DROPLET THEORY OF
PHASE TRANSITIONS

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ABSTRACT

We present a droplet theory for the Potts universality class, designed to describe the configurational properties of a system consisting of $q$ phases, on and close to its coexistence curve. The theory is based on the configurational energy for a single droplet of one phase, embedded in a background of different phase, as prescribed by the sum of surface tension and external (magnetic) field energy.

In order to capture critical point effects, the theory allows for droplet shape fluctuations and droplet nesting. The effects of shape fluctuations can be controlled, for dimensionality close to one, in a renormalisation group context, and the resulting single droplet statistical weight which we calculate to two loop order is shown to be dominated by essentially spherical droplets whose scale size is typically smaller than the correlation length $\xi$.

Subsequently, a properly scaling theory for the many droplet assembly of the Potts universality class is constructed. The renormalisation group argument utilised for this purpose rests on the observation that, near $d = 1$, the droplet boundaries remain controllably dilute; thus the problem of droplet nesting can be handled. The resulting exponents satisfy the standard scaling laws, and $\nu$ and $\beta$ are shown to be in reasonable agreement with the results of a
We then invoke the Kasteleyn-Fortuin relation and derive an expression for the generating function of the cluster probability distribution in the bond percolation problem, above the percolation threshold. A closed equation for the distribution is obtained and shown to yield scaling behaviour. The form of the distribution is determined analytically in the regimes of small and large cluster sizes yielding, respectively, power law fall-off and Kunz-Souillard exponential decay. The form of the distribution for intermediate cluster sizes is determined numerically.
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CHAPTER 1
INTRODUCTION

An important aspect of solid state physics is the study of how a given system responds to controlled variations in its parameters, like temperature, pressure or external field strength. Often, spectacular changes can be the results of very small parameter variations; for instance, a small change of temperature suffices to turn a gas into a liquid and vice versa. Phase diagrams specify the different phases and their domains of existence for a given system. Crossing a phase boundary implies that the system undergoes a phase transition.

There is a wide range of phase transitions, and it is one of the main tasks to identify a (measurable) order parameter which characterises the difference between the phases involved. For example, the order parameter for the liquid-gas transition is defined as the difference between the fluid and gas densities, because this quantity mirrors the change in the system as a discontinuity at the transition point. The main theoretical problem consists in the construction of mathematical models which reflect the suddenness of the phase transition and its basic features.

Let us consider a specific system. Fig. 1 shows the magnetic field against temperature phase diagram of a
Fig. 1
Phase diagram of a typical ferromagnet
typical ferromagnet. It exhibits a boundary line, along the horizontal axis $H = 0$, which ends in a critical point at $T = T_c$. For high temperatures and zero field, one observes a disordered (paramagnetic) phase which is characterised by zero magnetisation; as one lowers the temperature, a transition takes place at the critical point, and for $T < T_c$ one observes an ordered (ferromagnetic) phase which displays a non-zero spontaneous (i.e. zero field) magnetisation. The magnetisation is different according to whether $H \to 0^+$ or $0^-$; hence, the zero-field magnetisation represents a suitable order parameter. At $T_c$, it is a continuous function; the phase transition is therefore called a continuous (or second order) transition. We encounter a different situation on crossing the boundary line $0 < T < T_c$, $H = 0$ in the vertical direction: Here, the field-dependent magnetisation displays a discontinuity at $H = 0$; this type of behaviour is called a first order (or discontinuous) phase transition. In the case of the simple ferromagnet, it separates two ordered phases with opposite magnetisations which can coexist along the boundary line - hence the boundary is also called a coexistence curve. As the critical point is approached, the discontinuity decreases and eventually vanishes at the critical point. The magnetisation discontinuity at the first order transition is reminiscent of the density discontinuity in the liquid-gas system, and there exists a close analogy between external field and chemical potential.
Fluctuations play an important role in determining the change in the system from one phase to the other: in the vicinity of the transition point, one phase will be only marginally less stable than the other one, and regions of the less stable (unfavoured) phase may appear in the favoured phase. A measure of the average extension of those perturbations is given by the correlation length \( \xi \).

On approaching a critical point, the correlation length diverges, and this gives rise to the universality phenomenon which is well established in both the theoretical and experimental description of critical phenomena (Kadanoff 1971, Wilson and Kogut 1974, Fisher 1974, Bervillier 1976, Bruce 1981a). It relies on the fact that the ordering coordinates are strongly correlated even over large distances such that microscopic (i.e. short-range) characteristics of the mathematical model or physical system become irrelevant. Thus models or systems which are only distinguished from one another by their microscopic features display identical critical behaviour, modulo relatively trivial scale factors: they belong to the same universality class. The major macroscopic parameters which define a universality class are the space dimension and the dimension (i.e. the number of components) of the order parameter as well as the symmetry of the system; for an exhaustive list and a detailed discussion see e.g. Aharony (1976). It is generally believed that it is not essential for the critical behaviour of a model whether it is defined on a lattice or in the continuum (cf.
however Baker and Kincaid 1981). A further consequence of the infinite correlation length is the self-similarity of the system on all length scales large compared to some infrared cutoff, which typically takes the form of a lattice constant, molecular size, or range of interactions. More intuitively speaking, the (coarse-grained) configurations are universal, up to redefinitions of the order parameter and space coordinate scales, and are constructed self-similarly from a basic building block. It appears sensible, therefore, to found the description of a given universality class on its typical building block, once it has been identified.

Let us now investigate the system close to and on the coexistence curve. Here, the correlation length remains finite, and thus, for small non-zero external field \( \langle H \rangle > 0 \), say) and finite temperature, we expect to observe thermodynamic fluctuations, in the form of finite regions (droplets) of unfavoured phase sitting in favoured phase. If we change the external field to some negative value, we favour the previously unfavoured phase, and vice versa. Following the classical theory of first order phase transitions (Gibbs 1906) one distinguishes between two types of approach to the new ground state. If the initial state is unstable, its decay is characterised by an instability against non-localised, long wavelength fluctuations; the corresponding decay process is called spinodal decomposition. Alternatively, if the initial state is metastable, phase separation takes place via the formation of localised, finite droplets of the
new phase which grow and eventually fill the whole space. Due to the energy barrier caused by surface tension, this situation is characterised by a delay between the time of the field quench and the nucleation process, thus reflecting the metastability of the initial state. A complete study of both processes involves the analysis of dynamical phenomena, in particular since there appears to be no sharp distinction between unstable and metastable states, only a gradual transition from one type of behaviour to the other in the vicinity of the classical spinodal curve. For a general introduction and a discussion of non-equilibrium phenomena we refer to Gunton et al. (1983), Gunton and Droz (1983), and Gunton (1983). We shall here be concerned with the properties of the initial metastable and final equilibrium state.

Thus, droplets emerge naturally in studies of phase separation in the vicinity of the coexistence curve where they represent a localised, finite amplitude perturbation initiating the decay of a metastable state. Their importance has long been recognised (Becker and Döring 1935, Frenkel 1955). Simultaneously, it becomes apparent that standard perturbative approaches, specifically conventional field theory calculations, which focus explicitly on small fluctuations about an absolute minimum of the Hamiltonian, will experience difficulties in capturing transition properties due to finite or even large perturbations. Theories which are based on finite droplets as fundamental building blocks
attempt to take those effects into account; and it turns out that in the calculation of transition properties such as the singularity of the free energy on the coexistence curve the leading contributions can be traced to droplets which are large on the scale of the correlation length (Andreev 1963, Fisher 1967, Langer 1967, Günther et al. 1980). Low temperature series expansions are, in a sense, droplet expansions, as they attempt to assess the effect of configurations corresponding to finite perturbations (droplets) in otherwise homogeneous background. One might, however, envisage technical difficulties, since the preponderance of large droplets requires the evaluation of high order terms in the series; yet we stress that series-based studies (Baker and Kim 1980, Enting and Baxter 1980) are consistent with the most explicit droplet-based analysis of this phenomenon (Lowe and Wallace 1980).

The observation that the major contributions are made by large droplets has a number of simplifying consequences. Firstly, because of their size, those droplets remain essentially unaffected by microscopic properties of the system, and hence one expects to find some degree of universal behaviour. One should note, though, that in the present situation universality is not due to an infinite correlation length as in the case of critical phenomena. Secondly, large droplets are, to a very good approximation, spherical; and finally, they are rare, since the probability of finding them falls off exponentially with surface energy.
Consequently, earlier investigations of a neighbourhood of the coexistence curve (Fisher 1967, Langer 1967) emphasized the dominant role of large droplets by treating all droplets as spherical and the droplet ensemble as a dilute gas, thus neglecting excluded volume and droplet overlap effects. Yet, in spite of those two fundamental simplifications, qualitatively correct results were obtained; in particular, one can calculate the free energy of the system and study its analytic structure at zero magnetic field. Physically, the competition between surface tension and field energy determines the conditions for metastability. If one starts with a configuration in small external field and then reverses the field, droplets of the now favoured phase will grow if their radius is larger than some critical radius and thus pilot the phase transition, or otherwise decay. The critical radius is determined by the extremum of the total (single droplet) energy, where variations in the volume of the favoured phase will just balance the variations in surface energy. Clearly, for small external field, the critical radius will be very large. Hence the formation of a critical droplet will require a long time, reflecting the metastability of the system.

The major drawback of those simple droplet models resides in the fact that they are unsuited to capture critical point effects. These require the construction of a scale invariant picture. The corresponding changes in the configurational structure which invalidate the previous
simplifications are manifested in two ways:

Firstly, reflecting both decreasing spontaneous magnetisation and increasing correlation length, the system will be populated by droplets of all scale sizes, predominantly sizes small compared to the correlation length. Those small droplets cannot be treated a priori as spherical, and we have to allow for droplet shape fluctuations on all length scales down to some (lattice) cutoff. Secondly, a critical ensemble is not a dilute gas, hence some account of excluded volume and droplet overlap effects has to be taken. An intuitive visualisation of a system under those conditions has been given by Kadanoff (1976a) in his discussions of droplet nesting: He suggests that, as criticality is approached, one must allow for droplets to sit within larger droplets which are themselves embedded in even larger droplets, and so on. The correlation length determines the largest possible droplet size, whereas the infrared cutoff is reflected in a minimal allowed droplet size. It is only with the help of a corresponding decoration procedure that a fully self-similar droplet theory can be constructed.

It should be clear now that a droplet theory of the whole coexistence curve including the critical region needs to tackle two new problems: how to describe a fluctuating, droplet-like interface, and how to construct a self-similar, non-dilute droplet ensemble. Both tasks involve the study of critical fluctuations, i.e. fluctuations over many length
scales: Concerning the former, we expect the surface to become increasingly fuzzy, with surface tension between the two phases going to zero, whereas, regarding the latter, we have to incorporate the fluctuations in scale size over the whole range between the system size and the cutoff. The necessary tools are available in the form of renormalisation group (see e.g. Pfeuty and Toulouse 1975) and field theory (Amit 1978).

The Ising model constitutes the prototype description for the simplest ferromagnets. It possesses two main simplifying features: the order parameter is a scalar (one component) function, and the interactions are short ranged. The model exhibits a line of first order phase transitions, ending in a critical point, as depicted in Fig. 1; along the line, two phases of opposite magnetisations coexist. In its lattice version, the spin variables can assume only two values (+1 and -1, say, or "up" and "down"). The nearest neighbour interactions then take one value for parallel spins, and another for antiparallel spins. In a droplet description, this is reflected by the properties of surface energy: it is non-zero if we decorate one phase with a droplet of the other phase, whereas it is obviously zero for a homogeneous system. Further, the surface energy of a droplet is proportional to its surface area. The magnetic field energy, on the other hand, is proportional to the droplet volume, if we subtract off the ground state energy. Thus, the total energy of a single droplet can be written as the sum of
surface and field energy. If we add to this the assumption that the typical droplet is spherical and that the droplet ensemble is dilute we obtain a phenomenological droplet theory of the Ising model away from criticality. However, this cannot provide a description of the critical region, because shape and scale size fluctuations are not included.

Bruce and Wallace (1981, 1983; hereafter referred to as Bruce and Wallace) suggested a more elaborate model for the Ising universality class which remedies the shortfalls of the phenomenological theory. Their basic Hamiltonian is given by the surface energy of a single droplet whose shape is allowed to deviate from spherical; thus the effects of shape fluctuations are captured. As a first step, one may calculate the single droplet partition function and analyse its properties. As a second step, Bruce and Wallace developed a decoration procedure which enabled them to construct a properly scaling self-similar droplet ensemble and to evaluate its partition function in zero external field, thus capturing the effects of droplet nesting. As in similar problems, the spatial dimension plays a crucial role. It becomes apparent that, in low spatial dimension, both surface and scale size fluctuations can be controlled (in a field theory context), and that, moreover, both problems can be addressed separately. Hence one is able to distinguish between system properties due to interface fluctuations (like the thermal exponent $\nu$) on one hand, and scale size fluctuations (like the magnetic exponent $\beta$) on the other.
hand. Those results were subsequently refined by Sim (1984) and Sim and Bruce (1984) who analysed the effects of a magnetic field. The success of the model in low spatial dimension is plausible from a configurational viewpoint: in one dimension, the configurational physics is universally that prescribed by a gas of kinks (Krumhansl and Schrieffer 1975, Zinn-Justin 1981, Bruce 1981b); since the gas is dilute the spectrum of configurations and their collective behaviour can be easily determined. Our droplet theory may thus be regarded as the natural continuation to $1+\varepsilon$ dimensions of the one-dimensional kink theory (Bruce and Wallace 1983).

An obvious extension of the lattice Ising model is the Potts model (Potts 1952): at each lattice site, the spin is allowed to take on more than two orientations while keeping the interactions as simple as in the Ising model. Thus the interaction between two neighbouring spins still only assumes two values, depending on whether the two spins are in the same or a different state. The possible states of the ordering coordinate in the $q$-state Potts model may be represented by a set of $q$ vectors pointing to the corners of hypertetrahedron in $(q-1)$-dimensional space (Zia and Wallace 1975).

The Potts model has attracted much interest over recent years because, inspite of its relative simplicity, it possesses a richer and more general structure than the Ising model. For a detailed discussion, we refer to Wu (1982), and review
here only the main features of the phase diagram. At zero external field, the $q$-state Potts model has a phase transition at some "critical" temperature $T_c$ (whose value depends on the details of the underlying Hamiltonian). The nature of that transition has proved to be very interesting. Mean field theory predicts that the transition is first order for all $q > 2$, irrespective of dimension (Kihara et al. 1954, Mittag and Stephen 1974). There is, in fact, an exact result (Baxter 1973 for the square lattice, Baxter et al. 1978 for the triangular and honeycomb lattices) which shows that in two dimensions the transition is first order for $q > 4$. However, since the $q = 2$ Potts model is identical to the Ising model, there is also Onsager's exact result (Onsager 1944) which gives a continuous phase transition in two dimensions. Generally, one therefore expects the existence of a critical value $q_c(d)$ at which, in $d$ dimensions, the transition changes from continuous to first order. The value of $q_c(d)$ has been extensively studied. Apart from the results given above, we have $q_c(d) = 2$ for $d > 4$ (Aharony and Pytte 1981) and the renormalisation group results by Berker et al. (1980), Andelman and Berker (1981) and Nienhuis et al. (1981) which predict that $q_c(1+\varepsilon) = \exp{2/\varepsilon}$ for small $\varepsilon$. In other words, for small $q$ the transition will be continuous and only cross over to first order behaviour for very large $q$.

Let us now consider the effect of an external field on the phase diagram (Wu 1982, see also Straley and Fisher 1973). If the transition at $T_c$ is continuous, the ordered
phases can coexist along the line $H = 0$ and $T < T_c$. For a first order transition at $T_c$, the situation is somewhat more complex because at the transition point one observes coexistence of the $q$ ordered phases and the disordered phase.

Apparently, the domain of $q$ values with $q < q_c(d)$ where the transition at $T_c$ is continuous and where, below $T_c$, the $q$ ordered phases coexist along the line $H = 0$, shows a critical structure which is quite reminiscent of the Ising model. Moreover, for $d = 1+\varepsilon$ and $\varepsilon$ small, that domain stretches up to very large $q$ values. The first observation suggests that the Potts model might be amenable to a droplet description along the same lines as the Ising model, viewing regions of different phase as droplets which can now have $q$ (instead of two) "colours". The second observation implies that such a description might cover quite a large range of $q$ values, even if the approximation might be more appropriate for certain $q$ values than for others.

Another essential feature of the Potts model lies in the fact that it incorporates various other physical systems as limiting cases, for special values of $q$. Of particular interest to us will be the expression for the Potts free energy if $q \rightarrow 1$ because it coincides with the generating function of the cluster number distribution in the bond percolation problem (Kasteleyn and Fortuin 1969, Fortuin and Kasteleyn 1972).
The bond percolation problem is easily defined: Consider a $d$-dimensional lattice in which nearest neighbour bonds are present with a probability $p$, and absent with a probability $1-p$. Then the bonds form clusters which are labelled by the number of connected sites they contain; and one can study, for example, the probability for a given site to belong to an $s$-site cluster, as a function of $s$ and $p$.

Of course, the percolation of bonds does not exhaust the multitude of possible percolation problems. To name only a few of the other models which have attracted interest: in site percolation, one analyses the random occupation of lattice sites (rather than bonds); directed percolation attributes an orientation to the links between cluster sites, so that the connection can only be made in one direction; and continuum percolation models involve unit spheres (say) being distributed randomly in space.

Physical questions related to percolation include, for instance, the flow of fluid or charge in random networks, or the problem of non-magnetic dilutions in otherwise magnetic materials, or even biological problems like the spreading of an epidemic from one point to another.

The fundamental feature of all percolation problems is the following: For $p$ below some critical value $p_c$, there exist only clusters with a finite number of sites. However, if $p > p_c$, then there exists one infinite cluster in the
thermodynamic limit.

The attraction of the percolation problem resides in a number of properties: Firstly, it is easy to define and to visualise, yet the transition at $p_c$ contains many of the generic features in phase transitions. Secondly, since it is not a thermal problem, its simulation on a computer does not involve any problems with equilibration. And thirdly, as we have seen, it has a number of variants with important physical applications. All these points are elaborated in greater detail by Stauffer (1979) and Essam (1980) in their review articles, which also contain references to specific aspects.

It will turn out that the introduction of (many) more phases into the droplet description of the Potts model renders it less reliable; more specifically we observe that corrections to our expressions will typically be proportional to the number of phases, $q$. It may hence be anticipated that the theory might be best suited to describe the $q \to 1$ percolation limit of the Potts model. Invoking the Kasteleyn-Fortuin relation we establish a properly scaling droplet theory of the percolation problem, in low spatial dimension.

The organisation of this thesis is as follows: In the first section of the next chapter, we review the assumptions and results of the phenomenological droplet model, in order
to introduce some of the main concepts of the following work with the help of a simple model. In Section 2 of the same chapter we discuss more general interface Hamiltonians which allow for shape fluctuations. In particular, we shall analyse the relations between the familiar $\phi^4$-Hamiltonian and the Hamiltonians for fluctuating planar and spherical interfaces. Chapter 3 sees the calculation of the single droplet partition function up to and including two loops, as well as the renormalisation of that result. This extends and checks the one loop calculation by Bruce and Wallace which was not sufficient to prescribe the full one loop amplitude of the partition function within the droplet framework. Then we proceed to develop a droplet description for the Potts model in Chapter 4. The main point will be the calculation of the ensemble free energy, using a renormalisation group argument similar to the one employed by Bruce and Wallace in their study of the Ising model. Finally, we shall evoke the relation between the Potts and the percolation problem and obtain an expression for the free energy of bond percolation. This leads us to a study of the cluster size distribution, and we establish its scaling properties as well as its form over the whole range of cluster sizes. The thesis concludes with a critical assessment of our description and its relation to other results.
2.1 A simple droplet model

In order to set the scene and amplify some of the ideas mentioned in the introduction, let us briefly review a very elementary continuum version (Wallace 1980) of the simple droplet model which was introduced in its discrete version by Langer (1967) and Fisher (1967). This will also familiarise us with many of the themes which will recur in the presentation of the more sophisticated droplet model later on.

The basic line of thought is as follows: First of all, we write down the configurational energy (the "Hamiltonian") for a single, isolated droplet, thus providing a basis for the discussion of single droplet properties. This already implies a very fundamental assumption: that it is at all possible to identify an individual droplet and to treat it separately from all other droplets in the ensemble. More specifically, we assume that we may neglect interactions between droplets which typically take the form of droplet overlap and excluded volume effects. The neglect of interactions is justifiable if the droplet gas is sufficiently dilute. In that case, the calculation of the multi-droplet partition function presents no difficulties, and one may proceed to study its properties, checking the results against the assumptions
Let us then begin by looking at a single, isolated droplet, embedded in otherwise homogeneous background of different phase. The simplest single droplet Hamiltonian consists of the sum of two terms, the surface energy plus the volume dependent external field energy of a spherical droplet with radius $R$:

$$\mathcal{H} = \sigma R^{d-1} + h R^d$$ \hspace{1cm} \sigma, h > 0 \hspace{1cm} (2.1)$$

where all further proportionality factors have been absorbed into the surface tension $\sigma$ and the magnetic field $h$. The sign convention in eqn (2.1) corresponds to a configuration where the background consists of the phase favoured by the field and the droplet consists of unfavoured phase. The droplet thus represents a thermodynamic fluctuation, made possible by the thermal energy present in the system. If we reverse the field slowly to a small negative value, we reach a metastable state, described by the Hamiltonian eqn (2.1), with $h$ replaced by $-|h|$. In order to determine the criteria for the onset of nucleation we ask for the conditions under which a droplet grows or not. $\mathcal{H}$ becomes extremal if

$$\frac{d\mathcal{H}}{dR} = (d-1)\sigma R^{d-2} - |h| R^{d-1} = 0 \hspace{1cm} (2.2)$$

The solution to this equation is the critical radius $R_c$: 
If $R > R_c$, the droplet will grow because field energy wins over surface tension, and vice versa, if $R < R_c$. We observe that, for small $h$, nucleation requires the formation of a large droplet with $R_c \sim |h|^{-1}$, and is thus likely to occur only after a considerable time delay. Equivalently, the concentration of critical droplets is expected to remain small; consequently, the droplets which govern the system behaviour form a dilute gas, rendering the neglect of droplet interactions plausible.

This treatment can be pushed further by calculating the free energy of the stable phase. To begin with, consider the single droplet partition function obtained from the Hamiltonian eqn (2.1) by integrating over all droplet radii:

$$Z_1(h) = \int_0^\infty dR \exp -\mathcal{H}$$

(2.4)

This exhausts the (single droplet) configurational sum because the droplet shapes are restricted to spherical. The upper limit of the integral extends to infinity, thus allowing for droplets of arbitrary large sizes in the thermodynamic limit. We have simply set the lower bound equal to zero, in the expectation (which will be confirmed) that it is the large $R$ regime, specifically $R$ values with $R > R_c$, which determine the essential system properties. We note, however, that there is a certain degree of arbitrariness in the choice of
the lower bound: one may, for instance, cut off the contributions made by droplets with sizes smaller than or comparable to the correlation length, in recognition of the fact that the model (2.1) breaks down in that regime (Wallace 1980). We shall see presently that this only changes the real part of the free energy. The next step, the characterisation of a particular multi-droplet configuration in a dilute gas, is very easy: It suffices to specify the number of droplets in the configuration, and hence the sum over all configurations is equivalent to the sum over all droplet numbers. Due to the neglect of droplet interactions, the single droplet partition functions simply multiply, and a factor n! is needed to avoid overcounting. Thus,

\[ Z(h) = \sum_{n=0}^{\infty} \frac{1}{n!} [Z_i(h)]^n = e^{\mathcal{Z}_i(h)} \]

(2.5)

which yields for the free energy

\[ F(h) = \ln Z(h) = \int \! d\mathbf{R} \exp \left(-\mathcal{S}_0 \mathbf{R}^{d-1} + h \mathbf{R}^d \right) \]

(2.6)

This function can now be analysed for various values of h. For \( h > 0 \), it is well defined, since this corresponds to the thermodynamically stable phase. However, we observe that it possesses a singularity at \( h = 0 \) by examining the radius of convergence of its power series expansion which turns out to be zero.
With a singularity at the origin, we have to resort to analytic continuation if we want to study $F(h)$ for $h < 0$, i.e. in the metastable phase. Physically, analytic continuation of $F(h)$ corresponds to the smooth reversal of the magnetic field which we discussed earlier on. The idea is to rotate $h$ away from the positive real $h$-axis through complex $h$ until one reaches the negative real axis. When doing so, we need to ensure that $F(h)$ remains convergent, i.e. we demand that $\text{Re}(h R^d) > 0$. This requires a counterrotation of the integration contour, and if we choose $h R^d$ to remain real and positive, $\arg h = \pm \pi$ implies that $\arg R = \mp \pi/d$. For negative $h$, the contour now lies in the complex plane, and hence $F(h)$ will pick up an imaginary part. To leading order, for $|h|^{-1} \rightarrow \infty$, this imaginary part can be calculated by the method of steepest descent. In order to obtain $\text{Im} \ F(\arg h = \pm \pi)$ we choose the contour as in Fig. 2 and perform the half Gaussian integral around the saddle point $R_c$, whence we find the general structure

$$\text{Im} \ F(\arg h = \pm \pi) = \pm A |h|^{-b} \exp -B |h|^{-a} \cdot \left[ 1 + O(|h|^{d-1}) \right] \quad (2.7a)$$

The singularity has been investigated more carefully by Günther et al. (1980) who study the effect of surface fluctuations on $\text{Im} \ F$. The general structure of the result (2.7a) is confirmed, but the $h$-dependent prefactor and hence the exponent $b$ are renormalised by the surface fluctuations, giving
Fig. 2
Integration contour for steepest descent calculation
The general form (2.7a) and the exponents \( a \) and \( b \) are (presumably) universal, since they are simply associated with the geometry of a critical droplet of radius \( R_c \propto |h|^{-1} \). \( A \) and \( B \) are model-dependent (non-universal) constants. Hence \( F(h) \) possesses a branch cut along the negative real axis, with the purely imaginary discontinuity across the cut given by eqn (2.7). \( h = 0 \) is called a branch point singularity in the terminology of Zia (1981).

In conclusion of this section, let us summarise the main points. We have seen that, in a dilute gas approximation, it is justifiable to isolate an individual droplet and treat it separately from all other droplets in the ensemble. As a consequence, the single droplet partition function simply exponentiates to yield the ensemble partition function. The main shortfall of the model is that it presupposes purely spherical droplets which is sensible only for droplet sizes large compared to the correlation length. It is therefore not possible to capture critical point effects in this framework, which is further stressed by the breakdown of the dilute gas approximation, as \( T_c \) is approached. In the following, we shall present a more sophisticated model which will allow for shape fluctuations, while at the same time retaining the simple exponential relation between single

\[
\begin{align*}
  a &= d - 1 \\
  b &= \begin{cases} 
    \frac{(d-3)d/2}{7/3} & \text{if } 1 < d < 5, d \neq 3 \\
    d = 3 & \text{otherwise}
  \end{cases}
\end{align*}
\]
droplet and ensemble partition function in zero field. We shall see that it will be necessary to refine the dilute gas approximation so as to accommodate a non-dilute droplet population in the neighbourhood of the critical point.
In equilibrium statistical mechanics, the definition of any model requires a number of elements: First of all, the underlying space (which can be a lattice or the continuum) has to be fixed. Then we need to define the random variable whose values at each space point (or lattice site) define a particular configuration. Thirdly, a prescription is required for calculating the energy associated with a specific configuration. Finally, in order to obtain thermodynamic observables we need to perform the sum over all configurations, each weighted by the familiar Boltzmann factor.

Models for Ising systems fall into two major categories: lattice models, in which a spin variable, defined on the lattice, can take on two values, and their continuum limits, expressed appropriately by a corresponding scalar field theory. Which one to choose, is a question of technical convenience, at least ideally (cf. the corresponding remarks in the introduction), since critical phenomena are dominated by long wavelength fluctuations, and the universality principle is expected to hold. We shall find a field theory more amenable to our purposes.

Two types of interfaces are familiar from many physical instances: the planar interface, separating two phases in
equilibrium, at zero external field, and spherical interfaces associated with the decay of a metastable state. The latter case will eventually be of greater interest to us, yet we shall repeatedly invoke properties of the former in the discussion of spherical interfaces. Thus it seems worthwhile to introduce both types of interface simultaneously, specifically because both are designed to reflect properties of the Ising universality class. Many aspects of both cases have been extensively studied in the framework of lattice models, in particular in connection with exactly solvable models (cf. e.g. Fröhlich 1983), the solid-on-solid (SOS) approximation (Temperley 1952, Müller-Hartmann and Zittartz 1977) and equilibrium droplet shapes (Zia and Avron 1982, Avron et al. 1982, Rottman and Wortis 1981, cf. also the review by Zia 1983). Also, lattice representations are directly accessible to Monte Carlo simulations (see e.g. Binder 1979).

Intuitively, the energy of an interface in a continuum model, with respect to the fully ordered ground state, is determined by its surface energy. The justification of this assumption resides in the fact that, ultimately, we are interested in critical, i.e. long wavelength fluctuations. These are smooth, slowly varying deviations from the respective reference shape which are controlled by surface tension. In the critical limit, surface tension does not depend on orientation (if there is an anisotropy in the system, it becomes trivial and can be removed by a simple
rescaling, see e.g. Avron et al. (1982)); hence the energy of the interface is proportional to its total surface area.

With some geometry, we can obtain expressions for the surface area of \((d-1)\)-dimensional fluctuating planar and spherical interfaces in \(d\)-dimensional bulk. Their derivation requires only one specification: surface "overhangs" are neglected, so that the interface can be described by a single-valued field. Let us begin with the planar case (Fig. 3). Familiar physical examples are the interface between liquid and vapour at coexistence, or domain walls in uniaxial magnets or ferroelectrics. The position of the interface is given by a scalar field \(f(x)\) which specifies its distance from the \((x,0)\) plane at any point \(x=x_1,\ldots,x_{d-1}\). The total surface area, as a functional of \(f(x)\), reads:

\[
A = \int d^{d-1}x \left[ 1 + (\nabla f(x))^2 \right]^{1/2}
\]

where the root is the inverse of the direction cosine between the \(z\)-axis and the local unit normal to the surface. Eqn (2.8) gives rise to the Hamiltonian (Mandelstam 1913, Buff et al. 1965):

\[
\mathcal{H} = T_0^{-1} A
\]

where we have absorbed surface tension into the bare system temperature \(T_0\).
Fig. 3

Fluctuating planar interface

\[ f(x) \]
Examples of droplet shaped interfaces are familiar in systems below their critical temperature, close to the coexistence curve. A 'droplet' is a connected region of space occupied by one particular phase and separated from the background phase by a closed boundary, which is taken to be roughly spherical. We define a reference hypersphere with radius $R$, and let the field $f(\eta)$ denote the deviation from spherical in the direction specified by the radial unit vector $\eta$ (Fig. 4). Then, the Hamiltonian reads:

$$H = T_0^{-1} \int d\Omega (R + f)^{d-1} \left[ 1 + \frac{1}{2} (L_{ij} f)^2 (R + f)^{-2} \right]^{1/2} \quad (2.10a)$$

where $d\Omega$ denotes the solid angle element, and $L_{ij}$ is the rotation generator (angular momentum operator) in the $ij$-plane:

$$L_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \quad (2.10b)$$

Again, the root is the inverse of the direction cosine, this time between the local unit normal and the radial unit vector $\eta$; and $T_0$ is a measure of (inverse) surface tension. At this point, what we mean by 'radius' is not yet clearly defined. This will be done in eqn (3.8); for the moment it suffices to think of $R$ as some measure of the droplet scale size.

So far, we have outlined a very intuitive approach to interface Hamiltonians. There is, however, a somewhat more
Fig. 4

Geometry of the non-spherical droplet
technical - or, depending on one's philosophy, more rigorous approach to the problem which clarifies the connection with the familiar Landau-Ginsburg-Wilson Hamiltonian:

\[ \mathcal{H}[\phi] = \int d^d x \left[ \frac{1}{2} (\nabla \phi)^2 + V(\phi) - \mathcal{H} \phi \right] \quad (2.11) \]

\(\mathcal{H}[\phi]\) is a coarse-grained version of an Ising model close to \(T_c\) and possesses no memory of the underlying lattice anisotropies. \(\phi\) should be understood as a measure of the (coarse-grained) spin density, and \(\mathcal{H}\) is the external field. For \(V(\phi)\), we take the standard \(\phi^4\)-potential:

\[ V(\phi) = -\frac{1}{2} \mu \phi^2 + \frac{1}{4} q \phi^4 \quad \mu, q > 0 \quad (2.12) \]

yet most of the ensuing analysis will be applicable to a more general potential provided it possesses two degenerate minima.

In order to obtain an (almost) planar interface between two stable phases, we set \(\mathcal{H} = 0\). The two homogeneous phases then correspond to the minima of the potential (2.12):

\[ \phi_{\pm} = \pm \sqrt{\mu / q} \]

Let us choose the coordinate system such that the interface is perpendicular to the z-axis. The classical theory of the planar interface (Van der Waals 1894, cf. also Cahn and Hilliard 1958, Widom 1972 and Evans 1979) deals with a
configuration \( \phi_c(z) \), describing the interface profile and depending solely on \( z \), which is an extremum of \( \mathcal{H}[\phi] \):

\[
\frac{\delta \mathcal{H}}{\delta \phi}|_{\phi_c} = 0 \iff \frac{d^2 \phi_c(z)}{dz^2} = \frac{dV(\phi)}{d\phi}|_{\phi_c}
\]  

(2.13a)

and satisfies the boundary conditions

\[
\lim_{z \to \pm \infty} \phi_c(z) = \phi_{\pm} \quad \lim_{z \to \pm \infty} \phi'_c(z) = 0
\]  

(2.13b)

Then \( \phi_c(z) \) is the familiar kink solution which can be shown to exist for general degenerate double well potentials (Zinn-Justin 1980). For the \( \phi^4 \)-potential, the integration yields:

\[
\phi_c(z) = \phi_{\pm} \tanh \left[ \left( \frac{\mu}{2} \right)^{1/2} (z - z_0) \right]
\]  

(2.14)

The constant of integration \( z_0 \) specifies the position of the interface. Its arbitrariness reflects the translation invariance of \( V(\phi) \), whereas the freedom in the choice of the coordinate system reflects the rotation invariance of the potential. It is easy to show that, in the bulk phase, the correlation length \( \xi^{-1} = \sqrt{2}\mu \) to lowest order in \( g \). Hence we see from eqn (2.14) that the intrinsic width of the interface is proportional to \( \xi \), and thus diverges in the critical limit (see Ohta and Kawasaki 1977, Rudnick and Jasnow 1978 and Jasnow and Rudnick 1978).
Let us now proceed to discuss fluctuations about the classical solution. Let

\[ \phi = \phi_c + \hat{\phi} \]  

(2.15)

and expand \( \mathcal{H}[\phi] \):

\[ \mathcal{H}[\phi_c + \hat{\phi}] = \mathcal{H}[\phi_c] + \frac{\delta \mathcal{H}}{\delta \phi_c} \hat{\phi} + \frac{1}{2} \hat{\phi} \frac{\delta^2 \mathcal{H}}{\delta \phi_c^2} \hat{\phi} \]  

(2.16)

The first term on the right hand side is simply a constant and may be disregarded. The second term is identical to zero, since we expand about an extremum. Hence, to this order, we only have to study the eigenvalue spectrum of the differential operator

\[ M = \frac{\delta^2 \mathcal{H}}{\delta \phi_c^2} \phi_c = -\nabla^2 + V''(\phi_c) \]  

(2.17)

First of all, we find an eigenfunction with zero eigenvalue:

\[ M \frac{\partial}{\partial z} \phi_c(z) = 0 \]  

(2.18)

in which we recognise the Goldstone mode of the translation invariance broken by the interface. \( \partial / \partial z \) is the generator for infinitesimal translations in the z-direction, and eqn (2.18) is just the mathematical analogue of the physical observation that it does not cost any energy to translate the whole interface rigidly in the z-direction.
Further, we observe the existence of modes which become gapless in the long wavelength limit:

\[ M e^{i q x} \frac{\partial}{\partial z} \Phi_c(z) = q^2 e^{i q x} \frac{\partial}{\partial z} \Phi_c(z) \quad (2.19) \]

where \( x = x_1, ..., x_{d-1} \) are the \( d-1 \) transverse coordinates and \( q = q_1, ..., q_{d-1} \) is the corresponding \((d-1)\)-dimensional wave vector. These modes may be regarded as excitations of the zero mode, and they represent transverse fluctuations, away from planar, with wavevector \( q \). Their superposition, provided the amplitudes are small enough, describes a configuration

\[ \Phi(x, z) = \Phi_c(z) + \sum_q a_q e^{i q x} \frac{\partial}{\partial z} \Phi_c(z) \]

\[ \approx \Phi_c(z + f(x)) \quad (2.20a) \]

where

\[ f(x) = \sum_q a_q e^{i q x} \quad (2.21b) \]

Eqns (2.14) and (2.15) show that the configuration (2.20) corresponds to an interface which deviates from a reference plane by an amount \( f(x) \) at point \( x \), and the modes \( a_q \) represent the capillary waves on the interface. As a consequence, we recognise that the interface Hamiltonian (2.9) corresponds to the effective energy for the long wavelength fluctuations of the Goldstone field \( f(x) \). In principle, an effective Hamiltonian for the fluctuations involving \( f(x) \) can be obtained from the LGW model (2.11) by
averaging over all other fluctuations. This partial trace can be performed in the limit of long wavelengths and a deep well potential (corresponding to $\mu \to \infty$ while $\mu / g = \text{const}$ in eqn (2.12), leading indeed to the Hamiltonian (2.9) (Wallace 1980 and references therein, Lin and Lowe 1983).

Let us add a remark concerning the symmetries of the interface Hamiltonian (2.9). It is an established fact that effective Hamiltonians for Goldstone fields are invariant under the full original symmetry group; and that those transformations which correspond to spontaneously broken symmetries act non-linearly on the fields (see e.g. Peskin 1982 for a discussion). In our case, the full original symmetry group is $E(d)$, the Euclidean group of translations and rotations. The symmetries spontaneously broken by the interface are translation invariance in the $z$-direction and rotation invariance in the $z,x_1$-planes, corresponding to the transformations on $f$

$$f \to f + \alpha$$  \hspace{1cm} (2.21a)

for translation by $\alpha$ in the $z$-direction, and

$$f \to f - \theta (f \frac{\partial}{\partial x_1} f + x_1) + O(\theta^2)$$  \hspace{1cm} (2.21b)

for rotation by an infinitesimal angle $\theta$ in the $z,x_1$-plane. The invariance of the Hamiltonian (2.9) under those transformations may be explicitly checked.
A similar programme may be followed in order to obtain an effective Hamiltonian for fluctuations about a spherical interface (Günther et al. 1980). The LGW Hamiltonian eqn (2.11) now contains an external field term $-H_0 \delta^d x^{\phi}$ which ensures that $\Phi_+ (\Phi_-)$ is the stable phase for $H > 0$ ($H < 0$). Again, we are looking for a classical field configuration $\Phi_c$ for which the Hamiltonian eqn (2.11) becomes extremal; $\Phi_c$ thus corresponds to the critical droplet in the metastable phase. Hence we seek a spherically symmetric solution to the field equation

$$\frac{\delta H}{\delta \phi} |_{\phi_c} = 0 \quad (2.22a)$$

with boundary conditions

$$\Phi_c(0) = \Phi_+ \quad \Phi_c'(0) = 0 \quad \lim_{R \to \infty} \Phi_c(R) = \Phi_- \quad \lim_{R \to \infty} \Phi_c'(0) = 0 \quad (2.22b)$$

With the standard decomposition of $\nabla^2$,

$$\nabla^2 = \frac{d^2}{dR^2} + \frac{d-1}{R} \frac{d}{dR} + \frac{1}{2} \frac{L^2}{R^2}$$

where $L^2 = L_{ij} L_{ij}$ and $L_{ij}$ is defined in eqn (2.10b), the field equation reads

$$\frac{d^2 \phi_c}{dR^2} + \frac{d-1}{R} \frac{d \phi_c}{dR} + \frac{\partial}{\partial \phi} [V(\phi) - H\phi] |_{\phi_c} = 0 \quad (2.22c)$$

This may be interpreted as the equation of motion of a
particle with 'coordinate' $\phi_c$ in 'time' $R$, under the influence of the potential $V-H\phi$ and a damping term $(d-1)/R \, d\phi_c/dR$. The required solution starts from $\phi_+$ at $R = 0$, and dissipates just enough energy through friction that it reaches $\phi_-$ for $R \to \infty$. The mechanical analogue makes it plausible that such a solution always exists. However, even for the $\phi^4$-potential, $\phi_c$ is not an elementary function. But for small $H$, the radius of the critical droplet is much larger than the intrinsic width of the interface, and the profile of the spherical interface can be reasonably well approximated by the profile of the planar interface, eqn (2.14). This leads to an approximate expression for $\phi_c(R)$:

$$\phi_c(R) = \frac{1}{2} (\phi_- + \phi_+) + \frac{1}{2} (\phi_- - \phi_+) \tanh \left[ (\frac{\alpha}{2})^{(d-1)/2} (R-R_c) \right] \quad (2.23)$$

where $R_c$ is a parameter which locates the position of the interface: thus it plays the role of the critical radius, and is determined from the extremum of $H[\phi]$ as before. To study the effects of fluctuations, we set

$$\phi = \phi_c + \hat{\phi} \quad (2.24)$$

and expand $H[\phi]$ about $\phi_c$. From this, we get a classical contribution

$$H[\phi_c] = \text{Const} \, H^{-(d-1)/2} \left[ 1 + O(H^2) \right] \quad (2.25)$$

Since the linear term vanishes, the first non-trivial
contribution is quadratic in $\hat{a}$ and is governed by the differential operator

$$M = -\frac{d^2}{dR^2} - \frac{(d-1)}{R} \frac{d}{dR} - \frac{i}{2} \frac{\hbar^2}{R^2} + \frac{\partial^2 V}{\partial \phi^2} |\phi_c|$$  \hspace{1cm} (2.26)

whose eigenvalues and eigenvectors we need to analyse. The operator $L^2$ is the total angular momentum operator in $d$ dimensions whose eigenfunctions, the generalised spherical harmonics $Y^\alpha_{\lambda \nu}$ are described in Appendix A.

Then, first of all, since the droplet centre can be chosen arbitrarily, we have

$$M \frac{\partial}{\partial x_a} \phi_c = 0 $$ \hspace{1cm} (2.27)

corresponding to an eigenfunction $-\phi_c'(R) x_a / R$ with zero eigenvalue. Thus we recover the Goldstone mode of the translation invariance which is spontaneously broken by the droplet centre. It is $d$-fold degenerate, and, with

$$Y^\alpha_{\lambda \nu}(\eta) = \sqrt{d/S_d} \eta_\alpha$$

where $\eta$ is the radial unit vector, and $S_d$ denotes the volume of the $d$-dimensional unit sphere (cf. Appendix A), we can write it in the form

$$\frac{x_a}{R} \phi_c'(R) = \sqrt{S_d/d} \phi_c'(R) Y^\alpha_{\lambda \nu}(\eta)$$ \hspace{1cm} (2.28)
from which we see that it corresponds to the $l = 1$ mode of $M$. Its excitations $Y_1(\eta) \cdot \phi_c(R)$ are (approximate) eigenfunctions of $M$ with eigenvalues

$$E_\ell = \frac{(\ell-1)(\ell+d-1)}{R_c^2} \left[ 1 + O(\ell^2/R_c^2) \right]$$

(2.29)

In the limit of interest, $R_c \to \infty$, these modes become soft and describe deviations from the critical droplet $\phi_c$:

$$\Phi = \Phi_c(R) + \sum_{2,\alpha} a_{2,\alpha} Y_2^\alpha(\gamma) \phi_c'(R)$$

$$\approx \Phi_c(R + f(\gamma))$$

(2.30a)

where

$$f(\gamma) = \sum_{2,\alpha} a_{2,\alpha} Y_2^\alpha(\gamma)$$

(2.30b)

The $l = 0$ mode has negative energy, which gives rise to an imaginary part of the free energy in the context of a steepest descent calculation. Since $Y_0^0(\eta) = S^{-1/2}_c$ is simply a constant, the sum $R + f(\eta)$ in the argument of $\phi_c'$ eqn (2.30a), reflects the 'breathing' of the droplet, and the negative sign of $E_0$ expresses the tendency of the critical droplet to grow and eventually to nucleate. The phenomenological droplet model is thus recognised as a description which includes only the fluctuations of the zero mode, expressed by the integration over $R$ in the configurational sum eqn (2.4), and neglects all higher mode contributions.
As in the case of the planar interface, we wish to derive an effective Hamiltonian for the Goldstone field $f(\eta)$, and we know that this Hamiltonian has to be invariant under the full original symmetry group $\text{E}(d)$, including the spontaneously broken symmetries. These act non-linearly on the field:

$$f(\eta) \rightarrow f(\eta) + a\eta + \frac{a_i \eta_i \ln f(\eta)}{R + f(\eta)} + O(a^2)$$  \hspace{1cm} (2.31a)

describes an infinitesimal translation of the interface by $a$ and

$$f(\eta) \rightarrow f(\eta')$$  \hspace{1cm} (2.31b)

rotates the field from $\eta$ to $\eta'$. It can be shown explicitly, by checking its invariance under these transformations, that the interface Hamiltonian eqn (2.10) is indeed the desired effective Hamiltonian.

A few remarks are in order: We have seen that both interface Hamiltonians, eqns (2.9) and (2.10) are effective Hamiltonians for the fluctuations of the Goldstone field associated with broken translation invariance. In both cases, we have assumed that the interface can be described by a single-valued field, thus excluding overhangs and highly ramified droplets. We shall see later (Section 3.4) that this can be vindicated. Moreover, in the critical limit we are interested in short wavelength fluctuations, and intuitively
we do not expect the curvature of the surface to influence critical behaviour. Lin and Lowe (1983) confirm this statement in a field theory framework by showing that curvature terms (of the form $\nabla^2 f$, $(\nabla f)^2$, ...) are irrelevant interactions for the Hamiltonian (2.9). This is another version of the universality phenomenon: since both models have been constructed to describe interfaces in the Ising universality class, their (universal) critical behaviour must necessarily be identical. We shall find later on that this expectation is borne out by more detailed calculations.

Let us conclude this section with a remark about the renormalisability of the two interface Hamiltonians: By definition, the field has the dimension of length, and the Hamiltonian has the dimension of $(\text{length})^{d-1}$. It follows that $T_0$ too has dimension of $(\text{length})^{d-1}$. An expansion in powers of $T_0$ corresponds to an expansion in the number of loops (as we shall see more clearly later), and since each term in the expansion must have the same dimension, the extra loop integrals cancel the extra powers of $T_0$. Hence each loop integral has dimension $(\text{length})^{1-d}$ or $(\text{momentum})^{d-1}$, and will be convergent for $d < 1$ and divergent for $d > 1$, rendering the theory nonrenormalisable for $d > 1$, and demonstrating the need for some regularisation scheme. Conceptionally, it would be easiest to work on the lattice where the inverse lattice spacing provides a cutoff on the momenta; technically, however, it is easier to use dimensional regularisation ('t Hooft and Veltman 1972, 't Hooft 1973). We shall therefore
compute the loop integrals in $d = 1-\varepsilon$ dimensions, and then analytically continue to $d = 1+\varepsilon$, $\varepsilon > 0$. One would hardly expect this procedure to yield very reliable numerical results, in particular because a number of quantities depend very sensitively on $\varepsilon$. However, we shall see that the theory does contain valuable conceptual and quantitative information concerning the configurational structure of our system, which transcends the technical restrictions and may be trusted in two dimensions.
In this chapter, we are going to calculate the single droplet partition function to two loops. Up to and including one loop, these results were first obtained by Bruce and Wallace (1981, 1983). Their one loop calculation amounts to the evaluation of a Gaussian integral and therefore does not involve a discussion of the Feynman rules for our theory. However, this is clearly necessary if one wants to study the structure of higher order graphs. We shall in fact obtain a good picture of how divergences arise in this theory.

A further motivation for a two loop calculation resides in the fact that the one loop results are unsatisfactory: It turns out (as we shall see) that the one loop $\beta$-function is not sufficient to prescribe the correct amplitude for the one loop partition function, and the two loop $\beta$-function is needed to amend this shortfall. In the absence of an explicit two loop calculation of the droplet $\beta$-function, Bruce and Wallace invoked the universality principle according to which the universal parts (i.e. those related to universal quantities like the exponent $v$) of the $\beta$-functions of spherical and planar interface should be identical. So they substituted the (known) two loop planar $\beta$-function (Wallace and Zia 1979, Forster and Gabriunas 1981) for the (unknown) droplet quantity. In the explicit two loop calculation which we shall
present in this chapter, the identity of the two $\beta$-functions is confirmed and the universality principle vindicated.

Technically, the calculation will be interesting for a number of reasons. The Hamiltonian for a single, almost spherical droplet involves angular momentum operators acting on the field which describes the deviation from spherical. It will therefore be convenient to expand the field in the eigenfunctions of the differential operator. Associated with the shape fluctuations of the droplet are ultraviolet divergences which, as we have indicated in Chapter 2, can be controlled in $1+\varepsilon$ dimensions. An elegant way to do so is the dimensional regularisation scheme proposed by 't Hooft and Veltman ('t Hooft and Veltman 1972, 't Hooft 1973), conventionally followed by renormalisation through minimal subtraction. Thus emerges the need for generalised spherical harmonic functions, i.e. functions which are solutions to Laplace's equation in general dimension whose properties are discussed in Appendix A.

Techniques similar to the ones described here were used by various authors. We mention only a selection: Drummond (1975) discusses dimensional regularisation on the sphere for $\phi^3$ and $\phi^4$-theories. McKane and Wallace (1978) and Drummond and Shore (1979) investigate $\phi^4$-instantons in dimensional regularisation and are also led to a spherical formulation. These papers also contain further references.
For the sake of simplicity, we shall set the external field $h$ equal to zero in the subsequent calculation. Later on we shall indicate how the partition function in non-zero field is obtained and quote the result (Sim 1984, Sim and Bruce 1984).
3.1 Collective coordinate method

Let us begin by recalling the Hamiltonian in zero field

\[ H = T_0^{-1} \int d\Omega (R + f)^{d-1} \left[ 1 + \frac{1}{2} \left( L_{ij} f^2 (R + f)^2 \right)^{1/2} \right] \]  \hspace{1cm} (3.1)

In order to obtain the single droplet partition function we have to evaluate the functional integral

\[ Z_1 = \int Df e^{-H} \]  \hspace{1cm} (3.2)

up to the required order in perturbation theory. Following the standard procedure (see e.g. Amit 1978), first of all we have to locate an extremum of \( H \) and subsequently expand the field and the Hamiltonian about their extremum values. However, an extremal droplet of non-zero radius does not exist; but, once the scale size is fixed, surface energy is minimised by a purely spherical droplet shape. Further, a rigid translation of the droplet does not cost any energy, and therefore we also need to fix the droplet centre if we want to obtain a properly defined extremal configuration about which to do perturbation theory. The method which deals with those problems is known as the 'method of collective coordinates'; and in the following we describe how to treat the droplet scale size \( R \) and the coordinate of the droplet centre, \( x_\alpha \), as constrained collective variables.
In field theory, it is often convenient to express the field which is originally defined in real space by its Fourier components. Here we are dealing with a theory involving the angular momentum operators $L_{ij}$ and $L^2$, and therefore it is most appropriate to expand the field in generalised spherical harmonic functions $Y_1^\ell$ (see Appendix A) which are eigenfunctions of $L^2$:

$$f(\eta) = \sum_{\ell, \alpha} a_{\ell, \alpha} Y_1^\ell(\eta) \quad (3.3)$$

By a careful analysis of the change in measure it can be shown that under the transformation eqn (3.3) the functional integral becomes

$$Z_i = \prod_{\ell, \alpha} \int d a_{\ell, \alpha} e^{-\mathcal{H}} \quad (3.4)$$

With

$$Y_0^0(\eta) = S_d^{-1/2} \quad (3.5a)$$

we observe that the $l = 0$ mode corresponds to the breathing of the droplet, i.e. to changes in overall scale size. And since

$$Y_1^\ell(\eta) = \sqrt{d/S_d} \eta_\alpha \quad (3.5b)$$

the $l = 1$ mode corresponds to rigid translations of the droplet, i.e. to shifts of the droplet centre. We have seen
above that, in order to produce an expression for the partition function which can be handled by perturbation theory, we need to fix the droplet scale size and its centre. This is equivalent to the exclusion of the $l = 0$ and $l = 1$ modes from the expansion of $f$ in spherical harmonics, eqn (3.3).

Consider then

$$R + f(\eta) = R + S_d^{-1/2} a_{00} + \sum_{l \geq 1} a_{l\alpha} Y_l^\alpha(\eta) \tag{3.6}$$

The zero mode can be absorbed into the scale size $R$, replacing the integration over $a_{00}$ by an integration over $R$:

$$da_{00} = S_d^{1/2} dR$$

Hence our functional integral becomes

$$Z_1 = S_d^{1/2} \int dR \prod_{l \geq 1} \int da_{l\alpha} e^{-\mathcal{H}} \tag{3.7}$$

and from now onwards, the $l = 0$ mode is excluded from the expansion eqn (3.3) of $f$ so that, due to the orthonormality of the spherical harmonics, we obtain the relation:

$$\int d\Omega f(\eta) = 0 \tag{3.8}$$

This equation uniquely defines the scale size $R$ of any given (non-spherical) droplet as the zero mode contribution to the
total distance $R + f(\eta)$ between the droplet centre and a point on the surface.

The integration over the $l = 1$ mode, or, more precisely, over the $d$ coefficients $a_{1\alpha}$ can be replaced by an integration over the $d$ components $x_\alpha$ of the droplet centre coordinate. This can be seen as follows. Consider the identity

$$1 = \int d^d a \, \delta \left[ \frac{1}{L} \int d\Omega \, f_a(\eta) \, \eta \right] \cdot |\det \frac{\partial}{\partial \eta_\alpha} \int f_a(\eta) \, \eta_i \, d\Omega|$$

where

$$f_a(\eta) = f(\eta) + a\eta + \frac{a_i \eta_i L_{ij} f(\eta)}{R + f(\eta)} + O(a^2)$$

describes a droplet translated by $a$, eqn (2.31a). The integral in the argument of the $\delta$-function gives a "centre of mass" of the droplet surface. The $\delta$-function then fixes this "centre of mass" at the origin. Let us evaluate the second factor on the right hand side of eqn (3.9a) using the techniques of symmetric integration (Appendix B):

$$\frac{d}{da_\alpha} \int d\Omega f_a(\eta) \, \eta_i = \frac{S_d}{d} \left[ d_{i\alpha} + \frac{d}{S_d} \int d\Omega \, \eta_i \, \eta_i \frac{L_{ij} f}{R + f} \right]$$

and hence

$$|\det \frac{\partial}{\partial \eta_\alpha} \int d\Omega f_a(\eta) \, \eta_i | = \left( \frac{S_d}{d} \right)^d \exp \ln \left[ d_{i\alpha} + \frac{d}{S_d} \int d\Omega \, \eta_i \, \eta_i \frac{L_{ij} f}{R + f} \right]$$

where we have exploited elementary properties of
determinants as well as the identity, for a given matrix $A$,

$$\ln |\det A| = \text{tr} \ln A$$

Then, expanding the logarithm and taking the trace yields:

$$|\det \frac{\partial}{\partial a_k} \int d\Omega f_\alpha(\eta) \eta_i| =$$

$$= (\frac{S_d}{d})^d \exp \left[ \frac{d}{S_d} \int d\Omega \eta_i \eta_j \frac{L_i f_i}{R + f} - \frac{1}{2} (\frac{d}{S_d})^2 \int d\Omega \int d\Omega' \eta_i \eta_j \frac{L_i f_i}{R + f} \eta_i' \eta_j' \frac{L_i f_i'}{R + f'} + O(f^3) \right]$$

Since $\eta_i \eta_j$ is symmetric under exchange of $i$ and $j$ whereas $L_{ij}$ is antisymmetric, the first term on the right hand side of the last equation is zero. In the second term, we integrate by parts with respect to $d\Omega$ and $d\Omega'$, and with eqn (3.8) we finally obtain

$$|\det \frac{\partial}{\partial a_k} \int d\Omega f_\alpha(\eta) \eta_i| =$$

$$= (\frac{S_d}{d})^d \exp \left[ -\frac{1}{2} (\frac{d}{R S_d})^2 \int d\Omega \eta_i \eta_e f(\eta) \int d\Omega' \eta_i' \eta_e' f(\eta') + O(f^3) \right]$$

(3.10)

In the following it will become clear why it suffices to calculate this determinant up to $O(f^3)$. Now, our original identity eqn (3.9) reads

$$1 = \int d^d a \, \delta \left[ \int d\Omega f_\alpha(\eta) \eta \right] \cdot (\frac{S_d}{d})^d \cdot \exp \left\{ -\frac{1}{2} (\frac{d}{R S_d})^2 \int d\Omega \eta_i \eta_e f(\eta) \int d\Omega' \eta_i' \eta_e' f(\eta') \right\}$$

(3.11)

This is inserted into eqn (3.7):

$$Z_1 = S_d^{1/2} \int dR \prod_{\omega} \int d\xi_{\alpha} \cdot 1 \cdot e^{-\mathcal{H}}$$

(3.12)
Next we exchange the order of integration of \( d\eta \) and \( d\alpha \) and also change variables in the functional integral from \( f(\eta) \) to \( f_{-\alpha}(\eta) \). Under this transformation, both the measure \( Qf \) and the Hamiltonian \( H \) are invariant. Further, since 

\[
 f_{-\alpha}(\eta) \mapsto f_{-\alpha}(\eta) = f(\eta)
\]

the derivative \( \partial / \partial \alpha_k [ f_{-\alpha}(\eta) ] \) is calculated at \( \alpha = 0 \), hence the infinitesimal form of the translation as indicated in eqn (3.9b) is sufficient. Now let us look at the first factor on the right hand side of eqn (3.11). With eqn (3.5b) we recognise that only the \( l = 1 \) mode of \( f \) in the expansion eqn (3.3) contributes, and with symmetric integration we find

\[
\begin{align*}
\int d^d\alpha \delta \left[ \int d\Omega f(\eta) \eta_i \right] &= \\
&= \int d^d\alpha \delta \left[ \left( \frac{S_{d}}{d^2} \right)^{1/2} a_{1\alpha} \delta_{i\alpha} \right]
\end{align*}
\] (3.13)

The effect of the \( \delta \)-function is to set \( a_{1\alpha} \) equal to zero; and together with the integration over \( a_{1\alpha} \) contained in the functional integral, this yields

\[
Z_1 = S_{d}^{1/2} \left( \frac{S_{d}}{d^2} \right)^{d/2} \int d\alpha \int d^d\alpha \Pi_{\alpha,\beta} \int d\alpha_{\alpha} e^{-H - \Delta}
\] (3.14a)

where

\[
\begin{align*}
\Delta = \frac{1}{2} \left( \frac{d^2}{d\Omega} \right)^2 \int d\Omega \eta_i \eta_e f(\eta) \cdot \\
\int d\Omega' \eta_i' \eta_e' f(\eta')
\end{align*}
\] (3.14b)

and

\[
f(\eta) = \sum_{\alpha,\beta} a_{\alpha} Y_{\beta}^\alpha \eta(\eta)
\] (3.14c).
It is now trivial to perform the integral over all possible droplet centres $a_1$, since the integrand is independent of the integration variable. Hence this simply produces an overall factor $V$ where $V$ denotes the volume which is available to the droplet centre. In a single droplet picture, $V$ is just the total background volume; however, when we begin to discuss multi-droplet ensembles, the available volume $V$ will take account of the space which is filled with other droplets and therefore not available to new droplets. Thus,

$$Z_1 = V S_d v^2 (S_d / d)^{d/2} \int dR \prod_{\ell \neq 1} \int da_\ell \ e^{-\mathcal{H} - \Delta}$$

(3.15)

and we have obtained an expression which is directly accessible to standard perturbation theory.
3.2 Perturbative calculation

We wish to obtain an expression for $Z_1$ in the form of a double expansion. The first expansion parameter is the bare temperature $T_0$ which plays the role of a coupling constant; the second expansion parameter, $\epsilon = d-1$, is caused by the need to control the ultraviolet divergences of the theory with the help of dimensional regularisation. Both expansions are in fact coupled through the renormalisation group eqn (3.63) at which point it will become clear that an expansion to order $(T_0)^n$, to be consistent, must be accompanied by an expansion to order $\epsilon^n$.

Let us first of all analyse the orders of different terms in an expansion of the Hamiltonian (3.1) for small $\epsilon R$:

$$H = \frac{S_4 R^4}{T_0} + \frac{1}{2} \frac{R_{u-2}}{T_0} \int d\Omega [\varepsilon (\varepsilon - 1) \frac{1}{2} (L_{ij} f')^2] + \tilde{H}$$

$$= S_4 R^4 / T_0 + H_o + \tilde{H}$$

(3.16)

where $\tilde{H}$ contains all terms of higher than quadratic order. The first term in this expansion is the classical contribution where all fluctuations have been neglected, corresponding to the simple droplet model presented in Chapter 2. There is no linear term because the $l = 0$ mode has been excluded from $f(\eta)$ which led to eqn (3.8). Hence the quadratic term $H_o$ is the first correction to the classical contribution. After partial integration on $L_{ij}$, it takes the form
\[ \mathcal{H}_o = \frac{1}{2} T_o^{-1} R^{-2} \int dΩ \Omega f [\varepsilon (\varepsilon - 1) - \frac{1}{2} L^2] f \]

\[ = \frac{1}{2} T_o^{-1} R^{-2} \sum_{\ell \geq 2, \alpha} [\varepsilon (\varepsilon - 1) + \ell (\ell + \varepsilon - 1)] \alpha^2 \]

if we transform to angular momentum space and exploit the properties of the spherical harmonics. From this, we immediately read off the free propagator in angular momentum space

\[ G(\ell) = T_o R^{2-\varepsilon} \left[ \varepsilon (\varepsilon - 1) + \ell (\ell + \varepsilon - 1) \right]^{-1} \]

Here we observe again the importance of excluding the \( \ell = 0 \) and \( \ell = 1 \) modes, since, if \( d \to 1 \), \( G(\ell) \) would diverge for \( \ell = 0 \) or \( \ell = 1 \) and thus render a perturbation expansion impossible.

With the propagator eqn (3.18) and the Hamiltonian eqn (3.1) an expansion in powers of \( T_o \) is an expansion in the number of loops. If we neglect all interaction terms \( \tilde{\mathcal{H}} \), the remaining Hamiltonian is purely quadratic and produces the one loop corrections which are of \( O(1) \) in \( T_o \). In the corresponding (one loop) \( \varepsilon \)-expansion we keep terms of \( O(1) \) in \( \varepsilon \) to be consistent. The determinant \( \Delta \) does not contribute at the one loop level, since although it is quadratic it carries one power of \( T_o \) more than \( \mathcal{H}_o \) and hence only generates terms which are at least \( O(T_o) \). Thus, in order to obtain the full two loop form of the partition function, we need to extend the previous calculation to \( O(T_o) \) and \( O(\varepsilon) \), neglecting the (three loop) contributions \( O(T_o^2, \varepsilon T_o, \varepsilon^2) \).
Let us begin by investigating the interaction terms in $\bar{\mathcal{H}}$. They possess the generic form $\mathcal{F}^{m_1}[(L_{ij}f)^{2}]^{n}$, and thus in perturbation theory we have to evaluate expressions like

$$< \mathcal{F}^{m_1}[(L_{ij}f)^{2}]^{n} > \bigg|_{\mathcal{H}_0} = \int \mathcal{D}f \ e^{-\mathcal{H}_0} \mathcal{F}^{m_1}[(L_{ij}f)^{2}]^{n}$$

(3.19)

First of all, due to the symmetry properties of the functional integral under the transformation $f \rightarrow -f$, only terms with $m + 2n$ even contribute. Secondly, for the calculation of the partition function, we need not consider external legs, and eqn (3.19) corresponds to a vertex with $m + 2n$ internal legs. Thirdly, the standard Wick expansion holds for non-derivative as well as for derivative interactions, thus the average in eqn (3.19) may be represented in terms of three different types of contraction with their corresponding real space propagators:

$$< f(\eta)f(\eta') > = \sum_{\mathcal{L}_{2,\mathbb{R}}} G(\mathcal{E}) Y_{\mathcal{E}}(\eta) Y_{\mathcal{E}}(\eta')$$

$$< f(\eta)L_{ij}f(\eta') > = \sum_{\mathcal{L}_{2,\mathbb{R}}} G(\mathcal{E}) Y_{\mathcal{E}}(\eta) L_{ij} Y_{\mathcal{E}}(\eta')$$

(3.20)

$$< L_{ij}f(\eta)L_{pq}f(\eta') > = \sum_{\mathcal{L}_{2,\mathbb{R}}} G(\mathcal{E}) L_{ij} Y_{\mathcal{E}}(\eta) L_{pq} Y_{\mathcal{E}}(\eta')$$

In order to specify the $O(T_0)$ contributions, we need to identify all possible two loop graphs without external legs. It is easy to see that there are only three such graphs: the 'figure of eight', the 'London Transport' and the 'dumb-bell' graph (Fig. 5). Another contribution to this order in perturbation theory might come from the determinant $\Delta$, but
Fig. 5
Two loop graphs for the single droplet partition function
(a) Figure of eight
(b) London Transport
(c) Dumb-bell

(a)

(b)

(c)
repeating the argument which led to the neglection of $\Delta$ at the one loop level, we recognise that at the two loop level only the quadratic terms in $\Delta$ enter, and the $O(f^3)$ terms indicated in eqn (3.14b) may be neglected. Finally, we have to make sure that we capture the $O(\epsilon)$ corrections correctly.

In the following, we shall firstly obtain the contributions arising from the quadratic terms in the Hamiltonian and the determinant which amounts to the evaluation of a Gaussian integral. Then we shall discuss the contributions from the three two loop graphs, and we shall eventually end up with a full two loop expression for the single droplet partition function.
3.2.1 Gaussian contributions

In this section we want to calculate the Gaussian contributions to the partition function. Leaving aside the prefactors in eqn (3.15) this amounts to evaluating the expression

$$I := \prod_{\ell \neq l, \alpha} \int d\alpha_{\ell \alpha} e^{-\mathcal{H}_0 - \Delta}$$

(3.21)

where $\mathcal{H}_0$ is given in eqn (3.17) and the $O(f^3)$ terms in $\Delta$ are neglected:

$$\Delta = \frac{1}{2} \left( \frac{d^2}{R^2} \right)^2 \int d\Omega \; \eta_i \eta_j f(\eta) \cdot \int d\Omega' \eta_i' \eta_j' f(\eta')$$

The last expression can be simplified further. With the help of the information given on spherical harmonics in Appendix A, we can see that the product $\eta_i \eta_j$ which occurs in the expression for $\Delta$ can be written as the superposition of $l = 2$ spherical harmonics. Orthonormality then implies that only the $l = 2$ modes in $f$ contribute to $\Delta$. Using symmetric integration (Appendix B) and eqn (A.12) we obtain

$$\Delta = \frac{d^3}{S_d R^2 (d+2)} \sum_{\alpha} a_{2\alpha}^2$$

(3.22)

where the $a_{2\alpha}$ are the $l = 2$ expansion coefficients of $f(\eta)$, eqn (3.3). Now the integral in eqn (3.21) is easy to perform. We need the formula

$$\prod_i \int e^{-\frac{1}{2} \alpha_i x_i^2} dx_i = \prod_i \sqrt{2\pi / \alpha_i} = \exp - \frac{1}{2} \sum_i \ln \alpha_i / 2\pi$$

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whence we obtain

$$I = \exp -\frac{1}{2} \sum_{\ell \geq 2} \nu_{\ell}(d) \cdot \ln \left[ \frac{R^{\ell-2}}{2\pi T_0} \left( \epsilon (\epsilon - 1) + \ell (\ell + \epsilon - 1) \right) + \frac{d^3}{\pi S_d R^{\ell}(d+2)} \delta_{\ell,2} \right] \quad (3.23)$$

where the sum over \( \alpha \) has already been performed, resulting in the degeneracy factor \( \nu_1(d) \), eqn (A.8). So we are left with a sum over \( \ell \):

$$\Sigma = \sum_{\ell \geq 2} \nu_{\ell}(d) \ln \left[ \frac{R^{\ell-2}}{2\pi T_0} \left( \epsilon (\epsilon - 1) + \ell (\ell + \epsilon - 1) \right) + \frac{d^3}{\pi S_d R^{\ell}(d+2)} \delta_{\ell,2} \right] \quad (3.24)$$

which, as we discussed earlier on, must be evaluated up to and including terms of \( O(\epsilon) \). This will take us one step further than the first calculation by Bruce and Wallace. Rearranging eqn (3.24) yields

$$\Sigma = \sum_{\ell \geq 2} \nu_{\ell}(d) \ln \left[ \frac{R^{\ell-2}}{2\pi T_0} \left( \epsilon (\epsilon - 1) + \ell (\ell + \epsilon - 1) \right) \right] + \nu_2(d) \ln \left[ 1 + \frac{2d^3 T_0 R^{-\epsilon}}{S_d (d+2)(2 + \epsilon + \epsilon^2)} \right] \quad (3.25)$$

We find that the contributions from the determinant, i.e. the second term in eqn (3.25) are not only already \( O(T_0) \), but also pick up a factor \( \epsilon \) from

$$\nu_2(d) = \epsilon \Gamma(3 + \epsilon)/2$$

and may thus be neglected altogether. So we are left with
\[ \sum_{\ell \geq 2} = \ln \frac{R^{\varepsilon-1}}{2\pi T_0} \sum_{\ell \geq 2} \nu_2 (d) + \]
\[ + \sum_{\ell \geq 2} \nu_2 (d) \ln [ \ell (\ell + \varepsilon - 1) + \varepsilon (\varepsilon - 1) ] \]  
(3.26)

For the sum over degeneracies in the first term on the right hand side we find, with expression (A.8) for \( \nu_1 (d) \):

\[ \sum_{\ell \geq 2} \nu_2 (d) = -(d + 1) \]  
(3.27)

The second term is more complicated. Again inserting \( \nu_1 (d) \) we obtain

\[ \sum_{\ell \geq 2} \nu_2 (d) \ln [ \ell (\ell + \varepsilon - 1) + \varepsilon (\varepsilon - 1) ] = \]
\[ = \frac{\varepsilon}{\Gamma (d)} \sum_{\ell \geq 2} \frac{(2\ell - 1) \Gamma (\ell + \varepsilon - 1)}{(\ell + 1) \ln \ell (\ell - 1) + O(\varepsilon^2)} \]  
(3.28)

It is easy, yet somewhat tedious, to show that the neglected terms are in fact \( O(\varepsilon^2) \). In order to evaluate the remaining sum, we need a number of identities and approximations. First of all, we require an approximate expression for ratios of \( \Gamma \)-functions:

\[ \frac{\Gamma (\ell + \varepsilon)}{\Gamma (\ell)} = \ell^\varepsilon \left[ 1 - \frac{\varepsilon}{2 \ell} \left( 1 + \frac{1}{\ell^2} \right) + O(\varepsilon^3, \varepsilon \ell^{-3}) \right] \]  
(3.29)

Secondly, we note that the Riemann \( \zeta \)-function is defined by

\[ \zeta (z) = \sum_{n=1}^\infty n^{-z}, \quad z > 1 \]  
(3.30a)

and hence its derivative allows us to handle the logarithms occurring in eqn (3.28):
\[ \mathcal{S}'(z) = \frac{d}{dz} \mathcal{S}(z) = - \sum_{n=1}^{\infty} n^{-z} \ln n \]  \hspace{1cm} (3.30b)

Furthermore, near \( z = 1 \), we have (Erdélyi 1953):

\[ \mathcal{S}(z) = \frac{1}{z-1} + \gamma + \gamma_1 (z-1) + O((z-1)^2) \]  \hspace{1cm} (3.30c)

where \( \gamma = 0.577... \) is Euler's constant, and

\[ \gamma_1 = - \lim_{m \to \infty} \sum_{\ell=1}^{m} \left( \frac{\ln \ell}{\ell} - \frac{(\ln m)^2}{2} \right) \]
\[ = 0.0728166... \]

Thus

\[ \mathcal{S}'(1-\epsilon) = - \frac{1}{\epsilon^2} + \gamma_1 + O(\epsilon) \]  \hspace{1cm} (3.30d)

The evaluation of the sum in eqn (3.28) is simplified by the fact that \( v_1(d) \) already contains a factor \( \epsilon \), and using expressions (3.29) and (3.30) we finally obtain

\[ \sum_{i} = -(d+1) \ln \frac{b^{\epsilon-2}}{2\pi T_o} + \frac{4\epsilon}{\Gamma(d)} \left( \frac{1}{\epsilon^2} - \gamma_1 \right) - \frac{2\epsilon}{\Gamma(d)} C + O(\epsilon^2, \epsilon T_o) \]  \hspace{1cm} (3.31)

where

\[ C = \sum_{\ell \geq 2} \frac{\ln \ell}{\ell(\ell^2-1)} = 0.2346... \]

Returning to eqn (3.15) we find as the resulting expression for the partition function at this stage
\[
Z_1 = \sqrt[d]{S_d^{(d+1)/2}} \int d^d R \left( \frac{Re^{-2}}{2\pi \Delta_o} \right)^{(d+1)/2} \cdot \exp \left\{ - \frac{S_d}{\Delta_o} - \frac{2}{\epsilon \Gamma(d)} + \frac{\epsilon}{\Gamma^{\prime}(d)} (C + 2\gamma_1) \right\} [1 + O(T_o, \epsilon T_o \epsilon^2)]
\] (3.32)

Before commenting on this result, let us consider the additional contributions from the two loop graphs.
3.2.2. Quartic interactions: 'Figure of eight' graph

From the straightforward expansion of $\mathcal{H}$ for small $f$ indicated in eqn (3.16) we obtain three different quartic interactions:

$$
\tilde{\mathcal{H}}^{(4)} = -\frac{g^4}{\tau_0} \int d\Omega \left[ \frac{1}{32} (L_{ij}f)^2 (L_{pq}f)^2 - \left( \frac{3}{4} - \frac{\xi}{2} + \xi (\xi - 1) \right) f^2 (L_{ij}f)^2 - \left( \frac{\xi}{4} \right) f^4 \right] \tag{3.33}
$$

The 'figure of eight' represents the average of $\tilde{\mathcal{H}}^{(4)}$ with respect to $\mathcal{H}_0$:

$$
\langle \tilde{\mathcal{H}}^{(4)} \rangle = \int d\mathcal{H} \tilde{\mathcal{H}}^{(4)} e^{-\mathcal{H}_0} / \int d\mathcal{H} e^{-\mathcal{H}_0} \tag{3.34}
$$

It will not be necessary to treat all contributions from eqn (3.33) equally carefully, so let us only discuss the first term on the right hand side in more detail. From the Wick expansion we find

$$
\langle (L_{ij}f)^2 (L_{pq}f)^2 \rangle = \langle (L_{ij}f)^2 \rangle^2 + 2 \langle L_{ij}f L_{pq}f \rangle^2 \tag{3.35}
$$

which already generates the correct combinatoric factors. Invoking eqn (3.20) which translates the pair contractions into the corresponding propagators, the first contribution on the right hand side, including the integration over the solid angle, becomes
\[
\int d\Omega \langle (L_{ij} f)^2 \rangle^2 = \int d\Omega \sum_{\ell \geq 2, \alpha} \sum_{k \geq 2, \beta} G(\ell) G(k) \cdot \begin{pmatrix}
L_{ij} Y^\alpha_\ell (\eta) L_{ij} Y^\beta_k (\eta) L_{ij} Y^\alpha_\ell (\eta) L_{ij} Y^\beta_k (\eta)
\end{pmatrix}
\]

\begin{align*}
&= 4 \int d\Omega \sum_{\ell \geq 2, \alpha} \sum_{k \geq 2, \beta} G(\ell) G(k) \cdot \begin{pmatrix}
Y^\alpha_\ell (\eta) L(\ell + 1) Y^\beta_k (\eta) Y^\alpha_\ell (\eta) L(\ell + 1) Y^\beta_k (\eta)
\end{pmatrix}.
\end{align*}

(3.36)

after integrating by parts and exploiting the eigenvalue equation (3.3). The sums over \( \alpha \) and \( \beta \) can be performed with the help of the addition theorem given in Appendix A.

Setting \( \eta = \eta' \) in eqn (A.13), inserting into eqn (3.36) and performing the now trivial integration over \( d\Omega \) yields

\[
\int d\Omega \langle (L_{ij} f)^2 \rangle^2 = 4(T_0 R^2 - \varepsilon)^2 S_\alpha^{-1} \cdot \left[ \sum_{\ell \geq 2} Y^\alpha_\ell (\Delta) \right]^2 + O(\varepsilon T_0^2)
\]

(3.37)

The sum over degeneracies has already been evaluated in eqn (3.27), so finally

\[
\int d\Omega \langle (L_{ij} f)^2 \rangle^2 = 4(T_0 R^2 - \varepsilon)^2 S_\alpha^{-1} (d+1)^2 + O(\varepsilon T_0^2)
\]

(3.38)

An attempt to calculate the second term in eqn (3.35) in the same way fails, because, after rewriting the sum with the help of eqn (3.20), we observe that identical angular momentum operators act on spherical harmonics of different degree. Due to this structure it is not possible to generate a form similar to eqn (3.36) by straightforward partial integration. However, one can make use of the symmetry properties of \( \langle L_{ij} f \cdot L_{pq} f \rangle \). Averaging over all field configurations naturally eliminates all dependence on \( f \), but the tensorial structure of the indices \( i,j \) and \( p,q \) is
preserved: the result has to be antisymmetric under the interchange of \(i\) and \(j\), or \(p\) and \(q\); and symmetric under the interchange of the pairs \((i,j)\) and \((p,q)\). Accordingly, it must be of the form

\[
\langle L_{ij}f L_{pq}f \rangle = A(\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}) + \\
+ B (\eta_i \eta_p \delta_{jq} - \eta_i \eta_q \delta_{jp} - \eta_j \eta_q \delta_{ip} + \eta_j \eta_p \delta_{iq})
\]  
(3.39)

In order to determine the coefficients \(A\) and \(B\), we apply eqn (3.39) to two particular contractions, namely:

\[
\langle (L_{ij}f)^2 \rangle = (d-1)(dA + 2B)
\]

and

\[
\eta_i \eta_k \langle L_{ij}f L_{kj}f \rangle = (d-1)(A + B)
\]

Moreover, exploiting the definition of \(L_{ij}\), one finds

\[
\eta_i \eta_k \langle L_{ij}f L_{kj}f \rangle = \frac{1}{2} \langle (L_{ij}f)^2 \rangle
\]

Solving for \(A\) and \(B\) yields:

\[
A = 0 \quad B = \frac{1}{2} \langle (L_{ij}f)^2 \rangle / (d-1)
\]

Thus

\[
\langle L_{ij}f L_{pq}f \rangle^2 = \langle (L_{ij}f)^2 \rangle^2 / (d-1)
\]  
(3.40)
This result is crucial: it exposes the type of structure which leads to divergences in this theory. The simple contraction \( \langle (L_{ij} f)^2 \rangle^2 \), eqn (3.38), is \( 0(1) \) in \( \varepsilon \), whereas the mixed contraction \( \langle L_{ij} f L_{pq} f \rangle^2 \), eqn (3.40), picks up a factor \( \varepsilon^{-1} \). Generally speaking, divergences in more complicated graphs arise from the tensorial structure of contractions analogous to eqn (3.40).

With eqns (3.38) and (3.40), eqn (3.35), including all prefactors from eqn (3.33), finally leads to

\[
\frac{R^\varepsilon^{-4}}{32 \tau_0} \int d\Omega \langle (L_{ij} f)^2 (L_{pq} f)^2 \rangle = \frac{\tau_0 R^{-\varepsilon}}{S_\lambda} \left( \frac{1}{\varepsilon} + \frac{3}{2} \right) + O(\varepsilon \tau_0) \tag{3.41}
\]

Before we go on to discuss the other two contributions from eqn (3.33), let us invoke the analogous situation in the theory of the planar interface. The Hamiltonian reads, in keeping with the notation of Chapter 2,

\[
\mathcal{H} = \tau_0^{-1} \int d^{d-1} x \left\{ \left[ 1 + (\nabla f)^2 \right] V^2 + \frac{1}{2} \nu^2 f^2 \right\} \tag{3.42}
\]

Note that we have added a term \( \int d^{d-1} x \ m^2 f^2 \) to eqn (2.11). This term represents the effect of a gravitational field, for example, which pins the interface to a reference plane. Thus, we control the wandering of the interface which would otherwise render perturbation theory unapplicable. Depending on the circumstances, it might occasionally be more convenient to achieve the same objective with the
introduction of an infrared cutoff (which physically corresponds to placing the system into a finite box). Both methods exclude the $q = 0$ mode and thus ensure that we perturb about a proper extremum of the Hamiltonian. They constitute the planar analogue to the exclusion of the $l = 1$ mode in the droplet framework. It is a question of convenience (if the universality principle holds) whether one should use one or the other. With the Hamiltonian eqn (3.42), the free propagator reads:

$$G(q^2) = \mathcal{T}_0 (q^2 + m^2)^{-1}$$

(3.43)

At the two loop level of this theory, we encounter the contraction $\langle \frac{\partial}{\partial x_i} f(x) \frac{\partial}{\partial x_j} f(x) \rangle^2$ which is the direct analogue of $\langle L_{ij} f L_{pq} f \rangle^2$. In order to evaluate the corresponding loop integral in Fourier space we need the analogue of eqn (3.39):

$$\langle \frac{\partial}{\partial x_i} f \frac{\partial}{\partial x_j} f \rangle = \langle (\frac{\partial}{\partial x_k} f)^2 \rangle \delta_{ij} / (d-1)$$

Hence the loop integral simplifies considerably:

$$\int d^{d-1}q \int d^{d-1}k G(q^2) G(k^2) q_i q_j k_i k_j = \frac{1}{(d-1)^{-1}} \left[ \int d^{d-1}q G(q^2) q^2 \right]^2$$

(3.44)

It is easy to show that $\int d^{d-1}q G(q^2) q^2$ is $0(1)$ in $\varepsilon = d-1$. Consequently, we recover exactly the same structure as in eqn (3.40), and thus find the first explicit confirmation of the universality principle being applied to the critical
behaviour of spherical and planar interfaces.

The other two contributions to the figure of eight can now be dealt with more quickly. Following the same strategy as before, the second term in eqn (3.33) can be written as

\[
\int d\Omega \left< f^2 (L_{ij} f)^2 \right> = -T_0 R^{2-\delta} \sum_{\ell \leq 2} \nu_\ell (d) \sum_{\kappa \leq 2} G(\kappa) \nu_\kappa (d) \tag{3.45}
\]

It does not present any difficulties to show that the second sum in eqn (3.45) is \( O(\epsilon) \), such that the whole expression is \( O(\epsilon T_0) \) and may be neglected at the two loop level. The third contribution in eqn (3.33) is \( O(\epsilon^2 T_0) \) because it is essentially determined by the square of the second sum in eqn (3.45) and may also be neglected.

Thus we have derived the total contribution from the figure of eight:

\[
\left< \tilde{\mathcal{H}}^{(4)} \right> = -T_0 R^{-\delta} S_d^{-1} \left( \frac{1}{\epsilon} + \frac{3}{2} \right) + O(\epsilon T_0) \tag{3.46}
\]

We shall now proceed to calculate the other two graphs.
3.2.3 Cubic interactions: 'Dumb-bell' and 'London Transport' graph

The expansion of the Hamiltonian for small \( f \) yields two types of cubic interaction:

\[
\mathcal{H}^{(3)} = -\frac{R e^{-2}}{T_0} \int d\Omega [ \frac{e^{-2}}{4} (L_{ij} f)^2 f + (\frac{e}{3}) f^3 ]
\]

\[
= -\frac{R e^{-2}}{T_0} \int d\Omega [ \frac{2-e}{8} f^2 l^2 f + (\frac{e}{3}) f^3 ]
\]  \hspace{1cm} (3.47)

Trivially, the average over an odd number of fields with respect to a quadratic Hamiltonian is zero. However, the square of \( \mathcal{H}^{(3)} \) does give a contribution which will be represented by the London Transport or the dumb-bell graph depending on how the internal legs are connected.

Let us investigate the dumb-bell graph first, with an \( f^3 \) interaction on both vertices. Recalling eqn (3.20), the corresponding sum reads, up to various prefactors:

\[
\sum' = \sum_{l \geq 2, \mu} \sum_{k \geq 2, \nu} \sum_{p \geq 2, \lambda} G(\ell) G(k) G(p) \cdot \int d\Omega \int d\Omega' Y^\omega_\ell(\eta) Y^\rho_\eta(\gamma) Y^\lambda_k(\gamma') Y^\rho_p(\gamma')
\]

where we have exploited eqn (A.13). Since we have excluded the \( 1 = 0 \) mode, and since

\[
\int d\Omega \ Y^\omega_\ell(0) = \delta_{\ell,0} \ \delta_{\mu,0}
\]
we find that

\[ \sum \equiv 0 \]  

(3.48)

A derivative interaction \( f^2 L^2 f \) on one or both of the two vertices does not change the reasoning behind eqn (3.48): the operator \( L^2 \) acting on a spherical harmonic simply generates its eigenvalue as an additional factor in the numerator. As this does not influence the integration over \( dQ \) and \( dQ' \), these contributions vanish as well. This result can be understood in a more intuitive way: Angular momentum conservation only allows for \( l = 0 \) on the line connecting the two vertices. However, the zero mode has been excluded from all sums, and hence the contribution from this graph vanishes altogether.

Finally, we consider the London Transport graph, with two \( f^3 \) interactions on the vertices. All three internal lines start at one vertex and end at the other; thus we get the sum

\[
\sum_i := \sum_{k \geq 2, \alpha} \sum_{k' \geq 2, \beta} \sum_{p \geq 2, \gamma} G(\ell) G(k) G(p) \cdot \\
\left( \int d\Omega \int d\Omega' Y^*_{e}(\eta) Y^*_{e'}(\eta') Y^B_k(\eta) Y^B_k(\eta') Y^B_p(\eta) Y^B_p(\eta') \right) \\
= S_{d-2}^2 \sum_{k \geq 2} \sum_{k' = 2} \sum_{p \geq 2} G(\ell) v_k(d) G(k) v_{k'}(d) G(p) v_p(d) \cdot \\
\left[ C^e_\alpha(\ell) C^e_k(\ell) C^e_p(\ell) \right]^{-1} \int dt (1 - t^2)^{\frac{3}{2} - \frac{1}{2}} C^e_\alpha(t) C^e_k(t) C^e_p(t)
\]

(3.49)

where the last line has been obtained with the help of eqn...
(A.13) and $\lambda = (d-2)/2$. The integral occurring in eqn (3.49) was evaluated by Hsü (1938):

$$
\int_{-1}^{1} dt \left( 1 - t^2 \right)^{s-\frac{1}{2}} C_{\ell}^s(t) C_{\lambda}^l(t) C_{\lambda}^p(t) =
$$

$$
= \frac{2^{1-2\lambda}}{\Gamma(\lambda)} \prod_{j=0}^{s} \frac{\Gamma(s+\lambda)}{(s+j+1)} \frac{\Gamma(s-\ell+\lambda)}{(s-\ell+1)} \frac{\Gamma(s-k+\lambda)}{(s-k+1)} \frac{\Gamma(s-p+\lambda)}{(s-p+1)} \Delta(\ell,k,p).
$$

where

$$
2s = \ell + k + p
$$

and

$$
\Delta(\ell,k,p) = \begin{cases} 
1 & \text{if } \ell+k+p \text{ is even and } \ell,k,p \text{ can form the sides of a triangle} \\
0 & \text{otherwise} 
\end{cases}
$$

In three dimensions, the triangle rule, eqn (3.50c), governs orbital angular momentum conservation in quantum mechanics: it describes how two orbital angular momenta $k$ and $l$ couple to give a third, $p$.

It appears impracticable to evaluate $\Sigma$ from eqn (3.49). However, this is not necessary: To extract the $\epsilon$-dependence, it is sufficient to consider eqn (3.49) in the limit of large $l$, $k$ and $p$. With the approximations

$$
G(\ell) \approx T_0 \ell^{2-\epsilon} \ell^{-2} \left[ 1 + O(\ell, \ell^{-1}) \right]
$$

$$
\nu_{e(d)}/S_d \approx S_{d-1} \ell^{\epsilon-1} \left[ 1 + O(\ell, \ell^{-1}) \right]
$$
for large values of 1, and with eqn (3.29) for ratios of \( \Gamma \)-functions, eqn (3.49) can be simplified to

\[
\sum = -\frac{1}{4\pi} \int_{d} \int_{d-1} \int_{d} \int_{d} \int_{d} \int_{d} \frac{\ell^2}{k^2} G(\ell) \frac{1}{k} G(k) \frac{1}{p} G(p),
\]

where corrections of \( O(\epsilon) \) and \( O(l^{-1}, p^{-1}, k^{-1}) \) have been neglected, and where the sums in eqn (3.49) have been replaced by integrals. Now we shall proceed as follows:

Firstly, we shall invoke the planar interface again and investigate the London Transport graph in (d-1)-dimensional Euclidean space. Introducing an infrared cutoff the real space propagator reads

\[
G(x, x') = \int_{d-1} q^{1-d} (2\pi)^{-d} G(q) e^{i q(x-x')},
\]

where \( G(q) = T_0 q^{-2} \). Secondly, we shall show that the London Transport graph in Euclidean formulation can be cast in the form (3.51). Physically, this is again a confirmation of the expectation that, for large angular momenta, the spherical interface behaves essentially like a planar interface. Therefore, finally, we obtain the \( \epsilon \)-dependence of eqn (3.51) by evaluating the Euclidean expression using standard techniques.

So let us begin by writing down the expression for the London Transport graph in Euclidean space. Now, \( \ell, k \) and \( p \) stand for continuous, (d-1)-dimensional momenta:
Instead of integrating over \( x \) first, we integrate over the angular parts of \( J_1, J_2, J_3 \). This leads to the Poisson integral (Erdylyi 1953):

\[
\int d\Omega_q e^{iqx} = S_{d-2} \int_{-1}^{1} dt \left(1-t^2\right)^{\lambda-1} e^{iqxt} = S_{d-2} \Gamma(\lambda) J_{\lambda-\frac{1}{2}} (qx) \sqrt{\pi} \left(\frac{1}{2} qx\right)^{\frac{1}{2}-\lambda}
\]

where \( J_{\lambda-\frac{1}{2}} (qx) \) is a Bessel function of order \( \lambda-\frac{1}{2} \). Now the resulting integral over \( x \) in eqn (3.53) can be performed. Its angular part is trivial; hence we obtain

\[
I = S_{d-1} \int d\Omega_\ell J_{\lambda-\frac{1}{2}} (q\Omega_\ell) J_{\lambda-\frac{1}{2}} (p\Omega_\ell) x^{\frac{1}{2}-\lambda} = S_{d-1} \frac{2^{\lambda-3/2}}{\sqrt{\pi} \Gamma(\lambda)} (\ell k p)^{1-2\lambda} A^{2\lambda-2} \Delta(\ell, k, p)
\]

where again

\[
\Delta(\ell, k, p) = \begin{cases} 1 & \text{if } \ell, k, p \text{ can form the sides of a triangle} \\ 0 & \text{otherwise} \end{cases}
\]

and \( A \) is the area of the triangle formed by \( \ell, k, p \) (Watson 1922). If the perimeter of the triangle is denoted by \( 2s \), one has the relation

\[
A = \left[ s(s-\ell)(s-\kappa)(s-\rho) \right]^{1/2}
\]

so finally
which agrees up to a prefactor of $O(1)$ with eqn (3.51). Hence, eqn (3.53) may be invoked to calculate the $\varepsilon$-dependence of eqn (3.51). Similarly, if we have an $f^2 L^2$ interaction on one or both of the two vertices, we note that for large $L$ the spherical expression

$$L^2 Y^\alpha_\ell(\eta) \approx -2 L^2 Y^\alpha_\ell(\eta)$$

has to be replaced by the Euclidean analogue

$$\nabla^2 e^{i q \mathbf{x}} = -q^2 e^{i q \mathbf{x}}$$

which corresponds to an interaction $f^2 \nabla^2 f$ in the Euclidean formulation.

Thus, in the third and final item of our programme, we calculate the $\varepsilon$-dependence of the London Transport graph in the Euclidean formulation, with interactions $f^3$ or $f^2 \nabla^2 f$. This is completely straightforward, since we only have to deal with standard integrals which are well known in dimensional regularisation (confer 't Hooft 1973). Recalling that the graph carries an overall factor $\mathbf{T}_0$, we find that it is at most $O(\varepsilon \mathbf{T}_0)$, and thus does not contribute to the two loop partition function.
This concludes the discussion of two loop graphs. We have found only one contribution, arising from the figure of eight which exponentiates according to the well known theorem for connected vacuum graphs. Thus, with eqns (3.32) and (3.41), the full two loop partition function reads:

\[
Z_1 = S_d^{(d+1)/2} \int dR \left( \frac{R^2}{2\pi T_0} \right)^{(d+1)/2} \cdot \exp \left\{ -\frac{S_d R^e}{T_0} - \frac{2}{\epsilon \Gamma(d)} + \frac{T_0 R^{-\epsilon}}{\epsilon S_d} + \frac{\epsilon}{\Gamma(d)} (C+2\gamma) + \frac{3}{2} \frac{T_0 R^{-\epsilon}}{S_d} \right\} \cdot \left[ 1 + O(\epsilon^2, \epsilon T_0, T_0^2) \right]
\]

Some remarks are in order. The first term in the exponential is the classical contribution arising from purely spherical droplets of radius \( R \). It is followed by the corrections which take account of shape fluctuations. Both one and two loop contributions possess an \( \epsilon^{-1} \) divergence, due to large angular momentum contributions to the sums which constitutes an unsatisfactory feature of the result. Another problem is posed by the fact that the effective expansion parameter in eqn (3.56), \( T_0 R^{-\epsilon} \), becomes unboundedly large for \( R \to 0 \). This implies that we still cannot handle small droplets although this was one of our main objectives when we set out on this calculation. In the next section we shall see that the renormalisation of the result (3.56) takes care of both problems.
3.3 Renormalisation

We adopt a minimal subtraction scheme: The renormalised coupling $T$ is defined as a power series in the bare dimensionless coupling $T_0 R^{-\varepsilon}$ such that the coefficients contain only pure poles (including multi-poles) in $\varepsilon$ and absorb all divergences. If these conditions are satisfied, the coefficients are uniquely defined. Accordingly we write

$$(T_0 R^{-\varepsilon})^{-1} = T^{-1} + a + bT + O(T^2) \quad (3.57a)$$

which is equivalent to

$$T_0 R^{-\varepsilon} = T - aT^2 + (a^2 - b)T^3 + O(T^4) \quad (3.57b)$$

and the coefficients $a$ and $b$ need to be determined. Since we are now only interested in the $\varepsilon$ and $T_0$ dependence of the partition function, we abbreviate eqn (3.56) to

$$Z_1 = \text{const} \exp \left\{ -\frac{d+1}{2} \ln T_0 R^{-\varepsilon} - \right.$$  
$$ - \frac{S_d}{T_0 R^{-\varepsilon}} - \frac{2}{\varepsilon \Gamma(d)} + \frac{T_0 R^{-\varepsilon}}{\varepsilon S_d} + \frac{3T_0 R^{-\varepsilon}}{2 S_d} \right\}$$

Inserting eqn (3.57) we obtain

$$Z_1 = \text{const} \exp \left\{ -\frac{d+1}{2} \ln T(1-aT) - \right.$$  
$$ - S_d (T^{-1} + a + bT) - \frac{2}{\varepsilon \Gamma(d)} + \frac{T}{\varepsilon S_d} + \frac{3T}{2 S_d} \right\}$$
In order to eliminate the $\epsilon$-divergences, we compare coefficients order by order in $T$. At $O(1)$ we require

$$a \tilde{S}_d + \frac{2}{\epsilon \Gamma(d)} = O(1)$$

(3.58a)

and at $O(T)$

$$\frac{d+1}{2} a - b S_d + \frac{1}{\epsilon} S_d = O(1)$$

(3.58b)

resulting in

$$a = -\frac{1}{\epsilon} \quad b = -\frac{1}{4 \epsilon}$$

(3.59)

In principle, eqns (3.58) yield corrections of $O(1)$ to $a$ and $b$, as we have indicated on their right hand sides, but in a minimal subtraction scheme they are discarded since we want $a$ and $b$ to consist only of pure poles in $\epsilon$. So finally, the bare coupling depends on the renormalised coupling according to

$$(T_0 R^{-\epsilon})^{-1} = T^{-1} - \frac{1}{\epsilon} - \frac{1}{4 \epsilon} T + O(T^2)$$

(3.60a)

or, equivalently,

$$T_0 R^{-\epsilon} = T + \frac{1}{\epsilon} T^2 + \left( \frac{1}{4 \epsilon} + \frac{1}{\epsilon^2} \right) T^3 + O(T^4)$$

(3.60b)

Inserting eqn (3.60) into $Z_1$ yields
The replacement of $T_0 R^{-\varepsilon}$ by $T$ in the partition function eliminates the $\varepsilon$-divergences and thus evidently solves our first problem with regard to eqn (3.56). The second problem concerned the divergence of the bare coupling $T_0 R^{-\varepsilon}$ as $R \to 0$. Let us therefore investigate the renormalised coupling $T$ as a function of $R$. This information is contained in the $\beta$-function which determines the dependence of the renormalised $T$ on changes in length scale at fixed bare temperature $T_0$:

$$
\beta(T) \equiv \frac{\partial T}{\partial R} \bigg|_{T_0} = -\varepsilon T + T^2 + \frac{1}{2} T^3 + O(T^4)
$$

(3.62)

where the last equality has been obtained by substituting eqn (3.60). Now we are in a position to vindicate Bruce and Wallace's assumption concerning the identity of the universal parts of the $\beta$-functions of planar and spherical interfaces. Invoking the result for the planar two loop $\beta$-function (Wallace and Zia 1979)

$$
\beta_p(T) = -\varepsilon T + T^2 + \frac{1}{2} T^3 + O(T^4)
$$

and comparing this expression to eqn (3.62) for the spherical $\beta$-function we confirm again the (well founded) expectation that planar and spherical interfaces display the same critical behaviour.
The renormalisation group equation (3.62) has two fixed points:

\[ T = 0 \]

which is infrared stable: \( T \to 0 \) as \( R \to \infty \); and

\[ T_c = \varepsilon - \frac{1}{2} \varepsilon^2 + O(\varepsilon^2) \]  \hspace{1cm} (3.63)

which is ultraviolet stable: \( T \to T_c \) as \( R \to 0 \). Thus the second problem with regard to eqn (3.56), concerning the singular \( R \)-dependence of \( T_0 R^{-\varepsilon} \) for small \( R \), is also solved: The renormalised coupling \( T \) exhibits the desired non-singular behaviour for small \( R \), approaching the finite value \( T_c \) for \( R \to 0 \). Further, it now becomes clear that a perturbative expansion to \( O(T) \), say, requires a parallel \( \varepsilon \)-expansion to \( O(\varepsilon) \): since, to two loops, \( T \) is bounded above by \( T_c = \varepsilon - \varepsilon^2/2 \), terms of \( O(T) \) are at most \( O(\varepsilon) \), and neglecting \( O(T^2) \) and \( O(\varepsilon T) \) is consistent with neglecting \( O(\varepsilon^2) \).

Let us now introduce the \( R \)-independent length scale \( \xi \) which in every aspect plays the role of the correlation length (see, in particular, Bruce 1984):

\[ \xi = R \exp \{ - \int dT' / \beta(T') \} \]  \hspace{1cm} (3.64)

Substituting eqn (3.62) into eqn (3.64) and integrating we obtain the explicit \( R \)-dependence of \( T \) in the form
\[ T(R) \left[ \frac{1}{T_c} - \frac{T(R)}{T_c} \right]^{-\nu} \left[ 1 + O(\epsilon T) \right] = C_0 \left( \frac{\gamma}{R} \right)^\epsilon \]  \hspace{1cm} (3.65)  

where

\[ \nu^{-1} = \beta'(T_c) = \epsilon + \frac{1}{2} \epsilon^2 + O(\epsilon^3) \]  \hspace{1cm} (3.66)  

is the inverse of the thermal exponent and \( C_0 \) is an integration constant.

A few remarks are in order at this point: Firstly, replacing \( R \) by some fixed length scale \( L_0 \) which reflects a minimal droplet size or a lattice cutoff, we have

\[ \gamma = L_0 \left[ 1 - T(L_0)/T_c \right]^{-\nu} \left( \frac{T(L_0)}{C_0} \right)^{1/\epsilon} \]

Thus we identify \( T(L_0) \) as the "true" system temperature; and one obtains, for \( T(L_0) \to T_c \), the standard power law behaviour for the correlation length:

\[ \gamma \sim \left( \frac{T(L_0) - T_c}{T_c} \right)^{-\nu} \]  \hspace{1cm} (3.67)  

It is essential to note that the thermal exponent \( \nu \) already emerges from the single droplet picture and is thus determined from interface fluctuations alone. The fact that it contains no information about the ensemble structure of the system is a fundamental feature of our droplet theory, and we shall repeatedly return to it.

Secondly, for \( R \gg \epsilon \), we find from eqn (3.65) that
Recalling that $T_0$ and hence $T$ were defined to be a measure of surface tension, $T_0 = \sigma^{-1}$, the surface energy for a droplet of scale size $R$ large compared to $\xi$ is given by $S_d c_0^{-1}(R/\xi)^{d-1}$, and we recover the result (Fisk and Widom 1969, Widom 1972)

\[ \sigma \sim \xi^{-(d-1)} \tag{3.69} \]

We also note that the constant $c_0$ serves to bring surface tension and correlation length into correct correspondence. An independent argument has been used by Bruce (1984) to determine $c_0$, and his result

\[ c_0 = 2 \tag{3.70} \]

holds in two dimensions for Ising and Potts models alike. Thirdly and finally, eqn (3.65) implies that $T(R)$ only depends on $R$ through the ratio $R/\xi$. Thus the partition function may be written in the scaling form

\[ Z_1 = \sqrt{d} \int dR \, \nu^{-1}(R) \, \psi(R, \xi) \tag{3.71} \]

where $\nu(R) = S_d R^d / d$ is the volume of a spherical droplet of radius $R$ and

\[ \psi(R, \xi) = R^{-1} \tilde{\psi}(R/\xi) \tag{3.72a} \]
The function \( \psi(R, E) \) will play an important role in the ensuing discussion of multi-droplet properties, in particular its limits for large and small values of its argument \( z \): For \( z \gg 1 \), \( \hat{\psi}(z) \) decays exponentially

\[
\hat{\psi}(z) = \psi(z) \left( z e^{(d+1)/2} \right)^{-1/2} \exp \left\{ -\frac{S_d}{T_c} z + \frac{S_d}{e} e - \frac{2}{e} \frac{S_d}{T(\epsilon)} \right\}
\]

(3.73)

For \( z \ll 1 \), the parameter

\[
\Psi_0 \equiv \lim_{z \to 0} \hat{\psi}(z) = \left( \frac{2}{\pi} \frac{T_c}{T} \right)^{-(d+1)/2} \exp \left\{ -\frac{S_d}{T_c} + \frac{S_d}{e} e - \frac{2}{e} \frac{S_d}{T(\epsilon)} \right\}
\]

(3.74)

will be of major interest to us later on. Here the problem of Bruce and Wallace's one loop calculation becomes obvious:

To one loop, \( T_c = \epsilon + 0(\epsilon^2) \), and

\[
\Psi_0 = \left( \frac{2}{\pi} \frac{T_c}{e} \right)^{-(d+1)/2} \exp \left\{ -\frac{S_d}{T_c} + \frac{S_d}{e} e - \frac{2}{e} \frac{S_d}{T(\epsilon)} \right\} (1 + \mathcal{O}(\epsilon))
\]

Inserting the one loop \( T_c \) into \( \Psi_0 \) we find that the \( \mathcal{O}(1) \) terms in the exponential, given by

\[
\frac{S_d}{T_c} = \frac{S_d}{e(1+0(\epsilon))} = \frac{S_d}{e} + \mathcal{O}(1)
\]

which are essential for a full one loop expression cannot be
captured by a one loop $T_c$; thus we recognise that $T_c$ is required to two loops, even in the context of a one loop calculation. Inserting eqn (3.63) into (3.74) yields:

$$\psi_0 = \frac{2}{\pi} \varepsilon^{(d+1)/2} \exp\left[-\left(1+2\gamma+2/\varepsilon\right)\right] \cdot (1+O(\varepsilon))$$

which is the full one loop expression for $\psi_0$.

However, we observe again that the two loop $T_c$ is not sufficient to prescribe $\psi_0$ to two loops. Repeating Bruce and Wallace's procedure, one may invoke the three loop expression for $T_c$ from the planar interface (Forster and Gabriunas 1981):

$$T_c = \varepsilon - \frac{1}{2} \varepsilon^2 + \frac{3}{8} \varepsilon^3 + O(\varepsilon^4)$$

and, by inserting into eqn (3.74), obtain an explicit form of the parameter $\psi_0$ to two loops.
3.4. Mean droplet volume and surface area

It remains to be checked whether the assumption that droplets are essentially hyperspherical objects is consistent with the results so far. Two quantities which contain relevant information are the mean volume and the mean surface area of a single droplet with scale size $R$. The volume $V(R)$ of a droplet of scale size $R$ is given by

$$V(R) = d^{-1} \int d\Omega \left( R + f(\eta) \right)^d$$  \hspace{1cm} (3.77)

and the surface area $A(R)$ is already familiar from the Hamiltonian eqn (3.1). Their averages are defined as

$$\langle V(R) \rangle = Z_1^{-1} \int \mathcal{D} f \ V(R) e^{-\mathcal{H}}$$  \hspace{1cm} (3.78a)

and

$$\langle A(R) \rangle = Z_1^{-1} \int \mathcal{D} f \ A(R) e^{-\mathcal{H}} = T_0^2 \frac{\partial}{\partial T_0} \ln Z_1$$  \hspace{1cm} (3.78b)

respectively. Both quantities have already been obtained by Bruce and Wallace; and we shall just quote their results for completeness.

The average volume can be calculated in a straightforward perturbation expansion:
\[
\langle V(R) \rangle = \frac{S_d R^d}{d} \left\{ 1 + O(\epsilon^2) \right\} = \frac{S_d R^d}{d} \left\{ 1 + O(\epsilon^2) \right\}
\] (3.79)

since \( T(R) \) is bounded above by \( T_c = \epsilon - \epsilon^2/2 \). Hence the corrections to the hyperspherical volume can be safely neglected.

For the mean surface, the situation is somewhat more complex. Dimensional analysis shows that the bare operator eqn (3.78b) requires a renormalisation of the form

\[
a(R, L_0, T) = L_0^{-\epsilon} T^{-2} T_0^{-2} \frac{\partial T}{\partial T} \langle A(R) \rangle
\]

where \( a(R, L_0, T) \) is the renormalised area. A careful analysis is necessary concerning the behaviour of \( a(R, L_0, T) \) for \( R \gg \xi \) and \( R \ll \xi \). For \( R \gg \xi \), the surface area scales conventionally as \( R^{d-1} \), thus droplets which are large (on the scale of the correlation length) possess surfaces which are smooth and essentially spherical. For \( R \ll \xi \), however, the surface area picks up an anomalous dimension:

\[
a(R, L_0, T) = S_d R^\epsilon \left( R/L_0 \right)^{-\frac{d-1}{d}} (1 + O(\epsilon))
\] (3.80)

reflecting the fact that the surface fluctuations of a droplet which is small compared to \( \xi \) represent a critical problem, even though the correlation length may still be finite; the anomalous dimension captures the fractal character (Mandelbrot 1982) of those (critical) fluctuations.
3.5. Single droplet partition function in external field

So far, we have not considered the effect of an external magnetic field on our calculations. In the spin Ising model, we have the following situation: If N spins are parallel (antiparallel) to the magnetic field, the energy is decreased (increased) by an amount proportional to N. In the single droplet Hamiltonian, this is reflected by a field term which is proportional to the droplet volume eqn (3.77):

\[ H = T_o^{-1} A(R) + 2(-1)^\alpha h \nu(R) \]  

(3.81)

\( \alpha = 1 \) \((2)\) corresponds to a droplet consisting of favoured (unfavoured) phase. The factor 2 in front of the field term reminds us that \( H \) denotes the energy difference between the fully ordered ground state and a single droplet configuration.

Sim and Bruce (1984) have evaluated the single droplet partition function for an Ising system with the Hamiltonian (3.81). The field term plays the role of an additional interaction which can be treated in the perturbative framework developed in this chapter. They obtain the result, in the notation of eqn (3.71):

\[ Z_{\alpha}(h) = \nu_\alpha \int dR \nu^{-1}(R) \Psi(R, \xi) \cdot \exp \left\{ -(-1)^\alpha 2 h \nu(R) \left[ 1 + O(\epsilon^2) \right] \right\} \]  

(3.82)
$Z_{1\alpha}(h)$ denotes the single droplet partition function for a droplet of phase $\alpha$, embedded in background of phase $\beta$, in an external field $h$. We remark that, to this order, the field simply couples to the volume of a hyperspherical droplet. Also, the available volume $V_\alpha$ is now a function of $\alpha$. This feature will have important consequences for the calculation of the ensemble partition function, as we shall see in the next chapter.
In this chapter, we extend the droplet description of the Ising universality class to accommodate $q > 2$ phases (colours) in order to obtain a description of the Potts universality class. The symmetry of the lattice Potts model is reflected in the symmetry of surface tension which we shall take to be independent of colour. As a consequence, the generalisation of the single droplet description requires only minor changes. We shall then present a renormalisation group argument designed to construct a scale invariant droplet ensemble. Two aspects are particularly important: Firstly, at criticality, the volumes occupied by any of the $q$ phases must be equal, since the system disorders completely. Secondly, the familiar scaling properties of the critical region imply the self-similarity of the system on all length scales large compared to the lattice spacing; thus any droplet can serve as background volume for the embedding of a smaller droplet of different phase, and so on down to the (arbitrary) minimal droplet size.

This situation clearly involves the breakdown of the dilute gas approximation which formed an integral part of the phenomenological droplet description reviewed in Chapter 2. The present work goes beyond the simple model in that it is capable of handling a gas of droplets which is by no
means dilute. However, even though the droplets themselves are not dilute, there is a chance that the droplet boundaries still might be. The "dilute boundary" approximation will become the crucial assumption in our theory due to which we shall be able to calculate the multi-droplet partition function. Again, however, we shall have to check later whether the results for the droplet ensemble are in fact consistent with the inherent approximation. This will lead us back to the discussion of the dimensionality and the number of phases in the system.
4.1. The single droplet partition function

Let us begin by looking for a droplet representation of the Potts Hamiltonian which is in keeping with the symmetry of the Potts model. Consider a spin representation first: At each lattice site, we allow for \( q \) different spin states, but keep the nearest neighbour interactions restricted to only two values, as in the Ising model. These requirements are satisfied by a set of vectors \( \sigma^\alpha \), \( \alpha = 1, \ldots, q \), denoting the possible spin states, which point to the vertices of a symmetric hypertetrahedron in \( (q-1) \)-dimensional space. Normalising the \( \sigma^\alpha \)'s to unity, for convenience, yields a set of vectors with

\[
\sum_{\alpha=1}^{q} \sigma^\alpha = 0
\]

and

\[
\sigma^\alpha \sigma^\beta = \frac{q}{q-1} \delta^\alpha_\beta - \frac{1}{q-1}
\]

which clearly possesses the required symmetry (Zia and Wallace 1975).

The basic configuration for a droplet description of the Potts model will be a single droplet consisting of phase \( \alpha \), say, with scale size \( R \) and specified by the deviation field \( f(\eta) \), which is embedded in a background formed from different phase, \( \beta \) say. As in the Ising case, this involves the assumption that the interface can be isolated from its surroundings and that no account needs to be taken of the effects of other droplets upon interfacial fluctuations.
As before, the single droplet Hamiltonian is given by the energy difference of the single droplet configuration with respect to the fully ordered ground state. It consists of two terms: the first term models the short range interaction energy of the spin Hamiltonian whereas the second term takes an external ordering field into account.

The interaction part of the Hamiltonian is associated with the surface energy of the interface separating the two phases \( \alpha \) and \( \beta \). Thus it is proportional to the surface area \( A(R) \) of the droplet and hence identical to the Ising surface Hamiltonian eqn (2.10). \( T_0^{-1} \) is, as before, a measure of surface tension, and we take it to be independent of the specific phases \( \alpha \) and \( \beta \) involved, in order to respect the Potts symmetry.

The external field term arises from an ordering field which is chosen to favour one particular phase (labelled 1). In a lattice Potts model with spins defined by eqn (4.1), the energy of a region (droplet) of spins in state \( \sigma^\alpha \), embedded in a background of state \( \sigma^\beta \) under the influence of an ordering field parallel to \( \sigma^1 \), is given by

\[
\Delta E = \left\{ \sum_{\text{droplet}} \sigma^\alpha + \sum_{\text{backgr.}} \sigma^\beta - \left( \sum_{\text{droplet}} \sigma^\beta + \sum_{\text{backgr.}} \sigma^\beta \right) \right\} h \sigma^1
\]

\[
= \frac{q}{q-1} N_0 \left( \delta_{\alpha 1} - \delta_{\beta 1} \right) h
\]

where we have subtracted off the ground state energy. \( N_0 \) is the number of spins in the droplet. Thus, in a continuum
description, the external field term is taken to be directly proportional to the droplet volume, given by eqn (3.77). It will be convenient to adopt the normalisation inherent in eqn (4.1) which is reflected in the $q$-dependent prefactor of the field term in the above equation. It ensures that the Potts order parameter is normalised to unity at low temperatures.

Collecting both contributions, the full Potts single droplet Hamiltonian reads:

$$\mathcal{H} = T_0^{-1} A(R) + \hbar \frac{q}{q-1} (\delta_{\alpha,1} - \delta_{\beta,1}) V(R)$$

(4.2)

The functional integral over all single droplet configurations weighted by the Boltzmann factor implied by eqn (4.2) yields the single droplet partition function $Z_1(h, L_0, \xi)$. The generalisation of the Ising result eqn (3.82) is completely straightforward, and hence

$$Z_{1,\alpha,\beta}(h) = V_{\xi} \int dR \, v^{-1}(R) \psi(R, \xi) \cdot \exp \left\{ \frac{\hbar q}{q-1} (\delta_{\alpha,1} - \delta_{\beta,1}) v(R) \right\}$$

(4.3)

with $v(R)$ and $\psi(R, \xi)$ defined in eqns (3.71) and (3.72), respectively.

At this level of the theory, the partition function $Z_1(h, L_0, \xi)$ depends on the phases $\alpha$ and $\beta$ only through the field term. The function $\psi(R, \xi)$ which reflects the contributions of interface fluctuations is independent from $\alpha$. 

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and $\beta$, as a consequence of the colour independent character of surface tension. Accordingly, the results for single droplet properties in the Ising case apply also to the Potts model: The expressions for the mean volume and surface area of a single droplet, as well as the subsequent remarks, remain unchanged. The assumption that the droplets are essentially spherical is thus justifiable for the Potts case as well. We observe again that droplets small compared to the correlation length possess a surface area with fractal character. Eqns (3.65) and (3.66) for the correlation length $\xi$ and the thermal exponent $v$ can also be carried over. $v$ is thus manifestly independent of $q$, a point to which we shall return in the concluding chapter.
4.2 The multi-droplet free energy

Our task is to calculate the full Potts model partition function defined as the sum over all configurations weighted by the appropriate Boltzmann factor. The typical configurations in a droplet description are constructed from droplets of all q phases and all scale sizes, between the system size $L$ (which will eventually be taken to infinity in the critical limit), and the minimal droplet size $L_0$. The corresponding Boltzmann factor is the exponential of the total energy, namely the sum of all surface and field energies associated with the individual droplets that form the configuration. Denoting a particular configuration by $C(L,L_0)$ and its total energy by $\mathcal{H}(C(L,L_0))$ we have

$$Z(h, L_0, L) = \sum_{C(L,L_0)} \exp(-\mathcal{H}(C(L,L_0)))$$

(4.4)

Now, following Bruce and Wallace, let us split this sum up into the contributions made by the smallest droplets, ranging in size from $L_0$ to $L_0+dL_0$, and the contributions made by all other droplets:

$$\sum_{C(L,L_0)} e^{-\mathcal{H}(C(L,L_0))} = \sum_{C(L,L_0+dL_0)} e^{-\mathcal{H}(C(L,L_0+dL_0))} \cdot \sum_{C^*(L_0+dL_0,L_0)} e^{-\Delta \mathcal{H}(C^*(L_0+dL_0,L_0))}$$

(4.5)

where $\Delta \mathcal{H}(C(L_0+dL_0,L_0))$ denotes the energy difference between $\mathcal{H}(C(L,L_0))$ and $\mathcal{H}(C(L,L_0+dL_0))$, and the asterisk
reminds us that the configurations $C(L_0 + dL_0, L_0)$ are embedded in the configurations $C(L, L_0 + dL_0)$. The partial configurational sum over the smallest droplets can be expanded in the number of droplets that form a given configuration $C(L_0 + dL_0, L_0)$:

$$\sum_{C^* (L_0 + dL_0, L_0)} e^{-\Delta \mathcal{H}[C^* (L_0 + dL_0, L_0)]} = \sum_{n=0}^{\infty} \sum_{C_{n}^*} e^{-\Delta \mathcal{H}[C_{n}^* (L_0 + dL_0, L_0)]}$$

(4.6)

In order to evaluate this sum (approximately) we need to invoke the "dilute boundary" approximation: If the droplet boundaries are dilute, i.e. comparatively remote from each other, then a background droplet of size $L_0 + dL_0$ contains either no droplet at all or a single droplet with scale size $L_0$ whose boundary is sufficiently isolated from the boundary of the background droplet. Configurations which involve a background droplet containing two or more smaller ($L_0$-sized) droplets are ruled out, which amounts to neglecting corrections of order $(dL_0)^2$.

Thus, in the differential limit, we only need to evaluate two contributions to the configurational sum in eqn (4.6). The first contribution is the sum over all "zero-droplet" configurations which yields unity because there is only one such configuration and the energy $\Delta \mathcal{H}$ has been defined as an energy difference:

$$\sum_{C_{0}^* (L_0 + dL_0, L_0)} e^{-\Delta \mathcal{H}[C_{0}^* (L_0 + dL_0, L_0)]} = 1$$

(4.7)
The second contribution is the sum over all single droplet configurations which is completely identical to the calculation of the single droplet partition function as addressed in Chapter 3. We only have to take account of the various ways by which droplets of one phase can be embedded in any of the other \( q-1 \) phases. Let \( V \) denote the total system volume, and \( V_\alpha = V_\alpha[C(L,L_0+dL_0)] \) the volume which is occupied by droplets of phase \( \alpha \) with scale sizes between \( L \) and \( L_0+dL_0 \) in the background configuration \( C(L,L_0+dL_0) \). Then

\[
\sum_{C_\alpha^\ast} \exp[-\Delta \mathcal{H}[C_\alpha^\ast(L_0+dL_0,L_0)] = C_\alpha^\ast(L_0+dL_0,L_0) = \left[ (V-V_\alpha) e^{\frac{1}{\beta} \mathcal{H}(L_0)} + (q-1)V_\alpha e^{-\mathcal{H}(L_0)_{q-1}} + \sum_{\alpha=2}^{q} (V - (V_\alpha + V_\beta)) \right] \psi^{-1}(L_0) \psi(L_0, \xi) \tag{4.8}
\]

The first term on the right hand side reflects the embedding of the field favoured phase 1 into background formed by any other phase, giving rise to the appropriate field term multiplied by the available background volume \( V-V_\alpha \). The second term reflects the embedding of the \( q-1 \) other phases into the volume occupied by phase 1. The third term finally takes account of dressing phase \( \alpha = 2,3,...,q \) with phase \( \beta = 2,3,...,q; \alpha \neq \beta \). We now insert eqns (4.7) and (4.8) into (4.5), and rewrite in terms of the mean volume occupied by droplets of phase \( \alpha \) with scale sizes between \( L \) and \( L_0+dL_0 \):

\[
\bar{V}_\alpha(L_0+dL_0) = Z^{-1}(L_0+dL_0, \xi) \cdot \sum_{C(L,L_0+dL_0)} V_\alpha[C(L,L_0+dL_0)] e^{-\mathcal{H}[C(L,L_0+dL_0)]}
\]
The mean volumes \( \overline{\alpha}(h,R) \) satisfy two equations: Firstly, all \( \overline{\alpha}, \alpha = 2, ..., q \), are equal for symmetry reasons; and secondly, the sum over all individual volumes gives the total volume:

\[
\begin{align*}
\overline{\alpha} &= V_0, \quad \alpha = 2, ..., q; \\
\sum_{\alpha=1}^{q} \overline{\alpha} &= V
\end{align*}
\] (4.9)

It will be convenient to express the mean volumes by the system magnetisation. In analogy to the definition in spin models, we define

\[
Q(h,R, 3) = V^{-1} \sum_{\alpha=1}^{q} \overline{\alpha}(h,R) \sigma^\alpha
\] (4.10)

where \( \sigma^\alpha \) denotes the "magnetisation content" of a unit volume of phase \( \alpha \), and eqns (4.1) are satisfied. The inherent normalisation of \( Q(h,R,3) \) ensures that the magnetisation is given by the derivative of the free energy with respect to the ordering field. It is then straightforward to show that the magnetisation is parallel to \( \sigma^1 \) and its modulus is given by

\[
Q(h,R, 3) = 1 - q V_0(h,R)/V
\] (4.11)

Collecting all these results, eqn (4.5) can be cast in differential form

\[
Z^{-1}(h, L_0, 3) \frac{\partial Z(h, L_0, 3)}{\partial L_0} = -V \nu^{-1}(L_0) \psi(L_0, 3) \\
\cdot \frac{q-1}{q} \left\{ (1 - Q(h, L_0, 3)) e^{-q \nu(L_0)} + \right. \\
\left. + (1 + (q-1)Q(h, L_0, 3)) e^{-q \nu(L_0)} - \\
-(q-2)(Q(h, L_0, 3) - 1) \right\}
\] (4.12)
and solved with boundary condition $Z(h,L,E) = 1$:

$$
Z(h,L_o, \xi) = \exp \left\{ \frac{q-1}{q} \int_{L_o} dR \psi^{-1}(R) \psi(R, \xi) \right\} + \frac{q-1}{q} \left\{ e^{\frac{q-1}{q} \lambda_o(R)} \left( 1 - Q(h,R,\xi) \right) + \left( 1 + (q-1) Q(h,R,\xi) \right) e^{-\frac{q-1}{q} \lambda_o(R)} - (q-2)(Q(h,R,\xi) - 1) \right\} \right\}
$$

(4.13)

From this, we immediately read off the free energy density of the Potts droplet ensemble:

$$
F(h,L_o, \xi) = \frac{q-1}{q} \int_{L_o} dR \psi^{-1}(R) \psi(R, \xi) + \left\{ e^{\frac{q-1}{q} \lambda_o(R)} \left( 1 - Q(h,R,\xi) \right) + \left( 1 + (q-1) Q(h,R,\xi) \right) e^{-\frac{q-1}{q} \lambda_o(R)} - (q-2)(Q(h,R,\xi) - 1) \right\} \right\}
$$

(4.14a)

where

$$
Q(h,R,\xi) = \frac{\partial}{\partial h} F(h,R,\xi)
$$

(4.14b)

and we have taken the thermodynamic limit $L \to \infty$. Eqn (4.14) is the key result of this section and forms the essential basis for much of the remaining work. It is an implicit representation of the free energy of the Potts doplet model and thus a generalisation of Bruce and Wallace's result for the droplet Ising model which can be recovered from eqn (4.14) by setting $q = 2$. The minimum droplet size $L_o$ reflects the non-universal (and hence untrustworthy) character of the small scale configurational structures; we shall only be
interested in universal quantities (notably the exponents governing the singular behaviour of the derivatives of the free energy) which are characterised by their independence from \( L_0 \). Finally, we remark that by construction the droplet representation eqn (4.14) is valid below, but not above, the critical temperature \( T_c \) in a neighbourhood of the coexistence curve (see also Sim (1984) and Sim and Bruce (1984) for an elaboration of this point with regard to the Ising model).
4.3 Scaling of the free energy: critical exponents

In this section, we present a systematic method of solving eqn (4.14) for the free energy, order by order in the external field. The two lowest orders yield equations for the order parameter and the susceptibility whose critical behaviour can thus be determined. Collecting all orders together leads to a properly scaling form of the free energy.

Let us begin by recalling the equation for the free energy. We define

$$Q(h, L_0, \xi) = \sum_{n=0}^{\infty} Q_n(L_0, \xi) h^n$$  \hspace{1cm} (4.15)$$

and insert this into eqn (4.14). Comparing coefficients in \( h \) we obtain a closed hierarchy of equations for the set of functions \( \{Q_n(L_0, \xi)\} \).

The Potts order parameter is defined as the spontaneous (zero field) magnetisation of the system and is thus given by \( Q_o(L_0, \xi) \). To lowest order in \( h \), we obtain the equation:

$$Q_o(L_0, \xi) = 1 - q \int_{L_0}^{\infty} \psi(R, \xi) Q_o(R, \xi) \, dR$$  \hspace{1cm} (4.16a)$$

which has the solution

$$Q_o(L_0, \xi) = \exp \left\{ -q \int_{L_0}^{\infty} \psi(R, \xi) \, dR \right\}$$  \hspace{1cm} (4.16b)$$
To obtain the asymptotic critical behaviour of this quantity, we split the integral into two parts, one ranging from $L_o$ to the correlation length $\xi$, and the other one ranging from $\xi$ to infinity. Recalling the limiting behaviour of $\psi(R,\xi)$ as given in eqns (3.73) and (3.74) we observe that we can replace $\psi(R,\xi)$ by its limiting form for small $R$, $\psi_o R^{-1}$, in the first integral. In the second integral, $\psi(R,\xi)$ is replaced by its limiting form for large $R$. Since this falls off exponentially, this contribution affects only the amplitude of the power law resulting from eqn (4.16):

$$Q_o(L_o, \xi) \sim \left(\frac{\xi}{L_o}\right)^{-\beta/\nu}$$  \hspace{1cm} (4.17a)

with

$$\beta/\nu = q \psi_o$$  \hspace{1cm} (4.17b)

From the next order in $\hbar$, we obtain a relation for the (linear) susceptibility

$$\chi(L_o, \xi) = Q_1(L_o, \xi) = $$

$$= \int_{L_o} dR \psi(R, \xi) \left[ -2q Q_o(R, \xi) + 2 \nu(R) + \nu(R)(q-2)Q_o(R, \xi) \right]$$

which has the solution

$$Q_1(L_o, \xi) = Q_o^2(L_o, \xi) \int_{L_o} dR \nu(R) \psi(R, \xi) \cdot$$

$$\cdot \left[ 2 + (q-2)Q_o(R, \xi) \right] Q_o^{-2}(R, \xi)$$  \hspace{1cm} (4.18)

To obtain the critical behaviour of $Q_1(R,\xi)$ we note that
Q_0(R,E) vanishes in the critical limit, and thus we make the approximation

\[ 2 + (q-2) Q_0(R,\xi) \approx 2 \]

Then the asymptotic behaviour of the susceptibility has the form

\[ Q_1(L_0,\xi) \sim (\xi/L_0)^{\gamma/\nu} \quad (4.19a) \]

where

\[ \gamma/\nu = d - 2q \psi_0 \quad (4.19b) \]

\( \beta \) and \( \gamma \) satisfy the familiar hyperscaling law

\[ \gamma + 2 \beta = d \nu \quad (4.20) \]

It is completely straightforward to evaluate the higher order functions \( Q_n(L_0,\xi) \) and their asymptotic behaviour. The result is

\[ Q_n(L_0,\xi) \sim (\xi/L_0)^{-(n+1)q \psi_0} \xi^{nd} \quad (4.21) \]

The exponent clearly increases linearly with \( n \), which is an essential prerequisite of a proper scaling theory of the free energy. Collecting the results obtained so far, we may write the free energy close to criticality in the form

\[ F(L_0,\xi) = F(0,L_0,\xi) + \sum_{n=1}^{\infty} h^n Q_{n-1}(L_0,\xi)/n! \]
which yields the scaling form

\[ \tilde{F}(h, L, \xi) \approx \xi^{-d} \tilde{\xi} \tilde{F}(h \cdot \xi^{\psi_0}) + \text{analytic pieces} \]  

(4.22a)

with

\[ \delta = \frac{d}{\psi_0} - 1 = \frac{\nu}{\beta} + 1 \]  

(4.22b)

Eqn (4.22) recovers the familiar scaling form of the free energy with exponents that satisfy hyperscaling and all other standard scaling laws.
In this section, we present an approach to the droplet Potts model which is physically rather intuitive. However, it is limited to zero ordering field, as we shall presently see. Let us return to the calculation of the ensemble partition function in Section 4.2. In eqn (4.5) the full configurational sum, including droplets of all phases and all scale sizes between the system size \( L \) and the minimal droplet size \( L_0 \), is split up into two parts: one part which expresses the contributions made by the smallest droplets with sizes between \( L_0 + dL_0 \) and \( L_0 \), and another part containing the contributions from droplets with sizes between \( L \) and \( L_0 + dL_0 \).

Consider the partial configurational sum for the smallest droplets, eqn (4.8), in zero external field:

\[
\sum_{\mathcal{C}_i'(L_0 + dL_0, L_0)} -\Delta \mathcal{H}[\mathcal{C}_i'(L_0 + dL_0, L_0)] = (q-1)\sqrt{\psi^{-1}(L_0)} \psi(L_0, \frac{\tau}{2})
\]

(4.23)

Since, clearly, the volumes \( V_\alpha[\mathcal{C}(L, L_0 + dL_0)] \), \( \alpha = 1, \ldots, q \), satisfy the sum rule

\[
\sum_{\alpha=1}^{q} V_\alpha[\mathcal{C}(L, L_0 + dL_0)] = V
\]

and thus, in zero field, the configurational sum (4.23) is rendered independent of the sum over background configurations \( \mathcal{C}(L, L_0 + dL_0) \). Accordingly, we obtain for the full ensemble partition function...
\[ Z(L_0, \xi) = Z(L_0 + dL_0, \xi)[1 + (q-1)\psi(L_0)\psi(L_0, \xi) dL_0] \quad (4.24) \]

In conjunction with eqns (4.23) and (4.5) this may be interpreted as the decoration of a given ensemble consisting of droplets ranging from \( L \) down to \( L_0 + dL_0 \), with smaller droplets of size \( L_0 \), of all phases, in all possible ways, in order to obtain an expression for the extended ensemble which includes all droplet sizes between \( L \) and \( L_0 \). Generally, this dressing procedure starts with a fully ordered system which we take to be filled with phase 1, thus retaining the boundary condition (4.13); this system is then decorated with droplets of smaller and smaller sizes down to the minimal droplet size \( L_0 \). In this interpretation, the function \( \psi(R, \xi) \) prescribes the mean fraction of available space to be occupied by droplets of any one phase with scale sizes between \( R \) and \( R + dR \), \( dR < 0 \), at the corresponding stage of the decoration procedure. The fact that the function \( \psi(R, \xi) \) is the same for all phases again reflects the symmetry of the Potts model.

By contrast, in non-zero field, eqn (4.5) only allows for a much more restricted interpretation: It reflects the decoration not of a given ensemble, but of a given configuration \( C(L_0, dL_0) \) with smaller droplets to obtain the extended configuration \( C(L_0) \); the decoration argument may thus only be applied before the ensemble average is taken. Let us demonstrate the consequences of this complication with an example. Consider, for instance, the mean number
n(R,E) of droplets with size R, which is clearly a measure of the energy cost associated with such droplets. In zero external field, the energy cost is determined by the surface energy only, and is thus not changed by the subsequent dressing of our droplet with droplets of smaller sizes; accordingly, it is independent of the minimal droplet size \( L_0 \), as we shall see in eqn (4.30). In non-zero field, however, the additional field energy term does depend on the droplet substructure and hence on \( L_0 \). This is discussed further in Sim (1984) and Sim and Bruce (1984).

We can now use the zero-field interpretation of the function \( \psi(R,E) \) to evaluate the total volume fraction \( \psi(L,L_0) \) covered by phase \( k \), \( k=2,3,...,q \). Let us assume that we have already decorated the system from size \( L \) down to \( R \). If the decoration procedure is carried out one step further to include droplets of scale size \( R + dR, \ dR < 0 \), the corresponding change \( d\psi(L,R) \) in \( \psi(L,R) \) is the sum of two terms:

\[
d\psi(L,R) = - [1 - \psi(L,R)] \psi(R,F) \, dR + (q-1) \psi(L,R) \psi(R,F) \, dR
\]

In our sign convention, the first term describes the gain in \( k \)-volume, caused by decorating the volume occupied by all other phases with droplets of phase \( k \). The volume which is available for this purpose is given in the square brackets, and after multiplication with \( \psi(R,E) \), we obtain the fraction which is actually used. Similarly, the second term denotes
the loss, originating in the decoration of \( k \)-volume with
droplets of the other \( q-1 \) phases. The integration of this
equation yields

\[
\Psi(L, L_0) = \frac{1}{q} \left[ 1 - e^{q \int_{L_0}^L \psi(R, \xi) dR} \right] \tag{4.25b}
\]

The corresponding volume fraction for phase 1 is given by

\[
\Psi^{(1)}(L, L_0) = 1 - (q-1) \Psi(L, L_0) \tag{4.25c}
\]

In the critical limit, \( L \gg \xi \gg L_0 \), and with the help of the
limiting forms for \( \psi(R, \xi) \), eqns (3.73) and (3.74), we find:

\[
\Psi(L, L_0) \sim \frac{1}{q} \left[ 1 - (\xi/L_0)^{-q/\xi} \right] \tag{4.26a}
\]

and thus, precisely at criticality, we have

\[
\Psi(L, L_0) = \Psi^{(1)}(L, L_0) = \frac{1}{q} \tag{4.26b}
\]

Thus our droplet theory satisfies an essential requirement of
any sensible description of the critical region, namely that
all phases must occupy the same volume fraction at
criticality.

Eqn (4.25) may be used to rederive expression (4.16) for
the order parameter. Recalling that \( \sigma^a \) represents the spin
content of a unit volume filled with phase \( a \), we have, with
eqn (4.1):
\[ Q_0(\ell, L, \xi) = \sigma^1 \Phi(\ell, L, \xi) + \Phi(\ell, L, \xi) \sum_{\alpha=2}^{\eta} \sigma^\alpha. \quad (4.27) \]

\[ = 1 - \eta \phi(\ell, L, \xi) \]

and thus recognise our previous result (4.16). We may also ask for the volume fraction \( \psi(\ell, L, \xi) \) occupied by all outermost droplets of any one phase \( k=2, \ldots, q \) (outermost droplets of phase 1 are excluded, due to the boundary condition). Again we derive a differential equation for the change in \( \psi(\ell, L, \xi) \) when the decoration procedure is carried out one step further, and the solution reads:

\[ \psi(\ell, L, \xi) = \frac{1}{q-1} \left\{ 1 - \exp \left\{ - (q-1) \int \psi(R, \xi) dR \right\} \right\} \quad (4.28) \]

Then the actual connected volume of any given droplet of size \( \ell \), i.e. the volume remaining after subsequent dressing with smaller droplets, is given by

\[ V(\ell, L, \xi) = v(\ell) \left[ 1 - (q-1) \psi(\ell, L, \xi) \right] \quad (4.29a) \]

which is clearly smaller than the "bare" volume \( v(\ell) \). For \( \ell \gg L \), it displays the critical behaviour

\[ V(\ell, L, \xi) \sim \left( \frac{\xi}{L} \right)^{-(q-1)\psi} \quad (4.29b) \]

and hence vanishes in the critical limit; in other words, the outermost droplets contained in our given droplet fill it completely.
Another quantity of interest is the droplet size distribution, defined as the mean number $n(R,E)$ of $k$-phase droplets ($k=2,\ldots,q$) with scale size between $R$ and $R+dR$:

$$n(R,\xi) = \nu^{-1}(R) \int [1 - \Phi(L,R)] \psi(R,\xi)$$ (4.30a)

The square brackets denote the total available volume which is not yet covered by larger $k$-phase droplets, and multiplication with $\psi(R,E)$ generates the fraction which is actually utilised. Similarly, for droplets of phase 1, we have

$$n^{(1)}(R,\xi) = \nu^{-1}(R) [1 - \Phi^{(1)}(L,R)] \psi(R,\xi)$$ (4.30b)

With the help of those results we can now determine the mean droplet size, prescribed by the second moment of the droplet size distribution, and its asymptotic behaviour, determined by the limiting forms of $\psi(R,E)$, eqns (3.73) and (3.74):

$$S = \int_{L_0}^\infty dR \, n(R,\xi) V^2(R,\xi) \sim (\xi/L_0)^{\Theta/\nu}$$ (4.31a)

The exponent is identified as

$$\Theta/\nu = d - 2(q-1) \psi_0 = d - 2 \frac{q-1}{q} \beta/\nu$$ (4.31b)

This result is independent of which phase of droplets the distribution $n(R,E)$ is chosen to characterise.
The evaluation of the droplet number distribution also enables us to check the consistency of the "dilute boundary" approximation by establishing an expression for the density of droplet surfaces: As the dressing procedure progresses, the space which was initially filled homogeneously with phase 1, is decorated with droplets of all other phases and smaller sizes so that at the end of the day, the total remaining volume fraction of phase 1 is given by $\psi^{(1)}(L,L_0)$. The dressing procedure inherent in the derivation of the ensemble partition function will be consistent only if a dressing droplet, placed at random into the available background volume, will, with a high probability, not intersect the boundaries of that space. Let us then ask for the fraction $\Delta \psi^{(1)}$ of this fragmented volume which lies within a distance $L_0$ of an existing droplet surface:

$$\Delta \psi^{(1)} = L_0 \int \frac{dR}{L_0} \left[ (q-1) n(R, L) + n^{(1)}(R, L) \right] a(R, L, L_0) \sim (q-1) \psi_0$$

where $a(R, L, L_0)$ is the mean surface area of a droplet with scale size $R$ given in Section 3.4. The smallness of the parameter $(q-1)\psi_0$ as long as $q$ is not too large, justifies in the first instance the use of the function $\psi(R, L)$ to prescribe the appropriate volume fraction to be covered at each stage of the decoration procedure. Moreover, it vindicates the dilute boundary approximation which is instrumental to the calculation of both the single and the ensemble partition function: in the former case, it guarantees that a single droplet can be treated separately and that a single droplet
Hamiltonian can be meaningfully defined. In the latter case, it ensures the validity of the renormalisation group argument described in Section 4.2 and assigns physical meaning to the differential limit (4.12).
4.5 Migdal-Kadanoff renormalisation group

In this section, we present a real space renormalisation group calculation of the exponents $\nu$ and $\beta$, in order to compare the droplet results with results from a different approach.

The basic ingredient of renormalisation group theory consists in the observation that physical systems at (and very close to) criticality are self-similar on all length scales sufficiently large compared to the lattice spacing. Intuitively speaking, this means that different views of the system, seen through, say, magnifying glasses of different resolution, all show the same coarse-grained configurational structures - until we get down to the order of the lattice spacing where the microscopic features of the underlying space become apparent. The major problem of critical phenomena is thus the equal importance of the degrees of freedom associated with many length scales - in this situation, an ordinary analysis which singles out a few (presumably 'essential') degrees of freedom is bound to fail: A typical example is the phenomenological droplet model discussed in Chapter 2 which neglects all but the zero mode of the surface fluctuations. By contrast, the renormalisation group analysis attempts to take the effects of many length scales into account, at least approximately: for instance, the computation of the single droplet partition function in
Chapter 3 incorporated all the modes in the expansion of the deviation field \( f(\eta) \). Renormalisation group arguments come in many forms; in principle, one sets up a step-by-step procedure, where, at each stage, a fraction of the degrees of freedom is decimated, and the associated effects are retained in, for instance, an effective Hamiltonian for the remaining degrees of freedom. This is explicitly demonstrated by the derivation of the ensemble partition function in Chapter 4: the calculation of the partial configurational sum (4.6) corresponds to eliminating the degrees of freedom associated with the smallest droplets; the differential equation (4.12) is just the differential limit of the relation between the partition functions at two consecutive stages of the procedure. The general theory of the renormalisation group was implemented and developed by Kadanoff (1966), Wilson (1971), Wilson and Fisher (1972), and others. For a review and further references, we refer to Pfeuty and Toulouse (1975) and Wallace and Zia (1978).

Consider a lattice of interacting spins, with a lattice constant (short-distance cutoff) \( a \), and a Hamiltonian which contains various interaction terms and their couplings or external fields. As criticality is approached, the correlation length \( \xi \) becomes much larger than the lattice constant \( a \). One can collect groups of neighbouring spins into new "block spins" and calculate their effective interactions, thus obtaining a new Hamiltonian by decimating the old spins in favour of the new ones. For example, if we are dealing with
spins $\{\sigma\}$ and a Hamiltonian $\mathcal{H}[\sigma]$, and we want to decimate $p$ old spins in favour of one new spin variable $\sigma'$, then the new effective Hamiltonian $\mathcal{H}'[\sigma']$ is obtained by summing over all configurations of the old set of spins $\{\sigma\}$ while keeping the $\{\sigma'\}$ fixed:

$$e^{-\mathcal{H}'[\sigma']} = \sum_{\{\sigma\}} e^{-\mathcal{H}[\sigma]}$$ (4.33a)

Along with the decimation procedure goes a change in length scale, corresponding to changes in "magnification", since our new spins have a lattice constant which differs from the old one. Reexpressing all quantities in terms of the new (running) lattice spacing completes the prescription. The combination of decimation and rescaling defines a renormalisation group transformation. Repeating the procedure generates a sequence of Hamiltonians which define a trajectory in the space spanned by the coupling constants and external fields, i.e. the system parameter space. The decimation reduces the (dimensionless) correlation length: hence, if we start off with a finite correlation length, the blocking procedure will reduce it. However, if we start off with an infinite correlation length, it will clearly remain infinite. In parameter space, a trajectory starting on a surface with $\xi = \infty$ will stay on it, whereas a trajectory starting on a surface with finite $\xi$ will eventually move away from the surface with $\xi = \infty$. The surface $\xi = \infty$ is called the critical surface, and the corresponding temperature is the critical temperature of the system.

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A fixed point in parameter space is a point which maps onto itself under the renormalisation group transformation:

\[ \sum_{\{\sigma\}} e^{-\mathcal{H}[\sigma]} = e^{-\mathcal{H}[\sigma']} \]  

Eqn (4.33b), translated into our intuitive picture, implies that a change in magnification does not change the appearance of the system. We shall be particularly interested in fixed points on the critical surface, since their properties determine the critical behaviour of the system. Linearising the renormalisation group equations in a neighbourhood of such a fixed point leads to an analysis of the linear stability matrix. Its eigenvalues define those directions along which the parameters only change in scale under a renormalisation group transformation. If the length scale is changed by a factor \( s \) and a given parameter rescales according to \( s^y \), then \( y \) governs the critical behaviour of that parameter, i.e. it is essentially the corresponding critical exponent. For a given fixed point, one can define its region of attraction: all trajectories issuing from any point in that region will eventually converge towards the associated fixed point. In actual fact, this is the renormalisation group statement of the universality principle.

Different renormalisation schemes correspond to different prescriptions for calculating the block spins. We shall adopt the Migdal-Kadanoff scheme (Migdal 1975,
Kadanoff 1976b) since it is specifically tailored to systems whose dimensionality is close to 1: For \( d=1 \), it becomes exact.

We apply the scheme to a \( d \)-dimensional hypercubic lattice of Potts spins. It consists of two steps: Firstly, along a specified lattice direction, all bonds and on-site potentials (like the external field) are shifted to the edges of a lattice hypercube containing \( p^d \) spins. Hence the new block spins will be situated on the edges of the hypercube. After the shift, certain spins are now partly decoupled from their neighbours, and the remaining couplings have the form of a one-dimensional chain. In a second step, those spins are decimated, thus yielding a new effective Hamiltonian which governs the interactions between the block spins. The nature of this prescription, in particular of the first step, appears highly heuristic, but its application has proved to be reasonably successful (cf. Kadanoff 1976b), and technically feasible, due to the one-dimensional character of the decimation.

A generic problem of decimation schemes resides in the fact that they may generate new couplings in the Hamiltonian which were not included in the original model. In order to close the renormalisation group equations one either has to introduce the new couplings into the original Hamiltonian or otherwise devise an approximation scheme which allows to neglect them. In our case, we shall be able
to obtain a closed set of equations if we work with the Hamiltonian

\[
\hat{H} = -\sum_{\langle i,j \rangle} K \delta(\sigma_i, \sigma_j) - \sum_i \h \delta(\sigma_i, \sigma^1) - 2 \sum_{\langle i,j \rangle} J \delta(\sigma_i, \sigma^0) \delta(\sigma_i, \sigma_j)
\]

(4.34)

where \(\delta\) stands for the Kronecker symbol. The first term represents the nearest neighbour interactions, with an exchange constant \(K\) which is a measure of inverse temperature. Compared with equation (4.1) we have rescaled the nearest neighbour interaction energy, but this only gives rise to additional constants in the Hamiltonian about which we need not be concerned. The new definition will be more convenient for the subsequent calculations. The external field, in keeping with the previous section, is applied parallel to the state \(\sigma^1\) and thus produces contributions to \(\hat{H}\) only for those spins which are in state \(\sigma^1\). The last term is required to close the renormalisation group equations. We note that a \((q-1)\)-state lattice gas Potts model (Blume et al. 1971, Berker et al. 1978, Nienhuis et al. 1979) can be transformed into the Hamiltonian eqn (4.34), if the vacancies which are typical for lattice gas models are interpreted as an extra spin state (Wu 1982), corresponding to the state \(\sigma^1\) in our model.

The sequence of shift and decimation, and the arbitrariness in the choice of the direction of the shift generate an anisotropy in the couplings of the new Hamiltonian. However, this feature is negligible for \(d\) close
to 1, and since this corresponds to the dimensional range of interest to us we need not be concerned.

The first question which we address is the decimation of Potts spins along a one-dimensional chain. To perform the decimation, a transfer matrix approach proves useful (for a review, and further applications see e.g. Stanley 1971). The starting point is the Hamiltonian (4.34) in its one-dimensional form for a linear chain consisting of N spins. We define

\[ U(\sigma_i, \sigma_{i+1}) = -k \delta(\sigma_i, \sigma_{i+1}) - \frac{1}{2} h [\delta(\sigma_i, \sigma^1) + \delta(\sigma_{i+1}, \sigma^1)] - J \delta(\sigma_i, \sigma_{i+1}) [\delta(\sigma_i, \sigma^1) + \delta(\sigma_{i+1}, \sigma^1)] \]

and the transfer function

\[ f(\sigma_i, \sigma_{i+1}) = \exp - U(\sigma_i, \sigma_{i+1}) \]

The partition function can then be written as

\[ Z_N(\kappa, h, J) = \sum_{\{\sigma_i\}} \prod_{i=1}^{N} f(\sigma_i, \sigma_{i+1}) \]

The transfer matrix \( T \) is defined as

\[ T_{\alpha \beta} = f(\sigma_i = \sigma^\alpha, \sigma_{i+1} = \sigma^\beta) \]

By definition, \( T_{\alpha \beta} \) is a \( q \times q \) matrix for a \( q \)-state Potts model, and for our Hamiltonian (4.34) it has the form
First of all, we note that the transfer matrix can be used, for instance, to calculate the partition function. If we impose periodic boundary conditions we obtain

\[ Z_N(K, h, J) = \text{tr} \, T^N \]

Hence, the problem of calculating the partition function is equivalent to determining the trace of the Nth power of the transfer matrix. This can be done either explicitly by calculating the Nth power of \( T \) and summing the diagonal elements, or by finding the eigenvalues \( \{ \lambda \} \) of \( T \). Once these are known, the eigenvalues of \( T^N \) are simply \( \{ \lambda^N \} \), and the trace of \( T^N \) is the sum over its eigenvalues.

Let us turn to our original problem, the decimation of Potts spins along a linear chain. As we have seen, the decimation of groups of \( p \) spins \( \{ \sigma \} \) in favour of new spins \( \{ \sigma' \} \), amounts to the partial configurational sum

\[ e^{-\mathcal{H}'[\sigma']} = \sum_{\{ \sigma \}} e^{-\mathcal{H}[\sigma']} \]

This recursion relation can be cast in terms of the transfer matrices \( T' \) and \( T \), associated with \( \mathcal{H}'[\sigma'] \) and \( \mathcal{H}[\sigma] \), respectively. Noting that the configurational sum on the right hand side of the preceding equation can be expressed
with the help of $T^p$, we obtain the recursion relation for the transfer matrices:

$$ T' = e^{-c} T^p $$ (4.36)

where the constant $c$ takes care of spin-independent overall energy differences in the Hamiltonians. Hence our problem consists in calculating the $p$th power of the transfer matrix $T$. From eqn (4.35) it is clear that the elements of $T$ only assume four different values. Let us therefore rewrite eqn (4.35):

$$ T = \begin{pmatrix}
A^{(1)} & B^{(1)} & C^{(1)} & D^{(1)} \\
B^{(1)} & C^{(1)} & D^{(1)} & B^{(1)} \\
C^{(1)} & D^{(1)} & B^{(1)} & C^{(1)} \\
D^{(1)} & B^{(1)} & C^{(1)} & D^{(1)}
\end{pmatrix} $$ (4.37)

One may verify by inspection (or, more carefully, by induction) that the $p$th power of $T$ possesses the same structure, with elements $A^{(p)}, ..., D^{(p)}$. These are related to the elements $A^{(p-1)}, ..., D^{(p-1)}$ of $T^{p-1}$ through the recursion relation:

$$ A^{(p)} = e^{\kappa + h + 2\tau} A^{(p-1)} + (q-1) B^{(p-1)} $$

$$ B^{(p)} = e^{\kappa + h + 2\tau} B^{(p-1)} + e^\nu C^{(p-1)} + (q-1) e^\nu D^{(p-1)} $$

$$ C^{(p)} = e^\nu B^{(p-1)} + e^\kappa C^{(p-1)} + (q-2) D^{(p-1)} $$

$$ D^{(p)} = e^\nu B^{(p-1)} + C^{(p-1)} + (e^\kappa + q-3) D^{(p-1)} $$

The ansatz

$$ A^{(p)} = A^p \alpha^p, \quad \ldots \quad D^{(p)} = D^p \alpha^p $$ (4.39)
transforms the recursion relation into the eigenvalue equation for the $4 \times 4$ matrix

$$
M = \begin{pmatrix}
  e^{K+4+2J} & (q-1)e^{W_{12}} & 0 & 0 \\
  0 & e^{K+4+2J} & e^{W_{12}} (q-2)e^{W_{12}} & 0 \\
  0 & e^{W_{12}} & e^{K} & q-2 \\
  0 & e^{W_{12}} & 1 & e^{K+q-3}
\end{pmatrix}
$$

(4.40)

with $\alpha$ playing the role of the eigenvalues and $A, \ldots, D$ corresponding to the eigenvectors. The determination of the eigenvalues and eigenvectors is now absolutely straightforward. We find four distinct eigenvalues and the same number of eigenvectors, which are of course only defined up to a multiplicative constant. However, we have to fit our solution to the "initial condition" eqn (4.37) which is sufficient to eliminate the constants. Thus, the full solution reads

$$
\begin{pmatrix}
  A(w) \\
  B(w) \\
  C(w) \\
  D(w)
\end{pmatrix} = \frac{1}{q-1} \begin{pmatrix} 0 \\ 0 \\ q-2 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ q-2 \\ -1 \end{pmatrix} + \frac{2(q-1)e^K}{W+R} \begin{pmatrix} 0 \\ 0 \\ q-2 \\ -1 \end{pmatrix} + \frac{2(q-1)e^K}{W+R} \begin{pmatrix} 0 \\ 0 \\ q-2 \\ -1 \end{pmatrix}
$$

(4.41a)

with

$$
\alpha_{\pm} = \frac{1}{2} \left( e^{K+4+2J} + e^{K} + q-2 \pm R \right)
$$

$$
R = \left[ W^2 + 4(q-1)e^K \right]^{1/2}
$$

$$
W = e^{K+4+2J} - e^{K} - (q-2)
$$

(4.41b)

This has to be equated with the transfer matrix for the Hamiltonian $\mathcal{H}[\sigma]$, according to eqn (4.36): we require $T'$ to have the same structure as $T$, with parameters $K'$, $h'$, and $J'$. 122
Finally, in order to complete the renormalisation group transformation, we still have to include the effect of the bond shifts; and thus, we obtain the full Migdal-Kadanoff recursion relations

\[
\frac{K'}{p^{d-1}} = \ln \frac{C(p)}{D(p)}
\]

\[
\frac{h'}{2p^{d-1}} = \ln \frac{B(p)}{D(p)}
\]

\[
\frac{2J'}{p^{d-1}} = \ln \left[ \frac{A(p)}{C(p)} \left( \frac{D(p)}{B(p)} \right)^2 \right]
\]

with \(A(p), \ldots, D(p)\) as defined in eqn (4.41). As a check on this result we note that for general \(p\), and \(h = J = 0\) we recover Kadanoff's result for the q-state Potts model (Kadanoff 1976b).

The next step consists in locating the zero field fixed point which is of interest to us. There, \(h^* = J^* = 0\) holds exactly, and \(K^*\) obeys the equation:

\[
\exp(-K^*/p^{d-1}) = p \exp(-K^*) + (q-2) \binom{p}{2} \exp(-2K^*) + O[\exp(-3K^*)]
\]

Higher orders may be neglected because we expect \(K^*\) to be very large for \(d\) close to 1, with its inverse approaching zero as \(d\) goes to 1. Solving to first order, we obtain the inverse of the critical temperature:

\[
K^* = \frac{1}{e} + O(1)
\]
In order to evaluate critical exponents, the recursion relations (4.42) must be linearised about the fixed point. The resulting stability matrix has three eigenvalues one of which is less than zero and thus irrelevant. The other two eigenvalues yield the exponents $\nu$ and $\gamma_H$:

$$\nu^{-1} = \varepsilon + \frac{p-1}{2\ln p} \frac{q-2}{2} e^{-K^*} + O(e^{-2K^*}) \quad (4.44)$$

and

$$\gamma_H^{-1} = d - \frac{p^2-1}{6\ln p} q e^{-2K^*} + O(e^{-3K^*}) \quad (4.45a)$$

Using scaling laws, we find

$$\beta = \nu(d - \gamma_H^{-1})$$

$$= C \frac{p^2-1}{6\ln p} \frac{q}{\varepsilon} e^{-2L/e} \left[ 1 + O(e^{-1/e}) \right] \quad (4.45b)$$

where the $0(1)$ terms arising from $K^*$, eqn (4.42), are contained in the constant $C$. Let us recall the droplet results for $\nu$ and $\beta$ and compare them with the Migdal-Kadanoff results, eqns (4.44) and (4.45b). For the droplet thermal exponent $\nu$, we have eqn (3.66):

$$\nu^{-1} = \varepsilon + \frac{1}{2} \varepsilon^2 + O(\varepsilon^2)$$

which is found to be in accordance with eqn (4.44), to first order in $\varepsilon$. At that order, we also note that both results do not show any $q$-dependence. However, whereas the droplet $\nu$ is entirely independent of $q$, the Migdal-Kadanoff result
depends on $q$ via a factor $(q-2)$ in the second term on the right hand side of eqn (4.44). This is an interesting feature of the result, and we shall return to it in Chapter 6.

For the droplet magnetic exponent $\beta$, we have eqn (4.17b), which yields with eqn (3.75) for $\psi_0$

$$\beta = q \nu \psi_0 = \text{const} \left( \frac{q}{\epsilon^2} \right) e^{-2/\epsilon} \left[ 1 + O(\epsilon) \right]$$

In both results, the first non-vanishing term in an $\exp^{-1/\epsilon}$ expansion appears at $O(\exp^{-2/\epsilon})$, and, to this order, both results are simply proportional to $q$. We cannot, however, expect exactly the same prefactor structure, since the prefactor of the Migdal-Kadanoff result is manifestly non-universal (it depends on the decimation parameter $p$). However, we note that, as usual in the Migdal-Kadanoff scheme, the prefactor is finite for $p \rightarrow 1$.

In conclusion, we remark that the agreement of droplet exponents and Migdal-Kadanoff results is fairly good, considering the heuristic nature of the Migdal-Kadanoff prescription, and taking into account that the two approaches are based on rather different models: a field theory based on the configurational energy of surface tension on one hand, and a spin Hamiltonian on the other.
5.1 The Kastelevn-Fortuin relation

Let us begin with a few general remarks. We have already indicated in Chapter 1 that the bond percolation problem is concerned with the study of configurations which are generated by random distributions of bonds on a lattice. Bonds are present with a probability \( p \) and absent with probability \( 1-p \), and there exists a critical value \( p_c \) such that, above \( p_c \), an infinite cluster exists which is not the case below \( p_c \). We note that, for a finite lattice, the "infinite" cluster is a cluster whose size is of the order of the system size. A suitable order parameter for this problem is given by the probability \( P \) that a particular site belongs to the infinite cluster: It is zero for \( p < p_c \) but strictly positive for \( p > p_c \).

An important quantity in percolation is the cluster number distribution \( n(s) \) which specifies the mean number of finite clusters with \( s \) sites, per site. A 'one-site' cluster is defined to be a site which is not connected to any of its neighbours. Then the probability per site \( p(s) \) that a given site belongs to a (finite) cluster consisting of \( s \) sites is related to the cluster number distribution via
\[ p(s) = s \eta(s) \quad (5.1) \]

We shall always assume translation invariance so that \( p(s) \) is independent of the particular site chosen. Normalising the probabilities yields the sum rule

\[ 1 = P + \sum_{s=1}^{\infty} p(s) \quad (5.2) \]

It is very often convenient to express the information contained in the cluster number distribution with the help of a generating function

\[ f(h) = h + \sum_{s=1}^{\infty} \eta(s) e^{-hs} \quad (5.3) \]

Its derivative at zero field coincides with the order parameter

\[ \frac{\partial}{\partial h} f(h)|_{h=0} = 1 - \sum_{s=1}^{\infty} s \eta(s) = P \quad (5.4) \]

due to the sum rule eqn (5.2). We can see from this result that, formally, the generating function of the cluster number distribution plays a similar role for percolation as the free energy for a thermal phase transition, and we shall therefore occasionally refer to the parameter \( h \) as the "external field". Let us finally define the generating function of the cluster probability distribution which coincides with the field dependent derivative of eqn (5.3):
We shall now turn to a review of the connection between the free energy of the standard lattice Potts model and the generating function of the cluster number distribution of the bond percolation problem, defined on an identical lattice (Kasteleyn and Fortuin 1969, Fortuin and Kasteleyn 1972). Consider the Hamiltonian for a lattice Potts model in an external field:

\[ \mathcal{H} = K \sum_{\langle i,j \rangle} \left[ 1 - \delta(\sigma_i, \sigma_j) \right] - \frac{q}{q-1} \ln \sum_{\sigma} \left[ \delta(\sigma_i, \sigma^1) - \frac{1}{q} \right] \]

and the corresponding partition function

\[ Z = \sum_{\{\sigma\}} e^{-\mathcal{H}} \]

Due to the properties of the Kronecker \( \delta \), we may rewrite the exponential:

\[ e^{-\mathcal{H}} = \prod_{\langle i,j \rangle} \left[ (1-p) + p \delta(\sigma_i, \sigma_j) \right] \cdot \exp \frac{-\ln}{q-1} \sum_{\sigma} \left[ \delta(\sigma_i, \sigma^1) - \frac{1}{q} \right] \]

where

\[ p : = 1 - e^{-K} \]

The product over nearest neighbours can be reexpressed in terms of a graph picture: Consider a lattice on which a certain number of bonds is occupied and the rest are empty.
A particular configuration of occupied and empty bonds is called a graph. An expansion of the product over nearest neighbours generates the sum over all possible graphs: each graph corresponds to a term in the sum, and for any particular graph, an occupied bond between two spins $\sigma_i$ and $\sigma_j$ is represented by a factor $p\delta(\sigma_i, \sigma_j)$, whereas an empty bond yields a factor $1-p$. Let $P(G)$ denote the probability of generating a particular graph $G$. Then

$$\prod_{i,j} \left( p\delta(\sigma_i, \sigma_j) + (1-p) \right) = \sum_G P(G) W(G, \{\sigma\})$$

where

$$W(G, \{\sigma\}) = \begin{cases} 1 & \text{if } G \text{ consists only of bonds connecting spins parallel in } \{\sigma\} \\ 0 & \text{otherwise} \end{cases}$$

Each graph consists of one or more disjoint clusters. If $N$ is the total number of spins in the system, let $N_n(s, G)$ denote the number of finite clusters with $s$ sites in the graph $G$. Again, a 'one-cluster' is defined to be a site which is not connected to any other site. If $N_\infty(G)$ is the number of spins in the infinite cluster, we have

$$N = N_\infty(G) \Delta + N \sum_{s=1} N_n(s, G)$$

(5.7)

where

$$\Delta = \begin{cases} 1 & \text{if an infinite cluster exists} \\ 0 & \text{otherwise} \end{cases}$$
Since all sites in a given cluster are interconnected, the function $W(G, \{0\})$ ensures that the only contribution to the partition function is made by configurations in which all spins in a given cluster are parallel. In addition, the "external field" term only contributes when all spins in a given cluster are parallel and in state $0^1$. Hence

$$Z(h) = \sum_G P(G) \sum_{\{\sigma\}} W(G, \{\sigma\}) \exp \frac{qh}{q-1} \sum_i (\delta(\sigma_i, \sigma'_i) - \frac{1}{q})$$

$$= \sum_G P(G) \prod_{s \geq 1} \left( q - 1 + e^{\frac{qh}{q-1} s} \right)^{N_n(\xi, G)} \left( q - 1 + e^{\frac{qh}{q-1} s} \right)^{N_n(\xi, G)}$$

$$= e^{Nh} \sum_G P(G) \prod_{s \geq 1} \left( 1 + (q-1) e^{-\frac{qh}{q-1} s} \right)^{N_n(\xi, G)}$$

where we have used the fact that $N_\infty$ is very large. Thus we have obtained an expression for the partition function of the lattice Potts model in graph language. In the limit $q \to 1$ we have

$$Z(h) = e^{Nh} \sum_G P(G) \left\{ 1 + (q-1) \sum_{s \geq 1} N_n(\xi, G) e^{-\frac{qh}{q-1} s} \right\}$$

If we define the average density $n(s)$ of $s$-site clusters as

$$n(s) = \sum_G P(G) n(s, G)$$

we obtain the free energy density in this limit:

$$\mathcal{F}(h) = ln Z(h) = h + (q-1) \sum_{s \geq 1} n(s) e^{-\frac{qh}{q-1} s}$$

Hence we find the following relation between the Potts free
energy density and the generating function of the cluster number distribution of the percolation problem:

\[
\lim_{q \to 1} \frac{1}{q-1} \frac{d}{dh} F((q-1)h) = \\
= h + \sum_{s>1} n(s) e^{-hs} = f(h)
\]

(5.8a)

Equivalently, the same relation can be stated in its derivative form:

\[
\lim_{q \to 1} \frac{1}{q-1} \frac{d}{dh} F((q-1)h) = \\
= 1 - \sum_{s>1} p(s) e^{-hs} = q(h)
\]

(5.8b)

Both equations have to be read in conjunction with eqn (5.6). However, the status of the latter differs fundamentally from that of the former: Eqn (5.6) expresses a link between two specific lattice models; as such it has no immediate implications for universal quantities (the critical constants $K_C$ and $p_C$ being lattice-dependent), and is of no consequence to us. Eqns (5.8), although derived within the framework of a lattice model, clearly transcend this limitation: they express a link between two universality classes, and lead to the expectation that the universal features of the functions $f(h)$ and $g(h)$ will be captured correctly by eqns (5.8), no matter what particular representation of the Potts model is used in the calculation of the free energy density $F$.

Proceeding on this assumption we substitute into eqn (5.8b) our droplet based representation (4.14) for the Potts
free energy density. Taking the prescribed $q \to 1$ limit we find, in the notation of eqn (4.14),

$$g(h, l_0, \xi) = 1 + \int dR \psi(R, \xi) \cdot \left\{ (1 - g(h, R, \xi))e^{\nu_0(R) L} - e^{-\nu_0(R)} - \nu^{-1}(R) g'(h, R, \xi) (e^{\nu_0(R)} - 1) \right\}$$  

(5.9a)

where

$$g'(h, R, \xi) = \frac{\partial}{\partial h} g(h, R, \xi)$$  

(5.9b)

These equations define an implicit representation of the generating function for the percolation probability distribution $p(s, l_0, \xi)$. Eqn (5.8a) may be used to derive a similar representation for the generating function of the cluster number distribution; clearly, both generating functions contain the same information. We shall stick to eqns (5.9), because the cluster size distribution will eventually be of greater interest to us here.

A number of remarks are in order: Firstly, the dependence upon the minimum length scale $l_0$ indicated in eqn (5.9) is a reminder that our representation can be trusted to capture correctly only the truly universal features of the distribution, whose signature will be their independence of this arbitrary inner length. Secondly, our inclusion of the length $\xi$ amongst the arguments of the generating function anticipates the fact that the results we shall obtain will be parameterised by this length, whose behaviour near the
percolation threshold $p_C$ follows from (3.67) and (5.6) as

$$\xi \sim (p - p_C)^{-\nu}$$  \hspace{1cm} (5.10)

with $\nu$ as prescribed in eqn (3.66). Again, we need to register caution at the apparent $q$-independence of $\nu$, a point to which we shall return in the next chapter. Thirdly, we observe that, since our droplet representation of the free energy density, eqn (4.14), is valid below, but not above the critical temperature $T_c$, the implied representation for the generating function (5.9) is, in view of eqn (5.10), trustworthy above but not below the percolation threshold $p_C$. 

5.2 The moments of the cluster probability distribution

The moments of the cluster probability distribution are prescribed by the derivatives of the generating function \( g \) with respect to the "field" \( h \). Specifically

\[
\langle s^n \rangle = \int ds s^n p(s, L, \xi) = (-1)^{n+1} [ P_n(L, \xi) - \delta_{n,0} ]
\]

(5.11a)

where we define

\[
P_n(L, \xi) = \frac{\partial^n}{\partial h^n} g(h, L, \xi) \bigg|_{h=0}
\]

(5.11b)

An equation for the functions \( P_n \) is easily derived from the integral equation (5.9), as discussed for the corresponding Potts problem in Section 4.3. Neglecting terms which modify amplitudes by multiplicative factors of order \( 1 + \Psi_0 \) we find (for \( n > 0 \)):

\[
P_n(L, \xi) = \exp \left[ - (n+1) \int L \psi(R, \xi) \right] \cdot \int L \psi(R, \xi) \cdot \exp \left[ - \int R \psi(R', \xi) \right] \]

(5.12)

with

\[
P_0(L, \xi) = \exp \left[ - \int L \psi(R, \xi) \right]
\]

(5.13a)

which, with the aid of eqns (5.4) and (5.11), we identify with...
the order parameter $P$. The implied behaviour near the percolation threshold follows immediately

$$\mathcal{P}_0(L_0, \xi) \sim (L_0 / \xi)^{\beta/\nu}$$  \hspace{1cm} (5.13b)

with

$$\beta/\nu = \psi_0$$  \hspace{1cm} (5.13c)

The behaviour of the mean cluster size follows as the second moment of the number distribution which one may readily obtain from eqn (5.12):

$$\langle s \rangle = \int_0^\infty ds \ s \ p(s, L_0, \xi) = \mathcal{P}_1(L_0, \xi) \sim (\xi / L_0)^{\gamma/\nu}$$  \hspace{1cm} (5.14a)

where

$$\gamma/\nu = d - 2\psi_0 = d - 2(\beta/\nu)$$  \hspace{1cm} (5.14b)

Both (5.13c) and (5.14b) have interesting implications for the Ising problem to which we shall return in the final chapter. The behaviour of the general $n$-th moment follows in a similar way:

$$\mathcal{P}_n(L_0, \xi) \sim (\xi / L_0)^{-\frac{n+1}{\psi_0} - \frac{n}{\xi}}$$  \hspace{1cm} (5.15)

The fact that the exponent of the $n$-th order term varies linearly with $n$ guarantees scaling for the generating
function. Specifically, with eqns (5.11) and (5.15) we find

\[
g(h, L, \bar{g}) = \sum_{n=0}^{\infty} \mathcal{T}_n(L_0, \bar{g}) \frac{h^n}{n!} = \bar{g}^{-d+\beta \delta/\nu} \tilde{g}(h \bar{g}^{\beta \delta/\nu})
\]

(5.16a)

with

\[
\delta = \frac{a}{\psi_0} - 1 = \frac{\bar{g}}{\beta} + 1
\]

(5.16b)
5.3 The structure of the cluster probability distribution

To investigate the form of the cluster probability distribution we substitute into the defining relation

\[ p(s, L_0, \xi) = \frac{i}{2\pi} \int dh \, e^{ish} (1 - g(ih, L_0, \xi)) \]  

(5.17)

the integral representation for the generating function, eqn (5.9). Performing the integral on \( h \) we find

\[ p(s, \nu_0, \xi) = p_0(s, \xi) \Theta(s - \nu_0) + \]
\[ + s d^{-1} \int_{\nu_0}^{\infty} dv v^{-2} \Phi(v, \xi)[p(s + v, \nu, \xi) - p(s, \nu, \xi)] \]  

(5.18a)

We have changed the variable of integration from \( R \) to \( v(R) \) and, correspondingly, have chosen to reparameterise the short distance cutoff implicit in the probability distribution by a minimum volume scale \( \nu_0 = v(L_0) \) rather than the minimal length scale \( L_0 \) itself. We have further performed a partial integration on \( h \) and neglected the integrated term; this is justified because the probability distribution vanishes in the limit of very large clusters. We have also introduced the abbreviations

\[ p_0(s, \xi) := (ds)^{-1} \Phi(s, \xi) \]
\[ \Phi(\nu, \xi) = \tilde{\Psi} [ (\frac{\nu d}{s_d})^{1/d} / \xi ] \]  

(5.18b)

Equation (5.18a) is the central result of this section. It can
be solved analytically in the regions of sufficiently small, and sufficiently large $s$ values, as we now proceed to show.

We begin by recasting the equation in the differential form

$$\frac{\partial}{\partial \upsilon} p(s, \upsilon, \xi) = -p_0(s, \xi) \delta(s - \upsilon) -$$

$$- \frac{1}{d} \upsilon^{-1} s \phi(\upsilon, \xi) \left[ p(s + \upsilon, \upsilon, \xi) - p(s, \upsilon, \xi) \right]$$

Now we can observe that in the "scaling" regime characterised by $s$ values large compared to the ultraviolet cutoff $\upsilon$ the difference on the right hand side of eqn (5.19) may be replaced by a derivative with respect to the cluster size, yielding the partial differential equation

$$\frac{\partial}{\partial \upsilon} p(s, \upsilon, \xi) = - \frac{1}{d} \upsilon^{-1} s \phi(\upsilon, \xi) \frac{\partial}{\partial s} p(s, \upsilon, \xi)$$

(5.20)

The solutions to this equation are of the form

$$p(s, \upsilon, \xi) = p_\xi \left( \frac{s}{P_0(\upsilon, \xi)} \right)$$

(5.21)

where the functional form of $p_\xi$ remains to be determined and $P_0(\upsilon, \xi)$ obeys the differential equation

$$\frac{\partial}{\partial \upsilon} P_0(\upsilon, \xi) = (d \upsilon)^{-1} \phi(\upsilon, \xi) P_0(\upsilon, \xi)$$

(5.22)

which is recognisable as the equation satisfied by the order parameter eqn (5.13) with the ultraviolet cutoff
reparameterised as indicated above.

Thus far in the argument we have invoked no approximation other than that imposed by the requirement \( s \gg v \) needed to justify the replacement of (5.19) by (5.20). Now let us focus specifically on the region of large \( s \). By "large" we mean, in the first instance, \( s \) values for which \( s \gg v(E) \sim \epsilon \), although this condition will subsequently need to be refined. Given this additional constraint, one sees, firstly, that eqn (5.21) may be invoked even for \( v \) values as large as \( \alpha v(E) \), with \( \alpha \) some suitably small constant. Thus, explicitly,

\[
p(s, \alpha v(\xi), \xi) = p_x \left[ \frac{s}{\mathcal{P}_0(\alpha v(\xi), \xi)} \right]
\]  

(5.23a)

Moreover, we see from eqns (5.18a) and (5.13a) that, to within exponentially small corrections

\[
p(s, \alpha v(\xi), \xi) = p_0(s, \xi)
\]  

(5.23b)

\[
\mathcal{P}_0(\alpha v(\xi), \xi) = 1
\]  

(5.23c)

Matching equations (5.23) we conclude that

\[
p_x(s) = p_0(s, \xi)
\]  

(5.24)

whence, substituting back into eqn (5.21) and setting \( v = v_0 \)
\[ \rho(\xi, \nu_0, \xi) = \rho_0(\xi / \tilde{P}_0(\nu, \xi), \xi) \quad \xi \gg \xi \] (5.25)

Let us make the form of this result more explicit by recalling the large argument behaviour of the function $\psi$, eqn (3.73), obtaining finally

\[ \rho(\xi, \nu_0, \xi) = \psi(\infty)(d\nu(\xi))^{-1} \tilde{s}^{1-\Theta'} \exp\left\{ -\frac{C_d}{\xi_0} \tilde{s}^{\frac{d-1}{d}} \right\} \] (5.26a)

where

\[ \Theta' = \left( 1 + 4d - d^2 \right) / 2d \] (5.26b)

and $\tilde{s}$ is a scaled cluster size variable

\[ \tilde{s} = s / \left[ \tilde{P}_0(\nu_0, \xi) \nu(\xi) \right] \] (5.26c)

Now let us consider the regime of "small" $s$ values - that is, in the first instance, $s \ll v(\xi)$. In this case the differential equation (5.20) and the functional form (5.21) which it implies for the cluster probability distribution cannot be trusted in the region $v \ll v(\xi)$, since the "differential" approximation made in replacing (5.19) by (5.20) clearly breaks down at $v = s \ll v(\xi)$. Accordingly, we must search for some other regime of $v$ values which does lie within the domain of validity of the differential equation (5.20), and for which we may prescribe the form of the probability distribution. This aim is realised with the choice $v = \alpha s$, with $\alpha$ some suitably small constant. Clearly, such a choice guarantees the
validity of the differential approximation, and thus of the form (5.21), for this ν value:

\[ p(\xi, \alpha \xi, \xi) = p_\xi(s / \rho_\nu(\alpha \xi, \xi)) \]  

(5.27a)

Moreover we assert that for such a choice the probability distribution has the form

\[ p(\xi, \alpha \xi, \xi) = A(\alpha) / s \]  

(5.27b)

where \( A(\alpha) \) is some \( \alpha \)-dependent amplitude which we need not prescribe. To justify this assertion we consider an alternative approach to the solution of eqn (5.18): one may proceed to develop a hierarchy of approximate solutions to this equation by iteration in powers of the concentration function \( \psi \), taking as the first order solution the function

\[ p_1(s, \nu_0, \xi) = p_0(s, \xi) \theta(s - \nu_0) \]  

(5.28a)

which, in the small \( s \) region, simplifies to

\[ p_1(s, \nu_0, \xi) \approx (ds)^{-1} \psi_0 \theta(s - \nu_0) \]  

(5.28b)

Inserting this solution into (5.18) and iterating once yields, in the small \( s \) region:

\[ p_2(s, \nu_0, \xi) = \frac{\psi_0}{s(d - \psi_0)} \left[ 1 + \frac{\psi_0}{d} \ln \frac{\nu_0}{s} + O\left(\frac{s}{\nu_0(\xi)}\right) \right] \]  

(5.28c)
One discovers that the iteration procedure generates a perturbation expansion in powers of $\psi_0$. This expansion may be recognised as the $\psi_0$ expansion of a power law in $s$; however, for the present purposes it is sufficient to note that replacing the ultraviolet cutoff $v_0$ by $\alpha$ quells the logarithmically divergent corrections to the leading analytic behaviour displayed by the first order solution (5.28a), so that these corrections merely renormalise the amplitude of the $s^{-1}$ behaviour. Eqn (5.27b) is thus vindicated. With eqns (5.27a) and (5.27b) we find

\[ P_\xi(s | P_0(\alpha s, \xi)) \approx A(\alpha) / s \quad (5.29a) \]

Now, with the present parameterisation, the order parameter has the behaviour

\[ P_0(v, \xi) = B(v / \xi^d)^{\psi_0/d} \quad (5.29b) \]

where $B$ is a constant. In conjunction with eqn (5.29a) we find

\[ P_\xi(x) \approx A(\alpha) / \left[ \times B(\alpha / \xi^d)^{\psi_0/d} \right]^{d/(d-\psi_0)} \quad (5.29c) \]

Combining this result with eqn (5.21) we obtain

\[ P(s, v_0, \xi) \approx A(\alpha) (\alpha / v_0)^{-\psi_0(d-\psi_0) - d/(d-\psi_0)} s \quad (5.30) \]

The self-consistency of the argument requires that the
result be independent of the specific choice of the matching parameter $\alpha$ so that, finally

$$p(s, \nu_0, \xi) = A s^{-1} (\frac{s}{\nu_0})^{-\psi_0/(d-\psi_0)} ; \; s \ll \nu(\xi) \quad (5.31a)$$

where $A = A(1)$ is an amplitude which is independent of the ultraviolet cutoff $\nu_0$, but which this argument does not otherwise prescribe. In fact, the perturbative (iterative) approach discussed above, notably eqn (5.28c), shows that

$$A = \frac{\psi_0}{d - \psi_0} + O(\nu_0^2) \quad (5.31b)$$

Inspection of the two limiting cases (5.26) and (5.31) reveals that they are both consistent with the general scaling form

$$p(s, \nu_0, \xi) = \left(\frac{s}{\nu_0}\right)^{1-\tau} \tilde{p}\left(\frac{s}{(\nu_0 \xi)}\right) \quad (5.32a)$$

where

$$\tau = 2 + \frac{\psi_0}{(d - \psi_0)} \quad (5.32b)$$

and

$$s_{\xi} = \nu(\xi)^{-\psi_0} = S_d d^{-1} \xi^{-d-\beta/\nu} \quad (5.32c)$$

while $B_0$ is a non-universal parameter specifying the critical behaviour of the order parameter.
\[ P_0(\nu, \xi) = B_0 \xi^{\beta \nu} \]  

(5.32d)

Written in this way, the scaling function \( P \) is universal, given a specific convention for the definition of the correlation length.

Two remarks are now in order. Firstly we observe that one may establish the scaling form (5.32) more generally, and at the same time refine it somewhat, by substituting the scaling form of the function \( g \), eqn (5.16), into the defining relation for the probability distribution, eqn (5.17). The scaling form (5.32) is then found to hold in the critical region \( \xi >> \nu_0 \) where the moments scale according to eqn (5.15), up to an additional non-scaling and analytic contribution which contributes significantly only to the zeroth moment of the probability distribution function. Secondly, eqn (5.32) makes it plain that the dividing line between the "small s" and "large s" regions is set by the "typical cluster size" eqn (5.32c) rather than by \( \nu(\xi) \). This state of affairs is reflected in the form of the perturbation expansion discussed above through the appearance of terms which, though smaller than the leading contributions by powers of \( s/\nu(\xi) \), carry additional factors of \( \ln(s/\nu_0) \).

To complement the small \( s \) and large \( s \) solutions to eqn (5.18) which were determined analytically above we have obtained solutions for intermediate values of \( s \) by numerical integration of eqn (5.19), from the regime \( \nu >> \nu(\xi) \) where...
the first order perturbative solution \( p_1(s, v, \xi) \), eqn (5.28), is satisfactory down into the scaling regime \( v = v_0 \ll v(\xi) \). In performing the calculations we have set \( d = 2 \) and have chosen the parameter \( \psi_0 \) so as to reproduce the percolation exponent ratio \( \beta / \nu \), eqn (5.13c), as given by series expansions (Stauffer 1979): the value of \( \psi_0 \) is in fact prescribed by eqn (3.75) of our original droplet calculation, but the result depends so sensitively on \( \epsilon \) that the extrapolation to \( d = 2 \) is unreliable. We have also invoked eqn (3.70) to prescribe the value of the universal constant \( c_0 \).

The results of these calculations are presented in Fig. 6 which shows the logarithm of the scaling function \( \tilde{\varphi} \), defined by eqns (5.32), plotted as a function of the square root of the scaling variable:

\[
\tilde{s} = \frac{s}{\left[ T_0(v, \xi) \right]^{1/2}}
\]  

(5.33)

Clearly, the function evolves in a relatively structureless fashion between the limiting small \( \tilde{s} \) value, which is prescribed by the perturbative result (5.31b) and the large \( \tilde{s} \) behaviour

\[
\ln \tilde{\varphi}(\tilde{s}) \approx \tilde{s}^{1/2} - 0.804 \ln \tilde{s} + \text{const}
\]  

(5.34)

following from eqn (5.26). We remark, however, that the curvature of the function changes sign at \( \tilde{s} \approx 2 \). For higher \( \tilde{s} \) values the curvature is positive in accordance with the
Fig. 6
The negative of the logarithm of the $d = 2$ scaling function $\tilde{p}$ for the cluster probability distribution, plotted against the square root of the scaling variable $\tilde{s}$. 

\[
- \log \tilde{p}(\tilde{s})
\]
asymptotic form (5.34). For lower $\bar{s}$ values the curvature is negative. Indeed we believe that, as a function of $\bar{s}$, $\bar{p}$ approaches its $\bar{s} = 0$ value with infinite slope. Although we have not investigated this feature in detail, it seems likely that it should be attributed to the fact that, for fixed droplet scale size $R \ll E$, the droplet population function $\Phi(R, E)$ approaches $\psi_0$ non-analytically as a function of increasing $E$, though analytically as a function of decreasing $t \sim E^{-1/\nu}$.

Thus, we have obtained a closed equation, (5.18), for the cluster probability function from our theory. For clusters small compared to the typical cluster size $s_{\xi}$, eqn (5.32c), the distribution exhibits pure power law decay, eqn (5.31). For clusters large compared to $s_{\xi}$, the distribution exhibits an exponential fall-off, eqn (5.26), which bears out a prediction first made by Kunz and Souillard (1978), refining it in a number of instances. Specifically, the $s^{(d-1)/d}$-dependence of the argument of the exponential is seen to hold, in our theory, arbitrarily close to the percolation threshold, a region where the Kunz-Souillard theorem was not formally proven. This result confirms a conjecture by Klein and Stauffer (1980) and goes some way to allaying the doubts expressed on this issue by Nakanishi and Stanley (1980) on the basis of Monte Carlo studies to which we shall shortly return. As a corollary we see that the Kunz-Souillard form appears with parameters whose $E$-dependence ensures consistency with scaling (cf. also a renormalisation argument.
valid near $d = 6$, cited by Harris and Lubensky (1981)). The form of the exponential can be understood in terms of the underlying thermal problem if we note that

$$\xi^{(d-1)/d} \sim \xi^{-(d-1)} (s \xi^{\beta/\nu})^{(d-1)/d}$$

(5.35)

Recalling eqn (3.69), the first factor may be identified as a measure of surface tension; the factor $s \xi^{\beta/\nu}$ is a measure of the volume of space contained within the boundaries of a droplet of connected volume $s$ and fractal dimension (Mandelbrot 1982, Kapitulnik et al. 1983)

$$d_\nu = d - \beta/\nu$$

(5.36)

The factor $(s \xi^{\beta/\nu})^{(d-1)/d}$ thus measures the external surface area of such a droplet, and the argument of the exponential is simply the classical free energy of an essentially spherical droplet containing the prescribed connected volume.

Our calculation also captures the power law prefactor structure associated with the asymptotic (large $s$) form. This result, specifically the prediction for the exponent $\theta'$, eqn (5.26b), agrees with a calculation by Lubensky and McKane (1981). Their study, like our own, exploits the Potts representation of the percolation problem, and traces the asymptotic form of the distribution to the effects of large clusters ("droplets"). Their calculation, however, does not incorporate the critical (droplet-nesting) effects which
produce a properly scaling form, and thus yields simply the first order solution (5.28a) to our fundamental equation (5.18). On the other hand, the price we have paid to control critical point effects, i.e. the dependence upon dimensional regularisation in the calculation of $\psi(R,E)$, shows up in the failure of our equation (5.26b) to detect the anomaly in the exponent $\theta'$ in $d = 3$, observed by Lubensky and McKane. However, it is remarkable that the dimensionally regularised calculation proves correct, in this respect, for $d < 3$.

Finally, let us return to discuss the Monte Carlo studies of the cluster probability distribution by Nakanishi and Stanley (1980). At first sight, their calculations appear to be inconsistent with the Kunz-Souillard form; specifically, close to the percolation threshold a plot of minus the logarithm of the cluster probability versus the square root of the cluster number shows significant deviations from the linear behaviour expected for the "large" cluster numbers, in $d = 2$, and confirmed by the present calculation. A number of comments are in order here. Firstly, the existence of the prefactor structure in eqn (5.26a) shows that such a plot will have a finite intercept with the $S = 0$ axis and a positive curvature (decreasing as $1/\sqrt{S}$), if extrapolated back to $S = 0$. The former feature is indicated in the $d = 2$ Monte Carlo data but the curvature apparent in the data would appear to be negative. However, given the observed limiting slope of the data and using eqn (5.34) one may estimate the $E$ value appropriate for the $p$ values studied.
and estimate the range of values of the scaling variable \( \tilde{s} \), eqn (5.33), actually studied in the Monte Carlo simulation: one finds that the troublesome data is associated with values smaller than that at which the curvature in \(-\ln \bar{P}(\tilde{s})\) (as given by the present theory) changes sign. It thus seems quite likely that the Monte Carlo results are in fact consistent with the Kunz-Souillard form. However, a more precise comparison of the Monte Carlo data and the present predictions requires the determination of the (still unknown) amplitude connecting \( p_c - p \) and \( \xi \); one could thus fix the scale parameter between the observed cluster numbers \( s \) and the scaling variable \( \tilde{s} \), which is still adjustable at present.

Let us then conclude with an assessment of the reliability of the results summarised in Fig. 6. The behaviour of the scaling function for small cluster sizes depends significantly on the "dilute boundary" approximation at the heart of our theory. This dependence is somewhat mitigated, since we have parameterised \( \phi_0 \) by the (presumably) correct series value of the percolation ratio \( \beta/\nu \); still, the reliability of our prediction in this region remains open to question. By contrast, the validity of the predicted form for large clusters, specifically the structure of the exponential, very probably transcends that of the "dilute boundary" approximation. The \( s \)-dependent prefactor structure in \( \bar{P}(\tilde{s}) \) is also likely to be trustworthy in \( d = 2 \) where, we believe, our calculation correctly handles the droplet zero modes which control this factor.
CHAPTER 6
CONCLUDING REMARKS

In this thesis, we have presented a droplet theory for the Potts universality class, designed to describe the configurational properties of a system consisting of $q$ phases, on and close to its coexistence curve.

The theory is based on the configurational energy for a single droplet of one phase, embedded in a background of different phase, as prescribed by the sum of surface tension and external field energy.

In order to capture critical point effects, the theory allows for droplet shape fluctuations and droplet nesting. The effects of droplet shape fluctuations can be controlled, for dimensionality close to one, in a renormalisation group context, and we have seen that the resulting single droplet partition function is dominated by essentially spherical droplets whose scale sizes are typically smaller than the correlation length $\xi$. In order to establish a description of the many droplet assembly, one has to recognise in particular the self-similar structure of the system in the critical limit which implies that droplets of any one phase may house subdroplets of the other $q-1$ phases. Near $d = 1$, the droplet boundaries can be shown to remain controllably dilute, and consequently the problem of droplet nesting can be handled,
resulting in an expression for the ensemble free energy of the Potts universality class.

Invoking the Kasteleyn-Fortuin relation, the droplet representation of the Potts free energy may be utilised to derive an expression for the generating function of the cluster probability distribution in the bond percolation problem. Both the Potts free energy and the percolation generating function can be cast in a properly scaling form, with exponents which satisfy all the standard scaling laws.

One deficiency of our theory is immediately apparent and has been alluded to repeatedly in the above work: the droplet theory predicts the same exponent $v$ for systems with different $q$, which is at variance with established results. In two dimensions, the percolation ($q = 1$) value is $v = 1.35$ (see Stauffer 1979), while for the Ising ($q = 2$) case we have the exact result $v = 1$, and for $q = 3$ and $4$ we have $v = 0.85$ and 0.77 respectively (see Wu 1982). Also, as we have seen, $v$ does display a $q$-dependence in the Migdal-Kadanoff result, eqn (4.44).

The $q$-independence of $v$ in the droplet theory is a consequence of the fact that the index is derived from the behaviour of an artificially isolated interface; no account has been taken of the effects which the presence of other droplets might have on the interface. These effects include, for instance, the dressing of an interface separating two
phases with droplets of different phase (Selke and Pesch 1982, Selke and Huse 1983). As a consequence, one would expect $v$ and $\psi_o$ to pick up $q$-dependent corrections. More specifically, the exponent $v$ might be corrected by a factor $1 + O((q-2)\psi_o)$, originating in the dressing of an existing interface with droplets of the other $q-2$ phases. This structure seems indeed to be reflected in the Migdal-Kadanoff result for $v$, eqn (4.44). In any case, we expect the $q$-dependent corrections to $v$ and $\psi_o$ to increase with increasing $q$, and thus the interface-based calculations may well be most reliable in the low $q$ regime.

Let us now turn to the results for the exponent $\beta$, eqns (4.17b) and (5.13c). They illustrate the percolation aspects of the Ising phase transition: As the critical point is approached from below, in the Ising model, one finds that the volume of that portion of the originally homogeneously ordered phase left after decoration by the droplet hierarchy vanishes as $\xi^{-(\beta/v)_I}$ (cf. eqn (4.29b) with $q = 2$, and $R \to \infty$ in the critical limit) where

$$ (\beta/v)_I' = \frac{1}{2} (\beta/v)_I $$

Thus, with eqn (4.17b) for $q = 2$ and eqn (5.13c) we find that

$$ (\beta/v)_I' = (\beta/v)_p $$

and hence recognise that, implicit in the droplet
representation of the Ising problem, is a pure percolation problem in which the role of the infinite cluster is played by the residual undecorated volume of the ground state. This statement is further underlined by the results for the Ising droplet size exponent (eqn (4.31b), $q = 2$):

\[
(\theta / \nu)_I = d - (\beta / \nu)_I \quad (6.2a)
\]

which, combined with the scaling result for the cluster size exponent in percolation, eqn (5.14b), reveals that

\[
(\theta / \nu)_I = (\gamma / \nu)_p \quad (6.2b)
\]

Thus the mean droplet size (prescribed by the second moment of the droplet size distribution, eqn (4.31a)) in the Ising problem diverges with the same power of the thermal correlation length as does the mean cluster size in the percolation problem as a function of the connectedness length.

However, the link between Ising and percolation behaviour is not as simple as these results would suggest. Certainly, the Ising critical point is a percolation point in $d = 2$ (Coniglio et al. 1977), and the predictions (6.1a) and (6.2a) are consistent with the existing numerically established results (cf. Sykes and Gaunt 1976, Jan et al. 1982, and further discussion in Bruce and Wallace 1983). But the equations proffering explicit links between percolation
and Ising exponents, eqns (6.1b) and (6.2b), are irreconcilable with series-based results for percolation exponents (see Stauffer 1979). One might envisage a refined theory in which the droplet decoration of interfaces gives a $q$-dependent renormalisation of $v$ and $\psi$, while leaving the structure of the theory of the many droplet assembly essentially unaltered: the identifications (6.1a) and (6.2a) would then survive, whereas the relationships between exponents associated with systems of different $q$, eqns (6.1b) and (6.2b), would be modified.

In summary, the droplet theory presented here constitutes a configuration-based study of Potts (and Ising) systems below the critical point, and of the bond percolation problem above the percolation threshold. The theory is technically constrained to $1+\epsilon$ dimensions, due to the necessity of controlling critical fluctuations. As a consequence, the direct numerical evidence for critical exponents is not very reliable; however, we have seen that some of the more general predictions emerging from the theory do transcend this limitation and may be trusted in $d = 2$. Thus, the major value of the theory is to be found at a conceptual level, where it helps to illuminate the universal configurational features underlying critical point behaviour.
APPENDIX A

GENERAL SPHERICAL HARMONIC FUNCTIONS

Consider Laplace's equation in d dimensions:

$$\Delta g(x) = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} g(x) = 0 \quad (A.1)$$

We are looking for a solution $g(x)$ which is homogeneous of degree 1:

$$g(\lambda x) = \lambda^l g(x) \quad (A.2)$$

which implies that $g(x)$ can be written in the form

$$g(x) = r^l S_l(\eta) \quad (A.3)$$

where $r = |x|$ and $\eta = x/r$ and the function $S_l(\eta)$ is defined on the unit sphere.

Now consider the decomposition of $\Delta$ into a radial and an angular part:

$$\Delta = \frac{d^2}{dr^2} + \frac{d-1}{r} \frac{d}{dr} + \frac{l^2}{2r^2} \quad (A.4)$$

with

$$L^2 = L_{ij} L_{ij} ; \quad L_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$$

Inserting eqn (A.3) into Laplace's equation (A.1), with the decomposition (A.4), yields
\[ L^2 S_\ell (\gamma) = -2 \ell (\ell + d - 2) S_\ell (\gamma) \]  \hspace{1cm} (A.5)

and thus \( S_1(\gamma) \) is an eigenfunction of \( L^2 \) with eigenvalue 
\(-2(1+d-2)\).

For a given \( \ell \), there exist several such functions \( S_1(\gamma) \) which 
are linearly independent, and one may thus construct a basis 
of orthonormalised functions \( Y_\ell^\gamma(\gamma) \) so that 
\[ \int d\Omega Y_\ell^\alpha(\gamma) Y_{\ell'}^{\alpha'}(\gamma) = \delta_{\alpha\alpha'} \delta_{\ell\ell'} \]  \hspace{1cm} (A.6)

The basis functions \( Y_1^\gamma(\gamma) \) are called general spherical harmonic 
functions of degree 1 in \( d \) dimensions.

A particular representation of the \( Y_1^\gamma \)'s may be constructed as 
follows: Due to (A.2), \( g(x) \) can be written in the form

\[ g(x) = \nu \ell u_{i_1} \cdots i_\ell \eta_{i_1} \cdots \eta_{i_\ell} \]  \hspace{1cm} (A.7)

If \( g(x) \) is to satisfy eqn (A.1), the coefficients \( u_{i_1} \cdots i_\ell \) are 
symmetric traceless tensors of rank 1. The degeneracy \( \nu_1(d) \) 
of the spherical harmonics of degree 1 is then given by the 
number of linearly independent symmetric traceless tensors 
of rank 1:

\[ \nu_1(d) = \frac{\Gamma(\ell+d)}{\Gamma(d) \Gamma(\ell+1)} - \frac{\Gamma(\ell+d-2)}{\Gamma(d) \Gamma(\ell-1)} \]  \hspace{1cm} (A.8)

\[ = \frac{(2\ell+d-2)! \Gamma(\ell+d-2)}{\Gamma(d-1) \Gamma(\ell+1)} \]

It is possible to find a set of tensors \( u_{i_1} \cdots i_\ell \) such that the 
functions \( u_{i_1} \cdots i_\ell \eta_{i_1} \cdots \eta_{i_\ell} \) form an orthonormal basis for the 
space spanned by the functions \( S_1 \). Let us consider the cases
$l = 0, 1, 2$ which are important for the calculations presented in Chapter 3.

The case $l = 0$ is trivial. There exists only one spherical harmonic $Y_0^0$, which is a constant, and by orthonormality we find

$$Y_0^0 = S_d^{-1/2}$$

(A.9)

For $l = 1$, there are $d$ linearly independent functions $Y_1^\alpha(\eta)$, and clearly, the choice

$$Y_1^\alpha(\eta) = \sqrt{\frac{1}{d}} S_d \eta_\alpha$$

(A.10)

satisfies our requirements.

Finally, for $l = 2$, the $Y_2^\alpha(\eta)$'s have the form

$$Y_2^\alpha(\eta) = u_{ij}^{(\alpha)} \eta_i \eta_j$$

(A.11)

Orthonormality requires the relation

$$\delta_{\alpha \alpha'} = \int d\Omega \ Y_2^\alpha(\eta) Y_2^{\alpha'}(\eta) = u_{ij}^{(\alpha)} u_{pq}^{(\alpha')} \int d\Omega \eta_i \eta_j \eta_p \eta_q$$

$$= \frac{u_{ij}^{(\alpha)} u_{pq}^{(\alpha')}}{d(d+2)} (\delta_{ij} \delta_{pq} + \delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp})$$

which has been obtained with the help of Appendix B. Thus we find

$$\delta_{\alpha \alpha'} = \frac{S_d}{d(d+2)} \left[ u_{ii}^{(\alpha)} u_{pp}^{(\alpha')} + u_{ij}^{(\alpha)} u_{ij}^{(\alpha')} + u_{ij}^{(\alpha)} u_{ij}^{(\alpha')}ight]$$

$$= \frac{2 S_d}{d(d+2)} u_{ij}^{(\alpha)} u_{ij}^{(\alpha')}$$

(A.12)
which we have used in the derivation of eqn (3.22).

Let us return to the general case. Another important property of the spherical harmonics which we need for the calculation of the two loop graphs is the addition theorem:

\[
\sum_{\ell=1}^{\ell(d)} Y_\ell^m(\eta) Y_\ell^m(\eta') = \frac{\psi_{\ell(d)}(\eta)}{S_d} \frac{C_\ell^\lambda(\eta \eta')}{C_\ell^\lambda(1)} \tag{A.13a}
\]

where \( \lambda = (d-2)/2 \) and \( S_d \) denotes the volume of the \( d \)-dimensional unit sphere

\[
S_d = \frac{2\pi^{d/2}}{\Gamma(d)} \tag{A.13b}
\]

and the \( C_1^\lambda(t) \) are Gegenbauer polynomials of degree 1 and order \( \lambda \), with the normalisation

\[
C_1^\lambda(1) = \frac{\Gamma(\ell+2\lambda)}{\Gamma(\ell+1) \Gamma(2\lambda)} \tag{A.13c}
\]

For a more complete treatment, and proofs of our formulae, we refer to Erdélyi (1953).
APPENDIX B

SYMMETRIC INTEGRATION ON THE SPHERE

In the course of our work, we repeatedly encounter integrals of the type

\[ \int d\Omega \, \eta_i \cdot \eta_j \cdot \ldots \cdot \eta_e \]  

(B.1)

Let us evaluate two examples in order to demonstrate the principle. Firstly, by symmetry, we have

\[ \int d\Omega \, \eta_\alpha \, \eta_\beta = I \, \delta_{\alpha \beta} \]  

(B.2a)

With

\[ \int d\Omega \, \eta_\alpha \, \eta_\alpha = S_d \]

\[ = I \, \delta_{\alpha \alpha} = d \cdot I \]

we find that

\[ I = S_d / d \]

and hence

\[ \int d\Omega \, \eta_\alpha \, \eta_\beta = (S_d / d) \, \delta_{\alpha \beta} \]  

(B.2b)

Secondly, again by symmetry, one may write

\[ \int d\Omega \, \eta_i \, \eta_j \, \eta_p \, \eta_q = I' (\delta_{ij} \delta_{pq} + \delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}) \]  

(B.3a)
A special case of (B.3a) can be easily evaluated:

\[ \int d\Omega \left( \eta_1^2 + \ldots + \eta_d^2 \right)^2 = \mathcal{S}_d \]

\[ = I' d(d+2) \]

From the last equation follows that

\[ I' = \mathcal{S}_d / [d(d+2)] \]

and hence

\[ \int d\Omega \eta_i \eta_j \eta_k \eta_l = \frac{\mathcal{S}_d}{d(d+2)} \left( \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right) \quad (B.3b) \]

To conclude, we note that integrals over an odd number of \( \eta_i \)'s are identically zero. Other integrals over even numbers of \( \eta_i \)'s can be evaluated in analogy to the two examples discussed explicitly above.
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