamma \in \text{Hom}_\mathcal{S}(U, (R,S) N) \), \( \gamma M \) is a homomorphism from \( (\text{Fr}(U), U) \) to \( (\text{Fr}(S), S) \) in \( \mathcal{S} \), such that
\[ \gamma M / U = \gamma. \] Let \( \pi : (Fr(S), S) \to (R, S) \) be the natural homomorphism. Then \( \pi / S = I_S \). Therefore \( (\gamma M) \pi \) is a homomorphism from \( (Fr(U), U) \) to \( (R, S) \) in \( \mathcal{A} \). Now

\[(\gamma M) \pi \} \phi = \{(\gamma M) \pi \} / U \text{ in } \mathcal{S},\]

i.e. for each \( u \in U \),

\[
u \{[(\gamma M) \pi] \phi \} = u \{(\gamma M) \pi \}
= \{u(\gamma M)\} \pi
= \{u \gamma\} \pi
= u \gamma.
\]

So we get \( \{(\gamma M) \pi \} \phi = \gamma \), which proves that \( \phi \) is a surjection. Hence \( \phi \) is a bijection.

Let \( (U, (R, S)) \) and \( (T, (Q, V)) \in \mathcal{S} \times \mathcal{A} \), with homomorphisms \( a : U \to T \) in \( \mathcal{S} \) and \( \theta : (R, S) \to (Q, V) \), \( \nu : T M \to (R, S) \) in \( \mathcal{A} \). Then we get a commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{i_U} & U M \\
\downarrow \alpha & & \downarrow \alpha M \\
T & \xrightarrow{i_T} & T M \\
S = (R, S) & \xrightarrow{i_S} & (R, S) \\
\downarrow \theta & & \downarrow \theta \\
V = (Q, V) & \xrightarrow{i_V} & (Q, V)
\end{array}
\]
where $l_U, l_T, l_S, l_V$ are inclusion semigroup homomorphisms.

Now $\{(a M) v \} \phi = \{(a M) \nu \}/U$, i.e. for each $u \in U$,

\[
u \{\{(a M) v \} \phi \} = u \{(a M) \nu \} = \{u (a M)\} \nu = (u a) \nu = (u a)(\nu, \phi) = u \{a(\nu \phi)\}.
\]

Therefore $\{(a M) v \} \phi = a(\nu \phi)$.

We know that $(\nu \theta) \phi = (\nu \theta) / T$, i.e. for each $t \in T$,

\[
t \{(\nu \theta) \phi \} = t (\nu \theta) = (t \nu) \theta = \{t (\nu \phi)\} \theta = \{t (\nu \phi)\}(\theta N) = t \{(\nu \phi)(\theta N)\}.
\]

Therefore $(\nu \theta) \phi = (\nu \phi)(\theta N)$. This completes the proof that $N$ is an adjoint of $M$.

72
Chapter 5

Inverse Semigroups of Endomorphisms

5.1. Preliminaries

A semigroup $S$ is called regular if for each $x \in S$ there exists at least one element $y$ in $S$ such that $x y x = x$, and an element $y$ is said to be an inverse of $x$ if $x y x = x$, $y x y = y$. In a regular semigroup each element has an inverse which is not necessarily unique. An element $e \in S$ with $e^2 = e$ is called an idempotent. 0 and 1, if they are in $S$, are idempotents. In a regular semigroup $S$, for each $x \in S$, $x y$ and $y x$ are idempotents for every inverse $y$ of $x$.

We call a semigroup $S$ an inverse semigroup if each $x \in S$ possesses a unique inverse $y$ in $S$ and we denote this inverse by $x^{-1}$

Theorem 5.1.1. A semigroup $S$ is an inverse semigroup if and only if $S$ is regular and the idempotents of $S$ commute.

A lower semilattice is a partially ordered set in which each pair of elements has a greatest lower bound.

Theorem 5.1.2. Let $(E, \leq)$ be a lower semilattice. Then $(E, \wedge)$ is a commutative semigroup of idempotents and for all $a, b$ in $E$, $a \leq b$ if and only if $a \wedge b = a$.

Let $(E, .)$ be a commutative semigroup of idempotents. Then the relation $\leq$ on $E$ defined by, $a \leq b$ if and only if $a . b = a$,
is a partial order on \( E \) such that \((E, \preceq)\) is a lower semilattice.

In \((E, \preceq)\), \( a \wedge b = a \cdot b \).

This result shows that the notions of commutative semigroup of idempotents and lower semilattice are equivalent. So the term semilattice can be used with either meaning.

Since the set \( E \) of idempotents of an inverse semigroup \( S \) forms a commutative semigroup, we call it the semilattice of idempotents of \( S \).

Theorem 5.1.3. Let \( S \) be an inverse semigroup with semilattice \( E \) of idempotents. Then for all \( x, y \in S, e \in E \),

1) \((x^{-1})^{-1} = x ,

2) \(e^{-1} = e ,

3) \((xy)^{-1} = y^{-1} x^{-1} ,

4) \( xe x^{-1} \) and \( x^{-1} e x \in E \) .

Corollary 5.1.4. If \( x_1, \ldots, x_n \), \( n \) finite, are elements of an inverse semigroup \( S \), then

\[
(x_1 \ldots x_n)^{-1} = x_n^{-1} \ldots x_1^{-1} .
\]

In particular \((x^n)^{-1} = (x^{-1})^n\) for each \( x \in S \).

Theorem 5.1.5. Let \( S \) be an inverse semigroup with semilattice \( E \) of idempotents. Then

1) \( S = \bigcup \{ e S : e \in E \} \),
2) for \( e \in E, s \in S \), there exists an \( f \in E \) such that \( es = sf \).

Theorem 5.1.6. Every homomorphic image of an inverse semigroup is an inverse semigroup.

Remarks 5.1.7.

1) Sub-semigroups of an inverse semigroup need not be inverse.
2) \( s^{-1} \phi \) is the inverse in \( S \phi \) of \( s \phi \) for each \( s \in S \), where \( S \) is an inverse semigroup and \( \phi \) is a semigroup homomorphism.

Theorem 5.1.8. If \( \phi \) is a semigroup homomorphism from an inverse semigroup \( S \) to a group \( G \), then

1) \( e \phi = L_e \) for each \( e \in E \),
2) \( S \phi \) is a subgroup of \( G \).

We define a relation \( \preceq \) on an inverse semigroup \( S \) by

"\( x \preceq y \) if and only if there exists \( e \in E \) such that \( x = ey \)."

Then we have the following result:

Theorem 5.1.9. \((S, \preceq)\) is a partially ordered set.

The restriction of \( \preceq \) to \( E \) is easily seen to be the natural semilattice ordering on \( E \), "\( e \preceq f \) if and only if \( ef = e \)."

A semigroup \( S \) is called right reductive if for \( a, b \in S \), \( xa = xb \), for all \( x \in S \) implies that \( a = b \).

Theorem 5.1.10. An inverse semigroup is right reductive.

A congruence on a semigroup \( S \) is an equivalence relation \( \rho \)
on the set $S$ such that
\[(x, y), (a, b) \in \rho \text{ implies that } (xa, yb) \in \rho\]
for all $x, y, a, b \in S$.

A congruence $\rho$ on a semigroup $S$ is called a group congruence if $S/\rho$ is a group.

Theorem 5.1.11. Let $S$ be an inverse semigroup with semilattice $E$ of idempotents. Then the relation
\[\sigma = \{(x, y) \in S \times S : e x = e y \text{ for some } e \in E\}\]
is the minimum group congruence on $S$.

Definition 5.1.12. Let $S$ be a semigroup. $S$ is called a semilattice of groups if there exists a semilattice $Y$ such that
1) $\{S_a : a \in Y\}$ is a family of disjoint subgroups of $S$ indexed by $Y$;
2) for each pair $\alpha, \beta \in Y$ with $\alpha \geq \beta$ there exists a homomorphism $\phi_{\alpha, \beta} : S_{\alpha} \rightarrow S_{\beta}$ such that
   a) $\phi_{\alpha, \alpha}$ is the identity automorphism of $S_{\alpha}$ for each $\alpha \in Y$;
   b) $\phi_{\alpha, \beta} \phi_{\beta, \gamma} = \phi_{\alpha, \gamma}$ for every $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$.

Definition 5.1.13. A semigroup $S$ is called a Clifford semigroup if it is regular and its idempotents are central, i.e.,
$e s = s e$ for all $s \in S$, and for each idempotent $e \in S$. 

76
Theorem 5.1.14.

1) A semigroup $S$ is a semilattice of groups if and only if it is a Clifford semigroup.

2) An inverse semigroup $S$ is a semilattice of groups if and only if $ss^{-1} = s^{-1}s$ for all $s \in S$.

Definition 5.1.15. Let $G$ be a group. $G$ is called a semidirect product or split extension of its subgroups $H$ and $K$ if

1) $K$ is normal in $G$,

2) $H \cap K = \{0\}$,

3) $G = K \cdot H$.

Theorem 5.1.16 If a group $G$ is a semidirect product of its subgroup $H$ and a normal subgroup $K$ then there is a natural projection $\pi : G \longrightarrow H$ with $\ker \pi = K$.

Theorem 5.1.17. Let $S$ be an inverse semigroup with semilattice $E$ of idempotents and $G$ a group such that $S \subseteq \text{End}(G)$. Then for each $e \in E$, $G = \ker e + \text{Image } e$ is the semidirect product of its subgroups $\text{Image } e$ and $\ker e$. Moreover if $e < f$ then $\ker e \supset \ker f$ and $\text{Image } e \subseteq \text{Image } f$, and if $e < f$ then $\text{Image } e \subseteq \text{Image } f$, $\ker e \supset \ker f$.

Inverse semigroups arise naturally as sets of partial 1–1 mappings of a set. A partial 1–1 mapping of a set is a mapping whose
domain is a subset, possibly empty, of X. The set \( \mathcal{D}(X) \) of partial 1-1 mappings of X is a subset of \( \mathbb{B}(X) \), the set of binary relations on X. So the elements of \( \mathcal{D}(X) \) are multiplied by the law of composition of \( \mathbb{B}(X) \). We write \( \text{dom} \ a \) for domain of \( a \) and \( \text{ran} \ a \) for range of \( a \), \( a \in \mathcal{D}(X) \). If \( a, \beta \in \mathcal{D}(X) \) then 
\[(x, y) \in a \circ \beta \text{ if and only if there exists } z \in X \text{ such that } (x, z) \in a \text{ and } (z, y) \in \beta, \text{ i.e. if and only if there exists } z \in \text{ran} \ a \cap \text{dom} \ \beta \text{ such that } x \ a = z \text{ and } z \ \beta = y, \text{ i.e., if and only if } y = (x \ a) \ \beta, \text{ where } x \in (\text{ran} \ a \cap \text{dom} \ \beta) \ a^{-1}. \text{ Thus } a \circ \beta, \text{ usually written as } a \ \beta, \text{ is a partial 1-1 mapping with } \]
\[
\text{dom} \ a \ \beta = (\text{ran} \ a \cap \text{dom} \ \beta) \ a^{-1} \text{ and } \]
\[
\text{ran} \ a \ \beta = (\text{ran} \ a \cap \text{dom} \ \beta) \ \beta.
\]

One can prove the following result.

**Theorem 5.1.18.**

1) \( \mathcal{D}(X) \) is an inverse semigroup.

2) \( a \in \mathcal{D}(X) \) is an idempotent if and only if \( a = I_A \) for some subset \( A \) of X.

3) The semilattice of idempotents of \( \mathcal{D}(X) \) is isomorphic to the semilattice of subsets of X under intersection.

**Theorem 5.1.19.** (Vagner-Preston Representation Theorem)

If \( S \) is an inverse semigroup then there exists a set \( X \) and a
Definition 5.1.20. $\mathcal{I}(X)$ is called the symmetric inverse semigroup on $X$.

Definition 5.1.21. Let $G$ be a group. A partial 1-1 mapping of $G$, whose domain is a subgroup of $G$, and which is a homomorphism, is called a partial isomorphism.

Remark 5.1.22. The set of partial isomorphisms of a group $G$ forms a sub-semigroup $\mathcal{I}_G$ of $\mathcal{I}(G)$, the symmetric inverse semigroup on $G$. $\mathcal{I}_G$ is an inverse subsemigroup of $\mathcal{I}(G)$.

Let $S$ be any semigroup with identity. If for some $a, b$ in $S$, $ab = 1$ but $ba \neq 1$ then $a, b$ generate an inverse subsemigroup $\mathcal{C} = \mathcal{C}(a, b)$ of $S$. Each element of $\mathcal{C}$ is uniquely expressed as $b^m a^n$, for $m, n \geq 0$. \{ $b^m a^n : n \geq 0$ \} is the set of idempotents of $\mathcal{C}$.

Definition 5.1.23. $\mathcal{C}$ is called a bicyclic inverse semigroup.

Definition 5.1.24. A semilattice $(Y, \preceq)$ is called updirected if for each pair $x, y$ of elements of $Y$ there exists a $z \in Y$ such that $x \preceq z$ and $y \preceq z$.  

79
Example 5.2.1. Let \( \{ e, f, 0 \} \) be a semigroup such that each element of \( S \) is idempotent and products of distinct elements are 0.

Let \( (R, S) \) be a d.g. near-ring with 0 as the zero of \( (R, +) \) taken to be non-abelian.

Then we have \( e + f \in (R, S) \) such that

\[
(e + f) e = e^2 + fe = e + 0 = e, \\
(e + f) f = ef + f^2 = 0 + f = f,
\]

i.e., \( e + f \) is a left identity for \( e \) and \( f \) and hence for \( (R, S) \). Therefore by 1.6.9, \( (R, S) \) has a faithful representation on \( (R, +) \). Let \( K = \bigcap \{ \ker s : s \in S \} \).

Then \( K \) is not trivial, since \( e + f - e - f \in K \). Therefore \( K \supset \delta(R, +) \), the derived group of \( (R, +) \). So \( (R/K, +) \) is abelian. In this case \( K \) is an ideal of \( (R, S) \), by. 1.5.9, for

1) \( RK \subseteq K \) since \( (r k)s = r(ks) = ro = 0 \), for all \( s \in S \), \( k \in K \);

2) \( KS \subseteq K \), since \( ks = 0 \in K \) for \( k \in K \) and all \( s \in S \).

Now \( (R, S)/K \) is a d.g. near-ring with \( (R/K, +) \) abelian.

Therefore by 1.5.6.2 \( (R, S)/K \) is a ring.

Now we give an example of a d.g. near-ring over an inverse semigroup which does not have a left identity.

Example 5.2.2. Let \( E \) be an infinite, updirected semilattice of idempotents, which does not have a maximum element. Let
(Fr(E),E) be the free d.g. near-ring on E. For an element

\[ r = \epsilon e_1 + \cdots + \epsilon e_n \]

of \((Fr(E),E)\), using induction on \(n\), we can find an element \(e \in E\) such that \(e_i < e\) for \(i = 1, \ldots, n\).

Then

\[ re = (\epsilon e_1 + \cdots + \epsilon e_n)e \]

\[ = \epsilon e_1 e + \cdots + \epsilon e_n e \]

\[ = \epsilon e_1 + \cdots + \epsilon e_n \]

\[ = r, \]

\[ \neq e. \]

This proves that no element of \((Fr(E),E)\) can be a left identity.

Example 5.2.3. Let \(F\) be a free group on

\[ x = \{ x_1, x_2, \ldots, x_n, \ldots \}. \]

Define \(\alpha : X \rightarrow X\) by \(x_{2n-1} \rightarrow x_{2n}, x_{2n} \rightarrow x_{2n+1}\), and \(\beta : X \rightarrow X\) by \(x_1 \rightarrow x_1\) and \(x_{2n} \rightarrow x_{2n-1}\),

\[ x_{2n+3} \rightarrow x_{2n}, n > 1. \]

Then \(\alpha\) is a 1-1 mapping but not onto, while \(\beta\) is onto but not 1-1. By the property of free groups, \(\alpha\) and \(\beta\) can be extended to endomorphisms of \(F\), again denoted by \(\alpha\) and \(\beta\) respectively.

Also for \(n \geq 1\), we have

\[ x_{2n} \xrightarrow{\alpha} x_{2n+3} \xrightarrow{\beta} x_{2n} \]

\[ x_{2n-1} \xrightarrow{\alpha} x_{2n} \xrightarrow{\beta} x_{2n-1}. \]

Therefore \(\alpha \beta = I_X\) and hence \(\alpha \beta = I_F\). But since
we get \( \beta a \neq I_x \) and therefore \( \beta a \neq I_F \). Hence the sub-semigroup \( C = C(a, \beta) \) of the semigroup of endomorphisms of \( F \) is a bicyclic inverse semigroup. Each element of \( C \) is uniquely expressible as \( \beta^m a^n \) for \( m, n \geq 0 \). The inverse of \( \beta^m a^n \) is \( \beta^n a^m \) and \( \{ \beta^n a^n : n \geq 0 \} \) is the set of idempotents of \( C \) such that

\[
1 > \beta a > \beta^2 a^2 > \cdots > \beta^m a^n > \cdots \quad (5.2.3(a))
\]

By 5.1.17 we get the following infinite chains of subgroups of \( F \):

\[
0 \subseteq \text{Ker} \beta a \subseteq \text{Ker} \beta a^2 \subseteq \cdots \subseteq \text{Ker} \beta^n a^n \subseteq \cdots \quad (5.2.3(b))
\]

\[
F \supset \text{Im} \beta a \supset \text{Im} \beta a^2 \supset \cdots \supset \text{Im} \beta^n a^n \supset \cdots \quad (5.2.3(c))
\]

Now \( F \beta^n a^n \subseteq F a^n = F (a^n \beta^n) a^n \), as \( a^n \beta^n = I_F \)

\[
= F a^n (\beta^n a^n) \subseteq F \beta^n a^n,
\]

i.e. \( F \beta^n a^n = F a^n \). \quad (5.2.3(d))

Also \( 0 = \epsilon \beta^n a^n \) implies that \( 0 = \epsilon \beta^n a^n \beta^n = \epsilon \beta^n \), i.e.

\[
\text{Ker} \beta^n a^n \subseteq \text{Ker} \beta^n,
\]

and \( 0 = \epsilon \beta^n \) implies that \( 0 = \epsilon \beta^n a^n \), i.e.

\[
\text{Ker} \beta^n \subseteq \text{Ker} \beta^n a^n.
\]

Hence we get \( \text{Ker} \beta^n a^n = \text{Ker} \beta^n \). \quad (5.2.3(e))

Therefore the infinite series \((5.2.3(b))\) and \((5.2.3(c))\) can be
written as

\[ 0 \subset \text{Ker } \beta \subset \text{Ker } \beta^2 \subset \ldots \subset \text{Ker } \beta^n \subset \ldots \] (5.2.3(f))

\[ \mathbf{F} \supset \text{Im } \alpha \supset \text{Im } \alpha^2 \supset \ldots \supset \text{Im } \alpha^n \supset \ldots \] (5.2.3(g)).

Since \( \mathbf{F} \) is free, by the definition of \( \beta \), an element \( g \) of \( \mathbf{F} \) belongs to \( \text{Ker } \beta \) only if there are cancellations in \( g \beta \), and there can be cancellations in \( g \beta \) if and only if \( g \beta \) has pairs like

\[ \epsilon_i x_i \beta + \epsilon_{i+1} x_{i+1} \beta = \mp x_i \mp x_{i+1}. \]

This can happen if and only if \( \epsilon_i x_i + \epsilon_{i+1} x_{i+1} \) is \( \mp x_i \mp x_{i+1} \), or \( \mp x_i \mp x_i \), or \( \mp x_i \mp x_{i+1} \), or the inverses of such elements.

Case I \( g \beta \) is exhausted by the cancellations of these types.

Case II After such cancellations we are left with elements of the type \( h \beta - h \beta \).

In either case we have \( g \in \{ \text{Gr}(\mp x_i \mp x_i, \pm x_i \mp x_{i+1}) \}^F \).

Hence we have proved that

\[ \text{Ker } \beta = \{ \text{Gr}(\mp x_i \mp x_i, \pm x_i \mp x_{i+1}) \}^F \]

Similarly we can show that

\[ \text{Ker } \beta^2 = \{ \text{Gr}(\pm x_i \mp x_i, \pm x_i \mp x_i, \mp x_i \mp x_i, \mp x_i \mp x_i) \}^F \]

Now suppose that \( \text{Ker } \beta^n = \{ \text{Gr}(\pm x_i \mp x_i, \pm x_i \mp x_i, \pm x_i \mp x_i, \pm x_i \mp x_i, \pm x_i \mp x_i) \}^F \), for \( m \leq n \), and consider \( \text{Ker } \beta^{m+1} \);

\[ g \in \text{Ker } \beta^{m+1} \iff g \beta \in \text{Ker } \beta^m, \]

83
\[ g \beta \in \left( \text{Gp} \left\{ \pm x_{2i+1} \mp x_1, \pm x_{2i} \mp x_1 : i \leq m \right\} \right)^F, \]

by the induction hypothesis.

\[ g \in \left( \text{Gp} \left\{ \pm x_{2i+1} \mp x_1, \pm x_{2i} \mp x_1 : i \leq m+1 \right\} \right)^F, \]

as before.

Hence \( \text{Ker} \beta^{m+1} = \left( \text{Gp} \left\{ \pm x_{2i+1} \mp x_1, \pm x_{2i} \mp x_1 : i \leq m+1 \right\} \right)^F \).

This completes the induction argument and it is proved that for all finite \( n > 0 \),

\[ \text{Ker} \beta^n = \left( \text{Gp} \left\{ \pm x_{2i+1} \mp x_1, \pm x_{2i} \mp x_1 : i \leq n \right\} \right)^F. \]

Next we find the subgroups \( \{ \text{Im} \alpha^n : n > 0 \} \). From the definition of \( \alpha \) we see that \( x_1, x_3 \notin \text{Im} \alpha \), whereas the other \( x_i \)'s are in \( \text{Im} \alpha \). Therefore we get:

\[ \text{Im} \alpha = \text{Gp} \{ X \setminus \{ x_1, x_3 \} \}. \]

Similarly we can show the following:

\[ \text{Im} \alpha^2 = \text{Gp} \{ X \setminus \{ x_1, x_2, x_3, x_4 \} \}, \]

\[ \text{Im} \alpha^3 = \text{Gp} \{ X \setminus \{ x_1, x_2, x_3, x_4, x_5, x_7 \} \}, \]

\[ \text{Im} \alpha^4 = \text{Gp} \{ X \setminus \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \} \}. \]

Now suppose that for each \( m \leq n \), \( m \) even, we have;

\[ \text{Im} \alpha^{m-1} = \text{Gp} \{ X \setminus \{ x_i : i \leq 2(m-1) - 1, x_{2(m-1)+1} \} \}, \]

\[ \text{Im} \alpha^m = \text{Gp} \{ X \setminus \{ x_i : i \leq 2m \} \}. \]
We know that, \( x_{2i+3} = x_{2i} \) and \( x_{2i} = x_{2i-1} \). Since

\[
\{ x_{2i} , x_{2i-1} : i \leq m \} \not\subset \text{Im} \ a^m ,
\]
we get \( \{ x_{2i+3} , x_{2i} : i \leq m \} \not\subset \text{Im} \ a^{m+1} \). Therefore \( \{ x_{i+3} , x_i : i \leq 2m \} \not\subset \text{Im} \ a^{m+1} \). Also

\[
x_{2m+1} = x_{2(m-1)} \quad \text{and since} \quad x_{2(m-1)} \not\subset \text{Im} \ a^m , \quad x_{2m+1} \not\subset \text{Im} \ a^{m+1} .
\]

It is easy to see that the other \( x_i \)s belong to \( \text{Im} \ a^{m+1} \). Hence

\[
\text{Im} \ a^{m+1} = \text{Gp}( X | \{ x_i : i \leq 2m+1 , x_{2m+3} \} ) . \tag{III}
\]

Again from the definition of \( a \) and \( \text{Im} \ a^{m+1} \) we get that

\[
\{ x_i : i \leq 2(m+2) \} \not\subset \text{Im} \ a^{m+2} ,
\]
whereas the other \( x_i \)s are in \( \text{Im} \ a^{m+2} \). Thus we get

\[
\text{Im} \ a^{m+2} = \text{Gp}( X | \{ x_i : i \leq 2(m+2) \} ) . \tag{IV}
\]

Hence we have proved that (I) and (II) imply (III) and (IV).

This completes the induction argument and we get for all \( n > 0 \),

\[
\text{Im} \ a^n = \text{Gp}( X | \{ x_i : i \leq 2n \} ) , \quad \text{if} \ n \ \text{is even},
\]

\[
= \text{Gp}( X | \{ x_i : i \leq 2n-1 , x_{2n+1} \} ) , \quad \text{if} \ n \ \text{is odd} .
\]

85
Example 5.2.4. Let \((R,S)\) be a d.g. near-ring. Since \(S\) is a distributive semigroup, the elements of \(S\) are endomorphisms of \((R,+)\). But two distinct elements of \(S\) may define the same endomorphism of \((R,+)\).

Now let \((R,S)\) be a d.g. near-ring with \(S\) an inverse semigroup. By 5.1.10, for \(s \neq t\) in \(S\), there exists \(x \in S\) such that \(xs \neq xt\). Therefore \(s\) and \(t\) define two distinct endomorphisms of \((R,+)\). Hence in this case we get \(S \subseteq \text{End} (R,+)\).

From now onwards, in this chapter, unless otherwise stated, \(S\) will denote an inverse semigroup with semilattice \(E\) of idempotents.

5.3. Representations of Inverse Semigroups

Throughout this section \(G\) will be an additive group such that \(S \subseteq \text{End} (G)\). Therefore for each \(s \in S\), \(Gs, Gs^{-1}\) are subgroups of \(G\). Also

\[
Gs^{-1} \subseteq Gs^{-1} = Gs^{-1}s^{-1} \subseteq Gss^{-1},
\]

\[
Gs^{-1}s \subseteq Gs = Gss^{-1}s \subseteq Gs^{-1}s,
\]

i.e. \(Gs = Gs^{-1}s\) and \(G^{-1} = Gss^{-1}\).

We define \(\bar{s} = s/Gs^{-1}\). Then for all \(g \in G\) we have

\[
gs = g(s^{-1}s) = (gss^{-1}) \bar{s},
\]

and if \(gs^{-1} \neq 0\), then we get
Since $s^{-1}$ is an endomorphism of $G$, this implies that

$$(g s^{-1}) s^{-1} \neq 0.$$ 

Therefore $s$ is a 1-1 mapping from $G s^{-1}$ onto $Gs$. Also, being a restriction of a homomorphism, $s$ is a homomorphism. Hence each $s$ is a partial isomorphism of $G$. We write $G s^{-1} = \text{dom } s$ and $Gs = \text{ran } s$. Now we prove the following result:

Theorem 5.3.1. Let $S \subseteq \text{End}(G)$. Then $\{s : s \in S\}$ is a semigroup of partial isomorphisms of $G$, which is isomorphic to $S$.

Proof. We define $\psi : S \longrightarrow S$ by $s \psi = s^{-1}$. Then $\psi$ is well defined and onto. For $s, t \in S$, $s \psi = t \psi$ implies that $\text{dom } s = \text{dom } t$ and for all $x$ in this common domain $x s = x t$.

We know that for all $g \in G$,

$$g s = g (s s^{-1} s) = (g s s^{-1}) s = (g s s^{-1}) t = g s s^{-1} t.$$ 

Therefore $s = s s^{-1} t$, i.e. $s \leq t$. Similarly we can show that $t \leq s$. Hence we get $s = t$. Thus $\psi$ is 1-1.

Finally we prove that $\psi$ is a semigroup homomorphism. Since $G t^{-1} s^{-1} \subseteq G s^{-1}$, we have:

$$\text{dom } s t \subseteq \text{dom } s$$ 

for all $s, t \in S$. Now for all $g \in G$, as $S$ is inverse, we have

$$(g t^{-1} s^{-1}) s = g t^{-1} s^{-1} s = g t^{-1} t t^{-1} s = g t^{-1} s^{-1} s t t^{-1}.$$ 

Since right hand side is in $\text{dom } t$ we get $(\text{dom } s t) s \subseteq \text{dom } t$,  

87
and hence \( \text{dom } s \circ t \subseteq \text{dom } s \circ t \). Conversely, as

\[
\text{dom } s \circ t = \{ x \in \text{dom } s : x \circ s \in \text{dom } t \},
\]

for \( x \in \text{dom } s \circ t \), \( x = g \circ s^{-1} \) for some \( g \in G \) such that \( x \circ s = (g \circ s^{-1}) \circ s = h \circ t^{-1} \) for some \( h \in G \). Therefore

\[
x = g \circ s^{-1} = g \circ s^{-1} \circ s^{-1} = (g \circ s^{-1} \circ s^{-1}) = (h \circ t^{-1}) \circ s^{-1} = h \circ (t^{-1} \circ s^{-1}) \in \text{dom } s \circ t.
\]

This shows that \( \text{dom } s \circ t \subseteq \text{dom } s \circ t \). Hence we get \( \text{dom } s \circ t = \text{dom } s \circ t \).

For all \( g \in G \), 

\[
(g \circ t^{-1} \circ s^{-1}) \circ s \circ t = (g \circ t^{-1} \circ s^{-1} \circ s) \circ t = (g \circ t^{-1} \circ s^{-1}) \circ s \circ t = (g \circ t^{-1} \circ s^{-1}),
\]

i.e. \( x \circ s \circ t = x \circ s \circ t \) for all \( x \in \text{dom } s \circ t = \text{dom } s \circ t \). So 

\[
\text{dom } s \circ t = \text{dom } s \circ t \text{ or } (s \psi)(t\psi) = (s \circ t) \psi \text{ for all } s, t \in S.
\]

Hence \( \psi \) is a semigroup homomorphism. This completes the proof.

Note that \( s \circ t \) and \( s \circ t \) are partial isomorphisms of \( G \) with same domain and with same action on this common domain. Therefore \( s \circ t \) and \( s \circ t \) have the same range.

For \( e \in E \), \( \text{dom } e = G \circ e^{-1} = G \circ e = \text{ran } e \), and \( e \) is the identity automorphism of \( G \circ e \).

Corollary 5.3.2. The semilattice \( (E, \cdot) \) is isomorphic to the semilattice \( \{ (\text{dom } e, \cap) : e \in E \} \) of subgroups of \( G \).

Proof. From 5.3.1 and 5.1.1 we have

\[
\text{dom } e \circ f = \text{dom } e \circ f = \text{dom } f \circ e = \text{dom } f \circ e, \text{ and again by 5.3.1 }
\]

\[
\text{dom } e \circ f \subseteq \text{dom } e \cap \text{dom } f.
\]

88
Conversely, if \( x \in \text{dom } e \cap \text{dom } f \), then
\[
x = g \cdot e = h \cdot f,
\]
for some \( g, h \in \mathbb{G} \) and
\[
x = h \cdot f = h(ff) = (hf) \cdot f = (ge) \cdot f = g(ef) \in \text{dom } e \cdot f.
\]
Therefore \( \text{dom } e \cap \text{dom } f \subseteq \text{dom } e \cdot f \). Hence we get
\[
\text{dom } e \cap \text{dom } f = \text{dom } e \cdot f \tag{I}
\]
Also if \( \text{dom } e = \text{dom } f \), then for each \( g \in \mathbb{G} \), \( g \cdot e = h \cdot f \)
for some \( h \in \mathbb{G} \), and \( g(ef) = h(ff) = h \cdot f = ge \). This gives
us \( ef = e \), i.e. \( e \leq f \). Similarly we can show that \( f \leq e \),
so that \( e = f \). Therefore
\[
\text{dom } e = \text{dom } f \implies e = f \quad \forall e, f \in \mathbb{E} \tag{II}
\]
Now we define \( \phi : e \rightarrow \text{dom } e \), then (I) and (II) show that
\( \phi \) is an isomorphism of \( (\mathbb{E}, .) \) onto \( \{(\text{dom } e, \cap) : e \in \mathbb{E}\} \).
This completes the proof.

Theorem 5.3.3. For each \( s \in S \), there is a split extension
\[
\mathbb{G} = \text{Ker } s + \text{dom } s \text{ such that for all } s, t \in S, \text{ we have }
\]
1) \( \text{Ker } s \subseteq \text{Ker } st \),
2) \( (\text{Ker } st \cap \text{dom } s) \cdot s \subseteq \text{Ker } t \).
Proof. For \( s \in S \), if \( g \in (\text{Ker } s \cap \text{dom } s) \), then
\[
g = h \cdot s^{-1} = h(s^{-1}s^{-1}) = g(ss^{-1}) = 0s^{-1} = 0.
\]
Therefore \( \text{Ker } s \cap \text{dom } s = \{0\} \). Moreover, for each \( g \in \mathbb{G} \),
\[
(g - gs \cdot s^{-1})s = gs - gs \cdot s^{-1}s = gs - gs = 0.
\]
So \( g - gs \cdot s^{-1} = h \in \text{Ker } s \), and since \( gs \cdot s^{-1} \in \text{dom } s \),
\[ g = h + g s s^{-1} e(\text{Ker } s + \text{dom } s), \] i.e. \( G \subset \text{Ker } s + \text{dom } s \).

Hence \( G = \text{Ker } s + \text{dom } s \) is a split extension.

For each \( s \in S \), \( g \in \text{Ker } s \implies g(st) = 0 \) and \( \forall s \in S \),
\[ \implies g \in \text{Ker } st. \]

Therefore \( \text{Ker } s \subset \text{Ker } st \), for all \( s, t \in S \). This completes the proof of (1).

For \( s, t \in S \),
\[ g \in (\text{Ker } st \cap \text{dom } s) \implies g st = 0 \text{ and } g = h s^{-1}, \]
for some \( h \in G \).

And \( g st = (h s^{-1}) st = (h s^{-1}s) t = g st = 0 \).

Therefore \( g s \in \text{Ker } t \) and hence we get
\[ (\text{Ker } st \cap \text{dom } s) \subset \text{Ker } t. \]

This proves (2).

Corollary 5.3.4. For each \( s \in S \), \( \text{Ker } s = \text{Ker } e \) for some idempotent \( e \in E \).

Proof. For each \( s \in S \), we have from above
\[ \text{Ker } s \subset \text{Ker } s s^{-1} \subset \text{Ker } (s s^{-1}s) = \text{Ker } s. \]

Therefore \( \text{Ker } s = \text{Ker } s s^{-1} \), where \( s s^{-1} \) is an idempotent.

Remark 5.3.5. Corollary 5.3.4 is true even if \( S \) is a regular semigroup with \( S \subset \text{End } (G) \).

Corollary 5.3.6. If \( s s^{-1} \leq s^{-1}s \) (\( s^{-1}s \leq ss^{-1} \), \( s \in S \)),
then \( \text{Ker } s \supset \text{Ker } s^{-1} \) (\( \text{Ker } s^{-1} \supset \text{Ker } s \)).
Proof. \( ss^{-1} \leq s^{-1}s \implies \text{Ker} (ss^{-1}) \supseteq \text{Ker} (s^{-1}s) \) (by 5.1.17).

Therefore, by 5.3.4, we get

\[ \text{Ker} s = \text{Ker} ss^{-1} \supseteq \text{Ker} s^{-1}s = \text{Ker} s^{-1}. \]

Corollary 5.3.7. If \( S \) is a semilattice of groups, then

\[ \text{Ker} s = \text{Ker} s^{-1}, \text{for each } s \in S. \]

Proof. By 5.1.14.(2) we have in this case \( ss^{-1} = s^{-1}s \),

for each \( s \in S \). Therefore by 5.3.4,

\[ \text{Ker} s = \text{Ker} ss^{-1} = \text{Ker} s^{-1}s = \text{Ker} s^{-1}. \]

Let \( T \) be an inverse semigroup of partial isomorphisms of a group \( G \). For each \( t \in T \), we denote by \( D_t \) and \( R_t \) the domain and range of \( t \) respectively. Obviously \( D_{tu} \subseteq D_t \) and \( R_{tu} \subseteq R_u \) for all \( t, u \in T \). Now we prove the converse of 5.3.3 combined with 5.3.1.

Theorem 5.3.8. Let \( T \) be an inverse semigroup of partial isomorphisms of a group \( G \) such that for each \( t \in T \) there exists a normal subgroup \( K_t \) of \( G \) satisfying

1) \( G = K_t + D_t \) is a split extension,

2) \( K_t \subseteq K_{tu} \)

3) \( (K_t \cap D_t)t \subseteq K_u \)

Then \( T \) can be embedded in \( \text{End}(G) \).

Proof. For each \( t \in T \), by (1), there exists a natural projection \( \pi_t : G \longrightarrow D_t \). Then \( \pi_t \) is an endomorphism of \( G \).
We define \( \alpha : T \rightarrow \text{End}(G) \) by \( t \alpha = \pi_t \). Clearly \( \alpha \) is well defined. For \( t, u \in T \), by (i) each \( g \in G \) is expressed as \( g = x + y \), where \( x \in K_{tu} \) and \( y \in D_{tu} \). So

\[
g(tu) \alpha = g(\pi_{tu}) = (x + y)\pi_{tu} = y(tu). \quad (I)
\]

Now \( x = x_1 + y_1 \), where \( x_1 \in K_t \) and \( y_1 \in D_t \). Therefore

\[
y_1 = -x_1 + x \in (K_t + K_{tu}) \subseteq K_{tu} \quad \text{(by (2))}.
\]

So \( y_1 \in (K_{tu} \cap D_t) \) and hence by (i) \( y_1 t \in K_u \). Also

\[
(x_1 + y_1) \in (D_t + D_{tu}) \subseteq D_t.
\]

Therefore

\[
g = x + y = x_1 + (y_1 + y),
\]

where right hand side belongs to \( (K_t + D_t) \).

Now

\[
g(t \alpha)(u \alpha) = g(t \alpha)(u \alpha) = [g(t \alpha)](u \alpha)
\]

\[
= \left[ (x_1 + (y_1 + y)) \pi_t \right](\pi_u)
\]

\[
= \left[ (y_1 + y) \pi_t \right](\pi_u)
\]

\[
= (y_1 t + y t)(\pi_u), \text{ as } y_1 \text{ and } y \in D_t,
\]

\[
= (y t) u, \text{ as } y t \in D_u, y_1 t \in K_u,
\]

\[
= y(tu). \quad (II).
\]

Now (I) and (II) together imply that

\[
g(tu) \alpha = g(t \alpha)(u \alpha).
\]

Since this is true for all \( g \in G \), we get

\[
(tu) \alpha = (t \alpha)(u \alpha).
\]

Hence \( \alpha \) is a semigroup homomorphism.

To complete the proof we need to show that \( \alpha \) is 1-1.
For that let \( ta = ua \) for some \( t, u \in T \). Then
\[
\pi_t a = \pi_u a.
\]
Now since \( T \) is an inverse semigroup
\[
t^{-1} a = (ta)^{-1} = (ua)^{-1} = u^{-1} a.
\]
Therefore \( \pi_{t^{-1}} t^{-1} = \pi_{u^{-1}} u^{-1} \), so \( R_{t^{-1}} = R_{u^{-1}} \). But since
\[
R_{t^{-1}} = D_t, R_{u^{-1}} = D_u,
\]
we get \( D_t = D_u \). Also for all \( x \in D_t = D_u \),
\[
x t = x (\pi_t t) = x (\pi_u u) = x u.
\]
Therefore \( t = u \) and hence \( a \) is an injection.

Remark. It is easy to check that \( (T a) \psi = T \), where \( \psi \)
is the mapping defined in the proof of theorem 5.3.1.

Theorem 5.3.9. Let \( S \subseteq \text{End}(G) \), and let \( K = \cap \text{Ker} s \; s \in S \).
Then \( S \) is embedded in \( \text{End}(G/K) \) and the corresponding \( K \)
is trivial.

Proof. For each \( s \in S \) we define \( s a : G/K \longrightarrow G/K \)
by \( g + K \mapsto g s + K \). Then for \( g, h \in G \),
\[
g + K = h + K \quad \Rightarrow \quad (g - h) \in K
\]
\[
\Rightarrow (g - h)s = 0
\]
\[
\Rightarrow gs = hs
\]
\[
\Rightarrow g s + K = h s + K,
\]
\[
\Rightarrow (g + K)(s a) = (h + K)(s a).
\]
Therefore \( sa \) is well defined. Now

\[
\begin{align*}
\{(g + K) + (h + K)\}(sa) &= \{(g + h) + K\}(sa) \\
&= (g + h)s + K \\
&= (gs + hs) + K \\
&= (gs + K) + (hs + K) \\
&= (g + K)(sa) + (h + K)(sa).
\end{align*}
\]

Thus \( sa \), for each \( s \in S \), is an endomorphism of \( G / K \) and we get a mapping \( \alpha : s \mapsto sa \), from \( S \) to \( \text{End}(G / K) \). We now prove that \( \alpha \) is a semigroup homomorphism which is \( 1-1 \).

Let \( s, t \in S \). Then for all \( g + K \in G / K \) we have

\[
\begin{align*}
(g + K)(st)a &= \{g(st)\} + K \\
&= \{(gs)t + K\} \\
&= \{gs + K\}(ta) \\
&= (g + K)(sa)(ta).
\end{align*}
\]

Therefore \( (st)a = (sa)(ta) \), i.e. \( \alpha \) is a semigroup homomorphism.

Now let \( sa = ta \), for some \( s, t \in S \). Then for all \( g \in G \) we have \( (g + K)(sa) = (g + K)(ta) \). Therefore for all \( g \in G \), \( gs + K = gt + K \). This implies that \( gs - gt \) is in \( K \). Therefore \( 0 = (gs - gt)(s^{-1}s) = g(s s^{-1}s) - g(t s^{-1}s) \).

94
This implies that $gs = gts^{-1}s$, for all $g \in G$. Therefore $s = t s^{-1}s$ and hence $s \leq t$. Similarly we can show that $t \leq s$. Thus we get $s = t$. This proves that $\alpha$ is 1-1, and therefore $S \subseteq \text{End}(G/K)$.

Now let $\overline{K} = \cap \{ \text{Ker} sa : s \in S \} \subseteq G/K$.

$g + K \in \overline{K} \implies$ for all $s \in S$, $(g + K)(sa) = 0$,

$\implies gs + K = 0$,

$\implies gs \in K$,

$\implies (gs) s^{-1}s = 0$,

$\implies gs = 0$,

$\implies g \in K$,

$\implies g + K = 0$.

Hence $\cap \{ \text{Ker} sa : s \in S \} = \{0\}$ in $G/K$.

5.4. D.G. Near-rings over Inverse Semigroups

J.D.P. Meldrum conjectured that "a d.g. near-ring $(R, S)$ over an inverse semigroup $S$, is faithful with a faithful representation on $(R, +)$". Example 5.2.1 gave us a clue to prove this conjecture, which apparently looked difficult.

First we prove a few technical lemmas.

Lemma 5.4.1. Let $(R, S)$ be a d.g. near-ring. Then each elem-
ent \( e \in E \), commutes with each element of the subgroup \( G_\{p \mid E \} \) of \((R, +)\).

Proof. Let \( e \in E \) and \( r \in G_\{p \mid E \} \). Then

\[
r = e e + \cdots + e e,
\]

where \( e_i \in E \), \( i = 1, \ldots, n \). Now we have

\[
er = e (e e + \cdots + e e) = e e e + \cdots + e e e = (e e + \cdots + e e) e = r e.
\]

From example 5.2.2 we see that it is not necessary for a d.g. near-ring \((R, S)\) to contain a left identity. In the following lemma we prove that each sub near-ring of \((R, S)\), generated by a finite subset of \( E \), contains a left identity. We call these left identities "local left identities".

Lemma 5.4.2. Let \((R, S)\) be a d.g. near-ring. For every finite subset \( \{ e_1, \ldots, e_n \} \) of \( E \), there exists an element \( x \) in \( N.R. \{ e_1, \ldots, e_n \} \), the near-ring generated by this subset, such that

1) \( xe_i = e_i \) for \( i = 1, \ldots, n \)

2) \( x^2 = x \)

96
3) \( x e = e x \), for all \( e \in E \).

Proof. We use induction to prove this result. It is trivially true for all subsets of \( E \) which contain only one element. We consider a subset of \( E \), containing two elements only, say \( \{ e_1, e_2 \} \).

Let \( x = e_1 - e_1 e_2 + e_2 \). Then

\[
xe_1 = (e_1 - e_1 e_2 + e_2)e_1
\]

\[
= e_1^2 - e_1 e_2 e_1 + e_2 e_1
\]

\[
= e_1 - e_2 e_1 + e_2 e_1
\]

\[
= e_1 - e_2 e + e_2 e
\]

Similarly we can show that \( xe_2 = e_2 \). This proves (1) and it follows from here that \( x \) is a left identity for \( \mathbb{N}_R \{ e_1, e_2 \} \), and \( x \) belongs to this near-ring. Therefore \( x^2 = x \). This proves (2).

Since \( x \in \text{Gp}\{ E \} \), it follows from lemma 5.4.1 that \( x e = e x \) for all \( e \in E \). This completes the proof of the lemma for subsets of \( E \), which contain two elements.

Now we suppose that the lemma is true for all subsets of \( E \) which contain \( n - 1 \) elements, for \( n > 2 \), and consider a subset \( \{ e_1, \ldots, e_n \} \) of \( E \), containing \( n \) elements. By the induction hypothesis, there exists an element \( x \in \mathbb{N}_R \{ e_1, \ldots, e_{n-1} \} \) satisfying the three conditions of the lemma. Let

\[
y = x - e_n x + e_n
\]
Then for \( i = 1, \ldots, n - 1 \), by 5.4.1 and induction hypothesis, we have 

\[ ye_i = (x - e_n x + e_n) e_i \]

\[ = xe_i - e_n xe_i + e_ne_i \]

\[ = e_i - e_ne_i + e_ne_i \]

\[ = e_1 \]

and 

\[ ye_n = (x - e_n x + e_n) e_n \]

\[ = xe_n - e_n xe_n + e_ne_n \]

\[ = e_n \]

Therefore 

\[ ye_i = e_i \quad \text{for} \quad i = 1, \ldots, n \].

Hence, \( y \) is a left identity for \( N R \{ e_1, \ldots, e_n \} \). This gives us 

\[ y^2 = y \]

Since \( y \in \text{Gp}[E] \), by lemma 5.4.1, \( e y = ye \), for all \( e \in E \).

This completes the proof.

Corollary 5.4.6. Let \( S' \) be a finite inverse subsemigroup of \( S \). Then the sub \( d.g. \) near-ring \( (R', S') \) of \( (R, S) \) has a left identity.

Proof. Since \( S' \) is finite, the number of idempotents in \( S' \) is also finite. Let \( \{ e_1, \ldots, e_n \} \) be the set of idempotents of \( S' \). By lemma 5.4.2 there exists a left identity \( x \) in the near-ring generated by \( \{ e_1, \ldots, e_n \} \). We claim that \( x \) is a left identity of \( (R', S') \). For that let \( r \in (R', S') \). Then

\[ r = e_1 t_1 + \cdots + e_k t_k, \quad t_i \in S' \quad i = 1, \ldots, k. \]

Therefore 

\[ r = e_1 f_1 t_1 + \cdots + e_k f_k t_k, \quad \text{where} \quad f_i = t_i t_i^{-1} \quad \text{for each} \ i. \]

Since each \( f_i \in S' \), \( \{ f_1, \ldots, f_k \} \) is a subset.
We get

\[ x r = x ( \epsilon_1 f_1 t_1 + \ldots + \epsilon_k f_k t_k ) \]

\[ = \epsilon_1 x f_1 t_1 + \ldots + \epsilon_k x f_k t_k \]

\[ = \epsilon f_1 t_1 + \ldots + \epsilon f_k t_k \]

\[ = r. \]

Now we come to the main result of this section.

Theorem 5.4.4. If \( S \) is an inverse semigroup, then every \( \mathbb{Z} \)-near-ring \( (R,S) \) is faithful with a faithful representation on \( (R,+) \).

Proof. Obviously \( (R,S) \) has a right regular representation on \( (R,+) \). Let \( 0 \neq (\epsilon_1 e_1 s_1 + \ldots + \epsilon_n e_n s_n) = r \in (R,S) \).

By lemma 5.4.2, a left identity \( x \) exists in \( \mathbb{Z}.R.1 e_1, \ldots, e_n \).

As in the proof of 5.4.3 we can show that \( x r = r \), which proves that for each non-zero \( r \in (R,S) \) there exists an element \( x = x_r \), in \( (R,+) \), such that \( x r = r \). Hence by 1.6.11, \( (R,S) \) has a faithful representation on \( (R,+) \).

Corollary 5.4.5. Let \( K = \cap \{ \text{Ker } s : s \in S \} \) and let \( E_R \) be the set of idempotents of \( (R,+) \). Then \( K \cap E_R = \{0\} \).

Proof. Let \( r \in K \) then \( r = \epsilon_1 s_1 + \ldots + \epsilon_n s_n \), and

\[ r^2 = r (\epsilon_1 s_1 + \ldots + \epsilon_n s_n) \]

\[ = \epsilon_1 r s_1 + \ldots + \epsilon_n r s_n \]

\[ = 0 + \ldots + 0 = 0. \]
Thus \( r \in E_R \) only if \( r = 0 \). Hence we have proved the result.

Corollary 5.4.6. Let \((R,S)\) be a d.g. near-ring. Then for every ideal \( I \) of \((R,S)\), \((R,S)/I\) is faithful.

Proof. Since \( S \) is inverse, \((S+I)/I\) is an inverse semigroup by 5.1.6. Therefore, by 5.4.4, \((R,S)/I\) acts faithfully on \((R/I,+)\).

Theorem 5.4.7. Let \((R,S)\) be a d.g. near-ring and let

\[
I = \{ x \in (R,S) : x e = 0 \text{ for some } e \in E \}.
\]

Then \( I \) is an ideal of \((R,S)\).

Proof. Clearly \( I = \bigcup \{ \ker e : e \in E \} \) and \( \forall e, f \in E, \ker e \cup \ker f \subseteq \ker ef \). Therefore \( I \) is a normal subroup of \((R,+)\).

Let \( r \in (R,S), s \in S \), and \( x \in I \). Then there exists an \( e \) in \( E \) such that \( xe = 0 \). Now

\[
(rx)e = r(xe) = r0 = 0
\]

implies that \((rx) \in I\), i.e. \(RI \subseteq I\). Also, by 5.1.5 (a),

\[
0 = (xe)s = x(es) = x(sf) = (xs)f, \text{ for some } f \in E.
\]

It follows that \((xs) \in I\), i.e. \(IS \subseteq I\). Hence the result follows by 1.5.9.

Note that in general \( I \) is not a zero ideal, since in example 5.2.3, there are \( e \in E \) with \( \ker e \neq \{0\} \).

Let \( \theta : (R,S) \longrightarrow (R,S)/I \) be the natural homomorphism, where \( I \) is the ideal of \((R,S)\) defined in 5.4.7. Then we prove
Theorem 5.4.8. \( S \theta \) is a group.

Proof. \( [ \theta / S ] = \{ (s, t) \in S \times S : s \theta = t \theta \} \) is a congruence on \( S \). Also

\[
\begin{align*}
    s \theta = t \theta & \iff s + I = t + I \text{ in } (R, S)/I \\
    & \iff (s - t) \in I \\
    & \iff (s - t)e = 0 \text{ for some } e \in E \\
    & \iff s e = t e.
\end{align*}
\]

Hence \( [ \theta / S ] = \{ (s, t) \in S \times S : s e = t e \text{ for some } e \in E \} \), which is the minimum group congruence \( \sigma \) on \( S \). Thus

\[
S \theta = (S + I)/I = S/\left[ \theta / S \right] = S/\sigma,
\]

is a group, and \( \theta / S = \sigma^5 \).

Let \( C \) be the category of all d.g. near-rings over inverse semigroups and let \( \mathcal{D} \) be a subcategory of \( C \) which is the category of all d.g. near-rings over groups. Clearly \( (R, S)/I \) defined above is in \( \mathcal{D} \).

Theorem 5.4.9. \( (R, S)/I \) together with the natural homomorphism

\[
\theta : (R, S) \rightarrow (R, S)/I \text{ is the coreflection of } (R, S) \in C \text{ in } \mathcal{D}.
\]

Proof. Let \( \phi \) be a d.g. near-ring homomorphism from \( (R, S) \) to \( (T, G) \in \mathcal{D} \). Then, by 5.1.8 (2), \( S \phi \) is a subgroup of \( G \).

Let \( r \in I \), and let \( e \in E \) be such that \( re = 0 \). Then
Therefore \( r \in \text{Ker } \phi \). So there exists a unique d.g. near-ring homomorphism \( \psi : (R,S)/I \longrightarrow (T,G) \) such that \( \theta \psi = \phi \). This completes the proof.

Since each d.g. near-ring \( (R,S) \in \mathcal{C} \), has an associated d.g. near-ring in \( \mathcal{D} \), we get the following corollary.

**Corollary 5.4.10.** \( \mathcal{D} \) is a coreflective subcategory of \( \mathcal{C} \).

Suppose that the semilattice \( E \) of idempotents of \( S \) is updirected. Therefore, since \( e \leq e' \) in \( E \) implies that \( eS \subseteq e'S \) and conversely, for each pair \( eS, fS \) of sub-semigroups of \( S \) we get a sub-semigroup \( gS \) of \( S \), for some \( g \in E \), such that \( eS \subseteq gS, fS \subseteq gS \). Consider the diagram \( D \) in \( \mathcal{S} \) over the scheme \( (E,M,d) \), where for \( e \leq f, md = (e,f) \), \( m_{ef} : eS \longrightarrow fS \) is the inclusion map. Let \( \{ \theta_e : eS \longrightarrow S \}_{e \in E} \) be the family of inclusion homomorphisms. Then we prove the following lemma.

**Lemma 5.4.11.** \( \{ \theta_e : eS \longrightarrow S \}_{e \in E} \) is a colimit of \( D \) in \( \mathcal{S} \).

**Proof.** For all \( m \in M \) and all \( s \in S \) we have

\[
(eS)m_{ef}\theta_f = (eS)\theta_f = (eS)\theta_e,
\]

since \( eS \subseteq fS \) and \( \theta_f/(eS) = \theta_e \). Therefore \( m_{ef}\theta_f = \theta_e \)

for all \( m \in M \), i.e. \( \{ \theta_e : eS \longrightarrow S \}_{e \in E} \) is a cocompatible

102
family for $D$.  

Let $\{ \alpha_e : e S \longrightarrow T \}_{e \in E}$ be a cocompatible family for $D$ in $\mathcal{C}$. Define $\phi : S \longrightarrow T$ by $s \phi = s \alpha_s s^{-1}$. Note that $s = s s^{-1} s \in (s s^{-1}) S$. Moreover $s \phi = s \alpha_e$ for each $e$ in $E$ such that $s \in e S$, for then $(s s^{-1}) S \subseteq e S$. For $s, t$ in $S$, $s \in e S$, $t \in f S$, there is a sub-semigroup $g S$ for some $g \in E$, such that $e S \subseteq g S$, $f S \subseteq g S$. Then $s t \in g S$. 

Now $(s \phi)(t \phi) = (s \alpha_g)(t \alpha_g) = (s t) \alpha_g = (s t) \phi$. Hence $\phi$ is a semigroup homomorphism. For each $s \in S$, $e \in E$, we have 

$$(e s) \theta_e \phi = (e s) \phi = (e s) \alpha_e.$$ 

Therefore $\theta_e \phi = \alpha_e$ for each $e \in E$. Moreover it is clear that $\phi$ is unique with this property. Thus we have proved the lemma. 

Now $\{ (e R, e S) : e \in E \}$ is a family of sub $d.g.$ near-rings of $(R, S)$ each having a left identity, namely the corresponding $e$.

$\{ \theta_e : (e R, e S) \longrightarrow (R, S) \}_{e \in E}$ is a family of $d.g.$ near-ring inclusion homomorphisms. Moreover $e \leq f$ implies that $e R \subseteq f R$. So we get the corresponding diagram $D$ over $(E, M, d)$ in the category $\mathcal{A}$ of all $d.g.$ near-rings. In fact $D$ is in $\mathcal{B}$ since each $e S$ has a left identity namely $e$. Note that $\mathcal{B}$ is the category of all faithful $d.g.$ near-rings. 

Theorem 5.4.12. $\{ \theta_e : (e R, e S) \longrightarrow (R, S) \}_{e \in E}$ is a colimit of $D$ in $\mathcal{A}$. 

103
Proof. Obviously the given family of homomorphisms is cocompatible for \( D \). Let \( \{ a_e : (eR, eS) \rightarrow (T, U) \} \) be a cocompatible family for \( D \) in \( \mathcal{A} \). Then, since \( \{ a_e : eS \rightarrow U \} \) is cocompatible for \( D \) in \( \mathcal{B} \), by lemma 5.4.11, there exists a unique semigroup homomorphism \( \phi : S \rightarrow U \) such that \( \theta_e \phi = a_e \) for each \( e \in E \). To complete the proof we just need to show that \( \phi \) extends to a group homomorphism, then the result will follow from 1.5.13 and from the uniqueness of \( \phi \) with \( \theta_e \phi = a_e \), \( e \in E \), as semigroup homomorphisms. Let \( \varepsilon_1 s_1 + \cdots + \varepsilon_n s_n = 0 \) in \((R,S)\). Then

\[
0 = \varepsilon_1 s_1 + \cdots + \varepsilon_n s_n = \varepsilon_1 s_1 + \cdots + \varepsilon_n s_n .
\]

As in example 5.2.2, there exists \( g \in E \) such that \( e_i < g \) and hence \( e_i S \subseteq g S \) for \( i = 1, \ldots, n \). Therefore

\[
(\varepsilon_1 e_1 s_1 + \cdots + \varepsilon_n e_n s_n) \in (gR, gS) ,
\]

and hence is zero in \((gR, gS)\). Now

\[
\varepsilon_1 s_1 \phi + \cdots + \varepsilon_n s_n \phi = \varepsilon_1 e_1 s_1 a_1 + \cdots + \varepsilon_n e_n s_n a_n
\]

\[
= \varepsilon_1 s_1 a_g + \cdots + \varepsilon_n s_n a_g
\]

\[
= (\varepsilon_1 e_1 s_1 + \cdots + \varepsilon_n e_n s_n) a_g
\]

\[
= 0 a_g
\]

\[
= 0 ,
\]

as \( a_g \) is a d.g. near-ring homomorphism. Therefore \( \phi \) is well-defined and hence can be extended to a group homomorphism.
Remark. Since each member of the family \( \{(eR, eS) : e \in E \} \)
of d.£. near-rings, is in \( \mathfrak{B} \), the diagram \( D \), defined above, is in \( \mathfrak{B} \). We have seen, in chapter 2, that all colimits of diagrams in \( \mathfrak{B} \) exist in \( \mathfrak{B} \). Hence \((R, S)\) is in \( \mathfrak{B} \). Thus theorem 5.4.12 provides an elementary proof of faithfulness of a d.£. near-ring \((R, S)\) over an inverse semigroup \( S \) with updirected semi-lattice \( E \) of idempotents. Note from example 5.2.1 that \( E \) need not be updirected in general.
Chapter 6

Group d.g. near-rings

6.1. Preliminaries

Let \((R, S)\) be a faithful d.g. near-ring and let \(G\) be a multiplicative group. Let \(X\) be any set and define:

\[ Y = X \times G = \{(x, g) : x \in X, g \in G\}, \]

\[ X_g = \{(x, g) : x \in X\}. \]

Let \(F = \text{Fr}(Y, R, S)\) be the free \((R, S)\)-group on the set \(Y\) and let \(F_g = \text{Fr}(X_g, R, S)\) be the free \((R, S)\)-group on the set \(X_g\), for each \(g \in G\). Then we can consider each \(F_g\) as a subgroup of \(F\).

Moreover we have \(F = \ast \{ F_g : g \in G \}\), the free \((R, S)\)-product of \(\{ F_g : g \in G \}\). Since for all \(g, h \in G\), the free \((R, S)\)-generating sets \(X_g, X_h\) of \(F_g\) and \(F_h\) respectively, are in 1-1 correspondence, \(F_g\) and \(F_h\) are isomorphic as \((R, S)\)-groups. Therefore any permutation of \(G\) can be extended to an \((R, S)\)-automorphism of \(F\).

Now \(S\) is a semigroup and \(G\) is a group of endomorphisms of \(F\).

The d.g. near-ring generated by \(SG = \{ s g : s \in S, g \in G \}\) in \(E(F)\) is denoted by \((R(G), SG)\) and is called the group d.g. near-ring of \((R, S)\) on \(G\).

**Theorem 6.1.1.** If \(r \in (R, S), g \in G\), then \(rg = gr\) in \((R(G), SG)\).

**Theorem 6.1.2.** \(F\) is the free \((R(G), SG)\)-group on the set \(X\).

**Theorem 6.1.3.** \((R(G), +)\) considered as an \((R, S)\)-group, is an
orthogonal sum of a set \( \{ (R_g, +) : g \in G \} \) of \((R, S)_g\) groups, where \((R_g, +)\) is isomorphic to \((R, +)\) for each \(g \in G\).

6.2. A near-ring for a near-ring

Let \((R, S)\) be a near-ring and let \(G\) be a multiplicative group. Then we have the upper faithful near-ring \((R, S)\) together with a near-ring epimorphism \(\theta : (R, S) \twoheadrightarrow (R, S)\). For each \(g \in G\), we have groups \(R_g, R_g\) which are isomorphic copies of \((R, +)\) and \((R, +)\) respectively. Then \(R_g\) is isomorphic to \(R_g / I_g\) under the induced homomorphism \(\theta_g : R_g \twoheadrightarrow R_g\), obtained from \(\theta\).

Let \(* R_g\) and \(* R\) be the free products of the families \(\{ \overline{R}_g : g \in G \}\) and \(\{ R_g : g \in G \}\), respectively, of groups. Then there exists a unique epimorphism \(\theta^* : * R_g \twoheadrightarrow * R\)

\[ g \in G_g \]

such that \(\theta^* / R_g = \theta_g\), for each \(g \in G\). Let \(K\) be the normal closure of \(* I\) in \(* R\). It is easy to see that \(K\) is contained in \(\operatorname{Ker} \theta^*\).

From now onwards, we write \(* \overline{R}_g\) for \(* R_g\) and similar notation for the other free products, since these are convenient.

Lemma 6.2.1. \(K\) is the kernel of \(\theta^*\).

Proof. Let \(\pi : * \overline{R}_g \twoheadrightarrow (R_g / \operatorname{Ker} \theta_g)\) and \(\pi^* : * \overline{R}_g \twoheadrightarrow (* \overline{R}_g) / K\) be the natural homomorphisms. Since \(I_g \subseteq K\) for each \(g \in G\), \(\pi^*\) factors through \(* (R_g / I_g)\) uniquely.
as can be seen from the expression:

\[ \pi' = (\theta^* a) h, \quad (1) \]

where \( a \) is the isomorphism from \( \ast R_g \) onto \( \ast (\overline{R_g/\mathbb{I}_g}) \) and \( h : \ast (\overline{R_g/\mathbb{I}_g}) \rightarrow (\ast \overline{R_g})/K \) is an epimorphism.

Since \( \ast R_g \cong (\ast \overline{R_g})/\text{Ker} \theta^* \), denoting this epimorphism by \( b \) we get a unique homomorphism \( \kappa : (\ast \overline{R_g})/\text{Ker} \theta^* \rightarrow (\ast \overline{R_g})/K \)

such that

\[ b \kappa = ah \quad (2) \]

Since \( K \subseteq \text{Ker} \theta^* \), there exists a unique homomorphism

\[ \kappa' : (\ast \overline{R_g})/K \rightarrow (\ast \overline{R_g})/\text{Ker} \theta^* \], such that

\[ \pi' \kappa' = \pi \quad (3) \]

It is clear that \( \pi = \theta^* b \).

Thus we get a commutative diagram:

Now from (1), (2), (3) and (4) we get

\[ \pi \kappa \kappa' = \theta^* b \kappa \kappa' = \theta^* (ah) \kappa' = \pi' \kappa' = \pi, \quad (5) \]

\[ \pi' \kappa' \kappa = \pi \kappa = \theta^* b \kappa = \theta^* (ah) = \pi' \quad (6) \]

Since \( \pi, \pi' \) are epimorphisms, (5) and (6) respectively imply that
\[ k \kappa' = I(\ast \overline{R}_g) / \text{Ker } \theta^* , \quad (7) \]
\[ \kappa' \kappa = I(\ast \overline{R}_g) / K . \quad (8) \]

Hence \( \kappa \) is an isomorphism and we get \( K = \text{Ker } \theta^* \).

Let \( \overline{F} = F_r(Y, \overline{R}, S) \), where \( Y = X \times G \) and \( X \) is any set.

Then, as in § 6.1, we can define \( S \) as a semigroup of endomorphisms, and \( G \) as a group of \( (\overline{R}, S) \)-automorphisms of \( \overline{F} \) and then construct the group d.g. near-ring \( (\overline{R}(G), SG) \). Clearly \( (\overline{R}(G), +) \)
is a homomorphic image of \( \ast \overline{R}_G \) under the homomorphism

\[ \phi : r_{i, g_{1}} + \cdots + r_{n, g_{n}} \longmapsto r_{i, g_{1}} + \cdots + r_{n, g_{n}} \]
\[ r_{i} \in \overline{R}, \; g_{i} \in G . \]

Let \( J \) be the ideal of \( (\overline{R}(G), SG) \) generated by

\[ IG = \{ I g : g \in G \} . \]

Then by 1.5.10, \( J \) is the normal subgroup of \( (\overline{R}(G), +) \) generated by the set

\[ (\overline{R}(G)) IG (SG) = \{ \gamma a g , \gamma (a g)(s h) , (a g)(s h) , a g : \]
\[ \gamma \in \overline{R}(G) , a \in I , g , h \in G , s \in S \} . \]

Since \( I \) is an ideal of \( (\overline{R}, S) \) and \( r g = g r \) in \( (\overline{R}(G), SG) \) for all \( r \in \overline{R}, g \in G \), we get \( a g s h = a' g' \in IG \). Therefore \( J \) is the normal subgroup of \( (\overline{R}(G), +) \), generated by the set

\[ \{ \gamma a g , a g : \gamma \in \overline{R}(G) , a \in I , g \in G \} . \]

Now we get a quotient d.g. near-ring \( (\overline{R}(G), SG) / J \) with the natural
homomorphism $\pi : (\overline{R}(G), SG) \rightarrow (\overline{R}(G), SG)/J$.

Definition 6.2.2. We call $(\overline{R}(G), SG)/J$ the group d.g. near-ring of $(R, S)$ on $G$.

$(\overline{R}(G), SG)/J$ is independent of $X$, since $(\overline{R}(G), SG)$ is so.

Theorem 6.2.3. $(\overline{R}(G)/J, J^G)$ is a homomorphic image of $\ast R_g$.

Proof. It is easy to see that the homomorphism $\phi$ defined above, maps $\ast I_g$ into $Gp \{(\overline{R}(G) IG (SG)\}$ and hence $K$ into $J$. Therefore $K \subseteq \ker \phi$ and so there exists a unique homomorphism $\phi : \ast R_g \rightarrow (\overline{R}(G)/J, J^G)$ such that $\theta \ast \phi = \phi \pi$. Clearly $\phi$ is onto. This completes the proof.

Now for $(R, S)$ we have the lower faithful d.g. near-ring $(R, S)$ with a d.g. near-ring epimorphism $\theta : (R, S) \rightarrow (R, S)$. Again as in § 6.1, we can define $\triangleleft$ as a semigroup of endomorphisms and $G$ as a group of $(R, S)$-automorphisms of $F = Fr(Y, R, S)$, the free $(R, S)$-group on $Y$, and then construct the group d.g. near-ring $(\overline{R}(G), SG)$. Since $\theta \phi : (R, S) \rightarrow (R, S)$ is a d.g. near-ring homomorphism $F$ is an $(\overline{R}, S)$-group. Therefore there exists a unique $(\overline{R}, S)$-homomorphism $\mu : \overline{F} \rightarrow F$ such that $\mu / Y = I_Y$. Clearly $\mu$ is onto and maps $(x, g) \mapsto (x, g)(r(\theta \phi))$ for $r \in \overline{R}$.

Let $\alpha : SG \rightarrow SG$ be defined by $s g \mapsto \alpha g = (s \theta \phi) g$.

Then $\alpha$ is a semigroup homomorphism, because from above we get:

$$\{(s_1, g_1)(s_2, g_2)\} \alpha = \{(s_1, s_2)(g_1, g_2)\} \alpha = \{(s_1, s_2) \theta \phi \}(g_1, g_2)$$
Now we prove that $\alpha$ extends to a group homomorphism.

Let $\gamma = \epsilon_1 s_1 e_1 + \cdots + \epsilon_n s_n e_n$ be 0 in $(\mathbb{R}(G), +)$. Then $(x, g) \gamma = 0$ in $F$ for all $(x, g) \in Y$. Therefore, for each $(x, g) \in Y$, in $F$ we have:

$$0 = \{ (x, g) \gamma \} \mu$$

$$= \{ (x, g)(\epsilon_1 s_1 e_1 + \cdots + \epsilon_n s_n e_n) \} \mu$$

$$= \{ \epsilon_1 (x, g) s_1 e_1 + \cdots + \epsilon_n (x, g) s_n e_n \} \mu$$

$$= \{ \epsilon_1 (x, g) (s_1 g) e_1 + \cdots + \epsilon_n (x, g) (s_n g) e_n \}$$

$$= (x, g) \{ \epsilon_1 (s_1 g) a + \cdots + \epsilon_n (s_n g) a \}.$$  

Therefore $y \{ \epsilon_1 (s_1 g) a + \cdots + \epsilon_n (s_n g) a \} = 0$ in $F$ for each $y \in Y$. Hence $\epsilon_1 (s_1 g) a + \cdots + \epsilon_n (s_n g) a = 0$ in $(\mathbb{R}(G), sG)$. Therefore $\alpha$ extends to a group homomorphism $\alpha$ from $(\mathbb{R}(G), +)$ to $(\mathbb{R}(G), +)$. Hence $\alpha : (\mathbb{R}(G), sG) \rightarrow (\mathbb{R}(G), sG)$ is a d.g. near-ring homomorphism. It is clear that $\alpha$ is onto.

Now, since $\alpha \cdot a = (a \cdot g) g = (0 g) g = 0 g = 0$ for each $a \in I$ and each $g \in G$, we get $J \subseteq \text{Ker} \alpha$. So there exists a unique d.g. near-ring homomorphism

$$\beta : (\mathbb{R}(G), sG) / J \rightarrow (\mathbb{R}(G), sG).$$
such that $\pi \beta = \alpha$. Thus we have proved the following result:

Theorem 6.2.4. $(\mathbb{R}(G), SG)$ is a homomorphic image of $(\mathbb{R}(G), SG) / J$.

Now for each $g \in G$, $(\mathbb{R}_g + J)/J = \{ r_g + J : r \in \mathbb{R} \}$ is an additive subgroup of $(\mathbb{R}(G)/J, +)$ and

$$(\mathbb{R}(G)/J, +) = GP \{ (\mathbb{R}_g + J)/J : g \in G \}.$$ 

The groups $\{ (\mathbb{R}_g + J)/J : g \in G \}$ can be considered as $(R, S)$-groups in a natural way.

Lemma 6.2.5. For $r \in \mathbb{R}$, $g \in G$, $rg$ belongs to $J$ if and only if $r \in I$.

Proof. Clearly $r \in I$ implies that $(rg) \in I g \subseteq J$.

$$\sum \mathbb{R}_g,$$ the direct sum of $\{ \mathbb{R}_g : g \in G \}$, is a homomorphic image of $(\mathbb{R}(G), +)$. We denote this homomorphism by $\xi$. Also $\sum \mathbb{R}_g$ is isomorphic to $\mathbb{R}(G)/C$, where $C$ is the cartesian subgroup of $(\mathbb{R}(G), +)$ ($\S 1 [12]$). Clearly $\xi$ maps $J$ into $\sum I_g$ and therefore $J \cap \mathbb{R}_g$ into $I_g \subseteq \sum I_g$, for all $g \in G$. Hence $(rg)\xi = r_g = a_g$, for some $a \in I$, if $rg \in J$. Hence $r = a \in I$. This completes the proof.

Theorem 6.2.6. For each $g \in G$, $(\mathbb{R}_g + J)/J$ and $\mathbb{R}_g$ are isomorphic as $(R, S)$-groups.
Proof. Define \( \eta : (\overline{R}_g + J) / J \longrightarrow R_g \) by \( s g \longmapsto (s \theta) g \). By the above lemma \( \eta \) is well-defined and injective. Therefore it extends to a group homomorphism \( \eta : (\overline{R}_g + J) / J \longrightarrow R_g \), which is clearly onto. Hence we get \( (\overline{R}_g + J) / J \cong R_g \). It is easy to see that \( \eta \) is an \((R,S)_g\)-homomorphism.

We have proved that \((\overline{R}_G(J), +)\) is a group generated by a set of \((R_S)_g\)-groups. It may be interesting to see whether or not \((\overline{R}_G(J), +)\) is an \((R,S)_g\)-group itself.
References


