N-dimensional numerical solution of stochastic differential equations

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To my father and mother,
to whom I owe everything.
Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Qiming Li)
Abstract

To find approximate solution of general $nD$ Stochastic Differential Equations (SDEs), new SDEs approximation schemes are required. In this thesis, we introduce an order $\gamma \ (\gamma > 1/2)$ strong scheme and an improved weak scheme for the numerical approximation of solutions to SDEs, driven by $N$ Wiener processes. The strong scheme, called the $3/4$ Scheme, which is dependent on a differently constructed Brownian path, involves the area terms to bring better asymptotic accuracy than any numerical method based on classic constructed $N$ Dimension Brownian path. We demonstrate how to construct such a Brownian path, besides how to subdivide the Brownian path to get a sequence of approximations which converges pathwise as $h$-tends to 0. We prove that the convergence of such method is guaranteed if the time step size $h$ tends to 0.

We also present the Improved Weak Euler Scheme (IWES), whose sample error is much smaller than the classic Weak Euler Scheme's. The method reduces computation load and the sample error, which is generated during the Monte-Carlo(MC) approximation, by balancing the times of Euler iteration and MC simulation. A further improved IWES can be achieved by reusing the Brownian path. We prove that the extra sample error from reusing Brownian path is negligible in the latter method.
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Chapter 1

Introduction

1.1 Introduction

The original aim of this thesis was to obtain the Strong Approximation and Weak Approximation for SDEs. In the Chapter 1, we introduce SDEs and some useful mathematics tools. In the Chapter 2, 3, 4, 5, we discuss the 3/4 Scheme, a Strong Approximation, and in the Chapter 6, we discuss the Improved Euler Method, a Weak Approximation.

To get a strong approximation method of order 3/4 using an approximation to sums of area integrals, we construct the approximate Brownian path differently, whose single step-size is \( h \), so that one can approximate the area terms. The procedure of applying the 3/4 Method is similar to that of applying the Strong Taylor Scheme shown in Platen and Kloeden's book [17]: first, generate the approximate Brownian trajectory; second, apply the approximation scheme to get the approximate solution of the SDEs. The first step for 2D, 3D, nD cases will be shown in the initial construction parts in the Chapter 3, 4, 5 respectively. The second step is shown in the Chapter 2.

Furthermore, we show how to generate a sequence of Brownian trajectory approximations for the same Brownian path, using the 3/4 method for a decreasing sequence of stepsizes \( h \). As Gaines and Lyons [7] indicated, it will demand a lot more work and imagination to generate approximate Brownian trajectory \( (W^{(1)}, W^{(2)}) \) over steps of length \( h/4 \) given the corresponding approximate path for stepsize \( h \). One must subdivide the approximate Brownian trajectory with step size \( h \) into an approximate trajectory with step size \( h/4 \), keeping these two approximate trajectories consistent, such that the one with step size \( h/4 \) will be an approximation for the same Brownian trajectory but with smaller step size. We will reveal the subdivision process, for 2D, 3D, nD cases in Chapter 3, 4, 5 respectively. In each of these chapters, we will present theorems, proving the convergence of the Brownian path generated.

The Chapter 3 will articulate every step, from the initial construction of the approximate Brownian to the subdivision of this approximate trajectory, for the 2D case. There is nothing different from Davie's paper, but a bit more detail.

The Chapter 4 will sort out several issues which arise in 3D case. Although it shares the fundamental idea, the "Law of large numbers", with the 2D case, an
Numerical Approximation for SDE

adjust of constructing the Brownian trajectory is needed to approximate the area term properly. The subdivision process will be different from the methods in 2D case as well, because the structure of the 3D approximate Brownian trajectory is different from the 2D trajectory.

We will generalise the construction and subdivision of the Brownian trajectory in the Chapter 5. We provide theorems about the area terms, revealing their conditional expectation and covariance, which is essential when approximating these area terms. And general subdivision process is expressed in matrix form which will make the subdivision of the nD Brownian trajectory clear.

A different topic is discussed in the Chapter 6. We consider how to minimise the sample error when applying the Euler method to get the weak approximation. Because the step size of the approximate Brownian trajectory and the number of the Monte Carlo Simulations play different weight in generating sample error when we apply the Euler method. We try to balance these two factors, and minimise the sample error. An even improvement is achieved when we reuse the Brownian trajectories. A theorem proved during this procedure shows that the correlation between $(X_h - X_{h/2})$ and $(X_{h/2} - X_{h/4})$ is small, where $X_h$ and $X_{h/2}$ are the approximations for SDEs with step size $h$ and $h/2$ respectively, based on the same Brownian trajectory.

1.1.1 The Development of Stochastic Differential Equation

What is Stochastic Differential Equation (SDE)?

SDE can be seen as deterministic differential equations with an additional random noise term “dW_t”.

- **Deterministic Differential Equations:** Let $x_t$ be a $d \times 1$ vector $x_t = (x_t(1), \ldots, x_t(d))^T$, and $A(t, x_t)$ be a $d \times 1$-vector function $A : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$. Then one can have the $d$-Dimension deterministic differential equations.
  
  \[ dx_t = A(t, x_t) \, dt. \]

- **Stochastic Differential Equations:** Let $x_t$ be a $d \times 1$ vector $x_t = (x_t(1), \ldots, x_t(d))^T$, $dW$ be a $n \times 1$ Brownian vector $dW = (dW(1), \ldots, dW(n))^T$, $A(t, x_t)$ be a $d \times 1$-vector function $A : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, and $B$ be a $d \times n$-matrix function $B : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times n}$. Then one can have the $d$-Dimension stochastic differential equations driven by $n$-dimension Brownian trajectory.
  
  \[ dx_t = A(t, x_t) \, dt + B(t, x_t) \, dW_t. \]  \hspace{1cm} (1.1)

The main difference between Ordinary Differential Equation (ODE) and SDE is Brownian motion, which was named after the biologist Robert Brown. In the early 20th century, Bachelier, Einstein, and Wiener, developed the mathematical theory of Brownian motion. Wiener was the first to put Brownian motion on a
firm mathematical basis, hence we also call Brownian process with Wiener process, using \("W\) to stand for Wiener process. Meanwhile, Markov, Smoluchowski, and Pólya, among others, developed the general theory of random walk. Conditional expectation, Martingales, Riemann-Stieltjes Integral are also preliminary.

If \(\int_0^T f(t) \, dg(t)\) exists as a Riemann-Stieltjes integral for all continuous functions \(f\) on \([0,T]\), then \(g\) will need to have bounded variation. Because we need to consider the integrals \(\int_0^T f(B_t, t) \, dB_t\) for all continuous deterministic functions \(f\) on \([0,T]\), Riemann-Stieltjes integral will fail, since Brownian trajectory \(B\) does not have bounded variation on any finite interval. Hence, Itô integral \(\int_0^T f(B_t) \, dB_t\) and Stratonovich integral \(\int_0^T f(B_t) \circ dB_t\) are introduced by Itô and Stratonovich respectively. These two integrals can be transformed from one to another by the formula \(\int_0^T f(B_t) \circ dB_t = \int_0^T f(B_t) \, dB_t + \frac{1}{2} f'(B_t) \, dt\), although they use different chain rules. In this thesis, we consider the Itô form SDEs.

**What is the solution of SDE?**

There are two kinds of solutions to a SDE. These are call strong solution and weak solution. One can find the definitions from Kloeden and Platen’s book[17].

**Why we need numerical scheme for Stochastic Differential Equations?**

Analogous to the difficulty of the ODE, we have difficulty in finding the solution of SDE, although we could prove the uniqueness and existence in some cases. The Numerical Approximation becomes a practical way to face this difficulty. The SDE numerical approximation was fast developing since 1978. Milstein, Talay, and Tretyakov among others provide lots of works in numerical approximation of SDEs.

**How accurate shall we approximate?**

Before we could answer this question, we are stating the definition of Strong Convergence and Weak Convergence, which correspond to Strong Approximation and Weak Approximation respectively.

**Definition 1.1.1. (Strong Convergence)** Let \(N\) be an integer \(N > 0\), and \(x\) be SDEs defined in the equation (1.1). We shall say that a general time discrete approximation \(x_h(N)\) over time interval \([0,T]\) with step size \(h = T/N\) converges strongly to \(x(T)\) at time \(T\) if

\[
\lim_{h \to 0} E(|x(T) - x_h(N)|) = 0.
\]

In this thesis, we denote both \(x(t)\) and \(x_t\) the solution of the SDEs at time \(t\).

**Definition 1.1.2. (Weak Convergence)** Let \(N\) be an integer \(N > 0\), \(x\) be SDEs defined in the equation (1.1), \(h = T/N\) be step size for approximation, \(C\) be a class of smooth functions with polynomial growth. We shall say that a general time discrete approximation \(x_h(N)\) over time interval \([0,T]\) corresponding to a
time discretization \((\tau)_h\) converges weakly to \(x\) at time \(T\) as \(h \downarrow 0\) with respect to a class \(C\) of test function \(g : \mathcal{R}^d \rightarrow \mathcal{R}\) if we have

\[
\lim_{h \downarrow 0} |E(g(x(T))) - E(g(x_h(N)))| = 0,
\]

for all \(g \in C\).

Strong convergence is for problems involving direct simulation. The trajectories of the numerical solutions are close to the exact solution. One aim of the Strong Numerical Approximation is to visualize the sample paths of the solutions, which will show the behavior of the possible trajectory. Another aim of the Strong Approximation is to achieve reasonable approximations to the distributional characteristics of the solution to the SDEs.

Weak convergence is for problems considering the certain moments of the solution. Weak Numerical Approximation is important as the solutions, since they provide characteristics for a number of mathematical physics problems that are too complicated to permit an exact analysis, such as problems in the analysis of wave scattering in random media, the time reversed Kolmogorove backward equation, and the invariant measures.

Another important concepts is global order. Because of the difference of the strong convergence and weak convergence, we have different definitions for strong order and weak order.

**Definition 1.1.3. (order for strong convergence)** We call that a general discrete approximation \(x_h(N)\) with step size \(h\), \((T = Nh)\), converges strongly with order \(\gamma > 0\) to \(x(T)\) at time \(T\), if there is a finite \(h_0 > 0\), and a constant \(K\), which is independent to \(h\), such that

\[
E(|x_h(N) - x(T)|) \leq Kh^\gamma,
\]

for each \(0 < h < h_0\).

**Definition 1.1.4. (order for weak convergence)** Let \(C^1(R^d, R)\) be the space of \(1\) times continuously differentiable function \(\omega : R^d \rightarrow R\), \(C_P^1(R^d, R)\) be the subspace of functions \(\omega \in C^1(R^d, R)\) for which all partial derivatives up to order \(1\) have polynomial growth. We call that a general discrete approximation \(x_h(N)\) with step size \(h\), \((T = Nh)\), converges weakly with order \(\gamma > 0\) to \(x(T)\) at time \(T\), if for each \(g \in C_P^{2(\theta+1)}(R^d, R)\) there is a finite \(h_0 > 0\), and a positive constant \(K\), which doesn't depend on \(h\), such that

\[
|E(g(x_h(N))) - E(g(x(T)))| \leq Kh^\gamma,
\]

for each \(0 < h < h_0\).

Although lots of SDE approximation schemes share same idea with the ODE ones, the \(dW\) term brings many difficulties. Because we couldn't construct the \(nD\) Brownian path, which reveals enough detail information of the trajectory, we can not do much to improve the accuracy of the approximation.
How good can the strong approximation scheme be? The classical strong Euler and Milstein numerical scheme correspond to truncated order 1/2 and 1 strong Taylor scheme respectively. One can obtain higher order strong approximation schemes by including more stochastic integrals from the Taylor expansion. Hence it is vital to propose an algorithm for the simulation of the iterated Itô integrals and the Brownian trajectories, which are required by Milstein scheme. In high dimension case, if we had some way to generate $A_{h,k,l}^{(j)}$ ($k > l$), we accomplish the problem of generating the area term $A_{h,k,l}^{(j)}$ ($k < l$) as well. Why? Because we could obtain $A_{h,k,k}^{(j)}$ by the Itô integration as in the last paragraph, and $A_{h,k,l}^{(j)}$ ($k < l$) from $A_{h,k,l}^{(j)} = \Delta W_{h}^{(k)} \Delta W_{h}^{(l)} - A_{h,i,k}^{(j)}$. Notwithstanding Milstein Scheme has a higher order than Euler Scheme, in the general case of more than two dimensions and without the commutativity condition, it is difficult to apply the Milstein Scheme, because the character of the Wiener process makes it hard to approximate the Itô integrals $A_{h,k,l}^{(j)} = \int_{j}^{(j+1)h} \int_{j}^{t} dW^{(l)}(r) dW^{(k)}(t)$ and Brownian increments $\Delta W_{h}^{(k)} = W^{(k)}((j+1)h) - W^{(k)}(jh)$ and $\Delta W_{h}^{(l)} = W^{(l)}((j+1)h) - W^{(l)}(jh)$ at the same time.

So as to overcome this difficulty, some people try to approximate the Lévy stochastic area integrals, because the iterated Itô integrals

$$A_{h,k,l}^{(j)} = \int_{j}^{(j+1)h} \int_{j}^{t} dW^{(l)}(r) dW^{(k)}(t)$$

and the Lévy stochastic area integrals $A_{h,k,l}^{(levy)} = A_{h,k,l} - A_{h,i,k}$ are closely related. A formula for the characteristic function of the conditional distribution of $A_{h,k,l}^{(levy)}$ is given by Lévy [21]. Ryden [27] and Wiktorsson [33] approximate the area term based on Lévy's formula. Stump and Hill [29] develop a novel method for the evaluation of the Itô integrals using Sheffer systems of polynomials.

Gaines and Lyons suggest a 2D Stochastic Area Integrals method, which will lead to an order 1 method in 2D. Gaines and Lyons's 2D Stochastic Area Integrals method is based on "rectangle-wedge-tail" method, generalised to higher dimensions. Their 2D Stochastic Area Integrals method gives exact simulation of 2D approximate Brownian trajectory. However it seems hard to extend it to higher dimension, considering the use of an unacceptable amount of both calculation and storage.

In higher dimension, Kloeden, Platen and Wright [18] presented the algorithm based on truncation of an infinite series representation of the iterated Itô integrals. Ryden and Wiktorsson [27] investigate different simulation methods for the iterated Itô integrals in their paper. The simulation methods, which are considered, all generate the iterated Itô integrals conditioned on the Wiener increments. Ryden and Wiktorsson show mean-square convergence rates of these approximation methods and asymptotic normality of the remainder of the approximations. They also compare the mean error and the amount of random variables generated in these methods, showing that the computational load will be the same for Euler method, and Milstein method, which is combined with Kloeden-Platen-Wright's method, if one want to achieve the same accuracy. Because Euler method has
strong order $1/2$, $\epsilon \sim O(\sqrt{h})$, to achieve accuracy $\epsilon$, Euler method will need $O(1/h)$ iteration, hence will need $O(\epsilon^{-2})$ random variables. Ryden and Wiktorsson show that improvement can be achieved by applying the Milstein method combined with their new methods, $B'$ and $B''$. The $B'$ and $B''$ methods in Ryden and Wiktorsson's paper need to generate $O(\epsilon^{-3/2})$ random variables for accuracy $\epsilon$. They also confirm that the method of Gaines and Loyons is the fastest method within these methods, which only need to generate 14 standard normal variables for one iterated Itô integrals and two Wiener increments. Wiktorsson [33] also gives the way to simultaneously simulate the Itô integrals and Brownian increments. However, none of these paper on area integrals include method for subdivision, which is the way to generate the approximation for Brownian trajectory, $W$, with step size $h/2$, conditional on $W$'s approximate Brownian trajectory with stepsize $h$. And this demands a lot more work and imagination as Gaines and Lyons indicated in their paper [7].

Davie [1] gives a $3/4$ order scheme, which is briefly introduced at the second and third paragraphs of the section (1.1). This inspires a fresh idea to give a strong approximation for $nD$ SDEs. Because the $3/4$ Method will need to generate at most 4 random variables for one iterated Itô integrals and two Wiener increments, and have strong order $3/4$, $\epsilon \sim O(h^{3/4})$, to achieve accuracy $\epsilon$, one will need $O(1/h) \sim O(1/\epsilon^{3/2})$ iterations. Hence the $3/4$ method only need to generate $O(\epsilon^{-4/3})$ random variables. And one can apply the subdivision process to the approximate Brownian trajectory in $3/4$ Method, whereas there is no method to subdivide the approximate Brownian trajectory introduced by Gaines and Lyons.

How good can the weak approximation scheme be? The classical strong Euler and Milstein numerical scheme are order 1 and 2 respectively. One can obtain higher order schemes when one adds in more terms from the Itô-Taylor expansion. Pardoux and Talay [25], Pollard [26] and Kloeden and Platen [17] give good survey for weak approximation scheme.

1.2 Preliminaries

Now we are preparing our further discussion with more technical detail. Stochastic Differential Equation plays a key role in models, which include unpredictable factors, such as Filtering Theory, Mathematical Finance, and random excitations of physical phenomena in Physics. In the Filtering Theory, which is applied in orbit determination and reentry, the optimal continuous time Markov chain filter is the solution of the Itô stochastic differential equation. ([17] Chap 13. Selected Applications ) In the Financial Mathematics, the price of the random strike Asian option in the Black-Scholes Model requires the solution of some PDE with the boundary condition. ([16] Chap 11. Applications in Finance )

1.2.1 The Existence and Uniqueness of Solutions

The existence and uniqueness of solutions for a general $d$-dimension vector stochastic differential equation (1.1) is the premise for finding solutions of SDEs.
We borrow the definition and assumptions from Kloeden and Platen  (4.5 The Existence and Uniqueness of Strong Solutions)[17] and from Gard[9].

**Definition 1.2.1.** Solution of (1.1) is a process which satisfies the stochastic integral equation

\[ x_t = x_0 + \int_{t_0}^{t} A(t,x_t) \, dt + \int_{t_0}^{t} B(t,x_t) \, dW_t. \]

If any two solutions processes \( x_t \) and \( \tilde{x}_t \)

\[ P( \sup_{t_0 \leq t \leq T} |x_t - \tilde{x}_t| > 0) = 0, \]

we say that the solutions of SDEs (1.1) are **pathwise unique**.

To meet the existence and uniqueness of SDEs, we need some assumptions, most of which concern the vector coefficients \( A(t,x_t) \) and \( B(t,x_t) \). One can get these assumptions from [17] or [9].

**Assumption 1.2.2.** \( A(t,x_t) \) and \( B(t,x_t) \) are measurable with respect to \( t \) and \( x \) for \( t \in [0,T] \) and \( x \in \mathbb{R} \).

**Assumption 1.2.3.** There exists a constant \( K > 0 \) such that

\[ |A(t,x) - A(t,y)| \leq K|x - y| \]

and

\[ |B(t,x) - B(t,y)| \leq K|x - y|. \]

for all \( t \in [t_0,T], \ t \in \mathbb{R}, \) and \( x \in \mathbb{R}^d \), where \( | \cdot | \) is the matrix norms.

**Assumption 1.2.4.** There exists a constant \( K > 0 \) such that

\[ |A(t,x)|^2 \leq K^2(1 + |x|^2) \]

and

\[ |B(t,x)|^2 \leq K^2(1 + |x|) \]

for all \( t \in [t_0,T], \ t \in \mathbb{R}, \) and \( x \in \mathbb{R}^d \), where \( | \cdot | \) is the matrix norms.

**Assumption 1.2.5.** \( x_{t_0} \) is independent of \( W(t) \), for \( t > 0 \), and \( E(|x_{t_0}|^2) < \infty \)

Under assumptions (1.2.2)-(1.2.5), the stochastic differential equation (1.1) has a pathwise unique strong solution \( x_t \) on \( [t_0,T] \) with

\[ \sup_{t_0 \leq t \leq T} E(|x_t|^2) < \infty, \]

and a weak solution. This answers the existence and uniqueness of strong solutions. The detail proof could be found in "Numerical Solution of Stochastic Differential Equations"[17].
1.2.2 Strong Approximation of SDEs

Although we only consider SDEs that meet the existence and uniqueness conditions, it doesn’t guarantee a way to get the explicit solution. Without having explicit solution of the Stochastic Differential Equation in many cases, especially in two or higher dimensions SDEs, people had to find some way to obtain an approximated numerical solution. In the past decades, following the development of High Performance Computer, many methods, relying on high performance calculation came out, trying to get closed solution. However, it is still too difficult to get a satisfactory result in a short time because of the restriction of storage and computation speed.

Some people are interested in designing an efficient strong numerical scheme for the SDE (see definition of strong convergence). What is the order of the pathwise convergence of the widely used strong approximation schemes? Talay [31] gives an upper bound for the pathwise error of the Milstein method, because the stochastic differential equation and the Milstein scheme can be transformed to a random ordinary differential equation (ODE) and a corresponding approximation scheme respectively. Grüne and Kloeden [12] discuss the pathwise approximation of random ODE. Gyöngy [13] gives the order of the pathwise convergence for the explicit Euler-Maruyama scheme. Kloeden and Neuenkirch [19] present the relation between the order of convergence in the $p^{th}$ mean and the order of convergence in the pathwise sense. Then they apply this result to the Euler-Maruyama method for stochastic delay equations, Itô-Taylor schemes, and Stochastic Adams-Moulton-2 scheme. One can find the discussion about the strong convergence of truncated Itô-Taylor expansions in Kloeden and Platen's book[17] (Chapter 5. Stochastic Taylor Expansions) and Gard's book[9] (Chapter 7 Quantitative Theory of Stochastic Differential Equations: Sample Path Approximations). In Strong Taylor Approximation Scheme, one can obtain one approximate $x_h(N)$ to $x(T)$ corresponding to a specific Brownian trajectory $W$. Although the approximate $x_h(N)$ generated by truncated Itô-Taylor expansions, such as Euler Method, Milstein Method, and Order 1.5 Strong Taylor Scheme, varies with trajectory, the truncated Itô-Taylor expansion converges to Itô process $x(T)$ in the mean-square sense, having $E|x_h(N) - x(T)|^2 \leq Ch^{2\gamma}$, where $\gamma$ is the order of the strong approximation scheme.

There were lots of Strong Approximation Scheme, such as Euler, Milstein, Order 1.5 Strong Taylor Scheme, as long as we have adequate information from the Brownian Trajectory. However, this is difficult in many cases.

What is the ideal scheme? It is up to the restriction we have. General speaking, we want the scheme to be more accurate and have smaller computation load than others, when they are approximated with the same time step size $h$. We can see this by comparing the classic Euler Scheme and Milstein Scheme:

**Euler scheme vs Milstein scheme**

Suppose we have an SDE
Numerical Approximation for SDE

dx^{(i)}(t) = \sum_{k=1}^{n} b_{i,k}(x(t)) dW^{(k)}(t), \quad i = 1, \ldots, d

on an interval \([0, T]\), for a \(d\) dimensional vector \(x(t)\), with a \(n\) dimensional driving Brownian path \(W(t)\). \([0, T]\) is divided into \(N\) equal intervals of length \(h = T/N\). The coefficients \(b_{i,k}\) have continuous second partial derivatives. [1]

Considering the Euler approximation scheme ([17] Chapter 10. Strong Taylor Approximations), we have

\[
\tilde{x}^{(j+1)}_{[E],h} = \tilde{x}^{(j)}_{[E],h} + \sum_{k=1}^{n} b_{i,k}(\tilde{x}^{(j)}_{[E],h}) \Delta W^{(j)(k)}_{h},
\]

where

\[
\Delta W^{(j)(k)}_{h} = W^{(k)}((j + 1)h) - W^{(k)}(jh),
\]

\((j = 0, \ldots, N - 1; i = 1, \ldots, d)\).

Considering the Milstein approximation scheme ([17] Chapter 10. Strong Taylor Approximations), we have

\[
\tilde{x}^{(j+1)}_{[M],h} = \tilde{x}^{(j)}_{[M],h} + \sum_{k=1}^{n} b_{i,k}(\tilde{x}^{(j)}_{[M],h}) \Delta W^{(j)(k)}_{h} + \sum_{l,k=1} b_{i,k,l}(\tilde{x}^{(j)}_{[M],h}) A^{(j)}_{h,k,l},
\]

where

\[
\Delta W^{(j)(k)}_{h} = W^{(k)}((j + 1)h) - W^{(k)}(jh),
\]

\[
A^{(j)}_{h,k,l} = \int_{jh}^{(j+1)h} \{W^{(l)}(t) - W^{(l)}(jh)\} dW^{(k)}(t),
\]

\[
\rho_{i,k,l}(x) = \sum_{m=1}^{d} \frac{\partial b_{i,k}}{\partial x^{(m)}} b_{m,l}(x),
\]

and \((j = 0, \ldots, N - 1; i = 1, \ldots, d)\).

For well-behaved coefficients \(b_{i,k}\), the Euler scheme, which omits the term \(A^{(j)}_{h,k,l}\) having big error after several steps, only has order \(1/2\) in general, in the sense that \(E|\tilde{x}^{(j)} - x(jh)|^2 = O(h^1)\). However, this method is simple and fast calculating, so it is still widely used. The Milstein scheme with the \(A^{(j)}_{h,k,l}\) term has order \(1\), in the sense that \(E|\tilde{x}^{(j)} - x(jh)|^2 = O(h^2)\), whereas the key of implementing the Milstein scheme is to generate the area integrals, \(A^{(j)}_{h,k,l}\), which is difficult. In one-dimension case,

\[
A^{(j)}_{h,k,k} = \int_{jh}^{(j+1)h} \{W^{(k)}(t) - W^{(k)}(jh)\} dW^{(k)}(t).
\]
If we denote \( jh = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = (j+1)h \), then, when \( n \to \infty \), we have

\[
A_{h,k}^{(j)} = \lim_{n \to \infty} \sum_{i=0}^{n-1} \{ W^{(k)}(t_i^{(n)}) - W^{(k)}(jh) \} \{ (W^{(k)}(t_{i+1}^{(n)}) - W^{(k)}(t_i^{(n)})) \}.
\]

Therefore

\[
A_{h,k}^{(j)} = \frac{1}{2} \{ W^{(k)}((j+1)h) - W^{(k)}(jh) \}^2 - \frac{1}{2} h.
\]

The difficulty of generating area term, \( A_{h,k,i}^{(j)} \), is discussed in the section (1.1).

### 1.2.3 Weak Approximation of SDEs

In many situations, we are interested in \( E[g(x_T)] \), where \( g(x) \) is some function and \( x_T \) is the value of the Itô process at final time \( T \). Under assumptions of sufficient regularity, Weak Euler Approximation converges with weak order \( \gamma = 1.0 \) rather than \( \gamma = 1/2 \), which is the order of Strong Euler Approximation.

When we consider the error bound of the weak approximation, we need to consider the Kolmogorov formula and the lemma (1.2.7).

**Definition 1.2.6.** Consider the general SDEs (1.1) with coefficient matrix \( A \), \( B \), and define the \( d \times d \) matrix \( D(t,x) = B(t,x)B^T(t,x) \), or in component form \( d_{i,j}(t,x) = \sum_{l=1}^{m} b_{i,l}(t,x)b_{j,l}(x) \) for \( i,j = 1,2,\ldots,d \). The elliptic operator \( L \) is defined by

\[
L = \sum_{i=1}^{d} a_i(s,x) \frac{\partial}{\partial x(t)} + \frac{1}{2} \sum_{i,j=1}^{d} d_{i,j}(s,x) \frac{\partial^2}{\partial x(t) \partial x(j)}.
\]

**Lemma 1.2.7.** ([17] Theorem 4.8.6) Let \( C_P^{2(\gamma+1)} \) be subpace of functions as defined in definition (1.1.4). Suppose that \( f \in C_P^{2(\gamma+1)}(\mathbb{R}^d, \mathbb{R}) \) for some \( \gamma = 1,2,\ldots \) and that \( x(t) \) is the solution of the SDEs for which the drift vector and matrix components \( a_i, d_{i,j} \in C_P^{2(\gamma+1)}(\mathbb{R}^d, \mathbb{R}) \) with uniformly bounded derivatives. Then \( u(s,x) = E(f(x(T))|x(s) = x) : [0,T] \times \mathbb{R}^d \to \mathbb{R} \) satisfies \( \frac{\partial u}{\partial s} + Lu = 0 \) and \( u(T,x) = f(x) \), with \( \frac{\partial u}{\partial s} \) continuous and \( u(s,\cdot) \in C_P^{2(\gamma+1)}(\mathbb{R}^d, \mathbb{R}) \) for each \( 0 < s < T \).

### 1.2.4 Subdivision Process

Why we need the subdivision process? If we considered the \( E(|x_h(N) - x(T)|) \leq Ch^\gamma \) in the subsection (1.2.3), we can see that the accuracy of the approximate solution depends on the step size and the approximate scheme. To reveal the order of a scheme, we need to obtain a convergent sequence of approximations. Hence it is important to achieve the following two things: First, generate an approximation of a Brownian trajectory by time step size \( h \); Second, subdivide the same Brownian trajectory into an approximate trajectory by time step size \( h/4 \), \( h/16 \), and even smaller time step size. It is simple to generate an approximate
Brownian path, whereas it raises lots of problems to subdivide it in $nD$ case. Again, we explain this by comparing Euler Scheme with Milstein Scheme.

**Subdivision Process in Euler Scheme**

We only need to consider how to subdivide one Brownian component a time, when we consider subdivision process for the Euler Scheme. This is because the Euler Scheme for

$$dx(t) = \sum_{k=1}^{n} b_{i,k}(t, x) dW^{(k)}$$

is

$$x^{(i)}(N + 1) = x^{(i)}(N) + \sum_{k=1}^{n} b_{i,k}(t_N, x(N)) \Delta W^{(N)(k)}.$$  

To be precise, the following lemma indicates how to subdivide $\Delta W_h$ into mutually independent $\Delta W_{h,L}$ and $\Delta W_{h,R}$, such that $\Delta W_h = \Delta W_{h,L} + \Delta W_{h,R}$.

**Lemma 1.2.8.** Let $W = (W^{(1)}, W^{(2)}, \ldots, W^{(n)}), W^{(k)}(T) \sim N(0, T)$, be a standard $nD$ Brownian trajectory, $\Delta W^{(j)(k)} \sim N(0, h)$ be the known independent increment of the trajectory, and the variance of the unknown $\Delta W_L^{(j)(k)}, \Delta W_R^{(j)(k)}$ be $h/2$. Conditional on $\Delta W^{(j)(k)}$, we could obtain mutually independent increments $\Delta W_L^{(j)(k)}, \Delta W_R^{(j)(k)} \sim N(0, h/2)$ such that $\Delta W^{(j)(k)} = \Delta W_L^{(j)(k)} + \Delta W_R^{(j)(k)}$ by

\[
\begin{align*}
\Delta W_L^{(j)(k)} &= \frac{1}{2} \Delta W^{(j)(k)} + \frac{\sqrt{h}}{2} N \\
\Delta W_R^{(j)(k)} &= \Delta W^{(j)(k)} - \Delta W_L^{(j)(k)}
\end{align*}
\]

in which $N \sim N(0, 1)$ is an independent random value.

**Proof:**

Let $\Delta W_L^{(j)(k)}$ be generated with $\Delta W_L^{(j)(k)} = A \Delta W^{(j)(k)} + BN$. Conditional on $\Delta W^{(j)(k)}$, $E[\Delta W_L^{(j)(k)} \Delta W^{(j)(k)}] = E[(A \Delta W^{(j)(k)} + BN) \Delta W^{(j)(k)}] = Ah$, because $\Delta W^{(j)(k)}$ and $N$ are independent. And because $\Delta W_L^{(j)(k)}$ and $\Delta W_R^{(j)(k)}$ are mutually independent, $E[\Delta W_L^{(j)(k)} \Delta W_R^{(j)(k)}] = E[\Delta W_L^{(j)(k)} (\Delta W_L^{(j)(k)} + \Delta W_R^{(j)(k)})] = h/2$, we have $A = 1/2$. Because $\Delta W_L^{(j)(k)}$ and $\Delta W_R^{(j)(k)}$ are mutually independent, we have

$$E[\Delta W_L^{(j)(k)} \Delta W_R^{(j)(k)}] = 0,$$

and

$$E[\Delta W_L^{(j)(k)} \Delta W_R^{(j)(k)}] = E[\Delta W_L^{(j)(k)} (\Delta W^{(j)(k)} - \Delta W_L^{(j)(k)})]$$

$$= E[(A \Delta W^{(j)(k)} + BN)(\Delta W^{(j)(k)} - A \Delta W^{(j)(k)} - BN)]$$

$$= Ah - (A^2 h + B^2 h)$$

Hence we can obtain coefficient $B$ with $B = \sqrt{Ah - A^2 h} = \sqrt{h}/2$. 

\[\square\]
At this stage, we subdivide the approximation of the Brownian path from step size $h$ into $h/2$. Meanwhile, an pathwise approximation of an Itô process $x(T)$ by $X_{\frac{h}{2}}(2N)$ with time step size $h/2$ can be generated. This gives a subdivision process for the Euler Scheme perfectly.

**Subdivision Process in 3/4 Numerical Scheme**

However, the subdivision will be complicated, if we intend to apply the higher order scheme. Because it is impossible to subdivide the $\int W dY$. And if we do not involve this area term, the numerical scheme will not have a higher order than $1/2$.

We are going to introduce some tools that will be applied to generate the approximate Brownian trajectory and subdivide it into an approximate Brownian trajectory with smaller step size in the following part.

The lemma (1.2.10) will give us the method to generate multivariate normal distributed random variables such that they will have covariance matrix $C$. The lemma (1.2.9), (1.2.11), (1.2.12), and (1.2.13) tell us how to generate $X_1, \ldots, X_m$ ($X_i \sim N(0, \Sigma)$), such that $X_i$ and $X_j$ are independent ($i \neq j$), $X = X_1 + \ldots + X_m$, and $\Sigma = \Sigma_1 + \ldots + \Sigma_m$, conditional on $X \sim N(0, \Sigma)$.

**Lemma 1.2.9.** Let vector $X = (X_1, X_2, \ldots, X_m)^T \sim N(0, \Sigma)$ be random vector, vector $X_L = (X_{1,L}, X_{2,L}, \ldots, X_{m,L})^T \sim N(0, \Sigma^*)$ be unknown, and $\hat{\Sigma}$ be the matrix that $E[X_L X^T] = \hat{\Sigma}$. We know $X$, and want to generate $X_L$. Conditional on $X$, we could obtain the conditional expectation

$$E[X_L^T | X] = X^T D,$$

where $D^T = \hat{\Sigma} \Sigma^{-1}$, and conditional covariance

$$Cov[X_L, X_L^T | X] = \Sigma^* - \hat{\Sigma} \Sigma^{-1} \hat{\Sigma}^T$$

**Proof:**

The method to proof is standard. Let $Y = (X_{1,L}, \ldots, X_{m,L}, X_1, \ldots, X_m)^T$ be joint normal distributed with mean 0 and covariance matrix $E[YY^T]$. From the condition, we can obtain that $E[YY^T] = \begin{pmatrix} \Sigma^* & \hat{\Sigma} \\ \hat{\Sigma}^T & \Sigma \end{pmatrix}$. Hence, conditional on $X$, $X_L$ can be expressed with $X_L^T = X^T D + Z$, where $Z$ is independent to $X$ and $D$. We can work out the matrix $D$ by solving

$$E[X_L X^T] = E[(D^T X + Z^T)X^T] = D^T \Sigma E[Z^T] X^T = \hat{\Sigma}.$$

Hence $D^T = \hat{\Sigma} \Sigma^{-1}$, and $E[X_L^T | X] = X^T D$. Conditional on $X$, $X_L$ has covariance matrix

$$Cov[X_L, X_L^T | X] = E[X_L X_L^T] - \hat{\Sigma} \Sigma^{-1} \hat{\Sigma} = \Sigma^* - \hat{\Sigma} \Sigma^{-1} \hat{\Sigma}^T.$$
From time to time, we want to generate random variables $X_0, \ldots, X_j$ such that they will satisfy some covariance matrix. Here will give a lemma for this.

**Lemma 1.2.10.** Let $X = (X_1, \ldots, X_m)$ be the multivariate normal distributed random variables that we want to generate, $m \times m$ matrix $G$ be the covariance matrix. We want to generate $X$ such that $E[X^T X] = G$. The random variables $X$ can be given by

$$X = NM,$$

where $N = (N_1, \ldots, N_m)$ ($N_i \sim N(0, 1)$) are random variables. $M$ is Cholesky matrix that $G = M^T M$.

**Proof:** Because $X = NM$, so we have $E[X^T X] = E[M^T N^T NM] = E[M^T M] = G$. So we get the random variables satisfying our requirement.

Let's consider another useful result. This lemma (1.2.11) can be derived from lemma (1.2.9) and (1.2.10).

**Lemma 1.2.11.** Let the vector $X = (X_1, X_2, \ldots, X_m)^T \sim N(0, \Sigma)$ be random vector, vector $X_L = (X_{1L}, X_{2L}, \ldots, X_{mL})^T \sim N(0, \Sigma^*)$ be unknown, and $\Sigma$ be the covariance matrix that $E[X_L X^T] = \Sigma$. Conditional on $X$, we could obtain vectors $X_L \sim N(0, \Sigma^*)$, by

$$X_L^T = X^T D + N^T C,$$

where $D^T = \hat{\Sigma}^{-1}$, $C$ is lower triangular matrix, satisfying $C^T C = \Sigma^* - D^T \Sigma D$, and $N = (N_1, \ldots, N_m)^T \sim N(0, \Sigma)$ be independent random vector, having all components mutually independent.

**Proof:** The method to proof is standard. We have proven the lemma (1.2.9). Hence, conditional on $X$, we can set $X_L^T = X^T D + N^T C$, and have

$$E[X_L X^T] = E[(D^T X + C^T N)(X^T D + N^T C)] = D^T \Sigma + E[C^T N X^T] = \hat{\Sigma} \Sigma^{-1} \Sigma = \hat{\Sigma}.$$

We also have

$$E[X_L X_L^T] = E[(D^T X + C^T N)(X^T D + N^T C)] = E[D^T XX^T D + C^T NN^T C] = E[D^T \Sigma D + C^T C] = \Sigma^*.$$

That is $C^T C = \Sigma^* - D^T \Sigma D$. Let $\Sigma^* - D^T \Sigma D$ and $C$ be $G$ and $M$ in lemma (1.2.10) respectively. When we apply the method in lemma (1.2.10), we can obtain the Cholesky matrix $C$. Hence we can generate the $X_L$.

Let's consider a simple case.

**Lemma 1.2.12.** Let vector $X$ be $X = (X_1, X_2, \ldots, X_m)^T \sim N(0, \Sigma)$, $H$ be vector $H = (H_1, \ldots, H_m)$, and $X_L$ be vector $X_L = (X_{1L}, X_{2L}, \ldots, X_{mL})^T$, having
$X^T = X_T^T + N \times H$, where $N \sim N(0,1)$ is a random value, $Z$ be random variables $Z \sim N(0, 1 - H(\Sigma^{-1})H^T)$. Let $X_L$ and $N$ be independent, then $E[NX^T] = H$. Conditional on $X$, we could obtain variable $N \sim N(0, 1)$ by

$$N = X^TD + Z,$$

where $D^T = H\Sigma^{-1}$, and $\text{Var}[Z] = 1 - H(\Sigma^{-1})H^T$. Then one can get the vector $X_L$ in $X^T = X_L^T + N \times H$.

**Proof:** Conditional on $X$, we have $E[N|X] = X^TD$ and $\text{Cov}[N, N|X] = E(NN|X) - E^Z(N|X) = E(NN|X) - D^T\Sigma D$, where $D^T = H\Sigma^{-1}$, and $E(NN) = 1$, when we apply the lemma (1.2.9). Hence, one can generate the $N$ with $N = X^TD + Z$, where $D^T = H\Sigma^{-1}$, and $Z \sim N(0, 1 - H(\Sigma^{-1})H^T)$. Then one can obtain the vectors $X_L$ by $X_L^T = X^T - N \times H$. One can find an example in lemma (4.2.12).

Let’s generalise the subdivision with lemma (1.2.13), which is a useful tool in our later discussion. This lemma is following the lemma (1.2.9) and lemma (1.2.12).

**Lemma 1.2.13.** Let $E = \Sigma_1 + \ldots + \Sigma_m$, the vector $X \sim N(0, \Sigma)$ be the independent increment of a standard $nD$ Brownian process, and the variances of unknown vectors $X_1, \ldots, X_m$ be $\Sigma_1, \ldots, \Sigma_m$ respectively. Conditional on $X$, we could obtain mutually independent vectors $X_1 \sim N(0, \Sigma_1), X_2 \sim N(0, \Sigma_2), \ldots, X_m \sim N(0, \Sigma_m)$ s.t. $X = X_1 + \ldots + X_m$, by induction:

1. When generate $X_1 \sim N(0, \Sigma_1)$:

   $$X_1 = AX + BN,$$

   in which,

   $$\begin{cases}
   A = \Sigma_1(\Sigma)^{-1} \\
   BB^T = A\Sigma(1 - A^T) \\
   N \sim N(0, I) \text{ is independent to everything}
   \end{cases}$$

2. When generate $X_j$ ($j > 1$): set $X^* = X - \sum_{i=1}^{j-1} X_i$, then we know $X^* \sim N(0, \Sigma^*)$, where $\Sigma^* = \Sigma - \sum_{i=1}^{j-1} \Sigma_j$. We generate $X_j$ by

   $$X_j = AX^* + BN,$$

   in which

   $$\begin{cases}
   A = \Sigma_j(\Sigma_j + \ldots + \Sigma_m)^{-1} \\
   BB^T = A(\Sigma_j + \ldots + \Sigma_m)(1 - A^T) \\
   N \sim N(0, I) \text{ is independent to everything}
   \end{cases}$$

**Proof:** We can consider the generation of the $X_j$ case only. Because $X^* \sim N(0, \Sigma^*)$ is the remain part after partitioning $X_i$ ($i = 1, \ldots, j - 1$) off
$X$, conditional on $X^*$, we generate $X_j$ with $X_j = AX^* + BN$. Because $X^*$ and $N$ are independent,

$$E[X_j(X^*)^T] = E[(AX^* + BN)(X^*)^T] = E[AX^*(X^*)^T] = A\Sigma^*.$$

And because $X_i (i = j, \ldots, m)$ are mutually independent, we have

$$E[X_j(X^*)^T] = E[X_j(X_j + \ldots + X_m)^T] = E[X_jX_j^T] = \Sigma_j.$$

Therefore we can get $A = \Sigma_j(\Sigma^*)^{-1}$.

Because we want $X_j$ and $X^* - X_j$ mutually independent, we have $E[X_j(X^* - X_j)^T] = E[AX^*(X^*)^T - (AX^* + BN)(AX^* + BN)^T] = A\Sigma^* - A\Sigma^* A^T - BB^T = 0$. Hence we can obtain coefficient matrix $B$ with $BB^T = A\Sigma^*(1 - A^T)$.
Chapter 2

The 3/4 method

In this chapter, we will show some notations which are applied in chapter 2, 3, 4, 5, and reveal the 3/4 Method. So as to value this method, we also prove an error bound for a general scheme of this type as well.

2.1 Notations

To simplify our notations in this and in the following chapters, we shall use the following notations.

Method 2.1.1. (Number the instant of time) Divide \([a, b]\) into \(m\) equal intervals, \(J_0, \ldots, J_{m-1}\). Divide each interval \(J_r\) (\(r = 0, \ldots, m - 1\)) again into \(m\) equal subintervals \(J_0^{(r)} = [a_{rm}, a_{rm+1}], \ldots, J_s^{(r)} = [a_{rm+s}, a_{rm+s+1}], \ldots, J_{m-1}^{(r)} = [a_{rm+m-1}, a_{rm+m}]\). Denote \(h = a_{i+1} - a_i\), \((i = 0, \ldots, m^2 - 1)\), and \(N = m^2\), then we have \(h = \frac{T}{N}\), where \(a_0 = a\), and \(a_{mm} = b\). These are shown in the following graphic.

\[
\begin{array}{cccccccc}
& & J_0 & J_1 & \cdots & J_r & \cdots & J_{m-1} & b \\
a_0 & a_m & a_{2m} & \cdots & \cdots & a_{rm} & \cdots & a_{rm+m} \\
& a_{rm} & a_{rm+1} & \cdots & a_{rm+m-1} & a_{(r+1)m} \\
A_{h,i,k}^{(r,0)} + A_{h,i,k}^{(r,1)} + \cdots + A_{h,i,k}^{(r,m-1)} = A_{h,i,k}^{(r)} \sim \tilde{A}_{h,i,k}^{(r)} \\
\end{array}
\]

Based on this division of the time, we use the following symbols:
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The values in "()" indicate the instant of time.
The $i^{th}$ instant of time in segment $J_r$, where $t = (rm + j)h$

The values in "\{\}" indicate that it is the $k^{th}$ component of the vector $A$.

$h$ is the length of the discrete time

$k$ and $j$ indicate the components of the $A$. (see the example below)

The values in "()" indicate the instant of time.
The $i^{th}$ instant of time in the time segment $J_r$, where $t = (rm + i)h$.

The values in "\{\}" indicate that it is the $k^{th}$ component of the $W$.

The values in "()" indicate the instant of time.
The $j^{th}$ instant of time, where $t = jh$

$h$ is the length between two instants of discrete time

Now, we give some definitions of the notations which are based on the above notation method:

$$h = \frac{T}{N} = a_i - a_{i-1} \quad \text{where } i = 1, \ldots, N; N = m^2$$

$$A_{h,k,s}^{(r)} = \sum_{i=0}^{m-1} \int_{a_{rm+i}}^{a_{rm+i+1}} (W^{(s)}(t) - W^{(s)}(a_{rm+i})) dW^{(k)}(t) = \sum_{i=0}^{m-1} A_{h,k,s}^{(r,i)}$$

$$\Delta W_{h}^{(r,s)(k)} = W^{(k)}(a_{rm+i+1}) - W^{(k)}(a_{rm+i})$$

(2.1)

2.2 The Idea of 3/4 Scheme

Applying the notation (2.1), let's outline the basic idea.

How comes the 3/4 method?

It is shown in the section (1.2.2) that we have difficulty in approximating the area term $A$ in $nD$ SDEs. Because area term $A$ itself is difficult to get, we get the sum instead. That is we approximate $A_{h,k,s}^{(r)}$, the sum of $A_{h,k,s}^{(r,i)}$, rather than each $A_{h,k,s}^{(r,i)}$.

Why this is feasible? According to the "Law of Large Numbers", the distribution of the sum of the areas $A_{h,k,s}^{(r)}$ could be approximated, although it is hard to get the distribution of each area term $A_{h,k,s}^{(r,i)}$.

In the 3/4 Scheme, we apply the sum of area terms $A_{h,k,s}^{(r)}$ as a correction to Euler scheme, bringing better accuracy for $nD$ SDEs.

How to construct the Brownian trajectory?
It is proved that a strong numerical scheme based on an approximation of a Brownian trajectory by its values at times separated by an interval \( h \) will not have higher order than \( 1/2 \)[5]. To approximate the sum of area terms, we construct the Brownian trajectory with additional information \( hG_{h,k,s}^{(r,s)} \), which is the covariance of \( A_{h,k,s}^{(r)} \) and independent increment of the Brownian trajectory \( \Delta W_{h}^{(k)(r,i)} \) conditional on increment \( \Delta W_{h}^{(s)(r,i)} \).

### 2.3 The 3/4 Scheme

Let's outline the known conditions and the 3/4 method here.

**Assumption 2.3.1.** Suppose that we know variables \( \tilde{A}_{h,i,k}^{(r)} \) (\( r = 0, \ldots, m - 1 \)), which approximate \( A_{h,i,k}^{(r)} = \sum_{i=0}^{m-1} A_{h,i,k}^{(r,i)} \), where

\[
A_{h,i,k}^{(r,i)} = \int_{a_{mr+i+1}}^{a_{mr+i}} [W^{(k)}(t) - W^{(k)}(a_{mr+i})]dW^{(l)}(t),
\]

such that \( E[\tilde{A}_{h,i,k}^{(r)}] = 0, \ E[\tilde{A}_{h,i,k}^{(r)} - A_{h,i,k}^{(r)}] \leq Kh^{-2} \). Suppose that we know the Brownian path \( W \) at the instants of the time discretization \( (\tau)_h = \{\tau_n, n = 0,1,\ldots,N\} \), where \( \tau_n = a + nh, \ h = T/N \).

**Assumption 2.3.2.** Let's consider the \( d \)-Dimension SDEs (1.1) driven by \( n \)-Dimension Brownian trajectory,

\[
dx_t = B(t,x_t) \, dW_t,
\]

\( d \times n \)-matrix function \( B : [0,T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times n} \), and the coefficients \( B \) has bounded first and second order derivatives.

When the assumptions (2.3.1), (2.3.2), and (1.2.2)-(1.2.5) are satisfied, the sequence \( \{x_h^{(j)}, j = 0,1,\ldots,N\} \) of values of the 3/4 approximation at the instants of the time discretization \( (\tau)_h = \{\tau_n, n = 0,1,\ldots,N\} \) can be computed numerically:

**Method 2.3.3. (3/4 method)**

1. When \( j = 0, \ldots, m - 2; \ r = 0, \ldots, m - 1; \ i = 1, \ldots, d \)

\[
x_h^{(rm+j+1)(i)} = x_h^{(rm+j)(i)} + \sum_{k=1}^{n} b_{i,k}(x_h^{(rm+j)}) \Delta W_{h}^{(r,j)(k)}
\]

2. When \( j = m - 1; \ r = 0, \ldots, m - 1; \ i = 1, \ldots, d \)

\[
x_h^{(rm+j+1)(i)} = x_h^{(rm+j)(i)} + \sum_{k=1}^{n} b_{i,k}(x_h^{(rm+j)}) \Delta W_{h}^{(r,j)(k)} + \sum_{k,l=1}^{n} \rho_{i,k,l}(x_h^{(mr)}) \tilde{A}_{h,k,l}^{(r)}
\]
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\[ \rho_{i,k,l}(x) = \sum_{p=1}^{n} \frac{\partial b_{i,k}}{\partial x(p)} b_{p,l}(x). \]

We can obtain the \( \Delta W^{(j)}(k) \) \((j = 0, \ldots, N-1; k = 1, \ldots, n)\) and \( \tilde{A}^{(r)}_{h,k,l} \) \((r = 0, \ldots, m-1; k, l = 1, \ldots, n)\), where \( \tilde{A}^{(r)}_{h,k,l} \) approximates the \( A^{(r)}_{h,k,l} \) from the initial stage or subdivision stage, which will be mentioned later. The global error bound for 3/4 Scheme can be found in [1].

2.4 Global Error of the 3/4 Scheme

Before our approaching to generate the approximations, we are interested in the error bound for the approximate solution of the 3/4 Scheme. The new method will be worthless, if it did not bring improvement, when it is compared with the existed methods. Luckily, the theorems in Davie's paper affirm the improvement of the 3/4 Scheme.[1] Let's review theorems from Davie's paper.

Theorem 2.4.1. (Global Error of 3/4 Method) Let the assumptions (2.3.1), (2.3.2) and (1.2.2)-(1.2.5) be satisfied, \( x_{h}^{(j)} \) be the approximate solution with 3/4 Method. Let the \( b_{i,k} \) and their first and second-order derivatives are bounded. Then we have

\[ E|x_{h}^{(j)} - x(jh)|^2 \leq C N^{-3/2} \]

for \( j = 1, \ldots, N \), where \( C \) depends only on \( T, K \) and bounds for \( b_{i,k} \) and their derivatives.

Proof: Let \( C_{1}, C_{2}, \ldots \) be positive constants which depend only on \( T, K \) and bounds for \( b_{i,k} \) and their derivatives, \( \tilde{x}_{h}^{(j)} \) be the approximate SDEs solution with Milstein method.

The theorem (10.3.5) in section Milstein Scheme in Kloeden and Platen's book [17] shows that the Milstein method is order 1. Hence \( E|x(jh) - x_{h}^{(j)}|^2 \leq C N^{-3/2} \) is true, if we can prove that \( E|\tilde{x}_{h}^{(j)} - x_{h}^{(j)}|^2 \leq C N^{-3/2} \). We will prove \( E|\tilde{x}^{(j)}_{h} - x^{(j)}_h|^2 \leq C N^{-3/2} \) in the following discussion.

Let's consider the time segment \( J_{r} \) \((r = 0, \ldots, m-1)\). We define approximate solution \( \tilde{x}_{h}^{(j)} \) for time spot \( jh \) \((mr \leq j \leq m(r+1))\) with:

1. when \( j = mr \), \( \tilde{x}_{h}^{(mr)} = x_{h}^{(mr)} \)
2. when \( mr \leq j < (m+1)r \), \( \tilde{x}_{h}^{(j+1)(i)} = \tilde{x}_{h}^{(j)(i)} + \sum b_{i,k}(x_{h}^{(j)}) \Delta W^{(rj-rm)}(k) + \sum \rho_{i,k,l}(x_{h}^{(mr)} A_{h,k,l}^{(rj-rm)})]. \)

When we also consider the assumption (2.3.1) that \( E[\tilde{A}^{(r)}_{h,i,k} - A^{(r)}_{h,i,k}]^2 \leq K^2 \) and \( \rho_{i,k,l}(x_{h}^{(mr)}) < K_{1} \), where \( K \) and \( K_{1} \) are some constant, we have

\[ \tilde{x}_{h}^{(mr+1)}(i) - x_{h}^{(mr+1)}(i) = \sum \rho_{i,k,l}(x_{h}^{(mr)}) (A^{(r)}_{h,k,l} - \tilde{A}^{(r)}_{h,k,l}) \]

\[ E[\tilde{x}_{h}^{(mr+1)}(i) - x_{h}^{(mr+1)}(i)] = \sum \rho_{i,k,l}(x_{h}^{(mr)}) E(A^{(r)}_{h,k,l} - \tilde{A}^{(r)}_{h,k,l}) \]

\[ E|\tilde{x}_{h}^{(mr+1)} - x_{h}^{(mr+1)}|^2 \leq C_{1} m^{-4} \] (See assumption (2.3.1)) , (2.2)
where $A^{(r)}_{h,k,i}$ is the actual area that we cannot obtain, and $\tilde{A}^{(r)}_{h,k,i}$ is the approximate area that we are constructing in the following chapters.

Let $e_j$ be an error measure $e_j = E|x^{(j)}_h - \tilde{x}^{(j)}_{(M),h}|^2$, and $\hat{e}_j$ be an error measure $\hat{e}_j = E|\tilde{x}^{(j)}_h - \tilde{x}^{(j)}_{(M),h}|^2$. For a $j$ $(mr \leq j < m(r + 1) - 1)$, we have

$$
\begin{align*}
\tilde{x}^{(j+1)}_{h} &= x^{(j)}_h + \sum_{k=1}^{n} b_{i,k}(x^{(j)}_h) \Delta W^{(r,j-rm)}(k) \\
\tilde{x}^{(j+1)}_{(M),h} &= \tilde{x}^{(j)}_{(M),h} + \sum_{k=1}^{n} b_{i,k}(\tilde{x}^{(j)}_{(M),h}) \Delta W^{(r,j-rm)}(k) + \sum_{k,t=1}^{n} \rho_{i,k,t}(\tilde{x}^{(j)}_{(M),h}) A^{(r,j-rm)}_{h,k,t}.
\end{align*}
$$

And consequently, we have

$$
e_{j+1} = e_j + \sum_{i,k} \{b_{i,k}(x^{(j)}_h) - b_{i,k}(\tilde{x}^{(j)}_{(M),h})\}^2 h + \sum_{t} E\{\sum_{i} \rho_{i,k,t}(\tilde{x}^{(j)}_{(M),h}) A^{(r,j-rm)}_{h,k,t}\}^2 \leq e_j + C_2(e_j h + h^2).
$$

Iteration will bring us

$$
e_j < e_{mr} + C_3(m^{-1}e_{mr} + m^{-3})
$$

for $mr \leq j < m(r + 1)$.

Similarly, equations

$$
\tilde{x}^{(j+1)}_{h} = \tilde{x}^{(j)}_{h} + \sum_{k=1}^{n} b_{i,k}(x^{(j)}_h) \Delta W^{(r,j-rm)}(k) + \sum_{k,t=1}^{n} \rho_{i,k,t}(x^{(j)}_h) A^{(r,j-rm)}_{h,k,t}
$$

bring us $\hat{e}_{j+1} = \hat{e}_j + \sum_{i,k} \{b_{i,k}(x^{(j)}_h) - b_{i,k}(\tilde{x}^{(j)}_{(M),h})\}^2 h + \sum_{t} E\{\sum_{i} (\rho_{i,k,t}(\tilde{x}^{(j)}_{(M),h}) - \rho_{i,k,t}(x^{(mr)}_h)) A^{(r,j-rm)}_{h,k,t}\}^2$. Because $(x^{(mr)}_h - \tilde{x}^{(mr)}_{(M),h})$ only depends on the time $t \in [0, rmh]$ and $(\tilde{x}^{(j)}_{(M),h} - \tilde{x}^{(mr)}_{(M),h})$ only depends on the time $t \in (mrh, jh]$, they are independent. Because of $e_j = E|x^{(j)}_h - \tilde{x}^{(j)}_{(M),h}|^2$, $E|x^{(j)}_h - \tilde{x}^{(mr)}_{(M),h}|^2 \sim O(\frac{1}{m})$, and mutual independence of $(x^{(mr)}_h - \tilde{x}^{(mr)}_{(M),h})$ and $(\tilde{x}^{(j)}_{(M),h} - \tilde{x}^{(mr)}_{(M),h})$, we have

$$
E|\tilde{x}^{(j)}_{(M),h} - x^{(mr)}_h|^2 = E[(x^{(mr)}_h - \tilde{x}^{(mr)}_{(M),h}) + (\tilde{x}^{(mr)}_{(M),h} - \tilde{x}^{(j)}_{(M),h})]^2
$$

$$
= E[(x^{(mr)}_h - \tilde{x}^{(mr)}_{(M),h})^2 + (\tilde{x}^{(mr)}_{(M),h} - \tilde{x}^{(j)}_{(M),h})^2 + 2E[(x^{(mr)}_h - \tilde{x}^{(mr)}_{(M),h})(\tilde{x}^{(mr)}_{(M),h} - \tilde{x}^{(j)}_{(M),h})]]
$$

$$
= E|x^{(mr)}_h - \tilde{x}^{(mr)}_{(M),h}|^2 + E|\tilde{x}^{(j)}_{(M),h} - \tilde{x}^{(mr)}_{(M),h}|^2
$$

$$
\leq e_{mr} + C_4 m^{-1}
$$

Hence we obtain $\hat{e}_{j+1} \leq \hat{e}_j + C_5(e_j h + e_{mr} h^2 + m^{-5})$. And considering $e_j < e_{mr} + 30$
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For \( C_9(m^{-1}e_{mr} + m^{-3}) \), we have \( \hat{e}_{j+1} \leq \hat{e}_j + C_6(h e_{mr} + m^{-5}) \) for \( mr \leq j < m(r+1) \). Therefore, iteration will give us

\[
\hat{e}_{m(r+1)} \leq e_{mr} + C_7(m^{-1}e_{mr} + m^{-4}). \tag{2.4}
\]

Considering the martingale property that \( (\bar{x}_h^{(m(r+1))} - \bar{x}_h^{(mr)}) \) and \( (\bar{x}_h^{(m(r+1))} - \bar{x}_{[M],h}^{(mr)}) \) are independent with \( (x_h^{(mr)} - \bar{x}_h^{(mr)}) \) and \( (x_h^{(mr)} - \bar{x}_{[M],h}^{(mr)}) \), we can deduce that

\[
e_{m(r+1)} - e_{mr} = E|x_h^{(m(r+1))} - \bar{x}_h^{(m(r+1))}|^2 - E|x_h^{(mr)} - \bar{x}_h^{(mr)}|^2
= E|\hat{x}_h^{(m(r+1))} - \bar{x}_{[M],h}^{(m(r+1))} - (x_h^{(mr)} - \bar{x}_{[M],h}^{(mr)})|^2
= E|\hat{x}_h^{(m(r+1))} - \bar{x}_{[M],h}^{(m(r+1))} - x_h^{(mr)} + x_h^{(mr)} - \bar{x}_{[M],h}^{(mr)}|^2
= E|\hat{x}_h^{(m(r+1))} - \bar{x}_{[M],h}^{(m(r+1))} - (x_h^{(mr)} - \bar{x}_{[M],h}^{(mr)})|^2.
\]

Similarly, one can get

\[
\hat{e}_{m(r+1)} - e_{mr} = E|\hat{x}_h^{(m(r+1))} - \bar{x}_{[M],h}^{(m(r+1))} - (x_h^{(mr)} - \bar{x}_{[M],h}^{(mr)})|^2.
\]

Let \( \Lambda = \hat{x}_h^{(m(r+1))} - \bar{x}_{[M],h}^{(m(r+1))} - (x_h^{(mr)} - \bar{x}_{[M],h}^{(mr)}) \). We can have

\[
e_{m(r+1)} - e_{mr} = E[\Lambda - \hat{x}_h^{(m(r+1))} + x_h^{(m(r+1))}]^2
\leq 2E[\Lambda]^2 + 2E[\hat{x}_h^{(m(r+1))} - x_h^{(m(r+1))}]^2,
\]

from \( \hat{e}_{m(r+1)} - e_{mr} \) and \( e_{m(r+1)} - e_{mr} \). Applying inequality (2.4), \( \hat{e}_{m(r+1)} \leq e_{mr} + C_7(m^{-1}e_{mr} + m^{-4}) \), and inequality (2.2), \( E[\hat{x}_h^{(m(r+1))} + x_h^{(m(r+1))}]^2 \leq C_1m^{-4} \) to this inequality, one can get

\[
e_{m(r+1)} - e_{mr} \leq 2C_7(m^{-1}e_{mr} + m^{-4}) + 2C_1m^{-4}
\leq C_8e_{mr}m^{-1} + C_8m^{-4}.
\]

Consequently, we have \( e_{mr} \leq C_9m^{-3} \) for \( r = 0, \ldots, m - 1 \). Therefore (2.3) gives \( e_j \leq C_{10}m^{-3} \) for \( 0 \leq j \leq N \), which completes the proof.

In the following chapters, we describe methods of constructing Brownian process and generating the required approximations to the sums of area integrals, and introduce the subdivision process in which we can see how to decrease the time step size of the approximation. The Initial Method, and Subdivision Method, will be placed in Two, Three, and \( n \) Dimensions situations respectively. This will help us understand the idea of the \( 3/4 \) method.
Chapter 3

Pathwise Approximation of Solutions of Stochastic Differential Equations in 2 Dimension Case

The theorem (2.4.1) attests that the 3/4 Scheme will have higher order than the Euler Scheme. We need a scenario to cut our teeth on. Because the Subdivision Process in 2D case is fundamental to that in nD (n > 2) case, it would be easier for us to understand the method by applying the 3/4 method in 2D case first.

This chapter is outlined as follows. We start in Section "3.1 Initial Stage", constructing the approximate Brownian path with step size \( h \) and the areas, which are approximate to the sums of area integrals. The critical idea of the 3/4 method will be depicted in the "Initial Stage" as well. We then turn our attention to the Subdivision Process. In the section "3.2 The Subdivision Process", we are approximating the same Brownian trajectory with step size \( \frac{h}{4} \).

In the section (2.2), we mentioned that we only need to approximate the area term \( A_{k,j}^{(r)} \) (\( k > j \)). Therefore, in 2D case, we only need to generate \( A_{2,1}^{(r)} \), and could denote it \( A_{2,1}^{(r)} \) for short. \( A_{2,1}^{(r)} \) and \( A_{1,1}^{(r)} \) in this thesis correspond to \( A_{1,1}^{(r)} \) and \( A_{1,1}^{(r)} \) in Davie's paper [1] respectively. We also denote the Brownian path \( W_h^{(1)}(t) \) and \( W_h^{(2)}(t) \) with \( W_h(t) \) and \( Y_h(t) \) respectively.

The notations in this chapter are displayed in the Appendix.

3.1 Initial Stage

How to generate the Brownian path and area integrals? It is indicated in the section (2.2) that \( \tilde{A}_h^{(r)} \) will approximate the sum of area terms \( A_h^{(r)} = \sum_{i=1}^{m-1} A_{h}^{(r,i)} \), which is related to the independent Brownian paths \( W(t) \) and \( Y(t) \).

To construct such an approximate area \( \tilde{A}_h^{(r)} \), we need some preparation. Conditional on \( W(t) \), let \( hG_h^{(r,i)} \) denote the covariance of area integral \( A_h^{(r,i)} = \int_a^t (W(t) - W(a)) dy(t) \) and \( \Delta Y_h^{(r,i)} \). That is \( hG_h^{(r,i)} = E[\Delta Y_h^{(r,i)} A_h^{(r,i)} | W(t)] \). We list notations of this chapter in the Appendix at the back of the thesis. According
to the definition of the $G_h^{(r,i)}$, we know $G_h^{(r,i)}$ is random value depending on $Y(t)$ and $W(t)$. To be precise we have lemma (3.1.1).

**Lemma 3.1.1.** Let $\Delta W_h^{(r,i)}, \Delta Y_h^{(r,i)} \sim N(0, h)$ be the independent increment of the Brownian trajectory, $A_h^{(r)} = \sum_{i=0}^{m-1} A_h^{(r,i)}$ be the sum of area terms, and $hG_h^{(r,i)} = E[\Delta Y_h^{(r,i)}]A_h^{(r,i)}|W(t)]$ be the conditional covariance. Then $E[G_h^{(r,i)}] = 0$, $\text{Var}[G_h^{(r,i)}] = \frac{h}{3}$, and $E(\Delta W_h^{(r,i)}G_h^{(r,i)}) = \frac{h}{2}$.

**Proof:** Because $hG_h^{(r,i)}$ denotes $E[\Delta Y_h^{(r,i)}]A_h^{(r,i)}|W(t)]$, we have

$$hG_h^{(r,i)} = \int_{a_{rm+i}}^{a_{rm+i+1}} [W(t) - W(a_{rm+i})] dt.$$ 

Hence $E[hG_h^{(r,i)}] = 0$.

$$E[hG_h^{(r,i)}]^2 = 2E\left[\int_{I_r^{(r)}} \int_{a_{rm+i}}^{t} (W(s) - W(a_{rm+i}))^2 ds dt\right]$$

$$= 2 \int_{I_r^{(r)}} \int_{a_{rm+i}}^{t} (s - a_{rm+i}) ds dt = \frac{h^3}{3}. $$

$$E[\Delta W_h^{(r,i)}hG_h^{(r,i)}] = \int_{a_{rm+i}}^{a_{rm+i+1}} E[W(t) - W(a_{rm+i})]^2 dt = \frac{h^2}{2}.$$

We have difficulty in generating the $\text{Var}[A_h^{(r,i)}|W(t)]$, whereas we can get the $\text{Var}[A_h^{(r,i)}|W(t)] = \sum_{i=0}^{m-1} \text{Var}[A_h^{(r,i)}|W(t)]$, when $m \to \infty$.

**Lemma 3.1.2.** Let $\Delta W_h^{(r,i)}, \Delta Y_h^{(r,i)} \sim N(0, h)$ be the independent increment of the Brownian trajectory, $A_h^{(r)} = \sum_{i=0}^{m-1} A_h^{(r,i)}$ be the sum of area terms, and $hG_h^{(r,i)} = E[\Delta Y_h^{(r,i)}]A_h^{(r,i)}|W(t)]$ be the conditional covariance. Then we have $A_h^{(r)} \sim N(0, \frac{mh^2}{2})$,

$$E[A_h^{(r)}|W(t)] = 0$$

$$\text{Var}[A_h^{(r)}|W(t)] \simeq \frac{mh^2}{2}, \text{ when } m \text{ is large.}$$

**Proof:**

$$E[A_h^{(r)}|W(t)] = E\{\sum \int_{I_r^{(r)}} [W(t) - W(a_{rm+i})] dY(t)|W(t)\} = 0.$$ 

Because

$$\text{Var}[A_h^{(r)}|W(t)] = \sum_{i=0}^{m-1} \int_{a_{rm+i}}^{a_{rm+i+1}} E[(W(t) - W(a_{rm+i}))^2|W(t)] dt,$$
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and the \( \text{Var}[A_h^{(r)}|W(t)] \) is expected to be close to the mean when \( m \) is large, we have

\[
E[A_h^{(r)}|W(t)] \approx \sum_{i=0}^{m-1} \int_{a_{rm+i}}^{a_{rm+i+1}} (t - a_{rm+i}) \, dt = \frac{mh^2}{2}.
\]

These two lemmas show us the property of the \( G_h^{(r,t)} \sim N(0, h^2/3) \), \( \text{Var}[A_h^{(r)}|W(t)] \approx \frac{mh^2}{2} \) and \( A_h^{(r)} \sim N(0, mh^2/2) \).

**Explanation 3.1.3.** We are explaining the generation of the approximate \( \tilde{A}_h^{(r)} \) for \( A_h^{(r)} \) in this part.

Denote \( V_h^{(r)} = \text{var}[A_h^{(r)}|W, \Delta Y_h^{(r,0)}, \ldots, \Delta Y_h^{(r,m-1)}] \). We can therefore know the sum of area integrals \( A_h^{(r)} \) will be, conditionally on the path \( W(t) \) and \( \Delta Y_h^{(r,0)}, \ldots, \Delta Y_h^{(r,m-1)} \), normally distributed with mean \( \sum G_h^{(r,t)} \Delta Y_h^{(r,t)} \) and variance \( V_h^{(r)} = \sum \int_{a_{rm+i}}^{a_{rm+i+1}} [W(t) - W(a_{rm+i})]^2 \, dt - h \sum (G_h^{(r,t)})^2 \). The general situation is proved in lemma (1.2.9). When \( m \) is large, \( \sum \int_{a_{rm+i}}^{a_{rm+i+1}} [W(t) - W(a_{rm+i})]^2 \, dt \) and \( \sum (G_h^{(r,t)})^2 \) are expected to be close to their means. That is

\[
\sum \int_{a_{rm+i}}^{a_{rm+i+1}} [W(t) - W(a_{rm+i})]^2 \, dt \approx \sum \int_{a_{rm+i}}^{a_{rm+i+1}} (t - a_{rm+i}) \, dt = \frac{mh^2}{2} \quad \text{(See lemma (3.1.2))}
\]

\[
h^2 \sum (G_h^{(r,t)})^2 = 2 \sum \int_{a_{rm+i}}^{a_{rm+i+1}} \int_{a_{rm+i}}^{s} [W(t) - W(a_{rm+i})]^2 \, dt \, ds
\]

\[
\approx 2 \sum \int_{a_{rm+i}}^{s} (t - a_{rm+i}) \, dt \, ds = \frac{mh^3}{3}.
\]

Hence, let \( \tilde{V}_h^{(r)} = \frac{mh^2}{6} \)

\[
V_h^{(r)} = \sum \int_{a_{rm+i}}^{a_{rm+i+1}} [W(t) - W(a_{rm+i})]^2 \, dt - h \sum (G_h^{(r,t)})^2
\]

\[
\approx \tilde{V}_h^{(r)} = \frac{mh^2}{6}.
\]

So \( V_h^{(r)} \) is expected to be close to its mean \( \tilde{V}_h^{(r)} = mh^2/6 \).

We want to generate \( \tilde{A}_h^{(r)} \sim N(0, \frac{mh^2}{2}) \) as a good approximation to \( A_h^{(r)} \). So we generate a normal variable with the appropriate correlation with the \( \Delta Y_h^{(r,t)} \), and variance \( \frac{mh^2}{2} \). Because \( E[A_h^{(r)}|W(t), \Delta Y_h^{(r,0)}, \ldots, \Delta Y_h^{(r,m-1)}] = \sum G_h^{(r,t)} \Delta Y_h^{(r,t)} \) and \( \text{Var}[A_h^{(r)}|W(t), \Delta Y_h^{(r,0)}, \ldots, \Delta Y_h^{(r,m-1)}] \approx \frac{mh^2}{6} \). We want that \( \tilde{A}_h^{(r)} \) normally distributed with mean \( \sum G_h^{(r,t)} \Delta Y_h^{(r,t)} \) and variance \( \tilde{V}_h^{(r)} = \frac{mh^2}{6} \), conditional on the path \( W(t) \) and \( \Delta Y_h^{(r,0)}, \ldots, \Delta Y_h^{(r,m-1)} \). Conditional on the path

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$W(t), A_h^{(r)}$ normally distributed with mean 0, variance $\frac{mh^2}{2}$, covariance $G_h^{(r)}$. So we want $E[A_h^{(r)}|W(t)] = 0$, $\text{var}[A_h^{(r)}|W(t)] \simeq mh^2/2$, and $E(A_h^{(r)} \Delta Y_h^{(r)}|W(t)) = G_h^{(r)} h$.

Such a variable is given by $A_h^{(r)} = Z_h^{(r)} + \sum_{i=0}^{m-1} G_h^{(r,i)} \Delta Y_h^{(r,i)}$, where $Z_h^{(r)} \sim N(0, mh^2/6)$ is independent to $W, \Delta Y_h^{(r,i)}$ and $G_h^{(r,i)}$.

Furthermore, if we could prove the proposition $E(A_h^{(r)} - \bar{A}_h^{(r)})^2 \leq Km^{-4}$, where the actual area $A_h^{(r)} = \sum_{i=0}^{m-1} A_h^{(r,i)}$ is approximated by $\bar{A}_h^{(r)}$, then we could have error bound of the 3/4 Method from the theorem (2.4.1). We will construct such an approximate area $\bar{A}_h^{(r)}$ in the following method, and present the proof of the proposition in theorem (3.3.3). The Explanation (3.1.3) explains why we construct the approximate area $\bar{A}_h^{(r)}$ in such a method.

**Method 3.1.4. (Initial Method)** We explain the initial method based on the notation in method (2.1.1) and notations in Appendix. We generate the 3/4 Approximation for the 2 Dimension SDE on $[a, b]$ with step size $h = \frac{b-a}{m}$. So as to make things simple, we consider how to generate the $\Delta W_h^{(r,i)}, \Delta Y_h^{(r,i)}, G_h^{(r,i)}$, and $\bar{A}_h^{(r)} (i = 0, \ldots, m-1)$ in the interval $J_r = [a_r, b_r]$ only. That is:

1. **Generate Brownian path $W$ and $Y$:** Generate $N_{X,i}, N_{Y,i} \sim N(0,1)$, then we have $\Delta W_h^{(r,i)} = \sqrt{h} N_{X,i}, \Delta Y_h^{(r,i)} = \sqrt{h} N_{Y,i} \sim N(0, h)$ and $(i = 0, \ldots, m-1)$

2. **Generate $G$:** Generate $N_{G,i} \sim N(0,1)$, then $G_h^{(r,i)} = \sqrt{\frac{h}{2}} N_{X,i} + \sqrt{\frac{h}{12}} N_{G,i} \sim N(0, \frac{h^2}{6})$. In these two steps, we obtain $\Delta W_h^{(r,i)}, \Delta Y_h^{(r,i)} \sim N(0, h), G_h^{(r,i)} \sim N(0, \frac{h^2}{6})$, and $E[G_h^{(r,i)} \Delta W_h^{(r,i)}] = \frac{h}{2}$.

   The reason for generating the $\Delta W_h^{(r,i)}, \Delta Y_h^{(r,i)}$ and $G_h^{(r,i)}$ is stated in lemma (3.1.5). It is just a simple case of the lemma (1.2.10).

3. **Generate approximate area $\bar{A}_h^{(r)}$:**

   Generate $\tilde{Z}_h^{(r)} \sim N(0, \frac{mh^2}{6})$.

   We approximate the area term $A_h^{(r)}$, with $\bar{A}_h^{(r)}$, where

   $$\bar{A}_h^{(r)} = \tilde{Z}_h^{(r)} + \sum_{i=0}^{m-1} \Delta Y_h^{(r,i)} G_h^{(r,i)}.$$  

   The reason of generating the approximate area in this way is stated in Explanation (3.1.3).
Lemma 3.1.5. According to the definition of $\Delta W_h^{(r,i)} = W((mr + i + 1)h) - W((mr + i)h)$, $\Delta Y_h^{(r,i)} = Y((mr + i + 1)h) - Y((mr + i)h)$, and $G_h^{(r,i)} = h^{-1} \int_{a_{mr+i}}^{a_{mr+i+1}} [W(t) - W(a_{mr+i+1})] dt$, we have

$$E(G_h^{(r,i)} \Delta W_h^{(r,i)}) = \frac{h}{2}$$
$$E(G_h^{(r,i)} G_h^{(r,i)}) = \frac{h}{3},$$

from the lemma (3.1.1). When apply the method (3.1.4), we can generate $\Delta W_h^{(r,i)} \sim N(0, h)$, $\Delta Y_h^{(r,i)} \sim N(0, h)$, and $G_h^{(r,i)} \sim N(0, h/3)$, satisfying these conditions.

**Proof:** Generate $N_{X,i}, N_{Y,i}, N_{G,i} \sim N(0, 1)$, and obtain $\Delta W_h^{(r,i)}$, $\Delta Y_h^{(r,i)}$, $G_h^{(r,i)}$ with $\Delta W_h^{(r,i)} = \sqrt{h} N_{X,i}$, $\Delta Y_h^{(r,i)} = \sqrt{h} N_{Y,i} \sim N(0, h)$ and $G_h^{(r,i)} = \frac{\sqrt{h}}{2} N_{X,i} + \sqrt{\frac{h}{12}} N_{G,i}$ ($i = 0, \ldots, m - 1$).

$$E(G_h^{(r,i)} \Delta W_h^{(r,i)}) = E\{[N_{X,i} \frac{\sqrt{h}}{2} + N_{G,i} \sqrt{\frac{h}{12}}] N_{X,i} \sqrt{h}\} = \frac{h}{2}$$
$$E(G_h^{(r,i)} G_h^{(r,i)}) = E[N_{X,i} \frac{\sqrt{h}}{2} + N_{G,i} \sqrt{\frac{h}{12}}]^2 = \frac{h}{3}$$

We generate variables as required.

**Summary:** What shall we get from Lemmas in this section? The Lemma (3.1.1) and (3.1.2) shows the moments of covariance $G_h^{(r,i)}$ and sum of areas $A_h^{(r)}$. (3.1.5) shows that we could get proper $W$ and $G$ when applying the method (3.1.4). The Explanation (3.1.3) indicates that it is reasonable to approximate $A_h^{(r)}$ with $\tilde{A}_h^{(r)} = \tilde{Z}_h^{(r)} + \sum_{i=0}^{m-1} G_h^{(r,i)} \Delta Y_h^{(r,i)}$. Therefore, it is clear that we could approximate $A_h^{(r)}$ with $\tilde{A}_h^{(r)}$.

At this stage, we achieve to generate the approximate Brownian path $W((r + 1)mh) - W((r)mh)$, $Y((r + 1)mh) - Y((r)mh)$ with step size $h$ and the $\tilde{A}_h^{(r)}$, which approximates the $A_h^{(r)}$ ($r = 0, \ldots, m - 1$),

$$A_h^{(r)} = \sum_{i=0}^{m-1} \int_{a_{mr+i}}^{a_{mr+i+1}} [W(t) - W(a_{mr+i+1})] dY(t)$$

for the time interval $J_r$. Analogously, we are able to generate $\Delta W_h^{(r,i)}$, $\Delta Y_h^{(r,i)}$ and $\tilde{A}_h^{(r)}$ for other time intervals $J_0, \ldots, J_{m-1}$. Method (3.1.4) generates a single approximation to areas and Brownian paths.

### 3.2 The Subdivision Process

In the initial stage, we generate Brownian path and area term $A_h^{(r)}$, hence we could apply the 3/4 method. However, we know the 3/4 method is true when
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Figure 3.1: The J subdivision

\[ m \to \infty. \] Therefore, we need the subdivision method, which will decrease the time step size \( h \), and increase the number of time segments \( m \) in \( J_r \). In the subdivision step, we will divide the large interval \( J_r \) into two and small intervals \( I^{(r)}_t \) into four. This will make the initial setup similar to the setup which was subdivided, besides \( m \) replaced by \( 2m \). We achieve these with two steps: divide \( J_r \) into two \( J_{r,L} \) and \( J_{r,R} \), and split the \( I^{(r)}_t \) into two for twice.

### 3.2.1 The J Subdivision Process

We are only considering the big time interval \( J_r \), and indicate how to split \( J_r \) into two \( J_{r,L} \) and \( J_{r,R} \) below.

It is clear that \( I_t, G_h(r, i), \Delta W_h(r, i) \) and \( \Delta Y_h(r, i) \) are divided into two groups. However, we only have one \( Z_h(r) \) for a \( J_r \). Hence, when we split \( J_r \) into two, we also need to split \( Z_h(r) \) into two.

**Method 3.2.1.** Suppose we have a fixed \( J_r \), which is divided into \( m \) equal intervals, \( I_0^{(r)}, I_1^{(r)}, \ldots, I_{\frac{m}{2} - 1}^{(r)} \), with length \( h \), having associated random variables \( Z_h^{(r)}, \Delta H_h^{(r,i)} \), \( \Delta Y_h^{(r,i)} \), \( \Delta W_h^{(r,i)} \) and \( G_h^{(r,i)} \). The variance of the known \( Z_h^{(r)} \) is \( \Sigma Z_h \).

1. Divide segments into two groups. \( I_0^{(r)}, I_1^{(r)}, \ldots, I_{\frac{m}{2} - 1}^{(r)} \in J_{r,L} \) and \( I_{\frac{m}{2}}, \ldots, I_{m-1}^{(r)} \in J_{r,R} \). Divide \( \Delta Y_h^{(r,i)}, \Delta W_h^{(r,i)} \) and \( G_h^{(r,i)} \) into two groups. Hence \( \Delta W_h^{(r,i)}, \Delta Y_h^{(r,i)}, G_h^{(r,i)} \) (\( i = 0, \ldots, m/2 - 1 \)) for \( J_{r,L} \) and \( \Delta W_h^{(r,i)}, \Delta Y_h^{(r,i)}, G_h^{(r,i)} \) (\( i = m/2, \ldots, m - 1 \)) for \( J_{r,R} \). See the second line in figure (3.1).
2. Renumber \( J_{r,L} \) and \( J_{r,R} \) with \( J_{2r} \) and \( J_{2r+1} \) respectively. Renumber \( G_h^{(r,i)} \) \((i = 0, \ldots, m/2 - 1)\) in \( J_{r,L} \) with \( G_h^{(2r,i)} \) \((i = 0, \ldots, m/2 - 1)\). And renumber \( G_h^{(r,i)} \) \((i = m/2, \ldots, m - 1)\) in \( J_{r,R} \) with \( G_h^{(2r+1,i)} \) \((i = 0, \ldots, m/2 - 1)\). See the third line in the figure (3.1). So are the \( \Delta W_h^{(r,i)} \) and \( \Delta Y_h^{(r,i)} \).

3. Split the \( Z_h^{(r)} \sim N(0, \Sigma Z_h) \) into \( Z_h^{(r)} \sim N(0, 1/2 \Sigma Z_h) \) and \( Z_h^{(r)} \sim N(0, 1/2 \Sigma Z_h) \) such that \( Z_h^{(r)} \) and \( Z_h^{(r)} \) are mutually independent, and \( Z_h^{(r)} = Z_h^{(r)} + Z_h^{(r)} \) (See the lemma (1.2.8) or (1.2.13) for the way of splitting up.) Renumber \( Z_h^{(r)} \) and \( Z_h^{(r)} \) with \( Z_h^{(2r)} \) and \( Z_h^{(2r+1)} \).

\[ \square \]

3.2.2 The I Subdivision Process

After \( J \) subdivision, we are dividing the interval \( I^{(r)} \) in \( J \) into two. So are the \( \Delta Y_h^{(r,i)} \), \( \Delta W_h^{(r,i)} \), and \( G_h^{(r,i)} \). Please note that the \( J \) here is renumbered in the \( J \) subdivision. Let’s introduce the notation method, which will bring convenient to our further deduction.

Notation 3.2.2. (2DIS notation)

1. **Number the time instant** in \( J \) with \( a_i \) \((i = 0, \ldots, q)\). See the first line in figure (3.2). In the first I subdivision, the \( q = m/2 \), whereas the \( q = m - 1 \) in the second I subdivision process, since number of I in \( J \) had been double in the first subdivision of I.

2. Denote the time intervals in \( J \) with \( I_0, \ldots, I_q \) for short.

3. For time interval \( I_i \) \((i = 0, \ldots, q)\), we have corresponding \( G_h^{(r,i)}, \Delta W_h^{(r,i)} \), and \( \Delta Y_h^{(r,i)} \). (In the first I subdivision, \( q = m/2 \). In the beginning of the second I subdivision, \( q = m - 1 \). This is because, at the beginning of the second I subdivision, we denote \( G_h^{(r,0)} \), \( G_h^{(r,2)} \), \( G_h^{(r,4)} \), \( G_h^{(r,6)} \), \( G_h^{(r,8)} \) in order with \( G_h^{(r,1)} \). So are the \( \Delta W_h^{(r,i)} \), \( \Delta W_h^{(r,i)} \), \( \Delta Y_h^{(r,i)} \) and \( \Delta Y_h^{(r,i)} \). This is shown in the third line of the figure (3.2).)

4. Before the first I subdivision process, we have \( Z_h^{(r)} \). After the I subdivision process, we have \( Z_h^{(r)} \).

\[ \square \]

The numbered instant time is \( a_0 < a_1 < \ldots < a_q+1 \) before the first subdivision process of the \( I \). It becomes \( a'_0 < a'_1 < \ldots < a'_q+1 < a'_{2q+2} \) after the first \( I \) subdivision. After the first \( I \) subdivision, we renumber the instant time according to the notation system (3.2.2), so that the numbered instant time becomes \( a_0 < a_1 < \ldots < a_{2q+1} < a_{2q+2} \).
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Analogous, the $\Delta W_{h,i,k,L}^{(r,i)}$, $\Delta Y_{h,i,k,L}^{(r,i)}$, $G_{h,i,k,L}^{(r,i)}$ can be renumbered. Before the first $I$ subdivision, we have $\Delta W_{h,i,k,L}^{(r,i)}$, $\Delta Y_{h,i,k,L}^{(r,i)}$, $G_{h,i,k,L}^{(r,i)}$ ($i = 0, \ldots, \frac{m-2}{2}$). And we have $\Delta W_{h,i,k,R}^{(r,i)}$, $\Delta Y_{h,i,k,R}^{(r,i)}$, $G_{h,i,k,R}^{(r,i)}$ ($i = 0, \ldots, \frac{m-2}{2}$) after the first subdivision of $I$. Rename these $\Delta W_{h,i,k,L}^{(r,i)}$, $\Delta Y_{h,i,k,L}^{(r,i)}$, $G_{h,i,k,L}^{(r,i)}$, $\Delta W_{h,i,k,R}^{(r,i)}$, $\Delta Y_{h,i,k,R}^{(r,i)}$, $G_{h,i,k,R}^{(r,i)}$ in orders, we have new $\Delta W_{h,i,k,L}^{(r,i)}$, $\Delta Y_{h,i,k,L}^{(r,i)}$, $G_{h,i,k,L}^{(r,i)}$ ($i = 0, \ldots, m-1$), where

$$
\begin{align*}
&\left\{\begin{array}{l}
G_{h,i,k,L}^{(r,2i)} = G_{h,i,k,L}^{(r,i)} \\
G_{h,i,k,R}^{(r,2i)} = G_{h,i,k,R}^{(r,i)}
\end{array}\right. \\
&\left\{\begin{array}{l}
\Delta W_{h,i,k,L}^{(r,2i)} = \Delta W_{h,i,k,L}^{(r,i)} \\
\Delta W_{h,i,k,R}^{(r,2i)} = \Delta W_{h,i,k,R}^{(r,i)}
\end{array}\right. \\
&\left\{\begin{array}{l}
\Delta Y_{h,i,k,L}^{(r,2i)} = \Delta Y_{h,i,k,L}^{(r,i)} \\
\Delta Y_{h,i,k,R}^{(r,2i)} = \Delta Y_{h,i,k,R}^{(r,i)}
\end{array}\right.
\end{align*}
$$

So at the beginning of the second $I$ subdivision process, we have $\Delta W_{h,i,k,L}^{(r,i)}$, $\Delta Y_{h,i,k,L}^{(r,i)}$, $G_{h,i,k,L}^{(r,i)}$ ($i = 0, \ldots, m-1$). This process can be seen in the figure (3.2), which divides every $I_i^{(r)}$ ($i = 0, \ldots, \frac{m-2}{2}$) in the $J_r$ into two subintervals obtaining $I_i$ ($i = 0, \ldots, m-1$).

We want the variables after one $I$ subdivision satisfy the conditions:
Conditions 3.2.3.
\[
\begin{align*}
\Delta W^{(r,i)}_h &= \Delta W^{(r,i)}_{h,L} + \Delta W^{(r,i)}_{h,R} \\
\Delta Y^{(r,i)}_h &= \Delta Y^{(r,i)}_{h,L} + \Delta Y^{(r,i)}_{h,R} \\
E[\Delta W^{(r,i)}_h \Delta W^{(r,i)}_{h,L}] &= E[\Delta W^{(r,i)}_h \Delta W^{(r,i)}_{h,R}] = \frac{h}{2} \\
E[\Delta Y^{(r,i)}_h \Delta Y^{(r,i)}_{h,L}] &= E[\Delta Y^{(r,i)}_h \Delta Y^{(r,i)}_{h,R}] = \frac{h}{2} \\
G^{(r,i)}_h &= \frac{1}{2}[G^{(r,i)}_{h,L} + G^{(r,i)}_{h,R} + \Delta W^{(r,i)}_{h,L}] \quad \text{(can get from definition)} \\
E[\Delta W^{(r,i)}_h G^{(r,i)}_{h,L}] &= E[\Delta W^{(r,i)}_h G^{(r,i)}_{h,R}] = \frac{h}{4} \quad \text{(see lemma (3.1.1))} \\
E[\Delta Y^{(r,i)}_h G^{(r,i)}_{h,L}] &= E[\Delta Y^{(r,i)}_h G^{(r,i)}_{h,R}] = 0 \quad \text{($\Delta Y^{(r,i)}_h, G^{(r,i)}_{h,L}$ mutually independent)}
\end{align*}
\]

These conditions are necessary according to the definition of $\Delta W^{(r,i)}_h$, $\Delta Y^{(r,i)}_h$, $G^{(r,i)}_h$, $\Delta W^{(r,i)}_{h,L}$, $\Delta Y^{(r,i)}_{h,L}$, $G^{(r,i)}_{h,L}$, $\Delta W^{(r,i)}_{h,R}$, $\Delta Y^{(r,i)}_{h,R}$, $G^{(r,i)}_{h,R}$.

It is time to show how to subdivide $I$ now. We apply the following method to meet the condition (3.2.3)

**Method 3.2.4.** Suppose there are $q + 1$ time intervals $I_i$ ($i = 0, \ldots, q$) in $J$. We want the new values generated ($\Delta W^{(r,i)}_h, \Delta Y^{(r,i)}_h, G^{(r,i)}_h, \Delta W^{(r,i)}_{h,L}, \Delta Y^{(r,i)}_{h,L}, G^{(r,i)}_{h,L}, \Delta W^{(r,i)}_{h,R}, \Delta Y^{(r,i)}_{h,R}, G^{(r,i)}_{h,R}$) will meet the condition (3.2.3). Based on the notation (3.2.2), we apply the following steps to split up the $\Delta W^{(r,i)}_h$, $\Delta Y^{(r,i)}_h$, and $G^{(r,i)}_h$. The $Z^{(r)}$ will be corrected during this process.

1. **Get $\Delta W^{(r,i)}_{h,L}, G^{(r,i)}_{h,L}$:** Split up the $\Delta W^{(r,i)}_h \sim N(0, h)$ into mutually independent $\Delta W^{(r,i)}_{h,L} \sim N(0, h/2)$ and $\Delta W^{(r,i)}_{h,R} \sim N(0, h/2)$ such that $\Delta W^{(r,i)}_h = \Delta W^{(r,i)}_{h,L} + \Delta W^{(r,i)}_{h,R}$. Split up the $G^{(r,i)}_h$ into $G^{(r,i)}_{h,L}$ and $G^{(r,i)}_{h,R}$ such that $G^{(r,i)}_h = \frac{1}{2}[G^{(r,i)}_{h,L} + G^{(r,i)}_{h,R} + \Delta W^{(r,i)}_{h,L}]$, $E[\Delta W^{(r,i)}_{h,L} G^{(r,i)}_{h,L}] = E[\Delta W^{(r,i)}_{h,R} G^{(r,i)}_{h,R}] = \frac{h}{4}$, and $E[\Delta Y^{(r,i)}_{h,L} G^{(r,i)}_{h,L}] = E[\Delta Y^{(r,i)}_{h,R} G^{(r,i)}_{h,R}] = 0$. To be precise, one can get the detail instruction from the lemma (3.2.5).

2. **Get $H_i$:** Set
\[
H_i = \frac{1}{2}(G^{(r,i)}_{h,R} - G^{(r,i)}_{h,L} + \Delta W^{(r,i)}_{h,L}).
\]
One may note that the $H_i \sim N(0, \frac{h}{12})$.

3. **Get the $Q$:** Get the
\[
Q = \frac{12}{(q + 1)h} \sum_{i=0}^{q} H_i^2 - 1,
\]
such that $E(Q) = 0$, $EQ^2 = \frac{h}{q+1}$. The reason that we generate this value $Q$ is provided in explanation (3.2.6).
4. Generate $N_{2,0,i}$ and $\tilde{Z}_{\frac{i}{2}}^*$: We want to generate the $N_{2,0,i}$ and $\tilde{Z}_{\frac{i}{2}}^*$ in a way that

\[
\begin{align*}
\tilde{Z}_{\frac{i}{2}}^* &= \tilde{Z}_h - \sum_{i=0}^{q} N_{2,0,i} H_i \sqrt{h} \\
N_{2,0,j} &\sim N(0,1) \ (j = 0, \ldots, q)
\end{align*}
\]

The method of generating the $N_{2,0,i}$ will be demonstrated in method (3.2.7). One can get the reason for this split up process from the Explanation (3.2.6).

5. Generate $\Delta Y_{h,L}^{(r,i)}$ and $\Delta Y_{h,R}^{(r,i)}$:

\[
\begin{align*}
\Delta Y_{h,L}^{(r,i)} &= \frac{1}{2} [\Delta Y_{h}^{(r,i)} - N_{2,0,i} \sqrt{h}] \\
\Delta Y_{h,R}^{(r,i)} &= \frac{1}{2} [\Delta Y_{h}^{(r,i)} + N_{2,0,i} \sqrt{h}]
\end{align*}
\]

\[\square\]

**Lemma 3.2.5.** To split up the known $\Delta W_{h}^{(r,i)}$ and $G_{h}^{(r,i)}$ into $\Delta W_{h,L}^{(r,i)}$, $\Delta W_{h,R}^{(r,i)}$, $G_{h,L}^{(r,i)}$, and $G_{h,R}^{(r,i)}$, meeting the condition (3.2.3), we can generate them in this way:

\[
\begin{align*}
\Delta W_{h,L}^{(r,i)} &= -\frac{1}{4} \Delta W_{h}^{(r,i)} + \frac{3}{2} G_{h}^{(r,i)} + \frac{\sqrt{h}}{16} N_{1,0,i} \\
G_{h,L}^{(r,i)} &= -\frac{1}{4} \Delta W_{h}^{(r,i)} + G_{h}^{(r,i)} + \frac{\sqrt{h}}{48} N_{1,1,i} \\
\Delta W_{h,R}^{(r,i)} &= \Delta W_{h}^{(r,i)} - \Delta W_{h,L}^{(r,i)} \\
G_{h,R}^{(r,i)} &= 2G_{h}^{(r,i)} - G_{h,L}^{(r,i)} - \Delta W_{h,L}^{(r,i)}
\end{align*}
\]

where $i = 0, \ldots, q$, $N_{1,0,j} \sim N(0,1)$ and $N_{1,1,j} \sim N(0,1) \ (j = 0, \ldots, q)$ are independent variables.

**Proof:** In segment $J_i$ in which each small segments has length $h$, the $\Delta W_{h}^{(r,i)}$ and $G_{h}^{(r,i)}$ are related to $\Delta W_{h,L}^{(r,i)}$ and $G_{h,L}^{(r,i)}$. So we set

\[
\begin{align*}
\Delta W_{h,L}^{(r,i)} &= a \Delta W_{h}^{(r,i)} + b G_{h}^{(r,i)} + c N_{1,0,i} \\
G_{h,L}^{(r,i)} &= d \Delta W_{h}^{(r,i)} + e G_{h}^{(r,i)} + f N_{1,1,i}
\end{align*}
\]

where $a, b, c, d, e, f$ are coefficients, and $N_{1,0,i}, N_{1,1,i} \sim N(0,1)$ are independent values.

Apply these equations to meet the condition (3.2.3). We could obtain the coefficients, and consequently the equations for the $I$ subdivision as in the lemma.

\[\square\]

We are showing why and how we generate the value $Q$, which is the threshold value for generating $\Delta Y$.

**Explanation 3.2.6.** Here, we are explaining why the values $H$ and $Q$ are generated in the method (3.2.4).
Proof: After the $I$ subdivision process, the actual area $A^{(r)}_h = \sum_{i=0}^q A^{(r,i)}_h$ will become

$$A^{(r)}_h = A^{(r)}_{\frac{1}{2}} + \sum_{i=0}^q \Delta W^{(r,i)}_{h,L} \Delta Y^{(r,i)}_{h,R}.$$ 

On the other hand, let $\tilde{A}^{(r)}_h$ approximate $A^{(r)}_h$, then we have

$$\tilde{A}^{(r)}_h = \tilde{Z}^{(r)}_{\frac{1}{2}} + \sum_{i=0}^q \Delta Y^{(r,i)}_h G^{(r,i)}_h,$$  

(see Explanation (3.1.3))

where $\tilde{Z}^{(r)}_{\frac{1}{2}} \sim N(0, \frac{(q+1)h^2}{6})$ is independent to $\Delta W^{(r,i)}_{h,L}, \Delta Y^{(r,i)}_{h,R}, G^{(r,i)}_h$. After the $I$ subdivision, we split up $\Delta W^{(r,i)}_h$ and $G^{(r,i)}_h$ into $\Delta W^{(r,i)}_{h,L}, G^{(r,i)}_{h,L}, \Delta W^{(r,i)}_{h,R}$ and $G^{(r,i)}_{h,R}$. The $\tilde{A}^{(r)}_h$ and $\tilde{A}^{(r)}_{\frac{1}{2}}$ approximate $A^{(r)}_h$ and $A^{(r)}_{\frac{1}{2}}$ respectively. To have similar setup, $\tilde{A}^{(r)}_h = \tilde{Z}^{(r)}_{\frac{1}{2}} + \sum_{i=0}^q \Delta Y^{(r,i)}_h G^{(r,i)}_{h,L} + \sum_{i=0}^q \Delta Y^{(r,i)}_h G^{(r,i)}_{h,R}$.

Because we want $\tilde{A}^{(r)}_h$ to be a good approximate to $A^{(r)}_h$,

$$\tilde{A}^{(r)}_h \simeq \tilde{A}^{(r)}_{\frac{1}{2}} + \sum_{i=0}^q \Delta W^{(r,i)}_{h,L} \Delta Y^{(r,i)}_{h,R}$$

$$\tilde{Z}^{(r)}_{\frac{1}{2}} + \sum_{i=0}^q \Delta Y^{(r,i)}_h G^{(r,i)}_h \simeq \tilde{Z}^{(r)}_{\frac{1}{2}} + \sum_{i=0}^q (\Delta Y^{(r,i)}_h G^{(r,i)}_{h,L} + \Delta Y^{(r,i)}_h G^{(r,i)}_{h,R} + \Delta W^{(r,i)}_{h,L} \Delta Y^{(r,i)}_{h,R})$$

$$\tilde{Z}^{(r)}_{\frac{1}{2}} \simeq \tilde{Z}^{(r)} - \sum_{i=0}^q (G^{(r,i)}_h - G^{(r,i)}_{h,L})(\Delta Y^{(r,i)}_{h,R} - \Delta Y^{(r,i)}_{h,L}).$$

Because we want $\tilde{Z}^{(r)}_{\frac{1}{2}}$ to be independent to $\Delta Y^{(r,i)}_{h,L}, \Delta Y^{(r,i)}_{h,R}, G^{(r,i)}_h, G^{(r,i)}_{h,L}, G^{(r,i)}_{h,R}$, we split up $\tilde{Z}^{(r)}_{\frac{1}{2}}$ with the method (3.2.7).

Define value $H_i = G^{(r,i)}_{h,L} - G^{(r,i)}_{h,R}$ ( $H_i$ is the same as that in method (3.2.4), because of $G^{(r,i)}_{h,L} = 2G^{(r,i)}_h - G^{(r,i)}_{h,R} - \Delta W^{(r,i)}_h$. It should be $H_i \sim N(0, H/12)$, then we have random value $\sum_{i=0}^q (G^{(r,i)}_h - G^{(r,i)}_{h,L})(\Delta Y^{(r,i)}_{h,R} - \Delta Y^{(r,i)}_{h,L}) \sim N(0, \frac{(q+1)h^2}{12})$), because $G^{(r,i)}_h \sim N(0, h/3), G^{(r,i)}_{h,L} \sim N(0, 2h/3), E[G^{(r,i)}_h G^{(r,i)}_{h,L}] = h^2/24, \Delta Y^{(r,i)}_h \sim N(0, h), \Delta Y^{(r,i)}_{h,L} \sim N(0, h/2), E[\Delta Y^{(r,i)}_h \Delta Y^{(r,i)}_{h,L}] = h/2$, according to the definition. In such a case $\tilde{Z}^{(r)}_{\frac{1}{2}} \sim N(0, \frac{(q+1)h^2}{12})$.

So as to make sure $\text{var}[\sum_{i=0}^q (G^{(r,i)}_h - G^{(r,i)}_{h,L})(\Delta Y^{(r,i)}_{h,R} - \Delta Y^{(r,i)}_{h,L})] \simeq \frac{(q+1)h^2}{12}$, and consequently $\text{var} [\tilde{Z}^{(r)}_{\frac{1}{2}}] \simeq \frac{(q+1)h^2}{12}$, we set a threshold value $Q$. Define value $Q = \frac{12}{(q+1)h} \sum_{i=0}^q H_i^2 - 1$, then we have $E(Q) = 0$ and $E(Q^2) = \frac{2}{q+1}$. $Q$ in the method (3.2.4) will make sure that the subdivision of the $\Delta Y$ works correctly.

$|Q|$ is less than one. One may notice that the $\tilde{Z}^{(r)}_{\frac{1}{2}}$ is generated from $\tilde{Z}^{(r+2)}$, which is described in method (3.2.7). And the $\tilde{Z}^{(r+2)} \sim N(0, \frac{(q+1)h^2}{6} - h \sum_{i=0}^q H_i^2)$ leads to the restriction that $|Q| < 1$. If $|Q| > 1$, the algorithm will come up error when we apply the method (3.2.7), because the variance of $\tilde{Z}^{(r)}$ will be negative for some $i$ ($i = 1, \ldots, q+2$).
When $|Q| > 1/2$, we have $\var(\sum_{i=0}^{q}(C^{(r,i)}_{h} - C^{(r,i)}_{h,R})(\Delta Y_{h,R}^{(r,i)} - \Delta Y_{h,L}^{(r,i)}))$ far away from $(\delta^{(r+1)}h^{2})_{12}$. The subdivided Brownian trajectory with step size $h/2$ in such a case is not reasonable, so we subdivide the $\Delta Y_{h,R}^{(r,i)}$ in this part of Brownian trajectory with different way. The choose of the threshold value $\lambda = 1/2$ is arbitrary, whereas it is necessary that $\lambda \in (0, 1)$. It just makes the trajectory to be reasonable.

One can consider a more general situation. Conditional on $X \sim N(0, \sigma^2)$, we want to generate $X_0 \sim N(0, \sigma_0^2), \ldots, X_q \sim N(0, \sigma_q^2), Y \sim N(0, \sigma_Y^2)$, where $\sigma^2, \sigma_0^2, \ldots, \sigma_q^2, \sigma_Y^2$ are known, and $\sigma^2 = \sigma_0^2 + \ldots + \sigma_q^2 + \sigma_Y^2$. We want the $\sum_{i=0}^{q} \sigma_i^2$ and $\sigma_Y^2$ to be close to their expect values. So that they will not be far way from the values they should be.

The Explanation (3.2.6) reveals that it is closed related that splitting $\tilde{Z}^{(r)}_h$ and generating $N_{2,0,i}$. The $N_{2,0,i}$ is used to generate $\Delta Y_{h,L}^{(r,i)}$ and $\Delta Y_{h,R}^{(r,i)}$. We present how to generate $N_{2,0,i}$ below:

**Method 3.2.7. (Generating $N_{2,0,i}$)** We want to generate mutually independent $\tilde{Z}(q+2)$ and $N_{2,0,i} \sim N(0, 1)$ such that $\tilde{Z}(q+2) = \tilde{Z}^{(r)}_h - \sum_{i=0}^{q} N_{2,0,i} H_i \sqrt{h}$. We can achieve this by applying the method in lemma (1.2.12). Let's consider how to generate the $N_{2,0,i}$ only. Let $\tilde{Z}(i), \sigma^2, \tilde{Z}(i+1), N_{2,0,i}, H_{i-1} \sqrt{h}$ be $X, \Sigma, X_L, N, H$ in the lemma (1.2.12). Then one can generate the $N_{2,0,i}$ and $\tilde{Z}(i+1)$ as required. To be precise, we do the following things.

when $|Q| \leq \frac{1}{2}$:

1. In the 1st step, set the $\tilde{Z}(1) = \tilde{Z}^{(r)}_h$, the $\tilde{Z}(1)$'s variance $\sigma^2 = \frac{mh^2}{6}$.

2. Obtain $\tilde{Z}(q+1)$ and $N_{2,0,i}$: Let $R \sim N(0, 1)$ be an independent random variable. At the $i$th stage ($i = 1, \ldots, (q + 1)$), we can obtain the mutually independent $N_{2,0,i-1}$ and $\tilde{Z}(i+1)$ with:

\[
\begin{align*}
\left\{ \begin{array}{l}
N_{2,0,i-1} = \tilde{Z}(i) \frac{H_{i-1} \sqrt{h}}{\sigma^2} + R \sqrt{1 - \frac{H_{i-1}^2 h}{\sigma^2}} \\
\tilde{Z}(i+1) = \tilde{Z}(i) - N_{2,0,i-1} H_{i-1} \sqrt{h}
\end{array} \right.
\]

where $R$ is independent to $N_{2,0,i-1}$ and $\tilde{Z}(i+1)$. This process is explained with the following figure. Set $\tilde{Z}^{(1)}_h = \tilde{Z}^{(r)}_h$. 33
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\[ N_{2,0,0} \leftarrow \tilde{Z}^{(2)} = \tilde{Z}^{(1)} - N_{2,0,0}H_0\sqrt{h} \]
\[ N_{2,0,1} \leftarrow \tilde{Z}^{(3)} = \tilde{Z}^{(2)} - N_{2,0,1}H_1\sqrt{h} \]
\[ N_{2,0,2} \leftarrow \tilde{Z}^{(4)} = \tilde{Z}^{(3)} - N_{2,0,2}H_2\sqrt{h} \]

\[ \vdots \]

3. **Correct \( \tilde{Z}^h \):** After \((q+1)\) steps, we generate \( N_{2,0,0}, \ldots, N_{2,0,q}, Z^{(q+2)} \), and

\[ \text{Var}[Z^{(q+2)}] = \frac{(q+1)h^2}{6} - \sum_{i=0}^{q} H_i^2 h. \]

Set

\[ T = \frac{\text{Var}[\tilde{Z}^h]}{\text{Var}[Z^{(q+2)}]} = \frac{(q+1)h^2/12}{\frac{(q+1)h^2}{6} - \sum_{i=0}^{q} H_i^2 h} \]

The new \( \tilde{Z}^h \), which was corrected after the I **Subdivision** is

\[ \tilde{Z}^h = \tilde{Z}^{(q+2)} \sqrt{T} \]

\(|Q| \leq \frac{1}{2}\) is the more usual case, compared with \(|Q| > \frac{1}{2}\), and further discussion over \(|Q|\) will be in the theorem (3.9.1). When \(|Q| > \frac{1}{2}\), we generate the \( N_{2,0,j} \)

\((j = 0, \ldots, q)\) with the following steps:

when \(|Q| > \frac{1}{2}\):

1. Generate \( N_{2,0,i}^{(r)} \sim N(0, 1) \). \((i = 0, \ldots, q)\)

2. Generate \( \tilde{Z}^{(r)}_{\frac{h}{2}} \sim N(0, \frac{1}{2} \Sigma \tilde{Z}^h) \)

The method (3.2.4) and (3.2.7) provide a method of dividing \( \Delta W, \Delta Y, \) and \( G \), such that

\( \Delta W^{(r)}_{h,L} \sim N(0, h/2), \Delta W^{(r)}_{h,R} \sim N(0, h/6), \Delta Y^{(r)}_{h,L} \sim N(0, h/6), \) and \( \Delta Y^{(r)}_{h,R} \sim N(0, h/6) \) are mutually independent unconditionally.

**Lemma 3.2.8.** \( \tilde{Z}^h \sim N(0, \frac{(q+1)h^2}{12}) \) and \( N_{2,0,i} \sim N(0, 1) \) generated by the method (3.2.7) are independent of \( \Delta W^{(r)}_{h,L} \), \( \Delta W^{(r)}_{h,R} \), \( G^{(r)}_{h,L} \), \( G^{(r)}_{h,R} \).

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Proof: In each of the two cases in method (3.2.7), \( |Q| > 1/2 \) or \( |Q| < 1/2 \), we have, conditional on \( \Delta W_{h,L}^{(r,i)} \), \( \Delta W_{h,R}^{(r,i)} \), \( G_{h,L}^{(r,i)} \), \( G_{h,R}^{(r,i)} \), that \( \tilde{Z}_{h} \sim N(0, (q+1)K^2) \) and \( N_{2,0,i} \sim N(0, 1) \), and these variables are mutually independent and independent of \( \Delta Y_{(r,0)} \), \( \ldots \), \( \Delta Y_{(r,q)} \). So the (conditional) joint distribution of these variables is independent of \( \Delta W_{h,L}^{(r,i)} \), \( \Delta W_{h,R}^{(r,i)} \), \( G_{h,L}^{(r,i)} \), \( G_{h,R}^{(r,i)} \), which means they are independent of \( \Delta W_{h,L}^{(r,i)} \), \( \Delta W_{h,R}^{(r,i)} \), \( G_{h,L}^{(r,i)} \), \( G_{h,R}^{(r,i)} \) and these distributions hold unconditionally.\[1\]

These bring similar setup of the Brownian trajectory to the initial stage. And it also keeps the conditions (3.2.3).

### 3.3 The Difference of The Approximate Area

The previous discussion in this chapter provides method in generating and splitting the approximate area term \( \hat{A}_h^{(r)} \). However, how fast is the approximate area \( \hat{A}_h^{(r)} \) converging as time step size \( h \) decrease? Using the notation (3.2.2), we suppose \( q + 1 \) small time segment in \( J_r \) before the I subdivision, which is shown in the figure in Method 3.2.2. The following theorem reveals how the area of big interval \( J_r \) changes after one I subdivision.

**Theorem 3.3.1.** Let \( \hat{A}_h^{(r)} \) be the area for big interval \( J_r \), \( \hat{A}_h^{(r)} = \hat{A}_h^{(r)} + \sum W(a_i)\Delta Y_{(r,i)}^{(r,i)} = \hat{A}_h^{(r)} + \sum_{i=0}^q(G_{h,L}^{(r,i)} + W(a_i))\Delta Y_{(r,i)}^{(r,i)} \) and \( \hat{A}_h^{(r)} \) be the area for interval \( J_r \) after one I subdivision \( \hat{A}_h^{(r)} = \hat{A}_h^{(r)} + \sum W(a_i)\Delta Y_{(r,i)}^{(r,i)} = \hat{A}_h^{(r)} + \sum_{i=0}^q((G_{h,L}^{(r,i)} + W(a_{2i}))\Delta Y_{(r,i)}^{(r,i)} + (G_{h,R}^{(r,i)} + W(a_{2i+1}))\Delta Y_{(r,i)}^{(r,i)} \). See the second line of the figure (3.2) for the notations. Then

\[
E[\hat{A}_h^{(r)} - \hat{A}_h^{(r)}]^2 \leq 4h^2.
\]

**Proof:** From the I subdivision process, we have

\[
\hat{A}_h^{(r)} = \hat{Z}_h^{(r)} + \sum_{i=0}^q(G_{h,L}^{(r,i)} + W(a_{2i}))\Delta Y_{(r,i)}^{(r,i)} + (G_{h,R}^{(r,i)} + W(a_{2i+1}))\Delta Y_{(r,i)}^{(r,i)}
\]

because of \( a_{2i} = a_i \). So it is true that

\[
E[\hat{A}_h^{(r)} - \hat{A}_h^{(r)}]^2 = E[\hat{Z}_h^{(r)} - \hat{Z}_h] + \sum_{i=0}^q H_i N_{2,0,i+1} \sqrt{h}/2
\]

\[
= E[\hat{Z}_h - \hat{Z}(q+2)]^2
\]

\[
= E\{(\hat{Z}_h - \hat{Z}(q+2))_{X|Q| \leq 1}^2\} + E\{(\hat{Z}_h - \hat{Z}(q+2))_{X|Q| > 1}^2\}.
\]

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when we consider that $H_i = G_i^{(r,i)} - G_{h,L}^{(r,i)}$ and $G_{h,R}^{(r,i)} = 2G_i^{(r,i)} - G_{h,L}^{(r,i)} - \Delta W_{h,L}^{(r,i)}$. (See definition of $H_i$ in Method (3.2.4).)

Because $|Q| \leq 1/2$, we have

$$E[\sqrt{1 - Q} - 1]^2 = E[\frac{Q}{\sqrt{1 - Q + 1}}]^2 \leq E[\frac{Q}{\sqrt{1 - 1/2 + 1}}]^2$$

$$\leq E[\frac{\sqrt{2Q}}{1 + \sqrt{2}}]^2 \leq \frac{2}{5}E[Q]^2.$$ 

Hence, the first term on the right is

$$E\{(\tilde{Z}_h - Z^{(q+2)}_{x_{\{Q<1/2\}}}^2)\} = E\{((\tilde{Z}_h^{(q+2})^2(1 - (1 - Q)^{-\frac{1}{2}})^2)_{x_{\{Q<1/2\}}}\}$$

$$= E\{((\tilde{Z}_h^{(q+2)})^2(1 - (1 - Q)^{\frac{1}{2}})^2)_{x_{\{Q<1/2\}}}\}$$

$$= \frac{(q + 1)h^2}{12}E\{((1 - Q)^{\frac{1}{2}} - 1)^2)_{x_{\{Q<1/2\}}}\} = (Q, \tilde{Z}_h^{2} \text{ independent})$$

$$\leq \frac{(q + 1)h^2}{30}E(Q^2) \leq \frac{h^2}{15}.$$ 

See the explanation (3.2.6) for more detail about $Q$.

The second term on the right is

$$= E\{\tilde{Z}_h + \sqrt{h} \sum_{i=0}^{q} H_i N_{2,0,i+1}^2_{x_{\{Q>1/2\}}}\}$$

$$= E\{((q + 1)h^2 + \tilde{Z}_h^2 + h \sum_{i=0}^{q} H_i^2)_{x_{\{Q>1/2\}}}\}$$

$$= \frac{(q + 1)h^2}{4}P(|Q| > \frac{1}{2}) + \frac{(q + 1)h^2}{12}E\{[1 + Q]_{x_{\{Q>1/2\}}}\}$$

$$\leq \frac{(q + 1)h^2(1 + \frac{1}{12})P(|Q| > \frac{1}{2}) + \frac{(q + 1)h^2}{12}\{P(|Q| > \frac{1}{2})\}^\frac{1}{2}\{E[Q^2]\}^\frac{1}{2} \leq 3h^2.$$ 

Because of $E(Q^2) = \frac{2}{q+1}$, the consequent fact that $P(|Q| > \frac{1}{2}) \leq \frac{8}{q+1}$, and, in the second I subdivision, $q + 1 = m$, we have $E[\tilde{A}_h^{(r)} - \tilde{A}_h^{(r)}]^2 \leq 4h^2$.

The following theorem reveals how the area of $J_r$ changes after one $J$ subdivision. Suppose the time interval $J_r = [a_0, a_{q+1}]$ is divided into two equal segments $J_{r,L} = [a_0, a_{q+1}]$ and $J_{r,R} = [a_{q+1}, a_{q+1}]$. And then $J_{r,L}$ and $J_{r,R}$ are $I$ subdivided twice.

**Theorem 3.3.2.** Let $\tilde{A}_h^{(r)}$ be approximate to $\int_t [W(t) - W(a_0)] dY(t) + \sum_{i=0}^{q} W(a_i) \Delta Y_{h,(r,i)}$ having $\tilde{A}_h^{(r)} = \tilde{Z}_h^{(r)} + \sum_{i=0}^{q} (G_h^{(r,i)} + W(a_i)) \Delta Y_{h,(r,i)}$, $\tilde{A}_h^{(r)}$ be approximate to area $J_{r,L}$, $\tilde{A}_h^{(r)}_{h/4,L}$ be approximate to area $J_{r,R}$, $\Delta W_{J_{r,L}}$ be $\Delta W_{J_{r,L}} = W(a_{q+1}) - W(a_0)$, and $\Delta Y_{J_{r,L}} = \Delta Y_{J_{r,R}} = Y(a_{q+1}) - Y(a_{q+1})$. The definition of $\tilde{A}_h^{(r)}_{h/4,L}$ and $\tilde{A}_h^{(r)}_{h/4,R}$ are similar to the $\tilde{A}_h^{(r)}$, except that the time interval
Ii for $\hat{A}_{h/4,L}^{\text{(r)}}$ and $\hat{A}_{h/4,R}^{\text{(r)}}$ are $h/4$ rather than $h$. One I subdivision will behalf the length of the time segment. Then we have

$$E(\hat{A}_h^{\text{(r)}} - \hat{A}_{h/4,L}^{\text{(r)}} - \hat{A}_{h/4,R}^{\text{(r)}} - \Delta W_{h,L} \Delta Y_{J_r})^2 = O(2^{-4n}).$$

Proof: After we splitting the $J_r$ into $J_{r,L}$ and $J_{r,R}$, with the method (3.2.1), we have two approximate areas. Denote the corresponding approximate area integrals by $A_{h,L}^{\text{(r)}}$ and $A_{h,R}^{\text{(r)}}$, we have $\hat{A}_h^{\text{(r)}} = A_{h,L}^{\text{(r)}} + A_{h,R}^{\text{(r)}} + \Delta W_{h,L} \Delta Y_{J_r}$. Because $\hat{A}_{h/4,L}^{\text{(r)}}$ is the approximate area for $J_{r,L}$ after two I subdivisions and the conclusion of the theorem (3.3.1), we have $E[\hat{A}_{h/4,L}^{\text{(r)}} - A_{h,L}^{\text{(r)}}]^2 = O(2^{-4n})$. Similarly, we have $E[\hat{A}_{h/4,R}^{\text{(r)}} - A_{h,R}^{\text{(r)}}]^2 = O(2^{-4n})$. Therefore we can draw the conclusion that

$$E(\hat{A}_h^{\text{(r)}} - \hat{A}_{h/4,L}^{\text{(r)}} - \hat{A}_{h/4,R}^{\text{(r)}} - \Delta W_{h,L} \Delta Y_{J_r})^2 = O(2^{-4n}).$$

(3.1)

The following theorem reveals how the area of $J_r$ changes after one $J$ subdivision and two $I$ subdivisions.

Theorem 3.3.3. Suppose the construction method (3.2.4) has been carried out. Let $\hat{A}_{h,r}^{\text{(r)}}$ approximates the sum of the area integrals, $\sum \hat{A}_i$, having $\hat{A}_h^{\text{(r)}} = \hat{Z}_h^{\text{(r)}} + \sum G_{h,i}^{\text{(r)}} \Delta Y_h^{\text{(r)}},$ and $A_{h,r}^{\text{(r)}}$ is the actual sum of the area integrals in $J_r$. It will be

$$E(\hat{A}_h^{\text{(r)}} - A_{h,r}^{\text{(r)}})^2 = O(2^{-4n}).$$

Proof: To show this clearly, we use notation $J_i' (i = 0, \ldots)$ for big intervals after subdivisions. The notations shown in the picture below are only applied in this theorem. Let's consider the $J_r$ at the $n^\text{th}$ stage only.

- $\hat{A}_{h}^{\text{(r)}}$ approximates $A_J$, the area of $J_r$.
- $J_r$ for the $n^\text{th}$ stage (denote $J$ for short).

For each of the $2^n$ large intervals $J_r$ at the $n^\text{th}$ stage, we have an approximate area integral $\hat{A}_{h}^{\text{(r)}}$ as constructed in theorem (3.3.1). Now fix the interval $J$, and for $p \geq n$, define an approximate area integral $\hat{A}_{J}^{[p]} = \sum_{i=1}^{2^{p-n}} \{ \hat{A}_{J_i'} + [W(l_i) - W(l_{i-1})] \Delta Y_{J_i'} \},$ where $\hat{A}_{J_i'}$ is an area for $J_i'$, and $\hat{A}_{J}^{[p]}$ is the sum of areas over the $2^{p-n}$ large stage-$p$ intervals $J'$ contained in $J$. Here, we use the $l_i$ to indicate the left hand side instant times for $J_i'$. Please note that $\hat{A}_{J}^{[n]} = \hat{A}_h^{\text{(r)}}$ and $\hat{A}_{J}^{[n+1]} = \hat{A}_{h/4,L}^{\text{(r)}} + \hat{A}_{h/4,R}^{\text{(r)}} + \Delta W_{h,L} \Delta Y_{h,R}$.

Because the equation (3.1) told us that

$$E(\hat{A}_h^{\text{(r)}} - \hat{A}_{h/4,L}^{\text{(r)}} - \hat{A}_{h/4,R}^{\text{(r)}} - \Delta W_{h,L} \Delta Y_{h,R})^2 = O(2^{-4n}),$$

and there are $2^{p-n} J'$ areas in $J$, we have $E(\hat{A}_{J}^{[p+1]} - \hat{A}_{J}^{[p]})^2 = O(2^{p-n} 2^{-4p}) =
Numerical Approximation for SDE

Because \( E[\sum A_i^2] = O(2^{-p-n}) \) and \( \sum \{W(l_j) - W(l_j')\} \Delta Y_j \to A_j \) as \( p \to \infty \), we have \( \tilde{A}_j^p \to A_j \). When we consider \( E[\tilde{A}_j^{p+1} - \tilde{A}_j^p]^2 = O(2^{-3p-n}) \) as well, we have \( E[A_h^{(r)} - A_j]^2 = E[\tilde{A}_h^{(r)} - \tilde{A}_j]^2 \) as \( p \to \infty \), hence \( E[A_h^{(r)} - A_j]^2 = O(2^{-4n}) \).

It then follows that for the approximate sum of integrals
\[
\begin{align*}
\tilde{A}_h^{(r)} &= \tilde{Z}_h^{(r)} + \sum G_h^{(r,i)} \Delta Y_h^{(r,i)} = \tilde{A}_h^{(r)} - \sum W(l_i) \Delta Y_h^{(r,i)} \quad \text{and the actual sum of integrals} \quad A_h^{(r)} = A_j - \sum W(l_i) \Delta Y_h^{(r,i)},
\end{align*}
\]
we have \( E[A_h^{(r)} - A_h^{(r)}]^2 = O(2^{-4n}) \).

Remark: Theorem (3.3.3) proves \( E(\tilde{A}_h^{(r)} - A_h^{(r)})^2 = O(2^{-4n}) \), which satisfies the assumption (2.3.1), hence we can apply the theorem (2.4.1). This gives us the error bound of the 3/4 Method. In high dimension cases, the theorem (3.3.2) and (3.3.3) are still true.

In the 3D and nD cases, the theorem (3.3.2) and (3.3.3) still hold, although we need to slightly amend the theorem (3.3.1) for the different constructed area terms.

### 3.4 Illustrative example

Now we apply the method discussed in the previous section to a SDE problem. We generate the initial Brownian trajectory \((W, Y)\) with method (3.1.4) and the subdivided Brownian trajectory with method (3.2.4). Then the random variable \( x_h^{(N)} \) which approximates the \( x(T) \) for SDE at time \( T \), could be generated.

**Illustrative Example 3.4.1.** Consider the linear SDE problem

\[
\begin{align*}
dX^{(1)} &= dW \\
dX^{(2)} &= X^{(1)} \, dY
\end{align*}
\]

Let initial value \((X^{(1)}(0), X^{(2)}(0)) = (3, 4)\). The SDE is approximated with Euler and 3/4 Method on \([0, T]\) with time step size \( h, h/4, h/16, h/64, \) and \( h/256 \) where \( T = 10 \) and \( h = T/16 \). Approximate the \( E[X_h(T)] \) with \( M = 300 \) Monte Carlo simulations.

Then we have the numerical scheme which is characterized by an equal distant partition \( \tau_n \) of \([0, T]\):

\[
\tau_n : 0 = a_0 < a_1 < \ldots < a_N = T
\]

with mesh

\[
d = mesh(\tau_n) = a_{i+1} - a_i, \quad (i = 0, \ldots, N - 1).
\]

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The Approximation Function

1. When $N = m^2$ and $m = 2^k$, where $k$ is some constant, we have Euler approximation

\[
\begin{align*}
X^{(1)}(n) &= X^{(1)}(n - 1) + \Delta W^{(rm+i)}_h \\
X^{(2)}(n) &= X^{(2)}(n - 1) + X^{(1)}(n - 1)\Delta Y^{(rm+i)}_h
\end{align*}
\]

where $\delta = h = T/N$, $n = rm + i + 1$, $r = 0, \ldots, m - 2$.

2. We also have the $3/4$ approximation: When $n \not\equiv 0 \pmod{m}$

\[
\begin{align*}
X^{(1)}(n) &= X^{(1)}(n - 1) + \Delta W^{(rm+i)}_h \\
X^{(2)}(n) &= X^{(2)}(n - 1) + X^{(1)}(n - 1)\Delta Y^{(rm+i)}_h
\end{align*}
\]

When $n = m(r + 1)$ ($r = 0, 1, \ldots, m - 1$),

\[
\begin{align*}
X^{(1)}(n) &= X^{(1)}(n - 1) + \Delta W^{(rm+i)}_h \\
X^{(2)}(n) &= X^{(2)}(n - 1) + X^{(1)}(n - 1)\Delta Y^{(rm+i)}_h + A(r)
\end{align*}
\]

Approximate $E[X_h]$ 

Corresponding to $M$ trajectories, we have $M$ samples $X^{[i]}_h(N), \ldots, X^{[M]}_h(N)$, which approximate $x(T)$ for the SDE. Hence we can approximate the $E[X_h]$ with $\bar{X}_h(N) = \sum_{i=1}^{M} X^{[i]}_h(N)/M$. Similarly, we can approximate the $E[X_h - \frac{X}{4}]$ with $\sum_{i=1}^{M} |X^{[i]}_h(N) - \frac{X}{4}(4N)|/M$.

Order of The Scheme

For an order $\gamma > 0$ scheme, we have $E[X_h(N) - x(T)] \approx C h^\gamma$. Although we do not know $x(T)$ in general case, it is practicable to approximate it with relatively smaller step size approximation $X_{h/4}(4N)$. Define

\[
L_h = \frac{\log\{\sum_{i=1}^{M} |X_h(N) - X_{h/4}(4N)|/M\} - \log\{\sum_{i=1}^{M} |X_{h/4}(4N) - X_{h/16}(16N)|/M\}}{\log(4)}
\]

Then we have

\[
L_h \approx \frac{\log\{E[X_h(N) - x(T)]\} - \log\{E[X_{h/4}(4N) - x(T)]\}}{\log(4)} \approx \frac{\log\{C(h)^\gamma\} - \log\{C(h/4)^\gamma\}}{\log(4)} = \gamma
\]
Numerical Approximation for SDE

Simulation Result

The computer simulation gives us approximate $E[X_h - X_{\frac{h}{4}}]$, $E[X_{\frac{h}{4}} - X_{\frac{h}{16}}]$, $E[X_{\frac{h}{16}} - X_{\frac{h}{64}}]$, $E[X_{\frac{h}{64}} - X_{\frac{h}{256}}]$ by 3/4 Method,

<table>
<thead>
<tr>
<th>Step Size $h$</th>
<th>0.625</th>
<th>0.15625</th>
<th>0.0390625</th>
<th>$9.765625 \times 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[X_h^{(1)} - X_{\frac{h}{4}}^{(1)}]$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$E[X_h^{(2)} - X_{\frac{h}{4}}^{(2)}]$</td>
<td>0.96165</td>
<td>0.39315</td>
<td>0.15766</td>
<td>0.06160</td>
</tr>
</tbody>
</table>

and by Euler Method,

<table>
<thead>
<tr>
<th>Step Size $h$</th>
<th>0.625</th>
<th>0.15625</th>
<th>0.0390625</th>
<th>$9.765625 \times 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[X_h^{(1)} - X_{\frac{h}{4}}^{(1)}]$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$E[X_h^{(2)} - X_{\frac{h}{4}}^{(2)}]$</td>
<td>1.32406</td>
<td>0.57601</td>
<td>0.30048</td>
<td>0.14920</td>
</tr>
</tbody>
</table>

So we may approximate the Strong Scheme order $\gamma$ with $L_h$.

<table>
<thead>
<tr>
<th>Step Size $h$</th>
<th>0.15625</th>
<th>0.0390625</th>
<th>$9.765625 \times 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_h$ in Euler Method</td>
<td>0.60040</td>
<td>0.46940</td>
<td>0.50504</td>
</tr>
<tr>
<td>$L_h$ in 3/4 Method</td>
<td>0.64522</td>
<td>0.65915</td>
<td>0.67792</td>
</tr>
</tbody>
</table>

This result shows the 3/4 Method has higher order than the Euler one in the case that SDE was driven by two Wiener processes.
Chapter 4

Pathwise Approximation of Solutions of Stochastic Differential Equations in 3 Dimension Case

When understanding the 3/4 method in 2D situation, you may notice that the key of the method is how to generate and split up the area term $A_{h,k,s}^{(r)}$ ($k > s$) and Brownian paths $W(t)$ and $Y(t)$.

The idea of the 3/4 method is to generate approximation to certain sums of the areas, $\sum_{i=1}^{m} A_{h,k,s}^{(r)}$, according to the "Law of Large Numbers", although we could not generate each Area.[1]

Considering the notations in method (2.1.1) and equations (2.1), we are abbreviating them as what we did in 2D situation. We denote the components of the path by $W(t)$, $Y(t)$ and $R(t)$ rather than $W^{(1)}(t)$, $W^{(2)}(t)$, and $W^{(3)}(t)$.

According to the method (2.1.1), we suppose that $[a, b]$ is divided into $m$ equal intervals, $J_0, \ldots, J_{m-1}$, with each interval $J_r$ ($r = 0, \ldots, m-1$) being subdivided again into $m$ equal subintervals $I_0^{(r)} = [a_{rm}, a_{rm+1})$, $I_{m-1}^{(r)} = [a_{(r+1)m-1}, a_{(r+1)m}]$ with length $h$, where $a_0 = a$, and $a_{mm} = b$. We also have $\Delta W_h^{(r,i)} = W(a_{rm+i+1}) - W(a_{rm+i})$, $\Delta Y_h^{(r,i)} = Y(a_{rm+i+1}) - Y(a_{rm+i})$, $\Delta R_h^{(r,i)} = R(a_{rm+i+1}) - R(a_{rm+i})$.

Because in 3D situation, there will be three areas, so we denote them $A_{h,Y,W}^{(r)}$, $A_{h,R,W}^{(r)}$, $A_{h,R,Y}^{(r)}$ rather than $A_{h,1,1}^{(r)}$, $A_{h,3,1}^{(r)}$, $A_{h,3,2}^{(r)}$.

\[
A_{h,Y,W}^{(r)} = \sum_{i=r_m}^{r_m+m-1} \int_{a_i}^{a_{i+1}} (W(t) - W(a_i)) \, dY(t),
\]
\[
A_{h,R,Y}^{(r)} = \sum_{i=r_m}^{r_m+m-1} \int_{a_i}^{a_{i+1}} (Y(t) - Y(a_i)) \, dR(t),
\]
\[
A_{h,R,W}^{(r)} = \sum_{i=r_m}^{r_m+m-1} \int_{a_i}^{a_{i+1}} (W(t) - W(a_i)) \, dR(t),
\]
We denote $G^{(r,i)}_{h,W,1}$, $G^{(r,i)}_{h,W,2}$, $G^{(r,i)}_{h,Y,1}$ as below,

$$
G^{(r,i)}_{h,W,1} = \frac{1}{h} E[A^{(r)}_{h,Y,W} \Delta Y_{h}^{(r,i)} | W(t)]
= \frac{1}{h} E\{ \int_{a_{rm+i}}^{a_{rm+i+1}} (W(t) - W(a_{rm+i})) dY(t) | W(t) \}
= \frac{1}{h} \int_{a_{rm+i}}^{a_{rm+i+1}} (W(t) - W(a_{rm+i})) dt
$$

$$
G^{(r,i)}_{h,Y,1} = \frac{1}{h} E[A^{(r)}_{h,R,Y} \Delta R_{h}^{(r,i)} | Y(t)]
= \frac{1}{h} E\{ \int_{a_{rm+i}}^{a_{rm+i+1}} (Y(t) - Y(a_{rm+i})) dR(t) | Y(t) \}
= \frac{1}{h} \int_{a_{rm+i}}^{a_{rm+i+1}} (Y(t) - Y(a_{rm+i})) dt
$$

$$
G^{(r,i)}_{h,W,2} = \frac{1}{h} E\{ G^{(r,i)}_{h,Y,W} A^{(r)}_{h,W} | W(t) \}
= \frac{1}{h^2} E\{ \int_{a_{rm+i}}^{a_{rm+i+1}} (Y(t) - Y(a_{rm+i})) dt \int_{a_{rm+i}}^{a_{rm+i+1}} (W(t) - W(a_{rm+i})) dY(t) | W(t) \}
= \frac{1}{h^2} \int_{a_{rm+i}}^{a_{rm+i+1}} (a_{rm+i+1} - t)(W(t) - W(a_{rm+i})) dt.
$$

That is, conditional on the path $W(t)$, $A^{(r,i)}_{Y,W}$ has covariance $hG^{(r,i)}_{h,W,1}$ with $Y(t)$, and $A^{(r,i)}_{R,W}$ has covariance $hG^{(r,i)}_{h,W,1}$ with $R(t)$. Conditional on the path $Y(t)$, $A^{(r,i)}_{R,Y}$ has covariance $hG^{(r,i)}_{h,Y,1}$ with $R(t)$.

### 4.1 Initial Stage

In the 2D case, only one area term is needed, and in 3D case, we need to generate 3 area terms. Let’s see what property do $G^{(r,i)}_{h,W,1}$, $G^{(r,i)}_{h,W,2}$, $G^{(r,i)}_{h,Y,1}$ have.

**Lemma 4.1.1.** Let $\Delta W_{h}^{(r,i)}$, $\Delta Y_{h}^{(r,i)}$, $\Delta R_{h}^{(r,i)} \sim N(0,h)$ be the independent increments of the Brownian trajectory, $A_{h,Y,W}^{(r)}, A_{h,R,W}^{(r)}, A_{h,R,Y}^{(r)} \sim N(0, mh^2/2)$ be the sum of area terms, and $hG^{(r,i)}_{h,W,1}$, $hG^{(r,i)}_{h,Y,1}$, $hG^{(r,i)}_{h,W,2}$ be the conditional covariance, where

$$
\begin{align*}
    hG^{(r,i)}_{h,W,1} &= E[\Delta R_{h}^{(r,i)} A_{h,R,W}^{(r)} | W(t)] \\
    hG^{(r,i)}_{h,W,1} &= E[\Delta Y_{h}^{(r,i)} A_{h,Y,W}^{(r)} | W(t)] \\
    hG^{(r,i)}_{h,Y,1} &= E[\Delta R_{h}^{(r,i)} A_{h,R,Y}^{(r)} | Y(t)] \\
    hG^{(r,i)}_{h,W,2} &= E[ G^{(r,i)}_{h,Y,W} A_{h,Y,W}^{(r)} | W(t)].
\end{align*}
$$
Then

\[ E[\Delta W_{h}^{(r,i)} G_{h, W, 1}^{(r,i)}] = E[\Delta Y_{h}^{(r,i)} G_{h, Y, 1}^{(r,i)}] = \frac{h}{2} \]

\[ E[\Delta W_{h}^{(r,i)} G_{h, W, 2}^{(r,i)}] = \frac{h}{6} \]

\[ E[G_{h, W, 2}^{(r,i)} G_{h, W, 1}^{(r,i)}] = \frac{h}{8} \]

\[ G_{h, W, 2}^{(r,i)} \sim N(0, \frac{h}{20}), \quad G_{h, W, 1}^{(r,i)}, G_{h, Y, 1}^{(r,i)} \sim N(0, \frac{h}{3}). \]

**Proof:** Let's consider the interval \( I_{i}^{(r)} = [a_{rm+i}, a_{rm+i+1}] \) and its corresponding \( G_{h, W, 1}^{(r,i)}, G_{h, W, 2}^{(r,i)}, G_{h, Y, 1}^{(r,i)}. \) We have

\[ E[\Delta W_{h}^{(r,i)} G_{h, W, 1}^{(r,i)}] = E \left[ \frac{1}{h} \int_{I_{i}^{(r)}} (W(t) - W(a_{rm+i})) dt \int_{I_{i}^{(r)}} dW(t) \right] \]

\[ = \frac{1}{h} \int_{I_{i}^{(r)}} E(W(t) - W(a_{rm+i}))^2 dt = \frac{1}{h} \int_{I_{i}^{(r)}} (t - a_{rm+i}) dt = \frac{h}{2}, \]

\[ E[\Delta W_{h}^{(r,i)} G_{h, W, 2}^{(r,i)}] = E \left[ \frac{1}{h^2} \int_{I_{i}^{(r)}} (a_{rm+i+1} - t)(W(t) - W(a_{rm+i})) dt \int_{I_{i}^{(r)}} dW(t) \right] \]

\[ = \frac{1}{h^2} \int_{I_{i}^{(r)}} (a_{rm+i+1} - t)E[W(t) - W(a_{rm+i})]^2 dt \]

\[ = \frac{1}{h^2} \int_{I_{i}^{(r)}} [(a_{rm+i+1} - a_{rm+i})(t - a_{rm+i}) - (t - a_{rm+i})^2] dt \]

\[ = \frac{1}{h^2}(a_{rm+i+1} - a_{rm+i}) \frac{h^2}{2} - \frac{1}{h^3} \frac{h^3}{3} = \frac{h}{6}. \]

\[ E[G_{h, W, 2}^{(r,i)} G_{h, W, 1}^{(r,i)}] = \frac{1}{h^3} \int_{I_{i}^{(r)}} \int_{a_{rm+i}}^{s} (a_{rm+i+1} - t)E[W(t) - W(a_{rm+i})]^2 dt ds \]

\[ + \frac{1}{h^3} \int_{I_{i}^{(r)}} \{(a_{rm+i+1} - t) \int_{a_{rm+i}}^{t} E[W(s) - W(a_{rm+i})]^2 ds\} dt \]

\[ = \frac{h}{12} + \frac{h}{24} = \frac{h}{8}. \]

**Lemma 4.1.2.** Let \( \Delta W_{h}^{(r,i)}, \Delta Y_{h}^{(r,i)}, \Delta R_{h}^{(r,i)} \sim N(0, h) \) be the independent increment of the Brownian trajectory, and \( A_{h, Y, W}^{(r)}, A_{h, R, W}^{(r)}, A_{h, R, Y}^{(r)} \) be the sum of area terms. Then we have the expectation and variance of \( A_{h, Y, W}^{(r)}, A_{h, R, W}^{(r)}, A_{h, R, Y}^{(r)} \) which are shown in figure (4.1). Conditional on \( W(t) \) and \( Y(t) \), \( (A_{h, R, W}^{(r)}, A_{h, R, Y}^{(r)}, \Delta R_{h}^{(r,0)}), \ldots, \Delta R_{h}^{(r,i)} \) has joint normal distribution with mean 0 and \((m + 2) \times (m + 2)\)
<table>
<thead>
<tr>
<th></th>
<th>unconditional</th>
<th>conditional $W(t)$</th>
<th>conditional $Y(t)$</th>
<th>$m \to \infty$</th>
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<tr>
<td>$A_{h,R,Y}$</td>
<td>mean</td>
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<td>0</td>
<td>0</td>
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<tr>
<td>variance</td>
<td>$mh^2/2$</td>
<td>$V_{h,R,Y}$</td>
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<tr>
<td>$A_{h,Y,W}$</td>
<td>mean</td>
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<td>0</td>
<td>0</td>
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<tr>
<td>variance</td>
<td>$mh^2/2$</td>
<td>$V_{h,Y,W}$</td>
<td>$mh^2/2$</td>
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</table>

<table>
<thead>
<tr>
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<th>unconditional</th>
<th>conditional $W(t)$</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{h,R,W}^{(r)}$</td>
<td>$\sum_{i=0}^{m-1} \int_{t_{i(r)}}^{t_{i+1(r)}} (W(t) - W(a_{rm+i}))^2 dt$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_{h,R,Y}^{(r)}$</td>
<td>$\sum_{i=0}^{m-1} \int_{t_{i(r)}}^{t_{i+1(r)}} (Y(t) - Y(a_{rm+i}))^2 dt$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_{h,Y,W}^{(r)}$</td>
<td>$\sum_{i=0}^{m-1} \int_{t_{i(r)}}^{t_{i+1(r)}} (W(t) - W(a_{rm+i}))^2 dt$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.1: The expectation and variance of the area term

variance matrix $\Sigma_R =$

$$
\begin{pmatrix}
V_{h,R,W}^{(r)} & E(A_{h,R,Y}^{(r)} | W, Y) & hG_{h,Y,1}^{(r,0)} & hG_{h,Y,1}^{(r,i)} & hG_{h,Y,1}^{(r,m-1)} \\
E(A_{h,R,W}^{(r)} | W, Y) & V_{h,R,Y}^{(r)} & hG_{h,W,1}^{(r,0)} & hG_{h,W,1}^{(r,i)} & hG_{h,W,1}^{(r,m-1)} \\
hG_{h,W,1}^{(r,0)} & hG_{h,W,1}^{(r,i)} & hG_{h,W,1}^{(r,m-1)} & \cdot & \cdot \\
\end{pmatrix}
$$

Conditional on $W(t)$, $(A_{h,Y,W}^{(r)}, \Delta Y_h^{(r,0)}, \ldots, \Delta Y_h^{(r,i)}, G_{h,Y,1}^{(r,0)}, \ldots, G_{h,Y,1}^{(r,m-1)})$ has joint normal distribution with mean 0 and variance $\Sigma_Y$. ($\Sigma_Y$ is shown in Explanation (4.1.3)).

Proof: One can get the analogous proof from the lemma (3.1.2).

Analogous to the 2-D approximation, we could see that the sum of area integrals $A_{h,R,Y}^{(r)}$ and $A_{h,R,W}^{(r)}$ could be approximated with normal distribute variables

$$
\tilde{A}_{h,R,Y}^{(r)} = \tilde{Z}_{h,R,Y}^{(r)} + \sum G_{h,W,1}^{(r,i)} \Delta R_h^{(r,i)},
$$

$$
\tilde{A}_{h,R,W}^{(r)} = \tilde{Z}_{h,R,W}^{(r)} + \sum G_{h,W,1}^{(r,i)} \Delta R_h^{(r,i)},
$$

respectively, where $\tilde{Z}_{h,R,W}^{(r)}, \tilde{Z}_{h,R,W}^{(r)} \sim N(0, \frac{mh^2}{6})$ is independent random value. It might recall you the Explanation (3.1.3) in 2D case. In fact, it makes no difference.
between approximating the area term $A_h^{(r)}$ in 2D case and approximating area terms $A_{h,R,W}^{(r)}$ and $A_{h,R,Y}^{(r)}$. Because we could consider only $R(t)$ and $W(t)$, the approximation of the $A_{h,R,W}^{(r)}$ would be the same as in 2D case ($R(t)$ and $W(t)$). So are the $A_{h,R,Y}^{(r)}$. However, approximating the area term $A_{h,Y,W}^{(r)}$ is different.

**Explanation 4.1.3.** We explain the way of approximating $A_{h,Y,W}^{(r)}$ here.

The sum of area integrals $A_{h,Y,W}^{(r)} \sim N(0, \frac{m h^2}{2})$, will be conditionally on the path $W(t)$ and $\Delta Y_h^{(r,0)}$, $\Delta Y_h^{(r,m-1)}$, normal distributed with mean $\sum_{i=0}^{m-1} ((4G_{h,W,1}^{(r,i)} - 6G_{h,W,2}^{(r,i)})\Delta Y_h^{(r,i)} + (12G_{h,Y,1}^{(r,i)} - 6G_{h,Y,2}^{(r,i)})G_{h,Y,1})$ and variance $V_{h,Y,W} = \sum_{i=0}^{m-1} ((4G_{h,Y,1}^{(r,i)} - 6G_{h,Y,2}^{(r,i)})^2 h + (12G_{h,Y,2}^{(r,i)} - 6G_{h,Y,1}^{(r,i)})^2 \frac{h}{3}$.

Conditional on $W(t)$, $X = (A_{h,Y,W}^{(r)}, \Delta Y_h^{(r,0)}, \ldots, \Delta Y_h^{(r,m-1)}, G_{h,Y,1}, \ldots, G_{h,Y,1}^2)^T$ is joint normal distributed with mean 0 and $\Sigma_Y = E[XX^T] = 0$

\[
\begin{pmatrix}
\frac{m h^2}{2} & hG_{h,W,1}^{(r,0)} & hG_{h,W,1}^{(r,1)} & \ldots & hG_{h,W,1}^{(r,m-1)} & hG_{h,W,2}^{(r,0)} & \ldots & hG_{h,W,2}^{(r,m-2)} & hG_{h,W,2}^{(r,m-1)} \\
hG_{h,W,1}^{(r,0)} & h & 0 & \ldots & 0 & h/2 & 0 & \ldots & 0 \\
hG_{h,W,1}^{(r,1)} & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 \\
hG_{h,W,1}^{(r,m-2)} & 0 & 0 & \ldots & h & h/2 & 0 & \ldots & 0 & \ldots & 0 \\
hG_{h,W,1}^{(r,m-1)} & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
hG_{h,W,2}^{(r,0)} & h/2 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
hG_{h,W,2}^{(r,1)} & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
hG_{h,W,2}^{(r,m-2)} & 0 & 0 & \ldots & h/2 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
hG_{h,W,2}^{(r,m-1)} & 0 & 0 & \ldots & 0 & h/3 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
\end{pmatrix}
\]

So conditional on $W(t)$, the $A_{h,Y,W}^{(r)}$ can be expressed with

$$A_{h,Y,W}^{(r)} = \alpha_0 \Delta Y_h^{(r,0)} + \ldots + \alpha_{m-1} \Delta Y_h^{(r,m-1)} + \beta_0 G_{h,Y,1}^{(r,0)} + \ldots + \beta_{m-1} G_{h,Y,1}^{(r,m-1)} + Z.$$ 

Then conditional on $W(t)$, $\Delta Y_h^{(r,0)}$ and $G_{h,Y,1}^{(r,0)}$, we can get the coefficient $\alpha_0, \beta_0$ by solving $E[A_{h,Y,W}^{(r)} | W(t)] = hG_{h,W,1}^{(r,0)}$ and $E[A_{h,Y,W}^{(r)} | W(t)] = hG_{h,W,2}^{(r,0)}$. So are the other coefficients $\alpha_i, \beta_i$ ($i = 0, \ldots, m-1$). Hence, conditional on $W(t)$, $\Delta Y_h^{(r,0)}, \ldots, \Delta Y_h^{(r,m-1)}, G_{h,Y,1}, \ldots, G_{h,Y,1}^2$, we can have the mean and variance as stated.

Now if $m$ is large,

$$\sum_{i=0}^{m-1} \int_{I_i} (W(t) - W(a_{rm+i}))^2 dt \approx \frac{m h^2}{2}.$$

From the definition of $G_{h,W,1}^{(r,i)}, G_{h,W,2}^{(r,i)}, G_{h,Y,1}^{(r,i)}$, we have covariance matrix (see fig-
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<table>
<thead>
<tr>
<th>$\Delta W_h^{(r,i)}$</th>
<th>$G_{h,W,1}^{(r,i)}$</th>
<th>$G_{h,W,2}^{(r,i)}$</th>
</tr>
</thead>
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<tr>
<td>$h$</td>
<td>$h/2$</td>
<td>$h/6$</td>
</tr>
<tr>
<td>$G_{h,W,1}^{(r,i)}$</td>
<td>$h/2$</td>
<td>$h/8$</td>
</tr>
<tr>
<td>$G_{h,W,2}^{(r,i)}$</td>
<td>$h/6$</td>
<td>$h/8$</td>
</tr>
</tbody>
</table>

Figure 4.2: The covariances of $\Delta W_h^{(r,i)}$, $G_{h,W,1}^{(r,i)}$ and $G_{h,W,2}^{(r,i)}$

...ure (4.2)). Hence

$$
\sum_{i=0}^{m-1} ((4G_{h,W,1}^{(r,i)} - 6G_{h,W,2}^{(r,i)})^2 h + (12G_{h,W,2}^{(r,i)} - 6G_{h,W,1}^{(r,i)})^2 h = \frac{13mh^2}{30}.
$$

then $V_{h,Y,W}^{(r)}$ is expected to be close to its mean $\bar{V}_{h,Y,W}^{(r)} = \frac{mh^2}{15}$. Does the variance of $V$ obey some rule? The variance of $V$ in higher dimension situation will be proved in $n$ dimension case.

Hence we are approximating $A_{h,Y,W}^{(r)}$ with normal distribute variable

$$
\tilde{A}_{h,Y,W}^{(r)} = \tilde{Z}_{h,Y,W}^{(r)} + \sum_{i=1}^{m} ((4G_{h,W,1}^{(r,i)} - 6G_{h,W,2}^{(r,i)})\Delta Y_h^{(r,i)} + (12G_{h,W,2}^{(r,i)} - 6G_{h,W,1}^{(r,i)})G_{h,Y,1}^{(r,i)}),
$$

where $\tilde{Z}_{h,Y,W}^{(r)} \sim N(0, \frac{mh^2}{15})$ is independent to $G_{h,W,1}^{(r,i)}$, $G_{h,W,2}^{(r,i)}$, $G_{h,Y,1}^{(r,i)}$ and $\Delta Y_h^{(r,i)}$.

At this stage, we are well prepared to construct the initial stage.

Method 4.1.4. (Initial Method) We explain the initial method based on the notation in method (2.1.1). Suppose we have the $3/4$ Approximation for the 3 Dimension SDE on $[a, b]$ with step size $h = \frac{T}{m}$. So as to make things simple, we consider how to generate the values in the interval $J_r = [a_{rm}, a_{rm+m}]$ only. That is:

1. **Generate Brownian paths** $\Delta W_h^{(r,i)}$, $\Delta Y_h^{(r,i)}$, $\Delta B_h^{(r,i)}$ and $G_{h,Y,W}^{(r,i)}$, $G_{h,R,W}^{(r,i)}$, $G_{h,R,Y}^{(r,i)}$:

   We want to generate $\Delta W_h^{(r,i)}$, $\Delta Y_h^{(r,i)}$, $\Delta B_h^{(r,i)} \sim N(0, h)$, $G_{h,Y,1}^{(r,i)}$, $G_{h,Y,1}^{(r,i)} \sim N(0, h/3)$, $G_{h,W,2}^{(r,i)} \sim N(0, h/20)$ such that

   $$
   E[G_{h,W,1}^{(r,i)}\Delta W_h^{(r,i)}] = E[G_{h,Y,1}^{(r,i)}\Delta Y_h^{(r,i)}] = h/2,
   $$

   $$
   E[G_{h,W,1}^{(r,i)}G_{h,W,2}^{(r,i)}] = h/8, \text{ when we consider the lemma (4.1.1). One can get the detail procedure from the lemma (1.2.10).}
   $$

2. **Generate approximate area** $\tilde{A}_{h,h,R}^{(r)}$: Generate $\tilde{Z}_{h,R,W}^{(r)} \sim N(0, \frac{mh^2}{6})$, $\tilde{Z}_{h,R,W}^{(r)} \sim N(0, \frac{mh^2}{6})$ and $\tilde{Z}_{h,Y,W}^{(r)} \sim N(0, \frac{mh^2}{15})$. We are revealing the secret of $\tilde{Z}_h$'s variance in the $n$ dimension discussion.
We approximate the area term with

\[
\tilde{A}_{h,R,W}^{(r)} = \tilde{Z}_{h,R,W}^{(r)} + \sum_{i=0}^{m-1} G_{h,W_1}^{(r,i)} \Delta R_{h}^{(r,i)}
\]

\[
\tilde{A}_{h,R,Y}^{(r)} = \tilde{Z}_{h,R,Y}^{(r)} + \sum_{i=0}^{m-1} G_{h,Y_1}^{(r,i)} \Delta R_{h}^{(r,i)}
\]

\[
\tilde{A}_{h,Y,W}^{(r)} = \tilde{Z}_{h,Y,W}^{(r)} + \sum_{i=0}^{m-1} \{ [4G_{h,W_1}^{(r,i)} - 6G_{h,W_2}^{(r,i)}] \Delta Y_{h}^{(r,i)}
\]

\[
+ [12G_{h,W_2}^{(r,i)} - 6G_{h,W_1}^{(r,i)}] G_{h,Y_1}^{(r,i)} \}
\]

The reasons are given by the Explanation (4.1.3) and Explanation (3.1.3).

\[\square\]

At this stage, we generate independent \(\Delta W_{h}^{(r,i)}\), \(\Delta Y_{h}^{(r,i)}\), \(\Delta R_{h}^{(r,i)}\) \(\sim N(0, h)\) and values \(G_{h,W_1}^{(r,i)}, G_{h,Y_1}^{(r,i)} \sim N(0, \frac{h}{3})\), \(G_{h,W_2}^{(r,i)} \sim N(0, \frac{h}{25})\) such that \(E(\Delta W_{h}^{(r,i)} G_{h,W_1}^{(r,i)}) = \frac{h}{2}, E(\Delta Y_{h}^{(r,i)} G_{h,Y_1}^{(r,i)}) = \frac{h}{2}, E(\Delta W_{h}^{(r,i)} \Delta Y_{h}^{(r,i)}) = \frac{h}{6}, E(G_{h,W_2}^{(r,i)} G_{h,W_1}^{(r,i)}) = h/8\), satisfying those properties in lemma (4.1.1), for the time interval \(J_r\). Analogously, we are able to generate the Brownian paths for other time intervals \(J_0, \ldots, J_{m-1}\).

4.2 Subdivision Process

The last section, "Initial Stage", which considers the Area corresponding to the interval \(J_r\), is sufficient for generating an approximation for a given \(h\), but to exhibit convergence of sequence of approximations with \(h \to 0\) in the same Brownian Path, we need to divide the intervals with length \(h\). As we know, the approximate solution of the Areas, \(\tilde{A}_{h,Y,W}^{(r)}, \tilde{A}_{h,R,Y}^{(r)}\), and \(\tilde{A}_{h,R,W}^{(r)}\), are based on \(m \to \infty\) in the same Brownian Path. That is \(h \to 0\) in our case, since we are discussing the problem over the fixed length interval. Here we want to split up \(\Delta W_{h}^{(r,i)}, \Delta Y_{h}^{(r,i)}, \Delta R_{h}^{(r,i)}\), \(G_{h,W_1}^{(r,i)}, G_{h,Y_1}^{(r,i)}\), and \(G_{h,W_2}^{(r,i)}\), satisfying the conditions.

Hence, intervals with length \(h\) will then be split up into intervals with equal length \(\frac{h}{2}\) by doing this, and the length of intervals will tend to zero after several steps.

4.2.1 The \(J\) Subdivision Process

For a fixed \(J_r\), which is divided into \(I_0^{(r)}, I_1^{(r)} \ldots I_{m-1}^{(r)}\), the \(J_r\) have associated random variables \(Z_{h,R,W}^{(r)}, Z_{h,R,Y}^{(r)}, Z_{h,Y,W}^{(r)}, \Delta W_{i}^{(r,i)}, \Delta Y_{h}^{(r,i)}, \Delta R_{i}^{(r,i)}, G_{h,W_1}^{(r,i)}, G_{h,Y_1}^{(r,i)}\), \(G_{h,W_2}^{(r,i)}\).

Analogous to the 2D case, we split \(J\) into \(J_L\) and \(J_R\) to increase the number of \(J\). Therefore we separate segments into two groups as well as the random values, which are correspond to \(I_i^{(r)}\). And we also split \(Z_{h,R,W}^{(r)}, Z_{h,R,Y}^{(r)}, Z_{h,Y,W}^{(r)}\).
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<thead>
<tr>
<th>A</th>
<th>$J_0$</th>
<th>$J_1$</th>
<th>$J_{r-1}$</th>
<th>$J_r$</th>
<th>$J_{m-2}$</th>
<th>$J_{m-1}$</th>
<th>B</th>
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<td>$J_{1}^{(r)}$</td>
<td>$J_{2}^{(r)}$</td>
<td>$J_{m-2}^{(r)}$</td>
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<td>$J_r$</td>
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</tr>
<tr>
<td>$a_0$</td>
<td>$a_1$</td>
<td>$a_2$</td>
<td>$a_3$</td>
<td>$a_4$</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Figure 4.3: The J subdivision

#### Method 4.2.1.

Consider only a fixed $J_r$. 

1. Divide segments into two groups. $I_0^{(r)}, \ldots, I_{m-2}^{(r)} \in J_{r,L}$ and $I_1^{(r)}, \ldots, I_m^{(r)} \in J_{r,R}$. Divide $G_{h,Y,1}^{(r)}, G_{h,Y,2}^{(r)}, G_{h,Y,1}^{(r)},\Delta W_h^{(r,t)}, \Delta Y_h^{(r,t)}, \Delta R_h^{(r,t)}$ into two groups corresponding. See the second line of the figure (4.3).

2. Renumber $J_{r,L}$ and $J_{r,R}$ with $J_{2r}$ and $J_{2r+1}$ respectively. Renumber $G_{h,Y,1}^{(r)}, G_{h,Y,2}^{(r)}, G_{h,Y,1}^{(r)}$ in $J_{r,L}$ with $G_{h,Y,1}^{(r)}, G_{h,Y,2}^{(r)}, G_{h,Y,1}^{(r)}$. And Renumber $G_{h,Y,2}^{(r)}, G_{h,Y,1}^{(r)}$ in $J_{r,R}$ with $G_{h,Y,2}^{(r)}, G_{h,Y,1}^{(r)}$. This is shown in the third line of the figure (4.3).

3. Split $Z_{h,R,W}^{(r)} \sim N(0, mh^2/6)$ into mutually independent $Z_{h,R,W,L}^{(r)}, Z_{h,R,W,R}^{(r)} \sim N(0, mh^2/12)$. Split $Z_{h,R,Y}^{(r)} \sim N(0, mh^2/6)$ into $Z_{h,R,Y,L}^{(r)}, Z_{h,R,Y,R}^{(r)} \sim N(0, mh^2/12)$. Split $Z_{h,Y,W}^{(r)} \sim N(0, mh^2/15)$ into $Z_{h,Y,W,L}^{(r)}, Z_{h,Y,W,R}^{(r)} \sim N(0, mh^2/30)$ and $Z_{h,Y,W,R}^{(r)} \sim N(0, mh^2/30)$. Furthermore these variables should satisfy

$$
\begin{align*}
Z_{h,R,W}^{(r)} &= Z_{h,R,W,L}^{(r)} + Z_{h,R,W,R}^{(r)} \\
Z_{h,R,Y}^{(r)} &= Z_{h,R,Y,L}^{(r)} + Z_{h,R,Y,R}^{(r)} \\
Z_{h,Y,W}^{(r)} &= Z_{h,Y,W,L}^{(r)} + Z_{h,Y,W,R}^{(r)}.
\end{align*}
$$

One can get detail procedure in lemma (1.2.13). Rephrase $Z_{h,R,W,L}^{(r)},$
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4.2.2 Subdivision of Small Intervals $I_i$

Analogous to the 2D case, we consider only $J_r$ here. We have $J$ with $\frac{m}{2}$ segments and segment length $h$ before the first $I$ subdivision. And we have $J$ with $m$ segments and segment length $h/2$ after the first $I$ subdivision.

Notation 4.2.2. (3DIS notation)

1. Number the time instant in the $J_r$ (It is renumbered after $I$ subdivision) with $a_i$ ($i = 0, \ldots, q$). In the first $I$ subdivision, the $q = \frac{m-2}{2}$, whereas the $q = m - 1$ in the second $I$ subdivision process, since number of I segments in $J_r$ had been double in the first subdivision of $I$. Denote the time intervals $I_i(r)$ in $J_r$ with $I_0, \ldots, I_q$ for short. It is the same to the 2D situation shown in the first line of the figure (3.2).

2. For interval $I_i$, we have corresponding $G^{(r,i)}_{h,W,1}, G^{(r,i)}_{h,W,2}, G^{(r,i)}_{h,Y,1}, \Delta W^{(r,i)}_h, \Delta Y^{(r,i)}_h, \Delta R^{(r,i)}_h$, where $i = 0, \ldots, q$. (In the first $I$ subdivision, $q = \frac{m-2}{2}$. In the beginning of the second $I$ subdivision, $q = m - 1$.) This is because, at the beginning of the second subdivision, we denote $G^{(r,0)}_{h,W,1,L}, G^{(r,0)}_{h,W,1,R}, \ldots, G^{(r,\frac{m-2}{2})}_{h,W,1,L}, G^{(r,\frac{m-2}{2})}_{h,W,1,R}$ in order with $G^{(r,i)}_{h,W,1}$ ($i = 0, \ldots, m - 1$). This is similar to the 2D case, which is shown in figure (3.2). So are the $\Delta W^{(r,i)}_h, \Delta W^{(r,i)}_h, \Delta Y^{(r,i)}_h, \Delta Y^{(r,i)}_h, \Delta R^{(r,i)}_h, G^{(r,i)}_{h,Y,1,L}, G^{(r,i)}_{h,Y,1,R}, G^{(r,i)}_{h,W,2,L}, G^{(r,i)}_{h,W,2,R}$.

3. Before the $I$ subdivision, we have $Z^{(r)}_{h,R,Y}, Z^{(r)}_{h,Y,W}$ and $Z^{(r)}_{h,Y,W}$. New $Z^{(r)}_{\frac{h}{2},R,Y}, Z^{(r)}_{\frac{h}{2},Y,W}$ are generated at the end of the first $I$ subdivision process.

Basically, the 3D notation method is the same as the 2D one in (3.2.2). What we need to do next is to split each $I_i$ into two subintervals $I_i(L)$ and $I_i(R)$, having length $h/2$ each, and get the corresponding $\Delta W^{(r,i)}_h, \Delta W^{(r,i)}_h, \Delta Y^{(r,i)}_h, \Delta Y^{(r,i)}_h, \Delta R^{(r,i)}_h, G^{(r,i)}_{h,W,1,L}, G^{(r,i)}_{h,Y,1,L}, G^{(r,i)}_{h,Y,1,R}, G^{(r,i)}_{h,W,2,L}, G^{(r,i)}_{h,W,2,R}, Z^{(r)}_{h,R,W}$.
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<table>
<thead>
<tr>
<th>$\Delta W_{h,L}^{\text{r}(i)}$</th>
<th>$G_{h,W,1}^{\text{r}(i)}$</th>
<th>$G_{h,W,2}^{\text{r}(i)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta W_{h,R}^{\text{r}(i)}$</td>
<td>$3h/8$</td>
<td>$7h/48$</td>
</tr>
<tr>
<td>$G_{h,W,1,L}^{\text{r}(i)}$</td>
<td>$5h/24$</td>
<td>$17h/192$</td>
</tr>
<tr>
<td>$G_{h,W,2,L}^{\text{r}(i)}$</td>
<td>$7h/96$</td>
<td>$31h/960$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Delta Y_{h}^{\text{r}(i)}$</th>
<th>$G_{h,Y,1}^{\text{r}(i)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta Y_{h,L}^{\text{r}(i)}$</td>
<td>$h$</td>
</tr>
<tr>
<td>$G_{h,Y,1,L}^{\text{r}(i)}$</td>
<td>$h/2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Delta W_{h,L}^{\text{r}(i)}$</th>
<th>$G_{h,W,1,L}^{\text{r}(i)}$</th>
<th>$G_{h,W,2,L}^{\text{r}(i)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta W_{h,R}^{\text{r}(i)}$</td>
<td>$h/4$</td>
<td>$h/12$</td>
</tr>
<tr>
<td>$G_{h,W,1,L}^{\text{r}(i)}$</td>
<td>$h/6$</td>
<td>$h/16$</td>
</tr>
<tr>
<td>$G_{h,W,2,L}^{\text{r}(i)}$</td>
<td>$h/16$</td>
<td>$h/40$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Delta Y_{h,L}^{\text{r}(i)}$</th>
<th>$G_{h,Y,1,L}^{\text{r}(i)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta Y_{h,L}^{\text{r}(i)}$</td>
<td>$h/4$</td>
</tr>
<tr>
<td>$G_{h,Y,1,L}^{\text{r}(i)}$</td>
<td>$h/4$</td>
</tr>
</tbody>
</table>

Figure 4.4: The covariance of the values

$\tilde{Z}_{h,R,Y}, \tilde{Z}_{h,Y,W},$ where $\tilde{Z}_{h,R,W}, \tilde{Z}_{h,R,Y}, \tilde{Z}_{h,Y,W}$ are the new correction term for the new subdivided Area. We want the random values, which are generated in the $I$ subdivision, will satisfy the following condition:

**Conditions 4.2.3.**

\[
\Delta W_{h}^{\text{r}(i)} = \Delta W_{h,L}^{\text{r}(i)} + \Delta W_{h,R}^{\text{r}(i)} \\
\Delta Y_{h}^{\text{r}(i)} = \Delta Y_{h,L}^{\text{r}(i)} + \Delta Y_{h,R}^{\text{r}(i)} \\
\Delta R_{h}^{\text{r}(i)} = \Delta R_{h,L}^{\text{r}(i)} + \Delta R_{h,R}^{\text{r}(i)} \\
G_{h,W,1}^{\text{r}(i)} = \frac{1}{2}[G_{h,W,1,L}^{\text{r}(i)} + G_{h,W,1,R}^{\text{r}(i)} + \Delta W_{h,L}^{\text{r}(i)}] \\
G_{h,Y,1}^{\text{r}(i)} = \frac{1}{2}[G_{h,Y,1,L}^{\text{r}(i)} + G_{h,Y,1,R}^{\text{r}(i)} + \Delta Y_{h,L}^{\text{r}(i)}] \\
G_{h,W,2}^{\text{r}(i)} = \frac{1}{4}[G_{h,W,2,L}^{\text{r}(i)} + G_{h,W,2,R}^{\text{r}(i)} + G_{h,W,1,L}^{\text{r}(i)} + \frac{1}{2}\Delta W_{h,L}^{\text{r}(i)}]
\]

According to the definition of these random values, which are stated at the beginning of this chapter, we have their covariance in the figure (4.4).

We could proof the condition (4.2.3) should be satisfied, according to the definition of $\Delta W_{h}^{\text{r}(i)}, \Delta Y_{h}^{\text{r}(i)}, \Delta R_{h}^{\text{r}(i)}, G_{h,W,1}^{\text{r}(i)}, G_{h,Y,1}^{\text{r}(i)}, G_{h,W,2}^{\text{r}(i)}, \Delta W_{h,L}^{\text{r}(i)}, \Delta Y_{h,L}^{(\text{r}(i)}$, $\Delta R_{h,L}^{\text{r}(i)}$, $G_{h,W,1,L}^{\text{r}(i)}$, $G_{h,Y,1,L}^{\text{r}(i)}$, $G_{h,W,2,L}^{\text{r}(i)}$, $\Delta W_{h,R}^{\text{r}(i)}, \Delta Y_{h,R}^{\text{r}(i)}, \Delta R_{h,R}^{\text{r}(i)}$, $G_{h,W,1,R}^{\text{r}(i)}$, $G_{h,Y,1,R}^{\text{r}(i)}$, $G_{h,W,2,R}^{\text{r}(i)}$. Now let’s describe the $I$ subdivision process.

**Method 4.2.4.** Suppose there are $q + 1$ time intervals $I_i$ ($i = 0, \ldots, q$) with step size $h$ in $J$. Based on the notations in (4.2.2), we apply the following steps to split up the $I_i$

1. Get $G_{h,W,1,L}^{\text{r}(i)}$, $\Delta W_{h,L}^{\text{r}(i)}$, $G_{h,W,2,L}^{\text{r}(i)}$, $G_{h,W,1,R}^{\text{r}(i)}$, $\Delta W_{h,R}^{\text{r}(i)}$, $G_{h,W,2,R}^{\text{r}(i)}$. We want these variables generated will meet the condition (4.2.3). To be precise, one can get the detail procedure of getting $G_{h,W,1,L}^{\text{r}(i)}$, $\Delta W_{h,L}^{\text{r}(i)}$, $G_{h,W,2,L}^{\text{r}(i)}$ in...
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Then we can obtain $G_{h,W_1,R}^{(r,i)}, \Delta W_{h,R}^{(r,i)}, G_{h,W_2,R}^{(r,i)}$ with

\[
\begin{align*}
\Delta W_{h,R}^{(r,i)} &= \Delta W_{h}^{(r,i)} - \Delta W_{h,L}^{(r,i)} \\
G_{h,W_1,R}^{(r,i)} &= 2G_{h,W_1}^{(r,i)} - G_{h,W_1,L}^{(r,i)} - \Delta W_{h,L}^{(r,i)} \\
G_{h,W_2,R}^{(r,i)} &= 4G_{h,W_2}^{(r,i)} - G_{h,W_2,L}^{(r,i)} - G_{h,W_1,L}^{(r,i)} - \frac{1}{2} \Delta W_{h,L}^{(r,i)}.
\end{align*}
\]

2. Get $H$ and $Q$: Set $H_{Y,0}^{(r,i)}$ and $H_{Y,1}^{(r,i)}$.

\[
\begin{align*}
H_{Y,0}^{(r,i)} &= -4G_{h,W_1,L}^{(r,i)} - 2G_{h,W_1,R}^{(r,i)} + 6G_{h,W_2,L}^{(r,i)} + 6G_{h,W_2,R}^{(r,i)} + \Delta W_{h,L}^{(r,i)} \\
H_{Y,1}^{(r,i)} &= 6G_{h,W_1,L}^{(r,i)} - 6G_{h,W_1,R}^{(r,i)} - 12G_{h,W_2,L}^{(r,i)} + 12G_{h,W_2,R}^{(r,i)}.
\end{align*}
\]

Set $Q_{h,Y,W}$

\[
Q_{h,Y,W} = \frac{(q+1)h^2}{15} - \sum_{i=0}^{q} ((H_{Y,0}^{(r,i)})^2 \frac{h}{16} + (H_{Y,1}^{(r,i)})^2 \frac{h}{48})
\]

\[
Q = 1 - Q_{h,Y,W}.
\]

$Q$ is a threshold value for the subdivision. An explanation for this is in the Explanation (4.2.8). One could get $H_{Y,0}^{(r,i)} \sim N(0, \frac{2h}{15})$ and $H_{Y,1}^{(r,i)} \sim N(0, \frac{4h}{3})$.

3. Get $N_{2,0,i}, N_{2,1,i}$, and $\tilde{Z}_{\frac{1}{2},Y,W}^{(r)}$: Generate independent $N_{2,0,i}, N_{2,1,i} \sim N(0,1)$ $(i = 0, \ldots, q)$, having

\[
\begin{align*}
Z_{h,Y,W}^{(r,i)} &= \sum_{i=0}^{q} \left\{ \sqrt{\frac{h}{16}} H_{Y,0}^{(r,i)} N_{2,0,i} + \sqrt{\frac{h}{48}} H_{Y,1}^{(r,i)} N_{2,1,i} \right\} \\
\tilde{Z}_{\frac{1}{2},Y,W}^{(r)} &= \tilde{Z}_{\frac{1}{2},Y,W}^{(r)} / \sqrt{Q_{h,Y,W}}.
\end{align*}
\]

The method of generating the $N_{2,0,i}, N_{2,1,i}$, and $\tilde{Z}_{\frac{1}{2},Y,W}^{(r)}$ will be demonstrated in method (4.2.6). The correction from $\tilde{Z}_{\frac{1}{2},Y,W}^{(r)}$ to $\tilde{Z}_{\frac{1}{2},Y,W}^{(r)}$ is explain in the Explanation (4.2.9).

4. Get $\Delta Y_{h,L}^{(r,i)}, G_{h,Y_1,L}^{(r,i)}, G_{h,Y_1,R}^{(r,i)}$: We can generate $\Delta Y_{h,L}^{(r,i)}, G_{h,Y_1,L}^{(r,i)}$ by

\[
\begin{align*}
\Delta Y_{h,L}^{(r,i)} &= -\frac{1}{4} \Delta Y_h^{(r,i)} + \frac{3}{2} G_{h,Y_1}^{(r,i)} + \sqrt{\frac{h}{16}} N_{2,0,i} \\
G_{h,Y_1,L}^{(r,i)} &= -\frac{1}{4} \Delta Y_h^{(r,i)} + G_{h,Y_1}^{(r,i)} + \sqrt{\frac{h}{48}} N_{2,1,i}.
\end{align*}
\]

The Explanation (4.2.8) presents the Explanation for this.

5. Get threshold values $Q_{h,R,0}, Q_{h,R,1},$ and $Q_{h,R,0,R,1}$: Let $H_{Y,0}^{(r)}$ and $H_{Y,1}^{(r)}$
be

\[
H_{R,0}^{(r,i)} = G_{h,Y,0}^{(r,i)} - G_{h,Y,1,L}^{(r,i)},
\]
\[
H_{R,1}^{(r,i)} = G_{h,W,0}^{(r,i)} - G_{h,W,1,L}^{(r,i)}.
\]

Set \(Q_{h,R,0}, Q_{h,R,1}, Q_{h,R,0,R,1}\)

\[
Q_{h,R,0} = \frac{12 \sum_{i=0}^{q} (H_{R,0}^{(r,i)})^2}{(q + 1)h} - 1
\]
\[
Q_{h,R,1} = \frac{12 \sum_{i=0}^{q} (H_{R,1}^{(r,i)})^2}{(q + 1)h} - 1
\]
\[
Q_{h,R,0,R,1} = \frac{12 \sum_{i=0}^{q} (H_{R,0}^{(r,i)} H_{R,1}^{(r,i)})}{(q + 1)h}
\]

\(Q_{h,R,0}, Q_{h,R,1}, Q_{h,R,0,R,1}\) are threshold values for the subdivision. An explanation for this is in the Explanation (4.2.10).

6. Generate \(N_{3,0,i} \sim N(0,1), Z_{3/2,R,Y}^{(r)}\) and \(Z_{3/2,R,W}^{(r)}\): Apply the method stated in lemma (4.2.12), we can obtain \(N_{3,0,i} \sim N(0,1)\) and non-corrected \(Z\) as required. The method (4.2.11) explain the way to correct the \(Z_{R,Y}\) and \(Z_{R,W}\) meeting our need. And lemma (4.2.13) explain why method (4.2.11) is working.

7. Generate \(\Delta R_{h,R}^{(r,i)}\) and \(\Delta R_{h,L}^{(r,i)}\): \(\Delta R_{h,R}^{(r,i)} = \frac{1}{2}(\Delta R_{h}^{(r,i)} - N_{3,0,i}\sqrt{h}), \Delta R_{h,L}^{(r,i)} = \frac{1}{2}(\Delta R_{h}^{(r,i)} + N_{3,0,i}\sqrt{h})\).

Lemma 4.2.5. Split \(\Delta W_{h}^{(r,i)}, G_{h,W,1}^{(r,i)}, G_{h,W,2}^{(r,i)}\) with

\[
\Delta W_{h}^{(r,i)} = -\frac{1}{4} \Delta W_{h}^{(r,i)} + \frac{3}{2} G_{h,W,1}^{(r,i)} + \sqrt{\frac{h}{16}} N_{1,0,i}
\]
\[
G_{h,W,1}^{(r,i)} = \frac{1}{16} \Delta W_{h}^{(r,i)} - \frac{7}{8} G_{h,W,1}^{(r,i)} + \frac{15}{4} G_{h,W,2}^{(r,i)} + \sqrt{\frac{h}{768}} N_{1,1,i}
\]
\[
G_{h,W,2}^{(r,i)} = \frac{1}{16} \Delta W_{h}^{(r,i)} - \frac{5}{8} G_{h,W,1}^{(r,i)} + 2 G_{h,W,2}^{(r,i)} + \left(\sqrt{\frac{h}{2880}} N_{1,2,i} - \frac{1}{12} \sqrt{\frac{h}{16}} N_{1,0,i+1}\right).
\]

where \(N_{1,0,i}, N_{1,1,i}, N_{1,2,i} \sim N(0,1)\) are mutually independent. The \(\Delta W_{h}^{(r,i)}, G_{h,W,1}^{(r,i)}, G_{h,W,2}^{(r,i)}\) will meet the condition (4.2.3).

Proof: The general way of splitting random values as we required here could be found in lemma (1.2.11).

Consider this particular case only. In segment \(J\), in which each small segments has length \(h\), the \(\Delta W_{h}^{(r,i)}, G_{h,W,1}^{(r,i)}, G_{h,W,2}^{(r,i)}\) are related to \(\Delta W_{h}^{(r,i)}, G_{h,W,1}^{(r,i)}, G_{h,W,2}^{(r,i)}\).
when it is conditional on \( W(t) \). So we set

\[
\Delta W_{h,L}^{(r,i)} = a \Delta W_{h,L}^{(r,i)} + b G_{h,W,1}^{(r,i)} + c G_{h,W,2}^{(r,i)} + r N_{1,0,i}
\]

\[
G_{h,W,1}^{(r,i)} = d \Delta W_{h,L}^{(r,i)} - e G_{h,W,1}^{(r,i)} + f G_{h,W,2}^{(r,i)} + s N_{1,1,i}
\]

\[
G_{h,W,2}^{(r,i)} = g \Delta W_{h,L}^{(r,i)} - p G_{h,W,1}^{(r,i)} + m G_{h,W,2}^{(r,i)} + (u N_{1,0,i} + v N_{1,2,i}),
\]

where \( a, b, c, d, e, f, g, m, p, r, s, u, v \) are coefficients, and \( N_{1,0,i}, N_{1,1,i}, N_{1,2,i} \sim N(0,1) \) are mutually independent.

Apply these equations to meet the covariance in figure (4.4) in condition (4.2.3). We could obtain the coefficients, and consequently the equations for the I subdivision as in the lemma.

We are providing more detail in processing the step "Get \( N_{2,0,i}, N_{2,1,i}, \) and \( \bar{Z}_{h,Y,W} \)". This method is based on the lemma (1.2.13).

**Method 4.2.6.** We want the \( N_{2,0,i} \) and \( N_{2,1,i} \) to be mutually independent. So we do the following things. If \( |Q| \leq 1/2 \) (the usual case) We need to generate independent random values \( U(r,0), \ldots, U(r,q), P(r,0), \ldots, P(r,q) \), where \( U(r,i) \sim N(0, \frac{1}{16}(H_{Y,0}^{(r,i)})^2) \) and \( P(r,i) \sim N(0, \frac{1}{48}(H_{Y,1}^{(r,i)})^2) \) \( (i = 0, \ldots, q) \) are mutually independent. So as to achieve this, we generate \( q + 1 \) independent values \( N_{2,0,i}, N_{2,1,i} \sim N(0,1) \).

1. In the first step, we set \( \bar{Z}_{Y,W}^{(1)} = \bar{Z}_{h,Y,W} \), and \( \sigma^2 = \frac{(q+1)h^2}{15} \), which is the Variance of the \( \bar{Z}_{h,Y,W} \).

   In the \( i \)th \( (i = 2, \ldots) \) step, we set \( \bar{Z}_{Y,W}^{(i)} = \bar{Z}_{h,Y,W} - \sum_{k=0}^{i-2} (P(r,k) + U(r,k)) \), and \( \sigma^2 = \frac{(q+1)h^2}{15} - \sum_{k=0}^{i-2} ((H_{Y,0}^{(r,k)})^2 \frac{h}{16} + (H_{Y,1}^{(r,k)})^2 \frac{h}{48}) \).

2. Get \( U(r,i-1) \) at the \( i \)th stage. Say \( \sigma_i^2 = (H_{Y,0}^{(r,i-1)})^2 \frac{h}{16} \),

\[
d = \sigma_i^2/\sigma^2
\]

\[
e = \sqrt{d(1-d)\sigma^2}.
\]

Generate new independent values \( N_{i-1} \sim N(0,1) \), we obtain \( U(r,i-1) \) with \( U(r,i-1) = d \times \bar{Z}_{Y,W}^{(i)} + e \times N_{i-1} \).

3. Get \( P(r,i-1) \) at the \( i \)th stage.

   When \( i = 1 \)

\[
\begin{cases}
\bar{Z}_{Y,W}^{(1)} = \bar{Z}_{h,Y,W} - U(r,0) \\
\sigma^2 = \frac{(q+1)h^2}{15} - (H_{h,Y,0}^{(0)})^2 \frac{h}{16}
\end{cases}
\]

   When \( i > 1 \)

\[
\begin{cases}
\bar{Z}_{Y,W} = \bar{Z}_{h,Y,W} - \sum_{k=0}^{i-2} (P(r,k) + U(r,k)) - U(r,i-1) \\
\sigma^2 = \frac{(q+1)h^2}{15} - \sum_{k=0}^{i-2} ((H_{h,Y,0}^{(k)})^2 \frac{h}{16} + (H_{h,Y,1}^{(k)})^2 \frac{h}{48}) - (H_{h,Y,0}^{(i-1)})^2 \frac{h}{16}
\end{cases}
\]
Set $\sigma_i^2 = (H_{h,Y,W}^{(k)})^2 \frac{h}{48}$,

\[
\begin{align*}
  d &= \sigma_i^2 / \sigma^2 \\
  e &= \sqrt{d(1-d)\sigma^2}.
\end{align*}
\]

Generate new independent value $N_{i-1} \sim N(0,1)$, we have $P_{(r,i-1)}^{(r,i)}$ by $P_{(r,i-1)}^{(r,i)} = d \times \tilde{Z}_{Y,W}^{(i)} + e \times N_{i-1}$.

4. Obtain $N_{2,0,i}, N_{2,1,i}$

\[
\begin{align*}
  N_{2,0,i} &= \frac{P_{(r,i)}}{\sqrt{h/16H_{Y,W}^{(r,i)}}} \\
  N_{2,1,i} &= \frac{U_{(r,i)}}{\sqrt{h/48H_{Y,W}^{(r,i)}}}.
\end{align*}
\]

5. At the $q + 1$ step, we set $\tilde{Z}_{Y,W}^* = \tilde{Z}_{Y,W}^{(q+1)*}$

So we have $U_{(r,i)}$, $P_{(r,i)}$, satisfying the requirement, with the above process.

If $|Q| > 1/2$ (the exceptional case) We generate independent random values $N_{2,0,0}, N_{2,0,1}, N_{2,1,0}, \ldots, N_{2,1,q}, \tilde{Z}_{h,Y,W}$, where $N_{2,0,i} \sim N(0, \frac{h}{16}), N_{2,1,i} \sim N(0, \frac{h}{48}),$ and $\tilde{Z}_{h,Y,W} \sim N(0, \frac{(q+1)h^2}{2x_{15}})$.

The splitting up of the $\tilde{Z}_{h,Y,W}$ is based on the lemma (1.2.13). Let $\tilde{Z}_{h,Y,W}, \sqrt{\frac{h}{16}} N_{2,0,0}, \sqrt{\frac{h}{48}} N_{2,1,0}, \sqrt{\frac{h}{16}} N_{2,0,1}, \sqrt{\frac{h}{48}} N_{2,1,1}, \ldots, \sqrt{\frac{h}{16}} N_{2,0,q}, \sqrt{\frac{h}{48}} N_{2,1,q}$ be $X_1, X_2, \ldots, X_{2q+2}$, in the lemma (1.2.13), respectively, and $\tilde{Z}_{h,Y,W}$ be $X_{2q+3}$. Then we can apply the method in lemma (1.2.13) obtaining the random values. And you may also find the similar method in Method (3.2.7).

**Lemma 4.2.7.** Before I subdivision process, the area term is $A_{hY,W}^{(r)}$. After one I subdivision process, the area term becomes $A_{\frac{h}{2}Y,W}^{(r)}$. They satisfy equation

\[
A_{hY,W}^{(r)} = A_{\frac{h}{2}Y,W}^{(r)} + \sum_{i=0}^{2q+3} W_{h,Y}^{(r,i)} Y_{h,R}^{(r,i)}
\]

**Proof:** According to the definition of the area term $A_{hY,W}^{(r)} = \sum_{i=1}^{m/2} (W(t) - W(a_{i-1})) dY(t)$, we could deduce that

\[
A_{hY,W}^{(r)} = \sum_{i=0}^{q} \left\{ \int_{a_i}^{a_{i+1}} (W(t) - W(a_{i-1} + a_i/2)) dY(t) + \int_{a_{i-1}}^{a_i} (W(t) - W(a_{i-1})) dY(t) + W_{iL} Y_{iR} \right\}
\]

\[
= A_{\frac{h}{2}Y,W}^{(r)} + \sum_{i=0}^{q} W_{h,Y}^{(r,i)} Y_{h,R}^{(r,i)}
\]

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Explanation 4.2.8. One might be curious how comes the threshold value in method (4.2.4). We are explaining it here.

Considering the approximation and the I subdivision process, we have
\[ A^{(r)}_{h,Y,W} = A^{(r)}_{h,Y,W} + \sum_{i=0}^{q} W^{(r)}_{h,L,i} \gamma^{(r)}_{h,R} \]. According to the definition of \( \Delta Y^{(r)}_{h,L} \), \( G^{(r)}_{h,W,1} \), \( G^{(r)}_{h,W,2} \), we have the approximate sum of integrals \( \tilde{A}^{(r)}_{h,Y,W} \),

\[ \tilde{A}^{(r)}_{h,Y,W} \approx \tilde{Z}^{(r)}_{h,Y,W} + \sum_{i=0}^{q} [(4G^{(r)}_{h,W,1} - 6G^{(r)}_{h,W,2}) \Delta Y^{(r)}_{h,L} + (12G^{(r)}_{h,W,2} - 6G^{(r)}_{h,W,1})G^{(r)}_{h,Y,1}] \]

From the 2D subdivision discussion (see Explanation (3.1.3)), we know
\[ \Delta Y^{(r)}_{h,L} = -\frac{1}{4} \Delta Y^{(r)}_{h} + \frac{3}{2} G^{(r)}_{h,Y,1} + \sqrt{\frac{h}{16}} N_{2,0}, \]
\[ G^{(r)}_{h,Y,1,L} = -\frac{1}{4} \Delta Y^{(r)}_{h} + G^{(r)}_{h,Y,1} + \sqrt{\frac{h}{48}} N_{2,1}, \]

where \( N_{2,0}, N_{2,1} \sim N(0,1) \) are unknown yet. Replace \( G^{(r)}_{h,W,1}, G^{(r)}_{h,W,2}, G^{(r)}_{h,Y,1} \), \( \Delta Y^{(r)}_{h,L} \) in the formula \( \tilde{A}^{(r)}_{h,Y,W} \) with \( G^{(r)}_{h,W,1,L}, G^{(r)}_{h,W,2,L}, G^{(r)}_{h,Y,1,L}, \Delta Y^{(r)}_{h,L} \), \( G^{(r)}_{h,W,2,R}, G^{(r)}_{h,Y,1,R}, \Delta Y^{(r)}_{h,R} \) according to the definition, we have

\[ \tilde{A}^{(r)}_{h,Y,W} \approx \tilde{Z}^{(r)}_{h,Y,W} + \sum_{i=0}^{q} \{ \Delta W^{(r)}_{h,L} \Delta Y^{(r)}_{h,R} \} \]

\[ + \sum_{i=0}^{q} [(4G^{(r)}_{h,W,1,L} - 6G^{(r)}_{h,W,2,L}) \Delta Y^{(r)}_{h,L} + (12G^{(r)}_{h,W,2,L} - 6G^{(r)}_{h,W,1,L})G^{(r)}_{h,Y,1,L}] \]

\[ + \sum_{i=0}^{q} (H^{(i)}_{h,Y,0} \sqrt{\frac{h}{16}} N_{2,0} + H^{(i)}_{h,Y,1} \sqrt{\frac{h}{48}} N_{2,1}). \]

Please be aware that \( N_{2,0} \) and \( N_{2,1} \) are exactly the ones for generating \( G^{(r)}_{h,Y,1,L} \) and \( \Delta Y^{(r)}_{h,R} \). This explain the step 4 of the method (4.2.4).

Because it is true that \( \tilde{A}^{(r)}_{h,Y,W} \approx A^{(r)}_{h,Y,W} \) and \( \tilde{A}^{(r)}_{h,Y,W} \approx A^{(r)}_{h,Y,W} \), we could have

\[ A^{(r)}_{h,Y,W} \approx \tilde{Z}^{(r)}_{h,Y,W} + \sum_{i=0}^{q} \{ \Delta W^{(r)}_{h,L} \Delta Y^{(r)}_{h,R} \} \]

\[ + \sum_{i=0}^{q} [(4G^{(r)}_{h,W,1,L} - 6G^{(r)}_{h,W,2,L}) \Delta Y^{(r)}_{h,L} + (12G^{(r)}_{h,W,2,L} - 6G^{(r)}_{h,W,1,L})G^{(r)}_{h,Y,1,L}] \]

\[ + (4G^{(r)}_{h,W,1,R} - 6G^{(r)}_{h,W,2,R}) \Delta Y^{(r)}_{h,R} + (12G^{(r)}_{h,W,2,R} - 6G^{(r)}_{h,W,1,R})G^{(r)}_{h,W,2,R}), \]
where \( \tilde{Z}^{(r)}_{\frac{1}{2},Y,W} = \tilde{Z}^{(r)}_{h,Y,W} + \sum_{i=0}^{q} (H^{(i)}_{h,Y,0}) \frac{h}{16} N_{2,0,i} + H^{(i)}_{h,Y,1} \frac{h}{48} N_{2,1,i}). \) Conditional on \( \tilde{Z}^{(r)}_{h,Y,W}, H^{(i)}_{h,Y,0}, \) and \( H^{(i)}_{h,Y,1}, \) we obtain \( \tilde{Z}^{(r)}_{\frac{1}{2},Y,W} \sim N(0, (\frac{q+1)h^2}{15} - \sum_{i=0}^{q} ((H^{(i)}_{h,Y,0})^2 \frac{h}{16} + (H^{(i)}_{h,Y,1})^2 \frac{h}{48})). \) We indicated in Explanation (4.1.3) that the variance of \( \tilde{Z}^{(r)}_{\frac{1}{2},Y,W} \) should be \( \text{Var}[\tilde{Z}^{(r)}_{\frac{1}{2},Y,W}] = (\frac{q+1)h^2}{30} \simeq \text{Var}[A^{(r)}_{\frac{1}{2},Y,W} | W, \Delta Y^{(r,0)}, \ldots, \Delta Y^{(r,m-1)}]. \) One can get \( \frac{(q+1)h^2}{30} \) by replacing the \( h \) and \( m \) in Explanation with \( \frac{h}{2} \) and \( 2(q+1). \) So as to make sure the \( \frac{(q+1)h^2}{15} - \sum_{i=0}^{q} ((H^{(i)}_{h,Y,0})^2 \frac{h}{16} + (H^{(i)}_{h,Y,1})^2 \frac{h}{48}) \simeq (q+1)\overline{h}^{2}/30, \) we need the threshold value \( Q \) in method (4.2.4). And because of

\[ A^{(r)}_{h,Y,W} \simeq A^{(r)}_{\frac{1}{2},Y,W} + \sum_{i=0}^{q} \{\Delta W^{(r,i)}_{h,L} \Delta Y^{(r,i)}_{h,R}\}, \]

we have \( \tilde{Z}^{(r)}_{\frac{1}{2},Y,W} \simeq \tilde{Z}^{(r)}_{h,Y,W}. \)

This explain why we obtain the \( \tilde{Z}^{(r)}_{\frac{1}{2},Y,W} \) with the method (4.2.6).

\[ \square \]

**Explanation 4.2.9.** We explain why we need to correct the \( \tilde{Z}^{(r)}_{\frac{1}{2},Y,W} \) into \( \tilde{Z}^{(r)}_{h,Y,W} \) here.

From the lemma (3.1.2) and Explanation (4.1.3), we got the variance of \( \tilde{Z}^{(r)}_{h,Y,W}, \tilde{Z}^{(r)}_{h,R,W}, \tilde{Z}^{(r)}_{h,Y,R} \) already. The variance of \( \tilde{Z}^{(r)}_{\frac{1}{2},Y,W} \) is \( (\frac{(q+1)h^2}{15} - \sum_{i=0}^{q} ((H^{(i)}_{h,Y,0})^2 \frac{h}{16} + (H^{(i)}_{h,Y,1})^2 \frac{h}{48}) \) rather than \( \frac{(q+1)h^2}{30}, \) so we need to correct \( \tilde{Z}^{(r)}_{\frac{1}{2},Y,W} \)'s variance by \( Q_{h,Y,W}, \) making variance of \( \tilde{Z}^{(r)}_{\frac{1}{2},Y,W} = \frac{(q+1)h^2}{2x15} - \sum_{i=0}^{q} ((H^{(r,i)}_{Y,0})^2 \frac{h}{16} + (H^{(r,i)}_{Y,1})^2 \frac{h}{48}), \) become \( \frac{(q+1)h^2}{30}. \) Hence

\[ A^{(r)}_{\frac{1}{2},Y,W} = \tilde{Z}^{(r)}_{\frac{1}{2},Y,W} + \sum_{i=0}^{q} \{(4C^{(r,i)}_{h,W,1,L} - 6C^{(r,i)}_{h,W,2,L})\Delta Y^{(r,i)}_{h,L} + (12C^{(r,i)}_{h,W,2,L} - 6C^{(r,i)}_{h,W,1,L})C^{(r,i)}_{h,Y,1,L}\}
\]

\[ + (4C^{(r,i)}_{h,W,1,R} - 6C^{(r,i)}_{h,W,2,R})\Delta Y^{(r,i)}_{h,R} + (12C^{(r,i)}_{h,W,2,R} - 6C^{(r,i)}_{h,W,1,R})G^{(r,i)}_{h,W,2,R}\}. \]

where \( \tilde{Z}^{(r)}_{\frac{1}{2},Y,W} = \tilde{Z}^{(r)}_{\frac{1}{2},Y,W}/\sqrt{Q_1} \)

\[ \square \]

**Explanation 4.2.10.** We explain the threshold values \( Q_{h,R,0}, Q_{h,R,0}, Q_{h,R,0,R,1} \) indicated in the method (4.2.4) here.

We want to generate unknown independent value \( N_{3,0,i} \sim N(0,1) \) such that

\[ \tilde{Z}^{(r+1)}_{R,Y} = \tilde{Z}_{h,R,Y} - \sum_{i=0}^{q} H^{(r,i)}_{h,R,0} N_{3,0,i} \sqrt{h} \]

\[ \tilde{Z}^{(r+1)}_{R,W} = \tilde{Z}_{h,R,W} - \sum_{i=0}^{q} H^{(r,i)}_{h,R,1} N_{3,0,i} \sqrt{h}. \]
This split up process can be approached by applying lemma (1.2.13). After the split up process, one can get

\[ \text{Var}(\bar{Z}_{R,Y}^{(q+1)}) = \frac{(q+1)h^2}{6} - \sum_{i=0}^{q} (H_{h,R,0}^{(r,i)})^2 h \]

\[ \text{Var}(\bar{Z}_{R,W}^{(q+1)}) = \frac{(q+1)h^2}{6} - \sum_{i=0}^{q} (H_{h,R,1}^{(r,i)})^2 h. \]

It should be true that \( \text{Var}(\bar{Z}_{R,Y}^{(q+1)}) \sim \frac{(q+1)h^2}{12} \) and \( \text{Var}(\bar{Z}_{R,W}^{(q+1)}) \sim \frac{(q+1)h^2}{12} \) as the 2D situation. Hence we can have \( Q_{h,R,0}, Q_{h,R,0} \sim N(0,2/(q+1)) \).

Considering the coefficients for correcting \( \bar{Z}_{R,Y}^{(r,i)}, \bar{Z}_{R,W}^{(r,i)} \) in method (4.2.11), we can know that \( Q_{h,R,0,0,1} = 12 \sum_{i=0}^{q} H_{h,R,0}^{(r,i)} H_{h,R,1}^{(r,i)} / ((q+1)h) \) is required to be \( \left| Q_{h,R,0,0,1} \right| \leq \frac{1}{2} \). The explanation for general situation is given in Explanation (5.3.9).

So we have the threshold value \( Q_{h,R,0}, Q_{h,R,0}, Q_{h,R,0,0,1} \).

\[ \Box \]

**Method 4.2.11.** We generate \( \bar{Z}_{2,R,Y}^{(r)}, \bar{Z}_{2,R,W}^{(r)} \) and \( N_{3,0,i} \) with the following method:

**Case 1.** if \( \left| 1 - \frac{12(H_{h,R,0}^{(r,i)})^2}{(q+1)h} \right| \leq \frac{1}{2}, \left| 1 - \frac{12(H_{h,R,1}^{(r,i)})^2}{(q+1)h} \right| \leq \frac{1}{2} \) and \( \left| \frac{12H_{h,R,0}^{(r,i)}H_{h,R,1}^{(r,i)}}{(q+1)h} \right| \leq \frac{1}{2} \)

then we are generating independent random values \( N_{3,0,i} \), s.t

\[ N_{3,0,i} \sim N(0,h) \]

\[ \bar{Z}_{R,Y}^{(q+1)} = \bar{Z}_{h,R,Y} - \sum_{i=0}^{q} H_{h,R,0}^{(r,i)} N_{3,0,i} \]

\[ \bar{Z}_{R,W}^{(q+1)} = \bar{Z}_{h,R,W} - \sum_{i=0}^{q} H_{h,R,1}^{(r,i)} N_{3,0,i}. \]

The way to achieve this is shown in lemma (1.2.12) And obtain the correction term \( \bar{Z}_{2,R,Y}^{(q+1)} \) and \( \bar{Z}_{2,R,W}^{(q+1)} \) by

\[ \bar{Z}_{2,R,Y} = a \bar{Z}_{h,R,Y}^{(q+1)} \]

\[ \bar{Z}_{2,R,W} = b \bar{Z}_{h,R,W}^{(q+1)} + c \bar{Z}_{R,W}^{(q+1)} \]

in which

\[ a = \sqrt{\frac{(q+1)h}{2(q+1)h - 24 \sum_{i=1}^{(q+1)/2} (H_{h,R,0}^{(r,i)})^2}} \]
Numerical Approximation for SDE

\begin{align*}
b &= \sqrt{\frac{(q+1)^2}{24} (H_{R,0}^{(r,i)} H_{R,1}^{(r,i)})^2 h^2}{((H_{R,0}^{(r,i)})^2 h - (q+1)^2 12 - h(H_{R,1}^{(r,i)})^2 - (q+1)^2 12 - h(H_{R,0}^{(r,i)})^2 h (H_{R,0}^{(r,i)} H_{R,1}^{(r,i)})^2} \end{align*}

\begin{align*}c &= \sqrt{\frac{(q+1)^2}{24} (q+1)^2 12 - h(H_{R,0}^{(r,i)})^2}{((q+1)^2 12 - h(H_{R,0}^{(r,i)})^2 - (q+1)^2 12 - h(H_{R,0}^{(r,i)})^2 h (H_{R,0}^{(r,i)} H_{R,1}^{(r,i)})^2} \end{align*}

Case 2. if \( |1 - \frac{12H^{(2)}}{(q+1)h}| > \frac{1}{2} \), or \( |1 - \frac{12H^{(3)}}{(q+1)h}| > \frac{1}{2} \), or \( |\frac{12H^{(3)}}{(q+1)h}| > \frac{1}{2} \)
We generate independent random values \( N_{3,0,i} \sim N(0, h) i = 0, \ldots, q, \tilde{Z}_{h,R,Y} \sim N(0, \frac{(q+1)^2}{12}), \) and \( \tilde{Z}_{h,R,W} \sim N(0, \frac{(q+1)^2}{12}). \)

We prepare some notations which are only applied in for the next lemma (4.2.12).

\begin{align*}a &= \frac{(\sigma^{(i)}_c)^2 - \sigma^{(i)}_{R,Y} \sigma_{R,W} + h(H_{R,0}^{(r,i)})^2 \sigma_{R,W} - hH_{R,0}^{(r,i)} \sigma_{R,Y} - hH_{R,0}^{(r,i)} \sigma_{R,Y}}{\sigma_{R,W} \sigma_{R,Y} - (\sigma^{(i)}_c)^2} \\
b &= \frac{hH_{R,0}^{(r,i)} (H_{R,0}^{(r,i)} \sigma_{R,Y} - \sigma^{(i)}_c H_{h,R,0}^{(r,i)})}{\sigma_{R,W} \sigma_{R,Y} - (\sigma^{(i)}_c)^2} \\
d &= \sigma^{(i)}_{R,W} \sigma_{R,Y} (H_{h,R,0}^{(r,i)})^2 h - \sigma^{(i)}_{R,W} (H_{h,R,0}^{(r,i)})^2 h^2 - \sigma^{(i)}_{R,Y} (H_{h,R,1}^{(r,i)})^2 (H_{h,R,0}^{(r,i)})^2 h^2 \\
e &= 2 \sigma^{(i)}_c H_{h,R,1}^{(r,i)} (H_{h,R,0}^{(r,i)})^2 h^2 - (\sigma^{(i)}_c)^2 (H_{h,R,0}^{(r,i)})^2 h \\
c &= \sqrt{\frac{d + e}{\sigma_{R,W} \sigma_{R,Y} - (\sigma^{(i)}_c)^2}} \end{align*}

Lemma 4.2.12. Let \( \tilde{Z}_{R,Y}^{(i)} \sim N(0, \sigma_{R,Y}^{(i)}), \tilde{Z}_{R,W}^{(i)} \sim N(0, \sigma_{R,W}^{(i)}), \) and \( E[\tilde{Z}_{R,Y}^{(i)} \tilde{Z}_{R,W}^{(i)}] = \sigma^{(i)}_c (i = 1, \ldots, q + 1). \) Conditional on \( \tilde{Z}_{R,Y}^{(i)} \) and \( \tilde{Z}_{R,W}^{(i)} \), when generate \( \tilde{Z}_{R,Y}^{(i+1)}, \tilde{Z}_{R,W}^{(i+1)} \) with

\begin{align*}\tilde{Z}_{R,Y}^{(i+1)} &= a \tilde{Z}_{R,Y}^{(i)} + b \tilde{Z}_{R,W}^{(i)} + c N_{i}^{(8)} N_{i} \sim N(0, 1) \\
N_{3,0,i-1} &= \frac{(1 - a) \tilde{Z}_{R,Y}^{(i)} - b \tilde{Z}_{R,W}^{(i)} - c N_{i}^{(8)}}{H_{h,R,0}^{(r,i-1)}} \\
\tilde{Z}_{R,W}^{(i+1)} &= \frac{(H_{h,R,0}^{(r,i-1)} + b H_{h,R,1}^{(r,i-1)}) \tilde{Z}_{R,W}^{(i)} - H_{h,R,1}^{(r,i-1)} (1 - a) \tilde{Z}_{R,Y}^{(i)} + c H_{h,R,1}^{(r,i-1)} N_{i}^{(8)}}{H_{h,R,0}^{(r,i-1)}}, \end{align*}

where \( a, b, c \) are stated just before this lemma, we obtain the \( \tilde{Z}_{R,Y}^{(i+1)}, \tilde{Z}_{R,W}^{(i+1)} \) and \( N_{3,0,i-1} \sim N(0, 1), \) such that \( E(\tilde{Z}_{R,Y}^{(i+1)})^2 = E(\tilde{Z}_{R,Y}^{(i)})^2 - (H_{h,R,0}^{(r,i)})^2 h, E(\tilde{Z}_{R,W}^{(i+1)})^2 = E(\tilde{Z}_{R,W}^{(i)})^2 - (H_{h,R,1}^{(r,i)})^2 h. \)
Proof: We want to generate $N_{3,0,i} \sim N(0,1)$, $\tilde{Z}^{(i+1)}_{R,Y} \sim N(0, \sigma_{R,Y}^{(i)} - (H^{(r,i)}_{h,R,0})^2 h)$, $\tilde{Z}^{(i+1)}_{R,W} \sim N(0, \sigma_{R,W}^{(i)} - (H^{(r,i)}_{h,R,1})^2 h)$, such that

$$\tilde{Z}^{(i)}_{R,Y} = \tilde{Z}^{(i+1)}_{R,Y} + H^{(r,i)}_{h,R,0} N_{3,0,i} \sqrt{h}$$
$$\tilde{Z}^{(i)}_{R,W} = \tilde{Z}^{(i+1)}_{R,W} + H^{(r,i)}_{h,R,1} N_{3,0,i} \sqrt{h}$$

Set the $\tilde{Z}^{(i+1)}_{R,Y}$ to be

$$\tilde{Z}^{(i+1)}_{R,Y} = a \tilde{Z}^{(i)}_{R,Y} + b \tilde{Z}^{(i)}_{R,W} + c N^{(8)}_{i} \quad N^{(8)}_{i} \sim N(0,1)$$

$$\sqrt{h} N_{3,0,i-1} = \frac{(1-a) \tilde{Z}^{(i)}_{R,Y} - b \tilde{Z}^{(i)}_{R,W} - c N^{(8)}_{i}}{H^{(r,i-1)}_{h,R,0}}$$

$$\tilde{Z}^{(i+1)}_{R,W} = \frac{(H^{(r,i-1)}_{h,R,0} + b H^{(r,i-1)}_{h,R,1}) \tilde{Z}^{(i)}_{R,W} - H^{(r,i-1)}_{h,R,1} (1-a) \tilde{Z}^{(i)}_{R,Y} + c H^{(r,i-1)}_{h,R,1} N^{(8)}_{i}}{H^{(r,i-1)}_{h,R,0}}$$

where $a, b, c$ are unknown coefficients. Conditional on $\tilde{Z}^{(i)}_{R,Y}$ and $\tilde{Z}^{(i)}_{R,W}$, we have

$$\sigma_{R,Y}^{(i+1)} = a^2 \sigma_{R,Y}^{(i)} + b^2 \sigma_{R,W}^{(i)} + c^2 + 2abc \sigma_{c}^{(i)}$$

$$\sigma_{R,W}^{(i+1)} = (1 + \frac{H^{(r,i-1)}_{h,R,1}}{H^{(r,i-1)}_{h,R,0}}) b^2 \sigma_{R,W}^{(i)} + \frac{H^{(r,i-1)}_{h,R,1}}{H^{(r,i-1)}_{h,R,0}} (1-a)^2 \sigma_{R,Y}^{(i)} + \frac{c^2}{H^{(r,i-1)}_{h,R,0}}$$

$$hE(N_{3,0,i-1}) = \frac{(1-a) H^{(r,i-1)}_{h,R,0}}{H^{(r,i-1)}_{h,R,0}} \theta_{R,Y}^{(i)} + \frac{b}{H^{(r,i-1)}_{h,R,0}} \theta_{R,W}^{(i)} - 2 \frac{(1-a)b}{H^{(r,i-1)}_{h,R,0}} \theta_{c}^{(i)} + \frac{c}{H^{(r,i-1)}_{h,R,0}}$$

$$\sigma_{c}^{(i+1)} = \frac{H^{(r,i-1)}_{h,R,1}}{H^{(r,i-1)}_{h,R,0}} (a-1) \sigma_{R,Y}^{(i)} + (1 + \frac{H^{(r,i-1)}_{h,R,1}}{H^{(r,i-1)}_{h,R,0}}) b \sigma_{R,W}^{(i)}$$

$$+ (a + 2 \frac{H^{(r,i-1)}_{h,R,1}}{H^{(r,i-1)}_{h,R,0}} a b - H^{(r,i-1)}_{h,R,0}) \sigma_{c}^{(i)} + \frac{H^{(r,i-1)}_{h,R,1}}{H^{(r,i-1)}_{h,R,0}} c^2$$

On the other hand,

$$\sigma_{R,Y}^{(i+1)} = \sigma_{R,Y}^{(i)} - (H^{(r,i)}_{h,R,0})^2 h$$
$$\sigma_{R,W}^{(i+1)} = \sigma_{R,W}^{(i)} - (H^{(r,i)}_{h,R,1})^2 h$$

we could work out the coefficients $a, b, c$ by solving these equations.

Because the correction terms $\tilde{Z}^{(i+1)}_{h,R,Y} \sim N(0, \frac{2 \theta_{R,Y}^{(i)}}{6 \theta_{R,Y}^{(i)}})h^2$ and $\tilde{Z}^{(i+1)}_{h,R,W} \sim N(0, \frac{(\theta_{R,W}^{(i)})^2}{12})h^2$ are supposed to be independent to $\Delta W^{(r)}_{h}, \Delta Y^{(r)}_{h}, \Delta R^{(r)}_{h}, G^{(r)}_{h,W1}, G^{(r)}_{h,Y1}$ according to our set up. We are going to correct the $Z$ terms.
Lemma 4.2.13. Let

\[ \tilde{Z}^{(q+2)}_{\frac{1}{2}, R, Y} = a\tilde{Z}^{(q+2)}_{R, Y} \]

\[ \tilde{Z}^{(q+2)}_{\frac{1}{2}, R, W} = b\tilde{Z}^{(q+2)}_{R, Y} + c\tilde{Z}^{(q+2)}_{R, W}, \]

in which

\[ a = \sqrt{\frac{(q + 1)h}{2(q + 1)h - 24\sum_{i=1}^{(q+1)/2}(H^{(r,i)}_{R,0})^2}} \]

\[ b = \frac{(q+1)h^2}{24} \left( \frac{((H^{(r,i)}_{R,0})^2 h - (q+1)h^2)^2}{2(q+1)h^2} - h(H^{(r,i)}_{R,1})^2 \right) \]

\[ c = \frac{(q+1)h^2}{24} \left( \frac{(q+1)h^2 - h(H^{(r,i)}_{R,1})^2}{2(q+1)h^2} - h(h(H^{(r,i)}_{R,0})^2) \right). \]

\( \tilde{Z}^{(q+2)}_{\frac{1}{2}, R, Y}, \tilde{Z}^{(q+2)}_{\frac{1}{2}, R, W} \sim N(0, (q+1)h^2) \) will be mutually independent.

**Proof:** Because \( \tilde{Z}^{(q+2)}_{\frac{1}{2}, R, Y} \) only depends on \( \tilde{Z}^{(q+2)}_{R, Y} \), and \( \tilde{Z}^{(q+2)}_{\frac{1}{2}, R, W} \) will be affected by \( \tilde{Z}^{(q+2)}_{R, Y} \) and \( \tilde{Z}^{(q+2)}_{R, W} \), we can set

\[ \tilde{Z}^{(q+2)}_{\frac{1}{2}, R, Y} = a\tilde{Z}^{(q+2)}_{R, Y} \]

\[ \tilde{Z}^{(q+2)}_{\frac{1}{2}, R, W} = b\tilde{Z}^{(q+2)}_{R, Y} + c\tilde{Z}^{(q+2)}_{R, W}. \]

Consider the condition

\[ \tilde{Z}^{(q+2)}_{R, Y} \sim N(0, \frac{(q + 1)h^2}{6} - \sum_{i=0}^{q} (H^{(r,i)}_{h, R, 0})^2 h) \]

\[ \tilde{Z}^{(q+2)}_{R, W} \sim N(0, \frac{(q + 1)h^2}{6} - \sum_{i=0}^{q} (H^{(r,i)}_{h, R, 1})^2 h) \]

\[ E(\tilde{Z}^{(q+2)}_{R, Y} \tilde{Z}^{(q+2)}_{R, W}) = - \sum_{i=0}^{q} H^{(r,i)}_{h, R, 0} H^{(r,i)}_{h, R, 1} h \]

\[ \tilde{Z}^{(q+2)}_{\frac{1}{2}, R, Y}, \tilde{Z}^{(q+2)}_{\frac{1}{2}, R, W} \sim N(0, \frac{(q + 1)h^2}{12}) \]

\[ E\{\tilde{Z}^{(q+2)}_{\frac{1}{2}, R, Y}, \tilde{Z}^{(q+2)}_{\frac{1}{2}, R, W}\} = 0, \]

we can solve out the coefficients \( a, b, c \).
Further Subdivision

By doing these, there will be \( m \) segments in \( J \) now. We replace "\( q + 1 \)" with "\( m \)" and "\( h \)" with "\( \frac{b}{2} \)" in the formulas in the "Third Stage", and then repeat the process again, since the length of segment becomes \( \frac{m}{2} \) and number of segments becomes \( m \) after the former split up. Then there will be \( 2m \) small segments with length \( \frac{b}{4} \) in segment \( J \) after the second time split up.

4.3 The Difference of The Approximate Area

We only discuss the convergence of approximation of \( A^{(r)}_{h,Y,W} \) here, because \( A^{(r)}_{h,R,Y} \) and \( A^{(r)}_{h,R,W} \) are similar to the case in[1]. In segment \( J \), which has \( q \) small segments having length \( h \), we have

\[
\bar{Z}^{(q+2)}_{Y,W} = \bar{Z}_{h,Y,W} + \sum_{i=0}^{q} [H^{(i)}_{h,Y,0} N_{2,0,i} + H^{(i)}_{h,Y,1} N_{2,1,i}] \tag{4.1}
\]

where \( H^{(i)}_{h,Y,0} \) and \( H^{(i)}_{h,Y,1} \) are the same as those in method (4.2.4). Let \( \tilde{A}_J \) be the approximate area of \( J \) before \( I \) subdivision, \( \tilde{A}_J \) be the approximate area of \( J \) after one \( I \) subdivision.

\[
\hat{A}_J = \sum_{i=1}^{q} [A^{(r)}_{h,Y,W} + (W(l_i) - W(l_{j})) \Delta Y^{(r,i)}_h] \tag{4.2}
\]

\[
\alpha_i = 4G^{(r,i)}_{h,1} - 6G^{(r,i)}_{h,2} \quad \alpha_{i,L} = 4G^{(r,i)}_{h,1,L} - 6G^{(r,i)}_{h,2,L} \quad \alpha_{i,R} = 4G^{(r,i)}_{h,1,R} - 6G^{(r,i)}_{h,2,R}
\]

\[
\beta_i = 12G^{(r,i)}_{h,1} - 6G^{(r,i)}_{h,2} \quad \beta_{i,L} = 12G^{(r,i)}_{h,1,L} - 6G^{(r,i)}_{h,2,L} \quad \beta_{i,R} = 12G^{(r,i)}_{h,1,R} - 6G^{(r,i)}_{h,2,R}
\]

\[
\tilde{Z}^{(r)}_{Y,W} = \tilde{Z}_{h,Y,W} \quad U_0 = \tilde{Z}_{h,Y,W}^*
\]

\[
\tilde{A}_J = \tilde{Z}^{(r)}_{h,Y,W} + \sum_{i=1}^{q} [(\alpha_i \Delta Y^{(r,i)}_h + \beta_i G^{(r,i)}_{h,Y,1}) + (W(l_i) - W(l_{j})) \Delta Y^{(r,i)}_h], \tag{4.3}
\]

(See the figure (3.2).) the approximation given by subdivided version is

\[
\tilde{A}_J = \sum_{i=1}^{q} [A^{(r,i)}_{h,L} + A^{(r,i)}_{h,R} + (W(l_i) - W(l_{j})) \Delta Y^{(r,i)}_{h,L} + (W(l_i) - W(l_{j})) \Delta Y^{(r,i)}_{h,R}]
\]

\[
= \tilde{Z}^{(r)}_{h,Y,W} + \sum_{i=1}^{q} [(\alpha_{i,L} \Delta Y^{(r,i)}_{h,L} + \beta_{i,L} G^{(r,i)}_{h,Y,1,L}) + (W(l_i) - W(l_{j})) \Delta Y^{(r,i)}_{h,L}]
\]

\[+ \sum_{i=1}^{q} [(\alpha_{i,R} \Delta Y^{(r,i)}_{h,R} + \beta_{i,R} G^{(r,i)}_{h,Y,1,R}) + (W(l_i) - W(l_{j})) \Delta Y^{(r,i)}_{h,R}]\]

say \( Q \) and \( Q_{h,Y,W} \) the same as those in method (4.2.4), we have,

\[
E(\hat{A}_J - \tilde{A}_J)^2 = E\{(\tilde{Z} - U_0)^2_{x|Q|>\frac{b}{2}} + E\{(\tilde{Z} - U_0)^2_{x|Q|\leq\frac{b}{2}} \} \tag{4.4}
\]
Because

\[ Q = 1 - Q_{h,Y,W} = \frac{30}{(q + 1)h^2} \sum_{i=0}^{q} \left( (H^{(r,i)}_{h,Y,0})^2 \frac{h}{16} + (H^{(r,i)}_{h,Y,1})^2 \frac{h}{48} \right) - 1 \]  

(4.5)

so

\[
E(Q^2) = E\left\{ \frac{30^2}{(q + 1)^2 h^4} \sum_{i=0}^{q} \left[ (H^{(r,i)}_{h,Y,0})^4 \frac{h^2}{16^2} + (H^{(r,i)}_{h,Y,1})^4 \frac{h^2}{48^2} + 2(H^{(r,i)}_{h,Y,0})^2 (H^{(r,i)}_{h,Y,0})^2 \frac{h}{16} \frac{h}{48} \right] \right\} \\
+ \frac{60}{(q + 1)h^2} \sum_{i=0}^{q} \left[ (H^{(r,i)}_{h,Y,0})^2 \frac{h}{16} + (H^{(r,i)}_{h,Y,1})^2 \frac{h}{48} \right]
\]

so

\[
E(Q^2) \leq E\left\{ \frac{30^2}{(q + 1)^2 h^4} \sum_{i=0}^{q} \left[ (H^{(r,i)}_{h,Y,0})^4 \frac{h^2}{16^2} + (H^{(r,i)}_{h,Y,1})^4 \frac{h^2}{48^2} + 2(H^{(r,i)}_{h,Y,0})^2 (H^{(r,i)}_{h,Y,0})^2 \frac{h}{16} \frac{h}{48} \right] \right\} = \frac{1}{q + 1}
\]

(4.6)

i) when \(|Q| \leq \frac{1}{2} \)

\[
E\{(\hat{Z} - U_0)^2_{X|Q| \leq \frac{1}{2}}\} = E\left\{ \frac{U_0^2}{\sqrt{1 - Q}} (\sqrt{1 - Q} - 1)^2 \right\}_{X|Q| \leq \frac{1}{2}}
\]

then

\[
E\{(\hat{Z} - U_0)^2_{X|Q| \leq \frac{1}{2}}\} = \frac{(q + 1)h^2}{30} E\{(\sqrt{1 - Q} - 1)^2\}_{X|Q| \leq \frac{1}{2}}
\]

then

\[
E\{(\hat{Z} - U_0)^2_{X|Q| \leq \frac{1}{2}}\} \leq \frac{(q + 1)h^2}{30} \frac{2}{5} E(Q^2) = \frac{h^2}{75}
\]

ii) when \(|Q| > \frac{1}{2} \)

\[
E\{(\hat{Z} - U_0)^2_{X|Q| > \frac{1}{2}}\} = E\{\tilde{Z}_{h,Y,W} + \sum_{i=0}^{q} \left( (H^{(r,i)}_{h,Y,0})^2 \frac{h}{16} + (H^{(r,i)}_{h,Y,1})^2 \frac{h}{48} + \frac{(q + 1)h^2}{30} \right)_{X|Q| > \frac{1}{2}}\}
\]

then

\[
E\{(\hat{Z} - U_0)^2_{X|Q| > \frac{1}{2}}\} = \frac{(q + 1)h^2}{30} E\{(1 + Q)_{X|Q| > \frac{1}{2}}\} + \frac{(q + 1)h^2}{10} P(|Q| > \frac{1}{2}).
\]

Because \(E(Q^2) = \frac{1}{1+q} P(|Q| > \frac{1}{2}) \leq \frac{4}{q+1} \), we have

\[
E\{(\hat{Z} - U_0)^2_{X|Q| > \frac{1}{2}}\} \leq (q + 1)h^2 \left( \frac{1}{30} + \frac{1}{10} \right) P(|Q| > \frac{1}{2}) + \frac{(q + 1)h^2}{30} \left\{ P(|Q| > \frac{1}{2}) \right\}^{\frac{1}{2}} \left\{ E(Q^2) \right\}^{\frac{1}{2}}
\]

\[
\leq (q + 1)h^2 \left( \frac{4}{30 q + 1} + \frac{1}{30 q + 1} \right) = \frac{17h^2}{6}
\]
so
\[ E\{ (\hat{Z} - U_0)^2 \}_{|_{t > 1}} \leq \frac{17}{6} h^2 \]
in all
\[ E(\hat{A}_j - \tilde{A}_j)^2 \leq \frac{1}{75} h^2 + \frac{17}{6} h^2 < \frac{427}{150} h^2 \quad (4.7) \]

### 4.4 Illustrative Example

Let's consider a 3D SDEs problem here.

**Illustrative Example 4.4.1.** Consider the linear SDEs problem

\[
\begin{align*}
 dX^{(1)} &= dW \\
 dX^{(2)} &= X^{(1)} dY \\
 dX^{(3)} &= X^{(2)} dW + 2X^{(1)}X^{(2)} dR
\end{align*}
\]

Let initial value \((X^{(1)}, X^{(1)}, X^{(1)}) = (1, 2, 3)\). The SDE is approximate with Euler and 3/4 Method on \([0, T]\) with time step size \(h, h/4, h/16, h/64, \) and \(h/156\), where \(T = 10\) and \(h = T/8\). Approximate the \(E[X_h(T)]\) with \(M = 800\) Monte Carlo simulations.

Then we have the numerical scheme which is characterized by an equal distant partition \(\tau_n\) of \([0, T]\):

\[
\tau_N : 0 = a_0 < a_1 < \ldots < a_N = T
\]

with mesh
\[
\delta = mesh(\tau_N) = a_{i+1} - a_i, (i = 0, \ldots, N - 1).
\]

**The Approximation Function**

1. When \(N = m^2\) and \(m = 2^k\), where \(k\) is some constant, we have Euler approximation

\[
\begin{align*}
 X^{(1)}(n) &= X^{(1)}(n - 1) + \Delta W^{(r,i)}_h \\
 X^{(2)}(n) &= X^{(2)}(n - 1) + X^{(1)}(n - 1)\Delta Y^{(r,i)}_h \\
 X^{(3)}(n) &= X^{(3)}(n - 1) + X^{(2)}(n - 1)\Delta W^{(r,i)}_h + 2X^{(1)}(n - 1)X^{(2)}(n - 1)\Delta R^{(r,i)}_h
\end{align*}
\]

where \(\delta = h = T/N\), \(n = rm + i + 1\).

2. We also have the 3/4 approximation: When \(n \neq 0 \mod m\)

\[
\begin{align*}
 X^{(1)}(n) &= X^{(1)}(n - 1) + \Delta W^{(r,i)}_h \\
 X^{(2)}(n) &= X^{(2)}(n - 1) + X^{(1)}(n - 1)\Delta Y^{(r,i)}_h \\
 X^{(3)}(n) &= X^{(3)}(n - 1) + X^{(2)}(n - 1)\Delta W^{(r,i)}_h + 2X^{(1)}(n - 1)X^{(2)}(n - 1)\Delta R^{(r,i)}_h
\end{align*}
\]
When $n = m(r + 1)$ ($r = 0, 1, \ldots, m - 1$),

\[
\begin{align*}
X^{(1)}(n) &= X^{(1)}(n - 1) + \Delta W^{(r)}_h \\
X^{(2)}(n) &= X^{(2)}(n - 1) + X^{(1)}(n - 1)\Delta Y^{(r)}_h + A^{(r)}_{h,Y} \\
X^{(3)}(n) &= X^{(3)}(n - 1) + X^{(2)}(n - 1)\Delta W^{(r)}_h + 2X^{(1)}(n - 1)X^{(2)}(n - 1)\Delta R^{(r)}_h \\
&\quad + X^{(1)}(n - 1)\sum_{i=0}^{m-1} \Delta Y^{(r)}_h \Delta W^{(r)}_h - A^{(r)}_{h,R} + 2X^{(2)}(n - 1)A^{(r)}_{h,R} + 2X^{(1)}(n - 1)^2A^{(r)}_{h,Y}
\end{align*}
\]

Similar to the example in 2D, we present the simulation result below.

**Simulation Result**

The computer simulation gives us approximate $E[X_h - X_{h/4}]$, $E[X_{h/4} - X_{h/16}]$, $E[X_{h/16} - X_{h/64}]$, $E[X_{h/64} - X_{h/256}]$ by $3/4$ Method,

<table>
<thead>
<tr>
<th>Step Size $h$</th>
<th>0.15625</th>
<th>0.0390625</th>
<th>9.765625 x 10^{-3}</th>
<th>2.4414 x 10^{-3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[X_h^{(1)}] - X_{h/4}^{(1)}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$E[X_h^{(2)}] - X_{h/4}^{(2)}$</td>
<td>0.23464</td>
<td>0.08431</td>
<td>0.02608</td>
<td>0.00573</td>
</tr>
<tr>
<td>$E[X_h^{(3)}] - X_{h/4}^{(3)}$</td>
<td>14.04550</td>
<td>6.39965</td>
<td>2.30759</td>
<td>0.64186</td>
</tr>
</tbody>
</table>

and by Euler Method,

<table>
<thead>
<tr>
<th>Step Size $h$</th>
<th>0.15625</th>
<th>0.0390625</th>
<th>9.765625 x 10^{-3}</th>
<th>2.4414 x 10^{-3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[X_h^{(1)}] - X_{h/4}^{(1)}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$E[X_h^{(2)}] - X_{h/4}^{(2)}$</td>
<td>0.58290</td>
<td>0.31374</td>
<td>0.15086</td>
<td>0.07631</td>
</tr>
<tr>
<td>$E[X_h^{(3)}] - X_{h/4}^{(3)}$</td>
<td>17.70459</td>
<td>8.55891</td>
<td>4.52547</td>
<td>2.11426</td>
</tr>
</tbody>
</table>

So we may approximate the Strong Scheme order $\gamma^{(2)}_h$ and $\gamma^{(3)}_h$ with $L^{(2)}_h$ and $L^{(3)}_h$ respectively. The definition of the $L_h$ is given in the illustrative example (3.4.1). Because $L^{(2)}_h$ and $L^{(3)}_h$ approximate the order of 3/4 Method, we know the order of the 3/4 Method is approximate to 1.

<table>
<thead>
<tr>
<th>Step Size $h$</th>
<th>0.15625</th>
<th>0.0390625</th>
<th>9.765625 x 10^{-3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^{(2)}_h$</td>
<td>0.73834</td>
<td>0.84652</td>
<td>1.09285</td>
</tr>
<tr>
<td>$L^{(3)}_h$</td>
<td>0.56702</td>
<td>0.73580</td>
<td>0.92303</td>
</tr>
</tbody>
</table>

And we can approximate the Strong Scheme order with $L^{(2)}_h$ and $L^{(3)}_h$ for Euler Method, which derived from $X_h^{(1)}$ and $X_h^{(2)}$ respectively. Considering the order of Euler method, we will expect $L^{(2)}_h, L^{(3)}_h \approx \frac{1}{2}$.

<table>
<thead>
<tr>
<th>Step Size $h$</th>
<th>0.15625</th>
<th>0.0390625</th>
<th>9.765625 x 10^{-3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^{(2)}_h$</td>
<td>0.44684</td>
<td>0.52816</td>
<td>0.49161</td>
</tr>
<tr>
<td>$L^{(3)}_h$</td>
<td>0.52431</td>
<td>0.45968</td>
<td>0.54896</td>
</tr>
</tbody>
</table>

This result shows the 3/4 Method have higher order than the Euler one in the case that SDE was driven by three Wiener processes.

One may find that the approximate order of the 3/4 method is 1 rather than $3/4$. In fact, the simulation results show that $\gamma$, the order of 3/4 method, is closed to 1, when the step size $h$ becomes small. This might be cause by the fact
that the $|Q| > \frac{1}{2}$ is rare when $h$ is small, considering that $|Q| > \frac{1}{2}$ will lead to reconstruction of the approximate Brownian trajectory.
Chapter 5

Pathwise Approximation of Solutions of Stochastic Differential Equations in n Dimension Case

The previous discussion of 3/4 method gives us hope to get strong solutions for high dimension SDEs. We are going to show how to manage this.

5.1 Notation

We will adopt the following notation:

Method 5.1.1. (Number the instant of time) Divide $[a, b]$ into $m$ equal intervals, $J_0, \ldots, J_{m-1}$. Divide each interval $J_r$ ($r = 0, \ldots, m-1$) again into $m$ equal subintervals $I^{(r)}_0 = [a_{rm}, a_{rm+1}], \ldots, I^{(r)}_m = [a_{rm+m}, a_{rm+m+1}], \ldots, I^{(r)}_{m-1} = [a_{rm+m-1}, a_{rm+m}]$. Denote $h = a_{i+1} - a_i$, ($i = 0, \ldots, m^2 - 1$), and $N = m^2$, then we have $h = \frac{b-a}{N}$, where $a_0 = a$, and $a_{mm} = b$. These are shown in the following graphic.

Based on this division of the time, we use the following notation:

\[
\begin{array}{cccccccc}
J_0 & J_1 & \cdots & \cdots & J_r & \cdots & \cdots & J_{m-2} & J_{m-1} & b \\
\hline
a_0 & a_m & a_{2m} & & & & & a_{mm} \\
\hline
\end{array}
\]
The \( i^{th} \) instant of time in segment \( J_r \), where \( t = (rm + j)h \)

\( G_{h,k,j}^{(r,i)} \)

The values in "( )" indicate the instant of time.

\( h \) is the length of the discrete time.

\( k \) and \( j \) indicate the components of the \( G \). (see the example below)

\( \Delta W_{h}^{(r,i)\{k\}} \)

the \( k \) in "{}" indicates that it is the \( k^{th} \) component of the vector \( W \).

The values in "( )" indicate the instant of time.

\( X_{h}^{(j)\{k\}} \)

the \( k \) in "{}" indicates that it is the \( k^{th} \) component of the vector \( X \).

\( h \) is the length between two instants of discrete time

\( A_{h,k,j}^{(r,i)} \)

The values in "( )" indicate the instant of time.

\( k \) and \( j \) indicate the components of the \( A \). (see the example below)

Now, we give some definition of the notation in \( n \) \( D \) case, which is based on notation method (5.1.1). The method of denoting the \( A_{h,k,s}^{(r,i)} \) is similar to that of \( G_{h,k,s}^{(r,i)} \).

\[
h = \frac{T}{N} = a_i - a_{i-1} \quad \text{where} \quad i = 1, \ldots, N; N = m^2
\]

\[
A_{h,k,s}^{(r,i)} = \int_{arm+i+1}^{arm+i} (W^{(s)}(t) - W^{(s)}(arm+i+1))dW^{(k)}(t)
\]

\[
A_{h,k,s}^{(r)} = \sum_{i=0}^{m-1} A_{h,k,s}^{(r,i)}
\]

\[
\Delta W_{h}^{(r,i)\{k\}} = W^{(k)}(arm+i+1) - W^{(k)}(arm+i), \quad i = 0, \ldots, m - 1
\]

\[
G_{h,k,s}^{(r,i)} = W^{(k)}(arm+i+1) - W^{(k)}(arm+i), \quad (i = 0, \ldots, m - 1), \text{ when } s = 0
\]

\[
G_{h,k,s}^{(r,i)} = \frac{1}{h} E\{ A_{h,p,k}^{(r)} G_{h,p,(s-1)}^{(r,i)} W^{(k)} \}, \text{ when } s = 1, \ldots, n - k; \text{ in } nD \text{ case}
\]

\[
= \frac{1}{(s-1)!h^s} \int_{arm+i+1}^{arm+i+1} (arm+i+1 - t)^{s-1}(W^{(k)}(t) - W^{(k)}(arm+i+1))dt
\]

\[
G_{h,k,s}^{(r,i)} = 0 \quad \text{when} \quad s = n + 1 - k, \ldots, n - 1
\]

\[
G_{h,k,s}^{(r,i)} = \frac{1}{h} E\{ A_{h,p,k}^{(r)} G_{h,p,(s-1),R}^{(r,i)} W^{(k)} \}, \text{ where } s \geq 1
\]

\[
= \frac{1}{(s-1)\left(\frac{h}{2}\right)^s} \int_{arm+i+1}^{arm+i+1} [(arm+i+1 - t)^{s-1}]
\]

\[
(W^{(k)}(t) - W^{(k)}(arm+i+1)) \] }dt. \quad (5.1)

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5.2 Initial Stage

To bring convenient to our explanation, we introduce some vector variables here. Denote random values $G_{h,k}^{(r,i)}$ to be

$$G_{h,k}^{(r,i)} = (G_{h,k,0}^{(r,i)}, \ldots, G_{h,k,(n-k)}^{(r,i)})_{1 \times (n+1-k)}$$

Denote the area term $A_h^{(r)}$ to be

$$A_h^{(r)} = (A_{h,k,1}^{(r)}, \ldots, A_{h,k,(k-1)}^{(r)})_{1 \times (k-1)}$$

Denote the vector $b_{h,k,s}^{(r,i)}$ and matrix $B_{h,k}$ to be

$$b_{h,k,s}^{(r,i)} = (b_{h,k,s,0}^{(r,i)}, \ldots, b_{h,k,s,n-k}^{(r,i)})_{(n+1-k) \times 1}$$

$$B_{h,k}^{(r,i)} = (b_{h,k,1}^{(r,i)}, \ldots, b_{h,k,k-1}^{(r,i)})_{1 \times (k-1)}.$$ 

Denote the random variables $Z_{h,k}$ to be

$$Z_{h,k} = (Z_{h,k,1}, \ldots, Z_{h,k,k-1})_{1 \times (k-1)}$$

Denote the $(n+1-k) \times (n+1-k)$ covariance matrix $\Sigma_{h}^{G}$ and $(k-1) \times (k-1)$ matrix $\Sigma_{Z}^{h}$ to be

$$\Sigma_{h}^{G} = E((G_{h,k}^{(r,i)})^T G_{h,k}^{(r,i)})$$

$$\Sigma_{Z}^{h} = E(Z_{h,k}^T Z_{h,k}),$$

Denote a $(n+1-k) \times (n+1-k)$ matrix $M_k$

$$M_k = \begin{pmatrix} a_{0,0} & \ldots & a_{0,(n-k)} \\ \vdots & \ddots & \vdots \\ a_{(n-k),0} & \ldots & a_{(n-k),(n-k)} \end{pmatrix}_{(n+1-k) \times (n+1-k)}$$

$$a_{i,j} = \frac{h}{i!j!(i+j+1)}$$

Denote a $(n-s) \times (n-s)$ matrix $M_k^s$, (s indicates that the size of the matrix is $n-s$. k indicates that $M_k^s$ is similar to $M_k$, but making a shift.)

$$M_k^s = \begin{pmatrix} a_{1,1} & \ldots & a_{1,(n-s)} \\ \vdots & \ddots & \vdots \\ a_{(n-s),1} & \ldots & a_{(n-s),(n-s)} \end{pmatrix}_{(n-s) \times (n-s)}$$

$$a_{i,j} = \frac{h}{i!j!(i+j+1)},$$

when $i < n+1-k$ and $j < n+1-k$.

$$a_{i,j} = 0,$$

when $i \geq n+1-k$ or $j \geq n+1-k$.

Analogous to the 2D and 3D discussions, we are going to give some lemmas
revealing the property of the covariance terms and area terms.

Lemma 5.2.1. Let $hG_{h,k,s}^{(r)}$ be the conditional covariance $hG_{h,k,s}^{(r)} = E[A_{h,p,k}^{(r)} G_{h,p,s-1}^{(r)} W^{(k)}]$, and $G_{h,k}^{(r)}$ be a $1 \times (n + 1 - k)$ vector $(G_{h,k,0}, \ldots, G_{h,k,n-k}, 0, \ldots, 0)$. Then conditional covariance $G_{h,k}^{(r)}$ will have

$$\Sigma_{h,k}^{G} = E((G_{h,k}^{(r)})^T G_{h,k}^{(r)}) = M_k$$

$$E((G_{h,k}^{(r)})^T G_{h,k}^{(r)j}) = 0, (i \neq j).$$

(5.3)

**Proof:** According to the definition of the $G_{h,k,s}$ and $G_{h,k,p}$ in (5.1) equations, one can have

$$E[G_{h,k,s}^{(r)} G_{h,k,p}^{(r)}] = \frac{h}{s!p!(s + p + 1)},$$

when $s = 0, \ldots, n - k$ and $p = 0, \ldots, n - k$. $E[G_{h,k,s}^{(r)} G_{h,k,p}^{(r)}] = 0$ when $i \neq j$. So it comes to the conclusion. When we generate the random variables $G_{h,k,s}$, we want it have such a property as well.

□

Conditions 5.2.2. Let $G_{shift,k,p}^{(r)}$ be a $1 \times (n + 1 - p)$ vector

$$G_{shift,k,p}^{(r)} = (E[A_{h,p,k}^{(r)} G_{h,p,0}^{(r)} W^{(k)}(t)], \ldots, E[A_{h,p,k}^{(r)} G_{h,p,n-p}^{(r)} W^{(k)}(t)])_{1 \times (n+1-p)}.$$

Then $G_{shift,k,p}^{(r)} = h(G_{h,k,1}, \ldots, G_{h,k,n+1-p})_{1 \times (n+1-p)}$, having $(n+1-p) \times (n+1-p)$ matrix

$$E((G_{shift,k,p}^{(r)})^T G_{shift,k,p}^{(r)}) = h^2 M_k^{p-1}$$

(5.4)

**Proof:** From the definition of the $G_{h,k,s}$,

$$hG_{h,k,s}^{(r)} = E[A_{h,p,k}^{(r)} G_{h,p,s-1}^{(r)} W^{(k)}],$$

$(p = k + 1, \ldots, n; s = 1, \ldots, n + 1 - p)$, we have

$$G_{shift,k,p}^{(r)} = (hG_{h,k,1}, \ldots, hG_{h,k,n+1-p})_{1 \times (n+1-p)}.$$

Applying the lemma (5.2.1), we could have the conclusion.

□

Lemma 5.2.3. Let vector $A_{h,k}^{(r)}$ be $A_{h,k}^{(r)} = (A_{h,k,1}^{(r)}, \ldots, A_{h,k,k-1}^{(r)})_{1 \times (k-1)} \sim N(0, \Sigma^*)$, the $1 \times (n + 1 - k)$ vector $G_{h,k}^{(r)} (i = 0, \ldots, m - 1)$ be $G_{h,k}^{(r)} \sim N(0, \Sigma^* G_{h,k})$, $X$ be a vector $X = (G_{h,k}^{(r,0)}, \ldots, G_{h,k}^{(r,m-1)})$, $(k - 1) \times n$ covariance matrix $\Sigma_{h,k}$ to be $\Sigma_{h,k}^{G} = E[(A_{h,k}^{(r)})^T G_{h,k}^{(r)}]$, $(n + 1 - k) \times (k - 1)$ matrix $B_{h,k}^{(r)}$ be $(E_{h,k})^T = \Sigma_{h,k}^{G} (\Sigma_{h,k}^*)^{-1}$, and $(k - 1) \times (k - 1)$ matrix $\Gamma_k$ be conditional covariance matrix

$$\Gamma_k = Cov(A_{h,k}, A_{h,k} W^{(1)}(t), \ldots, W^{(k-1)}(t), X).$$
Consider the segment $J_r$. Conditional on the $W(t)$ $(s = 1, \ldots, k - 1)$ and $\Delta W(t)$ $(j = 0, \ldots, m - 1)$, $A_h$ has conditional expectation

$$E[A_h(t)^{(r)} W(t), \ldots, W^{(k-1)}(t)] = \sum_{i=0}^{m-1} G_h(t) B_h,$$

and conditional covariance matrix

$$\Gamma_h^{(r)} = E[(A_h(t)^{(r)} W(t), \ldots, W^{(k-1)}(t)] - \sum_{i=0}^{m-1} (B_h(t))^{T} \Sigma_h B_h.$$ 

**Proof:** Let vector $Y$ be $Y = (A_h(t), X)$. We have shown the conditional expectation and covariance matrix for 3D case in lemma (4.1.2) and Explanation (4.1.3), and that for 2D case in Explanation (3.1.3). Analogous to the 2D and 3D cases, conditional on $W(t)$ $(s = 1, \ldots, k - 1)$, $Y = (A_h(t), X)$ is joint normal distributed with mean 0.

Let the $A_h(t), \Sigma_h, G_h(t), B_h(t)$ be $X^T, \Sigma^T, \Sigma, D$ in the lemma (1.2.9), vector $P$ be $P = (W(t), \ldots, W^{(k-1)}(t), G_h(t), \ldots, G_h(t))$. Then, we can obtain the conditional expectation

$$E[A_h(t)^{(r)} W(t), \ldots, W^{(k-1)}(t)] = G_h(t) B_h(t),$$

$$E[A_h(t)^{(r)} P] = \sum_{i=0}^{m-1} G_h(t) B_h(t),$$

where $(B_h(t))^{T} = \Sigma_h(t)^{-1}.$

The lemma (1.2.9) also tells us the conditional covariance

$$\text{Cov}[A_h(t)^{(r)} | P] = E[(A_h(t)^{(r)} W(t), \ldots, W^{(k-1)}(t)] - \sum_{i=0}^{m-1} (B_h(t))^{T} \Sigma_h B_h(t),$$

$$= E[(A_h(t)^{(r)} W(t), \ldots, W^{(k-1)}(t)] - \sum_{i=0}^{m-1} \Sigma_h(t)^{-1} \Sigma_h(t)^{T}.$$

It would be helpful to take a look at the $B_h(t)$ more closely. Let $b_h(t)$ be $(n+1-k) \times 1$ vector, $B_h(t)$ be $(n+1-k) \times (k-1)$ matrix $B_h(t) = (b_h(t), \ldots, b_h(t))$. Because $A_h(t) = \sum_{i=0}^{m-1} G_h(t) b_h(t) + Z,$

$$E[A_h(t)^{(r)} W(t), \ldots, W^{(k-1)}(t)] = G_h(t),$$

we can have $1 \times (n+1-k)$ vector

$$E[A_h(t)^{(r)} G_h(t) W(t), \ldots, W^{(k-1)}(t)] = E[(G_h(t) b_h(t), \ldots, G_h(t) W(t), \ldots, W^{(k-1)}(t)]$$

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\[ G^{(r,i)}_{\text{shift},s,k} = (b^{(r,i)}_{h,k,s})^T E[(G^{(r,i)}_{h,k})^T G^{(r,i)}_{h,k} | W^{(1)}(t), \ldots, W^{(k-1)}(t)]. \]

Hence \( b^{(r,i)}_{h,k,s} \) be a \((n + 1 - k) \times 1\) vector

\[
(b^{(r,i)}_{h,k,s})^T = G^{(r,i)}_{\text{shift},s,k} (\Sigma_k^{G_h})^{-1} \tag{5.5}
\]

In the 3D case, we approximate the conditional covariance matrix \( \text{Cov}[A^{(r)}_{h,y} | W, \Delta Y] \) with \( E[(A^{(r)}_{h,y})^T A^{(r)}_{h,y}] - E^2[A^{(r)}_{h,y} | W] \), which is explained by the Explanation (4.1.3). Here, we will show the general situation.

Explanation 5.2.4. Let \( \Gamma^{(r)}_k \) be a matrix

\[
\begin{pmatrix}
\gamma_{1,1} & \cdots & \gamma_{1,k-1} \\
\gamma_{k-1,1} & \cdots & \gamma_{k-1,k-1}
\end{pmatrix}
\]

and vector \( P \) be \( P = (W^{(1)}(t), W^{(k-1)}(t), G^{(r,0)}_{h,k}, \ldots, G^{(r,m-1)}_{h,k}) \). Conditional on \( W^{(s)}(t), G^{(r)}_{h,k,0}, \ldots, G^{(r)}_{h,k,n-k} \) \((k = 2, \ldots, n; s = 1, \ldots, k - 1)\), we have the covariance matrix \( \Gamma^{(r)} \)

\[
\Gamma^{(r)} = \text{Cov}[A^{(r)}_{h,k}, A^{(r)}_{h,k} | P]
\]

When \( m \) is large, according to the Law of Large Number, we have

\[
\gamma_{s,j} = \text{Cov}[A^{(r)}_{h,k,s}, A^{(r)}_{h,k,j} | P] \approx E[(A^{(r)}_{h,k,s})^T A^{(r)}_{h,k,j}] - \sum_{i=0}^{m-1} (b^{(r,i)}_{h,k,s})^T \Sigma_k^{G_h} b^{(r,i)}_{h,k,j}.
\]

Because of \((b^{(r,i)}_{h,k,s})^T = G^{(r,i)}_{\text{shift},s,k} (\Sigma_k^{G_h})^{-1}\) we have

\[
\gamma_{s,j} = \text{Cov}[A^{(r)}_{h,k,s}, A^{(r)}_{h,k,j} | P] \approx E[(A^{(r)}_{h,k,s})^T A^{(r)}_{h,k,j}] - \sum_{i=0}^{m-1} G^{(r,i)}_{\text{shift},s,k} (\Sigma_k^{G_h})^{-1} (G^{(r,i)}_{\text{shift},j,k})^T.
\]

Consider the \( \gamma_{s,j} \) \((s \neq j)\).

When \( m \) is large, \( \sum_{i=0}^{m-1} G^{(r,i)}_{\text{shift},s,k} (\Sigma_k^{G_h})^{-1} (G^{(r,i)}_{\text{shift},j,k})^T \), will be closed to its expectation 0 and \( E[(A^{(r)}_{h,k,s})^T A^{(r)}_{h,k,j}] = 0 \). Hence we have \( \gamma_{s,j} \approx 0 \).

Consider the \( \gamma_{s,s} \).

Let the the conditional variance of \( A^{(r)}_{h,k,s} \) be \( V^{(r)}_{h,k,s} = \text{Var}[A^{(r)}_{h,k,s} | P] \). It is easy to get the unconditional variance of \( A^{(r)}_{h,k,s}, \text{Var}[A^{(r)}_{h,k,s}] = mh^2 / 2 \). Hence, when \( m \) is large, we have

\[
V^{(r)}_{h,k,s} = \gamma_{s,s} = E[A^{(r)}_{h,k,s} A^{(r)}_{h,k,s}] - \sum_{i=0}^{m-1} G^{(r,i)}_{\text{shift},s,k} (\Sigma_k^{G_h})^{-1} (G^{(r,i)}_{\text{shift},s,k})^T
\]

\[
\approx \frac{mh^2}{2} - \sum_{i=0}^{m-1} E[G^{(r,i)}_{\text{shift},s,k} (\Sigma_k^{G_h})^{-1} (G^{(r,i)}_{\text{shift},s,k})^T]. \tag{5.6}
\]
Theorem (5.2.7) reveals that the conditional variance
\[ V_{h,k,s}^{(r)} \approx \frac{mh^2(p-1)}{2(2p-1)(2p-3)} \quad \text{when } s = 1, \ldots, (k-1) \]
\[ V_{h,k,s}^{(r)} = 0 \quad \text{when } s = k, \ldots, n-1, \]
\[ p = n+2-k, \]
in the n dimension case, when m is large.

We explain the generation of the approximate sum of area terms here.

Explanation 5.2.5. Analogous to the 2D and 3D cases, \( \tilde{A}_{h,k,s}^{(r)} \) is a good approximation for \( A_{h,k,s}^{(r)} \), where
\[ \tilde{A}_{h,k,s}^{(r)} = \tilde{Z}_{h,k,s}^{(r)} + \sum_{i=0}^{m-1} G_{h,k}^{(r,i)} b_{h,k,s}^{(r,i)} \]

This approximate \( \tilde{A}_{h,k,s}^{(r)} \) has unconditional expectation 0, variance \( \frac{mh^2}{2} \), covariance \( \text{Cov}[A_{h,k}^{(r)}, G_{h,k}^{(r,i)}] = \Sigma_{h,k}^{(r,i)} = \text{Cov}[\tilde{A}_{h,k}^{(r)}, G_{h,k}^{(r,i)}] \), conditional expectation
\[ E[\tilde{A}_{h,k,s}^{(r)} | W^{(1)}(t), \ldots, W^{(k-1)}(t), X] = \sum_{i=0}^{m-1} G_{h,k}^{(r,i)} \tilde{b}_{h,k,s}^{(r,i)} \],
conditional covariance matrix
\[ \text{Cov}[\tilde{A}_{h,k,s}^{(r)}, \tilde{A}_{h,k,s}^{(r)} | W^{(1)}(t), \ldots, W^{(k-1)}(t), X] \approx \text{Cov}[A_{h,k}^{(r)}, A_{h,k}^{(r)} | W^{(1)}(t), \ldots, W^{(k-1)}(t), X], \]
and
\[ \text{Cov}[\tilde{A}_{h,k,s}^{(r)}, G_{h,k}^{(r,i)} | W^{(1)}(t), \ldots, W^{(k-1)}(t), X] \approx \text{Cov}[A_{h,k}^{(r)}, G_{h,k}^{(r,i)} | W^{(1)}(t), \ldots, W^{(k-1)}(t), X]. \]

Hence it is a good approximation to the \( A_{h,k,s}^{(r)} \).

We did not complete how to approximate the \( V_{h,k,s}^{(r)} \) in Explanation (5.2.4). We will complete it with the following theorems.

Theorem 5.2.6. Let vector \( G_{h,k}^{(r,i)} \) be \( G_{h,k}^{(r,i)} = (G_{h,k,0}^{(r,i)}, \ldots, G_{h,k,n-k}^{(r,i)})_1^{(n+1-k)} \), vector \( X = (G_{h,2}^{(r,0)}, \ldots, G_{h,2}^{(r,m-1)}) \), \( V_{h,2,1}^{(r)} \), be the conditional variance of \( A_{h,2,1}^{(r)} \) in \( n \) D (\( n \geq 3 \)) case, having
\[ \text{Var}[A_{h,2,1}^{(r)} | W^{(1)}(t), X] \approx E[A_{h,2,1}^{(r)} A_{h,2,1}^{(r)}] - E[G_{h,1}^{(r,i)} (\Sigma_{h,1}^{(r)})^{-1} (G_{h,1,1}^{(r)} X_{h,1}^{(r)})] \]

Then, when \( m \to \infty \), we have
\[ V_{h,2,1}^{(r)} \approx \frac{mh^2 p}{2(2p+1)(2p-1)} \]
\[ p = n-1. \]
in the n dimension case.
Proof: Let $\Sigma^G_\text{shift,i,h}$ be $\Sigma^G_\text{shift,i,h} = E[(C^{(r,i)}_\text{shift,i,k,h})^{T}(C^{(r,i)}_\text{shift,i,k,h})]$. 

In the $n$ dimension case, we have the conditional variance of $A^{(r)}_{h,2,1}$ driven by $nD$ Brownian trajectory from the equation (5.6), Explanation (5.2.4). That is

$$
\text{Var}[A^{(r)}_{h,2,1}|W^{(1)}_1, X] \simeq E[A^{(r)}_{h,2,1}A^{(r)}_{h,2,1}] - E[G^{(r)}_{\text{shift},i,2}(\Sigma^G_1)^{-1}(G^{(r)}_{\text{shift},i,2})^T] 
\simeq \frac{mh^2}{2} - \text{trace}((\Sigma^G_2)^{-1} \Sigma^G_1^\text{shift,i,h}),
$$

with $(n-1) \times (n-1)$ matrix, $(\Sigma^G_2)^{-1}$ and $\Sigma^G_1^\text{shift,i,h}$. Let $a_{ij}$ $(i, j = 1, \ldots, n)$ be components in $\Sigma^G_1$ or $\Sigma^G_1^\text{shift,i,h}$. According to the definition of $G_{\text{shift},k}$ and $G_{h,k}$, when $i = n$ or $j = n$, we should have $a_{ij} = 0$. The matrix $(\Sigma^G_2)^{-1}$ here is also a $(n-1) \times (n-1)$ matrix, having $a_{ij}$ when $i = n$ or $j = n$.

We are only interested in the $a_{ii}$ ($i = 1, \ldots, n-1$) components in the matrix $A$, since we want the trace$((\Sigma^G_2)^{-1} \Sigma^G_1^\text{shift,i,h})$. One can see $a_{n,n} = 0$ by investigating
the \((\Sigma^G_2)^{-1}\) and \(\Sigma^{G_{shift,h}}_1\). When \(i = j\), we have

\[
a_{ii} = \sum_{p=1}^{n-1} \left\{ b_{ip}^{(-1)} \frac{1}{p + i + 1} \frac{1}{i! p!} (i - 1)! (p - 1)! \right\}
\]

\[
= \frac{1}{(i + 1)i} \sum_{p=1}^{n-1} \left\{ b_{ip}^{(-1)} \left( \frac{1}{p} - \frac{1}{p + i + 1} \right) \right\}
\]

\[
= a_{ii1} - a_{ii2}
\]

\[
a_{ii1} = \frac{1}{(i + 1)i} \sum_{p=1}^{n-1} \left\{ b_{ip}^{(-1)} \frac{1}{p} \right\} = \begin{cases} 0 & \text{when } i \neq 1 \text{ and } i, j \leq n - 1 \\ 1/2 & \text{when } i = 1 \text{ and } i, j \leq n - 1, \end{cases}
\]

\[
a_{ii2} = \frac{1}{(i + 1)i} \sum_{p=1}^{n-1} \left\{ b_{ip}^{(-1)} \frac{1}{p + i + 1} \right\}.
\]

Because we could see \(\frac{1}{p}\) as \(\frac{1}{p + j - 1}\) with \(j = 1\). Similarly, \(a_{ii2}\) could be rewritten into

\[
a_{ii2} = \frac{1}{(i + 1)i} \sum_{p=1}^{n-1} \left\{ b_{ip}^{(-1)} \frac{1}{p + (i + 2) - 1} \right\}.
\]

Therefore, we could get \(a_{ii2} = 0\) when \(i \leq n - 3\) by formula (5.7).

With the above discussion, it comes to the conclusion

\[
\text{trace}(\Sigma^G_2) = a_{1,1} - a_{(n-2),(n-2)} - a_{(n-1),(n-1)},
\]

where \(a_{11} = \frac{1}{2}\),

\[
a_{(n-2),(n-2)} = \frac{1}{(n - 1)(n - 2)} \sum_{p=1}^{n-1} \left\{ b_{(n-2)p}^{(-1)} \left( \frac{1}{p + n - 1} \right) \right\},
\]

\[
a_{(n-1),(n-1)} = \frac{1}{(n - 1)n} \sum_{p=1}^{n-1} \left\{ b_{(n-1)p}^{(-1)} \left( \frac{1}{p + n} \right) \right\}.
\]

We are going to show the way to work out \(a_{(n-2),(n-2)}\) and \(a_{(n-1),(n-1)}\) in the following part.

Let

\[
f_i(x) = \left( \sum_{p=1}^{n-1} b_{ip}^{(-1)} \frac{1}{p + x - 1} \right) \prod_{p=1}^{n-1} (p + x - 1). \tag{5.8}
\]

\(f_i(x) = 0\) has \(n - 2\) solution, since equation (5.8). And equation (5.7) indicate \(f_i(j) = 0\) for \(j = 1 \ldots n - 1, j \neq i\), and \(f_i(x)\) is an \(n - 2\) degree polynomial. Therefore we have

\[
f_i(x) = C \prod_{j \neq i; j=1\ldots n-1} (x - j),
\]

where \(C\) will be get later. when \(x = i\), equation (5.8) shows \(f_i(i) = \prod_{p=1}^{n-1} (p + i - 1)\).
On the other hand, \( f_i(i) = C \prod_{j \neq i} (i - j) \). It comes out that

\[
C = \frac{\prod_{p=1}^{n-1} (p + i - 1)}{\prod_{j \neq i} (i - j)}.
\]

We have

\[
f_i(x) = \frac{\prod_{p=1}^{n-1} (p + i - 1)}{\prod_{j \neq i} (i - j)} \prod_{i \neq j} (x - j),
\]

where \( j = 1 \ldots n - 1 \).

When \( i = n - 2 \) and \( x = n, j = 1, \ldots, n - 3, n - 1 \), we have

\[
a_{(n-2),(n-2)} = -\frac{n - 2}{4(2n - 3)}.
\]

When \( i = n - 1; x = n + 1; j = 1, \ldots, n - 2 \), we have

\[
a_{(n-1),(n-1)} = \frac{n}{4[2n - 1]}.
\]

Hence we could draw the conclusion that

\[
V_{h,3,1} \simeq \frac{mh^2(n - 1)}{2(2n - 1)(2n - 3)}.
\]

**Theorem 5.2.7.** Let \( V_{h,k,s}^{(r)} \) be the conditional variance of \( A^{(r)}_{h,k,s} \) in the \( n \) \( (n \geq 3) \) dimension case. When \( m \to \infty \), the

\[
\tilde{V}_{h,k,s}^{(r)} \simeq V_{h,k,s}^{(r)}.
\]

where

\[
\tilde{V}_{h,k,s}^{(r)} = \frac{mh^2(p - 1)}{2(2p - 1)(2p - 3)} \text{ when } s = 1, \ldots, (k - 1)
\]

\[
\tilde{V}_{h,k,s} = 0 \text{ when } s = k, \ldots, n - 1,
\]

\[
p = n + 2 - k.
\]

**Proof:** Let \( 1 \times (n + 1 - k) \) vector \( G_{h,k}^{(r,i)} \) be \( G_{h,k}^{(r,i)} = (G_{h,k,0}^{(r,i)}, \ldots, G_{h,k,n-k}^{(r,i)})_{1 \times (n + 1 - k)} \), vector \( X \) be \( X = (G_{h,2}^{(r,0)}, \ldots, G_{h,2,n-1}^{(r,m-1)}) \). We have the conditional variance

\[
V_{h,2,1} = \text{Var}[A_{h,2,1}^{(r)}|W^{(1)}, G_{h,2,0}^{(r,i)}, \ldots, G_{h,2,n-2}^{(r,i)}], \quad (i = 0, \ldots, m - 1)
\]

\[
\simeq \frac{mh^2(n - 1)}{2(2n - 1)(2n - 3)}.
\]
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from the last theorem. $V_{h,k,s}$ in n dimension case can be considered as $V_{h,2,1}$ in $n + 2 - k$ dimension case

$$V_{h,k,s} = \text{Var}[A_{h,k,s}^{(r)}, W^{(s)}, G_{h,k,0}^{(r)}, \ldots, G_{h,k,n-k}^{(r)}], \quad (i = 0, \ldots, m - 1)$$

$$= \text{Var}[A_{h,k,s}^{(r)}, W^{(s)}, G_{h,k,0}^{(r)}, \ldots, G_{h,k,(n+2-k)-2}]$$

Hence we can draw the conclusion.

At this stage, we could present the method for generating the Brownian trajectory, which has enough information to approximate the sum of area terms.

**Method 5.2.8. (Initial Method)** We explain the initial method based on the notation in method (5.1.1). Suppose we have the $3/4$ Approximation for the n Dimension SDE on $[a, b]$ with step size $h = \frac{T}{n}$. That is:

1. **Generate Brownian path $W$ and relevant information $G$:** So as to make things simple, we consider how to generate the $W(rmh + mh) - W(rmh)$ and $G_{h,k,s}^{(r)}$ in the interval $I_r^{(r)}$ in $J_r = [a_{rm}, a_{rm+m}]$ only. And we only consider the approximate area $\bar{A}_{h,k}^{(r)}$ for $J_r = [a_{rm}, a_{rm+m}]$ in the next step.

   We generate such random variables $G_{h,k,s}^{(r)}$ $(i = 0, \ldots, m - 1)$ that

   $$E((G_{h,k}^{(r,i)})^T G_{h,k}^{(r,i)}) = M_k \quad (5.10)$$

   The lemma (5.2.1) explains how $M_k$ comes. The way to generate $G_{h,k}^{(r,i)}$ is shown in lemma (1.2.10). To be precise, let the vector $G_{h,k}^{(r,i)}$ and conditional covariance $\Sigma_k^{(r)}$ be $X$ and covariance $G$ in the lemma (1.2.10) respectively. One can obtain the random variables $G_{h,k}^{(r,i)}$. And we also have the Brownian sample path $W$ with $\Delta W_{h}^{(r,i)(k)} = G_{h,k}^{(r,i)}$.

   The reason why we approximate the $\Delta$ in this way is presented in explanation (5.2.5) and explanation (5.2.4).

2. **Generate approximate area $\bar{A}_{h,k}^{(r)}$:** Let vector $G_{shift,s,k}^{(r,i)}$ be $G_{shift,s}^{(r,i)} = (hG_{h,s,n+1-k}^{(r,i)}, \ldots, hG_{h,s,n-k}^{(r,i)}, 0, \ldots, 0)_{1 \times (n+1-k)}$, $(n + 1 - k) \times 1$ matrix $b_{h,k}^{(r,i)}$

   be $(b_{h,k,s}^{(r,i)})^T = G_{shift,s,k}^{(r,i)}(\Sigma_k^{(r)})^{-1}$. When $k > s$, generate $\tilde{Z}_{h,k,s}^{(r)} \sim N(0, \frac{mh^2(p-1)(2p-3)}{2(2p-1)(2p-3)})$, where $p = n + 2 - k$ ($k = 2, \ldots, n; s = 1, \ldots, k - 1$). Otherwise, when $k \leq s$, $\tilde{Z}_{h,k,s}^{(r)} = 0$.

   We approximate the area term $A_{h,k,s}^{(r)}$ $(k > s)$, with $\bar{A}_{h,k,s}^{(r)}$, where

   $$\bar{A}_{h,k,s}^{(r)} = \tilde{Z}_{h,k,s}^{(r)} + \sum_{i=0}^{m-1} G_{h,k,s}^{(r,i)}b_{h,k,s}^{(r,i)}$$

   The reason why we approximate the $A_{h,k,s}^{(r)}$ in this way is presented in explanation (5.2.5) and explanation (5.2.4).
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3. Now we achieve to generate the approximate Brownian path \( W(rm) - W((r-1)m) \) and the \( \tilde{A}^{(r)}_{h,k,s} \), which approximates the \( A^{(r)}_{h,k,s} (r = 0, \ldots, m-1) \),

\[
A^{(r)}_{h,k,s} = \sum_{i=0}^{m-1} \int_{a_{mr+i}}^{a_{mr+i+1}} [W^{(s)}(t) - W^{(s)}(a_{mr+i})] dW^{(k)}(t)
\]

for the time interval \( J_r \). Analogously, we are able to generate \( W \) and \( \tilde{A}^{(r)}_{h,k,s} \) for other time intervals \( J_0, \ldots, J_{m-1} \).

Remark: In the method, we generate three things: first, the increments of the Brownian path \( W \) over the steps of size \( h \); second, the conditional covariance term \( G^{(r,i)}_{h,k,s} (k = 1, \ldots, n; r, i = 0, \ldots, m-1; j = 0, \ldots, n - k) \); third, normal distributed random variable \( Z^{(r)}_{h,k,s} \sim N(0, \tilde{V}^{(r)}_{h,k,s}) (k = 2, \ldots, n; s = 1, \ldots, k - 1) \), whose variance approximates the variance of the \( A^{(r)}_{h,k,s} \). The explanation (5.2.5) and (5.2.4) explain the variance \( \tilde{V}^{(r)}_{h,k,s} \). Hence, we are able to approximate the sum of area terms, \( A^{(r)}_{h,k,s} \), with \( \tilde{A}^{(r)}_{h,k,s} \).

5.3 Subdivision Process

The last section reveal how to get the \( \tilde{A}^{(r)}_{h,k,s} \) term as well as the Brownian path \( W \). However it is complicated that splitting the approximate Brownian trajectory \( W_h \) over equal distant partition with step size \( h \) into step size \( h/2 \). We are going to achieve this in the following subsections.

5.3.1 The \( J \) Subdivision Process

Analogous to the 2D and 3D cases, we plan to split the time interval with two process: first, split the \( J_r \) \((r = 0, \ldots, m - 1)\) into \( J_{r,L} \) and \( J_{r,R} \) and rename them \( J_{2r} \) and \( J_{2r+1} \); second, split up the \( J_{i}^{(r)} \) in \( J_r \) into two pieces. By running the first process once and the second one twice, the step size and number of \( J \) intervals will change from \( h \) and \( m \) to \( \frac{h}{4} \) and \( 2m \) respectively. This process gives a similar setup with \( m \) replaced by \( 2m \) for the time interval setting.

OK! According to our plan, we subdivide the \( J_r \) \((r = 0, \ldots, m - 1)\) into two first. Such a process is presented in the figure (5.1).

Method 5.3.1. Suppose we have a fixed \( J_r \), which is divided into \( m \) equal intervals, \( J^{(r)}_0, J^{(r)}_1, \ldots, J^{(r)}_{m-1} \), with length \( h \), having associated random variables \( \tilde{Z}^{(r)}_{h,k,s} \) \((k = 2, \ldots, n; s = 1, \ldots, k - 1)\), \( \tilde{A}^{(r)}_{h,k,s} \) \((k = 2, \ldots, n; s = 1, \ldots, k - 1)\) and \( G^{(r)}_{h,k,s} \) \((k = 1, \ldots, n; s = 0, \ldots, n - k)\). (See first and second lines in figure (5.1).)

1. Divide segments into two groups. \( J^{(r)}_0, J^{(r)}_1, \ldots, J^{(r)}_{m-1} \in J_{r,L} \) and \( J^{(r)}_m, J^{(r)}_m, \ldots, J^{(r)}_m \in J_{r,R} \). Divide \( G^{(r)}_{h,k,s} \) into two groups. Hence
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\[ \begin{array}{|c|c|c|c|c|c|c|}
\hline
A & J_0 & J_1 & \cdots & J_r & \cdots & J_{m-2} & J_{m-1} & B \\
\hline
J_{r,L} & J_{r,R} & I_0^{(r)} & I_1^{(r)} & \cdots & I_{m-1}^{(r)} & \frac{m-1}{2} & \frac{m}{2} \\
\hline
a_0^{(r,0)} & a_1^{(r,1)} & G_{h,k,s}^{(r,0)} & G_{h,k,s}^{(r,1)} & \cdots & G_{h,k,s}^{(r,m-1)} & G_{h,k,s}^{(r,m/2-1)} & G_{h,k,s}^{(r,m+2/2)} \\
\hline
J_{2r} & J_{2r+1} & I_0^{(2r)} & I_1^{(2r)} & \cdots & I_{m-1}^{(2r)} & \frac{m-1}{2} & \frac{m}{2} \\
\hline
G_{h,k,s}^{(2r,0)} & G_{h,k,s}^{(2r,1)} & G_{h,k,s}^{(2r,m/2-1)} & G_{h,k,s}^{(2r,m+2/2)} & G_{h,k,s}^{(2r+1,0)} & G_{h,k,s}^{(2r+1,m/2-1)} \\
\hline
\end{array} \]

Figure 5.1: The J subdivision

\[ G_{h,k,s}^{(r,0)}, \ldots, G_{h,k,s}^{(r,m/2-1)} \text{ for } J_{r,L} \]
\[ \text{and } G_{h,k,s}^{(r,m/2)}, \ldots, G_{h,k,s}^{(r,m-1)} \text{ for } J_{r,R}. \] (See second line of the figure (5.1).)

2. Renumber \( J_{r,L} \) and \( J_{r,R} \) with \( J_{2r} \) and \( J_{2r+1} \) respectively. Renumber \( G_{h,k,s}^{(r,0)}, \ldots, G_{h,k,s}^{(r,m/2-1)} \) in \( J_{r,L} \) with \( G_{h,k,s}^{(2r,0)}, \ldots, G_{h,k,s}^{(2r,m/2-1)} \). And Renumber \( G_{h,k,s}^{(r,m/2)}, \ldots, G_{h,k,s}^{(r,m-1)} \) in \( J_{r,R} \) with \( G_{h,k,s}^{(2r+1,0)}, \ldots, G_{h,k,s}^{(2r+1,m/2-1)} \). (See the third line of the figure (5.1).)

3. Split \( \mathcal{Z}_{h,k,s,L}^{(r)} \) into \( \mathcal{Z}_{h,k,s,L}^{(r)} \) and \( \mathcal{Z}_{h,k,s,R}^{(r)} \), such that \( \mathcal{Z}_{h,k,s}^{(r)} = \mathcal{Z}_{h,k,s,L}^{(r)} + \mathcal{Z}_{h,k,s,R}^{(r)} \). Hence the \( J_{2r} \) and \( J_{2r+1} \) have corresponding \( \mathcal{Z}_{h,k,s}^{(2r)} \) and \( \mathcal{Z}_{h,k,s}^{(2r+1)} \).

Here we split \( J_r \) into \( J_{2r} \) and \( J_{2r+1} \). This process also brings us corresponding \( G_{h,k,s}^{(2r,0)} \), \( \mathcal{Z}_{h,k,s}^{(2r)} \), and \( G_{h,k,s}^{(2r+1,0)} \), \( \mathcal{Z}_{h,k,s}^{(2r+1)} \) \((i=0, \ldots, m/2 - 1)\) for the new \( J_{2r} \) and \( J_{2r+1} \) respectively.

### 5.3.2 Subdivision of I

After the J subdivision or the first I subdivision, we have the first line of the figure (5.2). According to our plan, we are dividing each interval \( I_i^{(r)} \) into two. So are the \( G_{h}^{(r)} \).

This process can be seen in the figure above, which subdivide every \( I_i^{(r)} \) \((i=0, \ldots, q)\) in the \( J_r \) into two obtaining \( I_i \) \((i=0, \ldots, 2q + 1)\). Renumber the variables, we get the third line of the figure (5.2). To explain the I subdivision precisely, we introduce the following notations.
Number the time instant in the $J_r$ with $a_i$ ($i = 0, \ldots, q + 1$). In the first $I$ subdivision, $q = \frac{m-2}{2}$, whereas $q = m - 1$ in the second $I$ subdivision process. This is because the number of $I$ segments in $J_r$ was doubled in the first $I$ subdivision. Denote the time intervals $I^{(r)}_i$ ($i = 0, \ldots, q$) in $J_r$ with $I_0, \ldots, I_q$ for short. See line one in figure (5.2).

2. For interval $I_i$, we have corresponding $G^{(r,i)}_{h,l,k}$ ($i = 0, \ldots, q; l = 1, \ldots, n; k = 0, \ldots, n - l$). (In the first $I$ subdivision, $q = \frac{m-2}{2}$. In the second $I$ subdivision $q = m - 1$. This is because, at the end of the first subdivision, which is shown in the 3rd line of the figure (5.2), we denote $G^{(r,0)}_{h,l,k,L}, G^{(r,0)}_{h,l,k,R}, \ldots, G^{(r,mz^2)}_{h,l,k,L}, G^{(r,mz^2)}_{h,l,k,R}$ in order with $G^{(r,i)}_{h,l,k} (i = 0, \ldots, m - 1).$)

3. Before the first $I$ subdivision process, we have $\tilde{Z}^{(r)}_{h,k,s}$ ($k = 2, \ldots, n; s = 1, \ldots, k - 1$). After one $I$ subdivision, we generate the $\tilde{Z}^{(r)}_{\frac{1}{2},k,s}$.

We will adopt the following notations: Let $N^{(r,i)}_{k,s}$ ($k = 1, \ldots, n$) be variable, and $N^{(r,i)}_k$ be vector,

\[ N^{(r,i)}_{k,s} \sim N(0,1) \quad \text{when} \quad s = 0, \ldots, n - k \]
\[ N^{(r,i)}_k = (N^{(r,i)}_{k,0}, \ldots, N^{(r,i)}_{k,(n-k-1)})_{1 \times (n+1-k)} \]
Let \( G_{h,k,s,R}^{(r,i)} \) be variables, \( G_{h,k,s,L}^{(r,i)} \) be vector \((i = 0, \ldots, q)\):

\[
G_{h,k,s,R}^{(r,i)} = \frac{1}{h} E\{A_{\frac{1}{2},p,k}^{(r)} G_{h,p,(s-1),R}^{(r,i)}|W^{(k)}\} , \text{ when } s = 1, \ldots, n - k
\]

\[
G_{h,k,0,R}^{(r,i)} = \Delta W^{(k)}(r,2^{+1})
\]

\[
G_{h,k,R}^{(r,i)} = (G_{h,k,0,R}^{(r,i)}, \ldots, G_{h,k,(n-k),R}^{(r,i)})_{1x(n+1-k)}
\]

Now we investigate some properties of the \( G_{h,k,s,a,R}^{(r,i)}, G_{h,k,s,L}^{(r,i)}, G_{h,k,a}^{(r,i)}, \) and \( A_{h,k,s}^{(r)} \) by the following lemma.

**Lemma 5.3.3.** Let \( A_{h,p,k}^{(r)} \) be area term defined in equations (5.1), \( A_{h,p,k}^{(r)} \) be sum of area terms for the segment \( J_r \) before the I Subdivision \( A_{h,p,k}^{(r)} = \sum_{i=0}^{q} A_{h,p,k}^{(r,i)} \), \( A_{\frac{1}{2},p,k}^{(r)} \) be sum of area terms for the segment \( J_r \) after one I Subdivision \( A_{\frac{1}{2},p,k}^{(r)} = \sum_{i=0}^{q+1} A_{\frac{1}{2},p,k}^{(r,i)} \) \( R_{k,s} \) be a \((n + 1 - k) \times (n + 1 - k)\) matrix \( R_{k,s} = E[G_{h,k,s,R}^{(r,i)} G_{h,k,s,R}^{(r,i)}]^T \), \( R_{j} \) be a \((n + 1 - k) \times (n + 1 - k)\) matrix \( R_{j} = E[(G_{h,j,R}^{(r,i)})^T G_{h,j,R}^{(r,i)}] \), having \( R_{k} = (R_{k,0}, \ldots, R_{k,n-k})^T \), \( D_{k} \) be a \((n + 1 - k) \times (n + 1 - k)\) matrix \( D_{k} = (R_{k,\Sigma_{k}^{-1}})^T \), \( D_{k} = (D_{k,0}, \ldots, D_{k,n-k})_{1x(n+1-k)} \), \( C_{k} \) be \( C_{k} = (C_{k,0}, \ldots, C_{k,n-k}) \), \( C_{k}^T C_{k} = \frac{1}{2} \Sigma_{k} - D_{k}^{-1} D_{k} \), and \( C_{k}^T C_{k} = \frac{1}{2} E[G_{h,k,s,R}^{(r,i)} G_{h,k,s,R}^{(r,i)}] - R_{k,s}(\Sigma_{k}^{-1}) R_{k,s} \).

We known \( G_{h,k}^{(r,i)} \) but do not know \( G_{h,k,s,R}^{(r,i)} \). Then, conditional on \( G_{h,k,s}^{(r,i)} (k = 2, \ldots, n \text{ and } s = 0, \ldots, n - k) \) we can generate \( G_{h,k,s,R}^{(r,i)} \) with

\[
G_{h,k,s,R}^{(r,i)} = G_{h,k}^{(r,i)} D_{k,s} + N_{k}^{(r,i)} C_{k,s}
\]

\[
G_{h,k,R}^{(r,i)} = G_{h,k}^{(r,i)} D_{k} + N_{k}^{(r,i)} C_{k}
\]

**Proof:** Let the \( R_{k}, G_{h,k}^{(r,i)}, G_{h,k,R}^{(r,i)}, D_{k}, C_{k}, \Sigma_{k}^h \) be \( \tilde{\Sigma}, X^T, X_L^T, D, C, \Sigma \) in the lemma (1.2.11).

Because \( G_{h,k,R}^{(r,i)} \) consider only the later half of the segment \([a_{i+1,1}, a_{i+1}]\) and \( G_{h,k,s}^{(r,i)} \) is over the segment \([a_{i}, a_{i+1}]\), we could know \( E[G_{h,k,s,R}^{(r,i)} G_{h,k,s,R}^{(r,i)}] = \frac{1}{2} E[(G_{h,k,s,R}^{(r,i)})^T G_{h,k,s,R}^{(r,i)}] \). Hence, we can let \( \frac{1}{2} \Sigma_{k}^{h} \) be the matrix \( \Sigma \) in lemma (1.2.11).

Applying the lemma (1.2.11), one can get \( G_{h,k,R}^{(r,i)} = G_{h,k}^{(r,i)} D_{k} + N_{k}^{(r,i)} C_{k} \).

Let the \( R_{k,s}, G_{h,k,s,R}^{(r,i)}, G_{h,k,s,R}^{(r,i)} \), \( D_{k,s}, C_{k,s}, \Sigma_{k}^{h} \) be \( \tilde{\Sigma}, X^T, X_L^T, D, C, \Sigma \) in the lemma (1.2.11). One can get \( G_{h,k,s,R}^{(r,i)} = G_{h,k}^{(r,i)} D_{k,s} + N_{k}^{(r,i)} C_{k,s} \).

\[\Box\]

**Lemma 5.3.4.** When \( s = 0 \), \( G_{h,k,s,L}^{(r,i)} = G_{h,k,s,L}^{(r,i)} - G_{h,k,s,R}^{(r,i)} \) \((i = 0, \ldots, q)\).

When \( s = 1 \), \( G_{h,k,s,L}^{(r,i)} = 2G_{h,k,s,R}^{(r,i)} - G_{h,k,0,R}^{(r,i)} - G_{h,k,s,R}^{(r,i)} \).

When \( s = 2, \ldots, n - k \),

\[
G_{h,k,s,L}^{(r,i)} = [s - 1]G_{h,k,s,L}^{(r,i)} - \Delta W_{h,L}^{(k)(r,i)} \left( \frac{h}{2} \right) \left( \frac{s}{s-1} \right) - G_{h,k,s,R}^{(r,i)} (s - 1)! \left( \frac{h}{2} \right) s
\]

\[
- \sum_{p=0}^{s-2} \left( \frac{s - 1}{p} \right) h^{s-1} p (p+1)! G_{h,k,(p+1),L}^{(r,i)} \left( \frac{h}{2} \right) \left( \frac{s}{s-1} \right)
\]

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Proof:

\[ G_{h,k,0,L}^{(r,i)} = G_{h,k,0}^{(r,i)} - G_{h,k,R}^{(r,i)} = \Delta W_{h}^{(k)(r,i)} - \Delta W_{h,R}^{(k)(r,i)} = \Delta W_{h,L}^{(k)(r,i)} \]

According to the definition of \( G_{h,j,k}^{(r,i)} \), we have

\[
G_{h,j,k}^{(r,i)} = \frac{1}{(k-1)!h^k} \int_{a_{rm+i-1}}^{a_{rm+i}} (a_{rm+i} - s)^{k-1} (W^{(j)}(s) - W^{(j)}(a_{rm+i-1})) ds
\]

\[
= \frac{1}{(k-1)!h^k} \sum_{p=0}^{k-1} \left( \frac{k-1}{p} \right) \left( \frac{h}{2} \right)^{k-p-1} p!(\frac{h}{2})^{p+1} G_{h,j,(p+1)\cdot L}^{(r,i)}
\]

\[ + G_{h,j,k,R}^{(r,i)} \frac{h}{2} \] + \( \Delta W_{h,L}^{(j)} \left( \frac{h}{2}, \frac{1}{k} \right) \),

then we can draw the result. We can write it in concise form

\[ G_{h,j,k}^{(r,i)} = G_{h,j,k}^{(r,i)} D_{k}^{(r,i)} + N_{k}^{(r,i)} C_{k}^{(r,i)} \] (5.11)

\[ \square \]

Lemma 5.3.5. In the segment \( J_{r} \), let \( A_{h,k,s}^{(r,i)} \) be area term defined in equations (5.1), \( A_{h,k,i}^{(r)} \) be the sum of area terms, which are not I subdivided, having \( A_{h,k,i}^{(r)} = \sum_{i=0}^{q} A_{h,k,i}^{(r,i)} \). \( A_{h,i,k}^{(r)} \) be the sum of area terms, which are I subdivided once, having \( A_{h,i,k}^{(r)} = \sum_{i=0}^{q} A_{h,i,k}^{(r,i)} \). Let the value \( F_{k,i}^{(r)} \) be \( F_{k,i}^{(r)} = \sum_{i=0}^{q} C_{h,k,0,R}^{(r,i)} G_{h,s,0,L}^{(r,i)} \). Then we have

\[ A_{h,k,i}^{(r)} = A_{h,i,k}^{(r)} + F_{k,i}^{(r)} \]

Proof: Let's consider the definition of the sum of area terms.

Area term before split: (The \( a_{i} \) in here are the renumbered ones which are given in method (5.3.1) and the third line of the figure (5.1).)

\[ A_{h,k,i}^{(r,i)} = \int_{a_{i}}^{a_{i+1}} \left( W^{(s)}(t) - W^{(s)}(a_{i}) \right) dW^{(k)}(t) \] (from equation (2.1))

\[ = \int_{a_{i}}^{a_{i+1}} \left( W^{(s)}(t) - W^{(s)}(a_{i}) \right) dW^{(k)}(t)
\]

\[ + \int_{a_{i+2}^{a_{i+1}}} \left( W^{(s)}(t) - W^{(s)}(a_{i}) \right) dW^{(k)}(t)
\]

\[ + [W^{(s)}(\frac{a_{i} + a_{i+1}}{2}) - W^{(s)}(a_{i})] [W^{(k)}(a_{i} + a_{i+1}) - W^{(k)}(\frac{a_{i} + a_{i+1}}{2})] \]

Area term after split: (See method (5.3.1) and the second line of the figure (5.2)
for the definition of $a_i'$)

$$A_{\frac{1}{2},k,s}^{(r)} = \int_{a_i'}^{a_{i+1}} (W(s)(t) - W(s)(a_i')) dW(k)(t)$$

where $\{a_{2i} = a_i, a_{2i+1} = \frac{a_i + a_{i+1}}{2}\}$.

So it is clear that

$$A_{h,k,s}^{(r)} = \sum_{i=0}^{q} A_{h,k,s}^{(r,i)} = \sum_{i=0}^{2q+1} [A_{\frac{1}{2},k,s}^{(r,i)} + G_{h,k,0,R}^{(r,i)} G_{h,s,0,L}^{(r,i)}] = A_{\frac{1}{2},k,s}^{(r)} + F_{k,s}^{(r)}$$

$$A_{h,k}^{(r)} = A_{\frac{1}{2},k}^{(r)} + F_{k}^{(r)}.$$ 

At this stage, we summarise the conditions that we want the variables will satisfy after subdivision:

**Conditions 5.3.6.** Let $A_{h,p,k}^{(r)} (p = 2, \ldots, n; k = 1, \ldots, p-1)$ be sum of area terms for the segment $J_r$ before the I Subdivision $A_{h,p,k}^{(r)} = \sum_{i=0}^{q} A_{h,p,k}^{(r,i)}$, $A_{\frac{1}{2},p,k}^{(r)}$ be sum of area terms for the segment $J_r$ after one I Subdivision $A_{\frac{1}{2},p,k}^{(r)} = \sum_{i=0}^{2q+1} A_{\frac{1}{2},p,k}^{(r,i)}$, $hG_{h,k,a}^{(r,i)} (k = 1, \ldots, n; s = 0, \ldots, n-k)$ be conditional covariance, $hG_{h,k,s}^{(r)}$ be conditional covariance. Then we have $(n+1-k) \times (n+1-k)$ matrix $E((G_{h,k,R}^{(r,i)})^{T} G_{h,k,R}^{(r,i)}) = \frac{1}{2} E((G_{h,k}^{(r,i)})^{T} G_{h,k}^{(r,i)})$,

$$G_{h,k,0,L}^{(r,i)} = G_{h,k,0}^{(r,i)} - G_{h,k,0,R}^{(r,i)} (i = 0, \ldots, q)$$

$$G_{h,k,1,L}^{(r,i)} = 2G_{h,k,1}^{(r,i)} - G_{h,k,0,L}^{(r,i)} - G_{h,k,1,R}^{(r,i)} (i = 0, \ldots, q)$$

$$G_{h,k,s,L}^{(r,i)} = [(s-1)G_{h,k,s}^{(r,i)} + \Delta W_{h,L}^{(r,i)}] \left(\frac{h}{2}\right)^{s-1} - G_{h,k,s,R}^{(r,i)}(s-1)\left(\frac{h}{2}\right)^{s-2}$$

$$ - \sum_{p=0}^{s-2} \binom{s-1}{p} \left(\frac{h}{2}\right)^{s-1-p} h^{p+1} G_{h,k,(p+1,L)}^{(r,i)} \frac{1}{s}$$

when $s = 2, \ldots, n-k$. And we also know the $R_{j,k} = E[G_{h,j,k,R}^{(r,i)}]$.

**Proof:** One can prove $E((G_{h,k,R}^{(r,i)})^{T} G_{h,k,R}^{(r,i)}) = \frac{1}{2} E((G_{h,k}^{(r,i)})^{T} G_{h,k}^{(r,i)})$ from the definition directly. One can also get the vector $R_{j,k}$ from the definition. The other results are presented in lemma (5.3.4).

We are introducing the I subdivision method and explain it in the following subsections.

**Method 5.3.7.** Suppose there are $q + 1$ time intervals $I_i (i = 0, \ldots, q)$ in $J$. Based on the notation method (5.3.2), we apply the following steps to split up the
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We will split up $G_{h_1,s}^{(r)} (s = 0, \ldots, n - 1)$ for a particular $r$ and $i$ first, then $G_{h_2,s}^{(r)}, G_{h_3,s}^{(r)}, \ldots, G_{h,n,s}^{(r)} (s = 0, \ldots, n - 1)$ one by one.

1. **Generate $G_{h,k,R}^{(r)}$ and $G_{h,k,L}^{(r)} (i = 0, \ldots, q)$:**
   - When $k = 1$, we generate independent $N_{k,s}^{(r)} \sim N(0, 1)$. When $k = 2, \ldots, n$, we obtain $N_k$ during the splitting of the $Z_{h,k,1}^{(r)}, \ldots, Z_{h,k,k-1}^{(r)}$, which will be stated later. We can generate the $G_{h,k,R}^{(r)}$ by
     \[
     G_{h,k,s}^{(r)} = \sum_{p=0}^{n-k} G_{h,k,p}^{(r)} D_{k,s,p} + \sum_{p=0}^{n-k} N_{k,p}^{(r)} C_{k,s,p} 
     \]
   \[
     \begin{align*}
     G_{h,k,s}^{(r)} &= G_{h,k,s}^{(r)} D_{k,s} + N_k^{(r)} C_{k,s}, \\
     G_{h,k,s}^{(r)} &= G_{h,k,s}^{(r)} D_{k,s} + N_k^{(r)} C_k
     \end{align*}
     \]  
   (5.12)

   where $C_{k,s}$ is a vector $C_{k,s} = (C_{k,s,0}, \ldots, C_{k,s,n-k})^T$, where $s = 0, \ldots, n - k$, $p = 0, \ldots, n - k$. Matrices $D_{k,s}$ and $C_{k,s}$ are shown in lemma (5.3.3).

   We can obtain the $G_{h,k,L}^{(r)}$ with $G_{h,k,L}^{(r)} = G_{h,k}^{(r)} D'_k + N_k C'_k$, where $D'_k$ and $C'_k$ are stated in lemma (5.3.4).

2. **Get $F_k$ and $H_k$:**
   - Let the variable $F_{k,s}^{(r)} = 0$ be
     \[
     F_{k,s}^{(r)} = 0, \text{ when } s = k, \ldots, n - 1,
     \]
     \[
     F_{k,s}^{(r)} = \sum_{i=0}^{q} G_{h,k,0,i} G_{h,s,0,L}^{(r)}, \text{ when } s = 1, \ldots, k - 1.
     \]

     $F_k^{(r)}$ be a vector $F_k^{(r)} = (F_{k,1}^{(r)}, F_{k,2}^{(r)}, \ldots, F_{k,k-1}^{(r)})$,
     $H_k^{(r)}$ be a vector $H_k^{(r)} = (H_{k,1}^{(r)}, \ldots, H_{k,s}^{(r)}, \ldots, H_{k,k-1}^{(r)})$ 1x$(k-1)$, $(i = 0, \ldots, q)$.

     When $s = 1, \ldots, k - 1$, solve the
     \[
     \begin{align*}
     N_k^{(r)} C_k H_{k,s}^{(r)} &= G_{h,k,s}^{(r)} b_{h,k,s}^{(r)} - G_{h,k,s,L}^{(r)} b_{h,k,s,L}^{(r)} - G_{h,k,R}^{(r)} b_{h,k,s,R}^{(r)} - F_{k,s}^{(r)} \\
     b_{h,k,s}^{(r)} &= ((\Sigma_k^{(r)})^{-1})^T (G_{h,k,s}^{(r)})^T \\
     b_{h,k,s,L}^{(r)} &= 2((\Sigma_k^{(r)})^{-1})^T (G_{h,k,s,L}^{(r)})^T \\
     b_{h,k,s,R}^{(r)} &= 2((\Sigma_k^{(r)})^{-1})^T (G_{h,k,s,R}^{(r)})^T \\
     C_{h,k,L}^{(r)} &= G_{h,k,L}^{(r)} D_k^{(r)} + N_k^{(r)} C_k^{(r)} \\
     G_{h,k,R}^{(r)} &= G_{h,k,R}^{(r)} D_k^{(r)} + N_k^{(r)} C_k
     \end{align*}
     \]

     to obtain the $H_{k,s}^{(r)}$. The reason for deriving the variable $H_{k,s}^{(r)}$ is given in the Explanation (5.3.8).

3. **Get the matrix $P$ and threshold value $Q$:**
   - Get the $(n+1-k) \times (n+1-k)$
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matrix \( P \), which satisfies

\[
\sum_{i=0}^{q} N_{k}^{(r,i)} C_k H_k^{(r,i)} \sim N(0, P(q + 1)h^2).
\]

Then we are able to get the threshold value

\[
Q = \sum_{i=0}^{q} (H_k^{(r,i)})^T C_k^T P^{-1} C_k (H_k^{(r,i)}) \frac{1}{(q + 1)h^2} - E_n,
\]

where \( E_n = (e_{i,j}) \) is a \((n + 1 - k) \times (n + 1 - k)\) unit matrix. The reason for generating the threshold value is given by the Explanation (5.3.9)

4. Generate \( N_k^{(r,i)} \): When \( |Q| < \frac{1}{2} \), we can apply the method in lemma (1.2.12) to generate the \( N_k^{(r,i)} \) in a way that

\[
\begin{align*}
N_k^{(r,i)} &\sim N(0, 1), \quad (k = 1, \ldots, n; \quad s = 0, \ldots, n - k; \quad i = 0, \ldots, q) \quad \text{and} \quad N_k^{(r,i)} = (N_k^{(r,i)}, \ldots, N_k^{(r,i)}, 0, \ldots, 0)_{1 \times n} \\
\hat{Z}_{h,k}^{(r)} &\sim \hat{Z}_{h,k}^{(r)} + \sum_{i=0}^{q} N_k^{(r,i)} C_k H_k^{(r,i)}
\end{align*}
\]

We can obtain the \( N_k^{(r,i)}, N_k^{(r,i)}, \ldots, (i = 0, \ldots, q) \) one by one by the method in lemma (1.2.12), and finally \( \hat{Z}_{h,k}^{(r)} = \hat{Z}_{h,k}^{(r)} + \sum_{i=0}^{q} N_k^{(r,i)} C_k H_k^{(r,i)} \). We generate \( N_k^{(r,1)}, N_k^{(r,1)}, \ldots, N_k^{(r,1)}(i = 2, \ldots, q) \) one by one first, and then \( N_k^{(r,1)}, N_k^{(r,1)}, \ldots, N_k^{(r,1)}(i = 2, \ldots, q) \). The equation \( \hat{Z}_{h,k}^{(r)} = \hat{Z}_{h,k}^{(r)} + \hat{Z}_{h,k}^{(r)} + \) gives us a proper correction term \( \hat{Z}_{h,k}^{(r)} \) having mean 0 and covariance matrix \( \frac{1}{2} \Sigma_{h}^{k} \). The reason for \( T_k \) is stated in Explanation (5.3.10).

When \( |Q| > \frac{1}{2} \), we generate \( \hat{Z}_{h,k}^{(r)} \sim \hat{Z}_{h,k}^{(r)} \) and \( N_{k,s} \sim N(0, 1) \) \((k = 1, \ldots, n; s = 0, \ldots, n - k; i = 0, \ldots, q)\).

We left several issues which did not explain in the method (5.3.7). The following Explanations and lemma will sort these out.

**Explanation 5.3.8.** We are explaining why we derive the variable \( H_{h,k}^{(r)} \) here.

Corresponding to the Explanation (3.2.6) and (4.2.8), we will need to derive the threshold value \( Q \). We also need to reveal the relationship between the sum of area terms \( A_{h,k}^{(r)} \) and the sum of area terms \( A_{h,k}^{(r)} \) being I subdivided.

Let the sum of area terms, which are not I subdivided, \( A_{h,k}^{(r)} \). Let the sum of area terms, which are I subdivided once, be \( A_{h,k}^{(r)} \). Let the value \( F_{h,k}^{(r)} \) be \( F_{h,k}^{(r)} = \sum_{i=0}^{q} G_{h,k,0,0}, G_{h,k,0,L}^{(r)} \). Then we have \( A_{h,k}^{(r)} = A_{h,k}^{(r)} + F_{h,k}^{(r)} \). One can find the detail proof of this in lemma (5.3.5).

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We are going to approximate $A_{r,k,s}$ and $A_{r,k,s}^{(r)}$ with $\tilde{A}_{r,k,s}^{(r)}$ and $\tilde{A}_{r,k,s}^{(r)}$ respectively. From the Explanation (5.2.5), we know that $A_{r,k,s}^{(r)} = \tilde{z}_{r,k,s}^{(r)} + \sum_{i=0}^{q} G_{h,k}^{(r,i)} b_{h,k,s,L}^{(r,i)} + G_{h,k}^{(r,i)} c_{h,k,s,R}^{(r,i)}$ is a good approximation for the $A_{r,k,s}^{(r)}$. We want $\tilde{A}_{r,k,s}^{(r)} = \tilde{z}_{r,k,s}^{(r)} + \sum_{i=0}^{q} [G_{h,k}^{(r,i)} b_{h,k,s,L}^{(r,i)} + G_{h,k}^{(r,i)} c_{h,k,s,R}^{(r,i)}]$ to be a good approximation for the $A_{r,k,s}^{(r)}$ as well. The $b_{h,k,s,L}^{(r,i)}$ and $b_{h,k,s,R}^{(r,i)}$ are similar to $b_{h,k,s,L}^{(r,i)}$, related the time segments $[a_i, a_{i+1}^2]$ and $[a_i, a_{i+1}^2, a_i]$ respectively, while $b_{h,k,s}$ is related to $[a_i, a_{i+1}^1]$. To understand this, one can see the second line of the figure (5.2).

Let $M$ be a variable $M = A_{r,k,s}^{(r)} - A_{r,k,s}^{(r)} - F_{k,s}^{(r)}$. Because $A_{r,k,s}^{(r)} = A_{r,k,s}^{(r)} + F_{k,s}^{(r)}$, the approximate areas should have $\tilde{A}_{r,k,s}^{(r)} = \tilde{A}_{r,k,s}^{(r)} + F_{k,s}^{(r)}$ too. And because equation (5.11) in Lemma (5.3.4) and equation (5.12) give $G_{h,k}^{(r,i)} C_k$ and $G_{h,k,L}^{(r,i)} = G_{h,k}^{(r,i)} D_k + N_k^{(r,i)} C'_k$, the variable $M$ should have

$$M = A_{r,k,s}^{(r)} - A_{r,k,s}^{(r)} - F_{k,s}^{(r)}$$

$$\simeq \tilde{z}_{r,k,s}^{(r)} + \sum_{i=0}^{q} G_{h,k}^{(r,i)} b_{h,k,s,L}^{(r,i)} - \tilde{z}_{r,k,s}^{(r)} - \sum_{i=0}^{q} G_{h,k,L}^{(r,i)} b_{h,k,s,L}^{(r,i)} - \sum_{i=0}^{q} G_{h,k}^{(r,i)} c_{h,k,L}^{(r,i)} - F_{k,s}^{(r)}$$

Because $b_{h,k,s,L}^{(r,i)} = ((\Sigma_{h,k}^{G,h/2})^{-1})^T (G_{shifL,s,k,L}^{(r,i)})^T$, $b_{h,k,s,R}^{(r,i)} = ((\Sigma_{h,k}^{G,h/2})^{-1})^T (G_{shifL,s,k,R}^{(r,i)})^T$, and $\Sigma_{h,k}^{G,h/2} = \frac{1}{2} \Sigma_{h,k}$, we have

$$M \simeq \left[ \tilde{z}_{r,k,s}^{(r)} - \tilde{z}_{r,k,s}^{(r)} \right] - F_{k,s}^{(r)} + \sum_{i=0}^{q} G_{h,k}^{(r,i)} ((\Sigma_{h,k}^{G,h})^{-1})^T (G_{shifL,s,k,L}^{(r,i)})^T$$

$$- \sum_{i=0}^{q} 2G_{h,k,L}^{(r,i)} ((\Sigma_{h,k}^{G,h})^{-1})^T (G_{shifL,s,k,L}^{(r,i)})^T - \sum_{i=0}^{q} 2G_{h,k}^{(r,i)} ((\Sigma_{h,k}^{G,h})^{-1})^T (G_{shifL,s,k,R}^{(r,i)})^T$$

$$= \left[ \tilde{z}_{r,k,s}^{(r)} - \tilde{z}_{r,k,s}^{(r)} \right] - F_{k,s}^{(r)} + \sum_{i=0}^{q} G_{h,k}^{(r,i)} ((\Sigma_{h,k}^{G,h})^{-1})^T (G_{shifL,s,k,L}^{(r,i)})^T$$

$$- 2G_{h,k,L}^{(r,i)} D_k ((\Sigma_{h,k}^{G,h})^{-1})^T (G_{shifL,s,k,L}^{(r,i)})^T - 2G_{h,k}^{(r,i)} D_k ((\Sigma_{h,k}^{G,h})^{-1})^T (G_{shifL,s,k,R}^{(r,i)})^T$$

$$- \sum_{i=0}^{q} 2N_k C_k ((\Sigma_{h,k}^{G,h})^{-1})^T (G_{shifL,s,k,L}^{(r,i)})^T + 2N_k C_k ((\Sigma_{h,k}^{G,h})^{-1})^T (G_{shifL,s,k,R}^{(r,i)})^T.$$

Conditional on $G_{h,k}^{(r)}$, $(\Sigma_{h,k}^{G,h})^{-1}$, $G_{shifL,s}^{(r)}$, $D_k'$, $D_k$, the expectation of $G_{h,k}^{(r)} ((\Sigma_{h,k}^{G,h})^{-1})^T (G_{shifL,s}^{(r)})^T - 2G_{h,k}^{(r)} D_k' ((\Sigma_{h,k}^{G,h})^{-1})^T (G_{shifL,s}^{(r)})^T - 2G_{h,k}^{(r)} D_k ((\Sigma_{h,k}^{G,h})^{-1})^T (G_{shifL,s}^{(r)})^T$ is zero. Additionally, because $E(M) = E[A_{r,k,s}^{(r)} - A_{r,k,s}^{(r)} - F_{k,s}^{(r)}] = 0$, $E\{2N_k C_k ((\Sigma_{h,k}^{G,h})^{-1})^T (G_{shifL,s,k,L}^{(r,i)})^T + $
2N^{(r,i)}_{k}C_{k}((\Sigma_{h}^{G_{h}})^{-1}T(G_{h,k,s,k,R}^{(r,i)})) = 0, E(F_{k,s}^{(r,i)}) = 0, we have
\[
\sum_{i=0}^{q}[G_{h,k}^{(r,i)}((\Sigma_{h}^{G_{h}})^{-1})T(G_{h,k,s,k,R}^{(r,i)}T - 2G_{h,k}^{(r,i)}D_{k}^{(r,i)}((\Sigma_{h}^{G_{h}})^{-1})T(G_{h,k,s,k,R}^{(r,i)}T
\]
\[
- 2G_{h,k}^{(r,i)}D_{k}((\Sigma_{h}^{G_{h}})^{-1})T(G_{h,k,s,k,R}^{(r,i)}T] \sim 0.
\]

Because $F_{k,s}^{(r,i)}$ is the product of $G^{(r,i)}_{h,k,0}$ and $G^{(r,i)}_{h,s,0}$ $(i = 0, \ldots, q)$, which depend on $N^{(r,i)}_{k}$, $M = 0$, and $M \sim [\tilde{Z}_{h,k,s}^{(r)} - \tilde{Z}_{h,k,s}^{L}] - \sum_{i=0}^{q}[2N^{(r,i)}_{k}C_{h,k,s,k,L}^{(r,i)}T(G_{h,k,s,k,L}^{(r,i)}T + 2N_{k}C_{h,k,s,k,R}^{(r,i)}T(G_{h,k,s,k,L}^{(r,i)}T - F_{k,s}^{(r,i)}),

we can have
\[
\tilde{Z}_{h,k,s}^{(r,i)} \sim \tilde{Z}_{h,k,s}^{(r)} + \sum_{i=0}^{q}N^{(r,i)}_{k}C_{h,k,s}^{(r,i)}(k = 2, \ldots, n; s = 1, \ldots, k - 1), (5.13)
\]
when choosing appropriate $H_{h,k,s}^{(r,i)}$. From the discussion, we can see that the generation of the $G^{(r,i)}_{h,k,s}$ is closed related to the splitting up of $Z_{h,k,s}^{(r)}$. 

The $H_{h,k,s}^{(r,i)}$ can be obtained by solving the equation
\[
N^{(r,i)}_{k}C_{h,k,s}^{(r,i)} = G_{h,k,s}^{(r,i)}B^{(r,i)}_{h,k,s} - G^{(r,i)}_{h,k,s,L}B^{(r,i)}_{h,k,s,L} - G^{(r,i)}_{h,k,s,R}B^{(r,i)}_{h,k,s,R} - F_{k,s}^{(r,i)},
\]
where $G^{(r,i)}_{h,k,s,L}$, $B^{(r,i)}_{h,k,s,L}$, $G^{(r,i)}_{h,k,s,R}$, and $B^{(r,i)}_{h,k,s,R}$ depends on $N^{(r,i)}_{k}$. The $H_{h,k,s}^{(r,i)}$ should only depends on $G_{h,p,s}^{(r,i)} (p = 1, \ldots, k - 1; s = 0, \ldots, n - p; i = 0, \ldots, q).$

**Explanation 5.3.9.** We are explaining why we generate the threshold value in this way here.

Analogous to the Explanation (4.2.8) and (4.2.10) in 3D case, and Explanation (3.2.6) in 2D case, we need a threshold value to make sure the approximate conditional variances of $A^{(r,i)}_{h,k,s}$, $V_{h,k,s}$, and approximate conditional variances of $\tilde{Z}_{h,k,s}^{(r,i)}$, $\tilde{V}_{h,k,s}^{(r,i)}$ will be $2\tilde{V}_{h,k,s}^{(r,i)} \sim \tilde{V}_{h,k,s}^{(r,i)}$, respectively.

After $\tilde{Z}_{h,k,s}^{(r,i)} \sim N(0, \tilde{V}_{h,k,s}^{(r,i)})$ being split up, the new generated $\tilde{Z}_{h,k,s}^{(r,i)}$ should be $\tilde{Z}_{h,k,s}^{(r,i)} \sim N(0, \tilde{V}_{h,k,s}^{(r,i)})$, having $2\tilde{V}_{h,k,s}^{(r,i)} \sim \tilde{V}_{h,k,s}^{(r,i)}$. The reason for splitting up $\tilde{Z}_{h,k,s}^{(r,i)}$ is given by the Explanation (5.3.8). However, when we apply the equation (5.13),
\[
\left(\begin{array}{c}
\tilde{Z}_{h,k,s}^{(r,i)} \\
\vdots \\
\tilde{Z}_{h,k,s}^{(r,i)}
\end{array}\right)_{(k-1)\times 1} = \left(\begin{array}{c}
\tilde{Z}_{h,k,s}^{(r,i)} \\
\vdots \\
\tilde{Z}_{h,k,s}^{(r,i)}
\end{array}\right)_{(k-1)\times 1} + \sum_{i=0}^{q}N^{(r,i)}_{k}C_{k}^{(r,i)}\left(\begin{array}{c}
H_{h,k,s}^{(r,i)} \\
\vdots \\
H_{h,k,s}^{(r,i)}
\end{array}\right)_{(k-1)\times 1}
\]
and the lemma (1.2.12), to generate $N_{k,s}^{(r,i)} (s = 0, \ldots, n - k; i = 0, \ldots, q)$ one by one, we could see that the new generated $\tilde{Z}_{h,k,s}^{(r,i)}$ will have conditional variance $\tilde{V}_{h,k,s}^{(r,i)} - \sum_{i=0}^{q}(H_{h,k,s}^{(r,i)})^{T}C_{k}^{(r,i)}C_{k}^{(r,i)}$. So as to distinguish from the $\tilde{Z}_{h,k,s}^{(r,i)}$.
which we will use to construct the approximate area $A^{(r)}_{1/2,k,s}$, we denote the $Z$ generated by splitting up $\tilde{Z}_{h,k,s}^{(r)}$ with $\tilde{Z}_{h,k,s}^{(r)}$. Such a conditional variance is not necessary to be closed to $1/2 \tilde{V}_{h,k,s}^{(r)}$ as we wished. Hence we construct a threshold value $Q = \sum_{i=0}^{q}(H_{k}^{(r,i)})^{T}C_{k}^{T}P^{-1}C_{k}H_{k}^{(r,i)} \frac{1}{(q+1)h^2} - E$, to make sure $\tilde{V}_{h,k,s}^{(r)} - \sum_{i=0}^{q}(H_{k}^{(r,i)})^{T}C_{k}^{T}C_{k}H^{(r,i)} \approx 1/2 \tilde{V}_{h,k,s}^{(r)}$, where the vector $\sum_{i=0}^{q}N_{k}^{(r,i)}C_{k}H_{k}^{(r,i)}$ is joint normal distributed with mean 0 and variance $P(q+1)h^2$ ($P$ a constant matrix).

**Explanation 5.3.10.** We are explaining the correction of $\tilde{Z}_{1/2,k,s}^{(r)}$ here.

From the Explanation (5.3.9), we know that the $\tilde{Z}_{h,k,s}^{(r)}$ generated from splitting up $\tilde{Z}_{h,k,s}^{(r)}$ has conditional variance $\tilde{V}_{h,k,s}^{(r)} - \sum_{i=0}^{q}(H_{k}^{(r,i)})^{T}C_{k}^{T}C_{k}H_{k}^{(r,i)}$ rather than $1/2 \tilde{V}_{h,k,s}^{(r)}$. We want the subdivided sum of area terms $A^{(r)}_{1/2,k,s}$ having conditional variance $1/2 \tilde{V}_{h,k,s}^{(r)}$, although we already apply the threshold value $Q$ to identify the subdivided approximate sum of areas in segments $J_r$, whose conditional variance $\tilde{V}_{h,k,s}^{(r)} - \sum_{i=0}^{q}(H_{k}^{(r,i)})^{T}C_{k}^{T}C_{k}H_{k}^{(r,i)}$ is closed to $1/2 \tilde{V}_{h,k,s}^{(r)}$. Hence we obtain a matrix $T_k$

$$
T_k = \begin{pmatrix}
T_{k11} & 0 & \ldots & 0 & 0 & 0 \\
T_{k21} & T_{k22} & 0 & \ldots & 0 & 0 \\
\vdots & & & \ddots & \vdots & \vdots \\
T_{k(k-1)(k-1)} & \ldots & T_{k(k-1)(k-1)} & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & & & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}
$$

by solving

$$
\frac{1}{2} \tilde{Z}_{h,k}^{(r)} = T_k^{T}[\Sigma_k^{(r)} - \sum_{i=0}^{q}(H_{k}^{(r,i)})^{T}C_{k}^{T}C_{k}H_{k}^{(r,i)}]T_k.
$$

Then we can obtain the corrected $\tilde{Z}_{h,k}^{(r)}$ with $\tilde{Z}_{h,k}^{(r)} = \tilde{Z}_{h,k}^{(r)}T_k$.

Why we generate the $T_k$ in this way? Because we can set $\tilde{Z}_{h,k}^{(r)} = \tilde{Z}_{h,k}^{(r)}T_k$, where $T_k$ is unknown. From the condition (5.3.6), we have

$$
E(\tilde{Z}_{1/2,k}^{(r)}\tilde{Z}_{1/2,k}^{(r)}) = \frac{1}{2} \Sigma_k - \frac{1}{2} E(\tilde{Z}_{h,k}^{(r)}\tilde{Z}_{h,k}^{(r)}) = T_k^{T}E[(\tilde{Z}_{h,k}^{(r)})^{T}\tilde{Z}_{h,k}^{(r)}]T_k,
$$

When we consider the conditional variance of $\tilde{Z}_{h,k}^{(r)}$ and the subdivided $\tilde{Z}_{1/2,k}^{(r)}$, we have

$$
E[(\tilde{Z}_{1/2,k}^{(r)})^{T}\tilde{Z}_{1/2,k}^{(r)}] = \Sigma_k^{(r)} - \sum_{i=0}^{q}(H_{k}^{(r,i)})^{T}C_{k}^{T}C_{k}H_{k}^{(r,i)}
$$

\[\square\]
5.4 Convergence

We still not yet answer the convergence of the 3/4 method, which is critical.

**Lemma 5.4.1.** Apply the notation method (5.3.2). Let $S = I - Q$, where we get $Q$ in Explanation (5.3.9), then we have

$$E(Q^T Q) \sim O\left(\frac{1}{(q + 1)}\right).$$

**Proof:** Let $S$ be

$$S = 2I - \sum_{i=0}^{q}(H_k^{(r,i)})^T C_k^T P^{-1} C_k H_k^{(r,i)} \frac{1}{(q + 1)h^2}.$$

Then we have

$$E(S) = E(2I - P(q + 1)h^2 P^{-1} \frac{1}{(q + 1)h^2}) = E(I)$$

$$E(Q) = E(P(q + 1)h^2 P^{-1} \frac{1}{(q + 1)h^2} - I) = 0$$

$$E(Q^T Q) = E\left[\frac{1}{(q + 1)^2 h^4} (P^{-1})^T \sum_{i=0}^{q}\{(H_k^{(r,i)})^T C_k^T C_k H_k^{(r,i)} (H_k^{(r,i)})^T C_k^T C_k H_k^{(r,i)} P^{-1}\} \right.$$

$$\quad + I - \frac{1}{(q + 1)h^2} (P^{-1})^T \sum_{i=0}^{q}(H_k^{(r,i)})^T C_k^T C_k H_k^{(r,i)}$$

$$\quad - \sum_{i=0}^{q}(H_k^{(r,i)})^T C_k^T C_k H_k^{(r,i)} P^{-1} \frac{1}{(q + 1)h^2}] < \frac{1}{(q + 1)^2 h^4} (P^{-1})^T \sum_{i=0}^{q}(H_k^{(r,i)})^T C_k^T C_k H_k^{(r,i)} (H_k^{(r,i)})^T C_k^T C_k H_k^{(r,i)} P^{-1}$$

In the last inequality, because there are $E(C_k^T C_k) \sim O(h)$, $E((H_k^{(r,i)})^T H_k^{(r,i)}) \sim O(h)$ and a $\sum_{i=1}^{q+1}$ term on the right hand side, we could get

$$E(Q^T Q) \sim O\left(\frac{1}{(q + 1)}\right).$$

\[\square\]

**Theorem 5.4.2.** Apply the notation method (5.3.2) and notation $\tilde{Z}_{\frac{r}{k}}^{(r)}$ (see step 4 of the method (5.3.7)). Let area term $\tilde{A}_{h,k}^{(r)}$ approximates the area of the $J_r$ before $I$ subdivision, $\tilde{A}_{\frac{r}{k},k}^{(r)}$ approximates the area of the $J_r$ after one $I$ subdivision,
having

\[ \hat{A}^{(r)}_{h,k,s} = \hat{A}^{(r)}_{h,k,s} + \sum_{i=0}^{q} (W^{(s)}(I_i) - W^{(s)}(I_{i+1})) \Delta W^{(r)}_h \]

\[ \hat{A}^{(r)}_{\frac{1}{2},k,s} = \hat{A}^{(r)}_{\frac{1}{2},k,s} + \sum_{i=0}^{2q} (W^{(s)}(I'_i) - W^{(s)}(I_{i+1})) \Delta W^{(r)}_{\frac{1}{2}}. \]

Then we have \( E\{(\hat{A}^{(r)}_{h,k} - \hat{A}^{(r)}_{\frac{1}{2},k})^2\} \sim O(h^2). \)

**proof:** To prove this, we only need to consider two situations: \(|Q| \leq \frac{1}{2}\) and \(|Q| > \frac{1}{2}\).

When \(|Q| \leq \frac{1}{2}\): \( I - Q \) is positive definite, so \( \exists T \), which is unique, symmetric and invertible, st. \( T^T T = E(I - Q) \). So, similar to the 2D case, we have

\[ E\{(Z_{\frac{1}{2},k}^{(r)} - \tilde{Z}_{\frac{1}{2},k}^{(r)})^2 \} \leq \frac{1}{2} \Sigma_k \frac{2}{5} E(Q^T Q) \sim O(h^2) \]

When \(|Q| > \frac{1}{2}\):

\[ E\{(Z_{\frac{1}{2},k}^{(r)} - \tilde{Z}_{\frac{1}{2},k}^{(r)})^2 \} \leq \frac{1}{2} \Sigma_k \frac{2}{5} E(Q^T Q) \sim O(h^2) \]

It comes that \( E\{(\hat{A}^{(r)}_{h,k} - \hat{A}^{(r)}_{\frac{1}{2},k})^2\} \sim O(h^2) \) with the above discussion.

Let \( \hat{A}^{(r)}_{h,k,s} \) approximate the actual area \( A^{(r)}_{k,s} \). Then one can get the \( E\{(\hat{A}^{(r)}_{h,k,s} - A^{(r)}_{k,s})^2\} \sim O(h^2) \), when one apply the theorem (3.3.2) and (3.3.3).
Chapter 6

Improved Euler Method in Weak Approximation

6.1 Introduction

In many cases, we need to get a weak approximation of Stochastic Differential Equation (SDE), which can be achieved by applying the Euler Scheme, and the higher-order Weak Taylor Scheme. A typical problem is to calculate \( E[f(x(T))] \) where \( x(t) \) is the solution of the SDE

\[
\frac{dx^i(t)}{dt} = a_i(t, x(t))dt + \sum_{k=1}^{d} b_{i,k}(t, x(t))dW^{(k)}(t), \quad i = 1, \ldots, q
\]

and \( f \) is some given function.

The goal of this Chapter is to explore the Weak approximation of the SDE. We will give an Improved Euler Method in subsection (6.2.1). Recently we find Improved Euler Method in subsection (6.2.1) is duplicated by the Multilevel Monte Carlo method from Giles' paper,[11] but the research is done by myself independently. We try a further improvement by reusing Brownian trajectory in subsection (6.2.2). For an approximate Brownian trajectory, we define \( X_h \), the Euler approximation for the solution of SDEs, \( x(T) \), with step size \( h \). In the section (6.4), we prove the correlation between \( (X_h - X_{\frac{h}{2}}) \) and \( (X_{\frac{h}{2}} - X_{\frac{h}{4}}) \) is small, in which \( X_h, X_{\frac{h}{2}} \) and \( X_{\frac{h}{4}} \) are generated by the same Brownian trajectory with different step size. This explains why not applying Richardson Extrapolation to estimate the SDEs for one Brownian trajectory. One may know that, in “The Law of the Euler Scheme for Stochastic Differential Equations: I. Convergence Rate of the Distribution Function”, Talay, Tubaro and Bally [2] have shown that the error \( Ef(x(T)) - Ef(X_h(N)) \) can be expanded in powers of \( h \), which permits to construct Romberg extrapolation procedures to accelerate the convergence rate. The situation discussed in the section (6.4) is not the same with the situation in Talay, Tubaro and Bally’s article.

Bit by bit, in the following subsections, we will show an Weak Euler Approximation Method (WEA), the idea of the Improved Weak Euler Approximation Method (IWEA) and the notations used in the later sections.
6.1.1 The Direct Weak Euler Approximation Method

One can use the Euler Approximation Method directly to get the \( f(X(T)) \)'s Weak Approximation, \( E[f(X_h)] \).

Say \( T \) is the final time, \( h \) is the step size, the number of time segments is \( N = \frac{T}{h} \), and the approximation for the \( x(T) \) is \( X_h(T) = X_h^{(N)} = X_h \).

**Method 6.1.1. direct WEA Method**

1) generate independent \( \Delta W \sim N(0, h) \) to get \( X_h(T) \), which is the Euler approximation to \( x(T) \), by

\[
\Delta X_h^{(i+1)} = A(X_h^{(i)})h + B(X_h^{(i)})\Delta W_h^{(i)}
\]

2) Then According to the Monte Carlo (MC) method, \( E[f(x(T))] \) could be approached by averaging a large number of independent \( f(X_h(T)) \)s.

The WEA method is very straightforward. Generally estimating, we can find that the total error of this method can be divided into two distinct species. There are the Weak Euler Approximation errors: the errors coming up when we apply the Weak Euler Approximation with step size \( h \); and the sample errors: the errors generated when we approximate the \( E(f(X_h)) \) by the Monte Carlo method.

Now, we can estimate its computational load, and analyse its error in detail. Suppose we want to implement the WEA method to get an approximation with accuracy \( \epsilon \). Then in the first step of the WEA method (6.1.1), we should take step size \( h = O(\epsilon) \) to obtain accuracy \( O(\epsilon) \), since Euler's method has weak order 1, in the sense that \( |E X_h - E x(T)| = O(h) \). On the other hand, in the second step of the (6.1.1) method, we need at least \( M \sim O(\epsilon^{-2}) \) independent runs to generate \( X_h^{[i]} \) \( (i = 1, \ldots, M) \), so as to make the sampling error \( O(\epsilon) \). That is, when \( M \sim O(\epsilon^{-2}) \)

\[
|\frac{1}{M} \sum_{i=0}^{M} f(X_h^{[i]}) - E[f(X_h)]| = O(\epsilon).
\]

Therefore, we need \( T/h \sim O(\epsilon^{-1}) \) steps to get the approximation of \( x(T) \) with the Euler method, and \( O(\epsilon^{-2}) \) independent runs to approximate \( E(f(x_h)) \). There is a total computational load of \( O(\epsilon^{-3}) \).

We can see that the computation load will rapidly increase as the expected bound of the error decreases, when one applies this method.

In the following part, we propose a technique, which would be effective, using the improved Euler method. Applying the 3/4 method, an even effective result can be achieved.

6.1.2 Idea of The Improved Weak Euler Approximation Method

The question can now be posed- exactly what are we approximating? Clearly, \( E[f(x(T))] \) is what we are looking for. As the Euler method gives us \( |E[f(x(T))] - E[f(X_h)]| \sim O(h) \), which bounds the error between \( E[f(x(T))] \) and \( E[f(X_h)] \), we only need to find an efficient way to approximate the \( E[f(X_h)] \).
Can we find another construction strategy to get the $E[f(X_h)]$? Our guess would be yes. In the WEA method, the bound of the approximation error $B$ for $E[f(X_h)]$ is suggested by the fixed time scale of the Euler method $h$, and the times of the Monte Carlo simulation $M$. Because the errors from these two parts play different weight, we are intuitive to vary the time scales and runs of the Monte Carlo simulations. This will balance the computational load obtained from the MC simulation, and that from the iteration of the Euler method.

We can describe our strategy in detail. Because of the

$$f(X_h) = [f(X_h) - f(X_{2h})] + \ldots + [f(X_{2k-1}h) - f(X_{2kh})] + f(X_{2kh}),$$

we can get the $E[f(X_h)]$ by adding up $E[f(X_{2i-1}h) - f(X_{2ih})]$ ($i = 1, \ldots, k, h = \frac{T}{2^k}$). That is

$$E[f(X_h)] = E[f(X_h) - f(X_{2h})] + \ldots + E[f(X_{2k-1}h) - f(X_{2kh})] + E[f(X_{2kh})].$$

As we mentioned in the last paragraph, errors generated are varied because of the times of the Monte Carlo simulation and the step size $2^i h$ ($i = 0, 1, \ldots, k$). For example, if we simulate $M$ times of the $f(X_h) - f(X_{2h})$ and $f(X_{2k-1}h) - f(X_{2kh})$ to approximate $E[f(X_h) - f(X_{2h})]$ and $E[f(X_{2k-1}h) - f(X_{2kh})]$ respectively, we may see that the former one will have a smaller second moment and sample error than the later one. Nevertheless, the simulation of the $[f(X_h) - f(X_{2h})]$ will require a much heavier computational load, because it has smaller time step size and the consequent more Euler iterations than the other one. So we shall simulate $f(X_h) - f(X_{2h})$ not as many times as $f(X_{2k-1}h) - f(X_{2kh})$, which will bring down the computational load significantly, while keeps its error still small.

### 6.1.3 The Notations for This Chapter

Let's give the basic notations, in which the statements and discussions will be dressed up in the following sections here.

- $T$ —— is the time scale we discussed
- $k$ —— is some integer
- $N = 2^k$ is the number of segments over $T$

The approximated solution to the $X = (X_t, t \in [0, T])$ is characterized by a partition $\tau_i$ ($i = 0, \ldots, k$) of $[0, T]$:

$$\tau_i : 0 = t_0 < t_1 < \ldots < t_{2^k-i-1} < t_{2^k-i} = T,$$

with mesh

$$\delta_i = \text{mesh}(\tau_i) = (t_j - t_{j-1}) = \Delta_j; (j = 1, \ldots, 2^k).$$

Hence, we also have

$$h = \delta_0 = \text{mesh}(\tau_0) = (t_j - t_{j-1}) = \Delta_j; (j = 1, \ldots, 2^k).$$
the $j$th instant of time, where $t = jh$

the $p$th sample Brownian path

the $j$th instant of time, where $t = jh$

the $p$th sample path

$mh$ is the length between two instants of discrete time

For simplicity, denote $W[j] = W[j](T) = W_h^{(2^k)[j]}$, $X^{[j]}_h = X^{[j]}_h(T) = X^{(2^{k-1})[j]}_h$, $X_{2^kh} = X_{2^k}^{(2^{k-1})}$, and $W^{(n)} = W^{(n)}_h$, where $0 < 2^kh = T$.

We say

$$f(h, W_h) \sim O(h^2),$$

if there is a finite constant $K$ and a positive constant $h_0$, such that $|E[f(h, W_h)f(h, W_h)]| \leq Kh_0^2$ for any time discretization with maximum step size $h \in (0, h_0)$.

we also replace some function $f(h, W_h)$, in which $f(h, W_h) \sim O(h^p)$, with $O(h^p)$, because in some cases we do not care the exact expression of the $f(h, W_h)$, but the bound of the function.

$f(X_h) \rightarrow$ is the approximation of the $E[f(X_h)]$ in the sense that

$$
\overline{f(X_h)} = \frac{\sum_{j=1}^{m_0}[f(x_h^{(2^k)[j]}) - f(x_2^{(2^{k-1})}[j])]}{m_0}
+ \ldots + \frac{\sum_{j=1}^{m_k}[f(x_h^{(2^k)[j]}) - f(x_2^{(1)}[j])] + \sum_{j=1}^{m_k}[f(x_2^{[j]}_h)]}{m_k}
$$

$\epsilon_s$ — the total sample error.

$\epsilon_w$ — the weak approximation error.

6.2 The Improved Weak Euler Approximation Method

We have the idea of the IWEA method and notations, now we need to answer two critical questions. How to hook them into a concrete IWEA method? And how good IWEA method can be? In the subsection (6.2.1), the algorithm IWEA (6.2.1) will be given. So as to answer the second question, we are going to reveal what is the sample error for this method, by Lemma (6.2.2) and (6.2.3), and how the computational load and the total error are related, by the theorem (6.2.4).

The Euler scheme attains strong order 0.5, and weak order 1 in the sense that

$$
E(f(X_h) - f(X_{2h}))^2 = O(h) \leq C_1 h \\
|E f(X_h) - E f(x(T))| = O(h) \leq C_2 h
$$
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\[ N = 2^k, \quad h = \frac{T}{N} \]

By making some changes over the WEA method described in the section (6.1.1), we have the IWEA Method.

### 6.2.1 Improved Weak Euler Approximation Algorithm

**Method 6.2.1.** One can approximate \( E[f(X_h)] \) by the following steps:

1. Generate \( m_i \) \((i = 0, \ldots, k)\) independent Brownian path \( W^{[j]} \), \((j = 1, \ldots, m_i)\), with the finest step size \( 2^i h = \frac{T}{2^i} \) \((k \text{ is some fixed natural number})\). That is, in \( m_i \) \((i = 0, 1, \ldots, k)\) independent Brownian path \( W^{[j]} \), \( \Delta W^{[j]} \sim N(0, 2^i h) \).

2. For one \( W^{[j]} \), \((j = 1, \ldots, m_i)\), we can get one \( X^{[j]}_{2^i h} \) and one \( X^{[j]}_{2^{i+1} h} \), which are the Euler approximation to the \( x(T) \) with time step size \( 2^i h \) and \( 2^{i+1} h \) respectively. That is

\[
\Delta X^{(q+1)}_{2^i h} = A(X^{[q]}_{2^i h}) \cdot 2^i h + B(X^{[q]}_{2^i h}) \cdot \Delta W^{[q]} \]
\[
\Delta X^{(q+1)}_{2^{i+1} h} = A(X^{[q]}_{2^{i+1} h}) \cdot 2^{i+1} h + B(X^{[q]}_{2^{i+1} h}) \cdot (\Delta W^{(q+1)} + \Delta W^{(2q+1)}) \]
\[
X^{[j]}_{2^i h} = X^{(2^i)}_{2^i h}
\]

\( m_i \) independent Brownian paths \( W \) give \( m_i \) independent \( f(X^{[j]}_{2^i h}) - f(X^{[j]}_{2^{i+1} h}) \).

3. By averaging \( m_i \) independent \( f(X^{[j]}_{2^i h}) - f(X^{[j]}_{2^{i+1} h}) \), we get the approximation of the \( E[f(X^{[j]}_{2^i h}) - f(X^{[j]}_{2^{i+1} h})] \) with sample error \( \frac{C_i}{m_i} \), in which \( i = 0, \ldots, k-1 \). We will discuss how to get this sample error in the lemma (6.2.3). In the other words, we define

\[
\overline{f(X_{2^i h})} - \overline{f(X_{2^{i+1} h})} = \frac{\sum_{j=1}^{m_i} [f(x^{(2^i)}_{2^i h}) - f(x^{(2^{i+1})}_{2^{i+1} h})]}{m_i},
\]

\[
\overline{f(X_{2^i h})} = \frac{\sum_{j=1}^{m_k} [f(x^{(i)}_{2^i h})]}{m_k},
\]

then we can get

\[
\overline{f(X_{2^i h})} \approx E[f(X_{2^i h}) - f(X_{2^{i+1} h})],
\]

\[
\overline{f(X_{2^i h})} \approx E[f(X_{2^{i+1} h})],
\]

having bound of the error \( \sqrt{\frac{C_i^{2^i h}}{m_i}} \), and \( \sqrt{\frac{C_i^{2^{i+1} h}}{m_k}} \) respectively.

4. When we define \( \overline{f(X_h)} \) as

\[
\overline{f(X_h)} = \sum_{i=0}^{k-1} \overline{f(X_{2^i h})} - f(X_{2^{i+1} h}) + \overline{f(X_{2^k h})},
\]
we can approximate \( E[f(X_h)] \) with the \( \tilde{f}(X_h) \), considering \( E[f(X_h)] = E[f(X_h) - f(X_{2h})] + \ldots + E[f(X_{2^{k-1}h}) - f(X_{2^kh})] + E[f(X_{2^kh})] \).

6.2.2 The error Bound of the IWEA method

After we get the detail algorithm, we need to know whether the IWEA method can achieve the same fixed bound of the approximation error, but using less computation than the WEA method.

As we notice in the subsection (6.1.2), the error of the Weak Euler Approximation consists of the sample error and Euler Approximation error. Because we already know what is the relation between the estimated error of the Euler Approximation and the time step size \( h \) by inequality (6.3), we still need to know what is the relation between the sample error and the computational load. Let's split this question into 3 pieces:

1. How the number of the MC simulation \( m_i \) affects the sample error \( \frac{C_12^i h}{m_i} \)?

2. What is the proper number of simulations for \( f(X_{2^ih}) - f(X_{2^i+1h}) \) (\( i = 0, \ldots, k - 1 \)) and \( f(X_{2^kh}) \), having time scale \( 2^ih \), and \( T \)?

3. And what is the computational load of the IWEA method?

To explain the question 1. and the sample error, \( \frac{C_12^i h}{m_i} \), mentioned in the step 3) of the method (6.2.1), we recall the sample error estimate in Lemma (6.2.2) first, and then prove it in Lemma (6.2.3).

**Lemma 6.2.2.** Suppose that \( y_{i}^{[1]}, y_{i}^{[2]}, \ldots, y_{i}^{[m_i]} \) \((m_i \in N)\) are independent samples, having \( E[(y_{i}^{[j]})^2] = \sigma^2 \), \((j = 1, \ldots, m_i)\). Then the mean of the samples, \( \bar{y}_i = \frac{1}{m_i} \sum_{j=1}^{m_i} y_{i}^{[j]} \), will have

\[
E(\bar{y}^2) = \frac{1}{m_i} \sigma^2.
\]

**Lemma 6.2.3.** Denote \( y_{i}^{[j]} = f(X_{2^ih}^{(2^i-1)[j]}) - f(X_{2^{i+1}h}^{(2^i-1)[j]}) \), \((j = 1 \ldots m_i)\), for some \( i = 0, \ldots, k - 1 \), where \( X_{2^ih}^{(2^i-1)[j]} \) and \( X_{2^{i+1}h}^{(2^i-1)[j]} \) are generated by Brownian path \( W_{[j]} \), \((j = 1, \ldots, m_i)\). The \( W_{[j]} \) and \( W_{[p]} \) \((j \neq p)\) are independent. Then

\[
E\left\{ \left[ \frac{1}{m_i} \sum_{j=1}^{m_i} |f(x_{2^ih}^{(2^i-1)[j]}) - f(x_{2^{i+1}h}^{(2^i-1)[j]})| \right]^2 \right\} \leq \frac{C_12^i h}{m_i}.
\]

**Proof:** \( y_{i}^{[j]} \) is independent to \( y_{i}^{[p]} \) for \((j \neq p)\), because of the independence of the trajectory of the \( W_{[j]} \). Hence by applying the Lemma (6.2.2) and the
inequality (6.2), we get to the result that the mean of the $y_t^{[i]}$ has

$$E[y_t^{[i]^2}] = E\left\{ \frac{1}{m_t} \sum_{j=1}^{m_t} \left[ f(X_{2^ih}^{(2^k-i-1)[j]}) - f(X_{2^{i+1}h}^{(2^k-i-1)[j]}) \right]^2 \right\}$$

$$= \frac{1}{m_t^2} \sum_{j=1}^{m_t} E\left\{ f(X_{2^ih}^{(2^k-i)[j]}) - f(X_{2^{i+1}h}^{(2^k-i-1)[j]}) \right\}^2$$

$$= \frac{1}{m_t^2} \sum_{j=1}^{m_t} E\left\{ f(X_{2^ih}^{[j]}) - f(X_{2^{i+1}h}^{[j]}) \right\}^2 \leq \frac{C_1 2^i h}{m_t}.$$

This gives us the sample estimate in the step 3) of the method (6.2.1), and answer the question, "How the number of the MC simulation $m_t$ affects the sample error $e_t$?"

Now we can consider the rest two questions together, which are raised at the beginning of this subsection, in the following theorem.

**Theorem 6.2.4.** Suppose the step size $h = T/2^k$, and $m_t$ independent runs of $f(X_{2^ih}) - f(X_{2^{i+1}h})$ give the estimate of $E[f(X_{2^ih}) - f(X_{2^{i+1}h})]$.

For a fixed computational load $L$, if we apply the method (6.2.1) to get $\widehat{f(X_h)}$, which is the estimate of the $E[f(x(T))]$, the bound of the error, $B(k, m_0, \ldots, m_k)$, achieves minimum, having

$$\epsilon^2 \leq B^2 \sim O\left( \frac{(\log_2 L)^2}{L} \right),$$

**Proof:** The conclusion, $\epsilon^2 \leq B^2 \sim O\left( \frac{(\log_2 L)^2}{L} \right)$, means there is a finite constant $K$, which is independent to $L$, such that $\epsilon^2 \leq B^2 \leq K(\log_2 L)^2$.

We shall follow a two-step process outlined here:

1. Find a proper strategy to allocate the computational load.

2. Find the right time step size $h$ to minimise the bound of the error.

**Step One: get the proper allocation strategy**

Actually, this is answering, what is the proper number of simulations for $f(X_{2^ih}) - f(X_{2^{i+1}h})$, having time scale $2^ih$, ($i = 0, \ldots, k$).

Suppose $f(X_{2^ih}^{[i]}) - f(X_{2^{i+1}h}^{[i]})$ and $f(X_{2^ih}^{[j]}) - f(X_{2^{i+1}h}^{[j]})$ ($i \neq j$) are independent, and $m_t$ independent runs of $f(X_{2^ih}^{[j]}) - f(X_{2^{i+1}h}^{[j]})$, ($j = 1, \ldots, m_t$), gives the
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estimate of $E[f(X_{2i+h}) - f(X_{2i+1+h})]$. It is true that

$$E[f(X_h)] = E[f(X_h) - f(X_{2h})] + \ldots + E[f(X_{k-1}h) - f(X_{kh})] + E[f(X_{kh})]$$

$$= \sum_{j=1}^{m_k} \left\{ f(x_{2h}^{(j)}) - f(x_{2h}^{(2)}) \right\} + \ldots + \left\{ f(x_{2h}^{(2k-1)}) - f(x_{2h}^{(1)}) \right\} + \sum_{j=1}^{m_k} f(x_{2h}^{(j)})$$

Because $f(X_{2i+h}) - f(X_{2i+1+h})$ and $f(X_{2i+h}) - f(X_{2i+1+h})$ ($i \neq j$) are independent, we have,

$$\text{var}\{f(X_h)\} = \text{var}\{f(X_h) - f(X_{2h})\} + \ldots + \text{var}\{f(X_{k-1}h) - f(X_{kh})\} + \text{var}\{f(X_{kh})\}$$

We shall get the Runs of the simulation, the bound of the error, and Sample Error as in the table, when we implement the method (6.2.1),

<table>
<thead>
<tr>
<th>Runs</th>
<th>$m_0$</th>
<th>$m_1$</th>
<th>$m_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error Bound</td>
<td>$C_1h$</td>
<td>$C_1(2^{k-1}h)$</td>
<td>$C_1(2^k h)$</td>
</tr>
<tr>
<td>Sample Error</td>
<td>$C_1h/m_0$</td>
<td>$C_1(2^{k-1}h)/m_k$</td>
<td>$C_1(2^k h)/m_k$</td>
</tr>
</tbody>
</table>

Here, Error Bound is based on the inequality (6.2) and the Sample Error is based on the Lemma (6.2.3).

According to the weak approximation bound in inequality (6.3) and the total sample error estimate in the table, we shall determine the total sample error $\epsilon_s$, and the weak approximation error $\epsilon_o$, with:

$$\epsilon_s^2 = \sum_{i=0}^{k} \frac{1}{m_i} C_1(2^i h) = \sum_{i=0}^{k} \frac{C_1 T 2^{-k}}{m_i}$$

$$\epsilon_o^2 = C_2 \left( \frac{T}{2^k} \right)^2.$$
And consequently, we obtain the computational load $L$, and error $\epsilon$: 

$$
\epsilon^2 = |E(f(x(T))) - \hat{f}(X_h)|^2 
\leq B^2(k, m_0, \ldots, m_k) = \epsilon_s^2 + \epsilon_o^2 = \sum_{i=0}^{k} C_1 T \frac{2^{i-k}}{m_i} + C_2 \left(\frac{T}{2^k}\right)^2 
$$

(6.4)

$$
L(k, m_0, \ldots, m_k) = \sum_{i=0}^{k} m_i \frac{T}{2^i h} = \sum_{i=0}^{k} m_i 2^{k-i}, \quad \text{ (6.5)}
$$

where $B(k, m_0, \ldots, m_k)$ is the bound of the error.

At this stage, we finally get the two essential tools, formulas (6.4) and (6.5), to find the proper allocation strategy. We can see that the computational load and total error are connected by MC runs and time step size.

Recall our goal in the Step One that we are going to see how to allocate $m_i$ ($i = 0, \ldots, k$) to minimise computational load $L$, if we fixed the bound of the error $B(k, m_0, \ldots, m_k)$ and $k$.

The formula (6.4) and (6.5) can be seen as a curve and a plane in vector space $(m_0, \ldots, m_k)$ respectively. Hence, $\forall k$ ($k = 1, 2, \ldots$), and fixed $B(k, m_0, \ldots, m_k)$, the $L$ achieves minimum, when $\nabla B(m_0, \ldots, m_i) = C \nabla L(m_0, \ldots, m_i)$, where $C$ is some constant.

That is, for any fixed $B(k, m_0, \ldots, m_k)$ and $k$, we can get the minimum $L$, when

$$
\frac{\partial B^2}{\partial m_i} 2^{i-k} = C, 
$$

where $C$ is some constant. Hence when

$$
m_i = 2^i m_0, \quad \text{ (6.6)}
$$

$L(k, m_0, \ldots, m_k)$ achieves minimum

$$
L(k, m_0) = \sum_{i=0}^{k} m_i 2^{k-i} = \sum_{i=0}^{k} 2^k m_0 = 2^k m_0 (k+1),
$$

$$
\iff m_0 = \frac{L}{2^k (k+1)}. \quad \text{ (6.7)}
$$

This safely answers what is the proper number of simulations for $f(X_{2^i h}) - f(X_{2^{i+1} h})$, ($i = 1, \ldots, k$).

**Step Two: Find the proper time step size $h$**

In the Step One, we optimise the allocation of the $m_i$ ($i = 0, \ldots, k$), so that it will make good use of the computation resource for a fixed error bound $B$ and a fixed $k$. Now, we bring the $k$ back in again. What happens if $k$ is a variable, which control the time step size $h$ and the number of iterations $N$. 

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Substituting \( m_i \) in equation (6.4) with (6.6) and (6.7), we can get
\[
\epsilon^2 \leq B^2(k, m_0, \ldots, m_k) = B^2(k, L) = \sum_{i=0}^{k} \frac{C_1 T}{L} (k + 1) + C_2^2 T^{2} 2^{-2k}.
\]

It follows from
\[
\frac{\partial B^2}{\partial k} = 0 \Rightarrow 2C_2^2 \left( \frac{T}{2^k} \right)^2 \ln 2 = \frac{(k + 2) C_1 T}{L}
\]
that the \( B(L, k) \) achieves the minimum
\[
\epsilon^2 \leq B^2(k, L) = C_2^2 \left( \frac{T}{2^k} \right)^2 (k + 1) \ln 2 + C_2^2 \left( \frac{T}{2^k} \right)^2,
\]
when the \( k \) satisfies
\[
L = \frac{(k + 1) C_1 2^k}{C_2^2 T \ln 2},
\]
for the fixed computational load \( L \). Because \( k = 0, 1, \ldots, \) and a smaller \( k \) leads to smaller \( L \) and bigger \( B \), we can find the \( k \leq \max\{0, [k^*] \in N\} \), where \( k^* \) is the solution of
\[
L = \frac{(k^* + 1) C_1 2^{k^*}}{C_2^2 T \ln 2}.
\]

Because the equations (6.9) and (6.10) give us inequality
\[
B^2(k, L) = C_2^2 \left( \frac{T}{2^k} \right)^2 (k + 1) \ln 2 + C_2^2 \left( \frac{T}{2^k} \right)^2 \leq C_B \frac{k + 1}{2^k},
\]
where \( C_L \) and \( C_B \) are some constants, we can have
\[
B^2(k, L) \leq C_L C_B \frac{(k + 1)^2}{L} \quad \log_2 L = \log_2 (k + 1) + 2k + \log_2 C_L.
\]
That is
\[
B^2(k, L) \leq C_L C_B \frac{(k + 1)^2}{L} = C_L C_B \frac{\left( \log_2 L - \log_2 (k + 1) - \log_2 C_L \right)^2}{L} + 1)^2 \\
\leq C_B' \frac{(\log_2 L)^2}{L} \sim O(\frac{(\log_2 L)^2}{L}),
\]
where \( C_B' \) is some independent constant to \( L \).

Summary:

At this stage, we completely answer all three questions presented at the beginning of the section. Choose \( k \) as in the equation (6.10), and simulate the
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\[ f(X_{2i+h}) - f(X_{2i+1h}) \] for \( m_i \) times (\( i = 0, \ldots, k \)) as in equations (6.6) and (6.7). Then \( f(X_h) \) approaches \( E[f(X_h)] \) with error \( B \sim O(\frac{\log L}{L}) \).

If we recall the WEA method, which have \( B \sim O(\epsilon) = O((\frac{1}{L})^{1/3}) \), surprising though it seems: for a computational load \( L \), IWEA method generated a much smaller error than the WEA one.

6.3 Reuse Values Weak Euler Approximation Method

A question now surfaces - Shall we improve the IWEA method further?

Within the IWEA method framework, one possible improvement could be generating less random values. But will the order of the computational load and error estimate still hold the same, if we replace ‘generate \( m_i \) independent \( f(X_{2i}) - f(X_{2i+1}) \)’ in the step 2 of the method 6.2.1 with combining each \( 2^i \) steps of the Brownian increment to get the Brownian path for \( X_{2i+h} \)?

6.3.1 Reuse Values Weak Euler Approximation Algorithm

Method 6.3.1. Reuse Values Weak Euler Approximation Method (RVWEA):

1. Generate \( m_0 \) independent Brownian path \( W_{h}^{[j]} \), \( (j = 1, \ldots, m_0) \), with the finest step size \( h = \frac{T_k}{2^k} \) (\( k \) is some fixed natural number). That is, in \( m_0 \) independent Brownian path \( W_{h}^{[j]} \), \( \Delta W_{h}^{[j]} \sim N(0, h) \). And Generate \( (m_i - m_{i-1}) \) independent Brownian path \( W_{2^i h}^{[j]} \), \( (j = m_{i-1} + 1, \ldots, m_i) \), with the finest step size \( 2^i h = \frac{T_k}{2^k-1} \). That is, in \( (m_i - m_{i-1}) \) independent Brownian path \( W_{2^i h}^{[j]} \) \( (j = m_i + 1, \ldots, m_{i+1}) \), we have \( \Delta W_{2^i h}^{[j]} \sim N(0, 2^i h) \).

2. For one \( W_{2^i h}^{[j]} \) \( (j = 1, \ldots, m_i; i = 0, \ldots, k) \), we can get one \( X_{2^i h}^{[j]} \), one \( X_{2^{i+1} h}^{[j]} \), ..., and one \( X_{2^k h}^{[j]} \), which are the Euler approximation to the \( z(T) \) with time step size \( 2^i h, 2^{i+1} h, \ldots, 2^k h \) respectively. That is

\[
\Delta X_{2^i h}^{(q+1)[j]} = A(X_{2^i h}^{(q)[j]} \cdot 2^i h) + B(X_{2^i h}^{(q)[j]} \cdot \Delta W_{2^i h}^{(q)[j]})
\]

\[
\Delta X_{2^{i+1} h}^{(q+1)[j]} = A(X_{2^{i+1} h}^{(q)[j]} \cdot 2^{i+1} h) + B(X_{2^{i+1} h}^{(q)[j]} \cdot (\Delta W_{2^k h}^{(2q)[j]} + \Delta W_{2^k h}^{(2q+1)[j]})
\]

\[...
\]

\[
\Delta X_T^{(1)[j]} = A(X_T^{(0)[j]} \cdot T) + B(X_T^{(0)[j]} \cdot W_{2^k h}^{[j]})
\]

\[
X_{2^i h}^{[j]} = X_{2^i h}^{(2^k-i)[j]}
\]

This process will give us as many as \( m_i \) of \( X_{2^i h}^{[j]} \) \( (j = 1, \ldots, m_i; i = 0, \ldots, k) \).

3. By averaging \( m_i \) of the \( f(X_{2^i h}) - f(X_{2^{i+1} h}) \), in which \( X_{2^i h} \) and \( X_{2^{i+1} h} \) are generated by the same Brownian trajectory \( W \), we get the approximation of the \( \mathbb{E}[f(X_{2^i h}) - f(X_{2^{i+1} h})] \), in which \( i = 0, \ldots, k \). In the other words, we
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\[ f(X_{2i+h}) - f(X_{2i+1+h}) = \sum_{j=1}^{m_i} \frac{f(x_{2i+1+h}) - f(x_{2i+h})}{m_i}, \]
\[ f(X_{2k+h}) = \sum_{j=1}^{m_k} \frac{f(x_{2k+h})}{m_k}, \]

then we can get
\[ f(X_{2i+h}) - f(X_{2i+1+h}) \approx E[f(X_{2i+h}) - f(X_{2i+1+h})], \]
\[ f(X_{2k+h}) \approx E[f(X_{2k+h})]. \]

4. When we define \( \bar{f}(X_h) \) as
\[ \bar{f}(X_h) = \sum_{i=0}^{k-1} f(X_{2i+h}) - f(X_{2i+1+h}) + f(X_{2k+h}), \]
we can approximate \( E[f(X_h)] \) with the \( \bar{f}(X_h) \), considering \( E[f(X_h)] = E[f(X_h) - f(X_{2h})] + \ldots + E[f(X_{2k-1+h}) - f(X_{2k+h})] + E[f(X_{2k+h})]. \)

One may question how can we obtain \( m_i X_{2i+h}^j \) \( (j = 1, \ldots) \) in step 2), while we only generate \( (m_i - m_{i-1}) W_{2i+h} \) in the first step. That is because we combine every \( 2^{p-i} \) \( (p > i) \) steps for \( W_{2i+h} \) to get a trajectory \( W_{2^{p}h} \), having step size \( 2^p \).

For example, we combine every 4 steps of a \( W_{1} \) \( (j = 1, \ldots, m_0) \) to get \( m_0 W_{4} \).
And we combine every 2 steps of a \( W_{1} \) \( (j = m_0 + 1, \ldots, m_1) \) to get \( (m_1 - m_0) W_{4} \)
\( (j = m_0 + 1, \ldots, m_1) \). Adding another \( (m_2 - m_1) \) of \( W_{4} \) \( (j = m_1 + 1, \ldots, m_2) \),
we get \( m_2 W_{4} \) \( (j = 1, \ldots, m_2) \) totally.

We will discuss the error generated by this method in the following theorem.

Here we introduce some notation for later use. Generally, for the Brownian trajectory \( W \), we have numerical approximation \( X_{h}(T) \), \( X_{2h}(T) \), \ldots, \( X_{2^{k}h}(T) \).
So we can have a numerical approximation \( X_{h}^{[j]}(T) \), \( X_{2h}^{[j]}(T) \), \ldots, \( X_{2^{k}h}^{[j]}(T) \), corresponding to a Brownian sample path \( W^{[j]} \). Denotes
\[ \xi_{0}^{[j]} = f(X_{0}^{[j]}) - f(X_{2h}) \]
\[ \xi_{i}^{[j]} = f(X_{i}^{[j]}) - f(X_{2i+h}^{[j]}), \]
\[ \xi_{k}^{[j]} = f(X_{2k+h}^{[j]}), \]
then we can get the sample size and the number of runs as in the table

<table>
<thead>
<tr>
<th>number of runs</th>
<th>samples</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_0$</td>
<td>$\xi_0^{[1]} \ldots \xi_0^{[m_0]}$</td>
</tr>
<tr>
<td>$m_1$</td>
<td>$\xi_1^{[1]} \ldots \xi_1^{[m_1]}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$m_k$</td>
<td>$\xi_k^{[1]} \ldots \xi_k^{[m_0]} \ldots \xi_k^{[m_1]} \ldots \xi_k^{[m_k]}$</td>
</tr>
</tbody>
</table>

when following the method (6.3.1). Without lost generality, say $i < j$, then we will have covariance

$$\sum_{p=1}^{m_i} E(\frac{1}{m_i} \xi_i^{[p]} \frac{1}{m_j} \xi_j^{[p]}) = \frac{1}{m_j} E(\xi_i \xi_j).$$

Consequently, one can get the relations of the sample size, covariance, and the sample error, with $m_i$, $E(\xi_i \xi_j)$, and $\epsilon_s$, respectively in the the table and formula below:

<table>
<thead>
<tr>
<th>$\frac{1}{m_0}$</th>
<th>$\frac{1}{m_1}$</th>
<th>$\frac{1}{m_2}$</th>
<th>$\ldots$</th>
<th>$\frac{1}{m_k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(\xi_0 \xi_0)$</td>
<td>$E(\xi_0 \xi_1)$</td>
<td>$E(\xi_0 \xi_2)$</td>
<td>$\ldots$</td>
<td>$E(\xi_0 \xi_{k-1})$</td>
</tr>
<tr>
<td>$0$</td>
<td>$E(\xi_1 \xi_1)$</td>
<td>$E(\xi_1 \xi_2)$</td>
<td>$\ldots$</td>
<td>$E(\xi_1 \xi_{k-1})$</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$E(\xi_2 \xi_2)$</td>
<td>$\ldots$</td>
<td>$E(\xi_2 \xi_{k-1})$</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$E(\xi_{k-1} \xi_{k-1})$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$E(\xi_{k-1} \xi_k)$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

$$\epsilon_s^2 = \sum_{i=0}^{k} \frac{1}{m_i} E(\xi_i \xi_i) + 2 \sum_{i=1}^{k} \frac{1}{m_i} \sum_{j=0}^{i} E(\xi_i \xi_j).$$

(6.11)

Analogous to the discussion of the method (6.2.1), it is important to know the computational load and error. Compared with the Independent case, the Reuse Random Value case need to consider the extra error generated by the random value reusing. One can estimate $E(\xi_i \xi_j)$ in formula (6.11) with Cauchy-Schwarz inequality, having $E(\xi_i \xi_j) \sim O(h)$. Nevertheless, it is possible to estimate this with a higher order version, having $E(\xi_i \xi_j) \sim O(h^{\frac{3}{2}})$, because of the relations of the $\xi_i$ and $\xi_j$. We will present the proof in the section (6.4).

Although the Proposition (6.4.1) provides that $\forall i, j \in [0, k], (i \leq j), \exists C_{j-i}^{\prime} \geq 0, \text{ st. } 2E(\xi_i \xi_j) \leq C_{j-i}^{\prime} 2^h$. Further discussion is required when we try to find out the bound of $\epsilon_s^2$ in the formula (6.11).

When we consider the sample error in formula (6.11) and $E(\xi_i \xi_j) \sim O((2^h)^{\frac{3}{2}})$, which could be expressed by the following table,
we have the total error, \( \varepsilon \), and computation load, \( L \)

\[
|E(f(x(T))) - \tilde{f}(X_i)|^2 = \varepsilon^2
\]

\[
\leq B^2(k, m_0, \ldots, m_k) = \varepsilon_0^2 + \varepsilon_1^2
\]

\[
= \sum_{i=0}^{k} C_{i} T \frac{2^i-k}{m_i} + C_2 \frac{T^{2}}{2^k} + \sum_{j=0}^{k-j} \frac{(2^i h)^{3/2}}{m_{i+j}}
\]

\[
L(k, m_0, \ldots, m_k)
\]

\[
= \sum_{i=0}^{k} m_i T \frac{2^i h}{2^i} = \sum_{i=0}^{k} m_i 2^{k-i}.
\]

In the case that \( \xi_i^{[p]} \) and \( \xi_i^{[q]} \) \( (j \neq p; \forall i, l) \) are independent, for a bound \( B \) and step size \( h = \frac{T}{2^k} \), the computational load \( L \) achieves minimum, when

\[
m_i = 2^i m_0 = 2^i \frac{C_1 T(k+1)}{B^2 2^k - C_2 T^{2} 2^{2-k}}.
\]

If we can prove the extra term,

\[
M = \sum_{j=1}^{k} C_j \sum_{i=0}^{k-j} \frac{(2^i h)^{3/2}}{m_{i+j}},
\]

is much smaller than the \( A = \sum_{i=0}^{k} \frac{C_1 T 2^{i-k}}{m_i} \) or \( B = C_2 \left( \frac{T}{2^k} \right)^2 \), it is reasonable to neglect the term \( M \), since it plays small weight on the error.

We introduce some notation for the later use. Denotes

\[
M = \sum_{j=1}^{k} C_j \sum_{i=0}^{k-j} \frac{(2^i h)^{3/2}}{m_{i+j}}
\]

\[
M_a = \sum_{i=0}^{k} C_1 T 2^{i-k} \frac{m_i}{m_i}
\]

\[
M_b = C_2 \left( \frac{T}{2^k} \right)^2
\]

**Theorem 6.3.2.** For a step size \( h = \frac{T}{2^k} \), implement the RVWEA method (6.3.1),
we can get the bound of the error estimate
\[ B^2(k, m_0, \ldots, m_k) = M_a + M_b + M. \]

\( \forall \epsilon^* > 0, \exists K^*, \text{ when } k > K^*, \text{ then we have} \)
\[ B \sim O\left(\frac{(\log_2 L)^2}{L}\right). \]

Proof: Say \( M = M_{k_0} + M_{\bar{k_0}} \), having
\[
M_{k_0} = \sum_{j=1}^{k_0} C'_j \sum_{i=0}^{k-j} \frac{(2^i h)^{3/2}}{m_{i+j}}, \text{ according to the Proposition (6.4.1)},
\]
\[
M_{\bar{k_0}} = \sum_{j=k_0+1}^{k} C'_j \sum_{i=0}^{k-j} \frac{(2^i h)^{3/2}}{m_{i+j}} = \sum_{j=k_0}^{k} \sum_{i=0}^{j-k_0} \frac{E[\xi_j]}{m_j},
\]
where \( k_0 \) is some fixed number having \( k_0 \in \mathbb{N}[1, k] \).

We are going to allocate the computational load as we did in the theorem (6.2.4), having
\[
m_i = 2^i m_0 \quad L = \sum_{i=0}^{k} 2^{k-i} m_i,
\]
and estimate the relative error of the \( M \) and \( M_a \).

part 1: relative error of the \( M_{k_0} \)

We can have a look at the \( M_{k_0} \) first.
\[
M_{k_0} = \sum_{j=1}^{k_0} (C'_j \sum_{i=0}^{k-j} \frac{(2^i h)^{3/2}}{m_{i+j}})
= \frac{h^{3/2}}{m_0} \sum_{j=1}^{k_0} [C'_j 2^{-j} \sum_{i=0}^{k-j} 2^{1/2}] \]
As for \( k_0 \) is some fixed number, so we can get the maximum value from \( C'_{0}, \ldots, C'_{k_0} \), having \( C'_m = MAX\{C'_0, \ldots, C'_{k_0}\} \). Then we can get
\[
M_{k_0} = \frac{h^{3/2} C'_m}{m_0 (\sqrt{2} - 1)} [\sqrt{2} 2^{1/2} (\frac{1 - 2^{-\frac{3}{2} k_0}}{2^{3/4} - 1} - (1 - 2^{-k_0})].
\]
Because when
\[
m_i = 2^i m_0,
\]
we have

\[ M_a = \frac{C_1 T(k + 1)}{2^k m_0}, \]

\[ \frac{M_{k_0}}{M_a} \sim \frac{h^{3/2} C_m}{m_0} \frac{m_0 2^k}{C_1 T(k + 1)} \sim \frac{C_m}{k + 1}. \]

part 2: relative error of the \( M_{k_0} \)

\[
\begin{align*}
M_{k_0} &= \sum_{j=0}^{k} \sum_{i=0}^{j-k_0} \frac{E[\xi_i \xi_j]}{m_j} \\
&\leq \sum_{j=0}^{k} \sum_{i=0}^{j-k_0} \frac{\sqrt{E[\xi_i]^2} \sqrt{E[\xi_j]^2}}{m_j}, \text{ from Cauchy-Schwarz} \\
&= \sum_{j=0}^{k} \sum_{i=0}^{j-k_0} \frac{C_1 (2^i h)^{1/2} (2^j h)^{1/2}}{2^i m_0} \\
&= \frac{C_1 h}{m_0 (\sqrt{2} - 1)} \left( (k - k_0 + 1) 2^{1-k_0} - \frac{2^{-k_0} (1 - 2^{-(k-k_0+1)})}{1 - 2^{-1}} \right), \\
&\sim O\left( \frac{1}{(k + 1) 2^{k_0/2}} \right). \\
\end{align*}
\]

part 3: conclusion

Hence, in all, when \( k \gg 1 \), \( M = M_{k_0} + M_{k_0} \ll M_a \). The bound of the error will be close to the method (6.2.1) case, having

\[ B \sim O\left( \frac{(\log_2 L)^2}{L} \right). \]

\( \square \)

From the theorem (6.3.2), we can find that when \( k \) becomes big, the sample error caused by the Random Values reusing will play a little role.

6.4 Estimate \( E[\xi_i \xi_j] \)

Looking more closely at \( E[\xi_i \xi_j] \), \( \xi_i = f(X_{2i+h}) - f(X_{2i+1+h}) \), where \( X_{2i+h} = X_{2i+h}(T) \), \( X_{2i+1+h} = X_{2i+1+h}(T) \), we shall get an estimate, better than the \textit{Cauchy-Schwarz inequality} one.
Proposition 6.4.1. When $\xi_i = f(X_{2i+1}^h) - f(X_{2i}^h)$, where $f(x)$, $a(x)$ and $b(x)$ are twice continuously differentiable, and $f(x)$, $\frac{\partial f(x)}{\partial x}$, $a(x)$, $\frac{\partial a(x)}{\partial x}$, $b(x)$, $\frac{\partial b(x)}{\partial x}$ satisfy a uniform Lipschitz condition, we can have the $E|\xi_i^2| \sim O((2^i h)^{3/2})$, ($i < j$).

That is to say, there is a constant $K$, s.t. $\forall h$, we always have $|E[\xi_i^2]| \leq K(2^i h)^{3/2}$. One may notice that it is different from the deterministic case, where Richardson Extrapolation method can be applied.

We need a simple case to cut our teeth on.

Proposition 6.4.2. When $\xi_i = f(X_{2i+1}^h) - f(X_{2i}^h)$, where $a(x)$ and $b(x)$ are twice continuously differentiable, $f(x) = x$, and $b(x)$, $1$ satisfy a uniform Lipschitz condition, the $E|\xi_0^2| \sim O(h^3)$. That is equivalent of the statement, there exists a constant $K > 0$, independent to $h$, such that for all $h$ we have

$$|E[\xi_0^2]| \leq K(h^3).$$

After proving and stating a series of lemma, we shall prove Proposition (6.4.2) at the end of the section.

Here, we give some definition of the notations referred to the following part:

**Definition:** $h = \frac{T}{2^k}$ ($k$, some constant), is the finest step size in this approximation method.

**Definition:** $X_{2i}^h$ — the Euler approximation of the

$$\int a(x,t) \, dt + b(x,t) \, dW,$$

corresponding to the Brownian path $W$, having step size $2^i h$ in time $[0,T]$.

So $X_h$, $X_{2h}$, $X_{4h}$ are the Euler approximated solution on the Brownian path $W$ with step size $h$, $2h$, and $4h$ respectively. And $\forall n = 1, 2, \ldots, 2^{k-2}$, it follows from the Euler method that

$$X_{4h}^{(4n)} = X_h^{(4n-1)} + a(X_h^{(4n-1)}, t) h + b(X_h^{(4n-1)}, t) \Delta W^{(4n-1)},$$

$$X_{2h}^{(2n)} = X_{2h}^{(2n-1)} + a(X_{2h}^{(2n-1)}, t) 2h + b(X_{2h}^{(2n-1)}, t) (\Delta W^{(4n-2)} + \Delta W^{(4n-1)}),$$

$$X_{4h}^{(n)} = X_{4h}^{(n-1)} + a(X_{4h}^{(n-1)}, t) 4h + b(X_{4h}^{(n-1)}, t) (\Delta W^{(4n-4)} + \Delta W^{(4n-3)} + \Delta W^{(4n-2)} + \Delta W^{(4n-1)}).$$
We also denote
\[ o(n) = X_{2n}^{(2n)} - X_h^{(4n)} \]
and
\[ f_1 = f(X_h^{(4n-4)}) \]
\[ g_1 = g(X_h^{(4n-4)}) \]
\[ b_1 = b(X_h^{(4n-4)}) \]
\[ a_1 = a(X_h^{(4n-4)}) \]
\[ a'_1 = \frac{\partial a(X_h^{(4n-4)})}{\partial x} \]
\[ b'_1 = \frac{\partial b(X_h^{(4n-4)})}{\partial x} \]
\[ a_4 = \frac{\partial a(X_{4h}^{(n-1)})}{\partial x} \]
\[ b_4 = \frac{\partial b(X_{4h}^{(n-1)})}{\partial x} \].

**Definition:** \( \xi_0(n) \) — the approximation of the \( \xi_0(n) = X_{2h}^{(2n)} - X_h^{(4n)} \), having \( \xi_0(0) = \xi_0(0) \).

\[
\xi_0(n) = \xi_0(n - 1) + 4\xi_0(n - 1)\alpha_1 h - b_1 b'_1 (\Delta W^{(4n-4)} \Delta W^{(4n-3)} + \Delta W^{(4n-2)} \Delta W^{(4n-1)}) \\
\quad + (\xi_0(n - 1)\alpha_1 + \frac{1}{2}\xi_0(n - 1)\beta_1^2) (\Delta W^{(4n-4)} + \Delta W^{(4n-3)} + \Delta W^{(4n-2)} + \Delta W^{(4n-1)}) \\
\quad - a_1 b_1 h (\Delta W^{(4n-3)} + \Delta W^{(4n-1)}) - a'_1 b'_1 h (\Delta W^{(4n-4)} + \Delta W^{(4n-2)}) \\
\quad + 2\xi_0(n - 1)\beta_1^2 + \xi_0(n - 1)\beta_1 (\Delta W^{(4n-4)} + \Delta W^{(4n-3)} \Delta W^{(4n-2)}) \\
\quad + \xi_0(n - 1)\beta_1 b'_1 (\Delta W^{(4n-4)} + \Delta W^{(4n-3)} \Delta W^{(4n-1)}) \\
\quad + \frac{1}{2} b_1^2 b'_1 [((\Delta W^{(4n-4)})^2 \Delta W^{(4n-3)} - (\Delta W^{(4n-4)})^2 \Delta W^{(4n-2)}) \\
\quad - (\Delta W^{(4n-3)})^2 \Delta W^{(4n-2)} - (\Delta W^{(4n-4)})^2 \Delta W^{(4n-1)}) \\
\quad - (\Delta W^{(4n-3)})^2 \Delta W^{(4n-1)} \Delta W^{(4n-1)}]
\]

**Definition:** \( \xi_1(n) \) — the approximation of the \( \xi_1(n) = X_{4h}^{(n)} - X_{2h}^{(2n)} \), having \( \xi_1(0) = \xi_1(0) \).

\[
\xi_1(n) = \xi_1(n - 1) + 4\xi_1(n - 1)\alpha_2 h - b_2 b'_2 (\Delta W^{(4n-4)} + \Delta W^{(4n-3)})(\Delta W^{(4n-2)} + \Delta W^{(4n-1)}) \\
\quad + (\xi_1(n - 1)\alpha_2 + \frac{1}{2}\xi_1(n - 1)\beta_2^2) (\Delta W^{(4n-4)} + \Delta W^{(4n-3)} + \Delta W^{(4n-2)} + \Delta W^{(4n-1)}) \\
\quad - 2a_2 b_2 (\Delta W^{(4n-4)} + \Delta W^{(4n-3)}) h - 2a_2 b'_2 (\Delta W^{(4n-2)} + \Delta W^{(4n-1)}) h \\
\quad - \frac{1}{2} b_2^2 b'_2 ((\Delta W^{(4n-4)})^2 \Delta W^{(4n-3)} + \Delta W^{(4n-4)})^2 \Delta W^{(4n-1)} + \Delta W^{(4n-1)})
\]

For the expression convenience, we write \( F(h) \sim O(h^p) \), if there exist a constant \( K \) and a positive constant \( h_0 \), such that \( |f(h)| \leq K h^p \), for any time discretization with maximum step size \( h \in (0, h_0) \). In the following sections, we even
replace some function $F(h)$, in which $F(h) \sim O(h^p)$, with $O(h^p)$, because we do not care the exact expression of the $F(h)$, but the bound of the function.

**Lemma 6.4.3.** Suppose $N = 2^k$ (k is some constant) and $h = \frac{T}{N}$, $\xi_0(n) = X_{2h}^{(2n)} - X_1^{(4n)}$, $\xi_1(n) = X_{2h}^{(n)} - X_{2h}^{(2n)}$. we can approximate $\xi_0(n)$ and $\xi_1(n)$ with $\xi_0(n)$ and $\xi_1(n)$ respectively. And

$$
\xi_0(n) = \frac{\xi_0(n)}{O(h^2)} + O(h^2)
$$

$$
\xi_1(n) = \frac{\xi_1(n)}{O(h^2)} + O(h^2)
$$

$$
\xi_0(n) \cdot \xi_1(n) = \frac{\xi_0(n) \cdot \xi_1(n)}{O(h^2)} + O(h^2)
$$

**Proof:** Because the Euler method gives

$$
X_{2h}^{(4n)} = X_h^{(4n-1)} + a(X_h^{(4n-1)}, t)h + b(X_h^{(4n-1)}, t)\Delta W^{(4n-1)},
$$

$$
X_{2h}^{(2n)} = X_h^{(2n-1)} + a(X_h^{(2n-1)}, t)2h + b(X_{2h}^{(2n-1)}, t)(\Delta W^{(4n-2)} + \Delta W^{(4n-1)}),
$$

$$
X_{4h}^{(n)} = X_{4h}^{(n-1)} + a(X_{4h}^{(n-1)}, t)4h + b(X_{4h}^{(n-1)}, t)(\Delta W^{(4n-4)} + \Delta W^{(4n-3)} + \Delta W^{(4n-2)} + \Delta W^{(4n-1)}),
$$

we can get

$$
\xi_0(n) = X_{2h}^{(2n-2)} - X_h^{(4n)}
$$

$$
= [X_{2h}^{(2n-2)} + a(x_{2h}^{(2n-2)}, t)2h + a(x_{2h}^{(2n-2)}, t)2h + b(x_{2h}^{(2n-2)}, t)(\Delta W^{(4n-4)} + \Delta W^{(4n-3)} + \Delta W^{(4n-2)} + \Delta W^{(4n-1)})]
$$

$$
+ [X_h^{(4n-4)} + a(x_h^{(4n-4)}, t)h + a(x_h^{(4n-4)}, t)h + a(x_h^{(4n-4)}, t)h + a(x_h^{(4n-4)}, t)h]
$$

$$
+ b(x_h^{(4n-4)}, t)\Delta W^{(4n-4)} + b(x_h^{(4n-4)}, t)\Delta W^{(4n-3)} + b(x_h^{(4n-4)}, t)\Delta W^{(4n-2)} + b(x_h^{(4n-4)}, t)\Delta W^{(4n-1)}]
$$

$$
\xi_1(n) = X_{4h}^{(n)} - X_{2h}^{(2n)}
$$

$$
= [X_{4h}^{(n-1)} + a(x_{4h}^{(n-1)}, t)4h]
$$

$$
+ b(x_{4h}^{(n-1)}, t)\Delta W^{(4n-4)} + b(x_{4h}^{(n-1)}, t)\Delta W^{(4n-3)} + b(x_{4h}^{(n-1)}, t)\Delta W^{(4n-2)} + b(x_{4h}^{(n-1)}, t)\Delta W^{(4n-1)}]
$$

On the other hand, for any well behave function $g(x)$, it will have up to order $\frac{3}{2}$ Taylor expansion for $g(X_{2h}^{(2n-2)}) - g(X_h^{(4n-4)})$, $g(X_{2h}^{(2n-2)}) - g(X_h^{(4n-3)})$, $g(X_{2h}^{(2n-1)}) - g(X_h^{(4n-2)})$, and $g(X_{2h}^{(2n-1)}) - g(X_h^{(4n-1)})$.

$$
g(X_{2h}^{(2n-2)}) - g(X_h^{(4n-4)})
$$

$$
= \frac{\partial g_1}{\partial X} (X_{2h}^{(2n-2)} - X_h^{(4n-4)}) + \frac{1}{2} \frac{\partial^2 g_1}{\partial X^2} (X_{2h}^{(2n-2)} - X_h^{(4n-4)})^2 + O(h^{\frac{3}{2}})
$$

$$
= \frac{\partial g_1}{\partial X} \xi_0 + \frac{1}{2} \frac{\partial^2 g_1}{\partial X^2} \xi_0^2 + O(h^{\frac{3}{2}})
$$
Numerical Approximation for SDE

\[ g(X_{2h}^{(2n-2)}) - g(X_h^{(4n-3)}) \]
\[ = [g(X_{2h}^{(2n-2)}) - g(X_h^{(4n-4)})] - [g(X_h^{(4n-3)}) - g(X_h^{(4n-4)})] \]
\[ = \left[ \frac{\partial g_1}{\partial X} \xi_0 + \frac{1}{2} \frac{\partial^2 g_1}{\partial X^2} \xi_0^2 \right] \]
\[ - \left[ \frac{\partial g_1}{\partial X} (a_1 h + b_1 \Delta W^{(4n-4)}) + \frac{1}{2} \frac{\partial^2 g_1}{\partial X^2} (b_1 \Delta W_0)^2 \right] + O(h^{\frac{3}{2}}) \]
\[ = \frac{\partial g_1}{\partial X} (\xi_0 - a_1 h - b_1 \Delta W^{(4n-4)}) + \frac{1}{2} \frac{\partial^2 g_1}{\partial X^2} \left[ \xi_0^2 + (b_1 \Delta W^{(4n-4)})^2 \right] + O(h^{\frac{3}{2}}) \]

\[ g(X_{2h}^{(2n-1)}) - g(X_h^{(4n-2)}) \]
\[ = [g(X_{2h}^{(2n-1)}) - g(X_h^{(4n-4)})] - [g(X_h^{(4n-2)}) - g(X_h^{(4n-4)})] \]
\[ = \left[ \frac{\partial g_1}{\partial X} \left( X_{2h}^{(2n-1)} - X_h^{(4n-4)} \right) + \frac{1}{2} \frac{\partial^2 g_1}{\partial X^2} \left( X_{2h}^{(2n-1)} - X_h^{(4n-4)} \right)^2 \right] \]
\[ - \left[ \frac{\partial g_1}{\partial X} \left( X_{2h}^{(4n-2)} - X_h^{(4n-4)} \right) + \frac{1}{2} \frac{\partial^2 g_1}{\partial X^2} \left( X_{2h}^{(4n-2)} - X_h^{(4n-4)} \right)^2 \right] + O(h^{\frac{3}{2}}) \]
\[ = \frac{\partial g_1}{\partial X} \left[ \xi_0 + \frac{\partial b_1}{\partial X} \xi_0 (\Delta W^{(4n-4)} + \Delta W^{(4n-3)}) \right] \]
\[ + b_1 \frac{\partial b_1}{\partial X} \Delta W^{(4n-4)} \Delta W^{(4n-3)} \]
\[ + \frac{1}{2} \frac{\partial^2 g_1}{\partial X^2} \left[ \xi_0^2 + 2b_1 \xi_0 (\Delta W^{(4n-4)} + \Delta W^{(4n-3)}) \right] \]
\[ - b_1^2 \left[ (\Delta W^{(4n-4)} + \Delta W^{(4n-3)})^2 \right] + O(h^{\frac{3}{2}}) \]

Replace \( g(x) \) in the Taylor expansion with the \( a(x) \) and then with the \( b(x) \), we can get the Taylor expansion of the functions \( a(x) \) and \( b(x) \). And then, we can obtain \( \xi_0(n) \) and \( \xi_1(n) \), the approximation of \( \xi_0(n) \) and \( \xi_1(n) \), by replacing functions \( a(x) \) and \( b(x) \) in \( \xi_0(n) \) and \( \xi_1(n) \) with the Taylor expansion of the \( a(x) \) and \( b(x) \).

Furthermore, according to the Euler Scheme, \( \xi_0(n) \sim O(h^{\frac{1}{2}}) \) and \( \xi_1(n) \sim O((2h)^{\frac{1}{2}}) \), any terms in \( \xi_0(n) \) and \( \xi_1(n) \), having higher order than \( O(h^{\frac{3}{2}}) \), can be omitted. Henceforth, the approximations of \( \xi_0(n) \) and \( \xi_1(n) \), within order \( \frac{3}{2} \) are...
\( \hat{\xi}_0(n) \) and \( \hat{\xi}_1(n) \). That is

\[
\begin{align*}
\hat{\xi}_0(n) &= \xi_0(n) + O(h^2) \\
\hat{\xi}_1(n) &= \xi_1(n) + O(h^2) \\
\hat{\xi}_0(n) \cdot \hat{\xi}_1(n) &= \hat{\xi}_0(n) \cdot \hat{\xi}_1(n) + O(h^{3/2})
\end{align*}
\]

This Lemma tells us only terms in \( \hat{\xi}_0(n) \) and \( \hat{\xi}_1(n) \) play big roles when we are trying to prove \( E[\xi_0(n)\xi_1(n)] \sim O(h^{3/2}) \).

\[ \square \]

We may see what happens here, when we replace \( \xi_0(n)/h^{3/2} \) and \( \xi_1(n)/h^{3/2} \) in

\[
\begin{align*}
\xi_0(n) &= \hat{\xi}_0(n) + O(h^2) \\
\xi_1(n) &= \hat{\xi}_1(n) + O(h^2),
\end{align*}
\]

with \( \hat{\xi}(n) \) and \( \hat{\xi}(n) \) respectively, getting

\[ \xi_0(n)/h^{3/2} = \hat{\xi}(n) \quad \xi_1(n)/h^{3/2} = \hat{\xi}(n). \tag{6.14} \]

Before our following discussion, we denote:

\[
\begin{align*}
D_{u1}(n) &= \left[ b^*_2(b'_2(DW^{(4n-4)} + DW^{(4n-3)} + DW^{(4n-2)} + DW^{(4n-1)}) \\
&\quad - b^*_2(b'_2(DW^{(4n-4)} + DW^{(4n-3)}))\Delta W^{(4n-2)} + DW^{(4n-1)}/(h^{3/2}) \right] \\
D_{u2}(n) &= \left[ b^*_2(b'_2+\Delta W^{(4n-4)} + DW^{(4n-3)} +DW^{(4n-2)} + DW^{(4n-1)}) \right. \\
&\quad \left. - \frac{1}{\sqrt{h}} b'_1 b'_2 \Delta W^{(4n-4)} \Delta W^{(4n-3)} - \Delta W^{(4n-2)} \Delta W^{(4n-1)} \right] \\
\delta u(n) &= \left[ b^*_2(b'_2(DW^{(4n-4)} + DW^{(4n-3)} + DW^{(4n-2)} + DW^{(4n-1)}) \\
&\quad - b^*_2(b'_2(DW^{(4n-4)} + DW^{(4n-3)}))\Delta W^{(4n-2)} + DW^{(4n-1)}/h \right] \\
&\quad + b^*_2(b'_2(DW^{(4n-4)} + DW^{(4n-3)}))\Delta W^{(4n-2)}/h^{3/2} \\
&\quad + (\Delta W^{(4n-4)} + \Delta W^{(4n-3)})\Delta W^{(4n-2)}/h^{3/2} \\
&\quad + b^*_2(b'_2(1)(\Delta W^{(4n-4)})^2\Delta W^{(4n-3)}/h) - \frac{1}{h}(\Delta W^{(4n-4)})^2\Delta W^{(4n-2)} \\
&\quad + \frac{1}{h}(\Delta W^{(4n-3)})^2\Delta W^{(4n-2)} - \frac{1}{h}(\Delta W^{(4n-4)})^2\Delta W^{(4n-1)} \\
&\quad + \frac{1}{h}(\Delta W^{(4n-3)})^2\Delta W^{(4n-1)} - \frac{1}{h}(\Delta W^{(4n-3)})^2\Delta W^{(4n-1)} \\
&\quad - a_1 b'_1(\Delta W^{(4n-3)} + \Delta W^{(4n-1)}) - a_1 b'_1(\Delta W^{(4n-4)} + \Delta W^{(4n-2)})
\end{align*}
\]

We can see \( D_{u1}(n), D_{u2}(n), \delta u(n), \delta v(n) \sim O(h^{3/2}) \). The substitution (6.14)
gives us
\[
\begin{align*}
\Delta X(n) &= a_4h + b_4(\Delta W^{(4n-4)} + \Delta W^{(4n-3)} + \Delta W^{(4n-2)} + \Delta W^{(4n-1)}) \\
\Delta \tilde{u}(n) &= 4\tilde{u}(n)a' + D_1(n) + h^{3/2}\delta u(n) + O(h^{3/2}) \\
\Delta \tilde{v}(n) &= 4\tilde{v}(n)a' + D_2(n) + h^{3/2}\delta v(n) + O(h^{3/2}).
\end{align*}
\]
(6.15)

We note that the formulas (6.15) can be interpreted as a weak approximation,
\[
\begin{align*}
\Delta X(n) &= a_4h + b_4(X,t)(\Delta W^{(4n-4)} + \Delta W^{(4n-3)} + \Delta W^{(4n-2)} + \Delta W^{(4n-1)}) \\
\Delta u(n) &= 4ua'_n + [u(n)b'_2(\Delta W^{(4n-4)} + \Delta W^{(4n-3)} + \Delta W^{(4n-2)} + \Delta W^{(4n-1)}) \\
&\quad - b_2b'_2(\Delta W^{(4n-4)} + \Delta W^{(4n-3)} + \Delta W^{(4n-2)} + \Delta W^{(4n-1)})/(h^{3/2}) \\
\Delta v(n) &= 4v(n)a' + [v(n)b'_2(\Delta W^{(4n-4)} + \Delta W^{(4n-3)} + \Delta W^{(4n-2)} + \Delta W^{(4n-1)}) \\
&\quad - b_1b'_1(\Delta W^{(4n-4)} + \Delta W^{(4n-3)} + \Delta W^{(4n-2)} + \Delta W^{(4n-1)})/(h^{3/2}),
\end{align*}
\]
(6.16)

having correction terms, \(\delta u(n)\) and \(\delta v(n)\), to the system
\[
\begin{align*}
\dot{d}X &= a_4 + b_4(X,t)dW_1 \\
\dot{d}u &= 4ua'_n dt + [ub'_2 dW_1 - bb'_2 dW_2] \\
\dot{d}v &= 4v(n)a'_n dt + [vb'_2 dW_1 - bb'_2 dW_3],
\end{align*}
\]
where \(Y(0) = (x_0, 0, 0)\), and \(W_1, W_2, W_3\) are Brownian process. If we ignored those correction terms, we may find (6.16) is more or less like Euler approximation. However, we need to consider those correction terms, and estimate their order of magnitude.

For the later use, we define some notation here.
\[
G = E\begin{pmatrix} 0 & \Delta X(n)\delta u(n) & \Delta X(n)\delta v(n) \\
\Delta X(n)\delta u(n) & 2\Delta u(n)\delta u(n) & \Delta u(n)\delta v(n) + \delta u(n)\Delta v(n) \\
\Delta X(n)\delta v(n) & \Delta u(n)\delta v(n) + \delta u(n)\Delta v(n) & 2\delta v(n)\Delta v(n) \end{pmatrix}
\]
\[
= \begin{pmatrix} g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33} \end{pmatrix},
\]

\(g_{11} = 0\)
\[
g_{12} = g_{21} = b_4b'_2u^2h - b_4b'_2b''h \\
g_{13} = g_{31} = b_4b'_2v^2h - a_1b_4b'_1h - a'_1b_1b_4h - b_4b'_2b''h \\
g_{22} = 2u^3b'_2b''h - 2ub'_2b''h \\
g_{23} = g_{32} = 2u^2ub'b''h - 2uab'b''h - 2ua'bb'h \\
&\quad - 4vb(b')^2h - 6vob'b''h - 2ub^2b'b''h + 2u^2vb'b''h \\
g_{33} = 2b'_1b''_1v^3h - 2a_1b_1b'_1vh - 2a'_1b_1b'_1vh - 2b'_2b''vvh.
\]

Numerical Approximation for SDE

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Lemma 6.4.4. The \((\Delta X(n), \Delta u(n), \Delta v(n))\) has second-moment matrix \(H = \)
\[
\begin{pmatrix}
16a^2h^2 + 4b^2h & 16a_4a_1'vh^2 + 4b_1b_2'vh \\
16a_4a_1'vh^2 + 4b_1b_2'vh & 16(ua_2h^2) + 4(b_2u)^2h + 4(b_1b_2')^2h \\
16a_4a_1'vh^2 + 4b_1b_2'vh & 16a_1a_2'h^2uv + 4b_1b_2'uvw \\
16a_4a_1'vh^2 + 4b_1b_2'vh & 16a_1a_2'h^2uv + 4b_1b_2'uvw \\
16a_4a_1'vh^2 + 4b_1b_2'vh & 16a_1a_2'h^2uv + 4b_1b_2'uvw \\
16(a_1'v)^2 + 4(b_1v)^2h + 2(b_1b_1')^2h
\end{pmatrix}
\]
The approximation system \((\Delta X(n), \Delta u(n), \Delta v(n)) = (\Delta X(n), \Delta u(n) + h^{\frac{1}{2}}\delta u, \Delta v(n)) + h^{\frac{1}{2}}\delta v\), having correction terms, will have up to \(O(h^{\frac{3}{2}})\) second-moment matrix
\[H + h^{\frac{1}{2}}G,\]
Proof: Because the formulas (6.15) show that \(\delta u(n) \sim O(h^{1/2}), \delta v(n) \sim O(h^{1/2})\), the matrix
\[
hE \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta u(n)\delta u(n) & \delta u(n)\delta v(n) \\ 0 & \delta u(n)\delta v(n) & \delta v(n)\delta v(n) \end{pmatrix} \sim O(h^2),
\]
will be order 2. Hence, the second-moment matrix within \(O(h^{3/2})\) for the system \((\Delta X(n), \Delta u(n), \Delta v(n))\) will be
\[
\begin{pmatrix}
\Delta X(n)\Delta X(n) & \Delta X(n)\Delta u(n) & \Delta X(n)\Delta v(n) \\
\Delta X(n)\Delta u(n) & \Delta u(n)\Delta u(n) & \Delta u(n)\Delta v(n) \\
\Delta X(n)\Delta v(n) & \Delta u(n)\Delta v(n) & \Delta v(n)\Delta v(n) \\
\end{pmatrix}
\]
\[+h^{\frac{1}{2}}E \begin{pmatrix} 0 & \Delta X(n)\delta u(n) & \Delta X(n)\delta v(n) \\ \Delta X(n)\delta u(n) & 2\Delta u(n)\delta u(n) & \Delta u(n)\delta v(n) + \delta u(n)\Delta v(n) \\ \Delta X(n)\delta v(n) & \Delta u(n)\delta v(n) + \delta u(n)\Delta v(n) & 2\delta v(n)\Delta v(n) \end{pmatrix}.
\]
Simple calculation will give the second-moment matrix for the systems of \((\Delta X(n), \Delta u(n), \Delta v(n)), H = \)
\[
\begin{pmatrix}
16a^2h^2 + 4b^2h & 16a_4a_1'vh^2 + 4b_1b_2'vh \\
16a_4a_1'vh^2 + 4b_1b_2'vh & 16(ua_2h^2) + 4(b_2u)^2h + 4(b_1b_2')^2h \\
16a_4a_1'vh^2 + 4b_1b_2'vh & 16a_1a_2'h^2uv + 4b_1b_2'uvw \\
16a_4a_1'vh^2 + 4b_1b_2'vh & 16a_1a_2'h^2uv + 4b_1b_2'uvw \\
16a_4a_1'vh^2 + 4b_1b_2'vh & 16a_1a_2'h^2uv + 4b_1b_2'uvw \\
16(a_1'v)^2 + 4(b_1v)^2h + 2(b_1b_1')^2h
\end{pmatrix}
\]
And we can also get the second-moment matrix up to \(O(h^{\frac{3}{2}})\) for the approximation system \((\Delta X(n), \Delta u(n), \Delta v(n)) = (\Delta X(n), \Delta u(n) + h^{\frac{1}{2}}\delta u, \Delta v(n)) + h^{\frac{1}{2}}\delta v\),
\[H + h^{\frac{1}{2}}G.
\]
Now, it will be good to see the error between the approximation (6.15) and the system, which the Euler approximation (6.16) work for.

The following discussions are based on the condition that \(a(x)\), and \(b(x)\) are moderately smooth functions. In these lemma, it means that \(a(x)\), and \(b(x)\) are three times differentiable, and \(a(x), \frac{\partial a(x)}{\partial x}, \frac{\partial^2 a(x)}{\partial x^2}, b(x), \frac{\partial b(x)}{\partial x}, \frac{\partial^2 b(x)}{\partial x^2}\) satisfy the uniform Lipschitz condition.

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Lemma 6.4.5. When \( a(x) \) and \( b(x) \) are moderately smooth functions, the \( \hat{u} \hat{v} \) having \( \hat{u} \) and \( \hat{v} \) as in formulas (6.15) is an order 0.5 weak approximation to the SDE system \( \hat{u} \hat{v} \), which is approximated with Euler method by the formulas (6.16). That is \( E[\hat{u}(T)\hat{v}(T)] - E[\hat{u}(T)\hat{v}(T)] = O(h^{0.5}) \).

**Proof.** Define: \( \varphi(t, Y) = E[\hat{u}(T)\hat{v}(T)|Y(t) = Y] \), that is \( \varphi(t, x, u, v) = E[\hat{u}(T)\hat{v}(T)|x(t) = x, u(t) = u, v(t) = v] \). Then

\[
\begin{align*}
\varphi(t, Y_N) &= E[\hat{u}(T)\hat{v}(T)|x(t) = x_N, u(t) = u_N, v(t) = v_N] \\
\varphi(0, Y_0) &= E[\hat{u}(T)\hat{v}(T)]
\end{align*}
\]

Because the \( a(x) \) and \( b(x) \) are moderately smooth functions, we can have that \( u(t, Y), v(t, Y) \) are twice differentiable, and that \( u(T, Y), \frac{\partial u(T, Y)}{\partial Y^i}, v(T, Y), \) and \( \frac{\partial v(T, Y)}{\partial Y^i}, \ (i = 1, 2, 3), \) satisfy the uniform Lipschitz condition. Consequently, \( f(t, Y) = u(T, Y)v(T, Y) \in \mathcal{C}^2 \). According to the Lemma (1.2.7), we can have \( \varphi(t, Y) \in \mathcal{C}^2 \).

We can have a look at the general situation first, and then apply the result to the approximation and SDE system we are considering.

When the system and approximation system have the same initial values at instant \( t_k \), we are going to see the difference between the system

\[
\begin{align*}
dZ &= A(Z, t) dt + B_1(Z, t) dR(t) \\
Z(t_k) &= Z_k = \hat{Y}_k
\end{align*}
\]

and the approximation system

\[
\begin{align*}
\Delta \hat{Y}_k &= A(\hat{Y}, t)h + B_1(\hat{Y}, t)\Delta R + h^{\frac{1}{2}} B_2(\hat{Y}, t)\Delta R \\
\hat{Y} &= Z_k
\end{align*}
\]

within time interval \([t_k, t_{k+1}].\)

First of all, let's solve

\[
\begin{align*}
dZ &= A(Z, t) dt + B_1(Z, t) dR(t) \\
Z(t_k) &= Z_k = \hat{Y}_k
\end{align*}
\]

According to the Itô-Taylor expansion:

\[
f(x_t) = f(x_{t_0}) + \int_{t_0}^{t} \left[ a(x_s) \frac{\partial}{\partial x} f(x) + \frac{1}{2} b^2(x_s) \frac{\partial^2}{\partial x^2} f(x_s) \right] ds + \int_{t_0}^{t} b(x_s) \frac{\partial}{\partial x} f(x_s) dW_s,
\]

when

\[
\begin{align*}
Y_t &= Y_{t_0} + \int_{t_0}^{t} A(Y_s, s) ds + \int_{t_0}^{t} B_1(Y_s, s) dR(s) \\
A(Y_s) &= A(Y_{t_0}) + \int_{t_0}^{s} [A(Y_r) \frac{\partial}{\partial Y} A(Y_r) + \frac{1}{2} B_1^2(Y_r) \frac{\partial^2}{\partial Y^2} A(Y_r)] dr \\
&\quad + \int_{t_0}^{s} B_1(Y_r) \frac{\partial}{\partial Y} A(Y_r) dR_r
\end{align*}
\]
\[
B_1(Y_t) = B_1(Y_0) \int_0^t \left[ A(Y_r) \frac{\partial}{\partial Y} B_1(Y_r) + \frac{1}{2} B_1^2(Y_r) \frac{\partial^2}{\partial Y^2} B_1(Y_r) \right] dr \\
+ \int_0^t B_1(Y_r) \frac{\partial}{\partial Y} B_1(Y_r) dR_r,
\]

we can get
\[
Z_t = Z_{t_0} + \int_{t_0}^t A(Z_s) ds + \int_{t_0}^t B_1(Z_s) dR(s) \\
+ \int_{t_0}^t \left\{ \int_{t_0}^s \left[ B_1(Z_r) \frac{\partial}{\partial Y} B_1(Z_r) \right] dR(r) \right\} dR(s) \\
\simeq Z_{t_0} + A(Z_{t_0}) \Delta t + B(Z_{t_0}) \Delta R \quad \text{(approximated within order } \frac{3}{2} \text{).}
\]

(6.17)

Because of
\[
\varphi(t_k, Z_k) = E[u(Y(T))v(Y(T))|Y(t_k) = Z_k] \\
= E\{E[u(Y(T))v(Y(T))|Y(t_{k+1}) = Z_{k+1}]|Y(t_k) = Z_k\} \\
= E\varphi(t_{k+1}, Z_{k+1}),
\]
it is true that the formula (6.17) gives the following statement
\[
\varphi(t_k, Z_k) = E\varphi(t_{k+1}, Z_{k+1}) \\
= \varphi(t_{k+1}, Z_k) + \sum_{i=1}^3 A_i(Z_k) \frac{\partial \varphi(t_{k+1}, Z_k)}{\partial Y(i)} h \\
+ \frac{1}{2} \sum_{i,q=1}^3 H_{iq}(Z_k) \frac{\partial^2 \varphi(t_{k+1}, Z_k)}{\partial Y(i) \partial Y(q)} + O(h^2),
\]

where \( H \) is the covariance matrix of \( B_1 dR \). The approximation system \( \tilde{u} \tilde{v} \) will have
\[
E[\varphi(t_{k+1}, \tilde{Y}_{k+1})] = \varphi(t_{k+1}, \tilde{Y}_{k+1}) \\
= \varphi(t_{k+1}, \tilde{Y}_k) + \sum_{i=1}^3 A_i(\tilde{Y}_k) \frac{\partial \varphi(t_{k+1}, \tilde{Y}_k)}{\partial Y(i)} h \\
+ \frac{1}{2} \sum_{i,q=1}^3 H_{iq} \frac{\partial^2 \varphi(t_{k+1}, \tilde{Y}_k)}{\partial Y(i) \partial Y(q)} \\
+ \frac{h^2}{2} \sum_{i,q=1}^3 G_{iq} \frac{\partial^2 \varphi(t_{k+1}, \tilde{Y}_k)}{\partial Y(i) \partial Y(q)} + O(h^2),
\]

where the \( G \) is the covariance matrix of \( B_1 dR \) and \( B_2 dR \). These two formulas give
\[
\varphi(t_k, Z_k) - E[\varphi(t_{k+1}, \tilde{Y}_{k+1})] \\
= \frac{h^2}{2} \sum_{i,q=1}^3 G_{iq} \frac{\partial^2 \varphi(t_{k+1}, \tilde{Y}_k)}{\partial Y(i) \partial Y(q)} + O(h^2).
\]

(6.18)
Now we can apply this general result on our problem. The last Lemma gives us the covariance matrix $H$ and $C$. Therefore, $\varphi(t_k, Z_k) - E[\varphi(t_{k+1}, \tilde{Y}_{k+1})] \sim O(h^{3/2})$, since $G_{tq} \sim O(h)$. Because $Z_0 = Y_0 = \tilde{Y}_0$, we can get the difference between systems (6.17) and (6.16) within $[t_0, t_N]$ by adding up the difference generated in each $[t_k, t_{k+1}]$, having $\varphi(0, Y_0) - E[\varphi(t_N, \tilde{Y}_N)] = O(h^{3/2})$. That is $E[u(T)v(T)] - E[\tilde{u}(T)\tilde{v}(T)] = O(h^{3/2})$.

We may observe that the Lemma (6.4.4) gives us

$$
\Delta u(n) \delta v(n) + \delta u(n) \Delta v(n) = 2v^2 u' b'' h - 2uab'b' h - 2ua'bb' h - 4v b'(b')^3 h - 6v b^2 b'' h - 2u b^2 b'' h + 2u^2 v b'' h,
$$

and $\frac{\partial \varphi}{\partial u} = 0, \frac{\partial \varphi}{\partial v} = 0, \frac{\partial \varphi}{\partial u} = 0, \frac{\partial \varphi}{\partial v} = 0, \frac{\partial \varphi}{\partial u} = 0, \frac{\partial \varphi}{\partial v} = 0$, so we can get

$$
\varphi(t_k, Z_k) - E[\varphi(t_{k+1}, \tilde{Y}_{k+1})] = \frac{h^{3/2}}{2} \left( 2v^2 u' b'' h - 2uab'b' h - 2ua'bb' h - 4v b'(b')^3 h - 6v b^2 b'' h - 2u b^2 b'' h + 2u^2 v b'' h \right) + O(h^2).
$$

**Corollary 6.4.6.** When $a(x), b(x)$ are three times differentiable, $a(x), \frac{\partial a(x)}{\partial x}$, $b(x), \frac{\partial b(x)}{\partial x}$, and $\frac{\partial^2 b(x)}{\partial x^2}$ satisfy the uniform Lipschitz condition, and the $\tilde{u}, \tilde{v}$ are implemented by formulas (6.15), then $E[\tilde{u}(T)\tilde{v}(T)] \sim O(h^{3/2})$.

**Proof:** According to the Lemma (6.4.5), $E[u(T)v(T)] - E[\tilde{u}(T)\tilde{v}(T)] = O(h^{3/2})$, henceforth $E[\tilde{u}(T)\tilde{v}(T)] \sim O(h^{3/2})$ becomes true, if we can proof $E(u(T)v(T)) = 0$.

So as to see it clearly, we define the Brownian process $W_1, W_2, W_3$ as below:

$$
\begin{align*}
\Delta W_1^{(n)} &= (\Delta W_{(4n-4)}^{(4n-4)} + \Delta W_{(4n-3)}^{(4n-3)} + \Delta W_{(4n-2)}^{(4n-2)} + \Delta W_{(4n-1)}^{(4n-1)}) \\
\Delta W_2^{(n)} &= (\Delta W_{(4n-4)}^{(4n-4)} + \Delta W_{(4n-3)}^{(4n-3)}) (\Delta W_{(4n-2)}^{(4n-2)} + \Delta W_{(4n-1)}^{(4n-1)}) / h^{1/2} \\
\Delta W_3^{(n)} &= (\Delta W_{(4n-4)}^{(4n-4)} \Delta W_{(4n-3)}^{(4n-3)} + \Delta W_{(4n-2)}^{(4n-2)} \Delta W_{(4n-1)}^{(4n-1)}) / h^{1/2}
\end{align*}
$$

Recall the formulas (6.16),

$$
dY = A(Y, t) dt + B_1(Y, t) dR, \quad Y(0) = (x_0, 0, 0),
$$

$$
\begin{align*}
dx &= a(x, t) dt + b(x, t) dW_1 \\
du &= 4u a' dt + u b' dW_1 - b b' dW_2 \\
dv &= 4v a' dt + v b' dW_1 - b b' dW_3 \\
x(0) &= x_0, \quad u(0) = 0, \quad v(0) = 0
\end{align*}
$$

we can see that, the Brownian path $(W_1, W_2, W_3)$, having increment $(dW_1, dW_2, dW_3)$, gives the SDE (6.16) solution, $(x(T), u(T), v(T))$. 

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However the path \((W_1, -W_2, W_3)\) with increment \((dW_1, -dW_2, dW_3)\) will make the SDE become

\[
\begin{aligned}
\frac{dY}{dt} &= A(Y, t) dt + B_1(Y, t) dR, \quad Y(0) = (x_0, 0, 0), \\
\frac{dx}{dt} &= a(x, t) dt + b(x, t) dW_1 \\
\frac{du}{dt} &= 4ua' dt + ub' dW_1 - bb'(-dW_2) \\
\frac{dv}{dt} &= 4va' dt + ub' dW_1 - bb' dW_3 \\
\end{aligned}
\]

Because \(u(0) = v(0) = 0\) and \(-dW_2\), the SDE gives \(dY = (dx_0, -du, dv)\), so it has solution \((x(T), -u(T), v(T))\).

Because \((W_1, W_2, W_3)\) is a Brownian path, the weak solution of the SDE (6.16) will be the same over the Brownian path \((W_1, -W_2, W_3)\), which has increment \((dW_1, -dW_2, dW_3)\). That is \(E[u(T)v(T)] = E[-u(T)v(T)]\). It follows from \(E[u(T)v(T)] + E[u(T)v(T)] = 0\) that

\[E[u(T)v(T)] = 0.\]

Therefore, it comes to the conclusion that \(E[\bar{u}(T)e(T)] \sim O(h^{3/2})\).

Now it is the right time to proof the Theorem (6.4.2).

**Proof of Proposition 6.4.2:** From formulas (6.14), we have \(E(\xi_0\xi_1) = hE[\bar{u}\bar{v}]\). So \(E(\xi_0\xi_1) \sim O(h^{3/2})\) follows from \(E[\bar{u}\bar{v}] \sim O(h^{3/2})\).

From the Proposition (6.4.2), we can go a further step.

**Proposition 6.4.7.** Suppose \(\xi_0 = f(X_{2h}) - f(X_h), \xi_1 = f(X_{4h}) - f(X_{2h}),\) where the \(f(x)\) is differentiable. \(a(x), b(x)\) are twice differentiable, and \(f(x), a(x), a'(x) b(x), b'(x)\) satisfy a uniform Lipschitz condition. Then we can have \(E(\xi_0\xi_1) \sim O(h^{3/2})\), in which \(\bar{X}\) is the approximation of \(X\).

**Proof:** Because

\[
\begin{align*}
\xi_0(n) &= \frac{df(X^{(2n-2)}_{2h}, X^{(4n-4)}_h)}{dx}(X^{(2n)}_{2h} - X^{(4n)}_h) + O(h) \\
\xi_1(n) &= \frac{df(X^{(n)}_{4h}, X^{(2n)}_{2h})}{dx}(X^{(n)}_{4h} - X^{(2n)}_{2h}) + O(h),
\end{align*}
\]

if we denote \(g(\bar{X}) = f'(X^{(2n-2)}_{2h}, X^{(4n-4)}_h) f'(X^{(n-1)}_{4h}, X^{(2n-2)}_{2h}),\) then we just need to prove the statement \(E[(X^{(2n)}_{2h} - X^{(4n)}_h)(X^{(n)}_{4h} - X^{(2n)}_{2h})g(\bar{X})] \sim O(h^{3/2})\) is true.

Analogous to the Lemma (6.4.5), say \(v(n) = \frac{X^{(2n)}_{2h} - X^{(4n)}_h}{h^{3/2}}, u(n) = \frac{X^{(n)}_{4h} - X^{(2n)}_{2h}}{h^{3/2}},\)
then we have

$$
\begin{cases}
\begin{align*}
dx &= a(x, t) dt + b(x, t) dW_1 \\
du &= 4ua' dt + ub' dW_1 - bb' dW_2 \\
dv &= 4va' dt + vb' dW_1 - bb' dW_3 \\
x(0) &= x_0, \quad u(0) = 0, \quad v(0) = 0 \\
dW_1, dW_2, dW_3 &\text{ are independent.}
\end{align*}
\end{cases}
$$

We define $\varphi(t, Y) = E[u(T)v(T)g(x)| Y(t) = Y]$, that is

$\varphi(t, x, u, v) = E[u(T)v(T)g(x(T))| x(t) = x, u(t) = u, v(t) = v]$. Then the system

$$
\begin{cases}
dZ &= A(Z, t) dt + B_1(Z, t) dR(t) \\
Z(t_k) &= Z_k = \bar{Y}_k,
\end{cases}
$$

and the approximation system

$$
\begin{cases}
\Delta \bar{Y}_k &= A(\bar{Y}, t)h + B_1(\bar{Y}, t)\Delta R + h^{\frac{1}{2}}B_2(\bar{Y}, t)\Delta R \\
\bar{Y}_k &= Z_k,
\end{cases}
$$

within time interval $[t_k, t_{k+1}]$, give us

$$
\varphi(t_k, Z_k) = E[\varphi(t_{k+1}, Z_{k+1})]
= \varphi(t_{k+1}, Z_k) + \sum_{i=1}^{3} A_i \frac{\partial \varphi(t_{k+1}, Z_k)}{\partial Y(i)} h
+ \frac{1}{2} \sum_{i,q=1}^{3} H_{iq} \frac{\partial^2 \varphi(t_{k+1}, Z_k)}{\partial Y(i)\partial Y(q)} + O(h^2),
$$

$$
E[\varphi(t_{k+1}, \bar{Y}_{k+1})] = \varphi(t_{k+1}, \bar{Y}_{k+1})
= \varphi(t_{k+1}, \bar{Y}_{k}) + \sum_{i=1}^{3} A_i \frac{\partial \varphi(t_{k+1}, \bar{Y}_{k})}{\partial Y(i)} h
+ \frac{1}{2} \sum_{i,q=1}^{3} H_{iq} \frac{\partial^2 \varphi(t_{k+1}, \bar{Y}_{k})}{\partial Y(i)\partial Y(q)}
+ h^{\frac{1}{2}} \sum_{i,q=1}^{3} G_{iq} \frac{\partial^2 \varphi(t_{k+1}, \bar{Y}_{k})}{\partial Y(i)\partial Y(q)} + O(h^2).
$$

So we can have

$$
\varphi(t_k, Z_k) - E[\varphi(t_{k+1}, \bar{Y}_{k+1})]
= h^{\frac{1}{2}} \sum_{i,q=1}^{3} G_{iq} \frac{\partial^2 \varphi(t_{k+1}, \bar{Y}_{k})}{\partial Y(i)\partial Y(q)} + O(h^2) \sim O(h^{\frac{3}{2}}).
$$

Hence, it is true that $E[u(T)v(T)g(x(T))] - E[u(T)v(T)g(x(T))] \sim O(h^{\frac{1}{2}})$, by adding up the difference in each $[t_k, t_{k+1}]$.  

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Similar to the Lemma (6.4.6),
\[
dY = A(Y, t) \, dt + B_1(Y, t) \, dR, \quad Y(0) = (x_0, 0, 0),
\]
\[
\begin{align*}
&dx = a(x, t) \, dt + b(x, t) \, dW_1 \\
&du = 4ua' \, dt + ub' \, dW_1 - bb' \, dW_2 \\
&dv = 4va' \, dt + vb' \, dW_1 - bb' \, dW_3 \\
&x(0) = x_0 \quad u(0) = 0 \quad v(0) = 0
\end{align*}
\]
\[
dW_1, \, dW_2, \, dW_3 \text{ are independent,}
\]
we can see that, the Brownian path \((W_1, W_2, W_3)\), having increment \((dW_1, dW_2, dW_3)\), gives the SDE (6.16) solution, \((x(T), u(T), v(T))\).

However the path \((W_1, -W_2, W_3)\) with increment \((dW_1, -dW_2, dW_3)\) will make the SDE become
\[
dY = A(Y, t) \, dt + B_1(Y, t) \, dR, \quad Y(0) = (x_0, 0, 0),
\]
\[
\begin{align*}
&dx = a(x, t) \, dt + b(x, t) \, dW_1 \\
&du = 4ua' \, dt + ub' \, dW_1 - bb' \, (-dW_2) \\
&dv = 4va' \, dt + vb' \, dW_1 - bb' \, dW_3 \\
&x(0) = x_0 \quad u(0) = 0 \quad v(0) = 0
\end{align*}
\]
\[
dW_1, -dW_2, \, dW_3 \text{ are independent}
\]
Because \(u(0) = v(0) = 0\) and \(-dW_2\), the SDE gives \(dY = (dx_0, -du, dv)\). Hence it has solution \((x(T), -u(T), v(T))\).

Because \((W_1, W_2, W_3)\) is a Brownian path, the weak solution of the SDE (6.16) will be the same over the Brownian path \((W_1, -W_2, W_3)\), which has increment \((dW_1, -dW_2, dW_3)\). That is \(E[u(T)v(T)g(X(T))] = E[-u(T)v(T)g(X(T))]\). It follows from \(E[u(T)v(T)] + E[u(T)v(T)] = 0\) that
\[
E[u(T)v(T)g(X(T))] = 0.
\]
Therefore, it comes to the conclusion that \(E[\tilde{u}(T)\tilde{v}(T)g(\tilde{X}(T))] \sim O(h^{\frac{3}{2}})\). That is \(E[\xi_0\xi_1g(\tilde{X}(T))] \sim O(h^{\frac{3}{2}})\)

\[\square\]

We can use the same method to get the following Proposition and Theorem.

**Proposition 6.4.8.** When \(\xi_i = f(X_{2i}h) - f(X_{2i+1}h)\), where the \(f(x) = x, a(x), \text{ and } b(x)\) are moderately smooth functions, the \(E[\xi_i\xi_j] \sim O(h^{\frac{3}{2}})\)

**Theorem 6.4.9.** When \(\xi_i = f(X_{2i+1}h) - f(X_{2i}h), \xi_j = f(X_{2j+1}h) - f(X_{2j}h)\), where the \(f(x) = x, g(x), a(x), \text{ and } b(x)\) are moderately smooth functions, we can have \(E[\xi_i\xi_jg(\tilde{X})] \sim O(h^{\frac{3}{2}})\), in which \(\tilde{X}\) is the approximation of \(X\).

Now we can proof the Proposition (6.4.1) with the result that we get from the Proposition (6.4.8) and the Theorem (6.4.9).
Proof of the Proposition (6.4.1):

\[ f(X_{2i+h}) - f(X_{2i+1+h}) \approx f(X_{2i+h}) - f(X_{2i+1+h}) \]

\[ = (X_{2i+h} - X_{2i+1+h})f' + (X_{2i+h} - X_{2i+1+h})^2 \frac{f''}{2} \]

\[ = [(X_{2i+h} - X_{2i+1+h})(X_{2i+h} - X_{2i+1+h})f''] + [(X_{2i+h} - X_{2i+1+h})^2 \frac{f''}{2}] \]

Because \((X_{2i+h} - X_{2i+1+h})\) and \((X_{2i+h} - X_{2i+1+h})\) have strong order \(O(h^{3/2})\), it is true to have \(E[(X_{2i+h} - X_{2i+1+h})^2 \frac{f''}{2}] \sim O(h^{3/2})\). And the Theorem (6.4.9) gives \(E[(X_{2i+h} - X_{2i+1+h})(X_{2i+h} - X_{2i+1+h})f'f'] \sim O(h^{3/2})\). Therefore, we have Proposition (6.4.1).

\[ \square \]

Therefore the Proposition (6.4.1) provides that \(\forall i, j \in [0, k], (i \leq j), \exists C_{j-i} \geq 0\) (\(C_{j-i}\) only depends on \(j - i\)), st. \(E(\xi_i \xi_j) \leq C_{j-i}^2 h\).

6.4.1 N Dimension Situation

We are going to consider the D-dimension system

\[ dX(t) = a(X, t) dt + b(X, t) dW, \]

\[ \begin{pmatrix}
    dX^{(1)}(t) \\
    \vdots \\
    dX^{(D)}(t)
\end{pmatrix} =
\begin{pmatrix}
    a^{(1)}(t) \\
    \vdots \\
    a^{(D)}(t)
\end{pmatrix} dt +
\begin{pmatrix}
    b_{1,1}(x) & \cdots & b_{1,M}(x) \\
    \vdots & \ddots & \vdots \\
    b_{D,1}(x) & \cdots & b_{D,M}(x)
\end{pmatrix}
\begin{pmatrix}
    W^{(1)}(t) \\
    \vdots \\
    W^{(D)}(t)
\end{pmatrix} \tag{6.19}
\]

where \(a\) is a \(D \times 1\) matrix and \(b\) is a \(D \times M\) matrices, \(dW = (dW^{(1)}, \ldots, dW^{(M)})^T\), and \(dX = (dX^{(1)}, \ldots, dX^{(D)})^T\), in this section. Denote

- the \(j\)th instant of time, where \(t = jh\)
- the \(q\)th component of the Wiener process \(W^{(q)}\)
- the \(p\)th sample Brownian path
- the \(q\) component of the Wiener process \(W = \{W_t, t > 0\}\)
- \(m\) \(h\) is the length between two instants of discrete time.

Similar to these notations, we use \(\{q\}\) to indicate the \(q\)th component in the vector, and \(i, j\) to indicate the \((i, j)\) component in the matrix. Denote \(a(x) = \)
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\[(a^{(1)}, \ldots, a^{(D)})^T, b_i(x) = (b_{i,1}, \ldots, b_{i,M})\] and

\[b(x) = \begin{pmatrix} b_{1,1}(x) & \cdots & b_{1,M}(x) \\ \vdots & \ddots & \vdots \\ b_{D,1}(x) & \cdots & b_{D,M}(x) \end{pmatrix}.

Now we are going to see the NDimension version of the proposition (6.4.2).

Proposition 6.4.10. When \(\xi_i^{(p)} = f^{(p)}(X_{2i \Delta t}) - f^{(p)}(X_{2i+1 \Delta t}), (p = 1, \ldots, D)\) where the \(f^{(p)}(x) = x^{(p)}, a(x) = (a^{(1)}, \ldots, a^{(D)})^T,\) and

\[b(x) = \begin{pmatrix} b_{1,1}(x) & \cdots & b_{1,M}(x) \\ \vdots & \ddots & \vdots \\ b_{D,1}(x) & \cdots & b_{D,M}(x) \end{pmatrix},

are moderately smooth functions, the \(E[\xi_i^{(p)}\xi_j^{(q)}] \sim O(h^{3/2}).\)

We can expand the \(\xi_0^{(1)}(n)\) and \(\xi_1^{(i)}(n)\) with Taylor - Expansion, and then consider \(\bar{X}(n), \bar{v}(n)\) and \(\bar{u}(n),\) where

\[\bar{v}(n) = \xi_0(n)/h^{3/2}, \quad \bar{u}(n) = \xi_1(n)/h^{3/2}.

Hence

\[\Delta \bar{v}^{(i)}(n) = \bar{v}^T \nabla b_i(\Delta W^{(4n-1)} + \Delta W^{(4n-2)} + \Delta W^{(4n-3)} + \Delta W^{(4n-4)})
+ 4\bar{v}^T \nabla (a^{(i)})_h - \frac{1}{\sqrt{h}}(b\Delta W^{(4n-4)} \cdot \nabla)b_i \Delta W^{(4n-3)}
- \frac{1}{\sqrt{h}}(b\Delta W^{(4n-2)} + \Delta W^{(4n-3)} + \Delta W^{(4n-4)}).
\]

We can further expand this expression for \(\Delta \bar{v}^{(i)}(n)\) and similarly for \(\Delta \bar{u}^{(i)}(n)\).
\[\begin{align*}
\Delta \hat{u}^{(1)}(n) &= \hat{u}^T(\nabla b_{\mathbf{i},1}, \ldots, \nabla b_{\mathbf{i},M})(\Delta W^{(4n-1)} + \Delta W^{(4n-2)} + \Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \\
&\quad + \frac{1}{2} h^{1/2}(\hat{u} \cdot \nabla) b_i(\Delta W^{(4n-1)} + \Delta W^{(4n-2)} + \Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \\
&\quad - \frac{1}{2} h^{1/2}[b(\Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \cdot \nabla] b_i(\Delta W^{(4n-2)} + \Delta W^{(4n-1)}) \\
&\quad - \frac{1}{2} h^{1/2}[b(\Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \cdot \nabla] b_i(\Delta W^{(4n-2)} + \Delta W^{(4n-1)}) \\
&\quad - h^{1/2}(\bar{a} \cdot \nabla) b_i(\Delta W^{(4n-2)} + \Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \\
&\quad + h^{1/2}[(\bar{a} \cdot \nabla)(\Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \cdot \nabla] b_i \Delta W^{(4n-2)} \\
&\quad + 2[(\bar{a} \cdot \nabla)(\Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \cdot \nabla] b_i \Delta W^{(4n-2)} \\
&\quad + (\bar{a} \cdot \nabla)[(\Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \cdot \nabla] b_i \Delta W^{(4n-2)} \\
&\quad + \frac{1}{2} (\Delta W^{(4n-2)} \cdot \nabla)^2 b_i(\Delta W^{(4n-3)} - \Delta W^{(4n-2)} - \Delta W^{(4n-1)}) \\
&\quad - \frac{1}{2} (\Delta W^{(4n-2)} \cdot \nabla)^2 b_i(\Delta W^{(4n-1)} + \Delta W^{(4n-2)}) \\
&\quad - \frac{1}{2} (\Delta W^{(4n-2)} \cdot \nabla)^2 b_i \Delta W^{(4n-1)}. \\
\end{align*}\]

Again, the \((\bar{X}(n), \bar{v}(n), \bar{u}(n))\) becomes the Euler approximation with correction term,

\[\begin{align*}
\delta \hat{u}^{(1)}(n) &= \frac{1}{2} h^{1/2}(\hat{u} \cdot \nabla) b_i(\Delta W^{(4n-1)} + \Delta W^{(4n-2)} + \Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \\
&\quad - \frac{1}{2} h^{1/2}[b(\Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \cdot \nabla] b_i(\Delta W^{(4n-2)} + \Delta W^{(4n-1)}) \\
&\quad - h^{1/2}(\bar{a} \cdot \nabla) b_i(\Delta W^{(4n-2)} + \Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \\
&\quad + h^{1/2}[(\bar{a} \cdot \nabla)(\Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \cdot \nabla] b_i \Delta W^{(4n-2)} \\
&\quad + 2[(\bar{a} \cdot \nabla)(\Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \cdot \nabla] b_i \Delta W^{(4n-2)} \\
&\quad + (\bar{a} \cdot \nabla)[(\Delta W^{(4n-3)} + \Delta W^{(4n-4)}) \cdot \nabla] b_i \Delta W^{(4n-2)} \\
&\quad + \frac{1}{2} (\Delta W^{(4n-2)} \cdot \nabla)^2 b_i(\Delta W^{(4n-3)} - \Delta W^{(4n-2)} - \Delta W^{(4n-1)}) \\
&\quad - \frac{1}{2} (\Delta W^{(4n-2)} \cdot \nabla)^2 b_i(\Delta W^{(4n-1)} + \Delta W^{(4n-2)}) \\
&\quad - \frac{1}{2} (\Delta W^{(4n-2)} \cdot \nabla)^2 b_i \Delta W^{(4n-1)}. \\
\end{align*}\]

to the system

\[\begin{align*}
\frac{dX}{dt} &= 4a \, dt + b \, dW_1 \\
\frac{du}{dt} &= 4u^T \nabla a_i \, dt + u^T \nabla b_i \, dW_1 - tr \{\nabla b_i \, dW_2 b^T \} \\
\frac{dv}{dt} &= 4v^T \nabla a_i \, dt + v^T \nabla b_i \, dW_1 - tr \{\nabla b_i \, dW_3 b^T \},
\end{align*}\]

where \(W_2, W_3\) are \(M \times M\) dimensions Brownian process, and \(W_1 = (W_1^{(1)}, \ldots, W_1^{(D)})^T\) is an \(D\)-Dimensional Wiener process. All components of \(W_2, W_3,\) and \(W_1\) are independent scalar Wiener processes.
Lemma 6.4.11. The Euler approximation,

\[
\Delta X(n) = 4ah + b(\Delta W(4n-4) + \Delta W(4n-3) + \Delta W(4n-2) + \Delta W(4n-1)) \\
\Delta u^{(i)}(n) = \tilde{u}^T \nabla b_i(\Delta W(4n-1) + \Delta W(4n-2) + \Delta W(4n-3) + \Delta W(4n-4)) + 4\tilde{u}^T \nabla a^{(i)} h \\
\Delta v^{(i)}(n) = \tilde{v}^T \nabla b_i(\Delta W(4n-1) + \Delta W(4n-2) + \Delta W(4n-3) + \Delta W(4n-4)) \\
- \frac{1}{\sqrt{h}} [b(\Delta W(4n-4) + \Delta W(4n-3)) \cdot \nabla b_i(\Delta W(4n-2) + \Delta W(4n-1))] \\
- \frac{1}{\sqrt{h}} (b\Delta W(4n-2) \cdot \nabla) b_i \Delta W(4n-1), (i = 1, \ldots, D),
\]

(6.23)

to the system (6.22), has second-moment matrix \( H = E[u(n)u^T(n)] \sim O(h) \), then the approximation (6.21), \((\tilde{X}(n), \tilde{v}(n), \tilde{u}(n)) = (\Delta X(n), \Delta v(n), \Delta u(n)) + \delta u(n), \delta v(n), \delta u(n))\), will have up to \( O(h^{3/2}) \) second-moment matrix \( H + h^{1/2} G \), where \( G \sim O(h) \), \( G \) equals to

\[
E \left( \begin{array}{ccc}
0 & \Delta X(n)[\delta u(n)]^T & \Delta X(n)[\delta v(n)]^T \\
\delta u(n)[\Delta X(n)]^T & \Delta u(n)[\delta u(n)]^T + \delta u(n)[\Delta u(n)]^T & \Delta u(n)[\delta v(n)]^T + \delta u(n)[\Delta v(n)]^T \\
\Delta X(n) \cdot \delta v(n) & \delta u(n)[\Delta u(n)]^T + \delta u(n)[\Delta v(n)]^T & \delta u(n)[\Delta v(n)]^T + \delta u(n)[\Delta v(n)]^T
\end{array} \right).
\]

The proof of this it is the same as the 1 - Dimension situation.

\(\square\)

Now we can have a look at the \( N \) Dimension version of Corollary (6.4.6).

Corollary 6.4.12. When \( a(x), b(x) \) are moderately smooth functions, and the \( \tilde{u}, \tilde{v} \) are implemented by formulas (6.21), then \( E[\tilde{u}(T)\tilde{v}(T)] \sim O(h^{3/2}) \).

Proof: When we define

\[
\varphi(t, Y) = E[u(T)v(T)|x(t) = x, u(t) = u, v(t) = v],
\]

the Lemma (6.4.5) gives us the general result

\[
\varphi(t_k, Z_k) - E[\varphi(t_{k+1}, \tilde{Y}_{k+1})] \\
= \frac{h^{1/2}}{2} \sum_{t,q=1}^{3} G_{tq} \frac{\partial^2 \varphi(t_{k+1}, \tilde{Y}_{k+1})}{\partial Y(t) \partial Y(q)} + O(h^2),
\]

in 1 - Dimension case. So when we define

\[
\varphi^{(i,j)}(t, Y) = E[u^{(i)}(T)v^{(j)}(T)|x(t) = x, u(t) = u, v(t) = v],
\]

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we can have
\[ \varphi^{(i,j)}(t_k, Z_k) - E[\varphi^{(i,j)}(t_{k+1}, \hat{Y}_{k+1})] = \frac{h^\frac{1}{2}}{2} \sum_{p,q=1}^{D} G_{ij} \frac{\partial^2 \varphi^{(i,j)}(t_{k+1}, \hat{Y}_k)}{\partial u_p \partial v_q} + O(h^2), \]
in the $D - Dimension$ case. And the Lemma stated in the last paragraph proves $G \sim O(h)$, so we can have the conclusion that $|E[\hat{\xi}^{(i)}(T)\hat{\xi}^{(j)}(T)] - E[u^{(i)}(T)v^{(j)}(T)]| \sim O(h^{\frac{3}{2}})$.

The method we used in the Corollary (6.4.6) is also applicable here, hence we can get $E\{u^{(i)}(T)v^{(j)}(T)\} = 0$, and then $E\{\hat{\xi}^{(i)}(T)\hat{\xi}^{(j)}(T)\} \sim O(h^{\frac{1}{2}})$.

\[ \square \]

**Proof of Proposition 6.4.10:** The Corollary (6.4.12) gives us $E\{\hat{\xi}^{(i)}(T)\hat{\xi}^{(j)}(T)\} \sim O(h^{\frac{3}{2}})$, so it is true that $E[\xi^{(i)}_0\xi^{(j)}_1] = hE\{\hat{\xi}^{(i)}(T)\hat{\xi}^{(j)}(T)\} \sim O(h^{\frac{3}{2}})$.

\[ \square \]

Similarly, we can have the $N$ Dimensions version of

**Corollary 6.4.13.** When $\xi^{(i)}_0 = f^{(i)}(X_{2h}) - f^{(i)}(X_h), \xi^{(j)}_1 = f^{(j)}(X_{4h}) - f^{(j)}(X_{2h})$, where the $f^{(i)}(x), g(x), a_i(x),$ and $b_{ij}(x)$ ($i, j = 1, \ldots, D)$ are moderately smooth functions, then we can have $E[\xi^{(i)}_0\xi^{(j)}_1g(\hat{X})] \sim O(h^{\frac{3}{2}})$, in which $\hat{X}$ is the approximation of $X$.

We can use the same method to get the following Proposition and Theorem.

**Proposition 6.4.14.** When $\xi^{(p)}_i = f^{(p)}(X_{2^i+1h}) - f^{(p)}(X_{2^i h}),$ where the $f^{(p)}(x) = x^{(p)}, a(x),$ and $b(x)$ are moderately smooth functions, the $E[\xi^{(p)}_i\xi^{(q)}_j] \sim O(h^{\frac{3}{2}})$

**Theorem 6.4.15.** When $\xi^{(p)}_i = f^{(p)}(X_{2^i+1h}) - f^{(p)}(X_{2^i h}), \xi^{(q)}_j = f^{(q)}(X_{2^j+1h}) - f^{(q)}(X_{2^j h})$, where the $f^{(p)}(x) = x^{(p)}, (p, q = 1, \ldots, D), g(x), a(x),$ and $b(x)$ are moderately smooth functions, we can have $E[\xi^{(p)}_i\xi^{(q)}_jg(\hat{X})] \sim O(h^{\frac{3}{2}})$, in which $\hat{X}$ is the approximation of $X$.

Hence we can prove the Proposition (6.4.10). This verifies the issue raised at the beginning of this section.

### 6.5 Simulation of Numerical Approximation

Can this method work practically? It is a good idea to test the method with computer simulation. We would like to choose some SDEs, and compared their explicit solution, the WEA solution, and the IWEA solution, according to 4 aspects: theoretical $E(X(T)), Var(X(T)), Sample E(\hat{X}_h), Var(\hat{X}_h)$. Because the discussion about RVWEA method shows that reusing the approximate Brownian trajectory will not significantly reduce the computational load, we only compare the simulation results of WEA and IWEA methods in this section.
Illustrative Example 6.5.1. Apply the WEA and IWEA method to the
\[ \dot{x} = x(dt + dw), \]
on the time interval \([0, 1]\) with initial value \(x_0 = 2\). Compute \(M = 3600\), simulations of the WEA approximation \(X_h\) with \(N = 4\) equidistant time steps with step size \(h = 0.25\). Compute \(m_0 = 1200\) simulations of the IWEA approximation \(X_h\) with \(N = 4\) equidistant time steps with step size \(h = 0.25\). What are the estimate expectation and variance of the \(X(T)\), which we obtain from the WEA and IWEA method? What are the sample expectation and variance of the \(X(T)\)'s mean, which we obtain from the WEA and IWEA method?

Explain:

The Expectation and Variance of the \(X(T)\) in WEA approximation:
We can work out the sample expectation and variance of the \(X_h\) which are generated by the WEA method. The direct WEA method gives
\[
X_h^{(i+1)} = X_h^{(i)}(1 + uh + \sigma \Delta W_h^{(i)}) \\
X_h^{(i+1)} = X_h^{(0)}(1 + uh + \sigma \Delta W_h^{(0)})(1 + uh + \sigma \Delta W_h^{(1)}) \cdots (1 + uh + \sigma \Delta W_h^{(N-1)})
\]
Hence, we shall see expectation, second moment, variance for the WEA:
\[
E[X_h(T)] = X_h^{(0)}(1 + uh)^N \\
E[X_h^2(T)] = (X_h^{(0)})^2[(1 + uh)^2 + \sigma^2 h]^N \\
\text{var}[X_h(T)] = (X_h^{(0)})^2\{[(1 + uh)^2 + \sigma^2 h]^N - (1 + uh)^{2N}\}. \quad (6.24)
\]

The Expectation and Variance of the estimate \(\overline{X(T)}\) in WEA approximation:
Because we already acquire the Expectation and Variance of the \(X(T)\) in WEA method, it is easy to get the Expectation and Variance of the estimate \(\overline{X(T)}\), in which \(\overline{X(T)}\) is the mean of the samples. Hence we can get the sample expectation \(E[X_h(T)]\) and sample variance \(\text{var}[X_h(T)]\),
\[
E[X_h(T)] = X_h^{(0)}(1 + uh)^N \\
\text{var}[\overline{X_h(T)}] = \frac{(X_h^{(0)})^2\{[(1 + uh)^2 + \sigma^2 h]^N - (1 + uh)^{2N}\}}{M}. \quad (6.25)
\]

The Sample Expectation and Variance of the IWEA approximation:
Expectation:
It is a bit complicated to obtain the sample expectation and variance of the \(\overline{X}_h\), but we can still solve it out according to the Euler approximation:
\[
X_h^{(i+1)} = X_h^{(i)}(1 + uh + \sigma \Delta W_h^{(i)}) \\
X_{2h}^{(i+1)} = X_{2h}^{(i)}(1 + 2uh + \sigma (\Delta W_h^{(2i)} + \Delta W_h^{(2i+1)})).
\]
We define \( \delta_h^{(i)} = X_h^{(2i)} - X_{2h}^{(i)} \), then

\[
\delta_h^{(i)} = \delta_h^{(i-1)}[1 + 2uh + \sigma(\Delta W_h^{(2i-1)} + \Delta W_h^{(2i-2)})] + X_h^{(2i-2)}[u^2h^2 + u\sigma(\Delta W_h^{(2i-2)} + \Delta W_h^{(2i-1)}) + \sigma^2\Delta W_h^{(2i-2)}\Delta W_h^{(2i-1)}].
\]

Hence, we obtain \( E[\delta_h^{(i)}] \) by inducting

\[
E[\delta_h^{(i)}] = E[\delta_h^{(i-1)}][1 + 2uh] + E[X_h^{(2i-2)}][u^2h^2]
\]

\[
E[X_h^{(2i-2)}] = X_0^{(2i-2)}[1 + uh]^{2i-2}
\]

\[
E[X_0^{(0)}] = X_0^{(0)} - X_2^{(0)} = 0.
\]

Consequently, we could get \( E(X_h(T) - X_{2h}(T)) \). Similarly, we can get \( E(X_{2h}(T) - X_{4h}(T)) \). Therefore, we can work out the \( E[X_h(T)] \) by

\[
E[X_h(T)] = E(X_h(T) - X_{2h}(T)) + E(X_{2h}(T) - X_{4h}(T)) + E(X_{4h}(T)).
\]

The Scilab program for \( E[X_h(T)] \) is indicated below:

```scilab
clear; k = 2; N = 2^k; u = 1; T = 1; x_0 = 2;
for j = 1 : k
    N = 2^j; h = T/N;
    for i = 1 : N/2;
        if (i == 1) x(1) = x_0(u^2*h^2);
        end
        if (i > 1) x(i) = x(i-1) * (1 + 2uh) + x_0 * (1 + uh)^2 * (u^2*h^2);
    end
    A(j) = x(N/2);
end
x_N = x_0 * (1 + u * T);
sum(A) + x_N
```

The Sample estimate of the expectation:

We can estimate the expectation \( E[X(T)] \) by the samples' mean

\[
E[X_h(T)] = E(X_h(T) - X_{2h}(T)) + E(X_{2h}(T) - X_{4h}(T)) + E(X_{4h}(T))
\]

\[
E[X_h(T)] = E[X_h(T) - X_{2h}(T)] + E[X_{2h}(T) - X_{4h}(T)] + E[X_{4h}(T)].
\]

Variance of the \( X_h(T) \) in the IWEA method:
We define

$$B_h(i) = B_h(i - 1)[1 + 4u^2h^2 + 2\sigma^2h + 4uh] + (X^{(0)})^2[(1 + uh)^2 + \sigma^2h^{2i-2}(u^4h^4 + 2u^2\sigma^2h^3 + \sigma^4h^2 + 4uh^2\sigma^2 + 2u^2h^2 + 4u^3h^3) - 2(X^{(0)})^2[(1 + 2uh)(1 + 2uh + u^2h^2) + 2uh\sigma^2 + 2u^2h^2\sigma^2]\text{\textsuperscript{2i-1}}(2u^2h^2\sigma^2 + u^2h^2 + 2u^3h^3)].$$

$$(i = 1, \ldots, \frac{N}{2}, h = \frac{T}{N})$$

$$B_h(0) = E[\delta_h^{(0)}]^2 = 0. \quad (6.27)$$

From the equation (6.26), we may obtain

$$E[\delta_h^{(i)}|^2 = E[\delta_h^{(i-1)}]^2[1 + 4u^2h^2 + 2\sigma^2h + 4uh] + E[X_h^{(2i-2)}]^2(u^4h^4 + 2u^2\sigma^2h^3 + \sigma^4h^2) + 2E[\delta_h^{(i-1)}X_h^{(2i-2)}](2uh^2\sigma^2 + u^2h^2 + 2u^3h^3).$$

Because equation (6.24) $E[X_h^{(2i)}]^2 = (X_h^{(0)})^2[(1 + uh)^2 + \sigma^2h]$, $E[X_h^{(2i)}] = E[X_h^{(i)}] - E[X_h^{(2i)}], \text{ and}$

$$E[X_h^{(2i)}] = E[X_h^{(i)}][1 + 2uh](1 + 2uh + u^2h^2) + 2uh\sigma^2 + 2u^2h^2\sigma^2\text{\textsuperscript{i}},$$

we can have

$$E[\delta_h^{(i)}]^2 = E[\delta_h^{(i-1)}]^2[1 + 4u^2h^2 + 2\sigma^2h + 4uh] + E[X_h^{(2i-2)}]^2(u^4h^4 + 2u^2\sigma^2h^3 + \sigma^4h^2 + 4uh^2\sigma^2 + 2u^2h^2 + 4u^3h^3)
- 2(X_h^{(0)})^2[(1 + 2uh)(1 + 2uh + u^2h^2) + 2uh\sigma^2 + 2u^2h^2\sigma^2\text{\textsuperscript{i}]}(2u^2h^2\sigma^2 + u^2h^2 + 2u^3h^3)
= B_h(i). \quad (6.28)$$

Induction of

$$E[\delta_h^{(i)}]^2 = B_h(i)$$

$$E[X_h^{(2i-2)}]^2 = (X_h^{(0)})^2[(1 + uh)^2 + \sigma^2h]\text{\textsuperscript{2i-2}}$$

$$E[\delta_h^{(0)}]^2 = B_h(0) = 0$$

gives us

$$E[X_h(T) - X_{2h}(T)]^2 = B_h\left(\frac{N}{2}\right)$$

$$\text{Var}[X_h(T) - X_{2h}(T)] = B_h\left(\frac{N}{2}\right) - E^2[X_h(T) - X_{2h}(T)].$$
Hence, in our case \( N = 4 \), we can bound the \( \text{Var}(X_h(T)) \) by

\[
\text{Var}(X_h(T)) = \text{Var}[X_h(T) - X_{2h}(T)] + \text{Var}[X_{2h}(T) - X_{4h}(T)] + \text{Var}[X_{4h}(T)]
\]

\[
= B_h\left(\frac{N}{2}\right) - E^2[\delta_h^{(\frac{N}{2})}] + B_{2h}\left(\frac{N}{4}\right) - E^2[\delta_{2h}^{(\frac{N}{4})}] + (X^{(0)})^2[\sigma^2T]
\]

Specifically, the Scilab program gives the

\[
B_h\left(\frac{N}{2}\right) - E^2[\delta_h^{(\frac{N}{2})}] + B_{2h}\left(\frac{N}{4}\right) - E^2[\delta_{2h}^{(\frac{N}{4})}] + (X^{(0)})^2[\sigma^2T]
\]

with:

```plaintext
clear;
k=2;N=2^k;u=1;\sigma=1;T=1;x_0=2;
for j=1:k
    N=2^j;h=T/N;
    for i=1:N/2
        if(i==1)
            B(i)=x_0.0.u(4h^4+2u^2\sigma^2h^3+\sigma^4h^2+4uh^2\sigma^2+2u^2h^2+4u^3h^3)
            -2x_0.0.u(2uh^2\sigma^2+u^2h^2+2u^3h^3);
        end
        if(i > 1)
            B(i)=B(i-1)(1+4u^2h^2+2\sigma^2h+4uh)
            +x_0.0.u((1+uh)^2+\sigma^2h)(2i-2)(u^4h^4+2u^2\sigma^2h^3+\sigma^4h^2+4uh^2\sigma^2
            +2u^2h^2+4u^3h^3)
            -2x_0.0.u((1+2uh)(1+2uh+u^2h^2)
            +2\sigma^2+2uh^2\sigma^2)(2i-1)(2uh^2\sigma^2+u^2h^2+2u^3h^3);
        end
        if(i == 1) x(1)=x_0.0.(u^2h^2);end
        if(i > 1) x(i)=x(i-1)(1+2uh)+x_0.0.(1+uh)(2i-2)(u^2h^2);end
    end
    A(j)=B(N/2)-(x(N/2))^2;
end
B_0=x_0.0.(\sigma^2T);
sum(A)+B_0
```

When \( N = 4 \), the theoretical sample expectation and variance, and the simulation mean and sample variance for the WEA and IWEA methods are list in the table:

<table>
<thead>
<tr>
<th>Method</th>
<th>Theoretical Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>WEA method</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theoretical</td>
<td>( E(X_h) = 4.8828 )</td>
<td>( \text{Var}(X_h) = 19.32714884 )</td>
</tr>
<tr>
<td>Compute simulation</td>
<td>4.81268</td>
<td>18.58129</td>
</tr>
<tr>
<td>IWEA method</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theoretical</td>
<td>4.8828</td>
<td>( 8.7646484 )</td>
</tr>
<tr>
<td>Compute simulation</td>
<td>4.93792</td>
<td>8.86838</td>
</tr>
</tbody>
</table>

The C++ program and the theoretical induction show that \( X_h(T) \), which generated by the IWEA method, has smaller variance than the one from the WEA method. According to the first step of the theorem (6.2.4), the computational allocation strategy should bring down the sample variance. We may check this by adapting the Scilab program in the last-paragraph.

the variance of the \( X_h(T) \)
According to the relation between variance and sample variance of the $X_h(T)$, we can get the sample variance for the IWEA method by

$$\text{var}(X_h(T)) = \text{var}[X_h(T) - X_{2h}(T)] + \text{var}[X_{2h}(T) - X_{4h}(T)] + \text{var}[X_{4h}(T)]$$

$$= \frac{B_h(\frac{N}{2}) - E^2\delta_h(\frac{N}{4})}{m_0} + \frac{B_{2h}(\frac{N}{4}) - E^2\delta_{2h}(\frac{N}{4})}{m_1} + \frac{(X(0))^2\sigma^2 T}{m_2},$$

where $m_1 = 2m_0$, $m_2 = 2^2m_0$.

When $N = 4$, the theoretical sample expectation and variance, and the simulation mean and sample variance for the WEA and IWEA methods are listed in the table:

<table>
<thead>
<tr>
<th>IWEA method</th>
<th>simulation num</th>
<th>steps</th>
<th>Variance</th>
<th>sample variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_h(T) - X_{2h}(T)$</td>
<td>$m_0 = 1200$</td>
<td>4</td>
<td>2.7646484</td>
<td>0.0023039</td>
</tr>
<tr>
<td>$X_{2h}(T) - X_{4h}(T)$</td>
<td>$m_1 = 2400$</td>
<td>2</td>
<td>0.0008333</td>
<td></td>
</tr>
<tr>
<td>$X_{4h}(T)$</td>
<td>$m_2 = 4800$</td>
<td>1</td>
<td>0.0008333</td>
<td></td>
</tr>
<tr>
<td>$X_h(T)$</td>
<td>$M = 3600$</td>
<td>4</td>
<td>19.327148</td>
<td>0.0053687</td>
</tr>
</tbody>
</table>

It is clear to see that the IWEA method will bring down the variance of the expectation. Therefore it will cause smaller sample error than the WEA method.

The computational load

The computational load $L_D$ of the WEA method is $L_D = \frac{T}{h} \times M$, in which $M$ is the number of the Monte Carlo simulation, and the $\frac{T}{h}$ is the number of iterations for Euler method. From the conditions we have above, it is easy to get $L_D = \frac{T}{h} \times M = \frac{1}{0.25} \times 3600 = 14400$.

According to the strategy (6.6), we have $m_0 = 1200$, $m_1 = 2400$, $m_2 = 4800$ simulations for $X_h - X_{2h}$, $X_{2h} - X_{4h}$ and $X_{4h}$ respectively. So we can obtain the mean and sample variance of $X_h - X_{2h}$, $X_{2h} - X_{4h}$ and $X_{4h}$ respectively. Hence we also have $\bar{X}_h$ and its variance by

$$\bar{X}_h = X_h - X_{2h} + X_{2h} - X_{4h} + X_{4h}$$

$$\text{var}(\bar{X}_h) = \text{var}(X_h - X_{2h}) + \text{var}(X_{2h} - X_{4h}) + \text{var}(X_{4h}).$$

The computational load $L_I$ of the IWEA method will be $L_I = m_0 \times N + m_1 \times \frac{N}{2^2} + m_2 \times \frac{N}{2^4} = 1200 \times 4 + 2400 \times 2 + 4800 \times 1 = 14400$. Hence these two methods have same computational load under our condition.

Illustrative Example 6.5.2. Let $\xi_0$ denote $f(X_h) - f(X_{2h})$. Suppose we investigate the $dx = a(dt + dW)$ on the time interval [0, 1] with the initial condition $X_0 = 2$. Repeat Illustrative Example 6.5.1 with these different conditions:

<table>
<thead>
<tr>
<th>WEA method</th>
<th>Number of Steps</th>
<th>Number of simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 3$;</td>
<td>$N = 2^k = 8$</td>
<td>12800</td>
</tr>
<tr>
<td>$k = 4$;</td>
<td>$N = 16$</td>
<td>40000</td>
</tr>
<tr>
<td>$k = 5$;</td>
<td>$N = 32$</td>
<td>115200</td>
</tr>
</tbody>
</table>
Numerical Approximation for SDE

<table>
<thead>
<tr>
<th>IWEA method</th>
<th>Number of Steps for $X_h$</th>
<th>Number of simulations for $\xi_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 3$</td>
<td>$N = 2^k = 8; 4; 2; 1$</td>
<td>3200</td>
</tr>
<tr>
<td>$k = 4$</td>
<td>$N = 16; 8; 4; 2; 1$</td>
<td>8000</td>
</tr>
<tr>
<td>$k = 5$</td>
<td>$N = 32; 16; 8; 4; 2; 1$</td>
<td>19200</td>
</tr>
</tbody>
</table>

Solve:

Analogous to the $N = 4$ situation, we can solve out the theoretical expectation and variance, and the simulation mean and variance for the WEA and IWEA methods as the table below:

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Variance</th>
<th>Comp Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>WEA</td>
<td>$E(X_h) = 5.1316$</td>
<td>$Var(X_h) = 29.609$</td>
<td>102400</td>
</tr>
<tr>
<td>$k = 3$</td>
<td>5.11930</td>
<td>29.03319</td>
<td></td>
</tr>
<tr>
<td>IWEA</td>
<td>5.131569</td>
<td>11.385959</td>
<td></td>
</tr>
<tr>
<td>$k = 3$</td>
<td>5.09685</td>
<td>11.22647</td>
<td></td>
</tr>
<tr>
<td>WEA</td>
<td>5.27585</td>
<td>38.08545</td>
<td>640000</td>
</tr>
<tr>
<td>$k = 4$</td>
<td>5.28761</td>
<td>38.08384</td>
<td></td>
</tr>
<tr>
<td>IWEA</td>
<td>5.275857</td>
<td>13.240223</td>
<td></td>
</tr>
<tr>
<td>$k = 4$</td>
<td>5.2603</td>
<td>13.12473</td>
<td></td>
</tr>
<tr>
<td>WEA</td>
<td>5.354</td>
<td>43.75197</td>
<td>3686400</td>
</tr>
<tr>
<td>$k = 5$</td>
<td>5.35517</td>
<td>43.97691</td>
<td></td>
</tr>
<tr>
<td>IWEA</td>
<td>5.3539803</td>
<td>14.334985</td>
<td></td>
</tr>
<tr>
<td>$k = 5$</td>
<td>5.34913</td>
<td>14.38318</td>
<td></td>
</tr>
</tbody>
</table>

We can also get the sample variance for these two methods:

<table>
<thead>
<tr>
<th></th>
<th>var</th>
<th>Sample Var</th>
<th>Simulation Num</th>
<th>Comp Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>WEA</td>
<td>29.609</td>
<td>0.0023132</td>
<td>12800</td>
<td>102400</td>
</tr>
<tr>
<td>$k = 3$</td>
<td>11.385959</td>
<td>0.0015636</td>
<td>$m_0 = 3200$</td>
<td></td>
</tr>
<tr>
<td>IWEA</td>
<td>38.085451</td>
<td>0.0009521</td>
<td>40000</td>
<td>640000</td>
</tr>
<tr>
<td>$k = 4$</td>
<td>13.240223</td>
<td>0.0005445</td>
<td>$m_0 = 8000$</td>
<td></td>
</tr>
<tr>
<td>WEA</td>
<td>43.75197</td>
<td>0.0003798</td>
<td>115200</td>
<td>3686400</td>
</tr>
<tr>
<td>$k = 5$</td>
<td>14.334985</td>
<td>0.0001705</td>
<td>$m_0 = 19200$</td>
<td></td>
</tr>
</tbody>
</table>

From these tables, we find that the IWEA method has smaller sample variance compared with the WEA one, when these two method have the same $N$ and computational load. We also find that the $X_h(T)$ in the IWEA method has smaller variance than the one in WEA method.

Illustrative Example 6.5.3. Suppose

$$dx = -2x \, dt + y \, dW$$
$$dy = -2y \, dt - x \, dW$$

(6.29)
on the time interval \([0, 1]\) with the initial condition \((X_0, Y_0) = (2.5, 11)\). What is the expectation and variance of the \(X(T)\), which is generated by the IWEA and WEA method? What is the sample expectation and variance?

Solve:

**Expectation and Variance of the \(X_h(T)\) of the WEA method**

If we define the matrices \(M_{(i-1)}\) and \(A\) as below, the Euler approximation can be written in this form:

\[
M_{(i-1)} = \begin{pmatrix} 1 - 2h & \Delta W^{(i-1)} \\ -\Delta W^{(i-1)} & 1 - 2h \end{pmatrix},
A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

\[
\begin{pmatrix} X_h^{(i)} \\ Y_h^{(i)} \end{pmatrix} = M_{i-1} \begin{pmatrix} X_h^{(i-1)} \\ Y_h^{(i-1)} \end{pmatrix} = M_{i-1} \ldots M_0 \begin{pmatrix} X_h^{(0)} \\ Y_h^{(0)} \end{pmatrix}.
\]

So we have the expectation of the Euler approximation, \(E[X_h^{(i)}], E[Y_h^{(i)}],\)

\[
\begin{align*}
E[X_h^{(0)}] &= (1 - 2h)X_h^{(0)}, \\
E[Y_h^{(0)}] &= (1 - 2h)Y_h^{(0)}.
\end{align*}
\]

(6.30)

So far we obtain the expectation of the \(X_h(T)\), which is generated by the WEA method, whereas it is tricky to get its second-moment matrix.

\[
\begin{pmatrix} X_h^{(i)} \\ Y_h^{(i)} \end{pmatrix} = M_{i-1} \ldots M_0 \begin{pmatrix} X_h^{(0)} \\ Y_h^{(0)} \end{pmatrix} M_0^T \ldots M_{i-1}^T
\]

\[
M_i = (1 - 2h)E + \Delta W^{(i)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (1 - 2h)E + \Delta W^{(i)}A
\]

So as to get the second moment matrix of the \((X_h^{(i)}, Y_h^{(i)})\), we want to know the matrix \(A^i\). We can get the eigenvalue of the matrix \(A\), and then get the matrices \(P\) and \(Q\), such that

\[
P = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix},
Q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
A = PQP^{-1}
\]

Let matrix \(M^r\) be \(M^r = (Q)^r\), and matrix \(S_r\) be

\[
S_r = \begin{pmatrix} S_{11}^{(r)} & S_{12}^{(r)} \\ S_{21}^{(r)} & S_{22}^{(r)} \end{pmatrix} = PM^rP^{-1} \begin{pmatrix} X_h^{(0)}X_h^{(0)} & X_h^{(0)}Y_h^{(0)} \\ X_h^{(0)}Y_h^{(0)} & Y_h^{(0)}Y_h^{(0)} \end{pmatrix} (PM^rP^{-1})^T.
\]

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It is easy to see that

\[
E[\left(\begin{array}{c}
X_h^{(N)} \\
Y_h^{(N)}
\end{array}\right) \left(\begin{array}{c}
X_h^{(N)} \\
Y_h^{(N)}
\end{array}\right)'] = \sum_{r=0}^{N-2} \binom{N}{r} (1 - 2h)^{2(N-r)} \prod_{j=1}^{r} \Delta W^{(i_j)} \Delta \sigma_r.
\]

In our case, we just want to know the \( S_{11}^{(r)} \) and \( S_{22}^{(r)} \). Calculation gives us:

\[
S_{11}^{(r)} = \begin{cases} 
(X_0^{(0)})^2 & \text{(when } r = 2K, K = 1, \ldots) \\
(Y_0^{(0)})^2 & \text{(when } r = 2K - 1, K = 1, \ldots) 
\end{cases}
\]

\[
S_{22}^{(r)} = \begin{cases} 
(Y_0^{(0)})^2 & \text{(when } r = 2K, K = 1, \ldots) \\
(X_0^{(0)})^2 & \text{(when } r = 2K - 1, K = 1, \ldots) 
\end{cases}
\]

Hence we can get

\[
E[\left(\begin{array}{c}
(X_h^{(N)})^2 \\
(Y_h^{(N)})^2
\end{array}\right)'] = \sum_{r=0}^{N-2} \binom{N}{r} (1 - 2h)^{2(N-r)} h_r \left(\begin{array}{c}
S_{11}^{(r)} \\
S_{22}^{(r)}
\end{array}\right). \tag{6.31}
\]

Cooperating with the equations (6.30), we can work out the variance of \( X_h(T) \) generated by the WEA method. So are the sample variance.

the Expectation and Variance of the \( X_h(T) \) of the IWEA method

Analogous to the WEA, we have

\[
\begin{pmatrix}
X_h^{(i)} \\
Y_h^{(i)}
\end{pmatrix} = \begin{pmatrix}
1 - 2h & \Delta W^{(i-1)} \\
-\Delta W^{(i-1)} & 1 - 2h
\end{pmatrix} \begin{pmatrix}
X_h^{(i-1)} \\
Y_h^{(i-1)}
\end{pmatrix} 
\]

\[
\begin{pmatrix}
X_{2h}^{(i)} \\
Y_{2h}^{(i)}
\end{pmatrix} = \begin{pmatrix}
1 - 4h & \Delta W^{(2i-2)} + \Delta W^{(2i-1)} \\
-\Delta W^{(2i-1)} - \Delta W^{(2i-2)} & 1 - 4h
\end{pmatrix} \begin{pmatrix}
X_h^{(i-1)} \\
Y_h^{(i-1)}
\end{pmatrix} .
\]

Define

\[
\begin{cases} 
\bar{X}_h^{(i)} = X_h^{(2i)} - X_{2h}^{(i)} \\
\bar{Y}_h^{(i)} = Y_h^{(2i)} - Y_{2h}^{(i)}
\end{cases}
\]
We can have the equations

\[
\begin{pmatrix}
\hat{X}_h^{(i)} \\
\hat{Y}_h^{(i)}
\end{pmatrix}
= \left[ (1 - 4h) + (\Delta W_{2i-1} + \Delta W_{2i-2}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{pmatrix}
\hat{X}_h^{(i-1)} \\
\hat{Y}_h^{(i-1)}
\end{pmatrix}
+ [4h^2 - 2h(\Delta W_{2i-1} + \Delta W_{2i-2}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \Delta W_{2i-1} \Delta W_{2i-2} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} ] \begin{pmatrix}
X_h^{(2i-2)} \\
Y_h^{(2i-2)}
\end{pmatrix}
\]

\[
E \left( \begin{pmatrix}
\hat{X}_h^{(i)} \\
\hat{Y}_h^{(i)}
\end{pmatrix} \right) = (1 - 4h)E \left( \begin{pmatrix}
\hat{X}_h^{(i-1)} \\
\hat{Y}_h^{(i-1)}
\end{pmatrix} \right) + (4h^2)E \left( \begin{pmatrix}
X_h^{(2i-2)} \\
Y_h^{(2i-2)}
\end{pmatrix} \right)
\]

These equations bring us the Expectation of the \((\hat{X}_h^{(i)}, \hat{Y}_h^{(i)})\). Now we go on to find the variance of the \((\hat{X}_h^{(i)}, \hat{Y}_h^{(i)})\).

\[
E \left( \begin{pmatrix}
\hat{X}_h^{(i)} \\
\hat{Y}_h^{(i)}
\end{pmatrix} \hat{X}_h^{(i)} \right) = (1 - 4h)^2 E \left( \begin{pmatrix}
\hat{X}_h^{(i-1)} \\
\hat{Y}_h^{(i-1)}
\end{pmatrix} \hat{X}_h^{(i-1)} \right) + 8h^2(1 - 4h)E \left( \begin{pmatrix}
X_h^{(2i-2)} \\
Y_h^{(2i-2)}
\end{pmatrix} \hat{X}_h^{(i-1)} \right)
+ 2hE \left( \begin{pmatrix}
\hat{Y}_h^{(i-1)} \\
\hat{X}_h^{(i-1)}
\end{pmatrix} \hat{X}_h^{(i-1)} \right) + (h^2 + 16h^4)E \left( \begin{pmatrix}
X_h^{(2i-2)} \\
Y_h^{(2i-2)}
\end{pmatrix} \begin{pmatrix}
X_h^{(2i-2)} \\
Y_h^{(2i-2)}
\end{pmatrix} \hat{X}_h^{(i-1)} \right)
+ 8h^3E \left( \begin{pmatrix}
Y_h^{(2i-2)} \\
X_h^{(2i-2)}
\end{pmatrix} \begin{pmatrix}
Y_h^{(2i-2)} \\
X_h^{(2i-2)}
\end{pmatrix} \hat{X}_h^{(i-1)} \right) - 8h^2E \left( \begin{pmatrix}
Y_h^{(2i-2)} \\
X_h^{(2i-2)}
\end{pmatrix} \begin{pmatrix}
Y_h^{(2i-2)} \\
X_h^{(2i-2)}
\end{pmatrix} \hat{X}_h^{(i-1)} \right)
\]

(6.32)

In the discussion of the Expectation and Variance of the \(X_h(T)\) of the WEA method, we have the result that

\[
E \left( \frac{(X_h^{(i)})^2}{(Y_h^{(i)})^2} \right) = \sum_{r=0}^{i} \binom{i}{r} (1 - 2h)^{2i-2r} h^r \left( \frac{S_{11}^{(r)}}{S_{22}^{(r)}} \right)
\]

(6.33)

\[
E \left( \frac{X_h^{(i)}}{Y_h^{(i)}} \right) = (1 - 2h)^i \left( \frac{X_h^{(0)}}{Y_h^{(0)}} \right)
\]

(6.34)

From the definition of \((X_h^{(2i)}, Y_h^{(2i)}, X_{2h}, Y_{2h})\), we can also obtain

\[
E \left( \frac{X_h^{(2i)} X_{2h}}{Y_h^{(2i)} Y_{2h}} \right) = 
\]

\[
E \left( \frac{Q_{2h}^{(i)} X_{2h}}{Q_{2h}^{(i)} Y_{2h}} \right) = 
\]

\[
E \left( \frac{(1 - 2h)^2(1 - 4h)(X_h^{(2i-2)} X_{2h}^{(i-1)}) + (1 - 2h)2h(Y_h^{(2i-2)} Y_{2h}^{(i-1)})}{(1 - 2h)^2(1 - 4h)(Y_h^{(2i-2)} Y_{2h}^{(i-1)}) + (1 - 2h)2h(X_h^{(2i-2)} X_{2h}^{(i-1)})} \right)
\]

(6.35)
Numerical Approximation for SDE

\[
E \left( \frac{X_h^{(2i)} Y_h^{(i)}}{Y_h^{(2i)} Y_h^{(i)}} \right) = E \left( \frac{(X_h^{(2i)})^2}{(Y_h^{(2i)})^2} \right) - E \left( \frac{X_h^{(2i)} Y_h^{(i)}}{Y_h^{(2i)} Y_h^{(i)}} \right) \quad (6.36)
\]

The induction of the equations (6.32), (6.33), (6.34), (6.35), (6.36) will bring us the variance of the \((X_h(T), Y_h(T))\), which is generated by the IWEA method. So is the sample variance of the \((X_h(T), Y_h(T))\).

A Scilab program induct the expectation and variance of the \((X_h(T), Y_h(T))\), besides the sample expectation and sample variance, is in the following example.

Illustrative Example 6.5.4. Apply the conclusion from the last example to the particular conditions in the following table. What is the expectation and variance of the \((X_h(T), Y_h(T))\)? What about the sample expectation and variance?

<table>
<thead>
<tr>
<th>WEA</th>
<th>k step numbers</th>
<th>N</th>
<th>time T</th>
<th>((X^{(0)}, Y^{(0)}))</th>
<th>simulation num</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>8</td>
<td>1</td>
<td>(5,10)</td>
<td></td>
<td>3200</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>1</td>
<td>(2,3)</td>
<td></td>
<td>8000</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>1</td>
<td>(7,11)</td>
<td></td>
<td>44800</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IWEA</th>
<th>k finest step num</th>
<th>time T</th>
<th>((X^{(0)}, Y^{(0)}))</th>
<th>simulation num</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>8</td>
<td>1</td>
<td>(5,10)</td>
<td>800</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>1</td>
<td>(2,3)</td>
<td>1600</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>1</td>
<td>(7,11)</td>
<td>6400</td>
</tr>
</tbody>
</table>

Solve:

We run the C++ program which applies the IWEA and WEA method on the function (6.29), and obtain the expectation and variance of the \((X_h(T), Y_h(T))\), which are generated by these two methods. We also run the Scilab program, which get the theoretical expectation and variance of the \((X_h(T), Y_h(T))\) for our particular function (6.29).

Let's take a look at the theoretical sample variance and expectation first.

<table>
<thead>
<tr>
<th>WEA</th>
<th>k</th>
<th>(\text{var}(X_h))</th>
<th>(\text{var}(Y_h))</th>
<th>(E(X_h))</th>
<th>(E(Y_h))</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.0014092</td>
<td>0.0010837</td>
<td>0.5005646</td>
<td>1.0011292</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.0000317</td>
<td>0.0000252</td>
<td>0.2361342</td>
<td>0.3542013</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.0000703</td>
<td>0.0000521</td>
<td>0.9175882</td>
<td>1.4419244</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IWEA</th>
<th>k</th>
<th>(\text{var}(X_h))</th>
<th>(\text{var}(Y_h))</th>
<th>(E(X_h))</th>
<th>(E(Y_h))</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.0543000</td>
<td>0.0306581</td>
<td>0.5005646</td>
<td>1.0011292</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.0013139</td>
<td>0.0009206</td>
<td>0.2361342</td>
<td>0.3542013</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.0011028</td>
<td>0.0007501</td>
<td>0.9175882</td>
<td>1.4419244</td>
<td></td>
</tr>
</tbody>
</table>
Data in the table indicates that sample variance of the IWEA is bigger than that of the WEA. Why? Let review the general situation:

\[
\begin{align*}
\text{Var}\{f(X_h)\} &= \text{Var}\{f(X_h) - f(X_{2h})\} + \ldots + \text{Var}\{f(X_{2^{k-1}h}) - f(X_{2^kh})\} \\
&+ \text{Var}\{f(X_{2^kh})\} \\
&\leq E\{|f(X_h) - f(X_{2h})|^2\} + \ldots + E\{|f(X_{2^{k-1}h}) - f(X_{2^kh})|^2\} \\
&+ E\{|f(X_{2^kh})|^2\} \\
&= E\left\{ \sum_{j=1}^{m_0} \frac{|f(x_{2^j}) - f(x_{2^{j-1}})|^2}{m_0} \right\} + \ldots + E\left\{ \sum_{j=1}^{m_k} \frac{|f(x_{2^k})|^2}{m_k} \right\}.
\end{align*}
\]

The principal idea of the IWEA method is to properly allocate the computation load so as to optimise the sample variance. In this particular case, it is clear that the final two terms \( E\left\{ \sum_{j=1}^{m_k} \frac{|f(x_{2^k})|^2}{m_k} \right\} \) contribute big sample error. We can see that the function is decreasing, hence, when its step size is big, it will cause large \( \text{var}(f(X(T))) \). So is the sample variance.

What shall we do? One method is that we choose a stage, which has more time steps, as final stage. That is to say we pick a \( j \) \((0 \leq j \leq k)\) such that

\[
h = \frac{T}{2^j}.
\]

\[
\text{Var}\{f(X_h)\} = \text{Var}\{f(X_h) - f(X_{2h})\} + \ldots + \text{Var}\{f(X_{2^{k-1}h}) - f(X_{2^kh})\} + \text{Var}\{f(X_{2^kh})\}.
\]

Now we change some conditions:

**Illustrative Example 6.5.5.** Apply the conclusion from the last example to the particular conditions in the following table. What is the expectation and variance of the \((X_h(T), Y_h(T))\)? What about the sample expectation and variance?

<table>
<thead>
<tr>
<th>WEA</th>
<th>k</th>
<th>step numbers N</th>
<th>time T</th>
<th>((X^{(0)}, Y^{(0)}))</th>
<th>simulation num</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>128</td>
<td>1</td>
<td>(5,10)</td>
<td>102400</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>1</td>
<td>(2,3)</td>
<td>51200</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>0.5</td>
<td>(7,11)</td>
<td>64000</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IWEA</th>
<th>k; j</th>
<th>big/small step num</th>
<th>time T</th>
<th>((X^{(0)}, Y^{(0)}))</th>
<th>simulation num ( m_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7; 4</td>
<td>128;16</td>
<td>1</td>
<td>(5,10)</td>
<td>25600</td>
<td></td>
</tr>
<tr>
<td>6; 3</td>
<td>64;8</td>
<td>1</td>
<td>(2,3)</td>
<td>12800</td>
<td></td>
</tr>
<tr>
<td>6; 2</td>
<td>64;4</td>
<td>0.5</td>
<td>(7,11)</td>
<td>12800</td>
<td></td>
</tr>
</tbody>
</table>

Solve:

Let’s do the experiment again. We display the theoretical sample expectation and variance in the following table.
Clearly, this change sharply decreases the sample variance. So is the sample error. And the following table indicates that the expectation and variance of the \((X_h(T), Y_h(T))\), which are generated by IEWA and WEA method, do not have big difference.

<table>
<thead>
<tr>
<th>WEA</th>
<th>k</th>
<th>\var(X_h)</th>
<th>\var(Y_h)</th>
<th>E(X_h)</th>
<th>E(Y_h)</th>
<th>computation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>7</td>
<td>0.0000236</td>
<td>0.0000153</td>
<td>0.6660758</td>
<td>1.3321516</td>
<td>13107200</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.0000046</td>
<td>0.0000035</td>
<td>0.2621681</td>
<td>0.3932521</td>
<td>3276800</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.0001492</td>
<td>0.0000886</td>
<td>2.5549057</td>
<td>4.0148518</td>
<td>4096000</td>
</tr>
<tr>
<td>IWEA</td>
<td>k; j</td>
<td>\var(X_h)</td>
<td>\var(Y_h)</td>
<td>E(X_h)</td>
<td>E(Y_h)</td>
<td>computation</td>
</tr>
<tr>
<td></td>
<td>7; 4</td>
<td>0.0000140</td>
<td>0.0000110</td>
<td>0.6660758</td>
<td>1.3321516</td>
<td>13107200</td>
</tr>
<tr>
<td></td>
<td>6; 3</td>
<td>0.0000035</td>
<td>0.0000033</td>
<td>0.2621681</td>
<td>0.3932521</td>
<td>3276800</td>
</tr>
<tr>
<td></td>
<td>6; 2</td>
<td>0.0000716</td>
<td>0.0000621</td>
<td>2.5549057</td>
<td>4.0148518</td>
<td>4096000</td>
</tr>
</tbody>
</table>

The Scilab program for the first case is expIWEA.sce. The Scilab program for the expectation and variance of the \(X_h(T)\) is expIWEA2D1.sce. And the Scilab program for the sample expectation and variance of the \(X_h(T)\) and \(X_h(T)\) is SamIWEA2D1.sce.
Appendix A

Notations

A.1 Number the Instant of Time for 3/4 Method

We summarise the notation that we used in the previous chapters here.

(Number the instant of time) Divide $[a, b]$ into $m$ equal intervals, $J_0, \ldots, J_{m-1}$. Divide each interval $J_r$ ($r = 0, \ldots, m - 1$) again into $m$ equal subintervals $I_0^{(r)} = [a_{rm}, a_{rm+1}], \ldots, I_s^{(r)} = [a_{rm+s}, a_{rm+s+1}], \ldots, I_{m-1}^{(r)} = [a_{rm+m-1}, a_{rm+m}]$. Denote $h = a_{i+1} - a_i$, ($i = 0, \ldots, m^2 - 1$), and $N = m^2$, then we have $h = \frac{b-a}{m^2}$, where $a_0 = a$, and $a_{mm} = b$. These are shown in the following graphic.

$$\begin{array}{cccccccc}
\quad & a_0 & a_m & a_{2m} & \cdots & a_{rm} & a_{rm+1} & a_{rm+m} & \cdots & a_{rm+m-2} & a_{rm+m-1} & a_{mm} \\
J_0 & J_1 & \cdots & J_r & \cdots & J_{m-2} & J_{m-1} & \cdots & J_{m-2} & J_{m-1} & \cdots \\
\quad & a_0 & a_m & a_{2m} & \cdots & a_{rm} & a_{rm+1} & a_{rm+m} & \cdots & a_{rm+m-2} & a_{rm+m-1} & a_{mm} \\
\end{array}$$

A.2 Notations for 2D Case

Because in 2D situation we only need to consider $hG_{(r,i)}^{(r,i)} = \int_{a_{mr+i}}^{a_{mr+i+1}} [W(t) - W(a_{mr+i})] dt$, which is the covariance of $A_h^{(r,i)}$ and $\Delta Y_h^{(r,i)}$ conditional on $W(t)$. We denote it $G_h^{(r,i)}$ for short,

$$G_h^{(r,i)} = \frac{1}{h} E[A_h^{(r,i)} \Delta Y_h^{(r,i)} | \Delta W_h^{(r,i)}] = \frac{1}{h} \int_{a_{mr+i}}^{a_{mr+i+1}} [W(t) - W(a_{mr+i})] dt.$$ 

$\Delta W_h^{(r,i)}$: Independent increment of the Brownian trajectory $W$ in time segment $I_i^{(r)}$.
The value $h$ indicate the step size of the approximate Brownian trajectory. The values in "()" indicate the instant of time.

The $i^{th}$ instant of time in segment $J_{i}$, where $t = (rm + j)h$

$J_{r}$: Time segment $J_{r}$, including $m$ small segments $I_{i}^{(r)}$. See the above picture.

$J_{i}^{(r)}$: Time segment. See the above picture.

$\Delta Y_{h}^{(r,i)}$: Independent increment of the Brownian trajectory $Y$ in time segment $I_{i}^{(r)}$

$A_{h}^{(r)}$: The sum of $m$ area term $A_{h}^{(r,i)}$, $A_{h}^{(r)} = \sum_{i=0}^{m-1} A_{h}^{(r,i)}$

$A_{h}^{(r,i)}$: The approximate area term for $A_{h}^{(r)}$

$V_{h}^{(r,i)}$: $V_{h}^{(r)} = Var[A_{h}^{(r)}Y_{h}^{(r)}]$, $i = 0, \ldots, m - 1$

$\bar{V}_{h}^{(r)}$: $\bar{V}_{h}^{(r)}$ approximates $V_{h}^{(r)}$

$G_{h}^{(r,i)}$: $G_{h}^{(r)} = \frac{1}{h} E[A_{h}^{(r)}Y_{h}^{(r)}W(t)] = \frac{1}{h} \int_{a_{mr+i}}^{a_{mr+i+1}} [W(t) - W(a_{mr+i})] dt$

$Z_{h}^{(r)}$: Random variable that $Z_{h}^{(r)} \sim N(0, \bar{V}_{h}^{(r)}$)

$Q$: The threshold value for $I$ subdivision process

### A.3 Notation for 3D Case

$\Delta W_{h}^{(r,i)}$: Independent increment of the Brownian trajectory $W$ in time segment $I_{i}^{(r)}$

The value $h$ indicates the step size of the approximate Brownian trajectory. The values in "()" indicate the instant of time.

The $i^{th}$ instant of time in segment $J_{i}$, where $t = (rm + j)h$

$J_{r}$: Time segment $J_{r}$, including $m$ small segments $I_{i}^{(r)}$. See the above picture.

$J_{i}^{(r)}$: Time segment. See the above picture.

$\Delta Y_{h}^{(r,i)}$: Independent increment of the Brownian trajectory $Y$ in time segment $I_{i}^{(r)}$

$A_{h,R}^{(r)}$: An area term $A_{h,R}^{(r)} = \int_{a_{mr+i}}^{a_{mr+i+1}} W(t) - W(a_{mr+i}) dY(t)$ for segment $I_{i}^{(r)}$

$A_{h,Y}^{(r)}$: An area term $A_{h,Y}^{(r)} = \int_{a_{mr+i}}^{a_{mr+i+1}} Y(t) - Y(a_{mr+i}) dR(t)$ for segment $I_{i}^{(r)}$

$A_{h,R,Y}^{(r)}$: An area term $A_{h,R,Y}^{(r)} = \int_{a_{mr+i}}^{a_{mr+i+1}} [W(t) - W(a_{mr+i})] dY(t)$ for segment $I_{i}^{(r)}$

$A_{h,R,W}^{(r)}$: The sum of $m$ area term $A_{h,R,W}^{(r)} = \sum_{i=0}^{m-1} A_{h,R,W}^{(r,i)}$

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$A_{h,Y,W}^{(r)}$: The sum of $m$ area term $A_{h,Y,W}^{(r)} = \sum_{i=0}^{m-1} A_{h,Y,W}^{(r,i)}$

$A_{h,R,Y,W}^{(r)}$: The approximate area term for $A_{h,R,Y}^{(r)}$. $A_{h,R,Y}^{(r)}$, $A_{h,Y,W}^{(r)}$ are similar.

$V_{h,R,W}^{(r)}$: $V_{h,R,W}^{(r)} = Var[A_{h,R,W}^{(r)}Y_{h}^{(r)}W_{h}^{(r)}]$, $i = 0, \ldots, m - 1$

$V_{h,R,Y}^{(r)}$: $V_{h,R,Y}^{(r)} = Var[A_{h,R,Y}^{(r)}Y_{h}^{(r)}|R_{h}^{(r)}]$, $i = 0, \ldots, m - 1$

$V_{h,Y,W}^{(r)}$: $V_{h,Y,W}^{(r)} = Var[A_{h,Y,W}^{(r)}Y_{h}^{(r)}W_{h}^{(r)}]$, $i = 0, \ldots, m - 1$

$\tilde{V}_{h,R,W}^{(r)}$: $\tilde{V}_{h,R,W}^{(r)}$ approximates $V_{h,R,W}^{(r)}$

$G_{h,Y,W}^{(r)}$: $G_{h,Y,W}^{(r)} = \frac{1}{h} E[A_{h,Y,W}^{(r)}Y_{h}^{(r)}W_{h}^{(r)}] = \frac{1}{h} \int_{a_{mr+i}}^{a_{mr+i+1}} [W(t) - W(a_{mr+i})] dt$

$G_{h,Y}^{(r)}$: $G_{h,Y}^{(r)} = \frac{1}{h} E[A_{h,Y}^{(r)}Y_{h}^{(r)}|R_{h}^{(r)}] = \frac{1}{h} \int_{a_{mr+i}}^{a_{mr+i+1}} [W(t) - W(a_{mr+i})] dt$

$G_{h,W}^{(r)}$: $G_{h,W}^{(r)} = \frac{1}{h} E[A_{h,W}^{(r)}Y_{h}^{(r)}|W_{h}^{(r)}] = \frac{1}{h} \int_{a_{mr+i}}^{a_{mr+i+1}} (a_{mr+i+1} - t)[W(t) - W(a_{mr+i})] dt$
\( \tilde{Z}^{(r)}_{h,R,W} \): Random variable that \( \tilde{Z}^{(r)}_{h,R,W} \sim N(0, \tilde{\sigma}^{(r)}_{h,R,W}) \). \( \tilde{Z}^{(r)}_{h,Y,W} \) and \( \tilde{Z}^{(r)}_{h,R,Y} \) are similar.

Q: The threshold value for subdivision process

A.4 General Notation for n D Case

Based on this division of the time, we use the following symbols:

- The values in "(\( )\)" indicate the instant of time.
- The \( i^{th} \) instant of time in segment \( J_r \), where \( t = (rm + j)h \)
- The \( k \) in "{}" indicates that it is the \( k^{th} \) component of the vector \( A \).
- \( A^{(r,i)}(k) \)
  - \( h \) is the length of the discrete time
  - \( j \) and \( k \) indicate the components of the \( A \). (see the example below)

- The values in "(\( )\)" indicate the instant of time.
- The \( i^{th} \) instant of time in segment \( J_r \), where \( t = (rm + j)h \)
- The \( k \) in "{}" indicates that it is the \( k^{th} \) component of the vector \( G \).
- \( G^{(r,i)}(k) \)
  - \( h \) is the length of the discrete time
  - \( j \) and \( k \) indicate the components of the \( G \). (see the example below)

- The values in "(\( )\)" indicate the instant of time.
- The \( i^{th} \) instant of time in the time segment \( J_r \), where \( t = (rm + i)h \).
- The \( k \) in "{}" indicates that it is the \( k^{th} \) component of the \( W \).
- \( W^{(r,i)}(k) \)
  - \( h \) is the length of the discrete time
  - \( j \) and \( k \) indicate the components of the \( W \).

- The values in "(\( )\)" indicate the instant of time.
- The \( j^{th} \) instant of time, where \( t = jh \)
- The \( k \) in "{}" indicates that it is the \( k^{th} \) component of the \( X \).
- \( X^{(j)}(k) \)
  - \( h \) is the length between two instants of discrete time

Now, we give some definition of the notation which is based on the above notation method:

\[
\begin{align*}
  h &= \frac{T}{N} = a_i - a_{i-1} \quad \text{where } i = 1, \ldots, N; N = m^2 \\
  A_{h,k,s}^{(r)} &= \sum_{i=0}^{m-1} \int_{a_{rm+i}}^{a_{rm+i+1}} (W^{(s)}(t) - W^{(s)}(a_{rm+i})) dW^{(k)}(t) = \sum_{i=0}^{m-1} A_{h,k,s}^{(r,i)} \\
  \Delta W_h^{(r,i)}(k) &= W^{(k)}(a_{rm+i+1}) - W^{(k)}(a_{rm+i}) \\
  G_{h,R,0}^{(r,i)} &= W^{(k)}(a_{rm+i+1}) - W^{(k)}(a_{rm+i}) \\
  G_{h,R,0}^{(r,i)} &= \frac{1}{h} E\{A_{h,p,k}^{(r,i)} G_{h,R}^{(r,i)} | G_{h,1,0}^{(r,i)}, \ldots, G_{h,h,0}^{(r,i)} \} \\
  &= \frac{1}{(s-1)!h^s} \int_{a_{rm+i}}^{a_{rm+i+1}} (a_{rm+i+1} - t)^{s-1} (W^{(k)}(t) - W^{(k)}(a_{rm+i})) dt
\end{align*}
\]
\[ G_{h,k,s,R}^{(r,i)} = \frac{1}{h} E\{A_{\frac{3}{2},p,k}^{(r,i)} C_{h,p,(s-1),R}^{(r,i)} | C_{h,1,0}^{(r,i)}, \ldots, C_{h,k,0}^{(r,i)} \} \]
\[ = \frac{1}{(s-1)!\left(\frac{3}{2}\right)^s} \int_{a_{rm+i}+\frac{1}{2}}^{a_{rm+i+1}} \left[ (a_{rm+i+1} - t)^{s-1} \right] dt. \]

(A.1)

A.5 Notation for Improved Euler Method in Weak Approximation

\(W_h^{(j)[p]}\) : the \(j\)th instant of time, where \(t = jh\)
- The value \(h\) indicates the step size of the approximate Brownian trajectory
- The value in "[ ]" indicate the number of the sample.
- The \(p\)th sample Brownian path

\(X_{mh}^{(j)[p]}\) the \(j\)th instant of time, where \(t = jh\)
- The value \(mh\) is the length between two instants of discrete time
- The value in "[ ]" indicate the number of the sample.
- The \(p\)th sample Brownian path

\(T\) is the time scale we discuss
\(k\) is some integer
\(N\) \(N = 2^k\) is the number of segments over \(T\)
\(\tau_i\) partition \(0 = t_0 < t_1 < \ldots < t_{2^{k-i}-1} < t_{2^k} = T\)
\(\delta_i\) mesh(\(\tau_i\)) = (\(t_j - t_{j-1}\)) = \(\Delta_j\); \(j = 1, \ldots, 2^k\)
\(f(X_h)\) is the approximation of the \(E[f(X_h)]\)
\(\varepsilon_o\) the total sample error.
\(\varepsilon_o\) the weak approximation error.
Bibliography


